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Bloch functions and asymptotic tail variance ☆

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A B S T R A C T

Let $\mathbf{P}$ denote the Bergman projection on the unit disk $\mathbb{D}$,

$$\mathbf{P}\mu(z) := \int_{\mathbb{D}} \frac{\mu(w)}{(1-z\bar{w})^2}\,\mathrm{d}A(w), \qquad z \in \mathbb{D},$$

where $\mathrm{d}A$ is normalized area measure. We prove that if $|\mu(z)| \le 1$ on $\mathbb{D}$, then the integral

$$I_\mu(a,r) := \int_0^{2\pi} \exp\left\{ a\frac{r^4|\mathbf{P}\mu(re^{\mathrm{i}\theta})|^2}{\log\frac{1}{1-r^2}} \right\} \frac{\mathrm{d}\theta}{2\pi}, \qquad 0<r<1,$$

has the bound $I_\mu(a,r) \le C(a) := 10(1-a)^{-3/2}$ for $0<a<1$, irrespective of the choice of the function μ. Moreover, for $a>1$, no such uniform bound is possible. We interpret the theorem in terms the *asymptotic tail variance* of such a Bergman projection $\mathbf{P}\mu$ (by the way, the asymptotic tail variance induces a seminorm on the Bloch space). This improves upon earlier work of Makarov, which covers the range $0<a<\frac{\pi^2}{64}=0.1542\ldots$. We then apply the theorem to obtain an estimate of the universal integral means spectrum for conformal mappings with a k-quasiconformal extension, for $0<k<1$. The estimate reads, for $t \in \mathbb{C}$ and $0<k<1$,

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$$\mathrm{B}(k,t) \le \begin{cases} \frac{1}{4}k^2|t|^2(1+7k)^2, & \text{for} \quad |t| \le \frac{2}{k(1+7k)^2}, \\ k|t| - \frac{1}{(1+7k)^2}, & \text{for} \quad |t| \ge \frac{2}{k(1+7k)^2}, \end{cases}$$

which should be compared with the conjecture by Prause and Smirnov to the effect that for real t with $|t| \le 2/k$, we should have $\mathrm{B}(k,t) = \frac{1}{4}k^2t^2$.

1. Introduction

1.1. Basic notation

We write $\mathbb{R}$ for the real line, $\mathbb{R}_+ :=]0, +\infty[$ for the positive semi-axis, and $\mathbb{C}$ for the complex plane. Moreover, we write $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$ for the extended complex plane (the Riemann sphere). For a complex variable $z = x + \mathrm{i}y \in \mathbb{C}$, let

$$\mathrm{d}s(z) := \frac{|\mathrm{d}z|}{2\pi}, \qquad \mathrm{d}A(z) := \frac{\mathrm{d}x\mathrm{d}y}{\pi},$$

denote the normalized arc length and area measures as indicated. Moreover, we shall write

$$\Delta_z := \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

for the normalized Laplacian, and

$$\partial_z := \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\right), \qquad \bar\partial_z := \frac{1}{2}\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right),$$

for the standard complex derivatives; then Δ factors as $\Delta_z = \partial_z\bar\partial_z$. Often we will drop the subscript for these differential operators when it is obvious from the context with respect to which variable they apply. We let $\mathbb{C}$ denote the complex plane, $\mathbb{D}$ the open unit disk, $\mathbb{T} := \partial\mathbb{D}$ the unit circle, and $\mathbb{D}_e$ the exterior disk:

$$\mathbb{D} := \{z \in \mathbb{C} : \ |z| < 1\}, \qquad \mathbb{D}_e := \{z \in \mathbb{C}_\infty : \ |z| > 1\}.$$

More generally, we write

$$\mathbb{D}(z_0, r) := \{z \in \mathbb{C} : \ |z - z_0| < r\}$$

for the open disk of radius r centered at z_0.

1.2. Dual action notation

We will find it useful to introduce the sesquilinear forms $\langle\cdot,\cdot\rangle_{\mathbb{T}}$ and $\langle\cdot,\cdot\rangle_{\mathbb{D}}$, as given by

$$\langle f,g\rangle_{\mathbb{T}} := \int_{\mathbb{T}} f(z)\bar{g}(z)\mathrm{d}s(z), \qquad \langle f,g\rangle_{\mathbb{D}} := \int_{\mathbb{D}} f(z)\bar{g}(z)\mathrm{d}A(z),$$

where, in the first case, $f\bar{g}\in L^1(\mathbb{T})$ is required, and in the second, we need that $f\bar{g}\in L^1(\mathbb{D})$.

1.3. The Bergman projection of bounded functions and the main result

For a function $f\in L^1(\mathbb{D})$, its *Bergman projection* is the function $\mathbf{P}f$, as defined by

$$\mathbf{P}f(z) := \int_{\mathbb{D}} \frac{f(w)}{(1-z\bar{w})^2}\,\mathrm{d}A(w), \qquad z\in\mathbb{D}. \tag{1.3.1}$$

The function $\mathbf{P}f$ is then holomorphic in the disk $\mathbb{D}$. We shall be concerned with the boundary behavior of holomorphic functions of the type $\mathbf{P}\mu$, where $\mu\in L^\infty(\mathbb{D})$, in which case $\mathbf{P}\mu$ is in the *Bloch space* (see Subsection 2.1). More precisely, we shall obtain the following result.

Theorem 1.3.1. *Suppose $g=\mathbf{P}\mu$, where $\mu\in L^\infty(\mathbb{D})$, and $\|\mu\|_{L^\infty(\mathbb{D})}\le 1$.*

(a) *If $0<a<1$, we then have the estimate*

$$\int_{\mathbb{T}} \exp\left\{a\frac{r^4|g(r\zeta)|^2}{\log\frac{1}{1-r^2}}\right\}\mathrm{d}s(\zeta)\le C(a), \qquad 0<r<1,$$

where $C(a)=10(1-a)^{-3/2}$.

(b) *If $1<a<+\infty$, there exists a $\mu_0\in L^\infty(\mathbb{D})$ with $\|\mu_0\|_{L^\infty(\mathbb{D})}=1$ such that with $g_0:=\mathbf{P}\mu_0$,*

$$\lim_{r\to 1^-}\int_{\mathbb{T}} \exp\left\{a\frac{r^4|g_0(r\zeta)|^2}{\log\frac{1}{1-r^2}}\right\}\mathrm{d}s(\zeta)=+\infty.$$

The proof is supplied in two installments: part (a) in Corollary 7.3.5, and part (b) in Corollary 4.1.4.

In the terminology of Section 3 on two notions of asymptotic variance (in the context of probabilistic modeling), the main aspects of this result may be formulated as follows: *The (uniform) asymptotic tail variance of the unit ball in* $\mathbf{P}L^\infty(\mathbb{D})$ *equals* 1.

Remark 1.3.2. (a) At this moment, it is not clear what happens at the critical parameter value $a = 1$.

(b) For small values of a, $0 < a < \frac{\pi^2}{64} = 0.1542\ldots$, the same bound with a different constant $C(a)$ can be obtained from an estimate found by Nikolai Makarov [38] (for details, see Pommerenke's book [47], Chapter 8, as well as Subsection 3.1 below). Later, Bañuelos [6] found an independent localized approach involving square functions which for Bloch functions gave more or less the same growth estimate as the one originally found by Makarov.

1.4. Comparison with the Dirichlet integral theorem

In [13], Alice Chang and Donald Marshall improve upon a classical theorem of Arne Beurling from the 1930s (see [10]). Their result is that for a positive real parameter a, *there exists a uniform finite integral bound*

$$\int_{\mathbb{T}} \exp\left\{a|f(\zeta)|^2\right\}\mathrm{d}s(\zeta) \le C(a)$$

if and only if $0 < a \le 1$, where f ranges over all holomorphic functions $f : \mathbb{D} \to \mathbb{C}$ with $f(0) = 0$ and

$$\int_{\mathbb{D}} |f'|^2\mathrm{d}A \le 1.$$

The finiteness for $0 < a < 1$ was covered by Beurling's work. At the superficial level, this is very much reminiscent of Theorem 1.3.1 above. However, we can neither derive Beurling's theorem from Theorem 1.3.1, nor can we derive Theorem 1.3.1 from the theorem of Chang and Marshall. To understand this, we consider the relation

$$f(\zeta) := \frac{r^2\zeta^2 g(r\zeta)}{\sqrt{\log\frac{1}{1-r^2}}}. \tag{1.4.1}$$

We observe the following:

(i) If $g = \mathbf{P}\mu$ where $\|\mu\|_{L^\infty(\mathbb{D})} = 1$, then the function f extends holomorphically to a disk of radius $1/r$ and hence has no chance of being an arbitrary element of the unit ball of the Dirichlet space.

(ii) Assuming only that $g = \mathbf{P}\mu$ where $\|\mu\|_{L^\infty(\mathbb{D})} = 1$, we cannot control the Dirichlet norm of f uniformly as r approaches 1. Indeed, the Dirichlet integral of f is

$$\int_{\mathbb{D}} |f'|^2\mathrm{d}A = \frac{1}{\log\frac{1}{1-r^2}} \int_{\mathbb{D}(0,r)} |(z^2g)'|^2\mathrm{d}A, \tag{1.4.2}$$

and since the construction of holomorphic functions with given growth is quite precise in [49] (see also, e.g., [25]), and the error term supplied by Proposition 4.3.1 is small in terms of its boundary contribution, we may find such a function $g_1 = \mathbf{P}\mu_1$ with $\|\mu_1\|_{L^\infty(\mathbb{D})} \le 1$, whose derivative grows so quickly that

$$\int_{\mathbb{D}(0,r)} |(z^2 g_1)'|^2 \mathrm{d}A \ge \epsilon_0 \frac{r^4}{1-r^2},$$

for some absolute constant $\epsilon_0 > 0$. With this choice $g := g_1$, the growth of the expression (1.4.2) is then at least as quick as

$$\epsilon_0 \frac{r^4}{(1-r^2)\log\frac{1}{1-r^2}},$$

which definitely tends to infinity as $r \to 1^-$.

1.5. Applications to exponential integrability

It might be more appropriate to compare Theorem 1.3.1 with the John–Nirenberg theorem on exponential integrability of BMO functions. We should also have in mind the Helson–Szegő theorem [28], which gives sharp exponential integrability for the Szegő projection of a bounded function (see also Garnett's book [17], and e.g. Wolff's paper [53]).

We first begin from the wrong end. Note that by the pointwise bound of Lemma 2.2.2, we know that if $g = \mathbf{P}\mu$ with $\|\mu\|_{L^\infty(\mathbb{D})} \le 1$, then

$$\frac{r^4|g(r\zeta)|^2}{\log\frac{1}{1-r^2}} \le r^2|g(r\zeta)|,$$

so that

$$\int_{\mathbb{T}} \exp\left\{a\frac{r^4|g(r\zeta)|^2}{\log\frac{1}{1-r^2}}\right\}\mathrm{d}s(\zeta) \le \int_{\mathbb{T}} \exp\left\{ar^2|g(r\zeta)|\right\}\mathrm{d}s(\zeta), \tag{1.5.1}$$

and the uniform boundedness (over $0 < r < 1$) of the right-hand side for small positive values of a would be very reminiscent of the John–Nirenberg theorem [32], except that we would need the dilates g_r to be in $\mathrm{BMO}(\mathbb{T})$ uniformly, which is not true for a general function $g = \mathbf{P}\mu$ (indeed, it is easy to cook up a μ such that the right-hand side in (1.5.1) tends to infinity as we let $r \to 1^-$, for any fixed positive a). This of course fits with the inequality in (1.5.1), which goes the wrong way if we want to derive consequences of Theorem 1.3.1. To obtain an estimate that works, we instead follow Marshall [43] who obtained the inequality (see (3.4.2))

$$\int_{\mathbb{T}} \left|\mathrm{e}^{tr^2 g(r\zeta)}\right| \mathrm{d}s(\zeta) \le (1-r^2)^{-|t|^2/(4a)} \int_{\mathbb{T}} \exp\left\{a \frac{r^4 |g(r\zeta)|^2}{\log \frac{1}{1-r^2}}\right\} \mathrm{d}s(\zeta), \qquad t \in \mathbb{C}. \tag{1.5.2}$$

For $|t| > 2a$, a better estimate can be obtained from a combination of (1.5.2) with the pointwise bound of Lemma 2.2.2 below (see Proposition 3.5.1 below):

$$\int_{\mathbb{T}} \left|\mathrm{e}^{tr^2 g(r\zeta)}\right| \mathrm{d}s(\zeta) \le (1-r^2)^{a-|t|} \int_{\mathbb{T}} \exp\left\{a \frac{r^4 |g(r\zeta)|^2}{\log \frac{1}{1-r^2}}\right\} \mathrm{d}s(\zeta), \qquad |t| > 2a. \tag{1.5.3}$$

It is well-known that $g = \mathbf{P}\mu$ is the restriction to he disk $\mathbb{D}$ of a function in two-dimensional BMO. In two dimensions, the John–Nirenberg theorem would say that $\exp(\lambda g)$ is locally in area-L^1, if $|\lambda|$ is small. In particular, $\exp(\lambda g)$ is integrable on the disk $\mathbb{D}$, and an argument involving subharmonicity and averages over disks gives that

$$\int_{\mathbb{T}} \left|\mathrm{e}^{\lambda g(r\zeta)}\right| \mathrm{d}s(\zeta) = \mathrm{O}((1-r^2)^{-1}), \quad \text{as } r \to 1^-, \tag{1.5.4}$$

again for small $|\lambda|$. Compared with (1.5.4), the estimates (1.5.2) and (1.5.3) are much more precise.

1.6. The type spectrum of a Bloch function

We need the concept of the *exponential type spectrum* of the function e^g, where $g : \mathbb{D} \to \mathbb{C}$ is holomorphic.

Definition 1.6.1. For a holomorphic function $g : \mathbb{D} \to \mathbb{C}$, let $\beta_g : \mathbb{C} \to [0, +\infty]$ be the function given by

$$\beta_g(t) := \limsup_{r \to 1^-} \frac{\log \int_{\mathbb{T}} |\mathrm{e}^{tg(r\zeta)}| \mathrm{d}s(\zeta)}{\log \frac{1}{1-r^2}}.$$

We call the function $\beta_g(t)$ the *exponential type spectrum* of the (zero-free) function e^g.

We may now derive an estimate from above of the exponential type spectrum $\beta_g(t)$, where $g = \mathbf{P}\mu$ and $\mu \in L^\infty(\mathbb{D})$, from the estimates (1.5.2) and (1.5.3), together with Theorem 1.3.1.

Corollary 1.6.2. *Suppose $g = \mathbf{P}\mu$, where $\mu \in L^\infty(\mathbb{D})$, with $\|\mu\|_{L^\infty(\mathbb{D})} \le 1$. Then*

$$\beta_g(t) \le \begin{cases} |t|^2/4, & |t| \le 2, \\ |t| - 1, & |t| \ge 2. \end{cases}$$

The proof of Corollary 1.6.2 is supplied in Subsection 3.5.

1.7. Control of moments

Makarov originally formulated his result in terms of moments; Theorem 1.3.1 implies a bound on the moments as well.

Corollary 1.7.1. *Suppose that $g = \mathbf{P}\mu$, where $\mu \in L^\infty(\mathbb{D})$. For $0 < q < +\infty$, we then have the estimate*

$$\int_{\mathbb{T}} |g(r\zeta)|^q \mathrm{d}s(\zeta) \le 10(3+q)^{3/2}\|\mu\|^q_{L^\infty(\mathbb{D})}\left(\frac{q}{2\mathrm{e}}\right)^{q/2}\left(\frac{1}{r^4}\log\frac{1}{1-r^2}\right)^{q/2}, \qquad 0<r<1.$$

The proof of Corollary 1.7.1 is supplied in Subsection 7.4. We should remark that in [31], Ivrii and Kayumov show how to control low order moments (i.e., for $0 < q \le (\log\frac{1}{1-r^2})^\delta$ for small positive δ) in terms of the uniform asymptotic variance of the unit ball of $\mathbf{P}L^\infty(\mathbb{D})$, which is smaller than 1, by [22]. It would be natural to combine the two estimates, using, e.g., the logarithmic convexity of the moments with respect to q.

1.8. Application to the universal quasiconformal extension spectrum

The exponential type spectrum may be defined analogously for a holomorphic function $g : \mathbb{D}_e \to \mathbb{C}$ as well:

$$\beta_g(t) := \limsup_{R\to 1^+} \frac{\log \int_{\mathbb{T}} |\mathrm{e}^{tg(R\zeta)}|\mathrm{d}s(\zeta)}{\log\frac{R^2}{R^2-1}}. \tag{1.8.1}$$

We recall the class Σ of conformal mappings $\psi : \mathbb{D}_e \to \mathbb{C}_\infty$, with asymptotics $\psi(z) = z + \mathrm{O}(1)$ as $z \to \infty$. For a parameter k with $0 < k < 1$, we denote by $\Sigma^{\langle k\rangle}$ the collection of all $\psi \in \Sigma$ that have a k-quasiconformal extension $\tilde\psi : \mathbb{C}_\infty \to \mathbb{C}_\infty$, by which we mean that $\tilde\psi$ is a homeomorphism of Sobolev class $W^{1,2}$ with dilatation estimate

$$|\bar\partial_z\tilde\psi(z)| \le k|\partial_z\tilde\psi(z)|, \qquad z\in\mathbb{C}.$$

The universal spectra $\mathrm{B}(k,t)$ for $0 < k \le 1$ and $t \in \mathbb{C}$ are defined to be

$$\mathrm{B}(1,t) := \sup_{\psi\in\Sigma} \beta_{\log\psi'}(t), \qquad \mathrm{B}(k,t) := \sup_{\psi\in\Sigma^{\langle k\rangle}} \beta_{\log\psi'}(t).$$

As a consequence of Theorem 1.3.1, we obtain an estimate of the universal spectrum $\mathrm{B}(k,t)$, which should be compared with the conjecture by Prause and Smirnov [48] that $B(k,t) = \frac{1}{4}k^2t^2$ for real t with $|t| \le 2/k$. Indeed, for small k, the estimate comes very close to the conjectured value.

Theorem 1.8.1. *We have the following estimate:*

$$\mathrm{B}(k,t)\le\frac{1}{4}\,k^2|t|^2(1+7k)^2,\qquad 0<k<1,\ \ t\in\mathbb{C}.$$

The proof of this theorem is supplied in Subsection 8.5. In comparison, the estimate of Prause and Smirnov [48] applies only to real $t\ge t_k$, where $t_k:=2/(1+\sqrt{1-k^2})$.

Remark 1.8.2. An argument based on the pointwise estimate (8.5.7) combined with Theorem 1.8.1 and Hölder's inequality, like in the proof of Corollary 1.6.2, shows that the estimate of Theorem 1.8.1 can be improved for big values of $|t|$, for $t\in\mathbb{C}$:

$$\mathrm{B}(k,t)\le k|t|-\frac{1}{(1+7k)^2},\quad \text{for}\quad |t|\ge\frac{2}{k(1+7k)^2}. \tag{1.8.2}$$

We may of course also combine this (Theorem 1.8.1 and (1.8.2)) with the estimate of Prause and Smirnov [48], using the convexity of the mapping $\mathbb{C}\ni t\mapsto \mathrm{B}(k,t)$ (convexity results from Hölder's inequality). The result is a sharper estimate for complex t near the real interval $[t_k,+\infty[$, where $t_k=2/(1+\sqrt{1-k^2})$.

1.9. Application to the Minkowski dimension of quasicircles

If in the setting of the preceding subsection, we have a conformal mapping $\psi\in\Sigma^{\langle k\rangle}$ for some $0<k<1$, which means that ψ has a k-quasiconformal extension that maps $\mathbb{C}\to\mathbb{C}$, it is of interest to analyze the fractal dimension of the boundary $\Gamma_\psi:=\psi(\mathbb{T})$ in terms of k. The fractal dimension of a curve Γ can be measured by (i) the upper Minkowski (or box-counting) dimension $\dim_M^+(\Gamma)$, (ii) the lower Minkowski (or box-counting) dimension $\dim_M^-(\Gamma)$, and (iii) the Hausdorff dimension $\dim_H(\Gamma)$. It is well-known that these dimensions are related:

$$\dim_H(\Gamma)\le\dim_M^-(\Gamma)\le\dim_M^+(\Gamma),$$

where each inequality may be strict. Let us go to the level of universal dimension bounds:

$$D_{M,1s}^+(k):=\sup_{\psi\in\Sigma^{\langle k\rangle}}\dim_M^+(\Gamma_\psi),\quad D_{H,1s}(k):=\sup_{\psi\in\Sigma^{\langle k\rangle}}\dim_H(\Gamma_\psi);$$

clearly, we have $D_{H,1s}(k)\le D_{M,1s}^+(k)$. Here, "1s" stands for one-sided, because ψ is conformal inside the exterior disk $\mathbb{D}_e$ and k-quasiconformal off it. A symmetrization procedure which goes back to Reiner Kühnau [35] (used by Stanislav Smirnov in [51]) permits us to remove the one-sidedness, and to identify the curves Γ_ψ where $\psi\in\Sigma^{\langle k\rangle}$ as k'-quasicircles (a k'-quasicircle is the image of a circle under a k'-quasiconformal map) where

$$k=\frac{2k'}{1+(k')^2}. \tag{1.9.1}$$

This allows us to say that

$$D^{+}_{M,1s}(k) = D^{+}_{M}(k'), \quad D_{H,1s}(k) = D_{H}(k'), \tag{1.9.2}$$

where the right-hand expressions are the optimal universal dimension bounds without one-sidedness. A result of Kari Astala [3] says that these dimension bounds are all the same:

$$D^{+}_{M,1s}(k) = D^{+}_{M}(k') = D_{H,1s}(k) = D_{H}(k').$$

Corollary 1.9.1. *We have that* $D_H(k') = D^{+}_{M}(k') \le 1 + (k')^2 + \mathrm{O}((k')^3)$ *as* $k \to 0^+$.

This is weaker than Smirnov's [51] bound on the Hausdorff dimension: $D_H(k') \le 1 + (k')^2$. For completeness, the proof is supplied in Subsection 9.1. In [30], Oleg Ivrii shows that actually $D_H(k') = 1 + \Sigma^2 (k')^2 + \mathrm{O}((k')^{5/2}$ as $k' \to 0$, where Σ^2 equals the asymptotic variance of the unit ball of $\mathbf{P}L^\infty(\mathbb{D})$. His approach is through a strengthening of the estimate of Theorem 1.8.1 for small k and $|t|$ to $\mathrm{B}(k,t) \sim \frac{1}{4}\Sigma^2 k^2 |t|^2$, which is based on the traditional Hardy identity method. In the implementation, he observes in the given situation, he needs only control an L^2 norm of the nonlinearity instead of the L^∞ norm, which can then be done in terms of the variance Σ^2. In [22], Hedenmalm shows that $\Sigma^2 < 1$, and moreover, that the actual value of Σ^2 is connected with hyperbolic geometry analogues of the Bose–Einstein condensates considered by Abrikosov.

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2. The Bloch space and duality

2.1. The Bloch space and the Bloch seminorm

The *Bloch space*, which is named after André Bloch [11], consists of the holomorphic functions $g : \mathbb{D} \to \mathbb{C}$ subject to the seminorm boundedness condition

$$\|g\|_{\mathcal{B}(\mathbb{D})} := \sup_{z\in\mathbb{D}} (1 - |z|^2)|g'(z)| < +\infty. \tag{2.1.1}$$

If, for $\zeta \in \mathbb{D}$, ϕ_ζ denotes the involutive Möbius automorphism of $\mathbb{D}$ given by

$$\phi_\zeta(z) := \frac{\zeta - z}{1 - \bar{\zeta} z},$$

then

$$\|g \circ \phi_\zeta\|_{\mathcal{B}(\mathbb{D})} = \|g\|_{\mathcal{B}(\mathbb{D})}, \qquad \zeta \in \mathbb{D},$$

which is easily obtained from the equality

$$\frac{1 - |\phi_\zeta(z)|^2}{|\phi'_\zeta(z)|} = 1 - |z|^2.$$

Together with the rotations, these Möbius involutions ϕ_ζ generate the full automorphism group, which makes the Bloch seminorm invariant under all Möbius automorphisms of $\mathbb{D}$. The subspace

$$\mathcal{B}_0(\mathbb{D}) := \big\{g \in \mathcal{B}(\mathbb{D}) : \lim_{|z| \to 1^-} (1 - |z|^2)|g'(z)| = 0\big\}$$

is called the *little Bloch space*. We shall be concerned here with the extremal growth properties of Bloch functions, where functions in the little Bloch space are seen to grow too slowly. In other words, the properties will take place in the quotient space $\mathcal{B}(\mathbb{D})/\mathcal{B}_0(\mathbb{D})$. An immediate observation we can make at this point is the following.

Lemma 2.1.1. *If $g \in \mathcal{B}(\mathbb{D})$ with $g(0) = 0$, then g enjoys the growth estimate*

$$|g(z)| \le \|g\|_{\mathcal{B}(\mathbb{D})} \int_0^{|z|} \frac{\mathrm{d}t}{1 - t^2} = \frac{1}{2}\|g\|_{\mathcal{B}(\mathbb{D})} \log \frac{1 + |z|}{1 - |z|}, \qquad z \in \mathbb{D}.$$

Proof. This is immediate from the integral formula

$$g(z) = g(z) - g(0) = \int_0^z g'(\zeta)\mathrm{d}\zeta,$$

where the path is chosen to be the line segment connecting 0 with z. □

2.2. The Bloch space as the dual of the integrable quadratic differentials

To a holomorphic quadratic differential $f(z)\mathrm{d}z^2$ on the unit disk $\mathbb{D}$ we supply the norm

$$\|f\|_{A^1(\mathbb{D})} := \int_{\mathbb{D}} |f(z)|\mathrm{d}A(z),$$

and identify the holomorphic quadratic differentials with finite norm with the Bergman space $A^1(\mathbb{D})$ (cf. [52], p. 85). Here, slightly more generally, for $0 < p < +\infty$, we write

$A^p(\mathbb{D})$ for the Bergman space of all holomorphic functions $f:\mathbb{D}\to\mathbb{C}$ subject to the condition

$$\|f\|_{A^p(\mathbb{D})} := \bigg(\int_{\mathbb{D}} |f(z)|^p \mathrm{d}A(z)\bigg)^{1/p} < +\infty.$$

Holomorphic quadratic differentials appear naturally in the context of Teichmüller theory. If $\phi:\mathbb{D}\to\mathbb{D}$ is a Möbius automorphism, while $f\in A^1(\mathbb{D})$ and $\mu\in L^\infty(\mathbb{D})$ are given, then

$$\langle f,\mu\rangle_{\mathbb{D}} = \int_{\mathbb{D}} f\bar{\mu}\mathrm{d}A = \int_{\mathbb{D}} (f\circ\phi)\,(\bar{\mu}\circ\phi)\,|\phi'|^2\mathrm{d}A = \langle f_\phi,\mu_\phi\rangle_{\mathbb{D}}, \tag{2.2.1}$$

where

$$f_\phi := (\phi')^2 f\circ\phi, \qquad \mu_\phi := \frac{\phi'}{\bar{\phi}'}\mu\circ\phi, \tag{2.2.2}$$

so that while f transforms as a quadratic differential, on the dual side μ transforms as a $\mathrm{d}z/\mathrm{d}\bar{z}$-form. That is, μ reverses the complex structure. In fact, it acts to send a $(0,1)$-differential to a $(1,0)$-differential:

$$h(z)\mathrm{d}\bar{z} \;\mapsto\; \mu(z)h(z)\mathrm{d}z.$$

For this reason, it will not come as a great surprise to us that such dual elements $\mu\in L^\infty(\mathbb{D})$ are related with Beltrami equations and quasiconformal theory. Not all $\mu\in L^\infty(\mathbb{D})$ give rise to nontrivial linear functionals on the space $A^1(\mathbb{D})$. The nontrivial part of $\mu\in L^\infty(\mathbb{D})$ may be represented by its Bergman projection $\mathbf{P}\mu$, given by (1.3.1). On the other hand, trivial such $\mu\in L^\infty(\mathbb{D})$ appear in the context of quasiconformal deformation in, e.g., [2], Lemma 1. It is well-known that $\mathbf{P}$ acts boundedly on $L^p(\mathbb{D})$ for each p with $1<p<+\infty$, and that $\mathbf{P}$ maps $L^\infty(\mathbb{D})$ onto the Bloch space $\mathcal{B}(\mathbb{D})$ (this result is from Coifman, Rochberg, and Weiss [15]; see also, e.g., the book [24]). This suggests that we should equip the space $\mathcal{B}(\mathbb{D})\cong\mathbf{P}L^\infty(\mathbb{D})$ with the alternative norm

$$\|g\|_{\mathbf{P}L^\infty(\mathbb{D})} := \inf\big\{\|\mu\|_{L^\infty(\mathbb{D})}:\ \mu\in L^\infty(\mathbb{D}) \quad\text{and}\quad g=\mathbf{P}\mu\big\}. \tag{2.2.3}$$

When we do so, we write $\mathbf{P}L^\infty(\mathbb{D})$ for the Bloch space. As such, $\mathbf{P}L^\infty(\mathbb{D})$ is isometrically isomorphic with the dual space of $A^1(\mathbb{D})$, with respect to the dual pairing $\langle\cdot,\cdot\rangle_{\mathbb{D}}$, which needs to be understood in a generalized sense. The reason is that for $g\in\mathbf{P}L^\infty(\mathbb{D})$ and $f\in A^1(\mathbb{D})$, it might happen that $f\bar{g}\notin L^1(\mathbb{D})$, which would leave the dual action $\langle f,g\rangle_{\mathbb{D}}$ undefined. To remedy this defect, we consider the dilates $f_r(z):=f(rz)$ for $0<r<1$, so that if $g=\mathbf{P}\mu$ where $\mu\in L^\infty(\mathbb{D})$, we may *define*

$$\langle f,g\rangle_{\mathbb{D}} := \lim_{r\to 1^-}\langle f_r,g\rangle_{\mathbb{D}},$$

because then

$$\langle f, g\rangle_{\mathbb{D}} = \langle f, \mathbf{P}\mu\rangle_{\mathbb{D}} = \langle f, \mu\rangle_{\mathbb{D}}, \tag{2.2.4}$$

as we see from the following calculation, which also justifies the existence of the limit:

$$\langle f, g\rangle_{\mathbb{D}} := \lim_{r\to 1^-} \langle f_r, g\rangle_{\mathbb{D}} = \lim_{r\to 1^-} \langle f_r, \mathbf{P}\mu\rangle_{\mathbb{D}} = \lim_{r\to 1^-} \langle \mathbf{P}f_r, \mu\rangle_{\mathbb{D}} = \lim_{r\to 1^-} \langle f_r, \mu\rangle_{\mathbb{D}} = \langle f, \mu\rangle_{\mathbb{D}}.$$

Here, we use that the Bergman projection $\mathbf{P}$ is self-adjoint on $L^2(\mathbb{D})$ and preserves $A^2(\mathbb{D})$, and that we have the norm convergence $f_r \to f$ as $r \to 1^-$ in the space $A^1(\mathbb{D})$. In conclusion, *we have identified the dual space of $A^1(\mathbb{D})$ with the space $\mathbf{P}L^\infty(\mathbb{D})$, isometrically and isomorphically, where the dual action is given by the sesquilinear form $\langle\cdot,\cdot\rangle_{\mathbb{D}}$.*

Recently, Antti Perälä [45] obtained the following estimate.

Lemma 2.2.1. (Perälä) *We have the inequality*

$$\|\mathbf{P}\mu\|_{\mathcal{B}(\mathbb{D})} \le \frac{8}{\pi}\|\mu\|_{L^\infty(\mathbb{D})}, \qquad \mu \in L^\infty(\mathbb{D}),$$

where the constant $8/\pi$ is best possible.

In the other direction, a less precise argument (see, e.g., Proposition 4.3.1 below) shows that a function $g \in \mathcal{B}(\mathbb{D})$ with $\|g\|_{\mathcal{B}(\mathbb{D})} \le 1$ can be written in the form $g = \mathbf{P}\nu_g + G$, where $\nu_g \in L^\infty(\mathbb{D})$ and $G \in H^\infty(\mathbb{D})$, with the (semi)norm bounds $\|\nu_g\|_{L^\infty(\mathbb{D})} \le 1$ and $\|G\|_{H^\infty(\mathbb{D})} \le |g(0)| + 6$. As the sharpness of Perälä's estimate also comes from boundary effects, it would appear that modulo bounded terms, the unit ball of $\mathcal{B}(\mathbb{D})$ can be mapped into the unit ball of $\mathbf{P}L^\infty(\mathbb{D})$, whereas the unit ball of $\mathbf{P}L^\infty(\mathbb{D})$ is mapped into $\frac{8}{\pi}$ times the unit ball of $\mathcal{B}(\mathbb{D})$.

As for pointwise bounds, the analogue of Lemma 2.1.1 for $\mathbf{P}L^\infty(\mathbb{D})$ runs as follows.

Lemma 2.2.2. *Suppose that $\mu \in L^\infty(\mathbb{D})$. Then*

$$|\mathbf{P}\mu(z)| \le \|\mu\|_{L^\infty(\mathbb{D})} \frac{1}{|z|^2}\log\frac{1}{1-|z|^2}, \qquad z \in \mathbb{D}.$$

Proof. This follows from the estimate

$$|\mathbf{P}\mu(z)| \le \int_{\mathbb{D}} \frac{|\mu(w)|}{|1-z\bar{w}|^2}\mathrm{d}A(w) \le \|\mu\|_{L^\infty(\mathbb{D})}\int_{\mathbb{D}} \frac{1}{|1-z\bar{w}|^2}\mathrm{d}A(w), \qquad z \in \mathbb{D},$$

by evaluation of the right-hand side integral. □

Remark 2.2.3. Both estimates of Lemmata 2.1.1 and 2.2.2 are optimal. Moreover, since

$$\lim_{r\to 1^-} \frac{\frac{1}{r^2}\log\frac{1}{1-r^2}}{\frac{1}{2}\log\frac{1+r}{1-r}} = 2,$$

the permitted boundary growth is about twice as big for an element of the unit ball of $\mathbf{P}L^\infty(\mathbb{D})$ as for the unit ball of $\mathcal{B}(\mathbb{D})$.

3. Two notions of asymptotic variance and Marshall's estimate

3.1. Gaussian modeling and notions of asymptotic variance

The standard normal rotationally invariant complex Gaussian distribution $N_{\mathbb{C}}(0,1)$ has the probability measure $\mathrm{e}^{-|z|^2}\mathrm{d}A(z)$ in the plane $\mathbb{C}$. More generally, we write $X \sim N_{\mathbb{C}}(m,\sigma^2)$ with mean $m \in \mathbb{C}$ and rotationally invariant standard deviation $\sigma > 0$ if

$$\frac{X-m}{\sigma} \sim N_{\mathbb{C}}(0,1).$$

If $X \sim N_{\mathbb{C}}(0,\sigma^2)$, we may recover *the variance* $\operatorname{var} X = \sigma^2$ from the formula

$$\operatorname{var} X := \mathbb{E}|X|^2, \tag{3.1.1}$$

where $\mathbb{E}$ stands for the expectation operation. But we may also recover the variance from the tail distribution behavior as follows: $\sigma^2 = \operatorname{tvar} X$, where $\operatorname{tvar} X$ is the *tail variance*

$$\operatorname{tvar} X = \sigma^2 = \inf\big\{\tau \in \mathbb{R}_+ :\ \mathbb{E}\mathrm{e}^{|X|^2/\tau} < +\infty\big\}. \tag{3.1.2}$$

Indeed, we see by direct inspection that

$$\mathbb{E}\mathrm{e}^{|X|^2/\tau} = \sigma^{-2}\int_{\mathbb{C}} \mathrm{e}^{|z|^2/\tau}\mathrm{e}^{-|z|^2/\sigma^2}\mathrm{d}A(z) = \frac{\tau}{\tau-\sigma^2}, \qquad \sigma^2 < \tau < +\infty, \tag{3.1.3}$$

which explodes as $\tau \to \sigma^2$. We might remark at this juncture that tail aspects of Gaussian densities are critical for the uncertainty principles for Fourier transform pairs considered by Hardy and Beurling (see [20,29,21]).

Makarov (see [38,40,41]) had the remarkable insight to model the boundary behavior of Bloch functions by Gaussian processes (for a more directly probabilistic perspective, see Lyons' paper [37]). For a typical Bloch function $g \in \mathcal{B}(\mathbb{D})$ with $g(0) = 0$ and given an r with $0 < r < 1$, he thought of the dilates $g_r(\zeta) = g(r\zeta)$, for $\zeta \in \mathbb{T}$, as an approximately rotationally invariant Gaussian stochastic variable, which in its turn evolves stochastically in time, where we think of time as related to the dilation parameter r via

$$t = \log\frac{1}{1-r^2}, \qquad \mathrm{d}t = \frac{2r\mathrm{d}r}{1-r^2}.$$

So, taking this into account, we normalize the dilate, and let $X_r = X_r[g]$ be the function

$$X_r(\zeta) := \frac{g(r\zeta)}{\sqrt{\log\frac{1}{1-r^2}}}, \qquad \zeta \in \mathbb{T},\ \ 0 < r < 1,$$

and since

$$\mathbb{E}X_r = \int_{\mathbb{T}} X_r(\zeta)\mathrm{d}s(\zeta) = \frac{g(0)}{\sqrt{\log \frac{1}{1-r^2}}} = 0,$$

we may calculate the variance from the formula

$$\operatorname{var} X_r[g] = \mathbb{E}|X_r|^2 = \frac{\int_{\mathbb{T}} |g(r\zeta)|^2 \mathrm{d}s(\zeta)}{\log \frac{1}{1-r^2}}.$$

The tail variance has no direct analogue, as the function g_r is bounded for fixed r (see Lemma 2.1.1). However, in view of the uniform control observed in (3.1.3), we can make sense of it asymptotically as $r \to 1^-$. Following Makarov and Curtis McMullen [44], we say that the Bloch function g has the *asymptotic variance*

$$\operatorname{av} g := \limsup_{r\to 1^-} \mathbb{E}|X_r|^2 = \limsup_{r\to 1^-} \frac{\int_{\mathbb{T}} |g(r\zeta)|^2 \mathrm{d}s(\zeta)}{\log \frac{1}{1-r^2}}, \tag{3.1.4}$$

and the *asymptotic tail variance*

$$\operatorname{atv} g := \inf \left\{\tau \in \mathbb{R}_+ : \limsup_{r\to 1^-} \mathbb{E}\,\mathrm{e}^{|X_r|^2/\tau} < +\infty\right\}, \tag{3.1.5}$$

where the indicated expectation is given by

$$\mathbb{E}\,\mathrm{e}^{|X_r|^2/\tau} := \int_{\mathbb{T}} \exp\left(\frac{|g(r\zeta)|^2}{\tau \log \frac{1}{1-r^2}}\right)\mathrm{d}s(\zeta). \tag{3.1.6}$$

These asymptotic formulae apply also in the case when $g(0) \neq 0$.

It is of interest to extend these notions of asymptotic variances to the setting of subsets $\mathcal{G} \subset \mathcal{B}(\mathbb{D})$. To this end, we let the *asymptotic (tail) variances* of $\mathcal{G}$ be the supremum of the individual asymptotic (tail) variances:

$$\operatorname{av} \mathcal{G} := \sup_{g\in\mathcal{G}} \operatorname{av} g, \qquad \operatorname{atv} \mathcal{G} = \sup_{g\in\mathcal{G}} \operatorname{atv} g, \tag{3.1.7}$$

but we also need uniform versions. We let the *uniform asymptotic variance* of $\mathcal{G}$ be the limit

$$\operatorname{av}_u \mathcal{G} = \limsup_{r\to 1^-} \sup_{g\in\mathcal{G}} \mathbb{E}|X_r[g]|^2 = \limsup_{r\to 1^-} \sup_{g\in\mathcal{G}} \frac{\int_{\mathbb{T}} |g(r\zeta)|^2 \mathrm{d}s(\zeta)}{\log \frac{1}{1-r^2}}, \tag{3.1.8}$$

and, analogously, the *uniform asymptotic tail variance* of $\mathcal{G}$ is defined to be

$$\operatorname{atv}_u \mathcal{G} := \inf \left\{\tau \in \mathbb{R}_+ : \limsup_{r\to 1^-} \sup_{g\in\mathcal{G}} \int_{\mathbb{T}} \exp\left(\frac{|g(r\zeta)|^2}{\tau \log \frac{1}{1-r^2}}\right)\mathrm{d}s(\zeta) < +\infty\right\}. \tag{3.1.9}$$

The way things are set up, we automatically have the inequalities

$$\operatorname{av}\mathcal{G} \le \operatorname{av}_u \mathcal{G}, \qquad \operatorname{atv}\mathcal{G} \le \operatorname{atv}_u \mathcal{G},$$

but we should expect that quite often, as a result of compactness, the above inequalities are equalities. Note that for $F \in H^\infty(\mathbb{D})$, we have that

$$\operatorname{av}_u F\mathcal{G} \le \|F\|^2_{H^\infty(\mathbb{D})} \operatorname{av}_u \mathcal{G}, \qquad \operatorname{atv}_u F\mathcal{G} \le \|F\|^2_{H^\infty(\mathbb{D})} \operatorname{atv}_u \mathcal{G},$$

where the uniformity may be removed (by considering the functions individually).

3.2. Metric properties of the notions of asymptotic variance

Let us consider the expressions, for $g \in \mathcal{B}(\mathbb{D})$,

$$\|g\|_{\mathrm{av}} := (\operatorname{av} g)^{1/2}, \quad \|g\|_{\mathrm{atv}} := (\operatorname{atv} g)^{1/2}. \tag{3.2.1}$$

Proposition 3.2.1. *The functionals $\|\cdot\|_{\mathrm{av}}$ and $\|\cdot\|_{\mathrm{atv}}$ given by (3.2.1) are seminorms on $\mathcal{B}(\mathbb{D})$.*

Remark 3.2.2. In particular, for two functions $g_1, g_2 \in \mathcal{B}(\mathbb{D})$, we have the following: (i) if $\operatorname{av}(g_1 - g_2) = 0$, then $\operatorname{av} g_1 = \operatorname{av} g_2$, and (ii) if $\operatorname{atv}(g_1 - g_2) = 0$, then $\operatorname{atv} g_1 = \operatorname{atv} g_2$. Examples of functions $g \in \mathcal{B}(\mathbb{D})$ with $\operatorname{av} g = \operatorname{atv} g = 0$ include elements of $H^\infty(\mathbb{D})$ as well as elements of the little Bloch space $\mathcal{B}_0(\mathbb{D})$.

Proof of Proposition 3.2.1. The homogeneity property of the norm follows from the corresponding property of the two asymptotic variances, which are easily verified by inspection:

$$\operatorname{av} \lambda g = |\lambda|^2 \operatorname{av} g, \quad \operatorname{atv} \lambda g = |\lambda|^2 \operatorname{atv} g,$$

for $g \in \mathcal{B}(\mathbb{D})$. To obtain the remaining property (subadditivity), that is,

$$\|g+h\|_{\mathrm{av}} \le \|g\|_{\mathrm{av}} + \|h\|_{\mathrm{av}}, \quad \|g+h\|_{\mathrm{atv}} \le \|g\|_{\mathrm{atv}} + \|h\|_{\mathrm{atv}}, \tag{3.2.2}$$

we begin with an elementary estimate. For complex numbers $\xi, \eta \in \mathbb{C}$, and a positive real α, we have the estimate

$$|\xi + \eta|^2 \le (1+\alpha)|\xi|^2 + \left(1 + \frac{1}{\alpha}\right)|\eta|^2. \tag{3.2.3}$$

It follows from (3.2.3) that

$$\frac{|(g+h)(r\zeta)|^2}{\log \frac{1}{1-r^2}} \le (1+\alpha)\frac{|g(r\zeta)|^2}{\log \frac{1}{1-r^2}} + \left(1+\frac{1}{\alpha}\right)\frac{|h(r\zeta)|^2}{\log \frac{1}{1-r^2}}. \tag{3.2.4}$$

If σ_1, σ_2 are positive reals such that $\mathrm{av}\, g < \sigma_1^2$ and $\mathrm{av}\, h < \sigma_2^2$, we integrate with respect to $\mathrm{d}s(\zeta)$ along $\mathbb{T}$ in (3.2.4), and take the limsup as $r \to 1^-$. The result is that

$$\mathrm{av}(g+h) \le (1+\alpha)\sigma_1^2 + \left(1+\frac{1}{\alpha}\right)\sigma_2^2 = (\sigma_1+\sigma_2)^2,$$

where in the last step, we made the optimal choice $\alpha = \sigma_2/\sigma_1$. By minimizing over σ_1, σ_2, the first subadditivity in (3.2.2) follows.

Next, we let σ_3, σ_4 be positive reals with $\mathrm{atv}\, g < \sigma_3^2$ and $\mathrm{atv}\, h < \sigma_4^2$, and observe that by (3.2.4),

$$\exp\left\{\frac{|(g+h)(r\zeta)|^2}{(\sigma_3+\sigma_4)^2 \log\frac{1}{1-r^2}}\right\} \le \exp\left\{\frac{(1+\alpha)|g(r\zeta)|^2}{(\sigma_3+\sigma_4)^2 \log\frac{1}{1-r^2}}\right\} \exp\left\{\frac{(1+\frac{1}{\alpha})|h(r\zeta)|^2}{(\sigma_3+\sigma_4)^2 \log\frac{1}{1-r^2}}\right\},$$

so that by Hölder's inequality,

$$\int_{\mathbb{T}} \exp\left\{\frac{|(g+h)(r\zeta)|^2}{(\sigma_3+\sigma_4)^2 \log\frac{1}{1-r^2}}\right\}\mathrm{d}s(\zeta) \le \left(\int_{\mathbb{T}} \exp\left\{\frac{(1+\alpha)p|g(r\zeta)|^2}{(\sigma_3+\sigma_4)^2 \log\frac{1}{1-r^2}}\right\}\mathrm{d}s(\zeta)\right)^{1/p}$$
$$\times \left(\int_{\mathbb{T}} \exp\left\{\frac{(1+\frac{1}{\alpha})p'|h(r\zeta)|^2}{(\sigma_3+\sigma_4)^2 \log\frac{1}{1-r^2}}\mathrm{d}s(\zeta)\right\}\right)^{1/p'},$$

where p, p' are dual exponents. The choice $\alpha = \sigma_4/\sigma_3$, $p = 1+\alpha$, and $p' = 1+\frac{1}{\alpha}$ gives

$$\frac{(1+\alpha)p}{(\sigma_3+\sigma_4)^2} = \frac{1}{\sigma_3^2}, \quad \frac{(1+\frac{1}{\alpha})p'}{(\sigma_3+\sigma_4)^2} = \frac{1}{\sigma_4^2},$$

and allows us to conclude that

$$\mathrm{atv}(g+h) \le (\sigma_3+\sigma_4)^2.$$

By minimizing over σ_3, σ_4, the second subadditivity in (3.2.2) follows as well. The proof is complete. □

Remark 3.2.3. Let $\mathcal{N}_{\mathrm{av}} := \{g \in \mathcal{B}(\mathbb{D}) : \mathrm{av}\, g = 0\}$ and $\mathcal{N}_{\mathrm{atv}} := \{g \in \mathcal{B}(\mathbb{D}) : \mathrm{atv}\, g = 0\}$ be the respective null subspaces for the seminorms (3.2.1). It would be natural to consider the Banach spaces which result from forming the completions of the quotient spaces $\mathcal{B}(\mathbb{D})/\mathcal{N}_{\mathrm{av}}$ and $\mathcal{B}(\mathbb{D})/\mathcal{N}_{\mathrm{atv}}$ with respect to the corresponding norms.

3.3. Makarov's growth estimate of a Bloch function

The following result is immediate from the work of Makarov (see [38], and [47], Theorem 8.9 and Exercise 8.5.2); more or less the same argument can also be found in the work of Clunie and MacGregor [14], but it is applied with less precision. It should be mentioned also that at about the same time, Boris Korenblum [34] found a cruder growth estimate than Makarov by studying dilatation as a map from the Bloch space to BMOA. In a sense, the present work may be viewed as a refinement of Korenblum's approach.

Theorem 3.3.1. (Makarov) *If $g \in \mathcal{B}(\mathbb{D})$ with $g(0) = 0$, then, for $0 < r < 1$, we have that*

$$\frac{\int_{\mathbb{T}} |g(r\zeta)|^2 \mathrm{d}s(\zeta)}{\log \frac{1}{1-r^2}} \le \|g\|^2_{\mathcal{B}(\mathbb{D})} \quad \textit{and} \quad \int_{\mathbb{T}} \exp\left\{\frac{|g(r\zeta)|^2}{\tau \log \frac{1}{1-r^2}}\right\} \mathrm{d}s(\zeta) \le \frac{\tau}{\tau - \|g\|^2_{\mathcal{B}(\mathbb{D})}},$$

provided that $\|g\|^2_{\mathcal{B}(\mathbb{D})} < \tau < +\infty$. In particular, it follows that

$$\operatorname{av} g \le \|g\|^2_{\mathcal{B}(\mathbb{D})} \quad \textit{and} \quad \operatorname{atv} g \le \|g\|^2_{\mathcal{B}(\mathbb{D})}.$$

Let $\mathcal{G}_1$ and $\mathcal{G}_2$ denote the two unit balls

$$\mathcal{G}_1 = \operatorname{ball}_0 \mathcal{B}(\mathbb{D}) := \big\{ g \in \mathcal{B}(\mathbb{D}) : \|g\|_{\mathcal{B}(\mathbb{D})} \le 1, \ \ g(0) = 0 \big\};$$

and

$$\mathcal{G}_2 = \operatorname{ball} \mathbf{P}L^\infty(\mathbb{D}) := \big\{ g = \mathbf{P}\mu : \|\mu\|_{L^\infty(\mathbb{D})} \le 1 \big\}.$$

In view of Proposition 3.2.1 as well as Proposition 4.3.1 below, we see that for all essential purposes, $\mathcal{G}_2$ is bigger than $\mathcal{G}_1$. Our main result, Theorem 1.3.1, establishes that $\operatorname{atv}_u \mathcal{G}_2 = \operatorname{atv} \mathcal{G}_2 = 1$. An inspiration for the present work is the paper [4] by Astala, Ivrii, Perälä, and Prause, where it was shown that $\operatorname{av} \mathcal{G}_2 \le 1$; later, it was discovered that $\operatorname{av}_u \mathcal{G}_2 < 1$ (see [22]). In particular, the asymptotic variance and the asymptotic tail variance do not always coincide. This may have some relation with dimension properties of quasicircles, see e.g. [36]. In comparison, Makarov's Theorem 3.3.1 obtains that $\operatorname{av}_u \mathcal{G}_1 \le 1$ and $\operatorname{atv}_u \mathcal{G}_1 \le 1$, which together with Perälä's Lemma 2.2.1 only leads to the following rather weak estimates: $\operatorname{av}_u \mathcal{G}_2 \le 64/\pi^2$ and $\operatorname{atv}_u \mathcal{G}_2 \le 64/\pi^2 = 6.484\ldots$.

3.4. Marshall's estimate of the exponential type spectrum

The following is the key observation of Marshall [43].

Proposition 3.4.1. (Marshall) *If $g \in \mathcal{B}(\mathbb{D})$ has $\operatorname{atv} g < \sigma^2$, where σ is a positive real number, then*

$$\int_{\mathbb{T}} |\mathrm{e}^{tg(r\zeta)}| \mathrm{d}s(\zeta) = \mathrm{O}\big((1-r^2)^{-\sigma^2 |t|^2/4}\big) \quad \textit{as} \quad r \to 1^-,$$

where the implied constant is uniform in $t \in \mathbb{C}$.

Proof. Marshall [43] expands the following modulus-squared, for a complex parameter $t \in \mathbb{C}$:

$$0 \le \left| \frac{g(r\zeta)}{\sigma\sqrt{\log\frac{1}{1-r^2}}} - \frac{\sigma\bar{t}}{2}\sqrt{\log\frac{1}{1-r^2}} \right|^2$$
$$= \frac{|g(r\zeta)|^2}{\sigma^2\log\frac{1}{1-r^2}} + \frac{\sigma^2|t|^2}{4}\log\frac{1}{1-r^2} - \mathrm{Re}(tg(r\zeta)). \qquad (3.4.1)$$

It is immediate from (3.4.1) that

$$\mathrm{Re}(tg(r\zeta)) \le \frac{|g(r\zeta)|^2}{\sigma^2\log\frac{1}{1-r^2}} + \frac{\sigma^2|t|^2}{4}\log\frac{1}{1-r^2},$$

and as we exponentiate both sides, and then integrate over the circle $\mathbb{T}$, we arrive at

$$\int_{\mathbb{T}} |\mathrm{e}^{tg(r\zeta)}|\mathrm{d}s(\zeta) \le (1-r^2)^{-\sigma^2|t|^2/4} \int_{\mathbb{T}} \exp\left\{\frac{|g(r\zeta)|^2}{\sigma^2\log\frac{1}{1-r^2}}\right\}\mathrm{d}s(\zeta), \qquad (3.4.2)$$

from which the assertion of the proposition is immediate. □

This has the following consequence for the exponential type spectrum $\beta_g(t)$ of the function e^g, where $g \in \mathcal{B}(\mathbb{D})$.

Corollary 3.4.2. (Marshall) *For $g \in \mathcal{B}(\mathbb{D})$ and $0 \le \sigma < +\infty$, we have the implication*

$$\mathrm{atv}\, g \le \sigma^2 \quad \Longrightarrow \quad \forall t \in \mathbb{C}: \;\; \beta_g(t) \le \frac{\sigma^2|t|^2}{4}.$$

3.5. The estimate from above of the exponential type spectrum associated with a function in the unit ball of $\mathbf{P}L^\infty(\mathbb{D})$

For a function $g = \mathbf{P}\mu$, where $\mu \in L^\infty(\mathbb{D})$, we need to estimate from above the exponential type spectrum. The key estimate is the following. To simplify the notation, we agree to write

$$I_g(a,r) := \int_{\mathbb{T}} \exp\left\{a\frac{r^4|g(r\zeta)|^2}{\log\frac{1}{1-r^2}}\right\}\mathrm{d}s(\zeta), \qquad 0 < r < 1. \qquad (3.5.1)$$

Proposition 3.5.1. *Let $g := \mathbf{P}\mu$, where $\mu \in L^\infty(\mathbb{D})$ with $\|\mu\|_{L^\infty(\mathbb{D})} \le 1$. If $I_g(a,r)$ is the integral in (3.5.1), we have the estimate*

$$\int_{\mathbb{T}} |\mathrm{e}^{tr^2 g(r\zeta)}|\mathrm{d}s(\zeta) \le \begin{cases} I_g(a,r)(1-r^2)^{-|t|^2/(4a)}, & |t| \le 2a, \\ I_g(a,r)(1-r^2)^{a-|t|}, & |t| > 2a. \end{cases}$$

Proof. In view of (3.4.2) and (3.5.1), we have for positive a that

$$\int_{\mathbb{T}} |e^{t_1 r^2 g(r\zeta)}| ds(\zeta) \le I_g(a,r)\,(1-r^2)^{-|t_1|^2/(4a)}, \qquad 0<r<1,\ t_1\in\mathbb{C}. \tag{3.5.2}$$

while, by the pointwise estimate of Lemma 2.2.2,

$$|e^{t_2 r^2 g(r\zeta)}| \le (1-r^2)^{-|t_2|}, \qquad 0<r<1,\ t_2\in\mathbb{C}. \tag{3.5.3}$$

For $t = t_1 + t_2 \in \mathbb{C}$, it follows from a combination of (3.5.2) and (3.5.3) that

$$\int_{\mathbb{T}} |e^{t r^2 g(r\zeta)}| ds(\zeta) = \int_{\mathbb{T}} |e^{t_1 r^2 g(r\zeta)} e^{t_2 r^2 g(r\zeta)}| ds(\zeta) \le I_g(a,r)\,(1-r^2)^{-|t_2|-|t_1|^2/(4a)}, \qquad 0<r<1, \tag{3.5.4}$$

where we are free to optimize over all decompositions $t = t_1 + t_2$. For $|t| \le 2a$, the best decomposition is $t = t_1 + 0$, that is, $t_1 = t$ and $t_2 = 0$, while for $|t| > 2a$, the best choice is $t_1 = \theta t$ and $t_2 = (1-\theta)t$, where $\theta := 2a/|t|$. After insertion into (3.5.4), we arrive at the claimed estimate. □

We may now supply the proof of Corollary 1.6.2 as an application of our main theorem, Theorem 1.3.1.

Proof of Corollary 1.6.2. By Theorem 1.3.1, we know that $I_g(a,r) \le C(a)$ for all $0 < r < 1$ and $0 < a < 1$. It follows from Proposition 3.5.1 that for all $0 < a < 1$,

$$\int_{\mathbb{T}} |e^{t r^2 g(r\zeta)}| ds(\zeta) \le \begin{cases} C(a)(1-r^2)^{-|t|^2/(4a)}, & |t| \le 2a, \\ C(a)(1-r^2)^{a-|t|}, & |t| > 2a. \end{cases}$$

The factor r^2 in the exponent tends to 1 as $r \to 1^-$, and nothing changes drastically if it gets replaced by 1. Finally, by letting a approach 1, we obtain the claimed estimate of the exponential type spectrum $\beta_g(t)$. □

4. Elementary properties of Bloch functions

4.1. The Bergman projection of an auxiliary bounded function

For the proof of part (b) of Theorem 1.3.1, we need to supply the Bergman projection of a special function $\mu_0 \in L^\infty(\mathbb{D})$.

Lemma 4.1.1. *The Bergman projection of the function* $\mu_0 \in L^\infty(\mathbb{D})$ *given by*

$$\mu_0(z) := \frac{1-\bar{z}}{1-z}, \qquad z \in \mathbb{D},$$

equals

$$\mathbf{P}\mu_0(z) = \int_{\mathbb{D}} \frac{1-\bar{w}}{(1-w)(1-z\bar{w})^2} \mathrm{d}A(w) = \frac{1}{z^2}\log\frac{1}{1-z} - \frac{1}{z}, \qquad z \in \mathbb{D}\setminus\{0\}.$$

The singularity at the origin is of course removable.

Proof of Lemma 4.1.1. This can be shown by direct computation of the Bergman integral. □

Remark 4.1.2. Along the segment $[0,1[\subset \mathbb{D}$, the function $\mathbf{P}\mu_0$ grows pretty much maximally quickly, compared with Lemma 2.2.2:

$$\mathbf{P}\mu_0(x) = \frac{1}{x^2}\log\frac{1}{1-x} - \frac{1}{x} = \frac{1}{x^2}\log\frac{1}{1-x^2} - \frac{1}{x^2}\big(x - \log(1+x)\big)$$
$$\geq \frac{1}{x^2}\log\frac{1}{1-x^2} - \frac{1}{2}, \qquad 0 < x < 1.$$

We will apply the following general estimate to the function $\mu = \mu_0$ of Lemma 4.1.1.

Proposition 4.1.3. *If* $\mu \in L^\infty(\mathbb{D})$ *with* $\|\mu\|_{L^\infty(\mathbb{D})} \leq 1$*, then, for* $0 < a < +\infty$*, we have*

$$(1-r^2)^{1/a} \int_{\mathbb{T}} \big|\mathrm{e}^{r^2\zeta^2\mathbf{P}\mu(r\zeta)}\big|^2 \mathrm{d}s(\zeta) \leq \int_{\mathbb{T}} \exp\left\{a\frac{r^4|\mathbf{P}\mu(r\zeta)|^2}{\log\frac{1}{1-r^2}}\right\}\mathrm{d}s(\zeta), \qquad 0 < r < 1.$$

Proof. This is immediate from Marshall's inequality (3.4.2), with $t := 2$, $a := 1/\sigma^2$, and $g(z) := z^2\mathbf{P}\mu(z)$. □

It is now easy to obtain the sharpness part (b) of Theorem 1.3.1.

Corollary 4.1.4. *If* $\mu_0 \in L^\infty(\mathbb{D})$ *is as in Lemma 4.1.1, and* $0 < a < +\infty$*, we have the estimate from below*

$$\int_{\mathbb{T}} \exp\left\{a\frac{r^4|\mathbf{P}\mu_0(r\zeta)|^2}{\log\frac{1}{1-r^2}}\right\}\mathrm{d}s(\zeta) \geq \mathrm{e}^{-2}(1-r^2)^{-(a-1)/a}.$$

Proof. By Lemma 4.1.1,

$$\mathrm{e}^{z^2\mathbf{P}\mu_0(z)} = \mathrm{e}^{-z}(1-z)^{-1},$$

so that

$$\int_{\mathbb{T}} \left|e^{r^2\zeta^2 \mathbf{P}\mu_0(r\zeta)}\right|^2 \mathrm{d}s(\zeta) \geq e^{-2} \int_{\mathbb{T}} |1-r\zeta|^{-2}\mathrm{d}s(\zeta) = e^{-2}(1-r^2)^{-1}.$$

The assertion of the corollary now follows rather immediately from Proposition 4.1.3. □

Remark 4.1.5. Theorem 1.3.1(b) now follows from the observation that

$$\lim_{r\to 1^-} e^{-2}(1-r^2)^{-(a-1)/a} = +\infty, \qquad 1 < a < +\infty.$$

4.2. The derivative of a Bloch function

We first supply an elementary estimate which applies to the derivative of a Bloch function.

Lemma 4.2.1. *Suppose $f : \mathbb{D} \to \mathbb{C}$ is holomorphic, with $(1-|z|^2)|f(z)| \leq 1$ on $\mathbb{D}$. Then we also have that*

$$(1-|z|^2)\omega(|z|)\left|\frac{f(z)-f(0)}{z}\right| \leq 1,$$

where $\omega(t) = \frac{1}{3}$ for $0 \leq t < \frac{1}{2}$, and $\omega(t) = t/(2-t^2)$ for $\frac{1}{2} \leq t < 1$.

Proof. The Cauchy integral formula applied to the dilate $f_r(\zeta) = f(r\zeta)$ gives that

$$\begin{aligned} f(rz) - f(0) = f_r(z) - f_r(0) &= \int_{\mathbb{T}} \left\{\frac{1}{1-z\bar{w}} - 1\right\} f_r(w)\mathrm{d}s(w) \\ &= \int_{\mathbb{T}} \left\{\frac{1}{1-z\bar{w}} - 1\right\} f_r(w)\mathrm{d}s(w) = z\int_{\mathbb{T}} \frac{\bar{w}}{1-z\bar{w}} f_r(w)\mathrm{d}s(w). \end{aligned}$$

As a consequence, we obtain the estimate

$$\begin{aligned} \left|\frac{f(rz)-f(0)}{rz}\right| &\leq \frac{1}{r}\int_{\mathbb{T}} \frac{1}{|1-z\bar{w}|}|f_r(w)|\mathrm{d}s(w) \leq \frac{1}{r(1-r^2)}\int_{\mathbb{T}} \frac{1}{|1-z\bar{w}|}\mathrm{d}s(w) \\ &\leq \frac{1}{r(1-r^2)}\frac{1}{|z|^2}\log\frac{1}{1-|z|^2}, \end{aligned}$$

for $z \in \mathbb{D}$ and $0 < r < 1$, which in its turn yields

$$(1-|rz|^2)\left|\frac{f(rz)-f(0)}{rz}\right| \leq \frac{1-|rz|^2}{r(1-r^2)|z|^2}\log\frac{1}{1-|z|^2}.$$

We plug in $r = 2^{-1/2}$:

$$(1 - |rz|^2)\left|\frac{f(rz) - f(0)}{rz}\right| \le 2^{1/2}\frac{2 - |z|^2}{|z|^2}\log\frac{1}{1 - |z|^2}, \qquad r = 2^{-1/2},$$

and check that the right-hand side expression is an increasing function in the variable $|z|$. By restricting our attention to $|z| \le 2^{-1/2}$, we find that

$$(1 - |\zeta|^2)\left|\frac{f(\zeta) - f(0)}{\zeta}\right| \le 2^{1/2}\frac{2 - \frac{1}{2}}{1/2}\log\frac{1}{1 - \frac{1}{2}} < 3, \qquad |\zeta| \le \frac{1}{2}.$$

It is of course elementary that the following estimate holds:

$$(1 - |\zeta|^2)\left|\frac{f(\zeta) - f(0)}{\zeta}\right| \le \frac{2 - |\zeta|^2}{|\zeta|}, \qquad 0 < |\zeta| < 1.$$

The assertion of the lemma follows from a combination of these two estimates. □

4.3. Decomposition of a Bloch function

It is well known that as sets, $\mathcal{B}(\mathbb{D}) = \mathbf{P}L^\infty(\mathbb{D})$. Here, we split a given Bloch function as an element of $\mathbf{P}L^\infty(\mathbb{D})$ with small norm plus a smooth remainder.

Proposition 4.3.1. *Suppose $g \in \mathcal{B}(\mathbb{D})$. Then there exists a $\nu_g \in L^\infty(\mathbb{D})$ with $\|\nu_g\|_{L^\infty(\mathbb{D})} \le \|g\|_{\mathcal{B}(\mathbb{D})}$, such that $g(z) = z^2\mathbf{P}\nu_g(z) + G(z)$, where $G \in H^\infty(\mathbb{D})$ has $G' \in \mathcal{B}(\mathbb{D})$, with the (semi)norm control*

$$\|G\|_{H^\infty(\mathbb{D})} \le |g(0)| + 6\|g\|_{\mathcal{B}(\mathbb{D})}, \quad \|G'\|_{\mathcal{B}(\mathbb{D})} \le 12\|g\|_{\mathcal{B}(\mathbb{D})}.$$

Proof. We put

$$\nu_g(z) := (1 - |z|^2)\omega(|z|)\frac{g'(z) - g'(0)}{z},$$

where $\omega(t)$ is as in Lemma 4.2.1, and observe that by the assertion of that lemma, $\|\nu_g\|_{L^\infty(\mathbb{D})} \le \|g\|_{\mathcal{B}(\mathbb{D})}$, as needed. If we write μ_g for the function

$$\mu_g(z) := (1 - |z|^2)\frac{g'(z) - g'(0)}{z},$$

then a direct calculation shows that

$$z^2\mathbf{P}\mu_g(z) = z^2\int_{\mathbb{D}}\frac{(1 - |w|^2)(g'(w) - g'(0))}{w(1 - z\bar{w})^2}\mathrm{d}A(w)$$

$$= \sum_{j=0}^{+\infty}(j+1)z^{j+2}\int_{\mathbb{D}}(1-|w|^2)\bar{w}^j\,\frac{g'(w)-g'(0)}{w}\mathrm{d}A(w)$$
$$= \sum_{j=0}^{+\infty}(j+1)z^{j+2}\,\frac{\hat{g}(j+2)}{j+1} = g(z)-g(0)-g'(0)z, \qquad z\in\mathbb{D}, \quad (4.3.1)$$

where the $\hat{g}(j)$ denote the Taylor coefficients of g. The difference $\mu_g - \nu_g$ may be written in the form

$$\mu_g(z)-\nu_g(z) = (1-|z|^2)(1-\omega(|z|))\frac{g'(z)-g'(0)}{z} = \frac{1-\omega(|z|)}{\omega(|z|)}\nu_g(z),$$

which immediately yields the estimate

$$|\mu_g(z)-\nu_g(z)| \le \frac{1-\omega(|z|)}{\omega(|z|)}\|\nu_g\|_{L^\infty(\mathbb{D})} \le \frac{1-\omega(|z|)}{\omega(|z|)}\|g\|_{\mathcal{B}(\mathbb{D})}.$$

A straightforward calculation tells us that

$$\int_{\mathbb{D}} \frac{1-\omega(|w|)}{\omega(|w|)}\,\frac{\mathrm{d}A(w)}{|1-z\bar{w}|^2} \le 5, \qquad z\in\mathbb{D},$$

and, that, as a consequence,

$$\|\mathbf{P}(\mu_g-\nu_g)\|_{H^\infty(\mathbb{D})} \le 5\|g\|_{\mathcal{B}(\mathbb{D})}.$$

Finally, we split g as $g = \mathbf{P}\nu_g + G$, where

$$G(z) := g(0)+g'(0)z+\mathbf{P}(\mu_g-\nu_g)(z),$$

and the claimed estimate $\|G\|_{H^\infty(\mathbb{D})} \le |g(0)| + 6\|g\|_{\mathcal{B}(\mathbb{D})}$ follows. The estimate for the Bloch seminorm of G' is obtained in a similar manner. □

5. Identities for dilates of harmonic functions

5.1. An identity involving dilates of harmonic functions

The following identity interchanges dilations, and although elementary, it is quite important.

Lemma 5.1.1. *Suppose $f, g : \mathbb{D} \to \mathbb{C}$ are two harmonic functions, which are area-integrable: $f, g \in L^1(\mathbb{D})$. Then we have that*

$$\int_{\mathbb{D}} f(rz)\bar{g}(z)\mathrm{d}A(z) = \int_{\mathbb{D}} f(z)\bar{g}(rz)\mathrm{d}A(z), \qquad 0<r<1.$$

Proof. Both integrals are well-defined, since $f, g \in L^1(\mathbb{D})$ and the dilates $f_r(z) = f(rz)$, $g_r(z) = g(rz)$, are bounded for $0 < r < 1$. If we consider also the dilates f_ϱ, g_ϱ for $0 < \varrho < 1$, we may use Fourier methods to establish the identities

$$\int_{\mathbb{D}} f(r\varrho z)\bar{g}(\varrho z)\mathrm{d}A(z) = \sum_{j\in\mathbb{Z}} \frac{\varrho^{2|j|}r^{|j|}}{|j|+1}\hat{f}(j)\overline{\hat{g}(j)}, \tag{5.1.1}$$

and

$$\int_{\mathbb{D}} f(\varrho z)\bar{g}(r\varrho z)\mathrm{d}A(z) = \sum_{j\in\mathbb{Z}} \frac{\varrho^{2|j|}r^{|j|}}{|j|+1}\hat{f}(j)\overline{\hat{g}(j)}, \tag{5.1.2}$$

so that

$$\int_{\mathbb{D}} f(r\varrho z)\bar{g}(\varrho z)\mathrm{d}A(z) = \int_{\mathbb{D}} f(\varrho z)\bar{g}(r\varrho z)\mathrm{d}A(z).$$

Here, we use $\hat{f}(j), \hat{g}(j)$ to denote the Fourier coefficients of the functions f, g, considered as distributions on the circle $\mathbb{T}$. The claimed identity now follows by letting $\varrho \to 1$, since $f_\varrho \to f$ and $g_\varrho \to g$ in $L^1(\mathbb{D})$, while $f_{r\varrho} \to f_r$ and $g_{r\varrho} \to g_r$ in $L^\infty(\mathbb{D})$. □

5.2. An identity involving dilates which connects the inner products on the circle and the disk

The following identity is key to our analysis.

Lemma 5.2.1. *Suppose $g, h : \mathbb{D} \to \mathbb{C}$ are functions, where g is holomorphic and h is harmonic. If $g \in L^1(\mathbb{D})$ and h is the Poisson integral of a function in $L^1(\mathbb{T})$, then we have that*

$$\langle g_r, \bar{z}h\rangle_{\mathbb{T}} = \langle g, (\partial h)_r\rangle_{\mathbb{D}},$$

where we write f_r for the dilate of the function f: $f_r(\zeta) = f(r\zeta)$.

Proof. As in the proof of Lemma 5.1.1, we let $\hat{g}(j), \hat{h}(j)$ denote the Fourier coefficients of the boundary distributions associated with g, h on $\mathbb{T}$. By the Plancherel identity, then, we know that

$$\langle g_r, \bar{z}h\rangle_{\mathbb{T}} = \int_{\mathbb{T}} \zeta g(r\zeta)\bar{h}(\zeta)\mathrm{d}s(\zeta) = \sum_{j=0}^{+\infty} \hat{g}(j)\overline{\hat{h}(j+1)}\, r^j.$$

On the other hand, since

$$(\partial h)(\zeta)=\sum_{j=0}^{+\infty}(j+1)\hat{h}(j+1)\zeta^j, \qquad \zeta\in\mathbb{D},$$

it follows from (5.1.1) by letting $\varrho\to 1^-$ that

$$\langle g,(\partial h)_r\rangle_{\mathbb{D}}=\int_{\mathbb{D}} g(z)\overline{(\partial h)(rz)}\mathrm{d}A(z)=\sum_{j=0}^{+\infty}\frac{r^j}{j+1}(j+1)\hat{g}(j)\overline{\hat{h}(j+1)}=\sum_{j=0}^{+\infty}\hat{g}(j)\overline{\hat{h}(j+1)}\,r^j.$$

The assertion of the lemma follows. □

6. Dilational reverse isoperimetry: Hardy and Bergman

6.1. The isoperimetric inequality of Carleman

The classical isoperimetric inequality says that the area enclosed by a closed loop of length L is at most $L^2/(4\pi)$. Torsten Carleman (see [12,52]) found a nice analytical approach to this fact which gave the estimate

$$\|f\|_{A^{2p}(\mathbb{D})}\le\|f\|_{H^p(\mathbb{D})}, \qquad f\in H^p(\mathbb{D}), \tag{6.1.1}$$

for $0<p<+\infty$. Here, $H^1(\mathbb{D})$ is the $p=1$ instance of the classical *Hardy space* $H^p(\mathbb{D})$, for $0<p\le+\infty$. For $0<p<+\infty$, $H^p(\mathbb{D})$ consists all holomorphic functions $f:\mathbb{D}\to\mathbb{C}$ subject to the norm boundedness condition

$$\|f\|^p_{H^p(\mathbb{D})}:=\sup_{0<r<1}\int_{\mathbb{T}}|f(r\zeta)|^p\mathrm{d}s(\zeta)<+\infty.$$

It is well-known that for $f\in H^p(\mathbb{D})$, the function has well-defined nontangential boundary values a.e., and that the norm is attained for $r=1$:

$$\|f\|^p_{H^p(\mathbb{D})}=\int_{\mathbb{T}}|f(\zeta)|^p\mathrm{d}s(\zeta).$$

6.2. A similar reverse inequality for dilates

The Carleman estimate (6.1.1) has a reverse if we take the Hardy norm of a dilate $f_r(\zeta):=f_r(\zeta)$ in place of the function. Here, we will not dwell on that matter, but instead look for a somewhat similar reverse estimate. We consider a nontrivial harmonic function $h:\mathbb{D}\to\mathbb{C}$, and obtain from the Cauchy–Schwarz inequality that

$$\|(\partial h)_r\|_{A^1(\mathbb{D})}=\int_{\mathbb{D}}|(\partial h)(r\zeta)|\mathrm{d}A(\zeta)=\frac{1}{r^2}\int_{\mathbb{D}(0,r)}|\partial h(z)|\mathrm{d}A(z)$$
$$\le\frac{1}{r^2}\Bigg(\int_{\mathbb{D}(0,r)}\frac{|h(z)|^\theta}{1-|z|^2}\mathrm{d}A(z)\Bigg)^{1/2}\Bigg(\int_{\mathbb{D}(0,r)}\frac{|\partial h(z)|^2}{|h(z)|^\theta}(1-|z|^2)\mathrm{d}A(z)\Bigg)^{1/2},\quad(6.2.1)$$

where θ is a real parameter, which we shall confine to interval $0\le\theta<1$.

7. Duality and the estimate of the uniform asymptotic tail variance

7.1. Green's formula

We recall Green's formula for the unit disk:

$$\int_{\mathbb{D}}(u\Delta v-v\Delta u)\mathrm{d}A=\frac{1}{2}\int_{\mathbb{T}}(u\partial_{\mathrm{n}}v-v\partial_{\mathrm{n}}u)\mathrm{d}s,$$

where u,v are both assumed C^2-smooth in the closed unit disk $\bar{\mathbb{D}}$, and the normal derivative is in the exterior direction. The constant $\frac{1}{2}$ is the result of our normalizations. If we choose $v(z):=1-|z|^2$, the formula simplifies to

$$\int_{\mathbb{D}}u\mathrm{d}A+\int_{\mathbb{D}}(1-|z|^2)\Delta u(z)\mathrm{d}A(z)=\int_{\mathbb{T}}u\mathrm{d}s.\quad(7.1.1)$$

Next, we let $h:\bar{\mathbb{D}}\to\mathbb{C}$ be C^2-smooth and harmonic in $\mathbb{D}$, let ϵ denote a small positive constant, and consider the function

$$u(z):=(|h(z)|^2+\epsilon)^s,$$

where it is assumed that $\frac{1}{2}<s\le1$. Then u is C^2-smooth on $\bar{\mathbb{D}}$, and we may calculate its Laplacian:

$$\Delta u=s\Big\{(|h|^2+\epsilon)^{s-1}\Delta|h|^2-(1-s)(|h|^2+\epsilon)^{s-2}|\partial|h|^2|^2\Big\}$$
$$=s(|h|^2+\epsilon)^{s-2}\Big\{(|h|^2+\epsilon)(|\partial h|^2+|\bar\partial h|^2)-(1-s)|\bar h\partial h+h\partial\bar h|^2\Big\}.\quad(7.1.2)$$

For complex numbers $a,b\in\mathbb{C}$ and a positive real number $t\in\mathbb{R}_+$, we know by direct algebraic manipulation that

$$|a+b|^2=|a|^2(1+t^2)+|b|^2(1+t^{-2})-|at-bt^{-1}|^2.$$

As we apply this identity in the setting of (7.1.2), with $a := \bar{h}\partial h$ and $b := h\partial\bar{h}$, we find that

$$\Delta u = s(|h|^2+\epsilon)^{s-2}\Big\{\epsilon(|\partial h|^2+|\bar\partial h|^2)+[1-(1-s)(1+t^2)]|h\partial h|^2 + [1-(1-s)(1+t^{-2})]|h\bar\partial h|^2+(1-s)|t\bar{h}\partial h - t^{-1}h\partial\bar{h}|^2\Big\} \qquad (7.1.3)$$

We choose t so that we may suppress the term with $|h\bar\partial h|^2$, which happens for $t := \sqrt{(1-s)/s}$. It now follows from (7.1.3) that

$$\Delta u = s(|h|^2+\epsilon)^{s-2}\Big\{\epsilon(|\partial h|^2+|\bar\partial h|^2)+\frac{2s-1}{s}\,|h\partial h|^2+s\Big|\frac{1-s}{s}\bar{h}\partial h - h\partial\bar{h}\Big|^2\Big\} \geq (2s-1)(|h|^2+\epsilon)^{s-2}|h\partial h|^2. \qquad (7.1.4)$$

As we insert this inequality into the identity (7.1.1), the result is that

$$\int_{\mathbb{D}}(|h|^2+\epsilon)^s\mathrm{d}A + (2s-1)\int_{\mathbb{D}}(1-|z|^2)(|h(z)|^2+\epsilon)^{s-2}|h(z)\partial h(z)|^2\mathrm{d}A(z) \leq \int_{\mathbb{D}} u\mathrm{d}A + \int_{\mathbb{D}}(1-|z|^2)\Delta u(z) = \int_{\mathbb{T}} u\mathrm{d}s = \int_{\mathbb{T}}(|h|^2+\epsilon)^s\mathrm{d}s. \qquad (7.1.5)$$

We recall the Hardy space $h^q(\mathbb{D})$ of harmonic functions. Since we will only be interested in exponents in the range $1 < q \leq 2$, these are just the Poisson extensions of boundary functions in $L^q(\mathbb{T})$: $h^q(\mathbb{D}) \cong L^q(\mathbb{T})$, isometrically and isomorphically.

Proposition 7.1.1. $(1 < q \leq 2)$ *Suppose $h : \mathbb{D} \to \mathbb{C}$ is the Poisson extension to the disk $\mathbb{D}$ of a boundary function in $L^q(\mathbb{T})$, also denoted by h. Then, unless h vanishes identically, it enjoys the estimate*

$$\int_{\mathbb{D}}|h|^q\mathrm{d}A + (q-1)\int_{\mathbb{D}}(1-|z|^2)\frac{|\partial h(z)|^2}{|h(z)|^{2-q}}\mathrm{d}A(z) \leq \int_{\mathbb{T}}|h|^q\mathrm{d}s.$$

Proof. We apply the estimate (7.1.5) to the dilates $h_r(z) = h(rz)$, for $0 < r < 1$, and use $s = q/2$. Then as $r \to 1^-$ and $\epsilon \to 0^+$,

$$\int_{\mathbb{T}}(|h_r|^2+\epsilon)^{q/2}\mathrm{d}s \to \int_{\mathbb{T}}|h|^q\mathrm{d}s,$$

and Fatou's lemma tells gives us the necessary control of the left-hand side. □

7.2. Hardy space methods and the weighted Bergman reverse Carleman isoperimetric inequality

We now turn the estimate (6.2.1) into a tool for effective control of a function h in the harmonic Hardy space $h^q(\mathbb{D})$ for $1 < q \le 2$. First, we implement (6.2.1) with $\theta = 2 - q$:

$$\|(\partial h)_r\|_{A^1(\mathbb{D})} \le \frac{1}{r^2}\Bigg(\int_{\mathbb{D}(0,r)} \frac{|h(z)|^{2-q}}{1-|z|^2}\mathrm{d}A(z)\Bigg)^{1/2}\Bigg(\int_{\mathbb{D}(0,r)} \frac{|\partial h(z)|^2}{|h(z)|^{2-q}}(1-|z|^2)\mathrm{d}A(z)\Bigg)^{1/2}, \tag{7.2.1}$$

for $0 < r < 1$, where all we ask of $h : \mathbb{D} \to \mathbb{C}$ is that it is harmonic in $\mathbb{D}$. Using polar coordinates, we see that

$$\int_{\mathbb{D}(0,r)} \frac{|h(z)|^{2-q}}{1-|z|^2}\mathrm{d}A(z) = \int_0^r \int_{\mathbb{T}} |h_\varrho(\zeta)|^{2-q}\mathrm{d}s(\zeta)\frac{2\varrho\mathrm{d}\varrho}{1-\varrho^2}$$
$$\le \sup_{0<\varrho<r} \|h_\varrho\|^{2-q}_{L^{2-q}(\mathbb{T})} \int_0^r \frac{2\varrho\mathrm{d}\varrho}{1-\varrho^2} = \sup_{0<\varrho<r} \|h_\varrho\|^{2-q}_{L^{2-q}(\mathbb{T})} \log\frac{1}{1-r^2} \tag{7.2.2}$$

We recognize on the right-hand side the harmonic Hardy space $h^{2-q}(\mathbb{D})$ quasinorm of the dilate h_r. From Hölder's inequality and the restriction $1 < q \le 2$, we know that $\|h_\varrho\|_{L^{2-q}(\mathbb{T})} \le \|h_\varrho\|_{L^1(\mathbb{T})}$, and, in addition, the norms $\|h_\varrho\|_{L^1(\mathbb{T})}$ are known to increase with the radius ϱ. So, it follows from (7.2.2) that

$$\int_{\mathbb{D}(0,r)} \frac{|h(z)|^{2-q}}{1-|z|^2}\mathrm{d}A(z) \le \|h_r\|^{2-q}_{L^1(\mathbb{T})} \log\frac{1}{1-r^2}. \tag{7.2.3}$$

Next, we assume h is the Poisson extension to the disk $\mathbb{D}$ of a function in $L^q(\mathbb{T})$, which we also denote by h. Then, by Proposition 7.1.1, we know that

$$\int_{\mathbb{D}} (1-|z|^2)\frac{|\partial h(z)|^2}{|h(z)|^{2-q}}\mathrm{d}A(z) \le \frac{1}{q-1}\Bigg\{\int_{\mathbb{T}} |h|^q\mathrm{d}s - \int_{\mathbb{D}} |h|^q\mathrm{d}A\Bigg\}, \tag{7.2.4}$$

and by inserting the estimates (7.2.3) and (7.2.4) into (7.2.1), we obtain that

$$\|(\partial h)_r\|_{A^1(\mathbb{D})} \le \frac{1}{r^2}\|h\|^{1-\frac{q}{2}}_{L^1(\mathbb{T})}\Bigg\{\frac{1}{q-1}\Bigg(\int_{\mathbb{T}} |h|^q\mathrm{d}s - \int_{\mathbb{D}} |h|^q\mathrm{d}A\Bigg)\Bigg\}^{1/2}\sqrt{\log\frac{1}{1-r^2}}. \tag{7.2.5}$$

We will refer to minus the differential entropy as the *differential anentropy*, and as the area-L^1 norm of the dilatation of the gradient gets controlled in terms of this quantity, we name the result accordingly.

Theorem 7.2.1. (differential anentropy bound) *Suppose $h : \mathbb{D} \to \mathbb{R}$ is the Poisson extension to the disk of a function in $L^p(\mathbb{T})$, for some p with $1 < p \le 2$. The boundary function is also denoted by h. If $h \ge 0$ on $\mathbb{D}$, and if $h(0) = 1$, then*

$$\|(\partial h)_r\|_{A^1(\mathbb{D})} \le \frac{1}{r^2}\bigg\{\int_{\mathbb{T}} h\log h\,\mathrm{d}s\bigg\}^{1/2}\sqrt{\log\frac{1}{1-r^2}}, \qquad 0<r<1.$$

Proof. The function

$$F(t,q) := \frac{t^q - t}{q-1}, \qquad 0 \le t < +\infty,\;\; 1<q<2,$$

is strictly decreasing as a function of q, with limit

$$F(t,1) := \lim_{q\to 1^+} F(t,q) = t\log t,$$

understood as $F(0,1) = 0$ for $t = 0$. Although $F(t,1)$ attains negative values for $0 < t < 1$, it is easy to see that $F(t,q) \ge F(t,1) \ge -\mathrm{e}^{-1}$. Since it is given that $h \ge 0$, we know that $|h| = h$, and by the subharmonicity of the function h^q,

$$\int_{\mathbb{D}} |h|^q \mathrm{d}A = \int_{\mathbb{D}} h^q \mathrm{d}A \ge h(0)^q = 1,$$

so that

$$\frac{1}{q-1}\bigg(\int_{\mathbb{T}} |h|^q \mathrm{d}s - \int_{\mathbb{D}} |h|^q \mathrm{d}A\bigg) \le \frac{1}{q-1}\bigg(\int_{\mathbb{T}} h^q \mathrm{d}s - 1\bigg) = \frac{1}{q-1}\int_{\mathbb{T}} (h^q - h)\mathrm{d}s = \int_{\mathbb{T}} F(h,q)\mathrm{d}s.$$

By the monotone convergence theorem applied to the positive functions $F(h,q) + \mathrm{e}^{-1}$, we see that

$$\lim_{q\to 1^+}\int_{\mathbb{T}} F(h,q)\mathrm{d}s = \int_{\mathbb{T}} F(h,1)\mathrm{d}s = \int_{\mathbb{T}} h\log h\mathrm{d}s$$

provided that $h \in L^p(\mathbb{T})$ for some p with $1 < p \le 2$, so that the left-hand side limit is finite. By letting $q \to 1^+$ in (7.2.5), the claimed estimate follows. □

7.3. Applications of duality techniques to the dilates of Bloch functions

In view of Lemma 5.2.1, combined with the equality (2.2.4), we have, for $\mu \in L^\infty(\mathbb{D})$, $g := \mathbf{P}\mu \in \mathcal{B}(\mathbb{D})$, and a harmonic function h on $\mathbb{D}$, which is the Poisson integral of an $L^1(\mathbb{T})$ function, also denoted by h,

$$\langle zg_r, h\rangle_{\mathbb{T}} = \langle g, (\partial h)_r\rangle_{\mathbb{D}} = \langle \mathbf{P}\mu, (\partial h)_r\rangle_{\mathbb{D}} = \langle \mu, (\partial h)_r\rangle_{\mathbb{D}},$$

for $0 < r < 1$. If, in addition, $h \geq 0$ on $\mathbb{D}$, $h(0) = 1$, and the boundary values are in $L^2(\mathbb{T})$, then Theorem 7.2.1 gives that

$$|\langle zg_r, h\rangle_{\mathbb{T}}| = |\langle \mu, (\partial h)_r\rangle_{\mathbb{D}}| \leq \|\mu\|_{L^\infty(\mathbb{D})}\|(\partial h)_r\|_{A^1(\mathbb{D})}$$
$$\leq \frac{\|\mu\|_{L^\infty(\mathbb{D})}}{r^2}\bigg\{\int_{\mathbb{T}} h\log h\,\mathrm{ds}\bigg\}^{1/2}\sqrt{\log\frac{1}{1-r^2}}, \qquad 0 < r < 1. \quad (7.3.1)$$

Theorem 7.3.1. *Suppose $g = \mathbf{P}\mu$, where $\mu \in L^\infty(\mathbb{D})$, and consider, for $0 < r < 1$ and $0 \leq \eta < +\infty$, the set*

$$\mathcal{E}(r, \eta) := \big\{\zeta \in \mathbb{T} :\ \mathrm{Re}(\zeta g(r\zeta)) \geq \eta\big\}.$$

Then $|\mathcal{E}(r, \eta)|_s$, the s-length of this set, enjoys the bound

$$|\mathcal{E}(r, \eta)|_s \leq \exp\bigg\{-\frac{r^4\eta^2}{\|\mu\|^2_{L^\infty(\mathbb{D})}\log\frac{1}{1-r^2}}\bigg\}.$$

Proof. We let h be the Poisson extension of the boundary function which equals $1/|\mathcal{E}(r, \eta)|_s$ on $\mathcal{E}(r, \eta)$ and vanishes off $\mathcal{E}(r, \eta)$. Then $h \geq 0$ on $\mathbb{D}$, and $h(0) = 1$, and the boundary function is in $L^\infty(\mathbb{T})$, so we are in a position to apply (7.3.1). As it turns out, the indicated estimate is a direct consequence of (7.3.1). □

Corollary 7.3.2. *Suppose $g = \mathbf{P}\mu$, where $\mu \in L^\infty(\mathbb{D})$, and consider, for $0 < r < 1$ and $0 \leq \eta < +\infty$, the set*

$$\mathcal{E}_N(r, \eta) := \big\{\zeta \in \mathbb{T} :\ \max_k \mathrm{Re}[\omega^k\zeta g(r\zeta)] \geq \eta\big\},$$

where $\omega := \mathrm{e}^{\mathrm{i}2\pi/N} \in \mathbb{T}$ is a root of unity, for some integer $N = 1, 2, 3, \ldots$. Then the s-length of this set enjoys the bound

$$|\mathcal{E}_N(r, \eta)|_s \leq N\exp\bigg\{-\frac{r^4\eta^2}{\|\mu\|^2_{L^\infty(\mathbb{D})}\log\frac{1}{1-r^2}}\bigg\}.$$

Proof. The assertion is immediate from Theorem 7.3.1, since the set $\mathcal{E}_N(r, \eta)$ may be split as the union of N sets, each of which may be estimated using Theorem 7.3.1. □

Lemma 7.3.3. *For $0 < r < 1$ and $0 \leq \eta < +\infty$, the set*

$$\mathcal{F}(r, \eta) := \big\{\zeta \in \mathbb{T} :\ |g(r\zeta)| \geq \eta\big\}$$

is contained in $\mathcal{E}_N(r, \eta')$, provided $N \geq 3$ and $\eta' = \eta\cos\frac{\pi}{N}$.

Proof. This follows from a geometric consideration which involves the inscription of a regular polygon with N edges inside a circle. □

Corollary 7.3.4. *Suppose $g = \mathbf{P}\mu$, where $\mu \in L^\infty(\mathbb{D})$, and consider, for $0 < r < 1$ and $0 \le \eta < +\infty$, the set*

$$\mathcal{F}(r,\eta) := \big\{\zeta \in \mathbb{T} : \ |g(r\zeta)| \ge \eta\big\}.$$

Then the s-length of this set enjoys the bound

$$|\mathcal{F}(r,\eta)|_s \le \min_{N\ge 3} N \exp\left\{-\frac{r^4\eta^2 \cos^2\frac{\pi}{N}}{\|\mu\|^2_{L^\infty(\mathbb{D})}\log\frac{1}{1-r^2}}\right\}.$$

Proof. The assertion is an immediate consequence of Corollary 7.3.2 together with Lemma 7.3.3. □

In its turn, this then leads to the following result, which constitutes part (a) of Theorem 1.3.1.

Corollary 7.3.5. *Suppose $g = \mathbf{P}\mu$, where $\mu \in L^\infty(\mathbb{D})$, and $\|\mu\|_{L^\infty(\mathbb{D})} \le 1$. Suppose that $0 < a < 1$. We then have the estimate*

$$\int_{\mathbb{T}} \exp\left\{a\frac{r^4|g(r\zeta)|^2}{\log\frac{1}{1-r^2}}\right\}\mathrm{d}s(\zeta) \le \frac{10}{(1-a)^{3/2}}, \qquad 0<r<1.$$

Proof. Let ν_r be the function defined by

$$\nu_r(\eta) := |\mathcal{F}(r,\eta)|_s, \qquad 0 \le \eta < +\infty,$$

where the set $\mathcal{F}(r,\eta)$ is as in Lemma 7.3.3. The function $\eta \mapsto \nu_r(\eta)$ is then a decreasing function which takes values in the interval $[0,1]$. We realize that

$$\int_{\mathbb{T}} \exp\left\{a\frac{r^4|g(r\zeta)|^2}{\log\frac{1}{1-r^2}}\right\}\mathrm{d}s(\zeta) = -\int_0^{+\infty} \exp\left\{a\frac{r^4\eta^2}{\log\frac{1}{1-r^2}}\right\}\mathrm{d}\nu_r(\eta),$$

and an application of integration by parts together with the estimate of Corollary 7.3.4 (for big enough N) shows that

$$\int_{\mathbb{T}} \exp\left\{a\frac{r^4|g(r\zeta)|^2}{\log\frac{1}{1-r^2}}\right\}\mathrm{d}s(\zeta) = 1 + \int_0^{+\infty} \exp\left\{a\frac{r^4\eta^2}{\log\frac{1}{1-r^2}}\right\}\frac{2ar^4\eta}{\log\frac{1}{1-r^2}}\nu_r(\eta)\mathrm{d}\eta. \tag{7.3.2}$$

In this step, we already used that $0 < a < 1$. Next, we let $N \ge 3$ be an integer so big that $0 < a < \cos^2\frac{\pi}{N}$ holds. The estimate of Corollary 7.3.4 applied to (7.3.2) leads to

$$\int_{\mathbb{T}} \exp\left\{a\frac{r^4|g(r\zeta)|^2}{\log\frac{1}{1-r^2}}\right\}\mathrm{d}s(\zeta)$$
$$\le 1+N\int_0^{+\infty}\exp\left\{-(\cos^2\tfrac{\pi}{N}-a)\frac{r^4\eta^2}{\log\frac{1}{1-r^2}}\right\}\frac{2ar^4\eta}{\log\frac{1}{1-r^2}}\mathrm{d}\eta = 1+\frac{aN}{\cos^2\frac{\pi}{N}-a}. \qquad (7.3.3)$$

It remains to choose N. We pick N to be *the smallest integer with*

$$N \ge \frac{\pi\sqrt{3}}{(1-a)^{1/2}};$$

then automatically, $N > 5$, and

$$\cos^2\frac{\pi}{N} - a > 1-a-\frac{\pi^2}{N^2} \ge 1-a-\frac{1-a}{3} = \frac{2}{3}(1-a).$$

At the same time, we have that

$$N \le 1+\frac{\pi\sqrt{3}}{(1-a)^{1/2}} \le \frac{1+\pi\sqrt{3}}{(1-a)^{1/2}},$$

and a combination with the above estimate shows that

$$1+\frac{aN}{\cos^2\frac{\pi}{N}-a} \le 1+\frac{\pi\sqrt{3}+1}{2/3}\,\frac{a}{(1-a)^{3/2}} \le 1+\frac{9a}{(1-a)^{3/2}} \le \frac{10}{(1-a)^{3/2}}.$$

The assertion of the corollary now follows from the estimate (7.3.3). □

Remark 7.3.6. In particular, Corollary 7.3.5 shows that atv $\mathbf{P}\mu \le \|\mu\|^2_{L^\infty(\mathbb{D})}$ for functions $\mu \in L^\infty(\mathbb{D})$.

7.4. The control of the moments of a Bloch function

We begin with the following easy lemma.

Lemma 7.4.1. *For $0 < s < +\infty$, we have the inequality*

$$y^s \le s^s \mathrm{e}^{-s+y}, \qquad 0 \le y < +\infty.$$

The proof is a calculus exercise and therefore omitted.

Proof of Corollary 1.7.1. Without loss of generality, we may assume $\|\mu\|_{L^\infty(\mathbb{D})} = 1$. We apply the above lemma with $s = q/2$ and

$$y = ar^4\frac{|g(r\zeta)|^2}{\log\frac{1}{1-r^2}},$$

where $0 < a < 1$, and obtain

$$a^{q/2} r^{2q} \frac{|g(r\zeta)|^q}{\big(\log \frac{1}{1-r^2}\big)^{q/2}} \le (q/2)^{q/2} \mathrm{e}^{-q/2} \exp\left\{ a r^4 \frac{|g(r\zeta)|^2}{\log \frac{1}{1-r^2}} \right\}.$$

After integration along the circle $\mathbb{T}$ in the ζ variable, we obtain

$$\begin{aligned}\int_{\mathbb{T}} |g(r\zeta)|^q \mathrm{d}s(\zeta) &\le \left(\frac{q}{2\mathrm{e}ar^4}\right)^{q/2} \left(\log \frac{1}{1-r^2}\right)^{q/2} \int_{\mathbb{T}} \exp\left\{ a r^4 \frac{|g(r\zeta)|^2}{\log \frac{1}{1-r^2}} \right\} \mathrm{d}s(\zeta) \\ &\le \frac{10}{a^{q/2}(1-a)^{3/2}} \left(\frac{q}{2\mathrm{e}r^4}\right)^{q/2} \left(\log \frac{1}{1-r^2}\right)^{q/2},\end{aligned}$$

where in the last inequality we implemented the estimate of Theorem 1.3.1. We are free to pick $0 < a < 1$, and with the choice $a := q/(q+3)$, we arrive at

$$\int_{\mathbb{T}} |g(r\zeta)|^q \mathrm{d}s(\zeta) \le 10(\mathrm{e}/3)^{3/2}(3+q)^{3/2} \left(\frac{q}{2\mathrm{e}}\right)^{q/2} \left(\frac{1}{r^4} \log \frac{1}{1-r^2}\right)^{q/2},$$

which gives the claimed estimate, since $\mathrm{e} < 3$. □

8. Conformal and quasiconformal mapping

8.1. Conformal mappings: the standard classes $\mathcal{S}$ and Σ

It is a central theme in Conformal Mapping to analyze the local dilation/contraction/rotation of the mapping in question. To be more specific, we introduce the standard class $\mathcal{S}$ of univalent functions $\varphi : \mathbb{D} \to \mathbb{C}$ subject to the normalizations $\varphi(0) = 0$ and $\varphi'(0) = 1$. We consider the function $h_\varphi(z) := \log \varphi'(z)$, which may be referred to as the *local complex distortion exponent*. A classical estimate of h_φ (due to Koebe and Bieberbach) is the inequality

$$\big|(1-|z|^2)h'_\varphi(z) - 2\bar{z}\big| = \left|(1-|z|^2)\frac{\varphi''(z)}{\varphi'(z)} - 2\bar{z}\right| \le 4, \qquad z \in \mathbb{D}. \tag{8.1.1}$$

In particular, h_φ is in the Bloch space, with seminorm estimate $\|h_\varphi\|_{\mathcal{B}(\mathbb{D})} \le 6$. On the other hand, *Becker's univalence criterion* asserts that if $\varphi : \mathbb{D} \to \mathbb{C}$ is a function which is locally univalent, that is, $\varphi'(z) \neq 0$ for all $z \in \mathbb{D}$, and, in addition, $\|h_\varphi\|_{\mathcal{B}(\mathbb{D})} \le 1$, then φ is necessarily univalent. Moreover, the bound 1 which appears here is best possible (see [8,9]). It seems that we are in a situation somewhat analogous to the what we found for $\mathbf{P}L^\infty(\mathbb{D})$ in Subsection 2.2: the set

$$h_{\mathcal{S}} := \{h_\varphi : \varphi \in \mathcal{S}\}$$

is contained in 6 times the unit ball of $\mathcal{B}(\mathbb{D})$, and every element g in the unit ball of $\mathcal{B}(\mathbb{D})$ with $g(0) = 0$ is in $h_{\mathcal{S}}$. One minor difference is that we cannot expect $h_{\mathcal{S}}$ to share the properties of a unit ball (convexity etc). The behavior of $h_\varphi = \log \varphi'$ may acquire additional boundary growth if the image domain $\varphi(\mathbb{D})$ is unbounded, because the derivative φ' is taken with respect to the Euclidean structure in the image $\varphi(\mathbb{D}) \subset \mathbb{C}$. To avoid taking such effects into consideration, we can pass to the univalent function $\psi : \mathbb{D}_e \to \mathbb{C}_\infty$ given by

$$\psi(\zeta) := \frac{1}{\varphi(1/\zeta)}, \qquad \zeta \in \mathbb{D}_e, \tag{8.1.2}$$

which has $\psi(\zeta) = \zeta + \mathrm{O}(1)$ as $\zeta \to \infty$ and hence is element of the class Σ. As for ψ, we know that the complement of the image domain $\psi(\mathbb{D}_e)$ is a compact continuum which does not divide the plane, contains the origin, and has diameter at most 4. The derivative of ψ evaluated at the point $1/z$ equals

$$\psi'(1/z) = \frac{z^2\varphi'(z)}{[\varphi(z)]^2}, \qquad z \in \mathbb{D}, \tag{8.1.3}$$

which encourages us to replace the study of h_φ by the study of

$$g_\varphi(z) := \log \frac{z^2\varphi'(z)}{[\varphi(z)]^2} = \log \psi'\left(\frac{1}{z}\right) = h_\psi\left(\frac{1}{z}\right), \qquad z \in \mathbb{D}. \tag{8.1.4}$$

The optimal pointwise estimate for the local complex distortion exponent $h_\psi(\zeta) = \log \psi'(\zeta)$ is (see [16], p. 123)

$$|h_\psi(\zeta)| = \big|\log \psi'(\zeta)\big| \le \log \frac{|\zeta|^2}{|\zeta|^2 - 1}, \qquad \zeta \in \mathbb{D}_e,$$

which in terms of the function $g_\varphi(z) = h_\psi(1/z)$ reads

$$|g_\varphi(z)| = |h_\psi(1/z)| = \left|\log \frac{z^2\varphi'(z)}{[\varphi(z)]^2}\right| \le \log \frac{1}{1 - |z|^2}, \qquad z \in \mathbb{D}. \tag{8.1.5}$$

8.2. Goluzin's inequality for the class Σ

There is an analogue of (8.1.1) found by Goluzin in 1943 (see [18], p. 132, as well as [1]) which applies to the class Σ, but contrary to first expectations, the estimate is not essentially better than for the class $\mathcal{S}$. Goluzin's inequality, which is sharp pointwise, reads as follows:

$$\left|\zeta h_\psi'(\zeta) + \frac{4|\zeta|^2 - 2}{|\zeta|^2 - 1} - \frac{4|\zeta|^2}{|\zeta|^2 - 1}\frac{E(1/|\zeta|)}{K(1/|\zeta|)}\right| \le \frac{4|\zeta|^2}{|\zeta|^2 - 1}\left(1 - \frac{E(1/|\zeta|)}{K(1/|\zeta|)}\right), \qquad \zeta \in \mathbb{D}_e, \tag{8.2.1}$$

where $h_\psi(\zeta) = \log \psi'(\zeta)$, and E and K denote the elliptic integrals

$$E(s) := \int_0^1 \sqrt{\frac{1 - s^2t^2}{1 - t^2}}\,\mathrm{d}t, \qquad 0 \le s \le 1,$$

and

$$K(s) := \int_0^1 \frac{\mathrm{d}t}{\sqrt{(1 - s^2t^2)(1 - t^2)}}, \qquad 0 \le s < 1.$$

The ratio $E(s)/K(s)$ tends to 0 as $s \to 1^-$, and it is elementary to obtain the estimates

$$1 - s^2 \le \frac{E(s)}{K(s)} \le 1, \qquad 0 \le s < 1;$$

as a consequence, we have that

$$0 \le \frac{4|\zeta|^2}{|\zeta|^2 - 1}\left(1 - \frac{E(1/|\zeta|)}{K(1/|\zeta|)}\right) \le \frac{4}{|\zeta|^2 - 1}, \qquad \zeta \in \mathbb{D}_e, \tag{8.2.2}$$

and

$$-\frac{2}{|\zeta|^2 - 1} \le \frac{4|\zeta|^2 - 2}{|\zeta|^2 - 1} - \frac{4|\zeta|^2}{|\zeta|^2 - 1}\frac{E(1/|\zeta|)}{K(1/|\zeta|)} \le \frac{2}{|\zeta|^2 - 1}, \qquad \zeta \in \mathbb{D}_e. \tag{8.2.3}$$

By inserting the estimates (8.2.2) and (8.2.3) into Goluzin's inequality (8.2.1), we arrive at

$$|\zeta h_\psi'(\zeta)| \le \frac{6}{|\zeta|^2 - 1}, \qquad \zeta \in \mathbb{D}_e, \tag{8.2.4}$$

which in terms of the function $g_\varphi(z) = h_\psi(1/z)$ in (8.1.4) reads

$$(1 - |z|^2)|g_\varphi'(z)| \le 6|z|, \qquad z \in \mathbb{D}. \tag{8.2.5}$$

Proposition 8.2.1. *Let $\nu_\varphi \in L^\infty(\mathbb{D})$ be the function*

$$\nu_\varphi(z) := (1 - |z|^2)\frac{g_\varphi'(z)}{z}, \qquad z \in \mathbb{D}.$$

Then $\|\nu_\varphi\|_{L^\infty(\mathbb{D})} \le 6$, and $z^2\mathbf{P}\nu_\varphi(z) = g_\varphi(z)$.

Proof. The norm estimate follows from (8.2.5), and the equality $z^2\mathbf{P}\nu_\varphi(z) = g_\varphi(z)$ is as in the calculation (4.3.1), since $g(0) = g'(0) = 0$ holds (compare, e.g., with the estimate (8.1.5)). □

8.3. Holomorphic motion, Beltrami equations, and quasiconformal extensions

We use the standard terminology of quasiconformal theory. So, for instance, if $\varphi : \Omega_1 \to \Omega_2$ is a homeomorphism of complex domains, and if k is a real parameter with $0 \le k < 1$, then φ is said to be *k-quasiconformal* if it of the Sobolev class $W^{1,2}$ locally, and enjoys the dilatation estimate

$$|\bar\partial_z \varphi(z)| \le k|\partial_z\varphi(z)|, \qquad z \in \Omega_1,$$

in the almost-everywhere sense. We will also need the notion of *holomorphic motion* (see [42,50], and the recent book [5]).

We recall that $\psi \in \Sigma$ means that $\psi : \mathbb{D}_e \to \mathbb{C}_\infty$ is univalent with $\psi(\zeta) = \zeta + \mathrm{O}(1)$ as $\zeta \to \infty$. Holomorphic motion will allow us to embed such a ψ with a quasiconformal extension $\mathbb{C} \to \mathbb{C}$ in a chain of conformal mappings indexed by a parameter $\lambda \in \mathbb{D}$. The procedure is somewhat analogous to the Loewner chain method, but the deformation is based on ideas from quasiconformal theory and Beltrami equations, and some aspects even rely on methods from Several Complex Variables.

We begin with a function $\mu \in L^\infty(\mathbb{D})$ of norm at most 1, that is, more formally, we have that $\mu \in L^\infty(\mathbb{C})$ with

$$|\mu(\zeta)| \le 1_{\mathbb{D}}(\zeta), \qquad \text{a.e. } \zeta \in \mathbb{C}_\infty. \tag{8.3.1}$$

Next, we obtain the so-called standard solution $\Psi : \mathbb{D} \times \mathbb{C} \to \mathbb{C}$ to the Beltrami equation

$$\bar\partial_\zeta \Psi(\lambda,\zeta) = \lambda\mu(\zeta)\,\partial_\zeta\Psi(\lambda,\zeta), \qquad (\lambda,\zeta) \in \mathbb{D}\times\mathbb{C}.$$

We need to describe in greater detail how to obtain this standard solution $\Psi(\lambda,\zeta)$. To this end, we need the Cauchy transform,

$$\mathbf{C}\mu(\zeta) := \int_{\mathbb{C}} \frac{\mu(w)}{\zeta - w}\mathrm{d}A(w), \tag{8.3.2}$$

as well as the Beurling transform

$$\mathbf{S}\mu(\zeta) := -\,\mathrm{pv}\int_{\mathbb{C}} \frac{\mu(w)}{(\zeta - w)^2}\mathrm{d}A(w). \tag{8.3.3}$$

If $\mathbf{M}_\mu$ stands for the multiplication operator $\mathbf{M}_\mu f(\zeta) := \mu(\zeta)f(\zeta)$, then

$$\Psi(\lambda,\zeta) = \zeta + \lambda\mathbf{C}\mu(\zeta) + \lambda^2\mathbf{C}\mathbf{M}_\mu\mathbf{S}\mu(\zeta) + \lambda^3\mathbf{C}\mathbf{M}_\mu\mathbf{S}\mathbf{M}_\mu\mathbf{S}\mu(\zeta) + \cdots. \tag{8.3.4}$$

As it turns out, for each fixed $\lambda \in \mathbb{D}$, $\Psi(\lambda,\cdot)$ is a quasiconformal mapping of $\mathbb{C}_\infty$, which preserves the point at infinity, and whose restriction to $\mathbb{D}_e$ is conformal and is in the class $\Sigma^{\langle|\lambda|\rangle}$. Since $\partial_\zeta\mathbf{C} = \mathbf{S}$, the complex derivative of $\Psi(\lambda,\cdot)$ equals

$$\partial_\zeta \Psi(\lambda,\zeta) = 1 + \lambda \mathbf{S}\mu(\zeta) + \lambda^2 \mathbf{S}\mathbf{M}_\mu \mathbf{S}\mu(\zeta) + \lambda^3 \mathbf{S}\mathbf{M}_\mu \mathbf{S}\mathbf{M}_\mu \mathbf{S}\mu(\zeta) + \cdots, \tag{8.3.5}$$

and if we take the logarithm, the result is

$$H(\lambda,\zeta) := \log \partial_\zeta \Psi(\lambda,\zeta) = \lambda \hat{H}_1(\zeta) + \lambda^2 \hat{H}_2(\zeta) + \lambda^3 \hat{H}_3(\zeta) + \cdots, \tag{8.3.6}$$

where

$$\hat{H}_1(\zeta) = \mathbf{S}\mu(\zeta), \quad \hat{H}_2(\zeta) = \mathbf{S}\mathbf{M}_\mu \mathbf{S}\mu(\zeta) - \frac{1}{2}\,[\mathbf{S}\mu(\zeta)]^2, \ldots, \tag{8.3.7}$$

and the power series (8.3.6) converges for $\lambda \in \mathbb{D}$ (at least for $\zeta \in \mathbb{D}_e$).

8.4. Beurling transform formulation of the main theorem

As the Beurling transform of μ is connected with the Bergman projection of $\mu^*(z) = \mu(\bar{z})$ via the relation

$$\mathbf{S}\mu\left(\frac{1}{z}\right) = -z^2 \mathbf{P}\mu^*(z), \qquad z \in \mathbb{D}, \tag{8.4.1}$$

Theorem 1.3.1(a) has a formulation involving the Beurling $\mathbf{S}$ in place of the Bergman projection $\mathbf{P}$:

$$\int_{\mathbb{T}} \exp\left\{ a \frac{|\mathbf{S}\mu(R\zeta)|^2}{\log \frac{R^2}{R^2-1}} \right\} \mathrm{d}s(\zeta) \le C(a), \qquad 1 < R < +\infty, \ \ 0 < a < 1, \tag{8.4.2}$$

where $C(a) = 10(1-a)^{-3/2}$. Moreover, by Theorem 1.3.1(b), no such bound is possible for $1 < a < +\infty$.

8.5. Estimate from above of the universal integral means spectrum of conformal mappings with quasiconformal extension

We denote by $\Sigma^{\langle k \rangle}$ the collection of all $\psi \in \Sigma$ that have a k-quasiconformal extension $\tilde{\psi} : \mathbb{C}_\infty \to \mathbb{C}_\infty$. Via holomorphic motion, any $\psi \in \Sigma^{\langle k \rangle}$ is such that for a suitable constant C_0, the function $\psi + C_0$ may be fitted into a standard Beltrami solution family $\Psi(\lambda, \cdot)$ at the parameter value $\lambda = k$. The correct value of the constant C_0 is $C_0 := \lim_{\zeta\to\infty} \zeta - \psi(\zeta)$. This means that we may focus our attention to standard Beltrami solution families $\Psi(\lambda\zeta)$, and think of k as $|\lambda|$. By the global estimate (8.2.4), which comes from Goluzin's inequality, we know that the function $H(\lambda,\zeta)$ defined by (8.3.6) meets

$$|\zeta \partial_\zeta H(\lambda,\zeta)| \le \frac{6}{|\zeta|^2 - 1}, \qquad \lambda \in \mathbb{D}, \ \ \zeta \in \mathbb{D}_e. \tag{8.5.1}$$

In terms of the function

$$G(\lambda,z):=\frac{H(\lambda,1/z)}{\lambda},\qquad \lambda,z\in\mathbb{D}\setminus\{0\},\tag{8.5.2}$$

where the singularities at $\lambda=0$ and $z=0$ are both removable, the estimate analogous to (8.5.1) reads (compare with (8.1.5))

$$(1-|z|^2)|\partial_z G(\lambda,z)|\le\frac{6|z|}{|\lambda|},\qquad \lambda,z\in\mathbb{D}.\tag{8.5.3}$$

The left-hand side of (8.5.3) is subharmonic in $\lambda\in\mathbb{D}$, so by the maximum principle, we may improve this estimate a little:

$$(1-|z|^2)|\partial_z G(\lambda,z)|\le 6|z|,\qquad \lambda,z\in\mathbb{D}.\tag{8.5.4}$$

The expansion (8.3.6) has an analogue for $G(\lambda,z)$:

$$G(\lambda,z)=\hat G_0(z)+\lambda\hat G_1(z)+\lambda^2\hat G_2(z)+\cdots,\tag{8.5.5}$$

where $\hat G_j(z):=\hat H_{j+1}(1/z)$, so that

$$\hat G_0(z)=\mathbf{S}\mu(1/z)=-z^2\mathbf{P}\mu^*(z),\quad \hat G_1(z)=\mathbf{S}\mathbf{M}_\mu\mathbf{S}\mu(1/z)-\frac{1}{2}\,[\mathbf{S}\mu(1/z)]^2,\ldots.\tag{8.5.6}$$

The function $G(\lambda,z)$ enjoys the global growth estimate

$$|G(\lambda,z)|\le\log\frac{1}{1-|z|^2},\qquad \lambda,z\in\mathbb{D},\tag{8.5.7}$$

which may be derived from (8.2.5) by an application of the maximum principle (or the Schwarz lemma). Next, let $\nu_\lambda\in L^\infty(\mathbb{D})$ be the function

$$\nu_\lambda(z):=(1-|z|^2)\frac{\partial_z G(\lambda,z)}{z},\qquad \lambda,z\in\mathbb{D},$$

which by (8.5.3) and Proposition 8.2.1 has $\|\nu_\lambda\|_{L^\infty(\mathbb{D})}\le 6$ and $z^2\mathbf{P}\nu_\lambda(z)=G(\lambda,z)$. We consider for a moment the function

$$F(\lambda,z):=\frac{G(\lambda,z)-\hat G_0(z)}{\lambda}=\frac{z^2\mathbf{P}\nu_\lambda(z)+z^2\mathbf{P}\mu^*(z)}{\lambda}=\frac{z^2\mathbf{P}(\nu_\lambda+\mu^*)(z)}{\lambda},\tag{8.5.8}$$

for $\lambda,z\in\mathbb{D}$, which is holomorphic across $\lambda=0$ in view of the expansion (8.5.5) and the formulae (8.5.6). Since $\|\mu\|_{L^\infty(\mathbb{D})}\le 1$ and $\|\nu_\lambda\|_{L^\infty(\mathbb{D})}\le 6$, we clearly have that

$$\left\|\frac{\nu_\lambda+\mu^*}{\lambda}\right\|_{L^\infty(\mathbb{D})}\le\frac{7}{|\lambda|},\qquad \lambda\in\mathbb{D}.\tag{8.5.9}$$

Let us choose a small number $0 < \epsilon < 1$, and let $N_{\lambda,\epsilon}$ denote the function which equals

$$N_{\lambda,\epsilon}(z) := \frac{\nu_\lambda + \mu^*}{\lambda}, \qquad z \in \mathbb{D},\ 1-\epsilon \le |\lambda| < 1,$$

whereas for $|\lambda| < 1-\epsilon$, the function $N_{\lambda,\epsilon}(z)$ is given as the Poisson extension to the interior of the boundary values on the circle $|\lambda| = 1-\epsilon$. By the maximum principle applied to (8.5.9), we find that

$$\|N_{\lambda,\epsilon}\|_{L^\infty(+D)} \le \frac{7}{1-\epsilon}, \qquad \lambda \in \mathbb{D}. \tag{8.5.10}$$

As taking the Poisson extension preserves the holomorphic functions, it is a consequence of (8.5.8) that $F(\lambda, z) = z^2 \mathbf{P} N_{\lambda,\epsilon}(z)$, and it follows from (8.5.8) that

$$G(\lambda, z) = \hat{G}_0(z) + \lambda F(\lambda, z) = -z^2 \mathbf{P}\mu^*(z) + \lambda z^2 \mathbf{P} N_{\lambda,\epsilon}(z) = z^2 \mathbf{P}(-\mu^* + \lambda N_{\lambda,\epsilon})(z).$$

From the estimate (8.5.10), we see that

$$\| - \mu^* + \lambda N_{\lambda,\epsilon}\|_{L^\infty(\mathbb{D})} \le 1 + \frac{7|\lambda|}{1-\epsilon},$$

and by Theorem 1.3.1, we get that

$$\operatorname{atv} G(\lambda, \cdot) \le \| - \mu^* + \lambda N_{\lambda,\epsilon}\|^2_{L^\infty(\mathbb{D})} \le \left(1 + \frac{7|\lambda|}{1-\epsilon}\right)^2.$$

The left hand side does not depend on the choice of ϵ, and we are free to let $\epsilon \to 0^+$:

$$\operatorname{atv} G(\lambda, \cdot) \le (1 + 7|\lambda|)^2.$$

Multiplying $G(\lambda, z)$ by the parameter λ results in multiplying the asymptotic tail variance by $|\lambda|^2$:

$$\operatorname{atv} \lambda G(\lambda, z) = |\lambda|^2 \operatorname{atv} G(\lambda, \cdot).$$

Finally, by Corollary 3.4.2, we may estimate the exponential type spectrum of the function $\exp(\lambda G(\lambda, \cdot))$:

$$\beta_{\lambda G(\lambda,\cdot)}(t) \le \frac{1}{4}|\lambda t|^2 (1 + 7|\lambda|)^2. \tag{8.5.11}$$

Proof of Theorem 1.8.1. Since

$$\lambda G(\lambda, 1/\zeta) = H(\lambda, \zeta) = \log \partial_\zeta \Psi(\lambda, \zeta),$$

and the exponential type spectrum estimate (8.5.11) is independent of the choice of dilatation coefficient μ in the unit ball of $L^\infty(\mathbb{D})$, this completes the proof of Theorem 1.8.1, since by holomorphic motion any $\psi \in \Sigma^{\langle k\rangle}$ can be fitted into a standard Beltrami solution family $\Psi(\lambda, \cdot)$ for the parameter value $\lambda = k$ (see, e.g., the book [5]). □

Remark 8.5.1. The source of loss of information in the proof of Theorem 1.8.1 is the fact that in the expansion (8.3.6), we can only effectively analyze the first term, $\lambda \hat{H}_1(\zeta)$. Much deeper understanding should result from an analysis of the rest of the terms, individually and put together. It is, however, known that each coefficient $\hat{H}_j(\zeta)$ is in planar BMO and hence its restriction to $\mathbb{D}_e$ is in the Bloch space (see Hamilton's paper [19], which builds on work of Reimann).

9. The dimension estimate for quasicircles

9.1. Pommerenke's Minkowski dimension bound

If we combine the estimate $\mathrm{B}(k,t) \le \frac{1}{4}k^2|t|^2(1+7k)^2$ from Theorem 1.8.1 with Pommerenke's Minkowski dimension estimate [46], which refines an estimate of Makarov [39], we obtain the following. Let $F(k,t)$ be the quadratic function

$$F(k,t) := \frac{1}{4}k^2t^2(1+7k)^2 - t + 1,$$

where $0 < k < 1$ and $1 < t < 2$ are considered. Then, *if $t = t_k$ is a root to $F(k,t) = 0$ with $1 < t_k < 2$, and if $\partial_t F(k,t)|_{t=t_k} < 0$, we have that $D^+_{M,1s}(k) \le t_k$.*

Proof of Corollary 1.9.1. For small k, more precisely, $0 < k < (\sqrt{15}-1)/14 = 0.205\ldots$, the equation $F(k,t) = 0$ has exactly one root in the interval $1 < t < 2$, and denote by it by $t = t_k$; explicitly, it is given by

$$t_k = \frac{2}{1+\sqrt{1-k^2(1+7k)^2}} = 1 + \frac{k^2}{4} + \mathrm{O}(k^3),$$

where the asymptotics is as $k \to 0^+$. Moreover, it is easy to verify that $\partial_t F(k,t)|_{t=t_k} < 0$, which means that $F(k,t)$ assumes negative values to the right of $t = t_k$. By Pommerenke's estimate (see above), then, it follows that

$$D^+_{M,1s}(k) \le t_k = 1 + \frac{k^2}{4} + \mathrm{O}(k^3),$$

and, in view of (1.9.1) and (1.9.2), we obtain

$$D^+_M(k') = D^+_{M,1s}\left(\frac{2k'}{1+(k')^2}\right) \le 1 + (k')^2 + \mathrm{O}((k')^3),$$

as claimed. □

10. On a conjecture of Marshall

10.1. Marshall's conjecture

The following conjecture is from Donald Marshall's notes [43]. Recall the notation $\operatorname{atv} g$ and $\operatorname{atv}_u \mathcal{G}$ for the asymptotic and uniform asymptotic tail variances of a Bloch function g and a collection of Bloch functions $\mathcal{G}$, respectively (see (3.1.5) and (3.1.9)).

Conjecture 10.1.1. (Marshall) *For every $\varphi \in \mathcal{S}$, we have that $\operatorname{atv} g_\varphi \le 1$, where g_φ is given by (8.1.4). Indeed, we should have that $\operatorname{atv}_u g_{\mathcal{S}} \le 1$, where $g_{\mathcal{S}} := \{g_\varphi : \ \varphi \in \mathcal{S}\}$.*

More explicitly, the first (weaker) part of Marshall's conjecture amounts to having

$$\limsup_{r\to 1^-} \int_{\mathbb{T}} \exp\left\{ a \frac{|g_\varphi(r\zeta)|^2}{\log \frac{1}{1-r^2}} \right\} \mathrm{d}s(\zeta) < +\infty \tag{10.1.1}$$

for every fixed a, $0 < a < 1$, and every conformal mapping $\varphi \in \mathcal{S}$. This goes beyond even Theorem 1.3.1, as we shall see.

A related approximate exponential quadratic integrability result was obtained by Baranov and Hedenmalm (for details, see [7], pp. 20–23). Following ideas developed by Jones and Makarov [33], it was derived from the well-known exponential integrability of the Marcinkiewicz–Zygmund integral.

10.2. Marshall's conjecture and standard Beltrami solution families

The strong form of Marshall's Conjecture 10.1.1 says that for every $0 < a < 1$, we have that

$$\limsup_{R\to 1^+} \sup_{\psi\in\Sigma} \int_{\mathbb{T}} \exp\left\{ a \frac{|h_\psi(R\zeta)|^2}{\log \frac{R^2}{R^2-1}} \right\} \mathrm{d}s(\zeta) < +\infty, \tag{10.2.1}$$

where $h_\psi(\zeta) = \log \psi'(\zeta)$, as before. We will analyze some implications of this conjecture in the setting of a standard Beltrami solution family $\Psi(\lambda, \zeta)$, as in Subsection 8.5, and we retain most of the notation from there. In this context, (10.2.1) entails that

$$\limsup_{R\to 1^+} \sup_{\lambda\in\mathbb{D}} \int_{\mathbb{T}} \exp\left\{ a \frac{|H(\lambda, R\zeta)|^2}{\log \frac{R^2}{R^2-1}} \right\} \mathrm{d}s(\zeta) < +\infty, \tag{10.2.2}$$

for every $0 < a < 1$. We use polar coordinates and write $\lambda = \rho\omega$, where $0 \le \rho < 1$ and $\omega \in \mathbb{T}$. From the Plancherel identity and the geometric–arithmetic mean inequality (or Jensen's inequality), we see that

$$
\begin{aligned}
\int_{\mathbb{T}} \exp\left\{a\sum_{j=1}^{+\infty}\frac{|\hat{H}_j(R\zeta)|^2}{\log\frac{R^2}{R^2-1}}\right\}\mathrm{d}s(\zeta)
&= \lim_{\rho\to 1^+}\int_{\mathbb{T}}\exp\left\{a\int_{\mathbb{T}}\frac{|H(\rho\omega,R\zeta)|^2}{\log\frac{R^2}{R^2-1}}\mathrm{d}s(\omega)\right\}\mathrm{d}s(\zeta)\\
&\le \lim_{\rho\to 1^+}\int_{\mathbb{T}}\int_{\mathbb{T}}\exp\left\{a\frac{|H(\rho\omega,R\zeta)|^2}{\log\frac{R^2}{R^2-1}}\right\}\mathrm{d}s(\zeta)\mathrm{d}s(\omega)\\
&\le \sup_{\lambda\in\mathbb{D}}\int_{\mathbb{T}}\exp\left\{a\frac{|H(\lambda,R\zeta)|^2}{\log\frac{R^2}{R^2-1}}\right\}\mathrm{d}s(\zeta). \qquad (10.2.3)
\end{aligned}
$$

It is now clear that Conjecture 10.1.1 amounts to a far-reaching extension of Theorem 1.3.1(a), since that theorem only involves the first term in expansion on the left-hand side of (10.2.3).

Remark 10.2.1. In [43], Marshall explains how under some additional uniformity of the constants involved, the implication arrow in Corollary 3.4.2 may be reversed. See also the paper by Hedenmalm and Kayumov [23], p. 2240, which relies on the approach of Hedenmalm and Shimorin, see [26], [27]. Marshall's conjecture implies the well-known conjectures of Binder, Kraetzer, Brennan, Carleson and Jones, and in a sense it may be thought of as equivalent to a strong form of the most extensive conjecture (that of Binder).

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