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BLOCH FUNCTIONS, ASYMPTOTIC VARIANCE, AND GEOMETRIC ZERO PACKING

By Haakan Hedenmalm

Abstract. Motivated by a problem in quasiconformal mapping, we introduce a problem in complex analysis, with its roots in the mathematical physics of the Bose-Einstein condensates in superconductivity. The problem will be referred to as *geometric zero packing*, and is somewhat analogous to studying Fekete point configurations. The associated quantity is a density, denoted $\rho_{\mathbb{C}}$ in the planar case, and $\rho_{\mathbb{H}}$ in the case of the hyperbolic plane. We refer to these densities as *discrepancy densities for planar and hyperbolic zero packing*, respectively, as they measure the impossibility of atomizing the uniform planar and hyperbolic area measures. The universal asymptotic variance Σ^2 associated with the boundary behavior of conformal mappings with quasiconformal extensions of small dilatation is related to one of these discrepancy densities: $\Sigma^2 = 1-\rho_{\mathbb{H}}$. We obtain the estimates $3.2\times 10^{-5} < \rho_{\mathbb{H}} \le 0.12087$, where the upper estimate is derived from the estimate from below on Σ^2 obtained by Astala, Ivrii, Perälä, and Prause, and the estimate from below is more delicate. In particular, it follows that $\Sigma^2 < 1$, which in combination with the work of Ivrii shows that the maximal fractal dimension of quasicircles conjectured by Astala cannot be reached. Moreover, along the way, since the universal quasiconformal integral means spectrum has the asymptotics $\mathrm{B}(k,t) \sim \frac{1}{4}\Sigma^2 k^2 |t|^2$ for small t and k, the conjectured formula $\mathrm{B}(k,t) = \frac{1}{4}k^2|t|^2$ is not true. As for the actual numerical values of the discrepancy density $\rho_{\mathbb{C}}$, we obtain the estimate from above $\rho_{\mathbb{C}} \le 0.061203\ldots$ by using the equilateral triangular planar zero packing, where the assertion that equality should hold can be attributed to Abrikosov. The value of $\rho_{\mathbb{H}}$ is expected to be somewhat close to that of $\rho_{\mathbb{C}}$.

1. Introduction.

1.1. Basic notation. We write $\mathbb{R}$ for the real line and $\mathbb{C}$ for the complex plane. Moreover, we write $\mathbb{C}_\infty := \mathbb{C}\cup\{\infty\}$ for the extended complex plane (the Riemann sphere). For a complex variable $z = x+\mathrm{i}y \in \mathbb{C}$, let

$$\mathrm{d}s(z) := \frac{|\mathrm{d}z|}{2\pi}, \quad \mathrm{d}A(z) := \frac{\mathrm{d}x\mathrm{d}y}{\pi},$$

denote the normalized arc length and area measures, as indicated. Moreover, we shall write

$$\varDelta_z := \frac{1}{4}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$$

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for the normalized Laplacian, and

$$\partial_z := \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\right), \quad \bar{\partial}_z := \frac{1}{2}\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right),$$

for the standard complex derivatives; then Δ factors as $\Delta_z = \partial_z\bar{\partial}_z$. Often we will drop the subscript for these differential operators when it is obvious from the context with respect to which variable they apply. We let $\mathbb{D}$ denote the open unit disk, $\mathbb{T} := \partial\mathbb{D}$ the unit circle, and $\mathbb{D}_e$ the exterior disk:

$$\mathbb{D} := \big\{z \in \mathbb{C}\colon |z| < 1\big\}, \quad \mathbb{D}_e := \big\{z \in \mathbb{C}_\infty\colon |z| > 1\big\}.$$

We will find it useful to introduce the sesquilinear forms $\langle\cdot,\cdot\rangle_{\mathbb{T}}$ and $\langle\cdot,\cdot\rangle_{\mathbb{D}}$, as given by

$$\langle f, g\rangle_{\mathbb{T}} := \int_{\mathbb{T}} f(z)\bar{g}(z)\mathrm{d}s(z), \quad \langle f, g\rangle_{\mathbb{D}} := \int_{\mathbb{D}} f(z)\bar{g}(z)\mathrm{d}A(z),$$

where, in the first case, $f\bar{g} \in L^1(\mathbb{T})$ is required, and in the second, we need that $f\bar{g} \in L^1(\mathbb{D})$. At times we use the notation 1_E for the characteristic function of a subset E, which equals 1 on E and vanishes off E.

As for distribution theory, a locally area-summable function u will be identified with the distribution acting on a test function φ according to

$$u(\varphi) = \int_{\mathbb{C}} f\varphi \mathrm{d}A.$$

The normalization in the area element $\mathrm{d}A$ is the reason why, e.g., $\Delta \log|z|$ equals $\frac{1}{2}$ times the unit point mass at the origin (and not $\frac{\pi}{2}$ times as would be the case with the standard area element).

1.2. The standard weighted Bergman spaces. For $0 < p < +\infty$ and $\alpha \in \mathbb{R}$, we introduce the scale of standard weighted Lebesgue spaces $L^p_\alpha(\mathbb{D})$ of (equivalence classes of) Borel measurable functions $f : \mathbb{D} \to \mathbb{C}$ with

$$\|f\|^p_{L^p_\alpha(\mathbb{D})} := \int_{\mathbb{D}} |f(z)|^p(1-|z|^2)^\alpha \mathrm{d}A(z) < +\infty.$$

We say that $f \in A^p_\alpha(\mathbb{D})$ if and only if f is holomorphic in $\mathbb{D}$ and $f \in L^p_\alpha(\mathbb{D})$. In this case, we will often write $\|\cdot\|_{A^p_\alpha(\mathbb{D})}$ in place of $\|\cdot\|_{L^p_\alpha(\mathbb{D})}$. The spaces $A^p_\alpha(\mathbb{D})$ are known as *the standard weighted Bergman spaces*. For $\alpha = 0$, we recover the Bergman spaces: $A^p_0(\mathbb{D}) = A^p(\mathbb{D})$. For $\alpha \le -1$, it is easy to see that the weighted Bergman space is trivial: $A^p_\alpha(\mathbb{D}) = \{0\}$. On the other hand, for, e.g., polynomials f,

$$\lim_{\alpha \to -1^+} (\alpha+1)\|f\|^p_{L^p_\alpha(\mathbb{D})} = \int_{\mathbb{T}} |f|^p \mathrm{d}s = \|f\|^p_{H^p(\mathbb{D})},$$

where on the right-hand side appears the Hardy space $H^p(\mathbb{D})$ norm (or quasinorm, if $0 < p < 1$), given by

$$\|f\|^p_{H^p(\mathbb{D})} := \sup_{0<r<1} \int_{\mathbb{T}} |f(r\zeta)|^p \mathrm{d}s(\zeta) < +\infty.$$

This means that in a sense, $H^p(\mathbb{D})$ appears as the limit of spaces $A^p_\alpha(\mathbb{D})$ as $\alpha \to -1^+$.

1.3. The Bloch space and the Bloch seminorm. The *Bloch space* consists of those holomorphic functions $g : \mathbb{D} \to \mathbb{C}$ that are subject to the seminorm boundedness condition

$$\|g\|_{\mathcal{B}(\mathbb{D})} := \sup_{z\in\mathbb{D}} \big(1 - |z|^2\big)|g'(z)| < +\infty. \tag{1.3.1}$$

Let $\mathrm{aut}(\mathbb{D})$ denote the group of sense-preserving Möbius automorphism of $\mathbb{D}$. By direct calculation,

$$\|g \circ \gamma\|_{\mathcal{B}(\mathbb{D})} = \|g\|_{\mathcal{B}(\mathbb{D})}, \quad \gamma \in \mathrm{aut}(\mathbb{D}),$$

which says that the Bloch seminorm is invariant under all Möbius automorphisms of $\mathbb{D}$. The subspace

$$\mathcal{B}_0(\mathbb{D}) := \Big\{g \in \mathcal{B}(\mathbb{D}) : \lim_{|z|\to 1^-} \big(1 - |z|^2\big)|g'(z)| = 0\Big\}$$

is called the *little Bloch space*. An immediate observation we can make at this point is that provided that $g(0) = 0$, we have the estimate

$$|g(z)| \le \|g\|_{\mathcal{B}(\mathbb{D})} \int_0^{|z|} \frac{\mathrm{d}t}{1-t^2} = \frac{1}{2}\|g\|_{\mathcal{B}(\mathbb{D})} \log\frac{1+|z|}{1-|z|}, \quad z \in \mathbb{D},$$

which is sharp pointwise.

1.4. The Bergman projection of bounded functions. For $f \in L^1(\mathbb{D})$, let

$$\mathbf{P}f(z) := \int_{\mathbb{D}} \frac{\mu(w)}{(1 - z\bar{w})^2} \mathrm{d}A(w), \quad z \in \mathbb{D},$$

be its *Bergman projection*. Restricted to $L^2(\mathbb{D})$, it is the orthogonal projection onto the subspace of holomorphic functions. In addition, it acts boundedly on $L^p(\mathbb{D})$ for each p in the interval $1 < p < +\infty$ (see, e.g., [20]).

By appealing to the Hahn-Banach theorem, we may identify the dual space of $A^1(\mathbb{D})$ isometrically and isomorphically with the space $\mathbf{P}L^\infty(\mathbb{D})$, with respect to the sesquilinear form $\langle\cdot,\cdot\rangle_{\mathbb{D}}$, provided $\mathbf{P}L^\infty(\mathbb{D})$ is equipped with the canonical norm

$$\|g\|_{\mathbf{P}L^\infty(\mathbb{D})} := \inf\big\{\|\mu\|_{L^\infty(\mathbb{D})} \colon \mu \in L^\infty(\mathbb{D}) \text{ and } g = \mathbf{P}\mu\big\}.$$

However, since for $f \in A^1(\mathbb{D})$ and $g \in \mathbf{P}L^\infty(\mathbb{D})$, it may happen that $f\bar{g}$ fails to be in $L^1(\mathbb{D})$, the identification via the sesquilinear form requires some care. The following calculation shows that $\langle f,g\rangle_{\mathbb{D}}$ remains meaningful for $f \in A^1(\mathbb{D})$ and $g = \mathbf{P}\mu$ with $\mu \in L^\infty(\mathbb{D})$ ($f_r(z) := f(rz)$ denotes the r-dilate of f):

$$\begin{aligned}\langle f,g\rangle_{\mathbb{D}} &:= \lim_{r\to1^-}\langle f_r,g\rangle_{\mathbb{D}} = \lim_{r\to1^-}\langle f_r,\mathbf{P}\mu\rangle_{\mathbb{D}} = \lim_{r\to1^-}\langle \mathbf{P}f_r,\mu\rangle_{\mathbb{D}} \\ &= \lim_{r\to1^-}\langle f_r,\mu\rangle_{\mathbb{D}} = \langle f,\mu\rangle_{\mathbb{D}}.\end{aligned} \tag{1.4.1}$$

Here, we use the facts that the Bergman projection $\mathbf{P}$ is self-adjoint on $L^2(\mathbb{D})$ and preserves $A^2(\mathbb{D})$, and that we have the norm convergence $f_r \to f$ as $r \to 1^-$ in the space $A^1(\mathbb{D})$.

It was shown by Coifman, Rochberg, and Weiss [11] that as a linear space, $\mathbf{P}L^\infty(\mathbb{D})$ equals the Bloch space $\mathcal{B}(\mathbb{D})$, but actually, the endowed norm differs substantially from the seminorm (1.3.1). Recently, Perälä [40] obtained the estimate

$$\|\mathbf{P}\mu\|_{\mathcal{B}(\mathbb{D})} \le \frac{8}{\pi}\|\mu\|_{L^\infty(\mathbb{D})}, \quad \mu \in L^\infty(\mathbb{D}), \tag{1.4.2}$$

and showed that the constant $8/\pi$ is best possible. As for lower bounds up to a little Bloch function, the best constant is not known, but it is easy to see that the constant 1 works. In conclusion, trying to understand the space $\mathbf{P}L^\infty(\mathbb{D})$ in terms of the Bloch seminorm involves a substantial loss of information.

1.5. Hyperbolic zero packing and the main result. We mention briefly the topic of optimal discretization of a given positive Riesz mass as the sum of unit point masses. The optimization is over the possible locations of the various point masses. While this problem has a classical flavor, it seems to have never been pursued in the precise context we now present. For r with $0 < r < 1$ and a polynomial f, we consider the function

$$\Phi_f(z) := \big((1-|z|^2)|f(z)|-1\big)^2, \quad z \in \mathbb{D},$$

which we call the *hyperbolic discrepancy function*. The function Φ_f cannot vanish on a nonempty open subset, because $\Phi_f(z) = 0$ means that $|f(z)| = (1-|z|^2)^{-1}$. This is not possible for holomorphic f as in the sense of distribution theory, $\Delta\log|f|$ is a sum of half unit point masses, whereas $\Delta\log\frac{1}{1-|z|^2} = (1-|z|^2)^{-2}$, which is a smooth positive Riesz density. We are interested in the quantity

$$\rho_{\mathbb{H}} := \liminf_{r\to1^-}\inf_f \frac{\int_{\mathbb{D}(0,r)}\Phi_f(z)\frac{\mathrm{d}A(z)}{1-|z|^2}}{\int_{\mathbb{D}(0,r)}\frac{\mathrm{d}A(z)}{1-|z|^2}} = \liminf_{r\to1^-}\inf_f \frac{\int_{\mathbb{D}(0,r)}\Phi_f(z)\frac{\mathrm{d}A(z)}{1-|z|^2}}{\log\frac{1}{1-r^2}}, \tag{1.5.1}$$

where the infimum runs over all polynomials f. The number $\rho_{\mathbb{H}}$, which obviously is confined to the interval $0 \le \rho_{\mathbb{H}} \le 1$, will be referred to as the *minimal discrepancy density for hyperbolic zero packing*. It measures how close the function Φ_f can be to 0, on average. There is also a more geometric interpretation (compare with Remark 6.1.2). A very similar density appeared in the context of the plane $\mathbb{C}$ in the work of Abrikosov (see [1, 2] for a more mathematical treatment) on Bose-Einstein condensates in superconductivity.

In connection with the universal asymptotic variance Σ^2 defined below, a variant of the density $\rho_{\mathbb{H}}$ is more appropriate, which we denote by $\rho_{\mathbb{H}}^*$. We write

$$\Phi_f(z,r) := \big((1-|z|^2)|f(z)| - 1_{\mathbb{D}(0,r)}(z)\big)^2, \quad z \in \mathbb{D},\ 0<r<1,$$

so that $\Phi_f(z,r) = \Phi_f(z)$ on $\mathbb{D}(0,r)$ while $\Phi_f(z,r) = (1-|z|^2)^2|f(z)|^2$ on the annulus $\mathbb{D}\setminus\mathbb{D}(0,r)$. The number $\rho_{\mathbb{H}}^*$ is defined by

$$\rho_{\mathbb{H}}^* := \liminf_{r\to 1^-}\inf_f \frac{\int_{\mathbb{D}} \Phi_f(z,r)\frac{\mathrm{d}A(z)}{1-|z|^2}}{\int_{\mathbb{D}} 1_{\mathbb{D}(0,r)}(z)\frac{\mathrm{d}A(z)}{1-|z|^2}} = \liminf_{r\to 1^-}\inf_f \frac{\int_{\mathbb{D}} \Phi_f(z,r)\frac{\mathrm{d}A(z)}{1-|z|^2}}{\log\frac{1}{1-r^2}}, \tag{1.5.2}$$

and we call it the *minimal discrepancy density for tight hyperbolic zero packing*. Clearly, we see that $\rho_{\mathbb{H}} \le \rho_{\mathbb{H}}^*$. In an earlier version of this paper, it was conjectured that $\rho_{\mathbb{H}}^* = \rho_{\mathbb{H}}$, and some hints were offered on how one might obtain this result based on the polynomial growth $\bar\partial$-techniques which were developed in the paper [4] by Ameur, Hedenmalm, and Makarov. Using the suggested approach, this was obtained recently by Wennman [53], so we now have a theorem.

THEOREM 1.5.1. (Wennman) *It holds that* $\rho_{\mathbb{H}}^* = \rho_{\mathbb{H}}$.

Actually, Wennman's theorem also gives some information regarding how big need be the degree of an approximately extremal polynomial. As a side remark we mention that if f_0 is extremal for the problem

$$\inf_f \frac{\int_{\mathbb{D}(0,r)} \Phi_f(z)\frac{\mathrm{d}A(z)}{1-|z|^2}}{\log\frac{1}{1-r^2}}$$

a variational argument which compares f_0 with $f_0+\epsilon h$ (where h is polynomial and $\epsilon\in\mathbb{C}$ tends to 0) shows that the extremal function f_0 meets

$$\begin{aligned}(1-r^2)f_0(z) + \frac{r^2}{z^2}\int_0^z \zeta f_0(\zeta)\mathrm{d}\zeta - \mathbf{P}_r\left[\frac{f_0}{|f_0|}\right](z) \\ = \mathbf{P}_r\left[(1-|z|^2)f_0(z) - \frac{f_0(z)}{|f_0(z)|}\right](z) = 0,\end{aligned} \tag{1.5.3}$$

where $\mathbf{P}_r$ denotes the Bergman projection corresponding to the disk $\mathbb{D}(0,r)$.

Remark 1.5.2. (a) The number $\arcsin(\rho_{\mathbb{H}}^{1/2})$ describes the asymptotic minimal angle between the two vectors $z\mapsto(1-|z|^2)|f(z)|$ and the constant function 1 along a family of weighted real Hilbert spaces, as can be seen from Lemma 4.1.1 below.

(b) To better explain geometric zero packing, we also explain the planar case where the expression $\Psi_f(z) := (|f(z)|\mathrm{e}^{-|z|^2} - 1)^2$ is the *planar discrepancy function*. We believe that the equilateral triangular lattice has a good chance to be extremal for planar zero packing, and we explain later how to evaluate the planar average of the corresponding Ψ_f as an integral over a single rhombus (which is the union of two adjacent triangles).

(c) The hyperbolic zero packing problem considered here belongs to a more extensive family of problems. Indeed, it is equally natural to consider, more generally, for positive α and β, the hyperbolic (α,β)-discrepancy function $\Phi_f^{\langle\alpha,\beta\rangle}(z) = ((1-|z|^2)^\alpha|f(z)|^\beta - 1)^2$. The instance $\alpha=\beta=2$ is related to the possible improvement in the application of the Cauchy-Schwarz inequality in [22, 23].

We now present the main result of this paper.

THEOREM 1.5.3. *The minimal discrepancy density for hyperbolic zero packing enjoys the following estimate:* $3.21\times10^{-5} < \rho_{\mathbb{H}} \le 0.12087$.

The proof of this theorem is supplied in Section 5. The importance of Theorem 1.5.3 comes from its consequences.

THEOREM 1.5.4. *Suppose* $g=\mathbf{P}\mu$, *where* $\mu\in L^\infty(\mathbb{D})$, *and if* g_r *denotes the dilate* $g_r(\zeta) := g(r\zeta)$, *then*

$$\limsup_{r\to1^-}\frac{\int_{\mathbb{T}}|g_r|^2\mathrm{d}s}{\log\frac{1}{1-r^2}} \le (1-\rho_{\mathbb{H}})\,\|\mu\|^2_{L^\infty(\mathbb{D})}.$$

In other words, with

$$\sigma^2(g) := \limsup_{r\to1^-}\frac{\int_{\mathbb{T}}|g_r|^2\mathrm{d}s}{\log\frac{1}{1-r^2}}$$

as McMullen's asymptotic variance [38], and

$$\Sigma^2 := \sup\left\{\sigma^2(g)\colon\, g=\mathbf{P}\mu,\ \|\mu\|_{L^\infty(\mathbb{D})}=1\right\}$$

as the universal asymptotic variance, we have that

$$\Sigma^2 \le 1-\rho_{\mathbb{H}}. \tag{1.5.4}$$

In fact, we have equality.

THEOREM 1.5.5. *We have that* $\Sigma^2 = 1-\rho_{\mathbb{H}}$.

In the paper [5] by Astala, Ivrii, Perälä, and Prause, the estimate $\Sigma^2 \geq 0.87913$ was obtained. As a consequence of the inequality (1.5.4), we obtain that $\rho_{\mathbb{H}} \leq 0.12087$. This is where the estimate from above of Theorem 1.5.3 comes from. This estimate is much smaller than the value $1-\frac{\pi}{4}=0.214\ldots$ which is the expected value of the discrepancy density for an appropriately tailored Gaussian Analytic Function (see Subsection 6.5).

Intuitively, the approximately extremal polynomial f for the definition (1.5.1) of the discrepancy density $\rho_{\mathbb{H}}$ should have its zeros as hyperbolically equidistributed as possible, with a prescribed density. Since it stands to reason that we may model these approximately minimizing polynomials by a single holomorphic function f in the disk $\mathbb{D}$, we could try to look for f which is a diffential of order 1 (or a character-diffential of the same order 1), periodic with respect to a Fuchsian group Γ such that $\mathbb{D}/\Gamma$ is a compact Riemann surface. The most natural choice would be to also ask that the zeros of f are located along a hyperbolic equilateral triangular lattice. For instance, we may compare with the analogous planar case the bound achieved by the unilateral triangular lattice is $\rho_{\mathbb{C}} \leq 0.061203\ldots$. However, the structure of hyperbolic lattices is more rigid than the corresponding planar one, and the relevant quantities are harder to evaluate.

Remark 1.5.6. McMullen's notion of asymptotic variance is very much related to Makarov's modelling of Bloch functions as martingales [33, 34, 35]. Compare also with Lyons' approach [32] to understand Bloch functions as maps from hyperbolic Brownian motion to a planar Brownian motion (but for it, the speed of the local variance is variable but at least bounded) [32].

We note in passing that in [19], the related notion of *asymptotic tail variance* was introduced.

1.6. The quasiconformal integral means spectrum and the dimension of quasicircles. For $0<k<1$, we consider the class $\mathbf{\Sigma}_k$ of normalized k-quasiconformal mappings $\psi:\mathbb{C}_\infty\to\mathbb{C}_\infty$, where $\mathbb{C}_\infty:=\mathbb{C}\cup\{\infty\}$ is the Riemann sphere, which preserve the point at infinity and are conformal in the exterior disk $\mathbb{D}_e$. The normalization is such that the mapping has a convergent Laurent expansion of the form

$$\psi(\zeta)=\zeta+b_0+b_1\zeta^{-1}+b_2\zeta^{-2}+\ldots,\quad |\zeta|>1.$$

The *integral means spectrum* for the function $h:=\log\psi'$ (which is defined in $\mathbb{D}_e$ only) is the function

$$\beta_h(t):=\limsup_{R\to1^+}\frac{\log\int_{\mathbb{T}}|\mathrm{e}^{th(R\zeta)}|\mathrm{d}s(\zeta)}{\log\frac{R^2}{R^2-1}},\quad t\in\mathbb{C}.$$

The *universal* integral means spectrum is obtained as $\mathrm{B}(k,t) := \sup_\psi \beta_h(t)$, where $h = \log \psi'$ and ψ ranges over $\mathbf{\Sigma}_k$. In [27], Ivrii obtains the following asymptotics for $\mathrm{B}(k,t)$.

THEOREM 1.6.1. (Ivrii) *The universal integral means spectrum enjoys the asymptotics*

$$\lim_{k\to 0^+}\lim_{t\to 0}\frac{\mathrm{B}(k,t)}{k^2|t|^2} = \frac{\Sigma^2}{4}.$$

Here, Σ^2 is the universal constant which appears in (1.5.4), so that $\Sigma^2 \le 1-\rho_{\mathbb{H}} < 1$. Hence a combination of Theorems 1.5.4 and 1.6.1 refutes the general conjecture to the effect that $\mathrm{B}(k,t) = \frac{1}{4}k^2|t|^2$ for real t with $|t| \le 2/k$ [28, 43].

We now comment on Ivrii's proof of his theorem. It is important for the proof that for small k, the function $\frac{1}{k}\log\psi'$ can be modelled by $\mathbf{S}\mu$ for some $\mu \in L^\infty(\mathbb{D})$ with $\|\mu\|_{L^\infty(\mathbb{D})}$, where $\mathbf{S}$ denotes the Beurling transform

$$\mathbf{S}\mu(z) = -\mathrm{pv}\int_{\mathbb{D}}\frac{\mu(w)}{(z-w)^2}\mathrm{d}A(w).$$

Moreover, after an inversion of the plane, $\mathbf{S}\mu$ essentially becomes $\mathbf{P}\mu$. While this is standard technology in quasiconformal theory, the first important observation Ivrii makes is the "box lemma", which says that for $g = \mathbf{P}\mu$ with $\|\mu\|_{L^\infty} \le 1$, the control of the right-hand side integral in

$$\int_{\mathbb{T}}|g(r\zeta)|^2\mathrm{d}s(\zeta) = |g(0)|^2 + r^2\int_{\mathbb{D}}|g'(r\zeta)|^2\log\frac{1}{|\zeta|^2}\mathrm{d}A(\zeta)$$

can be localized to a hyperbolic disk of large fixed radius instead. This is a kind of weak control of square function type (compare with e.g., Bañuelos [7]), which tells us we are in the right ballpark. A clever combination with the Lipschitz property of Bloch functions [20] then gives the control from above and below, more or less simultaneously.

Ivrii actually obtains slightly better control than stated above. In any case, he also derives the following dimension expansion via the Legendre transform formalism connecting the dimension and integral means spectra (see, e.g., [34, 35], and [42, p. 241]).

COROLLARY 1.6.2. (Ivrii) *For any $\epsilon > 0$, the maximal Minkowski (or Hausdorff) dimension $D(k)$ of a k-quasicircle has the asymptotic expansion*

$$D(k) = 1 + \Sigma^2 k^2 + \mathrm{O}(k^{8/3-\epsilon}) \quad \text{as } k \longrightarrow 0^+.$$

Here, a *k-quasicircle* is simply the image of the unit circle $\mathbb{T}$ under a k-quasiconformal mapping of the Riemann sphere $\mathbb{C}_\infty$. In particular, Astala's well-known conjecture $D(k) = 1 + k^2$ is incorrect. In fact, Prause made the observation that $D(k) < 1 + k^2$ holds for every $0 < k < 1$, based on a combination of Corollary 1.6.2 and the methods developed by Prause and Smirnov [43, 48]. It might be conjectured that the error term in the corollary, $\mathrm{O}(k^{8/3-\epsilon})$, may be improved to $\mathrm{O}(k^3)$.

1.7. Structure of the paper. In Section 2, some basic identities are mentioned, which are based on Green's formula as well an explicit calculation involving dilates of harmonic functions. In Section 3, we explore dilational Carleman reverse isoperimetry in a Bergman space setting, which later turns out to be closely connected with the calculation of the density $\rho_{\mathbb{H}}$. In Section 4, we begin with a seemingly elementary but powerful Hilbert space lemma, and then apply it repeatedly in the proofs of Theorems 1.5.4 and 1.5.5. In Section 5, we supply the proof of Theorem 1.5.3 by first obtaining a local statement, which is then made Möbius invariant, and finally, the estimate is obtained by integration over the hyperbolic area measure. In Section 6, we begin the semi-expository part of the paper, where we introduce geometric zero packing in the context of the plane and the hyperbolic plane. We also explore various relations with Gaussian analytic functions (GAFs) as well as with certain tilings of the plane and the hyperbolic plane, respectively. In Section 7, more general exponents β are considered, and a conjecture is made for planar zero packing which we attribute to Abrikosov. A relation with the LLL-equation is mentioned, which is the Bargmann-Fock analogue of the cubic Szegő equation. Finally, in Section 8, it is explained how to interpret the general zero packing problem for compact Riemann surfaces. The solution is expressed in terms of what we have decided to call "logarithmic monopoles" but is more commonly referred to as "Green functions" in the literature. The latter is an abuse of notation since classical Green functions are not available on compact surfaces (without boundary). In addition it is explained how the geometric zero packing problem differs from the classical Fekete configuration problem, in that the Fekete problem involves Dirichlet energy, while geometric zero packing instead involves Bergman energy (in the limit as $\beta \to 0$). Finally, in Section 9 a rather speculative connection is drawn using weighted heat flow to connect geometric zero packing on compact surfaces with β-deformed Fekete-type problems.

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2. Identities for dilates of harmonic functions.

2.1. Identities involving dilates of harmonic functions. The following identity interchanges dilations, and although elementary, it is quite important. We write f_r and g_r for the dilates $f_r(z) := f(rz)$ and $g_r(z) := g(rz)$, respectively.

LEMMA 2.1.1. *Suppose $f, g : \mathbb{D} \to \mathbb{C}$ are two harmonic functions, which are area-integrable: $f, g \in L^1(\mathbb{D})$. Then we have that*

$$\langle f_r, g\rangle_{\mathbb{D}} = \langle f, g_r\rangle_{\mathbb{D}}, \quad 0 < r < 1.$$

This is Lemma 5.1.1 in [19]. We also need the following identity.

LEMMA 2.1.2. *Suppose $g, h : \mathbb{D} \to \mathbb{C}$ are functions, where g is holomorphic and h is harmonic. If $g \in L^1(\mathbb{D})$ and h is the Poisson integral of a function in $L^1(\mathbb{T})$, then we have that*

$$\langle z g_r, h\rangle_{\mathbb{T}} = \langle g, (\partial h)_r\rangle_{\mathbb{D}},$$

where we write z for the coordinate function $z(\zeta) = \zeta$.

This is Lemma 5.2.1 in [19].

3. Dilational reverse isoperimetry for a Bergman space.

3.1. Carleman's isoperimetric inequality, dilates, and the L^1 Bergman space. The classical isoperimetric inequality says that the area enclosed by a closed loop of length L is at most $L^2/(4\pi)$. Torsten Carleman (see [10, 51]) found a nice analytical approach to this fact, which gave the estimate

$$\|f\|_{A^2(\mathbb{D})} \le \|f\|_{H^1(\mathbb{D})}, \quad f \in H^1(\mathbb{D}). \tag{3.1.1}$$

Here, $H^1(\mathbb{D})$ denotes the classical Hardy space, as in Subsection 1.2. As for (3.1.1), the geometrically relevant case is when f is the derivative of the conformal mapping from disk $\mathbb{D}$ to the domain enclosed by the loop. There is of course no converse to the isoperimetrical inequality, since for a given enclosed area, the length of the boundary may be infinite. However, if the boundary curve is regularized by replacing it with a level curve of the Green function, the reverse problem starts to make sense. We will not need here the appropriate regularized reverse version of (3.1.1), but instead the analogue where the Hardy space $H^1(\mathbb{D})$ is replaced by the corresponding Bergman space $A^1(\mathbb{D})$ of area-integrable holomorphic functions.

Since $H^1(\mathbb{D})$ may be thought of as an appropriate limit of the weighted Bergman spaces $A^p_\alpha(\mathbb{D})$ as $\alpha \to -1^+$, it stands to reason that Carleman's estimate (3.1.1) might be part of a more general estimate comparing the norm in $A^p_\alpha(\mathbb{D})$

with that of $A^{2p}_{\alpha+1}(\mathbb{D})$. We shall be interested in obtaining a reverse inequality after dilation, with $p=1$ and $\alpha=0$: Is it true that, for some positive constant $C_2(r)$,

$$\|f_r\|_{A^1(\mathbb{D})} \le C_2(r)\|f\|_{A^2_1(\mathbb{D})}, \quad f\in A^2_1(\mathbb{D})? \tag{3.1.2}$$

Here, $f_r(\zeta)=f(r\zeta)$ and $0<r<1$. The question at hand is to obtain in explicit form, or at least to estimate from above, the optimal constant $C_2(r)$, for $0<r<1$. By the Cauchy-Schwarz inequality, we have that

$$\begin{aligned}
\|f_r\|_{A^1(\mathbb{D})} &= \int_{\mathbb{D}} |f(r\zeta)|\mathrm{d}A(\zeta) = \frac{1}{r^2}\int_{\mathbb{D}(0,r)} |f(z)|\mathrm{d}A(z)\\
&\le \frac{1}{r^2}\left(\int_{\mathbb{D}(0,r)}\frac{\mathrm{d}A(z)}{1-|z|^2}\right)^{1/2}\left(\int_{\mathbb{D}(0,r)}|f(z)|^2(1-|z|^2)\mathrm{d}A(z)\right)^{1/2}\\
&= \frac{1}{r^2}\left(\log\frac{1}{1-r^2}\right)^{1/2}\left(\int_{\mathbb{D}(0,r)}|f(z)|^2(1-|z|^2)\mathrm{d}A(z)\right)^{1/2} \qquad (3.1.3)\\
&\le \frac{1}{r^2}\left(\log\frac{1}{1-r^2}\right)^{1/2}\left(\int_{\mathbb{D}}|f(z)|^2(1-|z|^2)\mathrm{d}A(z)\right)^{1/2}\\
&= \frac{1}{r^2}\left(\log\frac{1}{1-r^2}\right)^{1/2}\|f\|_{A^2_1(\mathbb{D})}.
\end{aligned}$$

This immediately shows that the optimal constant in (3.1.2) is at most

$$C_2(r)\le \frac{1}{r^2}\left(\log\frac{1}{1-r^2}\right)^{1/2}, \quad 0<r<1. \tag{3.1.4}$$

We intend to improve this estimate.

4. Calculation of asymptotic variance via hyperbolic zero packing.

4.1. Suboptimality of the Cauchy-Schwarz inequality. We need to analyze the degree of suboptimality in the Cauchy-Schwarz inequality in various situations. To this end, the following lemma is helpful.

LEMMA 4.1.1. *If $\mathcal{H}$ is an $\mathbb{R}$-linear Hilbert space, the following three conditions are equivalent for two given vectors $u,v\in\mathcal{H}$ and a real θ with $0\le\theta\le 1$:*
(a) $\forall c\in\mathbb{R}$: $\|u-cv\|_{\mathcal{H}}\ge\theta\|u\|_{\mathcal{H}}$,
(b) $\forall c\in\mathbb{R}$: $\|u-cv\|_{\mathcal{H}}\ge|c|\theta\|v\|_{\mathcal{H}}$, *and*
(c) $|\langle u,v\rangle_{\mathcal{H}}|\le(1-\theta^2)^{1/2}\|u\|_{\mathcal{H}}\|v\|_{\mathcal{H}}$.

Proof. If $v=0$, all the three conditions are trivially met. Next, we assume $v\ne 0$. By expanding the square, we find that

$$\|u-cv\|^2_{\mathcal{H}} = \|u\|^2_{\mathcal{H}} + c^2\|v\|^2_{\mathcal{H}} - 2c\langle u,v\rangle_{\mathcal{H}},$$

which for $v \neq 0$ attains its minimum for $c = \|v\|_{\mathcal{H}}^{-2}\langle u,v\rangle_{\mathcal{H}}$:

$$\inf_{c\in\mathbb{R}} \|u-cv\|_{\mathcal{H}}^2 = \|u\|_{\mathcal{H}}^2 - \frac{\langle u,v\rangle_{\mathcal{H}}^2}{\|v\|_{\mathcal{H}}^2}.$$

The equivalence of (a) and (c) for $n \neq 0$ is immediate from this formula. As for (b), we note that if introduce the reciprocal constant $c' = 1/c$, the inequality reads $\|c'u - v\|_{\mathcal{H}} \geq \theta\|v\|_{\mathcal{H}}$, which is the same as (a) if we switch the roles of u and v. Moreover, since (c) is preserved under such a switch, the equivalence of (b) and (c) now follows from the equivalence of (a) and (c). □

4.2. Asymptotic variance via hyperbolic zero packing. We now explain where the bound asserted in Theorem 1.5.4 comes from.

Proof of Theorem 1.5.4. We assume $0 < r < 1$. By the definition (1.5.2) of $\rho_{\mathbb{H}}^*$, and by using Wennman's theorem $\rho_{\mathbb{H}} = \rho_{\mathbb{H}}^*$, we see that there exists a parameter $\epsilon = \epsilon(r)$ with $0 \leq \epsilon \leq 1$ and $\epsilon(r) \to 0$ as $r \to 1^-$, such that for every polynomial f,

$$\int_{\mathbb{D}} \Phi_f(z,r)\frac{\mathrm{d}A(z)}{1-|z|^2} \geq (1-\epsilon)\rho_{\mathbb{H}}\log\frac{1}{1-r^2}. \tag{4.2.1}$$

If $\mathcal{H} = L^2_\alpha(\mathbb{D})$ with $\alpha = -1$, and u, v are the real-valued functions $u = 1_{\mathbb{D}(0,r)}$ and $v(z) = (1-|z|^2)|f(z)|$, then

$$\|u-v\|_{\mathcal{H}}^2 = \int_{\mathbb{D}} \Phi_f(z,r)\frac{\mathrm{d}A(z)}{1-|z|^2}, \quad \|u\|_{\mathcal{H}}^2 = \log\frac{1}{1-r^2},$$

and (4.2.1) expresses that

$$\|u-v\|_{\mathcal{H}}^2 \geq (1-\epsilon)\rho_{\mathbb{H}}\|u\|_{\mathcal{H}}^2. \tag{4.2.2}$$

Since f is an arbitrary polynomial we may freely replace v by cv in (4.2.2), provided that $c \geq 0$:

$$\|u-cv\|_{\mathcal{H}}^2 \geq (1-\epsilon)\rho_{\mathbb{H}}\|u\|_{\mathcal{H}}^2, \quad c \geq 0. \tag{4.2.3}$$

But then the inequality of (4.2.3) holds for all $c \in \mathbb{R}$, since by the proof of Lemma 4.1.1 the global minimum over c is attained at $c = \|v\|_{\mathcal{H}}^{-2}\langle u,v\rangle_{\mathcal{H}} \geq 0$ (the case when $\|v\|_{\mathcal{H}} = 0$ is trivial). Now, from the equivalence of the conditions (a) and (c) of Lemma 4.1.1, it follows that (4.2.3) is the same as having

$$(\langle u,v\rangle_{\mathcal{H}})^2 \leq (1-(1-\epsilon)\rho_{\mathbb{H}})\|u\|_{\mathcal{H}}^2\|v\|_{\mathcal{H}}^2. \tag{4.2.4}$$

When we write this out in terms of the chosen functions u, v, we obtain

$$\left(\int_{\mathbb{D}(0,r)} |f|\mathrm{d}A\right)^2 \leq \big(1-(1-\epsilon)\rho_{\mathbb{H}}\big)\log\frac{1}{1-r^2} \times \int_{\mathbb{D}} |f(z)|^2(1-|z|^2)\mathrm{d}A(z), \tag{4.2.5}$$

So far, f was assumed to be a polynomial, but by approximation, (4.2.5) holds for any holomorphic f in $\mathbb{D}(0,r)$ such that the right-hand side integral is finite. Next, we pick a bounded holomorphic function $h : \mathbb{D} \to \mathbb{C}$ with $h(0) = 0$, and apply Lemma 2.1.2 combined with (1.4.1):

$$\langle zg_r, h\rangle_{\mathbb{T}} = \langle g, (h')_r\rangle_{\mathbb{D}} = \langle \mathbf{P}\mu, (h')_r\rangle_{\mathbb{D}} = \langle \mu, (h')_r\rangle_{\mathbb{D}}.$$

It now follows that

$$|\langle zg_r, h\rangle_{\mathbb{T}}| \leq \|\mu\|_{L^\infty(\mathbb{D})}\|(h')_r\|_{A^1(\mathbb{D})} = \frac{\|\mu\|_{L^\infty(\mathbb{D})}}{r^2}\int_{\mathbb{D}(0,r)} |h'|\mathrm{d}A,$$

and we may combine this with the estimate (4.2.5), where $f = h'$, and arrive at

$$|\langle zg_r, h\rangle_{\mathbb{T}}|^2 \leq \big(1-(1-\epsilon)\rho_{\mathbb{H}}\big)\frac{\|\mu\|^2_{L^\infty(\mathbb{D})}}{r^4}\log\frac{1}{1-r^2} \times \int_{\mathbb{D}} |h'(z)|^2(1-|z|^2)\mathrm{d}A. \tag{4.2.6}$$

By the elementary inequality $1-|z|^2 \leq \log\frac{1}{|z|^2}$ and the standard Paley identity [15, p. 236] for the H^2 norm (which is a consequence of Green's formula), we know that

$$\int_{\mathbb{D}} |h'(z)|^2(1-|z|^2)\mathrm{d}A \leq \int_{\mathbb{D}} |h'(z)|^2\log\frac{1}{|z|^2}\mathrm{d}A = \int_{\mathbb{T}} |h|^2\mathrm{d}s = \|h\|^2_{H^2(\mathbb{D})},$$

where in the second last step, we used that $h(0) = 0$. We insert this into (4.2.6):

$$|\langle zg_r, h\rangle_{\mathbb{T}}|^2 \leq \big(1-(1-\epsilon)\rho_{\mathbb{H}}\big)\frac{\|\mu\|^2_{L^\infty(\mathbb{D})}\|h\|^2_{H^2(\mathbb{D})}}{r^4}\log\frac{1}{1-r^2}.$$

Finally, we plug in $h := zg_r$, where $g = \mathbf{P}\mu$, which yields

$$\|g_r\|^2_{H^2(\mathbb{D})} = \|zg_r\|^2_{H^2(\mathbb{D})} = |\langle zg_r, zg_r\rangle_{\mathbb{T}}| \leq \big(1-(1-\epsilon)\rho_{\mathbb{H}}\big)\frac{\|\mu\|^2_{L^\infty(\mathbb{D})}}{r^4}\log\frac{1}{1-r^2}.$$

Since $\epsilon = \epsilon(r) \to 0$ as $r \to 1^-$, the claimed estimate now follows. □

We now turn to the proof of Theorem 1.5.5.

Proof of Theorem 1.5.5. Since by Wennman's theorem $\rho^*_{\mathbb{H}}=\rho_{\mathbb{H}}$ holds, the number $\rho_{\mathbb{H}}$ can be understood as the largest nonnegative real such that (4.2.1) is valid. The analysis of the equivalence of (4.2.2) and (4.2.4) (which is the same as (4.2.5)) shows that

$$r^4\|f_r\|^2_{A^1(\mathbb{D})}\le\big(1-(1-\epsilon)\rho_{\mathbb{H}}\big)\|f\|^2_{A^2_1(\mathbb{D})}\log\frac{1}{1-r^2}, \tag{4.2.7}$$

for any $f\in A^2_1(\mathbb{D})$, where $\epsilon=\epsilon(r)\to 0$ as $r\to 1^-$. Again, the constant $\rho_{\mathbb{H}}$ is largest possible so that (4.2.7) holds. We now express the estimate (4.2.7) in terms of the following operator norm bound for the dilation $\mathbf{D}_r$ given by $\mathbf{D}_r f(z)=f_r(z)=f(rz)$:

$$\|\mathbf{D}_r\|^2_{A^2_1(\mathbb{D})\to A^1(\mathbb{D})}\le\big(1-(1-\epsilon)\rho_{\mathbb{H}}\big)r^{-4}\log\frac{1}{1-r^2}. \tag{4.2.8}$$

From the optimality of the constant $\rho_{\mathbb{H}}$ in (4.2.7) we see that

$$\limsup_{r\to 1^-}\frac{\|\mathbf{D}_r\|^2_{A^2_1(\mathbb{D})\to A^1(\mathbb{D})}}{\log\frac{1}{1-r^2}}=1-\rho_{\mathbb{H}}. \tag{4.2.9}$$

With respect to $\langle\cdot,\cdot\rangle_{\mathbb{D}}$, the dual space to the weighted Bergman space $A^2_1(\mathbb{D})$ is isometrically $H^2_*(\mathbb{D})$, which is just $H^2(\mathbb{D})$ but equipped with the equivalent norm

$$\|f\|^2_{H^2_*(\mathbb{D})}:=\|f\|^2_{H^2(\mathbb{D})}+\|f\|^2_{A^2(\mathbb{D})}.$$

With respect to the dual action $\langle\cdot,\cdot\rangle_{\mathbb{D}}$, Lemma 2.1.1 tells us that $\mathbf{D}_r^*=\mathbf{D}_r$, and we recall that isometrically, the dual space to $A^2_1(\mathbb{D})$ is $H^2_*(\mathbb{D})$ while the dual to $A^1(\mathbb{D})$ is $\mathbf{P}L^\infty(\mathbb{D})$. Since by basic functional analysis the norm of an operator and its adjoint are the same, we get from (4.2.9) that

$$\limsup_{r\to 1^-}\frac{\|\mathbf{D}_r\|^2_{\mathbf{P}L^\infty(\mathbb{D})\to H^2_*(\mathbb{D})}}{\log\frac{1}{1-r^2}}=1-\rho_{\mathbb{H}}. \tag{4.2.10}$$

For $\mu\in L^\infty(\mathbb{D})$ and $g=\mathbf{P}\mu$, we observe that $\|g_r\|_{A^2(\mathbb{D})}\le\|\mu\|_{L^2(\mathbb{D})}\le\|\mu\|_{L^\infty(\mathbb{D})}$, which shows that

$$\frac{\|g_r\|^2_{H^2_*(\mathbb{D})}}{\log\frac{1}{1-r^2}}=\frac{\|g_r\|^2_{H^2(\mathbb{D})}+\|g_r\|^2_{A^2(\mathbb{D})}}{\log\frac{1}{1-r^2}}=\frac{\|g_r\|^2_{H^2(\mathbb{D})}+\mathrm{O}(1)}{\log\frac{1}{1-r^2}}=\frac{\|g_r\|^2_{H^2(\mathbb{D})}}{\log\frac{1}{1-r^2}}+\mathrm{o}(1)$$

as $r\to 1^-$. It follows from this combined with (4.2.10) that

$$\limsup_{r\to 1^-}\frac{\|\mathbf{D}_r\|^2_{\mathbf{P}L^\infty(\mathbb{D})\to H^2(\mathbb{D})}}{\log\frac{1}{1-r^2}}=1-\rho_{\mathbb{H}}. \tag{4.2.11}$$

Now, the left-hand side expresses a uniform version of the asymptotic variance Σ^2. The relation (4.2.11) entails the following statement. There exists some (sparse) sequence of radii R_j such that $0 < R_j < 1$, $R_j \to 1$ as $j \to +\infty$, and R_j increases with j, as well as functions $\mu_j \in L^\infty(\mathbb{D})$ with $\|\mu_j\|_{L^\infty(\mathbb{D})} = 1$ such that if we put $g_j := \mathbf{P}\mu_j$ we have that

$$\lim_{j\to+\infty} \frac{\|(g_j)_{R_j}\|^2_{H^2(\mathbb{D})}}{\log \frac{1}{1-R_j^2}} = 1 - \rho_{\mathbb{H}}. \tag{4.2.12}$$

But we need to produce a single function g such that (4.2.12) holds with g_j replaced by g. How to do this? We make some preliminary observations. For $\mu \in L^\infty(\mathbb{D})$, let $\mu^{(r)} \in L^\infty(\mathbb{D})$ denote the function which equals $\mu^{(r)}(z) := \mu(z/r)$ for $|z| < r$ and $\mu^{(r)}(z) = 0$ elsewhere. A direct calculation verifies that

$$\mathbf{P}\mu^{(r)}(z) = \int_{\mathbb{D}} \frac{\mu^{(r)}(w)}{(1-z\bar{w})^2}\mathrm{d}A(w) = r^2 \int_{\mathbb{D}} \frac{\mu(w)}{(1-rz\bar{w})^2}\mathrm{d}A(w) = r^2\mathbf{P}\mu(rz),$$

so that in particular $\mathbf{P}\mu^{(r)}(rz) = r^2\mathbf{P}\mu(r^2z)$. We put $r_j := R_j^{1/2}$ and build the function $\mu \in L^\infty(\mathbb{D})$ with $\|\mu\|_{L^\infty(\mathbb{D})} = 1$ as follows:

$$\mu(z) = \mu_j^{(r_j)}(z) = \mu_j(z/r_j) \quad \text{for } r_{j-1} < |z| < r_j, \quad j = 2,3,4,\ldots,$$

whereas $\mu(z) := \mu_1(z/r_1)$ for $|z| < r_1$. Next, we rewrite $\mathbf{P}\mu$ in the form (since $r_j^2 = R_j$ and $g_j = \mathbf{P}\mu_j$)

$$\begin{aligned}\mathbf{P}\mu(z) &= \mathbf{P}\mu_j^{(r_j)}(z) + \mathbf{P}(\mu - \mu^{(r_j)})(z)\\ &= R_j g_j(r_j z) + \mathbf{P}(\mu - \mu_j^{(r_j)})(z), \quad r_{j-1} < |z| < r_j,\ j = 2,3,4,\ldots.\end{aligned}$$

By definition, $\mu - \mu_j^{(r_j)}$ vanishes on the annulus $r_{j-1} < |z| < r_j$, and $\mu - \mu_j^{(r_j)} = \mu$ on the annulus $r_j < |z| < 1$. We write $\mu_j^\sharp := (\mu - \mu_j^{(r_j)})1_{\mathbb{D}(0,r_{j-1})}$ and $\mu_j^\flat := (\mu - \mu_j^{(r_j)})1_{\mathbb{D}\setminus\mathbb{D}(0,r_j)}$ so that $\mu - \mu_j^{(r_j)} = \mu_j^\sharp + \mu_j^\flat$ holds area-almost everywhere on $\mathbb{D}$. With respect to $\mu_j^\sharp$ we apply the elementary estimate

$$\begin{aligned}\left|\mathbf{P}\mu_j^\sharp(z)\right| &= \left|\int_{\mathbb{D}(0,r_{j-1})} \frac{\mu(w) - \mu_j(w/r_j)}{(1-z\bar{w})^2}\mathrm{d}A(w)\right| \le 2\int_{\mathbb{D}(0,r_{j-1})} \frac{\mathrm{d}A(w)}{|1-z\bar{w}|^2}\mathrm{d}A(w)\\ &= \frac{2}{|z|^2}\log\frac{1}{1-R_{j-1}|z|^2}.\end{aligned}$$

With respect to $\mu_j^\flat$, on the other hand, we apply the alternative elementary estimate

$$\begin{aligned}\left|\mathbf{P}\mu_j^\flat(z)\right| &= \left|\int_{\mathbb{D}\setminus\mathbb{D}(0,r_j)} \frac{\mu(w)}{(1-z\bar{w})^2}\mathrm{d}A(w)\right| \le \int_{\mathbb{D}\setminus\mathbb{D}(0,r_j)} \frac{\mathrm{d}A(w)}{|1-z\bar{w}|^2}\mathrm{d}A(w)\\ &= \frac{1}{|z|^2}\log\frac{1-R_j|z|^2}{1-|z|^2}.\end{aligned}$$

Both estimates need to be applied for $|z| = r_j$:

$$\big|\mathbf{P}\mu_j^\sharp(z)\big| \le \frac{2}{R_j}\log\frac{1}{1-R_{j-1}R_j}, \quad \big|\mathbf{P}\mu_j^\flat(z)\big| \le \frac{1}{R_j}\log(1+R_j), \text{ if } |z| = r_j.$$

We now find that

$$g_{r_j}(z) = g(r_j z) = R_j g_j(R_j z) + \mathbf{P}\big(\mu - \mu_j^{(r_j)}\big)(r_j z),$$

where

$$\big|\mathbf{P}(\mu - \mu_j^{(r_j)})(r_j z)\big| \le \frac{2}{R_j}\log\frac{1}{1-R_{j-1}R_j} + \frac{1}{R_j}\log(1+R_j), \quad \text{if } |z| = 1.$$

In particular, since the $H^2(\mathbb{D})$-norm is dominated by the supremum norm on the circle $\mathbb{T}$, we obtain that

$$\big\|R_j^{-1} g_{r_j} - (g_j)_{R_j}\big\|_{H^2(\mathbb{D})} \le \frac{2}{R_j^2}\log\frac{1}{1-R_{j-1}R_j} + \frac{1}{R_j^2}\log(1+R_j),$$

so that

$$\frac{\big\|R_j^{-1} g_{r_j} - (g_j)_{R_j}\big\|_{H^2(\mathbb{D})}}{\sqrt{\log\frac{1}{1-R_j^2}}} \le \frac{2}{R_j^2}\frac{\log\frac{1}{1-R_{j-1}R_j}}{\sqrt{\log\frac{1}{1-R_j^2}}} + \frac{1}{R_j^2}\frac{\log(1+R_j)}{\sqrt{\log\frac{1}{1-R_j^2}}}. \tag{4.2.13}$$

By passing to a subsequence, we are free to make the sequence of radii R_j as sparse as we need. So we just pick them so that the right-hand side of (4.2.13) tends to 0 as $j \to +\infty$. Finally, by taking square roots in (4.2.12) we know that

$$\lim_{j\to+\infty}\frac{\|(g_j)_{R_j}\|_{H^2(\mathbb{D})}}{\sqrt{\log\frac{1}{1-R_j^2}}} = (1-\rho_{\mathbb{H}})^{1/2},$$

which together with the estimate (4.2.13) and the sparsity of the radii leads to

$$\lim_{j\to+\infty}\frac{\|(g)_{r_j}\|_{H^2(\mathbb{D})}}{\sqrt{\log\frac{1}{1-R_j^2}}} = (1-\rho_{\mathbb{H}})^{1/2},$$

which entails that

$$\limsup_{r\to1^-}\frac{\|(g)_r\|_{H^2(\mathbb{D})}}{\sqrt{\log\frac{1}{1-r^2}}} \ge (1-\rho_{\mathbb{H}})^{1/2},$$

since

$$\lim_{r\to1^-}\frac{\log\frac{1}{1-r^2}}{\log\frac{1}{1-r^4}} = 1.$$

As we have now found a single μ which does the job, the estimate $\Sigma^2 \geq 1-\rho_{\mathbb{H}}$ follows. Since the reverse inequality $\Sigma^2 \leq 1-\rho_{\mathbb{H}}$ was obtained in Theorem 1.5.4, the equality $\Sigma^2 = 1-\rho_{\mathbb{H}}$ is immediate. □

5. The proof of the estimate from below on $\rho_{\mathbb{H}}$.

5.1. Pointwise estimates. For a real parameter α, let $\mathrm{d}A_\alpha(z) = (1-|z|^2)^\alpha \mathrm{d}A(z)$ denote the standard weighted area measure on $\mathbb{D}$. We shall need the following estimate.

LEMMA 5.1.1. *We have the following pointwise estimates, for a holomorphic function $f:\mathbb{D}\to\mathbb{C}$ and $0<r<1$:*

$$|f(z)|^2 \leq r^{-2}\sum_{j=0}^{+\infty}\frac{(j+1)(j+2)}{1+(j+1)(1-r^2)}\left(\frac{|z|^2}{r^2}\right)^j\int_{\mathbb{D}(0,r)}|f|^2\mathrm{d}A_1, \quad z\in\mathbb{D}(0,r),$$

and

$$|f'(z)|^2 \leq r^{-4}\sum_{j=0}^{+\infty}\frac{(j+1)^2(j+2)(j+3)}{1+(j+2)(1-r^2)}\left(\frac{|z|^2}{r^2}\right)^j\int_{\mathbb{D}(0,r)}|f|^2\mathrm{d}A_1, \quad z\in\mathbb{D}(0,r).$$

Proof. Let $\mathcal{H}$ denote the Hilbert space of holomorphic functions f on $\mathbb{D}(0,r)$ that are L^2-integrable with respect to the measure $\mathrm{d}A_1$. The Bergman kernel representation of a function $f\in\mathcal{H}$ is

$$f(z) = \langle f, K_r(\cdot,z)\rangle_{\mathcal{H}} = \int_{\mathbb{D}(0,r)} K_r(z,w)f(w)\mathrm{d}A_1(w),$$

where K_r is the corresponding weighted Bergman kernel

$$K_r(z,w) := \sum_{j=0}^{+\infty} r^{-2j-2}\frac{(j+1)(j+2)}{1+(j+1)(1-r^2)}(z\bar{w})^j.$$

The corresponding representation for the derivative is

$$f'(z) = \langle f, \bar{\partial}_z K_r(\cdot,z)\rangle_{\mathcal{H}} = \int_{\mathbb{D}(0,r)} \partial_z K_r(z,w)f(w)\mathrm{d}A_1(w).$$

By elementary Hilbert space methods, the optimal estimate for the value and derivative are, respectively,

$$|f(z)|^2 \leq K_r(z,z)\|f\|_{\mathcal{H}}^2, \quad |f'(z)|^2 \leq \Delta_z K_r(z,z)\|f\|_{\mathcal{H}}^2,$$

for $z \in \mathbb{D}(0,r)$. Now, as for $K_r(z,z)$ we have that

$$K_r(z,z) = \sum_{j=0}^{+\infty} r^{-2j-2} \frac{(j+1)(j+2)}{1+(j+1)(1-r^2)} |z|^{2j}, \quad z \in \mathbb{D}(0,r).$$

Finally, as regards $\Delta_z K_r(z,z)$, we have that

$$\Delta_z K_r(z,z) = \sum_{j=1}^{+\infty} r^{-2j-2} \frac{j^2(j+1)(j+2)}{1+(j+1)(1-r^2)} |z|^{2j-2}, \quad z \in \mathbb{D}(0,r).$$

This completes the proof of the lemma. □

Let $\nabla := (\partial_x, \partial_y)$ stand for the usual gradient, if $z = x + \mathrm{i}y$ is the representation of the complex coordinate.

LEMMA 5.1.2. *Suppose $f : \mathbb{D} \to \mathbb{C}$ is holomorphic and nontrivial. Then the function $z \mapsto (1-|z|^2)|f(z)|$ has local minima only at the zeros of f. Moreover, for $0 < r < 1$, the gradient of this function enjoys the estimate*

$$\big|\nabla\big((1-|z|^2)|f(z)|\big)\big| \le A(r,z) \left\{ \int_{\mathbb{D}(0,r)} |f|^2 \mathrm{d}A_1 \right\}^{1/2}, \quad z \in \mathbb{D}(0,r),$$

where

$$A(r,z) := 2|z| K_r(z,z)^{1/2} + (1-|z|^2)(\Delta_z K(z,z))^{1/2},$$

and K_r is the reproducing kernel of the proof of Lemma 5.1.1.

Proof. Since

$$\Delta \log\{(1-|z|^2)|f(z)|\} = -(1-|z|^2)^{-2} < 0$$

holds away from the zeros of f, the critical points of the function $z \mapsto \log(1-|z|^2)|f(z)|$ can only be local maxima or saddle points, and this carries over to the function $z \mapsto (1-|z|^2)|f(z)|$ as well. The estimate of the gradient uses the estimates of Lemma 5.1.1 together with the product rule

$$\nabla[(1-|z|^2)|f(z)|] = -|f(z)|\nabla|z|^2 + (1-|z|^2)\nabla|f(z)|$$

and the facts that $|\nabla|z|^2| = 2|z|$ and $|\nabla|f|| \le |f'|$. The estimates of $|f(z)|$ and $|f'(z)|$ are from Lemma 5.1.1. □

5.2. The fundamental local estimate. We need to estimate the hyperbolic zero packing constant $\rho_{\mathbb{H}}$ from below. The hard part consists in obtaining the following local estimate.

PROPOSITION 5.2.1. *There exists an absolute constant ρ_1, with $0 < \rho_1 < 1$, such that for holomorphic $f : \mathbb{D} \to \mathbb{C}$,*

$$\rho_1 \le \int_{\mathbb{D}(0,\frac{1}{2})} \big(|f(z)|(1-|z|^2)-1\big)^2 \frac{\mathrm{d}A(z)}{1-|z|^2}.$$

For instance, $\rho_1 := 2.08 \times 10^{-5}$ will do.

Proof. We write $f = bf_0$, where b is a positive constant, where f_0 is normalized:

$$\int_{\mathbb{D}(0,\frac{1}{2})} |f_0(z)|^2(1-|z|^2)\mathrm{d}A(z) = 1. \tag{5.2.1}$$

So, we need to show that

$$\rho_1 \le \inf_{b>0} \int_{\mathbb{D}(0,\frac{1}{2})} \big(b|f_0(z)|(1-|z|^2)-1\big)^2 \frac{\mathrm{d}A(z)}{1-|z|^2}. \tag{5.2.2}$$

It is possible to verify that for $r = \frac{1}{2}$, the function $A(\frac{1}{2}, z)$ of Lemma 5.1.2 is radial and increasing with $|z|$, and plugging in $|z| = \frac{2}{5}$ we obtain from numerical work that

$$A\Big(\frac{1}{2}, z\Big) \le 67, \quad |z| \le \frac{2}{5}.$$

Now, from the normalization (5.2.1) we get from Lemma 5.1.2 that

$$\big|\nabla\big((1-|z|^2)|f_0(z)|\big)\big| \le 67, \quad z \in \mathbb{D}\Big(0, \frac{2}{5}\Big), \tag{5.2.3}$$

where we decided to estimate on a slightly smaller disk. Next, a straightforward calculus exercise shows that the minimum over $b > 0$ is attained at the value

$$b = b_{f_0} := \int_{\mathbb{D}(0,\frac{1}{2})} |f_0(z)|\mathrm{d}A(z).$$

Moreover, a well-known calculation shows that

$$\begin{aligned} \int_{\mathbb{D}(0,\frac{1}{2})} \big(b_{f_0}|f_0(z)|(1-|z|^2)-1\big)^2 \frac{\mathrm{d}A(z)}{1-|z|^2} &= \int_{\mathbb{D}(0,\frac{1}{2})} \frac{\mathrm{d}A(z)}{1-|z|^2} - b_{f_0}^2 \\ &= \log\frac{4}{3} - b_{f_0}^2, \end{aligned} \tag{5.2.4}$$

so that in particular, $b_{f_0}^2 \le \log\frac{4}{3}$. Next, we split our argument according to the size of b_{f_0}.

Case I. *Suppose that $b_{f_0}^2 \le \frac{1}{2}\log\frac{4}{3}$.* Then by (5.2.4), the claimed estimate holds whenever $\rho_1 \le \frac{1}{2}\log\frac{4}{3}$.

Case II. *Suppose that* $\frac{1}{2}\log\frac{4}{3} < b_{f_0}^2 \le \log\frac{4}{3}$. We let F be the function $F(z) := (1-|z|^2)|f(z)| = b_{f_0}(1-|z|^2)|f_0(z)|$, so that by (5.2.3), we know that

$$|\nabla F(z)| \le 67 b_{f_0} \le 36, \quad w \in \mathbb{D}\Big(0, \frac{2}{5}\Big). \tag{5.2.5}$$

For a positive real number ϵ, to be specified later, we consider the set $\Omega(f,\epsilon)$ given by

$$\Omega(f,\epsilon) := \Big\{ z \in \mathbb{D}\Big(0,\frac{1}{3}\Big) : \, (1-|z|^2)|f(z)| \ge \epsilon \Big\}.$$

We divide Case II further according to the properties of the set $\Omega(f,\epsilon)$.

Case II(a). *Suppose that* $\Omega(f,\epsilon) \ne \mathbb{D}(0,\frac{1}{3})$. Then $|F(z_0)| = (1-|z_0|^2)|f(z_0)| < \epsilon$ at some point $z_0 \in \mathbb{D}(0,\frac{1}{3})$, and, in view of (5.2.5) and convexity,

$$|F(z)| \le \epsilon + 36|z-z_0|, \quad z \in \mathbb{D}\Big(0,\frac{2}{5}\Big).$$

In particular, if $\epsilon \le \frac{1}{10}$, we see that

$$\begin{aligned} \int_{\mathbb{D}(0,\frac{1}{2})} \big(F(z)-1\big)^2 \frac{\mathrm{d}A(z)}{1-|z|^2} &\ge \int_{\mathbb{D}(z_0,\frac{1}{80})} \big(F(z)-1\big)^2 \frac{\mathrm{d}A(z)}{1-|z|^2} \\ &\ge \Big(1-\frac{36}{80}-\frac{1}{10}\Big)^2 \frac{1}{80^2} \ge 3.16\times 10^{-5}. \end{aligned} \tag{5.2.6}$$

Case II(b). *Suppose that* $\Omega(f,\epsilon) = \mathbb{D}(0,\frac{1}{3})$. To fit with the argument of Case II(a) we should assume that $\epsilon \le \frac{1}{10}$. In particular, $u := \log|f|$ is harmonic, and $F(z) = (1-|z|^2)\mathrm{e}^{u(z)}$. We will make use of the following elementary estimate:

$$(1-t)^2 \ge \frac{(1-\epsilon)^2}{(\log\frac{1}{\epsilon})^2}(\log t)^2, \quad \epsilon \le t < +\infty, \tag{5.2.7}$$

which follows from the monotonicity of the expression $\frac{1-\mathrm{e}^{-s}}{s}$ where $s = \log\frac{1}{t}$. It is immediate from (5.2.7) and from our assumption $\Omega(f,\epsilon) = \mathbb{D}(0,\frac{1}{3})$ that

$$\begin{aligned} &\int_{\mathbb{D}(0,\frac{1}{2})} \big(F(z)-1\big)^2 \frac{\mathrm{d}A(z)}{1-|z|^2} \\ &\ge \frac{(1-\epsilon)^2}{(\log\frac{1}{\epsilon})^2} \int_{\mathbb{D}(0,\frac{1}{3})} \big(\log F(z)\big)^2 \frac{\mathrm{d}A(z)}{1-|z|^2} \\ &= \frac{(1-\epsilon)^2}{(\log\frac{1}{\epsilon})^2} \int_{\mathbb{D}(0,\frac{1}{3})} \big(\log(1-|z|^2)+u(z)\big)^2 \frac{\mathrm{d}A(z)}{1-|z|^2} \\ &= \frac{(1-\epsilon)^2}{(\log\frac{1}{\epsilon})^2} \int_{\mathbb{D}(0,\frac{1}{3})} \Big(\big(\log(1-|z|^2)\big)^2 + 2u(z)\log(1-|z|^2) + u(z)^2\Big) \frac{\mathrm{d}A(z)}{1-|z|^2}. \end{aligned} \tag{5.2.8}$$

We calculate that

$$\int_{\mathbb{D}(0,\frac{1}{3})} \big(\log(1-|z|^2)\big)^2 \frac{\mathrm{d}A(z)}{1-|z|^2} = \frac{(\log\frac{9}{8})^3}{3},$$

and, moreover, by the mean value property of harmonic functions, that

$$\begin{aligned}\int_{\mathbb{D}(0,\frac{1}{3})} u(z)\log(1-|z|^2)\frac{\mathrm{d}A(z)}{1-|z|^2} &= u(0)\int_{\mathbb{D}(0,\frac{1}{3})} \log(1-|z|^2)\frac{\mathrm{d}A(z)}{1-|z|^2}\\ &= -\frac{(\log\frac{9}{8})^2}{2}u(0).\end{aligned}$$

Furthermore, since u is harmonic, the square u^2 is subharmonic, and hence

$$\int_{\mathbb{D}(0,\frac{1}{3})} u(z)^2\frac{\mathrm{d}A(z)}{1-|z|^2} \geq u(0)^2\int_{\mathbb{D}(0,\frac{1}{3})} \frac{\mathrm{d}A(z)}{1-|z|^2} = u(0)^2\log\frac{9}{8}.$$

Adding up the terms, we now obtain from (5.2.8) that

$$\begin{aligned}&\int_{\mathbb{D}(0,\frac{1}{2})} \big(F(z)-1\big)^2\frac{\mathrm{d}A(z)}{1-|z|^2}\\ &\quad\geq \frac{(1-\epsilon)^2}{(\log\frac{1}{\epsilon})^2}\bigg(\frac{(\log\frac{9}{8})^3}{3} - (\log\frac{9}{8})^2u(0) + u(0)^2\log\frac{9}{8}\bigg)\\ &\quad= \frac{(1-\epsilon)^2}{(\log\frac{1}{\epsilon})^2}\bigg(\Big(u(0)-\frac{1}{2}\log\frac{9}{8}\Big)^2\log\frac{9}{8} + \frac{1}{12}\Big(\log\frac{9}{8}\Big)^3\bigg)\\ &\quad\geq \frac{(1-\epsilon)^2}{12(\log\frac{1}{\epsilon})^2}\Big(\log\frac{9}{8}\Big)^3.\end{aligned} \tag{5.2.9}$$

Finally, we specify that $\epsilon := \frac{1}{10}$ so that (5.2.9) then gives that

$$\int_{\mathbb{D}(0,\frac{1}{2})} \big(F(z)-1\big)^2\frac{\mathrm{d}A(z)}{1-|z|^2} \geq 2.08\times 10^{-5}. \tag{5.2.10}$$

By comparing the estimates we obtained in Cases I, II(a), and II(b), we obtain that the assertion of the proposition holds with $\rho_1 = 2.08\times 10^{-5}$. □

Remark 5.2.2. We should mention that when asked, Borichev [9] came up with an absolute lower bound via a somewhat different argument.

5.3. Modification of the fundamental local estimate. As it turns out, we will need to compare locally not just with the constant 1 but with a family of functions whose logarithms are harmonic.

PROPOSITION 5.3.1. *There exists an absolute constant ρ_2 with $0<\rho_2<1$, such that for all holomorphic $f:\mathbb{D}\to\mathbb{C}$ and all points $\xi\in\mathbb{D}$,*

$$\rho_2 \le \int_{\mathbb{D}(0,\frac12)} \big(|f(z)|(1-|z|^2)-|1-\bar\xi z|^{-1}\big)^2 \frac{\mathrm{d}A(z)}{1-|z|^2}.$$

For instance, $\rho_2=\frac49\rho_1$ will do, where ρ_1 is the constant of Proposition 5.2.1.

Proof. We consider the auxiliary holomorphic function $g(z):=(1-\bar\xi z)f(z)$. An application of Proposition 5.2.1 with g in place of f gives that

$$\begin{aligned}\int_{\mathbb{D}(0,\frac12)} &\big(|f(z)|(1-|z|^2)-|1-\bar\xi z|^{-1}\big)^2 \frac{\mathrm{d}A(z)}{1-|z|^2}\\ &= \int_{\mathbb{D}(0,\frac12)} \big(|g(z)|(1-|z|^2)-1\big)^2|1-\bar\xi z|^{-2} \frac{\mathrm{d}A(z)}{1-|z|^2}\\ &\ge \frac49\int_{\mathbb{D}(0,\frac12)} \big(|g(z)|(1-|z|^2)-1\big)^2 \frac{\mathrm{d}A(z)}{1-|z|^2}\ge\frac49\rho_1,\end{aligned}$$

which expresses the asserted estimate. □

5.4. The global estimate from below. We now turn the local estimate into a global one.

Proof of Theorem 1.5.3. As mentioned in the introduction, the estimate from above $\rho_{\mathbb{H}}\le 0.12087$ follows from the work of Astala, Ivrii, Perälä, and Prause [5], so it remains to establish the estimate from below. Our starting point is Proposition 5.3.1, which tells us that there exists an absolute constant ρ_2, with $0<\rho_2<1$, such that for each $\lambda\in\mathbb{D}$ and each holomorphic function $h:\mathbb{D}\to\mathbb{C}$,

$$\rho_2 \le \int_{\mathbb{D}(0,\frac12)} \big(|h(z)|(1-|z|^2)-|1-\bar\lambda z|^{-1}\big)^2 \frac{\mathrm{d}A(z)}{1-|z|^2}. \tag{5.4.1}$$

Given $\lambda\in\mathbb{D}$, we introduce the mapping γ_λ given by

$$\gamma_\lambda(\zeta):=\frac{\lambda-\zeta}{1-\bar\lambda\zeta},$$

which is an involutive Möbius automorphism of the unit disk $\mathbb{D}$ (so that $\gamma_\lambda\circ\gamma_\lambda(\zeta)=\zeta$). Moreover, a direct calculation shows that the derivative of γ_λ equals

$$\gamma_\lambda'(\zeta)=-\frac{1-|\lambda|^2}{(1-\bar\lambda\zeta)^2}.$$

We make the auxiliary observation that

$$1-|\gamma_\lambda(\zeta)|^2=\frac{(1-|\lambda|^2)(1-|\zeta|^2)}{|1-\bar\lambda\zeta|^2}=(1-|\zeta|^2)|\gamma_\lambda'(\zeta)|. \tag{5.4.2}$$

Let h_λ denote the holomorphic function

$$h_\lambda(\zeta) := (-\gamma_\lambda'(\zeta))^{3/2} h\circ\gamma_\lambda(\zeta) = \frac{(1-|\lambda|^2)^{3/2}}{(1-\bar\lambda\zeta)^3}\, h\left(\frac{\lambda-\zeta}{1-\bar\lambda\zeta}\right),$$

and observe that by (5.4.2) and the change-of-variables formula,

$$\begin{aligned}
&\int_{\gamma_\lambda(\mathbb{D}(0,\frac12))} \big(|h_\lambda(\zeta)|(1-|\zeta|^2)-1\big)^2 \frac{\mathrm{d}A(\zeta)}{1-|\zeta|^2}\\
&= \int_{\gamma_\lambda(\mathbb{D}(0,\frac12))} \big(|h\circ\gamma_\lambda(\zeta)|(1-|\gamma_\lambda(\zeta)|^2)-|\gamma_\lambda'(\zeta)|^{-1/2}\big)^2 \frac{|\gamma_\lambda'(\zeta)|^2}{1-|\gamma_\lambda(\zeta)|^2}\mathrm{d}A(\zeta)\\
&= \int_{\mathbb{D}(0,\frac12)} \big(|h(z)|(1-|z|^2)-|\gamma_\lambda'(z)|^{1/2}\big)^2 \frac{\mathrm{d}A(z)}{1-|z|^2}\\
&= (1-|\lambda|^2)\int_{\mathbb{D}(0,\frac12)} \big(|\tilde h(z)|(1-|z|^2)-|1-\bar\lambda z|^{-1}\big)^2 \frac{\mathrm{d}A(z)}{1-|z|^2} \ge (1-|\lambda|^2)\rho_2,
\end{aligned} \tag{5.4.3}$$

where $\tilde h(z) = (1-|\lambda|^2)^{-1/2}h(z)$, and, in the last step, we invoked (5.4.1) with $\tilde h$ in place of h. If we write H in place of h_λ, we obtain from (5.4.3) that

$$\int_{\gamma_\lambda(\mathbb{D}(0,\frac12))} \big(|H(\zeta)|(1-|\zeta|^2)-1\big)^2 \frac{\mathrm{d}A(\zeta)}{1-|\zeta|^2} \ge (1-|\lambda|^2)\rho_2.$$

This inequality holds in fact for every holomorphic function $H:\mathbb{D}\to\mathbb{C}$, since for given H it is possible to write down h such that $H = h_\lambda$. We are of course free to integrate both sides with respect to a positive finite measure:

$$\begin{aligned}
&\int_{\mathbb{D}(0,r^4)}\int_{\gamma_\lambda(\mathbb{D}(0,\frac12))} \big(|H(\zeta)|(1-|\zeta|^2)-1\big)^2 \frac{\mathrm{d}A(\zeta)\mathrm{d}A(\lambda)}{(1-|\zeta|^2)(1-|\lambda|^2)^2}\\
&\qquad \ge \rho_2\int_{\mathbb{D}(0,r^4)} \frac{\mathrm{d}A(\lambda)}{1-|\lambda|^2} = \rho_2 \log\frac{1}{1-r^8}.
\end{aligned} \tag{5.4.4}$$

Moreover, we calculate that

$$\begin{aligned}
\int_{\mathbb{D}(0,r^4)} 1_{\gamma_\lambda(\mathbb{D}(0,\frac12))}(\zeta)\frac{\mathrm{d}A(\lambda)}{(1-|\lambda|^2)^2} &= \int_{\mathbb{D}(0,r^4)} 1_{\mathbb{D}(0,\frac12)}(\gamma_\zeta(\lambda))\frac{\mathrm{d}A(\lambda)}{(1-|\lambda|^2)^2}\\
&\le 1_{\mathbb{D}(0,r)}(\zeta)\log\frac{4}{3},
\end{aligned}$$

for $r_1 < r < 1$, provided $r_1 < 1$ is close enough to 1, where the bound by $\log\frac43$ is a consequence of hyperbolic invariance, and the fact the left-hand side vanishes is a consequence of a simple comparison of the hyperbolic lengths of the intervals $[0,\frac12]$ and $[r^4, r]$ (the latter interval is longer for $r_1 < r < 1$). It now follows from

(5.4.4) that

$$\text{(5.4.5)}\quad \frac{\rho_2}{\log\frac{4}{3}}\log\frac{1}{1-r^8}\le\int_{\mathbb{D}(0,r)}\big(|H(\zeta)|(1-|\zeta|^2)-1\big)^2\frac{\mathrm{d}A(\zeta)}{1-|\zeta|^2},\quad r_1<r<1.$$

Since with $\rho_2=\frac{4}{9}\rho_1$ and $\rho_1=2.08\times10^{-5}$, we have the inequality of constants

$$\frac{\rho_2}{\log\frac{4}{3}}>3.21\times10^{-5}.$$

Moreover, since

$$\lim_{r\to1^-}\frac{\log\frac{1}{1-r^8}}{\log\frac{1}{1-r^2}}=1,$$

the claimed assertion follows from (5.4.5). □

6. Geometric packing of zeros. In this section, we develop a rather general type of extremal problems in complex analysis, which we call *geometric zero packing problems*. We first explain the planar zero packing problem, and then turn to the hyperbolic zero packing problem, which was mentioned earlier.

6.1. A packing problem for zeros in the plane. We first study a packing problem for zeros pertaining to the Bargmann-Fock space of entire functions. It is well known that there is no entire function $f:\mathbb{C}\to\mathbb{C}$ such that $\log|f(z)|=|z|^2$. The reason is that in the sense of distribution theory, $\Delta\log|f|$ is a sum of half unit point masses located at the zeros of f (counting multiplicities), so that off the zeros, $\log|f|$ is harmonic, while $\Delta|z|^2=1$. In particular, the nonnegative function $(|f(z)|\mathrm{e}^{-|z|^2}-1)^2$ cannot vanish on a nonempty open set, and if $f(z)$ is a polynomial in z, then in particular $|f(z)|=\mathrm{O}(\mathrm{e}^{|z|^2})$ as $|z|\to+\infty$, and so we would know that the *discrepancy function*

$$\Psi_f(z):=(|f(z)|\mathrm{e}^{-|z|^2}-1)^2$$

is bounded. Note also that for the trivial function $f=0$, the discrepancy $\Psi_f=\Psi_0$ equals the constant 1. It is now a natural question to ask *how small the discrepancy Ψ_f can be, on average*, since it cannot vanish on nonempty open sets. So, we consider the minimal average of Ψ_f in a disk $\mathbb{D}(0,R)$ of large radius R:

(6.1.1)

$$\rho_{\mathbb{C}}(R):=\inf_f\frac{1}{R^2}\int_{\mathbb{D}(0,R)}\Psi_f(z)\mathrm{d}A(z)=\inf_f\frac{1}{R^2}\int_{\mathbb{D}(0,R)}(|f(z)|\mathrm{e}^{-|z|^2}-1)^2\mathrm{d}A(z),$$

where the infimum is taken over all polynomials f. Here, the use of the origin as the base point is inessential since in (6.1.1), we can take the infimum over all entire f

without changing the value of $\rho_{\mathbb{C}}(R)$, and, in addition, by the change-of-variables formula, we have for $a \in \mathbb{C}$ the translation invariance property

$$\frac{1}{R^2}\int_{\mathbb{D}(a,R)} \Psi_f(z)\mathrm{d}A(z) = \frac{1}{R^2}\int_{\mathbb{D}(0,R)} \Psi_{f_{\langle a\rangle}}(z)\mathrm{d}A(z),$$

where $f_{\langle a\rangle}$ denotes the Fock-space translate $f_{\langle a\rangle}(z) := \mathrm{e}^{-|a|^2-2\bar{a}z} f(a+z)$. In view of Lemma 4.1.1, this discrepancy density $\rho_{\mathbb{C}}(R)$ gives the best constant for the improved Cauchy-Schwarz inequality

$$\left\{\frac{1}{R^2}\int_{\mathbb{D}(0,R)} |f(z)|\mathrm{e}^{-|z|^2}\mathrm{d}A(z)\right\}^2 \le (1-\rho_{\mathbb{C}}(R))\frac{1}{R^2}\int_{\mathbb{D}(0,R)} |f(z)|^2\mathrm{e}^{-2|z|^2}\mathrm{d}A(z).$$

Definition 6.1.1. For the above problem, the *minimal discrepancy density for planar zero packing* is $\rho_{\mathbb{C}} := \liminf_{R\to+\infty} \rho_{\mathbb{C}}(R)$.

Remark 6.1.2. (a) The limsup might be considered as well, but we expect it to equal the liminf.

(b) In more geometric terms, the quantity $\rho_{\mathbb{C}}$ is a measure of how well the planar metric $\mathrm{d}s = |\mathrm{d}z|$ can be approximated by a metric obtained in the following manner: take the surface with the Gaussian metric $\mathrm{d}s = |f(z)|\mathrm{e}^{-|z|^2}|\mathrm{d}z|$, where f is a polynomial, which then has curvature

$$-4\Delta \log\left(|f(z)|\mathrm{e}^{-|z|^2}\right) = 4 - 2\sum_j \delta_{w_j},$$

where $\{w_j\}_j$ are the zeros of f, and δ_ξ is the unit mass delta function at $\xi \in \mathbb{C}$. The point masses in the curvature correspond to “branch” or “flabby cone” points with an opening of 4π in case of simple zeros, and more generally, an opening of $2(n+1)\pi$ for a zero of multiplicity n.

Since polynomials are determined up to a multiplicative constant by their zeros, we feel that the terminology “geometric zero packing” or “geometric packing of zeros” is appropriate.

Problem 6.1.3. Determine the value of $\rho_{\mathbb{C}}$. For which configurations of zeros of the polynomial f is it asymptotically attained? Is the equilateral triangular lattice optimal asymptotically?

In Conjecture 7.1.2 below we attribute the conjecture that the equilateral triangular lattice is optimal (in the more general context of an exponent β) to Abrikosov. We illustrate with an equilateral triangular tessellation in Figure 1.

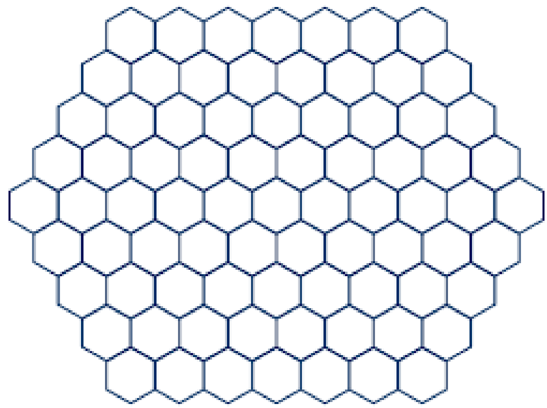

Figure 1. Illustration of the honeycomb lattice, with a zero to be placed at the center of each hexagon to produce the equilateral triangular tessellation.

The Weierstrass sigma function $\sigma(z)$, which arises in the analysis of the Weierstrass $\wp(z)$ function, can be used to analyze the asymptotic discrepancy density for the equilateral triangular lattice (see, e.g., the exposition of Ahlfors [3]). Let $\omega_1, \omega_2 \in \mathbb{C}$ be the periods associated with the lattice

$$\Lambda_{\omega_1,\omega_2} := \omega_1\mathbb{Z} + \omega_2\mathbb{Z},$$

where it is assumed that ω_1, ω_2 are $\mathbb{R}$-linearly independent. For simplicity, we suppose ω_1 is real with $\omega_1 > 0$, and that $\operatorname{Im}\omega_2 > 0$. The related Weierstrass function $\wp(z)$ then has the complex periods ω_1 and ω_2. We recall the formula for the associated sigma function (see, e.g., [3]):

$$\sigma(z) := z \prod_{0\neq\omega\in\Lambda_{\omega_1,\omega_2}} \left(1 - \frac{z}{\omega}\right) \exp\left\{\frac{z}{\omega} + \frac{z^2}{2\omega^2}\right\}.$$

The function $\sigma(z)$ is entire, with periodicity-type formulae

$$\sigma(z+\omega_j) = -\sigma(z)\exp\big(\lambda_j(z+\frac{\omega_j}{2})\big), \quad j = 1,2, \tag{6.1.2}$$

where the constants are $\lambda_j := 2\zeta(\frac{\omega_j}{2})$, as expressed in terms of the logarithmic derivative $\zeta(z) := \sigma'(z)/\sigma(z)$ known as the Weierstrass zeta function. The relationship with the classical Weierstrass function is $\zeta'(z) = -\wp(z)$. We consider in the planar zero packing problem the function f

$$f(z) := a\,\mathrm{e}^{\xi z+\eta z^2}\sigma(z),$$

where a is a positive amplitude constant, and $\xi, \eta \in \mathbb{C}$ are parameters to be determined. We would like the associated function

$$e^{-|z|^2}|f(z)| = a\,e^{-|z|^2+\mathrm{Re}(\xi z+\eta z^2)}|\sigma(z)| \tag{6.1.3}$$

to be periodic with the two complex periods ω_1, ω_2. This is only possible if the density of the lattice $\Lambda_{\omega_1,\omega_2}$ has a normalized area of the fundamental rhombus $\mathcal{D}$ which equals $\frac{1}{2}$. In terms of the periods, the requirement is that

$$\omega_1 \,\mathrm{Im}\,\omega_2 = \frac{\pi}{2},$$

which is related with the classical Legendre relation $\lambda_1\omega_2 - \lambda_2\omega_1 = \mathrm{i}2\pi$. Under this condition, it is indeed possible to specify values for the constants ξ, η such that the function (6.1.3) gets to be doubly periodic. If we choose

$$\omega_1 := \frac{\pi^{1/2}}{3^{1/4}}, \quad \omega_2 := \frac{\pi^{1/2}}{3^{1/4}} e^{\mathrm{i}\pi/3},$$

which makes $\Lambda_{\omega_1,\omega_2}$ an equilateral triangular tiling of the plane, this condition is fulfilled, and the appropriate values of the constants ξ, η are then

$$\xi = 0, \quad \eta = 1 - \frac{\zeta(\frac{\omega_1}{2})}{\omega_1} = \frac{\bar{\omega}_2}{\omega_2} - \frac{\zeta(\frac{\omega_2}{2})}{\omega_2}.$$

The asymptotic discrepancy density associated with this particular choice can then be calculated over a single fundamental rhombus $\mathcal{D}$ for the tiling $\mathbb{C}/\Lambda_{\omega_1,\omega_2}$,

$$\begin{aligned} &\lim_{R\to+\infty} \frac{1}{R^2} \int_{\mathbb{D}(0,R)} \left(e^{-|z|^2}|f(z)| - 1\right)^2 \mathrm{d}A(z) \\ &= \frac{1}{|\mathcal{D}|_A} \int_{\mathcal{D}} \left(e^{-|z|^2}|f(z)| - 1\right)^2 \mathrm{d}A(z) \\ &= \frac{1}{|\mathcal{D}|_A} \int_{\mathcal{D}} \left(a\,e^{-|z|^2+\mathrm{Re}(\eta z^2)}|\sigma(z)| - 1\right)^2 \mathrm{d}A(z), \end{aligned} \tag{6.1.4}$$

where we are free to minimize over the parameter a. Here, as mentioned previously, $|\mathcal{D}|_A = \frac{1}{2}$ is the normalized area of the fundamental rhombus. The right-hand side of (6.1.4) is in a natural sense the average of Ψ_f over the torus $\mathbb{C}/\Lambda_{\omega_1,\omega_2}$.

Remark 6.1.4. Numerical implementation of the above integral (6.1.4), minimized over the parameter a, was carried out by Wennman [54] using MATHEMATICA, which resulted in the value 0.061203..., so that in particular, $\rho_{\mathbb{C}} \le 0.061203\ldots$. We suggest that this inequality is actually an equality.

6.2. The stochastic minimization approach to planar zero packing. It is difficult to know offhand what kind of packing of zeros would be optimal for the calculation of the asymptotic minimal discrepancy density $\rho_{\mathbb{C}}$. A reasonable approach is to let a stochastic process do the digging for the optimal configuration, as in the so-called *Bellman function method*, exploited repeatedly in harmonic analysis (see, e.g., the survey [39]). First, we note that the assumption that the function f should be a polynomial in (6.1.1) is excessive, since polynomials are dense in many spaces of holomorphic functions. In particular, the minimal local density $\rho_{\mathbb{C}}(R)$ is unperturbed if we minimize, e.g., over all entire functions f. Here, we will replace f by a Gaussian analytic function (GAF) with close-to-optimal behavior. To set the notation, we let $N_{\mathbb{C}}(0,1)$ stand for the standard rotationally invariant Gaussian distribution with probability measure $\mathrm{e}^{-|\zeta|^2}\mathrm{d}A(\zeta)$ in the plane $\mathbb{C}$. We pick independent copies $\xi_j \in N_{\mathbb{C}}(0,1)$ for $j = 0,1,2,\ldots$, and let F be the GAF process [26]

$$F(z) := \sum_{j=0}^{+\infty} \frac{\xi_j}{\sqrt{j!}} 2^{j/2} z^j, \quad z \in \mathbb{C}.$$

The way things are set up,

$$F(z)\mathrm{e}^{-|z|^2}$$

has the same standard complex normal distribution $N_{\mathbb{C}}(0,1)$ irrespective of the point $z \in \mathbb{C}$. Given a positive amplitude constant b, we observe that the associated density

$$\rho_{bF}(R) := \frac{1}{R^2}\int_{\mathbb{D}(0,R)} \Psi_{bF}(z)\mathrm{d}A(z) = \frac{1}{R^2}\int_{\mathbb{D}(0,R)} (b|F(z)|\mathrm{e}^{-|z|^2} - 1)^2 \mathrm{d}A(z)$$

is stochastic, and we may ask for the number

$$\rho_{\mathbb{C}}^{\mathrm{prob}}(R) := \inf\big\{t > 0\colon \exists b > 0 \text{ such that } \mathbb{P}(\rho_{bF}(R) \le t) > 0\big\}$$

where $\mathbb{P}(e)$ stands for the probability of the event e. Then clearly, $\rho_{\mathbb{C}}(R) \le \rho_{\mathbb{C}}^{\mathrm{prob}}(R)$, and we actually have equality.

PROPOSITION 6.2.1. *We have that* $\rho_{\mathbb{C}}(R) = \rho_{\mathbb{C}}^{\mathrm{prob}}(R)$*, and hence that* $\rho_{\mathbb{C}} = \liminf_{R\to+\infty} \rho_{\mathbb{C}}^{\mathrm{prob}}(R)$.

Proof sketch. We will fix the parameter $b := 1$, which only makes things harder. Since every holomorphic f modulo $\mathrm{O}(z^{N+1})$ occurs with positive density in the process $F(z)$ (i.e., every finite sequence of the first N Taylor coefficients occurs with positive density in the stochastic sequence $2^{j/2}\xi_j/\sqrt{j!}$, $j = 0,\ldots,N$), and for fixed R, the infimum in (6.1.1) is almost achieved by polynomials of sufficiently high degree, we can conclude that $\rho_{\mathbb{C}}(R) = \rho_{\mathbb{C}}^{\mathrm{prob}}(R)$ should hold. The influence of the remaining stochastic Taylor coefficients $2^{j/2}\xi_j/\sqrt{j!}$ for $j > N$ to the stochastic integral $\rho_F(R) = \rho_{bF}(R)$ can be shown to be insignificant for big enough N. $\square$

Let $\mathbb{E}$ stand for the expectation, and observe that

$$\begin{aligned}\mathbb{E}\rho_{bF}(R) &= \frac{1}{R^2}\int_{\mathbb{D}(0,R)} \mathbb{E}\Psi_{bF}(z)\mathrm{d}A(z)\\ &= \frac{1}{R^2}\int_{\mathbb{D}(0,R)} (b^2\mathrm{e}^{-2|z|^2}\mathbb{E}|F(z)|^2 - 2b\mathrm{e}^{-|z|^2}\mathbb{E}|F(z)| + 1)\mathrm{d}A(z)\\ &= \frac{1}{R^2}\int_{\mathbb{D}(0,R)} (b^2 - b\sqrt{\pi} + 1)\mathrm{d}A(z) = b^2 - b\sqrt{\pi} + 1 = (b - \frac{1}{2}\sqrt{\pi})^2 + 1 - \frac{\pi}{4},\end{aligned}$$

which tells us that the expected value of $\rho_{bF}(R)$ is minimized for the amplitude $b = \frac{1}{2}\sqrt{\pi}$, and that the minimal expected value equals $1 - \frac{\pi}{4} = 0.214\ldots$. We obtain immediately an upper bound for $\rho_{\mathbb{C}}$:

PROPOSITION 6.2.2. *We have the following bounds:*

$$\rho_{\mathbb{C}} = \liminf_{R\to+\infty} \rho_{\mathbb{C}}^{\mathrm{prob}}(R) \le \liminf_{R\to+\infty} \min_{b>0} \mathbb{E}\rho_{bF}(R) = 1 - \frac{\pi}{4}.$$

It is of course naïve to believe that a simple expectation calculation would supply strong information. However, if we could get a grasp of the higher moments $\mathbb{E}(\rho_{bF}(R))^k$ for $k = 2, 3, 4, \ldots$ things would be different. This is related with the "moment support bounding problem".

Remark 6.2.3. By the planar analogues of the methods we develop in Section 5 for the hyperbolic setting, it can established that $\rho_{\mathbb{C}} > 0$.

6.3. Hyperbolic zero packing. We now return to the hyperbolic zero packing problem. It is as before related to the possible improvement in the Cauchy-Schwarz inequality, in line with Lemma 4.1.1. This time, the discrepancy is given by

$$\Phi_f(z) := \big((1-|z|^2)|f(z)| - 1\big)^2, \quad z \in \mathbb{D},$$

for a polynomial f, or more generally, f which is holomorphic in $\mathbb{D}$. Again, $\Phi_f(z) = 0$ is the same as the equality $(1-|z|^2)|f(z)| = 1$ which has no holomorphic solution f. The reason is the same as before: $\log|f|$ is harmonic off the zeros of f, while $\Delta \log \frac{1}{1-|z|^2} = (1-|z|^2)^{-2} > 0$. The average density of Φ_f with respect to the hyperbolic area element $\mathrm{d}A_{\mathbb{H}}(z) := (1-|z|^2)^{-2}\mathrm{d}A(z)$ is the ratio

$$\frac{\int_{\mathbb{D}(0,r)} \Phi_f \mathrm{d}A_{\mathbb{H}}}{\int_{\mathbb{D}(0,r)} \mathrm{d}A_{\mathbb{H}}} = \frac{\int_{\mathbb{D}(0,r)} \Phi_f \mathrm{d}A_{\mathbb{H}}}{\frac{r^2}{1-r^2}}$$

and we could consider the inf over f and then the liminf as $r \to 1^-$. However, since in hyperbolic geometry the length of boundary of $\mathbb{D}(0,r)$ is substantial, the cutoff is a bit rough. To reduce the boundary effects, we instead average further before taking the ratio (compare, e.g., with Seip's densities [46]),

$$\frac{\int_0^r \int_{\mathbb{D}(0,t)} \Phi_f \mathrm{d}A_{\mathbb{H}} \frac{\mathrm{d}t}{t}}{\int_0^r \int_{\mathbb{D}(0,t)} \mathrm{d}A_{\mathbb{H}} \frac{\mathrm{d}t}{t}} = \frac{\int_{\mathbb{D}(0,r)} \Phi_f(z)(1-|z|^2)\mathrm{d}A_{\mathbb{H}}(z)}{\log \frac{1}{1-r^2}}. \tag{6.3.1}$$

So, the minimal average discrepancy we are after is, for $0 < r < 1$,

$$\begin{aligned} \rho_{\mathbb{H}}(r) &:= \frac{1}{\log \frac{1}{1-r^2}} \inf_f \int_{\mathbb{D}(0,r)} \Phi_f(z) \frac{\mathrm{d}A(z)}{1-|z|^2} \\ &= \frac{1}{\log \frac{1}{1-r^2}} \inf_f \int_{\mathbb{D}(0,r)} \big((1-|z|^2)|f(z)|-1\big)^2 \frac{\mathrm{d}A(z)}{1-|z|^2}, \end{aligned} \tag{6.3.2}$$

where the infimum is over all polynomials f, or, which gives the same result, over all holomorphic functions $f: \mathbb{D} \to \mathbb{C}$. In view of Lemma 4.1.1, this discrepancy is the best constant for the improved Cauchy-Schwarz inequality

$$\left\{\int_{\mathbb{D}(0,r)} |f|\mathrm{d}A\right\}^2 \leq (1-\rho_{\mathbb{H}}(r)) \log \frac{1}{1-r^2} \times \int_{\mathbb{D}(0,r)} |f(z)|^2(1-|z|^2)\mathrm{d}A(z). \tag{6.3.3}$$

Definition 6.3.1. For the above problem, the *minimal discrepancy density for hyperbolic zero packing* is $\rho_{\mathbb{H}} := \liminf_{r\to 1^-} \rho_{\mathbb{H}}(r)$.

Problem 6.3.2. Determine the value of $\rho_{\mathbb{H}}$. For which configurations of zeros of the function f is it asymptotically attained? Is a lattice configuration optimal asymptotically?

Although the zero packing problem involves global issues, it probably has some analogies with the more local hyperbolic circle packing problems (see, e.g., [50]).

6.4. Hyperbolic Schäfli tilings. One strategy for hyperbolic zero packing would be to pack according to a lattice configuration, for instance given by a tiling of the disk by hyperbolic regular p-gons with q tiles meeting at each vertex (provided $p, q \geq 3$). We illustrate with a fourfold octagonal ($p = 8, q = 4$) tiling of Figure 2. Such a *Schäfli tiling* exists provided that $a_{p,q} := \frac{1}{4}(p-2-\frac{2p}{q}) > 0$, and then the hyperbolic $\mathrm{d}A_{\mathbb{H}}$-area of the p-gon is precisely $a_{p,q}$. A Schäfli tile is not always a fundamental domain for a Fuchsian group Γ, as this happens if and only if the Poincaré cycle condition is fulfilled (see [36]).

We are particularly interested in a Schäfli tiling which has normalized area $a_{p,q} := \frac{1}{4}(p-2-\frac{2p}{q}) = \frac{1}{2}$, because this is analogous to what we saw with the lattice tiling of Subsection 6.1, and would allow us to fit in exactly one zero per

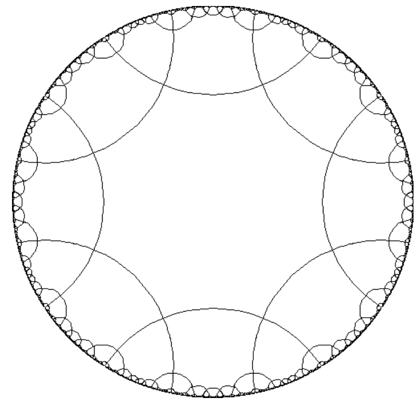

Figure 2. Illustration of the fourfold octagonal tiling $(p,q) = (8,4)$.

tile, located at the hyperbolic center point of each tile. This area condition can be written in the form

$$\frac{4}{p} + \frac{2}{q} = 1,$$

which has positive integer solutions (p,q) of the form $(5,10)$, $(6,6)$, $(8,4)$, and $(12,3)$, and generalized solutions $(4,\infty)$ and $(\infty,2)$. In particular, the $(8,4)$ tiling of Figure 2 has tiles with $\mathrm{d}A_{\mathbb{H}}$-area $\frac{1}{2}$. Such a tiling cannot correspond to a fundamental domain because the Poincaré cycle condition is not fulfilled. However, if we really want to, we can still glue together the edges of the octagon in the standard fashion (which means that every other edge gets glued pairwise, cyclically), but the resulting compact surface then obtains an irregular point with angle 4π around it (we might call it a *branching point*, a *ramified point*, or a *flabby cone point*). Another rather immediate way to see it is to use the Gauss-Bonnet theorem, which gives that the $\mathrm{d}A_{\mathbb{H}}$-area of a fundamental domain equals the integer $g-1 \geq 1$, where g is the genus of the corresponding compact Riemann surface.

6.5. The stochastic minimization approach to hyperbolic zero packing. As in the planar case, it is difficult to know offhand what kind of packing of zeros would be optimal for the calculation of the asymptotic minimal discrepancy density $\rho_{\mathbb{H}}$. Again, a reasonable approach is to let a stochastic process do the digging for the optimal configuration, and we look for an appropriate GAF process to supply random holomorphic functions in $\mathbb{D}$. As before, we pick independent copies $\eta_j \in N_{\mathbb{C}}(0,1)$ for $j = 0,1,2,\ldots$, and let G be the GAF process

$$G(z) := \sum_{j=0}^{+\infty} \eta_j \sqrt{j+1}\, z^j, \quad z \in \mathbb{C}.$$

It is well known that $(1-|z|^2)G(z)$ has complex normal distribution $N_{\mathbb{C}}(0,1)$ irrespective of the point $z \in \mathbb{C}$. Given a positive amplitude constant b, we observe that the associated density

$$\begin{aligned}\rho_{bG}(r) &:= \frac{1}{\log\frac{1}{1-r^2}} \int_{\mathbb{D}(0,r)} \Phi_{bG}(z) \frac{\mathrm{d}A(z)}{1-|z|^2} \\ &= \frac{1}{\log\frac{1}{1-r^2}} \int_{\mathbb{D}(0,r)} (b|G(z)|(1-|z|^2)-1)^2 \frac{\mathrm{d}A(z)}{1-|z|^2}\end{aligned}$$

is stochastic, and we may ask for the number

$$\rho_{\mathbb{H}}^{\text{prob}}(r) := \inf\big\{t > 0 \colon \exists b > 0 \text{ such that } \mathbb{P}(\rho_{bG}(r) \le t) > 0\big\}.$$

Then clearly, $\rho_{\mathbb{H}}(r) \le \rho_{\mathbb{H}}^{\text{prob}}(r)$, and in analogy with Proposition 6.2.1, we have equality.

PROPOSITION 6.5.1. *We have that* $\rho_{\mathbb{H}}(r) = \rho_{\mathbb{H}}^{\text{prob}}(r)$*, and hence* $\rho_{\mathbb{H}} = \liminf_{r\to 1^-} \rho_{\mathbb{H}}^{\text{prob}}(r)$.

The proof is essentially identical to that of Proposition 6.2.1, and left to the reader. As for the value of the asymptotic density $\rho_{\mathbb{H}}$, we observe that

(6.5.1)

$$\begin{aligned}&\mathbb{E}\rho_{bG}(r) \\ &= \frac{1}{\log\frac{1}{1-r^2}} \int_{\mathbb{D}(0,r)} \mathbb{E}\Phi_{bG}(z) \frac{\mathrm{d}A(z)}{1-|z|^2} \\ &= \frac{1}{\log\frac{1}{1-r^2}} \int_{\mathbb{D}(0,r)} \big(b^2(1-|z|^2)^2\mathbb{E}|G(z)|^2 - 2b\big(1-|z|^2\big)\mathbb{E}|G(z)|+1\big)\frac{\mathrm{d}A(z)}{1-|z|^2} \\ &= \frac{1}{\log\frac{1}{1-r^2}} \int_{\mathbb{D}(0,r)} \big(b^2 - b\sqrt{\pi}+1\big)\frac{\mathrm{d}A(z)}{1-|z|^2} = b^2 - b\sqrt{\pi}+1 = \Big(b-\frac{1}{2}\sqrt{\pi}\Big)^2 + 1 - \frac{\pi}{4},\end{aligned}$$

which tells us that the expected value of $\rho_{bG}(R)$ is minimized for the amplitude $b = \frac{1}{2}\sqrt{\pi}$, and that the minimal expected value equals $1-\frac{\pi}{4} = 0.214\ldots$. We obtain immediately an upper bound for $\rho_{\mathbb{H}}$, which is the same as in the planar case. This bound is substantially weaker than the one found by Astala, Ivrii, Perälä, and Prause in [5] ($\rho_{\mathbb{H}} \le 0.12087$).

PROPOSITION 6.5.2. *We have the following bounds:*

$$\rho_{\mathbb{H}} = \liminf_{r\to 1^-} \rho_{\mathbb{H}}^{\text{prob}}(r) \le \liminf_{r\to 1^-} \min_{b>0} \mathbb{E}\rho_{bG}(r) = 1 - \frac{\pi}{4}.$$

7. Geometric zero packing for exponent β. In this section, we introduce, for a positive real β, the β-exponent analogues of the planar and hyperbolic zero packing problems considered in Section 6.

7.1. Planar zero packing for exponent β. We introduce the β-exponent deformation of the density $\rho_{\mathbb{C}}$.

Definition 7.1.1. For a positive real β, let $\rho_\beta(\mathbb{C})$ be the density

$$\rho_\beta(\mathbb{C}) := \liminf_{R\to+\infty} \inf_f \frac{1}{R^2}\int_{\mathbb{D}(0,R)} \big(|f(z)|^\beta e^{-|z|^2} - 1\big)^2 dA(z), \tag{7.1.1}$$

where the infimum is taken over all polynomials f. We call this number $\rho_\beta(\mathbb{C})$ the *β-exponent minimal discrepancy density for planar zero packing*.

In view of Lemma 4.1.1, the density $\rho_\beta(\mathbb{C})$ may also be expressed as the minimal average ratio

$$\frac{1}{1-\rho_\beta(\mathbb{C})} = \liminf_{R\to+\infty} \inf_f \frac{R^{-2}\int_{\mathbb{D}(0,R)} |f(z)|^{2\beta} e^{-2|z|^2} dA(z)}{\big\{R^{-2}\int_{\mathbb{D}(0,R)} |f(z)|^{\beta} e^{-|z|^2} dA(z)\big\}^2} \tag{7.1.2}$$

where again the infimum is taken over all polynomials f. The instance $\beta = 2$ is the model problem considered by Aftalion, Blanc, and Nier [2]. This constitutes a mathematical simplification of the energy functional in the groundbreaking physical work of Abrikosov on Bose-Einstein condensates and type II superconductors (see [1]). For $\beta = 2$, it is shown rigorously in [2] that the equilateral triangular lattice (see Figure 1) is optimal among the lattices, and moreover, it is also shown that the corresponding Bargmann-Fock space function f solves the Bargmann-Fock analogue of the standing wave equation for the cubic Szegő equation (for the cubic Szegő equation, see, e.g., [16, 41]). This Bargmann-Fock analogue is known as the *lowest Landau level equation* (or LLL-equation), see, e.g., [17], but we might also suggest the term *cubic Bargmann-Fock equation*. We take a look at this matter in the following subsection (Subsection 7.2).

CONJECTURE 7.1.2. (Abrikosov) *The equilateral triangular lattice is optimal for β-exponent planar zero packing for each positive β.*

By a scaling transformation of the plane, it is easy to see that the density $\rho_\beta(\mathbb{C})$ can be written as

$$\rho_\beta(\mathbb{C}) = \liminf_{R\to+\infty} \inf_f \frac{1}{R^2}\int_{\mathbb{D}(0,R)} (|f(z)|^\beta e^{-\alpha|z|^2} - 1)^2 dA(z),$$

where the infimum is taken over all polynomials f, irrespectively of the value of the positive constant α. Using this observation, we see that Conjecture 7.1.2 maintains that

$$\rho_\beta(\mathbb{C}) = \frac{1}{|\mathcal{D}|_A}\int_{\mathcal{D}} (|f(z)|^\beta e^{-\beta|z|^2} - 1)^2 dA(z), \quad 0 < \beta < +\infty,$$

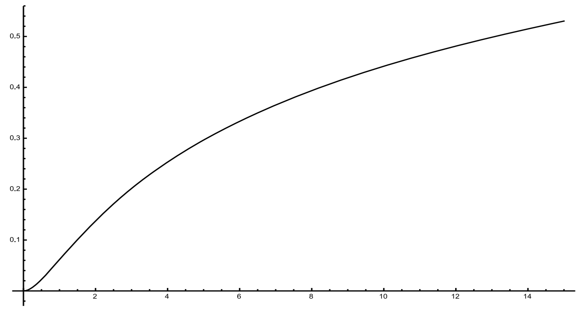

Figure 3. The graph of the density $\rho_\beta(\mathbb{C})$ as a function of β under Conjecture 7.1.2.

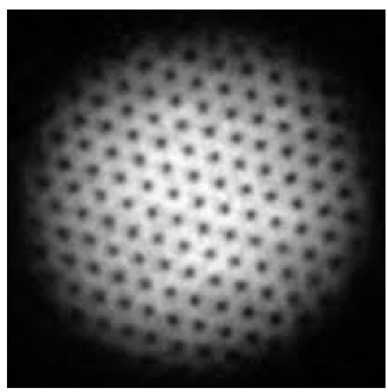

Figure 4. The equilateral triangular lattice (or honeycomb lattice) appears naturally in the physical context ($\beta = 2$). The associated zeros are located inside the grayish dots.

where $\mathcal{D}$ is the lattice rhombus and the function f is defined in terms of the Weierstrass sigma function, as in (6.1.4). We illustrate with the corresponding graph in Figure 3 communicated by Wennman [54]. For instance, the conjectured value for $\beta = 2$ is $\rho_2(\mathbb{C}) = 0.13763\ldots$, which corresponds to the number $\frac{1}{1-0.13763\ldots} = 1.1596\ldots$ mentioned in Theorem 1.4 of [2] for the hexagonal lattice (cf. Figure 4). Strictly speaking, Abrikosov did not quite go so far as to Conjecture 7.1.2, but he did suggest it should be enough to consider lattices and thought that the equilateral lattice was a natural candidate.

As for possible monotonicity in the parameter β, we notice that trivially $\rho_\beta(\mathbb{C}) \le \rho_{k\beta}(\mathbb{C})$ for $k = 1, 2, 3, \ldots$. We can actually do better.

PROPOSITION 7.1.3. *For positive reals β, β' with $\beta < \beta'$, we have that $\rho_\beta(\mathbb{C}) \le \rho_{\beta'}(\mathbb{C})$.*

Proof. It is elementary that for $0 < t < +\infty$,

$$(1 - t^\beta)^2 \le (1 - t^{\beta'})^2, \quad 0 < \beta < \beta' < +\infty,$$

and, consequently, it follows that for any polynomial f,

$$\frac{1}{R^2}\int_{\mathbb{D}(0,R)}(|f(z)|^{\beta}\mathrm{e}^{-\beta|z|^2}-1)^2\mathrm{d}A(z)$$
$$\leq \frac{1}{R^2}\int_{\mathbb{D}(0,R)}(|f(z)|^{\beta'}\mathrm{e}^{-\beta'|z|^2}-1)^2\mathrm{d}A(z), \quad 0<\beta<\beta'<+\infty.$$

Together with the above-mentioned scaling invariance of the density $\rho_\beta(\mathbb{C})$, this gives the asserted monotonicity in β. □

Remark 7.1.4. Given the monotonicity, it is a natural question to ask what is the limit of $\rho_\beta(\mathbb{C})$ as $\beta \to 0^+$ and as $\beta \to +\infty$. We believe that

$$\lim_{\beta\to 0^+}\rho_\beta(\mathbb{C})=0, \qquad \lim_{\beta\to+\infty}\rho_\beta(\mathbb{C})=1.$$

The first assertion is intuitively clear, since a sum of small point masses can approximate well a uniform distribution. This can probably form the backbone of a rigorous proof. The intuition behind the second assertion is that it should be impossible to reasonably approximate a uniform distribution using sums of very large point masses.

7.2. Planar zero packing for exponent $\beta = 2$ and the cubic Bargmann-Fock equation. Suppose f_0 is a minimizer of the right-hand side integral in (7.1.1) for fixed R. We then use a variational argument comparing f_0 with $f_0+\epsilon h$ for a polynomial h and an $\epsilon\in\mathbb{C}$ with $|\epsilon|$ tending to 0 to show that

$$\mathbf{\Pi}_1\left[\left(1-E_1|f_0|^{\beta}\right)\frac{|f_0|^{\beta}}{\bar{f}_0}1_{\mathbb{D}(0,R)}\right]=0.$$

This should be interpreted with some care for $0<\beta<1$ since it might then be the case that $|f_0|^{\beta}/\bar{f}_0$ develops bad singularities at multiple zeros of f_0 (alternatively, a separate argument would be needed to rule out multiple zeros). Here, $\mathbf{\Pi}_\alpha$ is the Bargmann-Fock projection on the plane $\mathbb{C}$ with the Gaussian weight $E_\alpha(z):=\mathrm{e}^{-\alpha|z|^2}$. More explicitly, $\mathbf{\Pi}_\alpha$ is given by

$$\mathbf{\Pi}_\alpha h(z):=\alpha\int_{\mathbb{C}}\mathrm{e}^{\alpha z\bar{w}}h(w)\,\mathrm{e}^{-\alpha|w|^2}\mathrm{d}A(w), \quad z\in\mathbb{C}.$$

Expecting some kind stability as $R\to+\infty$, we naturally look for entire solutions f_0 with

$$\mathbf{\Pi}_1\left[\left(1-E_1|f_0|^{\beta}\right)\frac{|f_0|^{\beta}}{\bar{f}_0}\right]=0. \tag{7.2.1}$$

For $\beta=2$, the equation (7.2.1) just says that

$$f_0=\mathbf{\Pi}_1[E_1 f_0|f_0|^2]. \tag{7.2.2}$$

The *cubic Bargmann-Fock equation* (or *LLL-equation*) we alluded to in the preceding subsection is

$$\mathrm{i}\partial_t u = \mathbf{\Pi}_1\big[E_1 u|u|^2\big], \tag{7.2.3}$$

where $u = u(t,z)$ is assumed differentiable in t and entire in z, and such that the integral expression defining the right-hand side of (7.2.3) is well defined. A *stationary wave* (= a traveling wave with zero speed) is a solution of the form $u(t,z) = \mathrm{e}^{-\mathrm{i}\omega t} f(z)$, where ω is a real constant and f is entire. The equation (7.2.3) then reduces to

$$\omega f = \mathbf{\Pi}_1\big[E_1 f|f|^2\big], \tag{7.2.4}$$

which for the value $\omega = 1$ we recognize as the equation (7.2.2). Note that for the right-hand side of (7.2.4) to be well defined, it is enough to assume that, e.g., $|f(z)| = \mathrm{O}(\mathrm{e}^{\eta|z|^2})$ as $|z| \to +\infty$ holds for some positive real $\eta < \frac{2}{3}$. We should also point out the possibility to include some higher Landau levels as well, as in [18]. Indeed, the corresponding *higher Landau level equation* analogous to (7.2.3) is

$$\mathrm{i}\partial_t u = \mathbf{\Pi}_1^{\langle N\rangle}\big[E_1 u|u|^2\big], \tag{7.2.5}$$

where $\mathbf{\Pi}_1^{\langle N\rangle}$ is the N-analytic Bargmann-Fock projection, for $N = 1,2,3,\ldots$. This is the orthogonal projection of the Gaussian weighted space $L^2(\mathbb{C},E_1)$ onto the subspace of N-analytic functions v, which solve the partial differential equation $\bar\partial^N v = 0$.

7.3. Hyperbolic zero packing for exponent β and field strength α. We turn to the β-exponent analogue of the hyperbolic zero packing problem, where we also introduce the positive real parameter α, which in a sense corresponds to *field strength*.

Definition 7.3.1. For a positive reals α,β, let $\rho_{\alpha,\beta}(\mathbb{H})$ be the density

$$\rho_{\alpha,\beta}(\mathbb{H}) := \liminf_{r\to 1^-}\inf_f \frac{1}{\log\frac{1}{1-r^2}}\int_{\mathbb{D}(0,r)}\big((1-|z|^2)^\alpha|f(z)|^\beta - 1\big)^2\frac{\mathrm{d}A(z)}{1-|z|^2},$$

where the infimum is taken over all polynomials f. We call this number $\rho_\beta(\mathbb{C})$ the *β-exponent minimal discrepancy density for hyperbolic zero packing with field strength α.*

The choice of parameters $\alpha = \beta = 1$ corresponds to the by now familiar density $\rho_{\mathbb{H}}$, that is, $\rho_{1,1}(\mathbb{H}) = \rho_{\mathbb{H}}$. The analogue of Proposition 7.1.3 in this hyperbolic context reads as follows.

PROPOSITION 7.3.2. *For positive reals $\alpha,\alpha',\beta,\beta'$ with $\beta<\beta'$ and $k\frac{\alpha'}{\alpha}=\frac{\beta'}{\beta}$ for some $k=1,2,3,\ldots$, we have that $\rho_{\alpha,\beta}(\mathbb{H})\le\rho_{\alpha',\beta'}(\mathbb{H})$.*

Proof. For $k=1$, the proof essentially amounts to a repetition of the argument used in Proposition 7.1.3. As for $k=2,3,4,\ldots$, we just need to observe that for a holomorphic function f, its power f^k is holomorphic as well, which gives the conclusion that $\rho_{\alpha,\beta}(\mathbb{H})\le\rho_{\alpha,k\beta}(\mathbb{H})\le\rho_{\alpha',\beta'}(\mathbb{H})$, where the last inequality follows from the $k=1$ case. □

Remark 7.3.3. It follows from Proposition 7.3.2 that the function $\beta\mapsto\rho_{\alpha\beta,\beta}(\mathbb{H})$ is monotonically increasing, for fixed positive α. It is then natural to ask for the limits as $\beta\to0^+$ and as $\beta\to+\infty$. We believe that

$$\lim_{\beta\to0^+}\rho_{\alpha\beta,\beta}(\mathbb{H})=0,\qquad\lim_{\beta\to+\infty}\rho_{\alpha\beta,\beta}(\mathbb{H})=1.$$

It is however less clear what happens to $\rho_{\alpha,\beta}(\mathbb{H})$ if we let $\alpha\to+\infty$ and keep β fixed.

CONJECTURE 7.3.4. *We believe that*

$$\lim_{\alpha\to+\infty}\rho_{\alpha,\beta}(\mathbb{H})=\rho_\beta(\mathbb{C}).$$

More intuitively, only local effects become important as we increase the field strength α.

Note that if we dilate the disk appropriately, the weight $(1-|z|^2)^\alpha$ becomes

$$\left(1-\frac{|z|^2}{\alpha}\right)^\alpha,$$

which has the limit $e^{-|z|^2}$ as $\alpha\to+\infty$. This shows the connection with the planar density.

Remark 7.3.5. There is a variant of (1.5.3) which applies for more general α,β. A minimizer f_0 for fixed r meets

$$\mathbf{P}_{\alpha-1,r}\left[\left((1-|z|^2)^\alpha|f_0|^\beta-1\right)\frac{|f_0|^\beta}{\bar f_0}\right]=0,$$

where $\mathbf{P}_{\alpha-1,r}$ is the weighted Bergman projection corresponding to the disk $\mathbb{D}(0,r)$ and the weight $(1-|z|^2)^{\alpha-1}$. As we noticed previously, the case when $0<\beta<1$ must be treated with additional care, as $|f_0|^\beta/\bar f_0$ may have nonintegrable singularities at zeros of f_0 of high multiplicity. Naturally, the instance $\alpha=\beta=1$ gives us back (1.5.3). If $\beta=2$, the above equation says that

$$f_0=\mathbf{P}_{\alpha-1,r}\left[(1-|z|^2)^\alpha f_0|f_0|^2\right],$$

and we are enticed to let $r \to 1^-$, and consider the equation

$$f_0 = \mathbf{P}_{\alpha-1}\big[(1-|z|^2)^\alpha f_0|f_0|^2\big],$$

where $\mathbf{P}_{\alpha-1}$ is the weighted Bergman projection on the unit disk $\mathbb{D}$ with the weight $(1-|z|^2)^{\alpha-1}$:

$$\mathbf{P}_{\alpha-1}h(z) := \alpha\int_{\mathbb{D}} \frac{(1-|w|^2)^{\alpha-1}}{(1-z\bar{w})^{\alpha+1}} h(w)\mathrm{d}A(w), \quad z\in\mathbb{D}.$$

As in the preceding subsection, there is a corresponding time evolution equation

$$\mathrm{i}\partial_t u = \mathbf{P}_{\alpha-1}\big[(1-|z|^2)^\alpha u|u|^2\big],$$

which we understand as a hyperbolic geometry analogue of the LLL-equation (7.2.3).

8. Geometric zero packing for compact Riemann surfaces using logarithmic monopoles. Our experience with geometric zero packing from Section 6 suggests a strong relation with regular configurations of lattice type, which suggests that the problem should be introduced on the quotient surface level, which should then be a compact Riemann surface. Moreover, the notion of a *logarithmic monopole* becomes very natural. It is the natural analogue of the Green function for the Laplacian in the context of compact surfaces.

8.1. Logarithmic monopoles for compact Riemann surfaces. We consider a compact Riemann surface $\mathcal{S}$ with genus g, where $g \ge 0$ is an integer. Then, by the uniformization theorem, $\mathcal{S}$ is has one of the following forms: (i) if $g = 0$, then $\mathcal{S}$ is topologically a sphere, which can be modelled by $\mathcal{S} = \mathbb{S}/\Gamma$ for a finite subgroup Γ of the automorphism group of the Riemann sphere $\mathbb{S}$, (ii) if $g = 1$, then $\mathcal{S}$ is a torus modelled by $\mathcal{S} = \mathbb{C}/\Lambda$ for a nontrivial lattice Λ, and (iii) if $g \ge 2$, then $\mathcal{S}$ is modelled by $\mathcal{S} = \mathbb{H}/\Gamma$, where $\mathbb{H}$ is the hyperbolic plane and Γ is a discrete subgroup of the automorphism group of $\mathbb{H}$. In each of the cases (i)–(iii), we have a complete Riemannian metric with constant curvature on the respective covering surfaces $\mathbb{S},\mathbb{C},\mathbb{H}$, which then induces a canonical Riemannian metric on the surface $\mathcal{S}$. In a similar fashion, the canonical normalized area measures $\mathrm{d}A_{\mathbb{S}},\mathrm{d}A,\mathrm{d}A_{\mathbb{H}}$ induce a normalized area measure on $\mathcal{S}$, which we denote by $\mathrm{d}A_{\mathcal{S}}$. The $\mathrm{d}A_{\mathcal{S}}$-area of the whole surface $\mathcal{S}$ is denoted by $a(\mathcal{S})$. The *logarithmic monopole* $U(z,w) = U_{\mathcal{S}}(z,w)$, for points $z,w\in\mathcal{S}$, is a real-valued function which for fixed w has

$$\varDelta^{\mathcal{S}} U(\cdot,w) = \frac{1}{2}\delta_w - \frac{1}{2a(\mathcal{S})},$$

where $\varDelta^{\mathcal{S}}$ is the normalized Laplace-Beltrami operator. The expression δ_w stands for the unit point mass at w, treated as a 2-form. The existence of this function is guaranteed by Corollary 8-2 of [49], which guarantees the existence of the corresponding logarithmic bipole $L(z,w,w')$ (see, e.g., [49, p. 213]), which has a source at w and a sink at w'. To obtain the monopole $U(z,w)$, we just average this bipole function $L(z,w,w')$ with respect surface area in the w' variable. It is unique up to an additive real constant.

For a nontrivial lattice Λ with two generators, the torus $\mathbb{C}/\Lambda$ is of course a compact Riemann surface with genus 1. In this case, the logarithmic monopole may be expressed explicitly in terms of the classical Weierstrass sigma function (see Subsection 8.4 below). We will also consider the spherical genus 0 case, as well as the (hyperbolic) genus ≥ 2 case.

8.2. Geometric zero packing on compact Riemann surfaces. We turn to the geometric zero packing problem for general compact surfaces. We introduce the notation

$$\langle f\rangle_{\mathcal{S}} := \frac{1}{a(\mathcal{S})}\int_{\mathcal{S}} f\,\mathrm{d}A_{\mathcal{S}}$$

for the surface average of a summable function $f:\mathcal{S}\to\mathbb{C}$. Let $U(z,w)=U_{\mathcal{S}}(z,w)$ denote the logarithmic monopole on the compact surface $\mathcal{S}$, normalized so that $\langle U(\cdot,w)\rangle_{\mathcal{S}}=0$.

Definition 8.2.1. For n points $w_1,\ldots,w_n\in\mathcal{S}$, we write

$$U^{\langle n\rangle}(z) := U(z,w_1)+\cdots+U(z,w_n).$$

For a positive real β, the *minimal average discrepancy for geometric β-zero packing* on $\mathcal{S}$ is the sequence of numbers

$$\rho_{n,\beta}(\mathcal{S}) := \inf_{b,w_1,\ldots,w_n}\left\langle\left(b\,\mathrm{e}^{\beta U^{\langle n\rangle}}-1\right)^2\right\rangle_{\mathcal{S}}, \tag{8.2.1}$$

where the infimum is over all positive reals b and all points $w_1,\ldots,w_n\in\mathcal{S}$. A collection of points $\{w_1,\ldots,w_n\}$ which realizes the infimum for some value of b is called an *equilibrium configuration* (for exponent β).

We observe that by standard Hilbert space methods,

$$\rho_{n,\beta}(\mathcal{S})^{1/2} = \inf_{b,w_1,\ldots,w_n}\sup_{g}\left\langle\left(b\,\mathrm{e}^{\beta U^{\langle n\rangle}}-1\right)g\right\rangle_{\mathcal{S}}$$

where the supremum runs over all real-valued functions g in the unit ball of $L^2(\mathcal{S})$. This is somewhat analogous to the Kantorovich-Wasserstein distance used in optimal transport [52]. The quantity $\rho_{n,\beta}(\mathcal{S})$ measures how evenly we can place the n points $w_1,\ldots,w_n$ on the surface so as to minimize the average discrepancy.

As for monotonicity issues, the approach used in Propositions 7.1.3 and 7.3.2 also shows the following. We suppress the analogous proof.

PROPOSITION 8.2.2. *Fix a compact Riemann surface $\mathcal{S}$ and an integer $n = 1,2,3,\ldots$. Then the function $\beta \mapsto \rho_{n,\beta}(\mathcal{S})$ is monotonically increasing:*

$$\rho_{n,\beta}(\mathcal{S}) \le \rho_{n,\beta'}(\mathcal{S}), \quad 0 < \beta < \beta' < +\infty.$$

Regarding the possible convergence as $n \to +\infty$ for a fixed exponent β, we suggest the following conjecture.

CONJECTURE 8.2.3. *We believe that*

$$\lim_{n\to+\infty} \rho_{n,\beta}(\mathcal{S}) \longrightarrow \rho_\beta(\mathbb{C})$$

for any fixed compact surface $\mathcal{S}$ and any fixed positive real β.

To arrive at an equilibrium configuration $\{w_1,\ldots,w_n\}$, we might try a numerical approach based on the gradient flow method. For a positive real $\gamma > 0$, we may think of

$$Z_\gamma(w_1,\ldots,w_n) := \langle \mathrm{e}^{\gamma U^{\langle n\rangle}} \rangle_{\mathcal{S}}$$

as a marginal partition function for the n-point β-ensemble on the surface $\mathcal{S}$ (see [30], also [31] for the sphere with $\gamma = 2$), and associate to it the probability measure on the surface $\mathcal{S}$

$$\frac{1}{Z_\gamma(w_1,\ldots,w_n)} \mathrm{e}^{\gamma U^{\langle n\rangle}} \frac{\mathrm{d}A_{\mathcal{S}}}{a(\mathcal{S})}. \tag{8.2.2}$$

We will return to this probability measure a little later, and meanwhile observe that the optimal value of b in the definition of $\rho_{n,\beta}(\mathcal{S})$ is

$$b = b(\beta) = \frac{\langle \mathrm{e}^{\beta U^{\langle n\rangle}} \rangle_{\mathcal{S}}}{\langle \mathrm{e}^{2\beta U^{\langle n\rangle}} \rangle_{\mathcal{S}}}. \tag{8.2.3}$$

Moreover, since

$$\inf_b \Big\langle \big(b\,\mathrm{e}^{\beta U^{\langle n\rangle}} - 1\big)^2 \Big\rangle_{\mathcal{S}} = \left\langle \left(\frac{\langle \mathrm{e}^{\beta U^{\langle n\rangle}} \rangle_{\mathcal{S}}}{\langle \mathrm{e}^{2\beta U^{\langle n\rangle}} \rangle_{\mathcal{S}}} \mathrm{e}^{\beta U^{\langle n\rangle}} - 1 \right)^2 \right\rangle_{\mathcal{S}} = 1 - \frac{\langle \mathrm{e}^{\beta U^{\langle n\rangle}} \rangle_{\mathcal{S}}^2}{\langle \mathrm{e}^{2\beta U^{\langle n\rangle}} \rangle_{\mathcal{S}}},$$

it is immediate that

$$\rho_{n,\beta}(\mathcal{S}) = 1 - \sup_{w_1,\ldots,w_n} \frac{\langle \mathrm{e}^{\beta U^{\langle n\rangle}} \rangle_{\mathcal{S}}^2}{\langle \mathrm{e}^{2\beta U^{\langle n\rangle}} \rangle_{\mathcal{S}}}.$$

The gradient flow method suggests that to climb the hill to the top we should always move in the direction of steepest ascent. In this situation, it is possible to calculate that direction at a given n-tuple $(w_1,\ldots,w_n)$. In particular, at the top where we stop the gradient vanishes. When we write this out, we obtain the following criterion.

PROPOSITION 8.2.4. *If $\{w_1,\dots,w_n\}$ is an equilibrium configuration for exponent β on the compact Riemann surface $\mathcal{S}$, then we have*

$$\frac{\left\langle e^{\beta U^{\langle n\rangle}}\partial_w U(\cdot,w)\right\rangle_{\mathcal{S}}}{\langle e^{\beta U^{\langle n\rangle}}\rangle_{\mathcal{S}}}=\frac{\left\langle e^{2\beta U^{\langle n\rangle}}\partial_w U(\cdot,w)\right\rangle_{\mathcal{S}}}{\langle e^{2\beta U^{\langle n\rangle}}\rangle_{\mathcal{S}}},\quad w\in\{w_1,\dots,w_n\}.$$

The expressions on the left-hand and right-hand sides of the displayed equation of the above proposition 8.2.4 are the expected values of the function $\partial_w U(\cdot,w)$ with respect to the probability measure (8.2.2) for $\gamma=\beta$ and $\gamma=2\beta$, respectively. Note that the necessary condition for an equilibrium configuration stated in Proposition 8.2.4 differs from the corresponding condition for Fekete configurations [45] as well as from that of spherical designs, e.g., [13]. Note that the space of n-tuples $(w_1,\dots,w_n)$ has complex dimension n, while the proposition supplies n complex nonlinear conditions (and hence $2n$ real conditions) which with some luck might have only a finite set of solutions.

Remark 8.2.5. (a) In principle, the definition of geometric β-zero packing (8.2.1) makes sense for negative exponents β as well, provided that the exponent is not too large negative: $-1<\beta<0$ is needed. It actually makes sense for $\beta=0$ as well, if we minimize instead the quantity

$$\rho_{n,\star}(\mathcal{S}):=\inf_{w_1,\dots,w_n}\left\langle\left(U^{\langle n\rangle}\right)^2\right\rangle_{\mathcal{S}},\tag{8.2.4}$$

which formally arises as the limit of $\beta^{-2}\rho_{n,\beta}(\mathcal{S})$ as $\beta\to 0$. In the context of (8.2.4) it is essential that we normalize the additive constant which we are allowed to add to $U(z,w)$ in such a way that

$$\langle U(\cdot,w)\rangle_{\mathcal{S}}=0,\quad w\in\mathcal{S}.\tag{8.2.5}$$

(b) Let us compare the minimization problem (8.2.4) with the Fekete problem which in the given setting asks for the configurations $w_1,\dots,w_n$ that achieve the maximum

$$\sup_{w_1,\dots,w_n}\sum_{j,k:j\neq k}U(w_j,w_k).$$

It will be convenient to cut off the singularity in the logarithmic monopole and replace it by a function $U_\epsilon(z,w)$ which is C^2-smooth in both variables and $\lim_{\epsilon\to 0^+}U_\epsilon(z,w)=U(z,w)$. If the approximation is done well, with controlled errors, we could arrange it so that for fixed $z \neq w$, we would have

$$\begin{aligned} U(z,w) &= 2\int_{\mathcal{S}} U(\cdot,w)\Delta^{\mathcal{S}} U_\epsilon(\cdot,z)\mathrm{d}A_{\mathcal{S}} + \mathrm{O}(\epsilon) \\ &= 2\int_{\mathcal{S}} U_\epsilon(\cdot,w)\Delta^{\mathcal{S}} U_\epsilon(\cdot,z)\mathrm{d}A_{\mathcal{S}} + \mathrm{O}(\epsilon) \\ &= -2\int_{\mathcal{S}} \partial^{\mathcal{S}} U_\epsilon(\cdot,z)\bar\partial^{\mathcal{S}} U_\epsilon(\cdot,w)\mathrm{d}A_{\mathcal{S}} + \mathrm{O}(\epsilon). \end{aligned}$$

Here we used the property (8.2.5), and in a second step, Green's formula on the surface. In terms of notation, the differential operators $\partial^{\mathcal{S}}$ and $\bar\partial^{\mathcal{S}}$ are modified to fit the surface (the conformal factor is inserted on the left-hand side). It now follows that

$$\begin{aligned} \sum_{j,k:j\neq k} U(w_j,w_k) = -2\int_{\mathcal{S}} \Big|\sum_j \partial^{\mathcal{S}} U_\epsilon(\cdot,w_j)\Big|^2 \mathrm{d}A_{\mathcal{S}} \\ + 2\sum_j \int_{\mathcal{S}} |\partial^{\mathcal{S}} U_\epsilon(\cdot,w_j)|^2 \mathrm{d}A_{\mathcal{S}} + \mathrm{O}(\epsilon), \end{aligned}$$

where the expression

$$\int_{\mathcal{S}} |\partial^{\mathcal{S}} U_\epsilon(\cdot,w_j)|^2 \mathrm{d}A_{\mathcal{S}} = \frac{1}{4}\int_{\mathcal{S}} |\nabla^{\mathcal{S}} U_\epsilon(\cdot,w_j)|^2 \mathrm{d}A_{\mathcal{S}}$$

tends to $+\infty$ with rather precise asymptotics as $\epsilon \to 0^+$ (at least if $U_\epsilon(z,w)$ is chosen correctly). If we add a term to neutralize each such contribution (for each point w_j), the Fekete problem essentially asks for configurations that minimize the Dirichlet energy

$$\int_{\mathcal{S}} |\nabla^{\mathcal{S}} U_\epsilon^{\langle n\rangle}|^2 \mathrm{d}A_{\mathcal{S}}, \quad \text{where } U_\epsilon^{\langle n\rangle} := U_\epsilon(\cdot,w_1) + \cdots + U_\epsilon(\cdot,w_n),$$

as $\epsilon \to 0^+$ (see, e.g., [47] for a more general situation). In comparison, the problem (8.2.4) asks us to minimize the corresponding Bergman energy (just the area-L^2 norm squared).

8.3. Spherical zero packing. We briefly mention what happens when the compact Riemann surface has *genus* 0. We will consider the Riemann sphere $\mathbb{S} := \mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ with the standard metric $\mathrm{d}s_{\mathbb{S}} := (1+|z|^2)^{-1}|\mathrm{d}z|$, which has constant positive Gaussian curvature. The associated spherical normalized area measure is $\mathrm{d}A_{\mathbb{S}}(z) := (1+|z|^2)^{-2}\mathrm{d}A(z)$. Moreover, the associated logarithmic monopole is the function

$$U(z,w) := \log|z-w| - \frac{1}{2}\log(1+|z|^2) + A(w), \quad z,w \in \mathbb{C},\ z \neq w,$$

where we are free to choose the real number $A(w)$. The choice $A(w) := -\frac{1}{2}\log(1+|w|^2)$ would seem to be the most appropriate, since it gives the symmetry property $U(z,w) = U(w,z)$, typical of Green functions. Then we may define $U(z,w)$ for $w = \infty$ as well: $U(z,\infty) = -\frac{1}{2}\log(1+|z|^2)$. After all, the basic property of the logarithmic monopole is that

$$\Delta^{\mathbb{S}} U(\cdot,w) = \frac{1}{2}\delta_w - \frac{1}{2}$$

in the sense of distribution theory. Here, it is of course important that the area of the sphere is normalized to be $a(\mathbb{S}) = 1$.

Let us look at the minimal average discrepancy for spherical β-zero packing, that is, the numbers $\rho_{n,\beta}(\mathbb{S})$, for $n = 1$ and $n = 2$. Since

(8.3.1)
$$\rho_{n,\beta}(\mathbb{S}) = \inf_{a,w_1,\ldots,w_n} \int_{\mathbb{S}} \left(a \frac{|z-w_1|^\beta \cdots |z-w_n|^\beta}{(1+|z|^2)^{n\beta/2}(1+|w_1|^2)^{\beta/2}\cdots(1+|w_n|^2)^{\beta/2}} - 1 \right)^2 \mathrm{d}A_{\mathbb{S}}(z),$$

where the infimum is over all positive reals a and all complex numbers $w_1, \ldots, w_n$, we calculate that (with $w_1 = 0$)

$$\rho_{1,\beta}(\mathbb{S}) = \inf_a \int_{\mathbb{S}} \left(\frac{a|z|^\beta}{(1+|z|^2)^{\beta/2}} - 1 \right)^2 \mathrm{d}A_{\mathbb{S}}(z) = \frac{\beta^2}{(2+\beta)^2}.$$

As for $n = 2$, it is intuitively clear that the two points should be antipodal in the optimal configuration, and we then pick $w_1 = 0, w_2 = \infty$. As a consequence,

$$\rho_{2,\beta}(\mathbb{S}) = \inf_a \int_{\mathbb{S}} \left(\frac{a|z|^\beta}{(1+|z|^2)^{\beta}} - 1 \right)^2 \mathrm{d}A_{\mathbb{S}}(z) = 1 - \frac{2^{-4\beta}\pi^2\Gamma(2+2\beta)}{(1+\beta)^2\Gamma(\frac{1+\beta}{2})^4},$$

which we may compare with $\rho_{1,\beta}(\mathbb{S})$. Computer work strongly suggests that $\rho_{1,\beta}(\mathbb{S}) \geq \rho_{2,\beta}(\mathbb{S})$ for all positive values of β (this is from a calculation made by Wennman [54]). For instance, with $\beta = 1$, we find that $\rho_{1,1}(\mathbb{S}) = 0.111\ldots$, whereas $\rho_{2,1}(\mathbb{S}) = 0.07472\ldots$, which is much smaller and considerably closer to the conjectured value of $\rho_{\mathbb{C}}$ (which is $0.061203\ldots$, see Remark 6.1.4). Here one might naïvely guess that the function $n \mapsto \rho_{n,\beta}(\mathbb{S})$ is decreasing for fixed β. While this may be true for small β, it is certainly false for large β, as evidenced by further numerical work for $n = 3$. Compare also with Conjecture 8.2.3.

8.4. Logarithmic monopoles for a torus. We turn to the case of a compact Riemann surface with *genus* 1. Such a Riemann surface is a torus, and can be modelled by $\mathbb{C}/\Lambda_{\omega_1,\omega_2}$, in the notation of Subsection 6.1. Note that if we take logarithms in (6.1.3), we obtain the real-valued function

$$\begin{aligned}U(z) &:= \log(\mathrm{e}^{-|z|^2}|f(z)|) = -|z|^2 + \log|f(z)| \\ &= -|z|^2 + \log a + \mathrm{Re}(\xi z + \eta z^2) + \log|\sigma(z)|,\end{aligned}$$

and, we may define, more generally, $U(z,w) := U(z-w)$. Since the positive constant a is free, the function $U(z,w)$ is real-valued and well defined up to an additive constant. Moreover, it is $\Lambda_{\omega_1,\omega_2}$-periodic in both z and w, with Laplacian

$$\Delta U(\cdot,w) = -1 + \frac{1}{2}\sum_{\lambda\in\Lambda_{\omega_1,\omega_2}} \delta_{\lambda+w}$$

in the sense of distribution theory, where δ_ξ is the unit point mass at the point $\xi \in \mathbb{C}$.

8.5. Logarithmic monopoles for higher genus surfaces and character-modular forms. We turn to the case when the Riemann surface $\mathcal{S}$ has genus $g \geq 2$. We then equip the surface with a metric of constant negative curvature, and use the hyperbolic plane $\mathbb{H}$ as the universal covering surface. We model the hyperbolic plane $\mathbb{H}$ by the unit disk $\mathbb{D}$ with the Poincaré metric. This gives us the identification $\mathcal{S} \cong \mathbb{D}/\Gamma$, where Γ is a Fuchsian group of Möbius automorphisms. We write D_Γ for a corresponding fundamental polygon bounded by hyperbolic geodesic segments. We denote by $a(\Gamma)$ the $\mathrm{d}A_{\mathbb{H}}$-area of D_Γ, which is the same as the corresponding area of the surface $a(\mathcal{S})$. We first relate two properties of periodicity type.

PROPOSITION 8.5.1. *Suppose $f : \mathbb{D} \to \mathbb{C}$ is holomorphic. Then, for real α, the following are equivalent:*

(a) *For all $\gamma \in \Gamma$, we have $|\gamma'(z)|^\alpha |f\circ\gamma(z)| = |f(z)|$ on the disk $\mathbb{D}$.*

(b) *The function $F(z) := (1-|z|^2)^\alpha |f(z)|$ is Γ-periodic, that is, $F\circ\gamma = F$ for all $\gamma \in \Gamma$.*

Proof. This is an immediate consequence of the identity

$$1-|\gamma(z)|^2 = (1-|z|^2)|\gamma'(z)|, \tag{8.5.1}$$

which holds for any Möbius automorphism $\gamma \in \mathrm{aut}(\mathbb{D})$. □

We want to analyze the property (a) of Proposition 8.5.1 more carefully. First, for an automorphism $\gamma \in \mathrm{aut}(\mathbb{D})$, the derivative γ' is nonzero, which permits us to define its logarithm $\log\gamma'$ holomorphically in $\mathbb{D}$ (any two choices will differ by an integer multiple of $\mathrm{i}2\pi$). The group Γ is finitely generated, and we pick generators $\gamma_1,\ldots,\gamma_m$, and choose the corresponding logarithms $\log\gamma_j'$ for $j = 1,\ldots,m$ as holomorphic functions any way we like (the freedom is up to constants in $\mathrm{i}2\pi\mathbb{Z}$). If the group Γ were free, we would then proceed to represent an arbitrary element

$\gamma \in \Gamma$ as a "word" (a finite composition of the generators), and let the logarithm $\log \gamma'$ be determined by the natural property that

$$\log(\tilde{\gamma} \circ \gamma)' = (\log \tilde{\gamma}') \circ \gamma + \log \gamma', \quad \gamma, \tilde{\gamma} \in \Gamma. \tag{8.5.2}$$

In a second step, the powers

$$(\gamma'(z))^\alpha = \exp(\alpha \log \gamma'(z)), \quad \gamma \in \Gamma, \tag{8.5.3}$$

would be defined as well, and we would have the property that for $\alpha \in \mathbb{C}$,

$$((\tilde{\gamma} \circ \gamma)')^\alpha = ((\tilde{\gamma}')^\alpha \circ \gamma)(\gamma')^\alpha, \quad \gamma, \tilde{\gamma} \in \Gamma. \tag{8.5.4}$$

However, typically Γ is not a free group, as there are finitely many so-called relations which need to be satisfied as well (a relation is the condition that a nontrivial combination of the generators equals the identity). These relations will put some restraints on our freedom of choosing the logarithms $\log \gamma_j'$ for our generators γ_j, and it may happen that this cannot be done in a manner consistent with (8.5.2). One way out is to express each group element $\gamma \in \Gamma$ uniquely as a combination of generators by picking the shortest "word" expressing the element (if there is competition, just pick one of the minimal "words"), and to use (8.5.2) along the "word" to define properly the logarithm of the derivative. Then we sacrifice the property (8.5.2) globally, and hence (8.5.4) need not hold for an arbitrary $\alpha \in \mathbb{C}$. However, equality in (8.5.4) holds automatically for integers $\alpha \in \mathbb{Z}$.

A Γ-*character* is a function $\chi : \Gamma \to \mathbb{T}$ with the multiplicative property $\chi(\tilde{\gamma} \circ \gamma) = \chi(\tilde{\gamma})\chi(\gamma)$ for all $\gamma, \tilde{\gamma} \in \Gamma$. We shall require a slightly generalized version of this notion, which we refer to as a (Γ, q)-root character, or simply a q-root character if the group Γ is taken for granted. For a positive integer q, we say that $\chi : \Gamma \to \mathbb{T}$ is a (Γ, q)-*root character* if and only if χ^q is a Γ-character (here, $\chi^q(\gamma) = (\chi(\gamma))^q$ is the q-th power). This generalizes the concept of the characters, since a character is automatically a q-root character for integers $q = 2, 3, 4, \ldots$.

PROPOSITION 8.5.2. *Suppose $f : \mathbb{D} \to \mathbb{C}$ is holomorphic. Then, for rational $\alpha = p/q$, where p, q are coprime integers and $q > 0$, the following are equivalent:*

(a) *For all $\gamma \in \Gamma$, we have $|\gamma'(z)|^\alpha |f \circ \gamma(z)| = |f(z)|$ on the disk $\mathbb{D}$.*

(b) *There exists a (Γ, q)-root character χ such that $(\gamma'(z))^\alpha f \circ \gamma(z) = \chi(\gamma) f(z)$.*

Proof. Clearly, (b)⇒(a), so we will obtain the remaining implication (a)⇒(b). We observe from (a) that for given $\gamma \in \Gamma$, the two holomorphic functions $(\gamma')^\alpha f \circ \gamma$ and f have the same modulus. This is only possible if one is a unimodular constant times the other, that is,

$$(\gamma')^\alpha f \circ \gamma = \chi(\gamma) f$$

holds for some *constant* $\chi(\gamma)$ of modulus 1. All that remains is to show that χ is a (Γ,q)-root character. To this end, we pick two elements $\gamma,\tilde{\gamma}\in\Gamma$, and observe that

$$\begin{aligned}\chi^q(\tilde{\gamma}\circ\gamma)f^q &= ((\tilde{\gamma}\circ\gamma)')^p f^q\circ\tilde{\gamma}\circ\gamma = (\gamma')^p((\tilde{\gamma}')^p\circ\gamma)\, f^q\circ\tilde{\gamma}\circ\gamma \\ &= \chi^q(\gamma)(\tilde{\gamma}')^p f^q\circ\tilde{\gamma} = \chi^q(\gamma)\chi^q(\tilde{\gamma})f^q,\end{aligned}$$

which shows that $\chi^q(\tilde{\gamma}\circ\gamma)=\chi^q(\gamma)\chi^q(\tilde{\gamma})$ and hence that χ is a (Γ,q)-root character. □

A function f which meets condition (b) of Proposition 8.5.2 is said to be *q-root character-periodic (or modular) of weight* α, with respect to the character $\chi:\Gamma\to\mathbb{T}$. Here, we could mention that when $\alpha=1$, they are character-periodic of weight 1, and called *Prym differentials*.

It is a natural question when there exist nontrivial functions f with property (a) of Proposition 8.5.2, for a given real number α. For instance, if this does not happen for irrational α, then Proposition 8.5.2 gives a rather complete picture. To sort this matter out, we consult Proposition 8.5.1, which asserts that the associated function $F(z):=(1-|z|^2)^\alpha|f(z)|\ge 0$ is Γ-periodic, and note that the (real-valued) logarithm

$$\log F(z)=\alpha\log(1-|z|^2)+\log|f(z)| \tag{8.5.5}$$

is Γ-periodic as well. We now turn to the logarithmic monopole $U(z,w)$ for the surface $\mathcal{S}\cong\mathbb{D}/\Gamma$, which has

$$(\varDelta^{\mathcal{S}}U(\cdot,w))\mathrm{d}A_{\mathcal{S}}=(\varDelta U(\cdot,w))\mathrm{d}A=-\frac{1}{2a(\Gamma)}\mathrm{d}A_{\mathbb{H}}+\frac{1}{2}\sum_{\gamma\in\Gamma}\delta_{\gamma(w)} \tag{8.5.6}$$

in the sense of distribution theory, where δ_ζ denotes is the unit point mass at ζ, considered as a 2-form. Note that on the right-hand side of (8.5.6), we have a $\frac{1}{2}$ point mass per tile (here, a tile is the image of D_Γ under an element of Γ), which is perfectly compensated on each tile by the hyperbolically uniform measure $-\frac{1}{2a(\Gamma)}\mathrm{d}A_{\mathbb{H}}$. As we apply the Laplacian to the relation (8.5.5), we find that

$$(\varDelta\log F)\mathrm{d}A=-\alpha\mathrm{d}A_{\mathbb{H}}+\frac{1}{2}\sum_{\zeta\in\mathrm{Z}(f)}\delta_\zeta, \tag{8.5.7}$$

where $\mathrm{Z}(f)$ denotes the zeros of f, counting multiplicities. Note that since F was Γ-periodic, the zero set $\mathrm{Z}(f)$ is Γ-periodic as well. As the surface $\mathcal{S}$ was compact, F (and equivalently f) can have only finitely many zeros in $\mathbb{D}/\Gamma$, say $\zeta_1,\ldots,\zeta_n\in\bar{D}_\Gamma$, where some of the points are allowed to be on the boundary of the fundamental polygon. We now form the function V_F,

$$V_F(z) := \sum_{j=1}^{n} U(z,\zeta_j),$$

which is Γ-periodic and has Laplacian

$$(\varDelta V_F)\mathrm{d}A = \sum_{j=1}^{n}(\varDelta U(\cdot,\zeta_j))\mathrm{d}A = -\frac{n}{2a(\Gamma)}\mathrm{d}A_{\mathbb{H}} + \frac{1}{2}\sum_{j=1}^{n}\sum_{\gamma\in\Gamma}\delta_{\gamma(\zeta_j)},$$

in the sense of distribution theory. Moreover, since the points $\gamma(\zeta_j)$, with $j = 1,\ldots,n$ and $\gamma \in \Gamma$, run through the zero set $\mathrm{Z}(f)$, the above relation simplifies to

$$(\varDelta V_F)\mathrm{d}A = -\frac{n}{2a(\Gamma)}\mathrm{d}A_{\mathbb{H}} + \frac{1}{2}\sum_{\zeta\in\mathrm{Z}(f)}\delta_\zeta, \tag{8.5.8}$$

which we may compare with (8.5.7). The difference of V_F and $\log F$ is Γ-periodic, with Laplacian

$$(\varDelta(V_F - \log F))\mathrm{d}A(z) = \left(\alpha - \frac{n}{2a(\Gamma)}\right)\mathrm{d}A_{\mathbb{H}}, \tag{8.5.9}$$

which expression has constant sign. Since a subharmonic function on a compact Riemann surface must be constant, we conclude that this is only possible if $V_F - \log F$ is constant and hence $\alpha = \frac{n}{2a(\Gamma)}$. Finally, an application of the Gauss-Bonnet theorem gives that $a(\Gamma) = g-1$, where $g \geq 2$ is the genus of the surface $\mathcal{S} \cong \mathbb{D}/\Gamma$, so that in particular $\alpha = \frac{n}{2(g-1)}$ is rational. We gather these simple observations in a proposition.

PROPOSITION 8.5.3. *Let $\mathcal{S} \cong \mathbb{D}/\Gamma$ be a compact Riemann surface of genus $g \geq 2$. Then the $\mathrm{d}A_{\mathbb{H}}$-area of the fundamental polygon D_Γ equals $a(\Gamma) = g-1$. Suppose $f : \mathbb{D} \to \mathbb{C}$ is holomorphic and such that the associated function $F(z) := (1-|z|^2)^\alpha |f(z)|$ is Γ-periodic for some real parameter α. If f is nontrivial, then $n := 2a(\Gamma)\alpha$ is a nonnegative integer, and F takes the form*

$$F = C\exp\left(\sum_{j=1}^{n} U(\cdot,\zeta_j)\right), \tag{8.5.10}$$

where $U(z,w)$ is the logarithmic monopole for $\mathcal{S}$, C is a positive constant, and $\zeta_1,\ldots,\zeta_n$ enumerate the zeros of F in $\mathbb{D}/\Gamma$. Moreover, any function of the form (8.5.10) *can be written as $F(z) = (1-|z|^2)^\alpha |f(z)|$ for some holomorphic function f on $\mathbb{D}$ if $\alpha = \frac{n}{2a(\Gamma)}$.*

Proof. All the assertions are settled by the arguments preceding the statement of the proposition, except that it remains to show that a function F given by (8.5.10) can be written as $F(z) = (1-|z|^2)^\alpha |f(z)|$ with $\alpha = \frac{n}{2a(\Gamma)}$, for some holomorphic f. We see from (8.5.10) that the purported f should have $|f(z)| = (1-|z|^2)^{-\alpha} F(z)$ and hence

$$\begin{aligned}\log\big|f(z)\big| &= -\alpha \log\big(1-|z|^2\big) + \log F(z)\\ &= -\alpha \log\big(1-|z|^2\big) + \log C + \sum_{j=1}^{n} U\big(z,\zeta_j\big).\end{aligned}$$

Taking Laplacians on both sides we get that

$$\begin{aligned}\Delta \log|f(z)| &= \frac{\alpha}{(1-|z|^2)^2} + \sum_{j=1}^{n} \Delta_z U(z,\zeta_j) = \frac{\alpha - \frac{n}{2a(\Gamma)}}{(1-|z|^2)^2} + \frac{1}{2}\sum_{j=1}^{n}\sum_{\gamma\in\Gamma} \delta_{\gamma(\zeta_j)}\\ &= \frac{1}{2}\sum_{j=1}^{n}\sum_{\gamma\in\Gamma} \delta_{\gamma(\zeta_j)},\end{aligned}$$

in the sense of distribution theory, where we use that $\alpha = \frac{n}{2a(\Gamma)}$. This just asks for f to have zeros (counting multiplicities) along the sequence of points $\gamma(\zeta_j)$, with $\gamma \in \Gamma$ and $j = 1,\ldots,n$. A version of the Weierstrass factorization theorem (see, e.g., [44]) assures us that there exists a holomorphic function $h : \mathbb{D} \to \mathbb{C}$ with precisely the zeros prescribed for f, and then the difference $u := \log|f| - \log|h|$ must be harmonic. By forming the harmonic conjugate to u we obtain a holomorphic function $U : \mathbb{D} \to \mathbb{C}$ with real part equal to u. Finally, we realize that the choice $f := \mathrm{e}^U h$ is holomorphic with the right modulus so that $F(z) = (1-|z|^2)|f(z)|$ holds. □

Remark 8.5.4. For some related geometric complex analysis on compact Riemann surfaces, involving forms and sections, see the textbooks [8, 14, 24].

8.6. Character-modular forms, ergodic geodesic flow, and hyperbolic zero packing. We keep the setting of a compact Riemann surface $\mathcal{S}$ with genus ≥ 2, so that $\mathcal{S} \cong \mathbb{D}/\Gamma$ for a Fuchsian group with a fundamental domain D_Γ. Then, as a matter of definition (see (8.2.1)),

$$\rho_{n,\beta}(\mathcal{S}) = \inf_{b,w_1,\ldots,w_n} \Big\langle \big(b\mathrm{e}^{\beta U^{\langle n\rangle}} - 1\big)^2 \Big\rangle_{\mathcal{S}},$$

where b is a positive real and $w_1,\ldots w_n \in \mathcal{S}$. We would like to see how this fares compared with the hyperbolic densities $\rho_{\alpha,\beta}(\mathbb{H})$ defined in Subsection 7.3. We use Proposition 8.5.3 to see that there exists a holomorphic function f with zeros (counting multiplicities) exactly at the points $\gamma(w_j)$ when $j = 1,\ldots,n$ and $\gamma \in \Gamma$, such that

$$b\mathrm{e}^{\beta U^{\langle n\rangle}(z)} = (1-|z|^2)^{\frac{n\beta}{2a(\Gamma)}}|f(z)|^\beta,$$

and both sides express Γ-periodic functions. Now, for the right-hand side we could try to compute the discrepancy density with respect to the disk $\mathbb{D}$ as well:

$$\rho_{\alpha',\beta}(f) := \liminf_{r\to 1^-} \frac{1}{\log\frac{1}{1-r^2}} \int_{\mathbb{D}(0,r)} \big((1-|z|^2)^{\alpha'}|f(z)|^\beta - 1\big)^2 \frac{\mathrm{d}A(z)}{1-|z|^2},$$

with $\alpha' := \frac{n\beta}{2a(\Gamma)}$. The following result tells us that the above average can be achieved by integrating over one tile with respect to the Fuchsian group Γ.

PROPOSITION 8.6.1. *In the above setting, with $f : \mathbb{D}\to\mathbb{C}$ holomorphic and the associated function $z \mapsto (1-|z|^2)^{\alpha'}|f(z)|^\beta$ assumed Γ-periodic, we have that*

$$\lim_{r\to 1^-} \frac{1}{\log\frac{1}{1-r^2}} \int_{\mathbb{D}(0,r)} \big((1-|z|^2)^{\alpha'}|f(z)|^\beta - 1\big)^2 \frac{\mathrm{d}A(z)}{1-|z|^2}$$
$$= \frac{1}{a(\Gamma)} \int_{D_\Gamma} \big((1-|z|^2)^{\alpha'}|f(z)|^\beta - 1\big)^2 \mathrm{d}A_{\mathbb{H}}.$$

Proof. To simplify the notation, we write

$$\Phi_{\alpha',\beta,f}(z) := \big((1-|z|^2)^{\alpha'}|f(z)|^\beta - 1\big)^2,$$

which is Γ-periodic and hence a well-defined function on $\mathcal{S}\cong\mathbb{D}/\Gamma$. The claim can now be expressed in the form

$$\lim_{r\to 1^-} \frac{\int_{\mathbb{D}(0,r)} \Phi_{\alpha',\beta,f}(z)\frac{\mathrm{d}A(z)}{1-|z|^2}}{\log\frac{1}{1-r^2}} = \frac{\int_{\mathcal{D}_\Gamma} \Phi_{\alpha',\beta,f}(z)\mathrm{d}A_{\mathbb{H}}(z)}{\int_{\mathcal{D}_\Gamma}\mathrm{d}A_{\mathbb{H}}}, \tag{8.6.1}$$

that is, averages formed in two different ways coincide. Such assertions remind us of ergodic theory. Indeed, it follows from the well-known ergodicity of geodesic flow on compact hyperbolic surfaces, originally due to Hopf and later extended to higher-dimensional manifolds by Anosov (see, e.g., Hopf's expository paper [25]). The "time average" of Φ_f over the geodesic ray $z = t\zeta$ with $|\zeta| = 1$ and radial parameter t with $0 < t < r$, is

$$\frac{\int_0^r \Phi_f(t\zeta)\frac{\mathrm{d}t}{1-t^2}}{\int_0^r \frac{\mathrm{d}t}{1-t^2}} = \frac{\int_0^r \Phi_f(t\zeta)\frac{\mathrm{d}t}{1-t^2}}{\frac{1}{2}\log\frac{1+r}{1-r}}, \tag{8.6.2}$$

while the "space average" is expressed by the right-hand side of (8.6.1). The limit of the ratio (8.6.2) as $r\to 1^-$ is clearly unperturbed if we replace $\Phi_f(t\zeta)$ by $t\Phi_f(t\zeta)$, and hence (8.6.1) results from (8.6.2) by integration over the circle $\mathbb{T}$

in ζ. Here, we used the elementary observation that

$$\lim_{r\to 1^-} \frac{\log \frac{1}{1-r^2}}{\log \frac{1+r}{1-r}} = 1.$$

The proof is complete. □

Remark 8.6.2. Using more refined control of the error term in the ergodicity of geodesic flow, it is possible to obtain a comparison of the densities $\rho_{n,\beta}(\mathcal{S})$ and $\rho_{\alpha',\beta}(\mathbb{H})$, to the effect that

$$\rho_{\alpha',\beta}(\mathbb{H}) \le \rho_{n,\beta}(\mathcal{S}) \quad \text{where } \alpha' = \frac{n\beta}{2a(\Gamma)} \text{ and } \mathcal{S} \cong \mathbb{D}/\Gamma.$$

The question comes to mind if, for fixed α' and β, the right-hand side expression can be made arbitrarily close to the left-hand side expression by varying suitably the surface $\mathcal{S}$ and the number of points n. This may not be the case.

9. Fekete configurations, geometric zero packing, and heat flow on compact Riemann surfaces. This section takes the form of a rather extensive remark on heat flows and the relation between Fekete-type problems and geometric zero packing on a compact Riemann surface $\mathcal{S}$ supplied with a metric of constant Gaussian curvature. The particular normalizations and notational conventions are kept as before.

9.1. Integration along heat flow. We begin with the comparison of Remark 8.2.5, in the context of heat flow on a compact Riemann surface $\mathcal{S}$. First, let

$$U^{\langle n\rangle}(z) = U(z,w_1) + \cdots + U(z,w_n), \quad z \in \mathcal{S}, \tag{9.1.1}$$

be a sum of logarithmic monopoles, each normalized so that (8.2.5) holds. For $t>0$, we let $U_t^{\langle n\rangle}(z)$ denote the function of (z,t) for $z\in\mathcal{S}$ and $t>0$ which solves the *heat equation*

$$\partial_t U_t^{\langle n\rangle}(z) = \Delta^{\mathcal{S}} U_t^{\langle n\rangle}(z), \quad z\in\mathcal{S}, \tag{9.1.2}$$

with initial datum at $t=0$ given by $U^{\langle n\rangle}(z)$. Note that in view of (8.2.5), it follows that $U_t^{\langle n\rangle} \to 0$ as $t\to+\infty$. Then according to Remark 8.2.5 the Fekete configuration problem is concerned with minimizing the Dirichlet energy

$$\int_{\mathcal{S}} \left|\nabla^{\mathcal{S}} U_t^{\langle n\rangle}\right|^2 \mathrm{d}A_{\mathcal{S}} \tag{9.1.3}$$

in the limit as $t \to 0^+$ over all point configurations $w_1, \ldots, w_n$, where the gradient and area elements are geometrically adjusted. On the other hand, the $\beta \to 0$ geometric zero packing problem (8.2.4) minimizes instead the Bergman energy

$$\int_{\mathcal{S}} \big(U_t^{\langle n \rangle}\big)^2 \mathrm{d}A_{\mathcal{S}} \tag{9.1.4}$$

for $t = 0$. We would like to relate, if possible, these two problems. Suppose we are lucky and the minimizing configuration for the Dirichlet energy (9.1.3) as $t \to 0^+$ *actually minimizes for all* $t > 0$ *at the same time*. Then *the same configuration is also minimizing for the Bergman energy as well*. To see this, we calculate as follows:

$$\begin{aligned} \int_0^{+\infty} \int_{\mathcal{S}} |\nabla^{\mathcal{S}} U_t^{\langle n \rangle}|^2 \mathrm{d}A_{\mathcal{S}} \mathrm{d}t &= -4 \int_0^{+\infty} \int_{\mathcal{S}} U_t^{\langle n \rangle} \Delta^{\mathcal{S}} U_t^{\langle n \rangle} \mathrm{d}A_{\mathcal{S}} \mathrm{d}t \\ &= -4 \int_0^{+\infty} \int_{\mathcal{S}} U_t^{\langle n \rangle} \partial_t U_t^{\langle n \rangle} \mathrm{d}A_{\mathcal{S}} \mathrm{d}t \\ &= -2 \int_{\mathcal{S}} \int_0^{+\infty} \partial_t \{(U_t^{\langle n \rangle})^2\} \mathrm{d}t \mathrm{d}A_{\mathcal{S}} \\ &= 2 \int_{\mathcal{S}} (U^{\langle n \rangle})^2 \mathrm{d}A_{\mathcal{S}}. \end{aligned} \tag{9.1.5}$$

In any case, the formula (9.1.5) shows that the Bergman energy is the time integral of the Dirichlet energy along the heat flow. We now search for an analogue of (9.1.5) which applies for more general values of the parameter β.

9.2. Integration along β-deformed heat flow. Let $V_{b,\beta}^{\langle n \rangle}$ denote the function

$$V_{b,\beta}^{\langle n \rangle} := \frac{b\mathrm{e}^{\beta U^{\langle n \rangle}} - 1}{\beta},$$

where b is positive and real and $U^{\langle n \rangle}$ is as in (9.1.1); the minimum

$$\inf_b \Big\langle \big(V_{b,\beta}^{\langle n \rangle}\big)^2 \Big\rangle_{\mathcal{S}},$$

which is of interest in connection with the problem of geometric β-zero packing, is then attained for

$$b = b(\beta) := \frac{\langle \mathrm{e}^{\beta U^{\langle n \rangle}} \rangle_{\mathcal{S}}}{\langle \mathrm{e}^{2\beta U^{\langle n \rangle}} \rangle_{\mathcal{S}}}.$$

To simplify the notation, we write $V_\beta^{\langle n\rangle} := V_{b(\beta),\beta}^{\langle n\rangle}$. After all, the geometric zero packing problem involves the same minimization over all constants b and all point configurations $w_1,\ldots,w_n$ (which determine the function $U^{\langle n\rangle}$). We then observe that $V_\beta^{\langle n\rangle} \to U^{\langle n\rangle}$ as $\beta\to 0$, which suggests that we may think of $V_\beta^{\langle n\rangle}$ as a (non-linear) β-deformation of the function $U^{\langle n\rangle}$. The average of $V_\beta^{\langle n\rangle}$ equals

$$\langle V_\beta^{\langle n\rangle}\rangle_{\mathcal{S}} = \frac{1}{\beta}\left(\frac{\langle \mathrm{e}^{\beta U^{\langle n\rangle}}\rangle_{\mathcal{S}}^2}{\langle \mathrm{e}^{2\beta U^{\langle n\rangle}}\rangle_{\mathcal{S}}} - 1\right)$$

which vanishes only when $\beta = 0$. This means that for $\beta \neq 0$, after infinite time the heat limit with initial datum $V_\beta^{\langle n\rangle}$ equals a nonzero constant, which makes the approach underlying the formula (9.1.5) difficult to carry out. However, it is indeed the case that a weighted average of $V_\beta^{\langle n\rangle}$ vanishes:

$$\big\langle V_\beta^{\langle n\rangle}\mathrm{e}^{\beta U^{\langle n\rangle}}\big\rangle_{\mathcal{S}} = 0.$$

To use this fact, we equip the surface $\mathcal{S}$ with the β-deformed average

$$\langle f\rangle_{\beta,\mathcal{S}} := \big\langle f\mathrm{e}^{2\beta U^{\langle n\rangle}}\big\rangle_{\mathcal{S}}$$

which we understand as a deformation of the area element, and correspondingly we may consider the deformed geometric Laplacian

$$\Delta^{\beta,\mathcal{S}} := \mathrm{e}^{-2\beta U^{\langle n\rangle}}\Delta^{\mathcal{S}}.$$

Associated with this β-deformed Laplacian we have a β-deformed heat flow, which we express in terms of the operator $\mathbf{H}_{\beta,t} = \mathrm{e}^{t\Delta^{\beta,\mathcal{S}}}$. Here, to be more precise, $v_t = \mathbf{H}_{\beta,t}v_0$ means that v_t solves the geometrically deformed heat equation

$$\partial_t v_t = \Delta^{\beta,\mathcal{S}} v_t$$

with initial datum v_0 at $t=0$. Let $\tilde V_\beta^{\langle n\rangle} := \mathrm{e}^{-\beta U^{\langle n\rangle}} V_\beta^{\langle n\rangle}$, and observe that

$$\lim_{t\to+\infty}\mathbf{H}_{\beta,t}\tilde V_\beta^{\langle n\rangle} \longrightarrow \big\langle \tilde V_\beta^{\langle n\rangle}\big\rangle_{\beta,\mathcal{S}} = \big\langle \mathrm{e}^{\beta U^{\langle n\rangle}} V_\beta^{\langle n\rangle}\big\rangle_{\mathcal{S}} = 0,$$

which allows us to obtain a β-deformed analogue of the identity (9.1.5):

(9.2.1)
$$\begin{aligned}
\frac{1}{2}\int_0^{+\infty} &\Big\langle \big|\nabla^{\mathcal{S}}\mathbf{H}_{\beta,t}\tilde{V}_\beta^{\langle n\rangle}\big|^2\Big\rangle_{\mathcal{S}}\mathrm{d}t \\
&= -2\int_0^{+\infty}\Big\langle \big(\mathbf{H}_{\beta,t}\tilde{V}_\beta^{\langle n\rangle}\big)\Delta^{\mathcal{S}}\mathbf{H}_{\beta,t}\tilde{V}_\beta^{\langle n\rangle}\Big\rangle_{\mathcal{S}}\mathrm{d}t \\
&= -2\int_0^{+\infty}\Big\langle \big(\mathbf{H}_{\beta,t}\tilde{V}_\beta^{\langle n\rangle}\big)\Delta^{\beta,\mathcal{S}}\mathbf{H}_{\beta,t}\tilde{V}_\beta^{\langle n\rangle}\Big\rangle_{\beta,\mathcal{S}}\mathrm{d}t \\
&= -2\int_0^{+\infty}\Big\langle \big(\mathbf{H}_{\beta,t}\tilde{V}_\beta^{\langle n\rangle}\big)\partial_t\mathbf{H}_{\beta,t}\tilde{V}_\beta^{\langle n\rangle}\Big\rangle_{\beta,\mathcal{S}}\mathrm{d}t \\
&= -\int_0^{+\infty}\Big\langle \partial_t\Big\{\big(\mathbf{H}_{\beta,t}\tilde{V}_\beta^{\langle n\rangle}\big)^2\Big\}\Big\rangle_{\beta,\mathcal{S}}\mathrm{d}t \\
&= \Big\langle \big(\tilde{V}_\beta^{\langle n\rangle}\big)^2\Big\rangle_{\beta,\mathcal{S}} = \Big\langle \big(V_\beta^{\langle n\rangle}\big)^2\Big\rangle_{\mathcal{S}}.
\end{aligned}$$

Again, if for each $t>0$, the Dirichlet energy

(9.2.2)
$$\Big\langle \big|\nabla^{\mathcal{S}}\mathbf{H}_{\beta,t}\tilde{V}_\beta^{\langle n\rangle}\big|^2\Big\rangle_{\mathcal{S}}$$

is minimized for one and the same function $U^{\langle n\rangle}$ of the form (9.1.1), then the same goes for the Bergman norm on the right-hand side. This would suggest the introduction of a β-deformed Fekete problem, which asks for the minimization of (9.2.2) in the limit as $t\to 0^+$ (after renormalization to remove the infinities which arise from the singularities at $w_1,\ldots,w_n$). As a side remark, if we recall that

$$\tilde{V}_\beta^{\langle n\rangle} = \mathrm{e}^{-\beta U^{\langle n\rangle}}V_\beta^{\langle n\rangle} = \frac{b-\mathrm{e}^{-\beta U^{\langle n\rangle}}}{\beta},$$

and note that constants are preserved under weighted heat evolution, it might be reasonable to try to express the weighted heat evolution in the form

$$\mathbf{H}_{\beta,t}\tilde{V}_\beta^{\langle n\rangle}(z) = \frac{b-\mathrm{e}^{-\beta\Theta(z,t)}}{\beta}.$$

The function $\Theta(z,t)$ then evolves according to the nonlinear heat equation

$$\partial_t\Theta = \mathrm{e}^{-2\beta U^{\langle n\rangle}}\Big(\Delta^{\mathcal{S}}\Theta - \frac{\beta}{4}\big|\nabla^{\mathcal{S}}\Theta\big|^2\Big),$$

with initial datum $\Theta(z,0)=U^{\langle n\rangle}(z)$ for $t=0$. The above equation has at least some superficial similarity with the KPZ (Kardar-Parisi-Zhang) equation if we let the configuration $\{w_1,\ldots,w_n\}$ (which determines $U^{\langle n\rangle}$ by (9.1.1)) be stochastic and perhaps time-dependent. For the KPZ equation, see, e.g., the survey paper [12] and the original contribution by Kardar, Parisi, and Zhang [29].

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