Grunsky inequalities and beyond

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Aim

The Grunsky inequalities give detailed information about the distortion behavior of conformal mappings. We first describe how the Grunsky inequalities appear, and then indicate possible directions to extend the scope of the method.

Notation

- Cauchy operator $\mathfrak{C}_{\mathbb{C}}$:

$$\mathfrak{C}_{\mathbb{C}}[f](z) = \int_{\mathbb{C}} \frac{f(w)}{w - z} \, \mathrm{d}A(w).$$

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- Ahlfors-Beurling operator $\mathfrak{B}_{\mathbb{C}}$:

$$\mathfrak{B}_{\mathbb{C}}[f](z) = \mathrm{pv} \int_{\mathbb{C}} \frac{f(w)}{(w - z)^2} \, \mathrm{d}A(w).$$

Ambient space

We discuss properties of conformal mappings $\varphi : \mathbb{D} \to \Omega$, where $\Omega \subset \mathbb{C}$ is open and simply connected $\Omega \neq \mathbb{C}$). The ambient space $\mathbb{C}$ (in which Ω lies) is the source of rigidity. We must use some property of the plane $\mathbb{C}$ which encodes appropriately the needed information.

Fact

The Ahlfors-Beurling operator is a singular integral operator which acts isometrically (unitarily) on $L^2(\mathbb{C})$.

Consequence

Let $\varphi : \mathbb{D} \to \Omega \subset \mathbb{C}$ be a conformal mapping. Then the transferred Ahlfors-Beurling operator

$$\mathfrak{B}_\varphi[f](z) = \mathrm{pv} \int_{\mathbb{D}} \frac{\varphi'(z)\varphi'(w)}{(\varphi(z) - \varphi(w))^2} \, f(w) \, \mathrm{d}A(w), \qquad z \in \mathbb{D},$$

acts contractively on $L^2(\mathbb{D})$.

The Grunsky identity and inequalities

Let e denote the function $e(z) = z$. Also, let $\mathfrak{P}$ denote the orthogonal projection onto the subspace of $L^2(\mathbb{D})$ consisting of analytic functions; likewise, $\bar{\mathfrak{P}}$ is the orthogonal projection onto the subspace of antianalytic functions. Then we have the *Grunsky identity*:

$$(GID) \qquad \mathfrak{B}_\varphi - \mathfrak{B}_e = \mathfrak{P}\mathfrak{B}_\varphi = \mathfrak{B}_\varphi\bar{\mathfrak{P}} = \mathfrak{P}\mathfrak{B}_\varphi\bar{\mathfrak{P}}.$$

The Grunsky inequalities are equivalent to the statement that

$$(GIN) \qquad \big\|(\mathfrak{B}_\varphi - \mathfrak{B}_e)[f]\big\|_{L^2(\mathbb{D})} \le \|f\|_{L^2(\mathbb{D})}.$$

Ahlfors-Beurling transform on L^p

It is known that $\mathfrak{B}_{\mathbb{C}}$ is bounded $L^p(\mathbb{C}) \to L^p(\mathbb{C})$, for all p with $1 < p < +\infty$. Let $K(p)$ be the optimal positive constant such that

$$(B-Lp) \qquad \|\mathfrak{B}_{\mathbb{C}} f\|_{L^p(\mathbb{C})} \le K(p)\,\|f\|_{L^p(\mathbb{C})}, \qquad f \in L^p(\mathbb{C}).$$

The value of $K(p)$ is known only for $p = 2$ $(K(2) = 1)$. A conjecture of Tadeusz Iwaniec claims that $K(p) = p^* - 1$, where $p^* = \max\{p, p'\}$, and $p' = p/(p-1)$ is the dual exponent.

Transferred skewed Ahlfors-Beurling transform

For $0 \le \theta \le 2$, we introduce the *θ-skewed Ahlfors-Beurling transform*, as defined by

$$\mathfrak{B}_\varphi^\theta[f] = \mathrm{pv} \int_{\mathbb{D}} \frac{\varphi'(z)^\theta \varphi'(w)^{2-\theta}}{(\varphi(z)-\varphi(w))^2} f(w)\, \mathrm{d}A(w).$$

It follows from (B-Lp) that

$$\left\| \mathfrak{B}_\varphi^{2/p} f \right\|_{L^p(\mathbb{D})} \le K(p) \, \|f\|_{L^p(\mathbb{D})}, \qquad f \in L^p(\mathbb{D}),$$

for all p with $1 < p < +\infty$. In the symmetric case $\theta = 1$, we note that $\mathfrak{B}_\varphi^1 = \mathfrak{B}_\varphi$.

Skewed Grunsky identity

Let $\mathfrak{D}'$ denote the operator

$$\mathfrak{D}'[f](z) = \int_{\mathbb{D}} \frac{f(w)}{(w-z)(1-\bar{z}w)}\, \mathrm{d}A(w),$$

and $\mathfrak{M}_F$ the operator of multiplication by F. For $0 < \theta < 2$, we have the operator identity

$$(GGID) \qquad \mathfrak{B}_{\varphi}^{\theta} - \mathfrak{B}_e + (\theta - 1)\mathfrak{M}_{1-|z|^2}\mathfrak{M}_{\varphi''/\varphi'}\mathfrak{D}' = \mathfrak{B}_{\varphi}^{\theta}\bar{\mathfrak{P}}.$$

Skewed Grunsky inequality

We obtain

$$\left\|\left(\mathfrak{B}_{\varphi}^{2/p}-\mathfrak{B}_{e}+(2/p-1)\mathfrak{M}_{1-|z|^2}\mathfrak{M}_{\varphi''/\varphi'}\,\mathfrak{D}'\right)[f]\right\|_{L^p(\mathbb{D})}\leq K(p)\,\|f\|_{L^p(\mathbb{D})},$$

for f in the subspace of $L^p(\mathbb{D})$ consisting of the antianalytic functions.

Applications

By choosing a particular f in the skewed Grunsky inequality, we may obtain

$$\int_{\mathbb{D}} \left| \frac{\varphi'(z)^{2/p}\varphi'(w)^{2-2/p}}{(\varphi(z)-\varphi(w))^2} - \frac{1}{(z-w)^2} + \left(\frac{2}{p}-1\right)\frac{\varphi''(z)(1-|z|^2)}{(w-z)(1-\bar{z}w)\varphi'(z)} \right|^p \mathrm{d}A(z) \le K(p)^p\, {}_2F_1\big(p,p;2;|w|^2\big), \qquad w \in \mathbb{D},$$

where $K(p)$ is as in (B-Lp), and ${}_2F_1$ is Gauss' hypergeometric function. The case $p = 2$ is an invariant version of Grönwall's classical area theorem.

Pointwise estimates

We get the pointwise estimate

$$\bigg|\frac{\varphi'(z)^{2/p}\varphi'(w)^{2-2/p}}{(\varphi(z)-\varphi(w))^2}-\frac{1}{(z-w)^2} \\ +\left(\frac{2}{p}-1\right)\frac{1}{(z-w)^2}\log\frac{\varphi'(w)}{\varphi'(z)}\bigg| \\ \le K(p)\left\{{}_2F_1\left(p,p;2;|w|^2\right)\right\}^{1/p}\left\{{}_2F_1\left(p',p';2;|z|^2\right)\right\}^{1/p'},$$

for $z,w\in\mathbb{D}$, where $p'=p/(p-1)$ is the dual exponent.

General scheme

We now turn to a possible generalization to several complex variables. We begin with a singular integral operator

$$\mathfrak{S}[f](z) = \int_{\mathbb{C}^d} S(z,w)\, f(w)\, \mathrm{d}V_d(w),$$

where $\mathrm{d}V_d$ is the volume element in $\mathbb{C}^d$. We assume that $\mathfrak{S}$ acts contractively on $L^2(\mathbb{C}^d)$. We restrict to a domain $\Omega \subset \mathbb{C}^d$ and obtain a contractive operator $\mathfrak{S}_\Omega$ acting on $L^2(\Omega)$. We have a (canonical) domain $\mathcal{D} \subset \mathbb{C}^d$ and a biholomorphic mapping $\varphi : \mathcal{D} \to \Omega$. Let f live on Ω and g on $\mathcal{D}$, and connect them via

$$g = \bar{J}_\varphi\, f \circ \varphi,$$

where J_φ is the associated Jacobian.

General scheme cont

f and g have the same norms, in $L^2(\Omega)$ and $L^2(\mathcal{D})$, respectively. Next, let $\mathfrak{S}_\varphi$ be the operator acting on $L^2(\mathcal{D})$ induced by $\mathfrak{S}_\Omega$:

$$\mathfrak{S}_\varphi[g](z) = J_\varphi\, \mathfrak{S}[f] \circ \varphi;$$

it is then a contraction. We write it out as

$$\mathfrak{S}_\varphi[g](z) = \int_{\mathcal{D}} S(\varphi(z), \varphi(w))\, J_\varphi(z)\, J_\varphi(w)\, g(w)\, \mathrm{d}V_d(w).$$

The statement

$$\|\mathfrak{S}_\varphi[g]\|_{L^2(\mathcal{D})} \le \|g\|_{L^2(\mathcal{D})},$$

especially applied to antiholomorphic g, is a straightforward generalization of the Grunsky inequality.

General scheme: remarks

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- It would be worthwhile to express $\mathfrak{S}_\varphi \bar{\mathfrak{P}}$ even in dimension $d = 1$, for some interesting unitary singular integral operator like

$$\mathfrak{S}[f](z) = \frac{1}{2} \int_{\mathbb{C}} \frac{f(w)}{|w - z|(w - z)} \, \mathrm{d}A(w),$$

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  whose square is the Ahlfors-Beurling operator.
- It would be of interest to see how the scheme would apply to quasiconformal mappings in d real dimensions. Then the Jacobians must be modified, and the subspaces of holomorphic or antiholomorphic functions may be replaced by one, consisting of harmonic functions.