A BEURLING-RUDIN THEOREM FOR $H^\infty$

BY

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0. Introduction

Let $H^\infty$ be the Banach algebra of bounded analytic functions on the open unit disc $D = \{ z \in \mathbb{C} : |z| < 1 \}$, supplied with the uniform norm. It is well known that we may regard $H^\infty$ as a closed subalgebra of $L^\infty$, the uniform algebra of (equivalence classes of) essentially bounded Lebesgue measurable functions on the unit circle $T = \partial D$. Consider the linear space

$$H^\infty + C = \{ f + g : f \in H^\infty, g \in C = C(T) \};$$

Donald Sarason [8, p. 377] has shown that this is a closed subalgebra of $L^\infty$. In connection with $H^\infty + C$, Sarason introduced $QC$, the closed subalgebra of $H^\infty + C$ consisting of all functions whose complex conjugates also lie in $H^\infty + C$, and its analytic subalgebra $QA = QC \cap H^\infty$. Expressed differently, $QC$ is the largest $C^*$ algebra contained in $H^\infty + C$.

In the second section of this paper, we give a complete description of the closed ideals in $QA$. This result is hardly surprising, for Arne Beurling’s (unpublished) and Walter Rudin’s [15] independently obtained description of the closed ideals in the disc algebra $A = C \cap H^\infty$ and several results of Thomas Wolff [20], [21] suggest that this is possible. In fact, Srinivasan and Wang’s proof [16] of Beurling’s and Rudin’s result can be extended to $QA$. We use this result in the third section of this paper to obtain some rather surprising results about closed ideals of $H^\infty$. We will show that an arbitrary closed nonzero ideal in $H^\infty$ has the form $u(J \cap H^\infty)$, where $u$ is an inner function, and $J$ is a closed ideal in $H^\infty + C$; this is what we mean by a Beurling-Rudin theorem for $H^\infty$. Moreover, we shall see that the quotient algebras $H^\infty/J \cap H^\infty$ and $(H^\infty + C)/J$ are canonically isomorphic. And in $H^\infty + C$, as opposed to $H^\infty$, a theorem of Šilov [9, §45] and a later refinement of it due to Errett Bishop and Irving Glicksberg [7, p. 61] give us quite a lot of information about the closed ideals. The first result in this direction was obtained by Håkan Hedenmalm [11], and later generalized by Raymond

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Mortini [14]. For a closed $H^\infty$-ideal $I$ whose hull $h(I)$ is contained in the Šilov boundary $\mathcal{M}(L^\infty)$ (for these concepts, see Section 1 below), we can say a great deal more: there exists a closed $L^\infty$-ideal whose intersection with $H^\infty$ equals $I$, and since a theorem of Šilov [9, §36] tells us that all closed ideals in $L^\infty = C(\mathcal{M}(L^\infty))$ are intersections of maximal ones, the same can be said about $I$. We will also show that

$$H^\infty / I \cong \hat{h}^\infty |_{h(I)} = C( h(I) ).$$

Most of the results that appear in this paper extend easily to $H^\infty(\Omega)$ on finitely connected domains with the techniques of [11].

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1. Basic concepts

The bilinear form linking any Banach space $A$ with its dual space $A^*$ will always be denoted by $\langle \cdot, \cdot \rangle$.

All Banach algebras are assumed complex, commutative, and unital. For a Banach algebra $B$, we denote by $\mathcal{M}(B)$ its maximal ideal space; the elements of $\mathcal{M}(B)$ are the nonzero complex homomorphisms on $B$. With the Gelfand topology, $\mathcal{M}(B)$ is a compact Hausdorff space. The Gelfand transform, always denoted by $\hat{\cdot}$, defines a continuous homomorphism $B \to C(\mathcal{M}(B))$. The algebra $B$ is said to be semisimple if the Gelfand transform is injective. A uniform algebra on a compact Hausdorff space $X$ is a closed unital subalgebra of $C(X)$. A uniform algebra is a semisimple Banach algebra whose image under the Gelfand transform is a uniform algebra on the maximal ideal space. A $C^*$ algebra is a uniform algebra which is closed under complex conjugation. The Stone-Weierstrass theorem allows us to conclude that a $C^*$ algebra $B$ is isomorphic to $C(\mathcal{M}(B))$. Let $B$ be a uniform algebra on a compact Hausdorff space $X$. A closed subset $E$ of $X$ is a peak set if there is a function $p \in B$ such that $p|_E = 1$ and $|p| < 1$ on $X \setminus E$; we call $p$ a peaking function. A weak peak set is an intersection of peak sets. An interpolation set $E$ is a closed subset of $X$ such that $B|_E = C(E)$. Unless $X$ is specified, $X$ is tacitly assumed to be $\mathcal{M}(B)$.

For an ideal $I$ in a Banach algebra $B$, its hull is the closed set

$$h(I, B) = \{ m \in \mathcal{M}(B) : m(x) = 0 \text{ for all } x \in I \}.$$ 

If $I$ is closed, it is well known [7, p. 12] that we may identify $\mathcal{M}(B/I)$ with $h(I, B)$. Whenever possible, we will write $h(I)$ for $h(I, B)$. 
A corollary of Beurling’s famous invariant subspace theorem [8, p. 85] states that every weak * closed ideal in $H^\infty$ other than $\{0\}$ has the form $uH^\infty$, where $u$ is an inner function. The weak * topology on $H^\infty$ is the one inherited from $L^\infty = (L^1(T))^*$. A weak * closed subspace of $H^\infty$ is an ideal if (and only if) it is invariant under multiplication by the coordinate function $z$; hence the weak * closure in $H^\infty$ of an ideal in $H^\infty$ or QA is an $H^\infty$-ideal. So, given an ideal $I$ in $H^\infty$ or QA other than $\{0\}$, we find an inner function $u$ such that the weak * closure of $I$ equals $uH^\infty$. We shall call the function $u$, which is unique except for a (unimodular) constant factor, the inner factor of $I$. It could also be described as the greatest common divisor of the inner factors of the functions in $I$ [8, pp. 83–84].

Using the fact that $C \subset QC \subset H^\infty + C$, one can show that $QC = QA + C$. It follows from this that $\mathcal{M}(QA) = \mathcal{M}(QC) \cup D$ [21]. It is standard to identify $\mathcal{M}(H^\infty + C)$ with $\mathcal{M}(H^\infty) \setminus D$ and $\mathcal{M}(L^\infty)$ with the Šilov boundary of $\mathcal{M}(H^\infty)$. Let

$$
\Gamma: \mathcal{M}(L^\infty) \to \mathcal{M}(QC) \quad \text{and} \quad \gamma: \mathcal{M}(H^\infty + C) \to \mathcal{M}(QC)
$$

be the respective restriction mappings. Because $QC$ is a $C^*$ algebra, a theorem of Šilov [9, §12] tells us that $\Gamma$ and $\gamma$ are surjective. Let $D_m = \gamma^{-1}(\{m\})$ and $E_m = \Gamma^{-1}(\{m\})$ denote the $QC$ level sets corresponding to the point $m \in \mathcal{M}(QC)$. The restriction of $H^\infty$ to $E_m$ (or $D_m$) is a uniform algebra with maximal ideal space $D_m$ and Šilov boundary $E_m$.

2. Closed ideals in QA

Let $\phi$ be an element of the dual space of $QC = C(\mathcal{M}(QC))$. By the Riesz Representation Theorem, $\phi$ has the form

$$
\langle f, \phi \rangle = \int_{\mathcal{M}(QC)} \hat{f}d\mu, \quad f \in QC,
$$

where $\mu$ is a regular Borel measure on $\mathcal{M}(QC)$. In particular, the continuous linear functional $\phi_0$ defined by

$$
\langle f, \phi_0 \rangle = \int_T f d\lambda, \quad f \in QC,
$$

where $\lambda$ is normalized Lebesgue measure on $T$, defines a measure $\sigma$ on $\mathcal{M}(QC)$ such that

$$
\int_{\mathcal{M}(QC)} \hat{f}d\sigma = \int_T f d\lambda, \quad f \in QC,
$$
which we will call lifted Lebesgue measure. Lemmas 2.1 and 2.2 below are due to Thomas Wolff [20].

**Lemma 2.1.** The Gelfand transform on QC has a unique extension to an isometric isomorphism $L^1(T, \lambda) \to L^1(\mathcal{M}(QC), \sigma)$ (still denoted by ) such that

$$\int_T fg \, d\lambda = \int_{\mathcal{M}(QC)} \hat{f} \hat{g} \, d\sigma, \quad f \in L^1(T, \lambda), \ g \in QC.$$

The following is the QA analogue of the classical F & M Riesz theorem.

**Lemma 2.2.** If $\mu \in QC^*$ annihilates QA, then $\mu$ is absolutely continuous with respect to $\sigma$, that is, $d\mu = f d\sigma$ for some $f \in L^1(\mathcal{M}(QC), \sigma)$.

The easiest way to see that the above statement is true is to realize that the quotient spaces $C/A$ and $QC/QA = (QA + C)/QA$ are canonically isomorphic, and therefore must have the same dual space $H^1_Q = \{ f \in H^1 : f(0) = 0 \} \subset L^1(T, \lambda)$.

**Lemma 2.3.** Let $f \in QA$ have the factorization $f = ug$, where $u$ is an inner function and $g \in H^\infty$. Then $g \in QA$, and if $m(f) = 0$ for some $m \in \mathcal{M}(QC)$, $m(g) = 0$.

**Proof.** Since we are assuming $g \in H^\infty$, and $\bar{g} = uf \in H^\infty + C$ because $\bar{f} \in QC$, we see that $g \in QA$. To prove the remaining part of the assertion, find $\phi \in \mathcal{M}(L^\infty)$ such that $\phi|_{QC} = m$; this is possible because the restriction mapping $\Gamma: \mathcal{M}(L^\infty) \to \mathcal{M}(QC)$ is onto. Since $u$ is inner, $|\phi(u)| = 1$, and hence $|\phi(g)| = |\phi(u)\phi(g)| = |\phi(ug)| = |\phi(f)| = |m(f)| = 0$. Since $g \in QA$, $m(g) = 0$.

**Remark 2.4.** Lemma 2.3 states that QA has the $f$-property in the sense of Havin, answering one of the two questions raised by Milne Anderson in [3]. The fact that QA has this property has been shown previously by Pamela Gorkin [10].

In [15] Walter Rudin described all closed ideals in the disc algebra. Srinivasan and Wang [16] later gave a proof that relied less on function theory. It is this proof that we shall modify to prove a similar result for QA. For a closed set $E \subset \mathcal{M}(QC)$, introduce the notation

$$I(E, QA) = \{ f \in QA : \hat{f}|_E = 0 \}.$$

Clearly, $I(E, QA)$ is a closed ideal in QA. Thomas Wolff [20] has shown that a closed subset of $\mathcal{M}(QC)$ is contained in the zero set of a nonidentically vanishing QA function if and only if it has lifted Lebesgue measure 0. Hence
If \( I(E, QA) \neq \{0\} \) if and only if \( \sigma(E) = 0 \). Assume for the moment that \( \sigma(E) = 0 \). Clearly, the inner factor of \( I(E, QA) \) must be 1, in view of Lemma 2.3. If \( u \) is an inner function such that the set of all \( m \in \mathcal{M}(QC) \) for which \( u|_E \) is nonconstant is contained in \( E \), \( uI(E, QA) \) will be contained in \( QA \), because the functions in \( uI(E, QA) \) are constant on all \( QC \) level sets; here we use a theorem of Šilov [9, §44]. Because \( u \) is inner, this is a closed ideal in \( QA \). Our theorem below states that all (nontrivial) closed ideals in \( QA \) arise in this fashion.

**THEOREM 2.5.** Let \( I \) be a closed \( QA \)-ideal other than \( \{0\} \). Then there exist an inner function \( u \) and a closed subset \( E \) of \( \mathcal{M}(QC) \) with \( \sigma(E) = 0 \) such that \( I = uI(E, QA) \).

**Proof.** Let \( u \) be the inner factor of \( I \) (see Section 1), and let

\[
J = \{ f \in QA : uf \in I \}.
\]

First observe that because \( I \) is a closed \( QA \)-ideal, \( J \) is, too. In view of Lemma 2.3, we are done if we can show that \( J = I(E, QA) \), where \( E = \{ m \in \mathcal{M}(QC) : m(f) = 0 \text{ for all } f \in I \} \). Clearly, \( J \) is contained in \( I(E, QA) \), again by Lemma 2.3. To show the reverse inclusion, let \( \mu \) be a regular Borel measure on \( \mathcal{M}(QC) \) annihilating \( J \). Since \( J \) is an ideal, for any \( f \in J \) and \( g \in QA \),

\[
\int_{\mathcal{M}(QC)} \hat{f} \hat{g} d\mu = 0.
\]

Thus \( \hat{f} d\mu \) annihilates \( QA \). By Lemma 2.2, \( \hat{f} d\mu \) is absolutely continuous with respect to \( \sigma \) and \( \mu_s \) where \( \mu_a \) is absolutely continuous with respect to \( \sigma \) and \( \mu_s \) is singular, we see that \( \hat{f} d\mu_s = 0 \). Thus \( \text{supp} \mu_s \subset \{ m \in \mathcal{M}(QC) : m(f) = 0 \text{ for all } f \in I \} \) for any \( f \in J \) and hence \( \text{supp} \mu_s \subset E \). Now it is clear that \( \mu_s \) annihilates \( I(E, QA) \). Since \( J \) is contained in \( I(E, QA) \), both \( \mu_s \) and \( \mu \) annihilate \( J \), so \( \mu_s \) does, too. Since the inner factors of \( J \) and \( I(E, QA) \) must be 1, we see that \( J \) and \( I(E, QA) \) are both weak * dense in \( H^\infty \). Recall that the weak * topology of \( H^\infty \) is the weak * topology of \( L^\infty = (L^1(T))^* \) restricted to \( H^\infty \). Since \( \mu_a \) is absolutely continuous with respect to \( \sigma \), Lemma 2.1 tells us that \( \mu_a \) belongs to the predual of \( L^\infty \), and hence the fact that \( \mu_a \perp J \) implies \( \mu_a \perp I(E, QA) \). Now both \( \mu_a \) and \( \mu_s \) annihilate \( I(E, QA) \), so \( \mu \) does, too. Thus \( J \perp \subset I(E, QA) \perp \), so \( I(E, QA) \subset J \), as desired.

In this context, the \( QA \) analogue of the Rudin-Carleson theorem, which is due to Thomas Wolff [20], [21], takes the following form. An epimorphism is a surjective homomorphism.

**THEOREM 2.6.** Let \( E \subset \mathcal{M}(QC) \) be a closed set of lifted Lebesgue measure 0. Then there exists a unique continuous epimorphism

\[
QC \rightarrow QA/I(E, QA)
\]
which is canonical on QA; its kernel is the closed QC-ideal
\[ I(E, QC) = \{ f \in QC : f|_E = 0 \}. \]

**Proof.** Wolff's theorem [20], [21] states that \( QA|_E = C(E) \). Because QA is a logmodular algebra on its Šilov boundary \( \mathcal{M}(QC) \), \( E \) is a weak QA peak set [17, p. 216]. Hence we may deduce that
\[ h(I(E, QA), QA) = E. \]

It is standard to identify \( QA/I(E, QA) \) with \( QA|_E \) [17, p. 117]. So, by letting \( L \) be the restriction mapping
\[ QC \to QC|_E = C(E) \cong QA/I(E, QA), \]
we obtain a continuous epimorphism that is canonical on QA, and clearly, its kernel is \( I(E, QC) \). Observe that as a consequence, \( QC = QA + I(E, QC) \).

It remains to be shown that \( L \) is unique. For the algebra \( QC = C(\mathcal{M}(QC)) \), a theorem of Šilov [9, §36] specializes to show that the closure of the QC-ideal generated by \( I(E, QA) \) equals \( I(E, QC) \), because \( h(I(E, QA), QA) = E. \)

Hence any other continuous epimorphism \( L' : QC \to QA/I(E, QA) \) which is canonical on QA must vanish on \( I(E, QC) \), but since \( QC = QA + I(E, QC) \), \( L' \) must coincide with \( L \), which is our desired conclusion.

These are some of the many ways in which QA acts like the disc algebra. But there are some differences. Rudin [15] showed that every closed ideal in the disc algebra is the closure of a principal ideal. This is not true in QA. Take for example the maximal ideal \( \{ f \in QA : m(f) = 0 \} \) for \( m \in \mathcal{M}(QC) \); indeed, using the corona theorem for QA and Sundberg's and Wolff's description of QA interpolating sequences [19], it is not hard to check that any QA function vanishing at \( m \) must vanish on a set homeomorphic to \( \beta \mathbb{N} \setminus \mathbb{N} \), where \( \beta \mathbb{N} \) denotes the Stone-Čech compactification of the nonnegative integers \( \mathbb{N} \). However, for countably generated closed ideals, we have a result analogous to that for the disc algebra [6, p. 73]. Namely, every countably generated closed ideal in QA is the principal ideal generated by a finite Blaschke product. This follows from a result of Dietrich [6, p. 72], after the observation that \( \mathcal{M}(QC) \) is connected.

### 3. Closed ideals in \( H^\infty \)

A Douglas algebra is a closed subalgebra of \( L^\infty \) containing \( H^\infty \). For certain Douglas algebras \( B \) and closed \( H^\infty \)-ideals \( I \) we construct a continuous epimorphism (i.e., surjective homomorphism) \( L_{I, B} : B \to H^\infty /I \) which is canonical on \( H^\infty \). Whenever possible we write \( L_I \) for \( L_{I, B} \). This epimorphism will enable us to study closed ideals in \( H^\infty \) by replacing them with closed ideals in \( B \).

The following lemma, which is probably known, will prove useful.
**Lemma 3.1.** For a uniform algebra $B$ on a compact Hausdorff space $X$, let $E$ be a peak set with peaking function $p$. Then the closure of the $B$-ideal generated by $1 - p$ equals \( \{ f \in B : f|_E = 0 \} \).

**Proof.** Clearly, \( \{ f \in B : f|_E = 0 \} \) is a closed ideal in $B$, and it contains the function $1 - p$. Let $f \in B$ vanish on $E$. It is easy to check that

\[
(1 - p^n)f \to f \quad \text{as } n \to \infty,
\]

and since $1 - p^n = (1 - p)(1 + p + \cdots + p^{n-1})$, the assertion follows.

The proposition below is what makes everything in this section work. It uses Wolff's remarkable result [21] that every $L^\infty$ function can be multiplied into $QC$ with an outer $QA$ function.

For a closed set $E \subset \mathcal{M}(QC)$, $I(E, H^\infty)$ denotes the closed ideal

\[
\{ f \in H^\infty : f|_{\gamma^{-1}(E)} = 0 \}.
\]

**Proposition 3.2.** Let $I \neq \{0\}$ be a closed ideal in $H^\infty$ with inner factor 1. Then $I$ contains $I(E, H^\infty)$ for some QA peak set $E \subset \mathcal{M}(QC)$ (which has lifted Lebesgue measure 0, of course).

**Proof.** Thomas Wolff [21] has shown that to a given function $f \in L^\infty$, we can find an outer function $q \in QA$ such that $qf \in QC$. In particular, this result can be applied to the functions in $I$, showing that $I \cap QA \neq \{0\}$. Since $q$ was outer, we can say even more, namely that the closed $QA$-ideal $I \cap QA$ has inner factor 1. By our characterization of the closed ideals in $QA$ (Theorem 2.5), we find a closed set $E_0 \subset \mathcal{M}(QC)$ with $\sigma(E_0) = 0$ such that $I \cap QA = I(E_0, QA)$. We already noticed in the proof of Theorem 2.6 that $E_0$ is a weak $QA$ peak set, meaning that it is an intersection of peak sets. So, we can find a $QA$ peak set $E \subset \mathcal{M}(QC)$ containing $E_0$, and such sets have $\sigma(E) = 0$ according to Wolff [20]. Let $p$ be a $QA$ function that peaks on $E$. Then $p$ is also a peaking function for the set

\[
\gamma^{-1}(E) \subset \mathcal{M}(H^\infty + C) = \mathcal{M}(H^\infty) \setminus D
\]

in the algebra $H^\infty$, and Lemma 3.1 tells us that the closure of the $H^\infty$-ideal generated by $1 - p$ coincides with $I(E, H^\infty)$. Since

\[
1 - p \in I(E, QA) \subset I(E_0, QA) = I \cap QA,
\]

the assertion follows.

**Theorem 3.3.** Let $I \neq \{0\}$ be a closed ideal in $H^\infty$ with inner factor 1. Then there exists a unique continuous epimorphism $H^\infty + C \to H^\infty/I$ which is canonical when restricted to $H^\infty$. 
Proof. Let us do the uniqueness part first. Observe that any such epimorphism \( L: H^\infty + C \to H^\infty / I \) must have \( L(z^{-n}f) = (z + I)^{-n}(f + I) \) for \( f \in H^\infty \) and \( n \geq 0 \), and since such functions span a dense subspace of \( H^\infty + C \), continuity shows that \( L \) must be unique whenever it exists.

By Proposition 3.2, we can find a QA peak set \( E \) in \( \mathcal{M}(QC) \) such that \( I \supseteq I(E, H^\infty) \). In the proof of Proposition 3.2, we mentioned that \( y^{-1}(E) \) is a peak set for \( H^\infty \), so by [17, p. 117], \( H^\infty |_{y^{-1}(E)} \), which is isomorphic to \( \mathcal{A}(y^{-1}(E)) \), is a closed subalgebra of \( C(y^{-1}(E)) \). Our epimorphism will be the composition of the following maps:

\[
H^\infty + C \to H^\infty + C |_{y^{-1}(E)} = H^\infty |_{y^{-1}(E)} \cong H^\infty / I(E, H^\infty) \to H^\infty / I.
\]

The first map is the restriction of the Gelfand transform. The equality sign holds because of Wolff's interpolation theorem \( QA \leq C(E) \) [20], [21]. The last map is well defined because \( I \supseteq I(E, H^\infty) \).

We now arrive at our main result.

**Theorem 3.4.** Let \( I \neq \{0\} \) be a closed ideal in \( H^\infty \) with inner factor \( u \). Then \( I \subseteq I(E, H^\infty) \), where \( E \) is a closed ideal in \( H^\infty + C \). Also, the quotient algebras \( H^\infty / J \cap H^\infty \) and \( (H^\infty + C) / J \) are canonically isomorphic.

Proof. Clearly, \( I_0 = \{ f \in H^\infty : uf \in I \} \) is a closed \( H^\infty \)-ideal because \( I \) is, and \( I = uI_0 \). By construction, \( I_0 \) has inner factor 1. Theorem 3.3 gives us a continuous epimorphism

\[
L_{I_0}: H^\infty + C \to H^\infty / I_0
\]

which is canonical on \( H^\infty \), so putting \( J = \ker L_{I_0} \) we obtain a closed \((H^\infty + C)\)-ideal whose intersection with \( H^\infty \) is \( I_0 \). The map \( L_{I_0} \) induces a topological isomorphism \((H^\infty + C) / J \to H^\infty / I_0 \), which is the inverse of the canonical homomorphism \( H^\infty / I_0 \to (H^\infty + C) / J \) because \( L_{I_0} \) is canonical on \( H^\infty \).

An antisymmetric set for \( H^\infty + C \) is a set \( S \subset \mathcal{M}(L^\infty) \) such that whenever \( f \in H^\infty + C \) and \( f|_S \) is real valued, then \( f|_S \) is constant. Bishop's antisymmetric decomposition theorem for ideals, which is due to Glicksberg, tells us that Theorem 3.4 has the following corollary [7, p. 61], [17, p. 115].

**Corollary 3.5.** Let \( I \neq \{0\} \) be a closed \( H^\infty \)-ideal with inner factor 1. Then an \( H^\infty \) function \( f \) is an element of \( I \) if (and only if) \( f|_S \in I|_S \) for all maximal antisymmetric sets \( S \) for \( H^\infty + C \).

Since \( QC \) level sets are unions of maximal antisymmetric sets for \( H^\infty + C \), Corollary 3.5 has the following consequence. Corollary 3.6 may also be
deduced from Theorem 3.4 by applying Šilov's decomposition theorem for ideals [9, §45].

**Corollary 3.6.** Let \( I \neq \{0\} \) be a closed \( H^\infty \)-ideal with inner factor 1. Then an \( H^\infty \) function \( f \) is an element of \( I \) if (and only if) \( f|_{E_m} \in I|_{E_m} \) for all QC level sets \( E_m \).

**Corollary 3.7.** If \( I \neq \{0\} \) is a closed ideal in \( H^\infty \) with inner factor 1, then \( I \cap QA = I(E, QA) \), where \( E = \gamma(h(I, H^\infty)) \).

**Proof.** By Proposition 3.2, the inner factor of \( I \cap QA \) is 1, and by our description of the closed ideals in \( QA \), \( I \cap QA = I(E, QA) \) for some closed set \( E \subset \mathcal{M}(QC) \) of lifted Lebesgue measure 0, which clearly must contain \( \gamma(h(I, H^\infty)) \). On the other hand, if \( f \in QA \) vanishes on \( \gamma(h(I, H^\infty)) \), then \( f|_{E_m} \in I|_{E_m} \) trivially for all QC level sets \( E_m \), since the maximal ideal space of \( H^\infty|_{E_m} \) is \( D_m = \gamma^{-1}(\{m\}) \). Corollary 3.6 now tells us that \( f \in I \). That is, \( I \cap QA = I(\gamma(h(I, H^\infty)), QA) \), as asserted.

One may wonder whether the ideal \( J \) in the formulation of Theorem 3.4 is uniquely determined by \( I \). This turns out to be the case, indeed, \( J \) is the closure of the \((H^\infty + C)\)-ideal generated by \( I \cap H^\infty \). Here is our precise statement.

**Theorem 3.8.** The mapping \( J \mapsto J \cap H^\infty \) is one-to-one from the set of all closed \((H^\infty + C)\)-ideals with \( \sigma(\gamma(h(J, H^\infty + C))) = 0 \) onto the set of all closed \( H^\infty \)-ideals with inner factor 1. Also, if \( \sigma(\gamma(h(J, H^\infty + C))) > 0 \), \( J \cap H^\infty = \{0\} \).

**Proof.** If \( \sigma(\gamma(h(J, H^\infty + C))) > 0 \), \( J \) cannot contain any nonidentically vanishing \( QA \) function [20], and hence \( J \cap H^\infty \) must equal \( \{0\} \), by Wolff's generalized Fatou theorem [21].

Let \( J \) be a closed \((H^\infty + C)\)-ideal such that the lifted Lebesgue measure of \( E = \gamma(h(J, H^\infty + C)) \) is zero. We shall show that \( J \) contains the ideal

\[
I(E, H^\infty + C) = \{ f \in H^\infty + C : f|_{\gamma^{-1}(E)} = 0 \}.
\]

To this end, observe that if \( m \in \mathcal{M}(QC) \setminus E \), then \( J|_{D_m} = J|_{\gamma^{-1}(\{m\})} \) is not contained in any maximal ideal of \( \mathcal{M}(H^\infty + C|_{D_m}) = D_m \), so \( J|_{D_m} = H^\infty + C|_{D_m} \). An application of Šilov's decomposition theorem for ideals [9, §45] yields \( J \supset I(E, H^\infty + C) \), as desired. Using this we have \( J \supset I(E, QA) \). Since we already know that \( I(E, QA) \) has inner factor 1, we see that \( J \cap H^\infty \) is a closed \( H^\infty \)-ideal with inner factor 1. If \( I \neq \{0\} \) is a closed ideal in \( H^\infty \) with inner factor 1, taking the kernel of the epimorphism of Theorem 3.3 provides us with a closed \((H^\infty + C)\)-ideal \( J \) whose intersection with \( H^\infty \) equals \( I \). By the above remark, \( \sigma(\gamma(h(J, H^\infty + C))) = 0 \). Hence the mapping \( J \mapsto J \cap H^\infty \)
is onto, as asserted. What remains to be shown is that it is one-to-one. To this end, let $I \neq \{0\}$ be an arbitrary closed ideal in $H^\infty$ with inner factor 1, and let $J$ and $J'$ be two closed ideals in $H^\infty + C$ such that $J \cap H = J' \cap H = I$. We wish to show that $J = J'$. According to what we have done so far, the lifted Lebesgue measure of the sets

$$E = \gamma\left(h(J, H^\infty + C)\right)$$

and

$$E' = \gamma\left(h(J', H^\infty + C)\right)$$

must be zero. From our work above we know that $J \supset I(E, H^\infty + C)$ and $J' \supset I(E', H^\infty + C)$. By Wolff's interpolation result $QA|_E = C(E)$ [20], [21], we may conclude that $H^\infty + I(E, H^\infty + C) = H^\infty + C$; just take an arbitrary function $f = g + h \in H^\infty + C$, find a $q \in QA$ with $q|_E = h|_E$, and observe that

$$f = (g + q) + (h - q) \in H^\infty + I(E, H^\infty + C).$$

Hence $H^\infty + J = H^\infty + C$, and in the same fashion, $H^\infty + J' = H^\infty + C$. Elementary algebra now tells us that

$$H^\infty / I = H^\infty / J \cap H^\infty \cong (H^\infty + J) / J = (H^\infty + C) / J$$

and

$$H^\infty / I = H^\infty / J' \cap H^\infty \cong (H^\infty + J') / J' = (H^\infty + C) / J'$$

algebraically, and hence topologically, by the open mapping theorem. These isomorphisms induce two continuous epimorphisms $H^\infty + C \to H^\infty / I$ that are canonical on $H^\infty$, with kernels $J$ and $J'$, respectively. Theorem 3.3 tells us that these two epimorphisms must be identical, and hence $J = J'$. The proof of the theorem is complete.

Whenever we can get a continuous epimorphism $L_I: L^\infty \to H^\infty / I$ which is canonical on $H^\infty$, we can say much more about the closed $H^\infty$-ideal $I$. If such a map exists, then $L^\infty / \ker L_I \cong H^\infty / I$. Thus we may identify the maximal ideal spaces of these algebras. But $\mathcal{M}(L^\infty / \ker L_I)$ may be identified with $h(\ker L_I, L^\infty)$, and $\mathcal{M}(H^\infty / I)$ may be identified with $h(I, H^\infty)$ [7, p. 12]. From this it follows that a necessary condition for the existence of such a map is that $h(I, H^\infty)$ be contained in $\mathcal{M}(L^\infty)$. Surprisingly enough, it is also sufficient. We obtain this result as a corollary of the following theorem, the proof of which is based on Sheldon Axler's neat result on factorization of $L^\infty$ functions [4], and its proof, as in [18].

**THEOREM 3.9.** Let $I$ be an ideal in $H^\infty + C$ such that

$$h(I, H^\infty + C) \subset \mathcal{M}(L^\infty).$$
Then there exists a unique epimorphism $\mathcal{L}_I: L^\infty \to (H^\infty + C)/I$ which is canonical on $H^\infty + C$. If $I$ is closed, $\mathcal{L}_I$ is continuous.

Proof. By Axler’s theorem, for each $f \in L^\infty$, there exists a Blaschke product $b$ such that $bf \in H^\infty + C$. We define $\mathcal{L}_I$ as follows:

$$\mathcal{L}_I(f) = (b + I)^{-1}(bf + I).$$

Note that $(b + I)^{-1}$ exists since $h(I, H^\infty + C) \subset \mathcal{M}(L^\infty)$ and $|\hat{b}| = 1$ on $\mathcal{M}(L^\infty)$. We will show that the choice of the Blaschke product $b$ does not affect the definition of $\mathcal{L}_I$. Suppose that we have $f \in L^\infty$ and Blaschke products $b$ and $c$ such that $bf$ and $cf$ are in $H^\infty + C$. Then

$$(b + I)^{-1}(bf + I) = (b + I)^{-1}(c + I)^{-1}(bcf + I)$$

$$= (b + I)^{-1}(c + I)^{-1}(b + I)(cf + I)$$

$$= (c + I)^{-1}(cf + I).$$

Thus $\mathcal{L}_I$ is well defined, and in the same way one checks that it is a homomorphism. That $\mathcal{L}_I$ is canonical on $H^\infty + C$ is obvious.

For the uniqueness, let $L: L^\infty \to (H^\infty + C)/I$ be an arbitrary epimorphism that is canonical on $H^\infty + C$. Then for $f \in L^\infty$ and a Blaschke product $b$ with $bf \in H^\infty + C$,

$$(b + I)L(f) = L(b)L(f) = L(bf) = bf + I,$$

so $L = \mathcal{L}_I$.

From now on, we assume $I$ is closed. The kernel of $\mathcal{L}_I$ is the ideal

$$J = \{ f \in L^\infty : bf \in I \text{ for some Blaschke product } b \}.$$

From the proof of Axler’s factorization theorem (see [18]) it follows that $J$ is closed. The map $\mathcal{L}_I$ induces an algebraic isomorphism

$$L^\infty/J \to (H^\infty + C)/I$$

which is the inverse of the canonical homomorphism. Since the canonical homomorphism $(H^\infty + C)/I \to L^\infty/J$ is continuous, the open mapping theorem states that this isomorphism must be topological, and hence that $\mathcal{L}_I$ is continuous. The proof of the theorem is complete.

Corollary 3.10. Let $I$ be a closed ideal in $H^\infty$ such that $h(I, H^\infty) \subset \mathcal{M}(L^\infty)$. Then there exists a unique continuous epimorphism $L^\infty \to H^\infty/I$ which is canonical on $H^\infty$. 
Proof. An inner function has modulus 1 on the Šilov boundary $\mathcal{M}(L^\infty)$; therefore if it is not invertible in $H^\infty$, then it must vanish somewhere else in $\mathcal{M}(H^\infty) \setminus \mathcal{M}(L^\infty)$. Consequently, the ideal $I$ must have inner factor 1. Theorem 3.3 provides us with a continuous epimorphism

$$L_I: H^\infty + C \to H^\infty / I$$

which is canonical on $H^\infty$, and if we let $J$ denote its kernel, we obtain a closed ideal in $H^\infty + C$ with

$$h(J, H^\infty + C) = h(I, H^\infty) \subset \mathcal{M}(L^\infty).$$

Defining $L_{I, L^\infty} = \tilde{L}_I \circ \mathcal{L}_I$, where $\tilde{L}_I$ is the isomorphism $(H^\infty + C)/J \to H^\infty / I$ induced by $L_I$ and $\mathcal{L}_I$ is as in Theorem 3.9, we get the desired epimorphism. For the uniqueness, observe that any such epimorphism must coincide with $L_{I, L^\infty}$ on quotients of inner functions, and since such functions span a dense subspace of $L^\infty$ by the Douglas-Rudin theorem [8, pp. 192–195], it must by continuity coincide with $L_{I, L^\infty}$ everywhere. The proof is complete.

Corollary 3.11. Let $I$ be a closed ideal in $H^\infty$ with $h(I, H^\infty) \subset \mathcal{M}(L^\infty)$. Then $I$ is an intersection of maximal ideals, and $h(I, H^\infty)$ is a weak peak interpolation set for $H^\infty$.

Proof. Corollary 3.10 gives us an epimorphism $L_{I, L^\infty}: L^\infty \to H^\infty / I$, and since it is canonical on $H^\infty$, the intersection of its kernel $J = \ker L_{I, L^\infty}$ and $H^\infty$ equals $I$. Also, since $H^\infty / I$ and $L^\infty / J$ are canonically isomorphic, they have the same maximal ideal spaces $h(J, L^\infty) = h(I, H^\infty)$. By a theorem of Šilov [9, §36], $J$ is an intersection of maximal ideals, and hence the same can be said about $J \cap H^\infty = I$. Thus the fact that $H^\infty / I$ and $L^\infty / J$ are canonically isomorphic can be restated as $H^\infty |_{h(I, H^\infty)} = L^\infty |_{h(I, H^\infty)}$. Now $L^\infty |_{h(I, H^\infty)} = C(h(I, H^\infty))$, making $h(I, H^\infty)$ an $H^\infty$ interpolation set, and since $H^\infty$ is logmodular on its Šilov boundary $\mathcal{M}(L^\infty)$, it follows that $h(I, H^\infty)$ is also a weak peak set [17, p. 216].

At this point, it is certainly reasonable to conjecture that if $I \neq \{0\}$ is a closed ideal in $H^\infty$ with inner factor 1, such that $h(I, H^\infty) \subset \mathcal{M}(B)$ for a Douglas algebra $B$, then there exists a continuous epimorphism $L_{I, B}: B \to H^\infty / I$ which is canonical on $H^\infty$. We shall give an example to show that this is not true in general. Before we do so, we introduce some new terminology and notation.

A sequence $\{z_n\}$ of points of $D$ is said to be thin if

$$\lim_{n \to \infty} \prod_{k \neq n} |(z_n - z_k)/(1 - \bar{z}_k z_n)| = 1.$$
A Blaschke product associated to a thin sequence is called a thin Blaschke product. Any point \( m \in \mathcal{M}(H^\infty) \setminus \mathbb{D} \) which is in the closure of a thin sequence is called a thin point. We let \( \mathcal{F} \) denote the collection of all thin points in \( \mathcal{M}(H^\infty) \). It is well known that \( \mathcal{F} \) is a union of nontrivial Gleason parts (see for instance [12]). In [13], Kenneth Hoffman introduced for every \( m \in \mathcal{M}(H^\infty) \) an analytic mapping \( L_m: \mathbb{D} \to \mathcal{M}(H^\infty) \) varying continuously with \( m \), the image of which is the Gleason part \( \mathcal{P}(m) \) containing \( m \). Hoffman showed among other things if \( m \in \mathcal{F} \), then \( L_m \) is a homeomorphism. In fact, if \( b \) is a thin Blaschke product whose zero sequence captures \( m \) in its closure, then \( \hat{b} \circ L_m(z) = \lambda z, z \in \mathbb{D} \), for some unimodular constant \( \lambda \), which we can take to be 1 by a change of \( b \). It is now also clear that \( \hat{\hat{b}} \circ L_m = H^\infty \), because if \( f \in H^\infty \), then \( f \circ \hat{b} \in H^\infty \) is a function such that \( f \circ \hat{b} \circ L_m = f \). We are now ready to give our example of a closed ideal \( I \) in \( H^\infty \) with inner factor 1 and a Douglas algebra \( B \) such that \( h(I, H^\infty) \subset \mathcal{M}(B) \), but no continuous homomorphism of \( B \) onto \( H^\infty/I \) that is canonical on \( H^\infty \) exists. Let \( k \) denote the singular inner function

\[
k(z) = \exp((z + 1)/(z - 1)), \quad z \in \mathbb{D}.
\]

**Example 3.12.** The ideal \( kH^\infty \) is closed in \( H^\infty \). Define the ideal \( I \) by

\[
I = \{ f \in H^\infty : f \circ L_m \in kH^\infty \}
\]

for some \( m \in \mathcal{F} \); then \( I \) is a closed ideal in \( H^\infty \) with inner factor 1. Since \( \hat{\hat{h}} \circ L_m = H^\infty, \hat{f} \circ L_m = kH^\infty \). Let \( B \) be the smallest Douglas algebra containing the complex conjugates of all thin Blaschke products. In [12] it is shown that \( \mathcal{M}(B) = \mathcal{M}(H^\infty) \setminus (\mathcal{F} \cup \mathbb{D}) \). We first show that \( h(I, H^\infty) \subset \mathcal{M}(B) \). Observe that \( I \) contains the closed ideal

\[
J = \{ f \in H^\infty : \hat{f} \circ L_m = 0 \},
\]

so \( h(I, H^\infty) \subset h(J, H^\infty) \), but since \( \hat{I} \circ L_m = kH^\infty \),

\[
h(I, H^\infty) \subset h(J, H^\infty) \setminus \mathcal{P}(m).
\]

The formula

\[
L_m(\phi)(f) = \phi(\hat{f} \circ L_m), \quad f \in H^\infty, \phi \in \mathcal{M}(H^\infty),
\]

extends \( L_m \) to a continuous mapping \( \mathcal{M}(H^\infty) \to \mathcal{M}(H^\infty) \). Our next step is to show that \( h(J, H^\infty) = L_m(\mathcal{M}(H^\infty)) \). To this end, let \( \phi \in h(J, H^\infty) \). Since \( \hat{\hat{h}} \circ L_m = H^\infty \), the formula \( \psi(\hat{f} \circ L_m) = \phi(f), f \in H^\infty \), defines a nonzero complex homomorphism \( \psi \) on \( H^\infty \) such that \( \phi = L_m(\psi) \), as desired. We will now show that every thin Blaschke product has modulus 1 on

\[
h(J, H^\infty) \setminus \mathcal{P}(m) = L_m(\mathcal{M}(H^\infty) \setminus \mathbb{D}),
\]
thereby ensuring that

\[ h(I, H^\infty) \subset h(J, H^\infty) \setminus P(m) \subset \mathcal{M}(B) \quad [8, \text{p. 375}]. \]

By [12], for a thin Blaschke product \( \hat{b} \circ L_m \), \( \hat{b} \circ L_m \) is identically a unimodular constant, or

\[ \hat{b} \circ L_m(z) = \frac{z - \alpha}{1 - \bar{\alpha}z}, \quad z \in \mathbb{D}, \]

for some \( \alpha \in \mathbb{D}, \lambda \in \mathbb{T} \). In either case, \( |\hat{b} \circ L_m| = 1 \) on \( \mathcal{M}(H^\infty) \setminus \mathbb{D} \).

Now suppose a continuous homomorphism \( L_I : B \to H^\infty/I \) that is canonical on \( H^\infty \) exists. Let \( b \) be a thin Blaschke product such that \( m(b) = 0 \), normalized so that \( \hat{b} \circ L_m(z) = z, z \in \mathbb{D} \). Then \( b_\zeta = (b - \zeta)/(1 - \bar{\zeta}b) \) is thin for all \( \zeta \in \mathbb{D} \) [12], and hence \( b_\zeta = b_\zeta^{-1} \in B \). Therefore

\[ \|L_I(b_\zeta^{-1})\| = \|(b_\zeta + I)^{-1}\| \leq \|L_I\| \cdot \|\hat{b}_\zeta\| = \|L_I\| = C \]

for all \( \zeta \in \mathbb{D} \). Thus, there exists \( h_\zeta \in H^\infty \) such that \( \|h_\zeta + I\| \leq C \) and \( h_\zeta b_\zeta - 1 \in I \). Without loss of generality, we may assume \( \|h_\zeta\| \leq 2C \). Now

\[ (h_\zeta \hat{b}_\zeta) \circ L_m(z) - 1 = (h_\zeta \circ L_m(z))(z - \zeta)/(1 - \bar{\zeta}z) - 1 = k(z)f_\zeta(z), \quad z \in \mathbb{D}, \]

for some \( f_\zeta \in H^\infty \). But \( \|h_\zeta \circ L_m\| \leq 2C \), so \( \|f_\zeta\| \leq 2C + 1 \) for all \( \zeta \in \mathbb{D} \). Plugging in \( \zeta = z \), we see that the fact that \( k(z) \to 0 \) as \( z \to 1 \) along the real axis makes this impossible.

**Remarks 3.13.** (a) There is a wide variety of closed subalgebras of \( H^\infty \) containing \( QA \) for which the techniques of this section are applicable. For instance, the algebra \( QA_B = B \cap H^\infty \) for a Douglas algebra \( B \) is of this kind. More explicitly, the analogue of Theorem 3.4 states that every closed ideal of \( QA_B \) has the form \( u(J \cap QA_B) \), where \( u \) is an inner function, and \( J \) is a closed ideal in the algebra \( QA_B + C = \overline{B} \cap (H^\infty + C) \). This description is possible because an argument similar to that of Lemma 2.3 shows that \( QA_B \) has the \( f \)-property in the sense of Havin. However, the obvious analogue of Corollary 3.10, which would state that if \( I \) is a closed \( QA_B \)-ideal with inner factor 1 and \( h(I, QA_B) \subset \mathcal{M}(Q_B) \), where \( Q_B = B \cap \overline{B} \), there is a continuous epimorphism \( Q_B \to QA_B/I \) that is canonical on \( QA_B \), turns out not to be true in general. A counterexample can be produced along the lines of Example 3.12. Let \( COP \) be the closed subalgebra of \( H^\infty \) consisting of those functions that are constant on all Gleason parts other than \( \mathbb{D} \). It is easy to check that
COP + C is a closed subalgebra of $L^\infty$, knowing that $H^\infty + C$ is. In COP, every closed ideal with inner factor 1 has the form $J \cap \text{COP}$, where $J$ is a closed ideal in COP + C. We cannot get the general statement for arbitrary closed ideals, because COP does not have the $f$-property [2].

(b) One may wonder if in Corollary 3.5 we may localize to the Gleason parts instead of the maximal antisymmetric sets for $H^\infty + C$. We shall see that this is not possible, and indeed, one may not even localize to the COP level sets, which are those sets in $\mathcal{M}(H^\infty)$ where COP functions are constant. Donald Sarason has observed that COP contains an infinite Blaschke product $b$ (see [2]). Let $J$ be the closed principal ideal $b^2(H^\infty + C)$, which is proper because $b$ is not invertible in $H^\infty + C$, and put $I = J \cap H^\infty$, which has inner factor 1. Clearly, $b|_S \subseteq I|_S$ on every COP level set $S$, but since $b$ isn’t invertible in $H^\infty + C$, $b$ cannot be in $J = b^2(H^\infty + C)$.

(c) An open problem of Norman Alling [1] asks for a complete description of all closed prime ideals in $H^\infty$. It is conjectured that if $P$ is a closed prime ideal in $H^\infty$, other than $\{0\}$, then $P$ is either maximal, or there exists an $m \in \mathcal{M}(H^\infty) \setminus \mathcal{D}$ with nontrivial Gleason part $\mathcal{P}(m)$ such that $P = \{ f \in H^\infty : f|_{\mathcal{P}(m)} = 0 \}$. Using Theorem 3.4 and certain facts about uniform algebras it is possible to show that the following is true. If $P$ is a closed nonzero nonmaximal prime ideal in $H^\infty$, then there exists a unique maximal antisymmetrical set $S$ for $H^\infty + C$ such that if $f|_S \subseteq P|_S$, then $f \in P$ for any $f \in H^\infty$. A proof of this can be devised using Bishop’s construction of maximal antisymmetrical sets [5] together with Theorem 3.4 and the observation that if $A_{a}$ is a uniform algebra on $X_a$ and $P_a$ is a closed prime ideal of $A_{a}$, then there is a unique $A_{a} \cap \overline{A_{a}}$ level set $F \subset X_{a}$ such that $f(F) = 0$ implies $f \in P_a$.

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