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Large Bergman spaces: invertibility, cyclicity, and subspaces of arbitrary index

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Abstract

In a wide class of weighted Bergman spaces, we construct invertible non-cyclic elements. These are then used to produce z-invariant subspaces of index higher than one. In addition, these elements generate non-trivial bilaterally invariant subspaces in anti-symmetrically weighted Hilbert spaces of sequences.

Keywords: Bergman spaces; Cyclic vectors; Index of invariant subspaces; Bilaterally invariant subspaces; Weighted Hilbert spaces of sequences

1. Introduction

Consider the operation which sends a complex-valued sequence $\{a_n\}_n$ to the shifted sequence $\{a_{n-1}\}_n$; if the index set is the collection of all integers, all is fine, and if it is the non-negative integers, we should specify that we need the rule $a_{-1} = 0$. This shift operation is a linear transformation, and we denote it by $\mathbf{S}$. Sometimes, it is convenient to work with formal Laurent or Taylor series instead of sequence spaces, because of the simple form $\mathbf{S}$ takes, as it just corresponds to the

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multiplication by the formal variable z:

$$\sum_n a_n z^n \mapsto \sum_n a_n z^{n+1}.$$

For an analyst, it makes sense to restrict the shift to some Hilbert space like ℓ^2, and then to use invariant subspaces to better understand the operator. We recall that a linear subspace $\mathscr{M}$ is *shift invariant* if it is closed with respect to the given topology, and if $x \in \mathscr{M} \Rightarrow \mathbf{S}x \in \mathscr{M}$. The invariant subspaces offer the possibility of studying the action of $\mathbf{S}$ on smaller pieces. However, generally speaking, it may not be possible to reconstruct the operator as a mosaic of really small and understandable pieces. Beurling [7] found a complete characterization of the shift invariant subspaces in ℓ^2 on the non-negative integers in terms of so-called inner functions. A consequence of the theorem is that if a sequence (on the non-negative integers) is in ℓ^2, and its convolution inverse is as well, then the smallest invariant subspace containing it is all of ℓ^2. Another consequence is that every non-zero shift invariant subspace has index 1; this means that $\mathscr{M} \ominus \mathbf{S}\mathscr{M}$ is one-dimensional. If we instead consider weighted ℓ^2 spaces, with weights that make the space bigger than unweighted ℓ^2, we generally encounter a new kind of invariant subspaces, having indices that exceed 1; in fact, the dimension of $\mathscr{M} \ominus \mathbf{S}\mathscr{M}$ can assume any (integer) value between 1 and $+\infty$ [3]. Beurling later returned to shift invariant subspaces in the setting of weights, but he softened the topology somewhat, so instead of considering a weighted ℓ^2 space, he looked at a union of such, with the property of being an algebra, with respect to convolution of sequences, which corresponds to ordinary multiplication of Taylor series. He found that there is a critical topology such that on the one side—with relatively small spaces—the singular inner function for a point mass generates a non-trivial invariant subspace, whereas on the other side of the borderline—with relatively large spaces—it generates the whole space as an invariant subspace. Later, Nikolski [23, Chapter 2] showed that the dichotomy is even deeper: we may replace the atomic singular inner function by any zero-free function in the space. That is, in the case of relatively large spaces, every zero-free function generates the whole space as an invariant subspace.

Here, we focus on hard topology spaces, that is, weighted ℓ^2 spaces on the non-negative integers. We show that there is no such dichotomy as in Beurling's situation. In fact, we find analytic functions in the disk that belong to the given space along with their reciprocals, while the shift invariant subspaces they generate fail to be the whole space. We also use these functions to build concrete examples of invariant subspaces of high index. Furthermore, we find that if we extend the weight and the space to the collection of all integers, in such a way that the logarithm of the weight becomes an odd function, then the non-trivial shift invariant subspace generated by our invertible function extends to a bilaterally shift invariant subspace, in the sense that the intersection of the bilaterally invariant subspace with the analytic part just returns us our initial shift invariant subspace.

2. Description of results

2.1. Invertibility versus cyclicity

In the Gelfand theory of commutative Banach algebras with unit, an element generates a dense ideal if and only if it is invertible, in which case its Gelfand transform has no zeros, and the ideal it generates is the whole algebra. Let $\mathscr{X}$ be a Banach space (or a quasi-Banach space, that is, a complete normed space with a p-homogeneous norm, for some $0<p<1$) of holomorphic functions on the unit disk $\mathbb{D}$. We assume that the point evaluations at points of $\mathbb{D}$ are continuous functionals on $\mathscr{X}$, and that $\mathbf{S}f \in \mathscr{X}$ whenever $f \in \mathscr{X}$, where $\mathbf{S}f(z) = zf(z)$ is the operator of multiplication by z. This means that for each $f \in \mathscr{X}$, all the functions $\mathbf{S}f, \mathbf{S}^2 f, \mathbf{S}^3 f, \dots$ are in $\mathscr{X}$ as well. Let $[f] = [f]_{\mathscr{X}}$ denote the closure in $\mathscr{X}$ of the finite linear span of the vectors $f, \mathbf{S}f, \mathbf{S}^2 f, \dots$; we say that f is *cyclic* in $\mathscr{X}$ provided $[f]_{\mathscr{X}} = \mathscr{X}$. If $\mathscr{X}$ is a Banach (or quasi-Banach) algebra containing a unit element (the constant function 1), then all the polynomials belong to $\mathscr{X}$. If in fact the polynomials are dense in $\mathscr{X}$, then $f \in \mathscr{X}$ is cyclic if and only if it is invertible. In other words, in *spaces* $\mathscr{X}$, the concept of cyclicity generalizes that of invertibility, provided that the polynomials belong to and are dense in $\mathscr{X}$. It is then of interest to compare cyclicity with genuine invertibility. Consider, for instance, the space $\mathscr{X} = H^2$, the Hardy space on the unit disk $\mathbb{D}$. By Beurling's invariant subspace theorem, a function is cyclic if and only if it is an outer function. The invertible functions in H^2 are all outer, so for this space, all invertible elements are cyclic. However, by the examples provided by Borichev and Hedenmalm in [13], this fails for the Bergman spaces $B^p(\mathbb{D})$, $0<p<+\infty$, consisting of pth power area-summable holomorphic functions on $\mathbb{D}$. For an earlier example of a Banach space of analytic functions where this phenomenon occurs, we refer to [26].

Given a continuous strictly positive area-summable function ω on $\mathbb{D}$—referred to as a weight—we form the space $B^p(\mathbb{D}, \omega)$ (for $0<p<+\infty$) of holomorphic functions f on $\mathbb{D}$ subject to the norm bound restriction

$$\|f\|_{\omega,p} = \left(\int_{\mathbb{D}} |f(z)|^p \, \omega(z) \, dm_{\mathbb{D}}(z) \right)^{1/p} < +\infty,$$

where $dm_{\mathbb{D}}(z) = \pi^{-1} \, dx \, dy$ is normalized area measure ($z = x + iy$). It is a Banach space of holomorphic functions for $1 \leqslant p < +\infty$, and a quasi-Banach space for $0<p<1$. We shall be concerned exclusively with radial weights ω: from now on, $\omega(z) = \omega(|z|)$ holds for all $z \in \mathbb{D}$. One nice thing about radial weights is that the polynomials are guaranteed to be dense in $B^p(\mathbb{D}, \omega)$. Let us form the soft topology space $\mathscr{A}(\mathbb{D}, \omega) = \bigcup_{0<p<+\infty} B^p(\mathbb{D}, \omega)$, supplied with the inductive limit topology. Spaces of this kind were studied by Beurling [8] and Nikolski [23, Chapter 2]. Under natural regularity conditions on ω, the following dichotomy holds: if

$$\int_0^1 \sqrt{\frac{\log \frac{1}{\omega(t)}}{1-t}}\, dt < +\infty,$$

then there exist a non-cyclic function in $\mathscr{A}(\mathbb{D},\omega)$ without zeros in $\mathbb{D}$ (the singular inner function for an atomic measure will do), whereas if the above integral diverges, that is,

$$\int_0^1 \sqrt{\frac{\log \frac{1}{\omega(t)}}{1-t}}\, dt = +\infty,$$

then each function in $\mathscr{A}(\mathbb{D},\omega)$ lacking zeros in $\mathbb{D}$ is cyclic in $\mathscr{A}(\mathbb{D},\omega)$. One is then led to wonder whether there exists a similar dichotomy for the hard topology spaces $B^p(\mathbb{D},\omega)$. Some progress has already been made on this matter. Nikolski constructed in [23, Section 2.8] a special class of weights ω that vanish at the boundary arbitrarily fast, and such that $B^p(\mathbb{D},\omega)$ contains zero-free non-cyclic elements. Hedenmalm and Volberg [21] proved that $B^2(\mathbb{D},\omega)$, for

$$\omega(z) = \exp\left(-\frac{1}{1-|z|}\right), \quad z \in \mathbb{D},$$

contains invertible (and hence zero-free) non-cyclic elements. Atzmon [4] produced zero-free S-invariant subspaces of index 1 (for the notion of index see the next subsection) in $B^2(\mathbb{D},\omega)$ for all ω satisfying some weak regularity conditions.

We formulate our first result. *We shall assume that the* (*positive*) *weight function* ω *decreases*, *and that* $\omega(t)\to 0$ *as* $t \to 1$ *so quickly that for some* ε_0, $0<\varepsilon_0<1$,

$$\lim_{t\to 1} (1-t)^{\varepsilon_0} \log\log \frac{1}{\omega(t)} = +\infty; \tag{2.1}$$

in other words, the speed is at least as fast as two exponentials. Without loss of generality, we can assume ω is C^1-smooth as well as decreasing, and that the values are taken in the interval $(0, 1/e)$. No further assumptions of growth or regularity type will be made.

Theorem 2.1. *For weights* ω *that meet condition* (2.1), *let* $\tilde{\omega}$ *be the associated weight*

$$\log \frac{1}{\tilde{\omega}(z)} = \log \frac{1}{\omega(z)} - \left[\log\log \frac{1}{\omega(z)}\right]^2, \quad z \in \mathbb{D}, \tag{2.2}$$

which decreases slightly more slowly than ω *to* 0 *as we approach the boundary*, *so that* $B^p(\mathbb{D},\tilde{\omega})$ *is contained in* $B^p(\mathbb{D},\omega)$. *There exists a function* $F \in B^1(\mathbb{D},\omega)$ *without zeros in* $\mathbb{D}$ *which is non-cyclic in* $B^1(\mathbb{D},\omega)$, *and whose reciprocal* $1/F$ *is in* $B^1(\mathbb{D},\tilde{\omega})$. *Moreover*, *we can get* F *such that in addition*, $F^{1/p}$ *is non-cyclic in* $B^p(\mathbb{D},\omega)$ *for each* p, $0<p<+\infty$.

Note that both the growth and the decay of $|F|$ are somewhat extremal. Indeed, were $F^{-1-\delta}$ to belong to $B^1(\mathbb{D},\omega)$ for some $\delta>0$, then, for sufficiently regular ω we would have $F^{1+\delta/2}\in B^1(\mathbb{D},\omega)$. This follows from a uniqueness theorem for harmonic functions [10], which improves a result by Nikolski [23, Section 1.2]. An elementary argument due to Shapiro [27] would then show that F is cyclic in $B^1(\mathbb{D},\omega)$.

It is interesting to note that we may strengthen the assertion of Theorem 2.1 to the following.

Theorem 2.2. *There exists a function F satisfying the conditions of Theorem* 2.1 *and such that for $f=F^{1/2}$, we have*

$$\int_{\mathbb{D}}\int_{\mathbb{D}}\frac{1}{|f(z)|^2}\frac{|f(z)-f(w)|^2}{|z-w|^2}\,\omega(z)\omega(w)\,dm_{\mathbb{D}}(z)\,dm_{\mathbb{D}}(w)<+\infty.$$

2.2. Subspaces of large index

The functions F we construct are also extremal in a different sense. Fix a p, $0<p<+\infty$. By the above theorem, there exists a non-trivial invariant subspace of $B^p(\mathbb{D},\omega)$ which is generated by a function in $B^p(\mathbb{D},\omega)$ whose reciprocal belongs to the slightly smaller space $B^p(\mathbb{D},\tilde{\omega})$. Here we use standard terminology: a closed linear subspace $\mathscr{M}$ of $B^p(\mathbb{D},\omega)$ is *invariant* if $\mathbf{S}f\in\mathscr{M}$ whenever $f\in\mathscr{M}$. Note that for every $f\in B^p(\mathbb{D},\omega)$, $\|\mathbf{S}f\|\geqslant C\|f\|$, for some constant $C=C(p,\omega)$, $0<C<1$. Therefore, for every invariant subspace $\mathscr{M}$, the set $\mathbf{S}\mathscr{M}$ is a closed subspace of $\mathscr{M}$. The dimension of the quotient space $\mathscr{M}/\mathbf{S}\mathscr{M}$ we shall call the *index of* $\mathscr{M}$ and denote by $\operatorname{ind}\mathscr{M}$. Given two invariant subspaces $\mathscr{M}_1$ and $\mathscr{M}_2$, we can form $\mathscr{M}_1\vee\mathscr{M}_2$, the smallest invariant subspace containing both $\mathscr{M}_1$ and $\mathscr{M}_2$. This definition naturally extends to more general collections of invariant subspaces, with more than two elements. The index function is then subadditive, in the following sense: the index of $\mathscr{M}_1\vee\cdots\vee\mathscr{M}_n$ is less than or equal to the sum of the individual indices for $\mathscr{M}_1,\ldots,\mathscr{M}_n$.

Apostol et al. [3] proved that $B^2(\mathbb{D},\omega)$ contains invariant subspaces of arbitrary index. Their is a pure existence theorem, but later on, concrete examples of invariant subspaces of large index were given in [1,9,18,20]. Usually, subspaces of index bigger than 1 have somewhat strange properties. For example, they cannot contain multipliers [25]; in $B^p(\mathbb{D})$, $1<p<+\infty$, if $\operatorname{ind}[f,g]=2$ (where $[f,g]=[f]\vee[g]$ is the closed invariant subspace generated by f and g), then f/g has finite non-tangential limits almost nowhere on the unit circle [2, Corollary 7.7]. It turns out that the invertible functions F we construct could be used to produce invariant subspaces of higher index.

Theorem 2.3. *Let us assume that the weight ω satisfies property* (2.1). *Then there exist invertible functions F_j in $B^1(\mathbb{D},\omega)$ for $j=1,2,3,\ldots$, such that for every $0<p<+\infty$*

and for every positive integer n, the subspace

$$[F_1^{1/p}, \dots, F_n^{1/p}] = [F_1^{1/p}] \vee \cdots \vee [F_n^{1/p}]$$

has maximal index in $B^p(\mathbb{D}, \omega)$, *namely n. Moreover, the assertion holds also when n assumes the value* $+\infty$:

$$[F_1^{1/p}, F_2^{1/p}, \dots]$$

has infinite index in $B^p(\mathbb{D}, \omega)$.

2.3. The induced bilateral Hilbert space

The space $B^2(\mathbb{D}, \omega)$ is a Hilbert space, and it is possible to describe it as a weighted ℓ^2 space on the set of non-negative integers $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$. At times, we also need the collection of negative integers $\mathbb{Z}_- = \{-1, -2, -3, \dots\}$. We note that a holomorphic function

$$f(z) = \sum_{n=0}^{+\infty} \hat{f}(n) z^n, \quad z \in \mathbb{D},$$

is in $B^2(\mathbb{D}, \omega)$ if and only if

$$\sum_{n=0}^{+\infty} |\hat{f}(n)|^2 \, \Omega(n) < +\infty, \tag{2.3}$$

where

$$\Omega(n) = \int_{\mathbb{D}} |z|^{2n} \omega(|z|) \, dm_{\mathbb{D}}(z) < +\infty, \quad n = 0, 1, 2, 3, \dots \, . \tag{2.4}$$

The function Ω is log-convex, that is, $\Omega(n)^2 \leqslant \Omega(n-1)\Omega(n+1)$ holds for all $n = 1, 2, 3, \dots$, and it has the property

$$\lim_{n \to +\infty} \Omega(n)^{1/n} = 1. \tag{2.5}$$

The left hand side of (2.3) equals the norm squared of f in $B^2(\mathbb{D}, \omega)$. With the usual Cauchy duality

$$\langle f, g \rangle = \sum_{n=0}^{+\infty} a_n b_n,$$

where g is the convergent Laurent series

$$g(z) = \sum_{n=0}^{+\infty} b_n z^{-n-1}, \quad 1 < |z| < +\infty,$$

we can identify the dual space $B^2(\mathbb{D}, \omega)^*$ with the space of Laurent series g with norm

$$||g||^2_{\omega^*} = \sum_{n=0}^{+\infty} \frac{|b_n|^2}{\Omega(n)} < +\infty. \tag{2.6}$$

We then form the sum space $\mathfrak{L}^2(\mathbb{T}, \omega) = B^2(\mathbb{D}, \omega) \oplus B^2(\mathbb{D}, \omega)^*$ of formal Laurent series

$$h(z) = \sum_{n=-\infty}^{+\infty} c_n z^n, \quad |z| = 1,$$

with norm

$$||h||^2_{\mathfrak{L}^2(\mathbb{T},\omega)} = \sum_{n=0}^{+\infty} |c_n|^2 \Omega(n) + \sum_{n=0}^{+\infty} \frac{|c_{-n-1}|^2}{\Omega(n)}.$$

Note that although we have indicated the unit circle as the domain of definition, in general the formal series converges nowhere. If we extend the weight Ω to negative integers by

$$\Omega(n) = \frac{1}{\Omega(-n-1)}, \quad n = -1, -2, -3, \ldots,$$

then the norm on $\mathfrak{L}^2(\mathbb{T}, \omega)$ takes on a more pleasant appearance:

$$||h||^2_{\mathfrak{L}^2(\mathbb{T},\omega)} = \sum_{n=-\infty}^{+\infty} |c_n|^2 \Omega(n).$$

On the other hand, given a positive log-convex function Ω_0 such that

$$\lim_{n \to +\infty} \Omega_0(n)^{1/n} = 1,$$

we can find a weight function ω on [0,1) such that the function Ω defined by ω via formula (2.4) is equivalent to Ω_0, written out $\Omega_0 \asymp \Omega$ (see [12, Proposition B.1]). In concrete terms, this means that for some positive constant C, $C^{-1}\Omega_0 \leqslant \Omega \leqslant C\Omega_0$ on the non-negative integers. Let us see what condition (2.1) requires in terms of the weight Ω. Suppose we know that for some $0 < \alpha < +\infty$,

$$\Omega(n) \leqslant \exp\left[-\frac{n}{(\log(2+n))^\alpha}\right], \quad n = 0, 1, 2, \ldots\ . \tag{2.7}$$

Then the decreasing weight ω automatically satisfies (2.1). Indeed, by (2.4) we have

$$2\omega(x)\int_0^x r^{2n+1}\,dr \leqslant 2\int_0^1 r^{2n+1}\omega(r)\,dr \leqslant \exp\left[-\frac{n}{(\log(n+2))^\alpha}\right],$$

for $n = 0, 1, 2, 3, \ldots$, and hence

$$\omega(x) \leqslant \min_n \frac{(n+1)\exp\left[-\frac{n}{(\log(n+2))^\alpha}\right]}{x^{2n+2}}, \quad 0 < x < 1,$$

from which (2.1) follows.

The shift operator $\mathbf{S}$ extends to $\mathfrak{L}^2(\mathbb{T}, \omega)$,

$$\mathbf{S}h(z) = zh(z) = \sum_{n=-\infty}^{+\infty} c_{n-1} z^n,$$

but now $\mathbf{S}$ is an *invertible* bounded operator. It is therefore natural to consider closed subspaces that are invariant with respect to both the forward shift and the backward shift $\mathbf{S}^{-1}$. We call them bilaterally invariant, and the shift in both direction the bilateral shift.

Let $\mathscr{N}$ be a bilaterally invariant subspace of $\mathfrak{L}^2(\mathbb{T}, \omega)$, and $\mathscr{M}$ a subset of $B^2(\mathbb{D}, \omega)$. We consider the intersection $\mathscr{N}_{\mathbb{D}} = \mathscr{N} \cap B^2(\mathbb{D}, \omega)$, and the extension $\mathscr{M}_{\mathbb{T}}$ of $\mathscr{M}$, the closed linear span of $\bigcup_{n\in\mathbb{Z}} \mathbf{S}^n\mathscr{M}$ in $\mathfrak{L}^2(\mathbb{T}, \omega)$. Then $\mathscr{N}_{\mathbb{D}}$ is a forward invariant subspace of index 1 in $B^2(\mathbb{D}, \omega)$, and $\mathscr{M}_{\mathbb{T}}$ is a bilaterally invariant subspace of $\mathfrak{L}^2(\mathbb{T}, \omega)$.

If the weight ω is sufficiently regular and

$$\int^1 \log\log\frac{1}{\omega(x)}\,dx < +\infty,$$

or, equivalently,

$$\sum_{n=0}^{+\infty} \frac{\log\frac{1}{\Omega(n)}}{n^2+1} < +\infty,$$

then the classical result of Wermer [28] implies that $\mathfrak{L}^2(\mathbb{T}, \omega)$ possesses non-trivial (bilaterally) invariant subspaces. It seems that the existence of non-trivial bilaterally invariant subspaces $\mathscr{N}$ in $\mathfrak{L}^2(\mathbb{T}, \omega)$ is unknown for general ω (for the current status, see [6,5]); however, for sufficiently regular weights ω, this was proved by Domar [14].

In his example, we have $\mathscr{N}_{\mathbb{D}} = \{0\}$. Let us sketch an argument to this effect. Recall that $\mathfrak{L}^2(\mathbb{T}, \omega)$ is isometrically isomorphic to the space $\ell^2(\mathbb{Z}, \Omega)$ of complex-valued sequences $\{c_n\}_{n\in\mathbb{Z}}$ with

$$\|\{c_n\}_n\|^2_{\ell^2(\mathbb{Z},\Omega)} = \sum_{n\in\mathbb{Z}} |c_n|^2\Omega(n) < +\infty,$$

provided that Ω is extended "anti-symmetrically" to the negative integers:

$$\Omega(n) = \frac{1}{\Omega(-n-1)}, \quad n<0.$$

Note that

$$0 < \inf_{n\in\mathbb{Z}_+} \frac{\Omega(n-1)}{\Omega(n)} \leqslant \sup_{n\in\mathbb{Z}_+} \frac{\Omega(n-1)}{\Omega(n)} < +\infty,$$

so that for all essential purposes, we can think of $\log\Omega(n)$ as an anti-symmetric function of n. Domar constructs a non-trivial entire function f of exponential type a, with $0<a<\frac{1}{2}\pi$, such that

$$|f(n)|^2 \leqslant \frac{1}{(n^2+1)\Omega(n)}, \quad n\in\mathbb{Z}.$$

Then

$$\hat{f}(z) = \sum_{n\in\mathbb{Z}} f(n)z^n \in \mathfrak{L}^2(\mathbb{T},\omega),$$

and $[\hat{f}]_{\mathbb{T}} \neq \mathfrak{L}^2(\mathbb{T},\omega)$ (here, of course, $[\hat{f}]_{\mathbb{T}}$ is the bilaterally invariant subspace generated by $\hat{f}$ in $\mathfrak{L}^2(\mathbb{T},\omega)$). To show that $[\hat{f}]_{\mathbb{T}} \cap B^2(\mathbb{D},\omega) = \{0\}$, we suppose that for a sequence of entire functions f_k of exponential type a we have $f_k|_{\mathbb{Z}} \to h$ in $\ell^2(\mathbb{Z},\Omega)$, where $h|_{\mathbb{Z}_-} \equiv 0$, $h(0)\neq 0$. Then $F_k(z) = f_k(z)f_k(-z) \to H(z) = h(z)h(-z)$ in $\ell^1(\mathbb{Z})$. Applying the Cartwright theorem [22, Section 21.2] and the Phragmén–Lindelöf theorem in the upper and the lower half-planes, we obtain that as $k\to+\infty$, F_k converges to an entire function F of exponential type at most $2a$, and that $F|_{\mathbb{Z}} = H|_{\mathbb{Z}}$. Note that $H(n) = 0$ on $\mathbb{Z}\setminus\{0\}$; but this requires H to have at least type π, which is not possible, by the assumption that $a<\frac{1}{2}\pi$.

In the related work [15–17] Esterle and Volberg proved that for asymmetrically weighted Hilbert spaces of sequences $\ell^2(\mathbb{Z},\Omega)$, with $\Omega(n) \ll \Omega(-n)^{-1}$ as $n\to+\infty$, and

$$\sum_{n=1}^{+\infty} \frac{\log\Omega(-n)}{n^2} = +\infty,$$

(i) every bilaterally invariant subspace $\mathcal{N}$ is generated by its analytic part $\mathcal{N}\cap\ell^2(\mathbb{Z}_+,\Omega)$, where $\ell^2(\mathbb{Z}_+,\Omega)$ is the subspace of $\ell^2(\mathbb{Z},\Omega)$ consisting of all sequences vanishing on the negative integers $\mathbb{Z}_-$, and

(ii) $\mathcal{N} + \ell^2(\mathbb{Z}_+,\Omega) = \ell^2(\mathbb{Z},\Omega)$.

On the other hand, for anti-symmetrically weighted $\ell^2(\mathbb{Z}, \Omega)$, with sufficiently regular Ω, such that for some positive value of the parameter ε,

$$n^{1+\varepsilon} \leqslant \Omega(n)$$

for all big n, Borichev [11] proved that for every bilaterally invariant subspace $\mathcal{N}, \mathcal{N} \cap \ell^2(\mathbb{Z}_+, \Omega) = \{0\}$.

Let us give here an application of our Theorem 2.1, which actually requires a slightly stronger property of the function F. If Ω is a log-convex weight function satisfying the conditions of Theorem 3-3 in [16], such that (2.7) holds for some $\alpha > 0$, then $\ell^2(\mathbb{Z}_+, \Omega)$ contains a non-trivial zero-free shift invariant subspace of index 1; thus, by Theorem 3-3 of [16], $\ell^2(\mathbb{Z}, \Omega)$ contains non-trivial bilaterally invariant subspaces.

We construct bilaterally invariant subspaces in $\mathfrak{L}^2(\mathbb{T}, \omega)$ with properties similar to (i) and (ii).

Theorem 2.4. *Let us assume that the weight ω satisfies property* (2.1). *Then there exists a function F satisfying the conditions of Theorem* 2.1 *and such that $F^{1/2}$ is non-cyclic in $\mathfrak{L}^2(\mathbb{T}, \omega)$ with respect to the bilateral shift. Furthermore, if $\mathcal{M} = [F^{1/2}]$, then*

$$\left.\begin{aligned} (\mathcal{M}_{\mathbb{T}})_{\mathbb{D}} &= \mathcal{M}, \\ \mathcal{M}_{\mathbb{T}} + B^2(\mathbb{D}, \omega) &= \mathfrak{L}^2(\mathbb{T}, \omega). \end{aligned}\right\} \tag{2.8}$$

Remark 2.5. (a) In view of the theorem and the above observations, if Ω is a log-convex weight function satisfying (2.5) and (2.7) for some $\alpha > 0$, and extended to negative indices n by $\Omega(n) = 1/\Omega(-n-1)$, then $\ell^2(\mathbb{Z}, \Omega)$ contains a singly generated bilaterally invariant subspace $\mathcal{N}$ such that

(i) $\mathcal{N}$ is generated by $\mathcal{N} \cap \ell^2(\mathbb{Z}_+, \Omega)$, and
(ii) $\mathcal{N} + \ell^2(\mathbb{Z}_+, \Omega) = \ell^2(\mathbb{Z}, \Omega)$.

(b) It is interesting to contrast the situation depicted in part (a) with the case $\Omega(n) \equiv 1$, when each bilaterally shift invariant subspace $\mathcal{N}$ of $\ell^2(\mathbb{Z})$ is given by a common zero set on the unit circle $\mathbb{T}$, and we have $\mathcal{N} \cap \ell^2(\mathbb{Z}_+) = \{0\}$ unless $\mathcal{N} = \ell^2(\mathbb{Z})$. An intermediate case between these opposites would be the Bergman–Dirichlet situation, with $\Omega(n) = 1/(n+1)$ for $n = 0, 1, 2, \ldots$ and $\Omega(n) = -n$ for $n = -1, -2, -3, \ldots$. Here, it is not known whether there exist non-trivial bilaterally invariant subspaces $\mathcal{N}$ with property (i) above.

2.4. *The spectra associated with bilaterally invariant subspaces*

Given an invariant subspace $\mathcal{M}$ in $B^2(\mathbb{D}, \omega)$, the operator $\mathbf{S}$ induces an operator $\mathbf{S}[\mathcal{M}] : B^2(\mathbb{D}, \omega)/\mathcal{M} \to B^2(\mathbb{D}, \omega)/\mathcal{M}$ defined by $\mathbf{S}[\mathcal{M}](f + \mathcal{M}) = \mathbf{S}f + \mathcal{M}$; clearly,

$\mathbf{S}[\mathscr{M}]$ has norm less than or equal to that of $\mathbf{S}$, which equals 1. Similarly, given a bilaterally invariant subspace $\mathscr{N}$ in $\mathfrak{L}^2(\mathbb{T},\omega)$, the operator $\mathbf{S}$ induces an operator $\mathbf{S}[\mathscr{N}]:\mathfrak{L}^2(\mathbb{T},\omega)/\mathscr{N}\to\mathfrak{L}^2(\mathbb{T},\omega)/\mathscr{N}$ defined by $\mathbf{S}[\mathscr{N}](f+\mathscr{N})=\mathbf{S}f+\mathscr{N}$. Again, $\mathbf{S}[\mathscr{N}]$ has norm less than or equal to that of $\mathbf{S}$, and $\mathbf{S}[\mathscr{N}]^{-1}$ has norm less than or equal to that of $\mathbf{S}^{-1}$. We do not indicate the underlying space here in the notation, because we feel that no confusion is possible. In the situation indicated in Theorem 2.4, the operators $\mathbf{S}[\mathscr{M}]$ and $\mathbf{S}[\mathscr{M}_\mathbb{T}]$ are canonically similar. For, it is easy to check that the canonical mapping

$$j_\mathscr{M}:B^2(\mathbb{D},\omega)/\mathscr{M}\to\mathfrak{L}^2(\mathbb{T},\omega)/\mathscr{M}_\mathbb{T}$$

given by $j_\mathscr{M}(f+\mathscr{M})=f+\mathscr{M}_\mathbb{T}$ is an isomorphism, and we have the relationship $\mathbf{S}[\mathscr{M}_\mathbb{T}]=j_\mathscr{M}\circ\mathbf{S}[\mathscr{M}]\circ j_\mathscr{M}^{-1}$. In particular, the operators $\mathbf{S}[\mathscr{M}]$ and $\mathbf{S}[\mathscr{M}_\mathbb{T}]$ have the same spectrum. The spectrum of $\mathbf{S}[\mathscr{M}_\mathbb{T}]$ is a compact subset of $\mathbb{T}$, because the unit circle is the spectrum of the bilateral shift on $\mathfrak{L}^2(\mathbb{T},\omega)$. Generally speaking, the spectrum of $\mathbf{S}[\mathscr{M}]$ is the common zero set of $\mathscr{M}$ on the open unit disk plus a generalized zero set on the unit circle (see [19], which treats the unweighted Bergman space case). It is a consequence of the next theorem that the above situation depicted in Theorem 2.4 cannot occur if $\mathbf{S}[\mathscr{M}]$ has countable spectrum (contained in the unit circle).

Theorem 2.6. *Suppose $\mathscr{N}$ is a bilaterally shift invariant subspace of $\mathfrak{L}^2(\mathbb{T},\omega)$. Then the spectrum $\sigma(\mathbf{S}[\mathscr{N}])$ of $\mathbf{S}[\mathscr{N}]$ is a perfect set, that is, a closed subset of $\mathbb{T}$ without isolated points. In particular, if $\sigma(\mathbf{S}[\mathscr{N}])$ is countable, then $\sigma(\mathbf{S}[\mathscr{N}])=\emptyset$, and $\mathscr{N}=\mathfrak{L}^2(\mathbb{T},\omega)$.*

The proof of the above theorem in Section 3 does not use any of the strong assumptions made earlier on the weight, but rather holds in a much more general context. In view of the above theorem, it is natural to ask what kinds of sets may actually occur as spectra of $\mathbf{S}[\mathscr{N}]$, if $\mathscr{N}$ is a bilaterally shift invariant subspace of $\mathfrak{L}^2(\mathbb{T},\omega)$. It is possible to verify that the bilaterally invariant subspace $\mathscr{M}_\mathbb{T}$ appearing in Theorem 2.4 has spectrum $\mathbb{T}$, that is, the induced operator $\mathbf{S}[\mathscr{M}_\mathbb{T}]$ has spectrum $\mathbb{T}$. But perhaps there are other bilaterally invariant subspaces with more complicated spectra? At the present moment in our investigation, we do not even know if the space $\mathfrak{L}^2(\mathbb{T},\omega)$ always possesses a bilaterally invariant subspace whose spectrum is a non-trivial closed arc of $\mathbb{T}$. Possible candidates might be the subspaces constructed by Domar [14]. Once this question is resolved, it is natural to ask what it means for the spectrum of a bilaterally invariant subspace $\mathscr{N}$ if we add the requirement that the analytic part of $\mathscr{N}$ is non-trivial, that is, $\mathscr{N}\cap B^2(\mathbb{D},\omega)\neq\{0\}$.

2.5. *The idea of the proof of Theorem 2.1*

The largeness of a function sometimes implies its non-cyclicity. At first glance this is counter-intuitive, because in Hardy spaces, for example, outer functions are the "largest" ones, but they are cyclic. However, in spaces of analytic functions determined in terms of growth, the largeness does imply non-cyclicity. One can get a flavor of the proof by first looking at the following "toy picture". Let a class of analytic functions f be defined by the condition $|f(z)| \leqslant C\varrho(|z|)$, where ϱ is a (radial) weight of growth type, that is $\varrho(x) \nearrow +\infty$, $x \to 1$. Furthermore, let F be in this class, and let it be "maximally large" in the sense that $|F(z)| \geqslant \varrho(|z|)$ for all $z \in \mathcal{Q} \subset \mathbb{D}$. If the set $\mathcal{Q}$ is massive enough, here is what will happen. Any sequence of polynomials q_n such that $|q_n(z)F(z)| \leqslant C\varrho(|z|)$ will obviously satisfy the estimate $|q_n(z)| \leqslant C$, $z \in \mathcal{Q}$. Now the massiveness of $\mathcal{Q}$ guarantees the uniform boundedness of the family $\{q_n\}_n$ (here massiveness may mean, for example, that almost every point of the circle can be approached by points from $\mathcal{Q}$ in a non-tangential way). The uniform boundedness of the family $\{q_n\}_n$ should be combined with a property of F (which one has to establish in advance) to tend to zero along a certain sequence. Together, these two properties show that the products q_nF cannot converge to a non-zero constant uniformly on compact subsets of the disk. Thus, the non-cyclicity of F follows.

This kind of idea was used to construct non-cyclic functions in the paper of Borichev and Hedenmalm [13]. The same idea will also be used in the present article. However, we shall not be able to prove the uniform boundedness of q_n. Instead, we shall prove the normality of the family $\{q_n\}_n$, with effective uniform estimates on the growth of $|q_n(z)|$. The difference with the "toy problem" is that now, F will be of maximal largeness in the integral sense, that is, on average, rather than pointwise as above. In its turn, this will imply that q_n are not uniformly bounded on a massive set as before, but rather have some integral estimates; more precisely, the weighted sum of the absolute values of q_n along a lattice-like sequence of points in $\mathbb{D}$ will satisfy some effective estimates (independent of n). Unlike in the previous consideration, where the massiveness of $\mathcal{Q}$ was used (essentially) via the harmonic measure estimates, we will use the Lagrange interpolation theorem.

To describe the idea of "integral largeness" in more details, we start with the weight ω of decrease type from Section 2.1. We will construct F in the unit ball of $B^1(\mathbb{D}, \omega)$, which is maximally large in a certain sense. What this amounts to is

$$\int_{\mathbb{D}} |F(z)|\omega(|z|)\, dm_{\mathbb{D}}(z) \leqslant 1$$

and (the inequality below is what we mean by "integral maximal largeness")

$$\int_{\mathcal{D}_{n,k}} |F(z)|\omega(|z|)\, dm_{\mathbb{D}}(z) \asymp \frac{1-r_n}{n^2},$$

where r_n is a sequence of radii rapidly converging to 1, and $w_{n,k}$, with $0 \leqslant k < N_n \asymp \frac{1}{1-r_n}$, are the points equidistributed over the circle of radius r_n about the origin, and finally, $\mathcal{D}_{n,k}$ are the disks

$$\mathcal{D}_{n,k} = \{z \in \mathbb{C}\colon |z - w_{n,k}| < (1-r_n)^2\}.$$

Summing up over k, n, we notice that the integral of $|F(z)|\omega(|z|)$ over the union of our small disks is proportional to the integral over the whole unit disk. A big part of the mass of $|F(z)|\omega(|z|)\, dm_{\mathbb{D}}(z)$ lies inside a tiny set, which is the union of small disks. In other words, a big part of the norm of F is concentrated on a set which is tiny in the sense of area, but which is sufficiently "widespread".

Suppose that for a sequence of polynomials q_m,

$$||q_m F||_{B^1(\omega)} \leqslant A,$$

where A is a positive constant. Then the "integral maximal largeness" inequality shows that for some points $z_{n,k}$ (the points will depend on the choice of q_m, they are the points of minimum for $|q_m(z)|$ on the corresponding small disks $\mathscr{D}_{n,k}$) we have the weak type estimate

$$\sum_{k=0}^{N_n-1} |q_m(z_{n,k})| \leqslant Cn^2 N_n.$$

Applying the results of Section 9, we get for some positive constant C independent of m that

$$|q_m(z)| \leqslant C \exp\left[\frac{1}{1-|z|}\right], \quad z \in \mathbb{D}.$$

This is the effective estimate on normality we mentioned above. Next, we have to guarantee that F tends to zero sufficiently rapidly along a sequence of points. This, together with the last estimate, implies that the products $q_m F$ cannot converge to a non-zero constant uniformly on compact subsets of the disk. Thus, the non-cyclicity of F follows.

2.6. The plan of the paper

First, in Section 3, we prove Theorem 2.6. Then, in Section 4, we show how to use Theorem 2.2 to deduce Theorem 2.4, applying a method similar to that used in [16]. After this, we turn to the more technical aspects of the paper. In Section 9, we construct some elements in $B^1(\mathbb{D}, \omega)$ of "maximal possible growth". The constructions use

(a) a regularization procedure for ω described in Section 6,
(b) estimates for auxiliary harmonic functions established in Sections 7 and 8, and
(c) a special Phragmén–Lindelöf-type estimate obtained in Section 5.

Theorems 2.1, 2.2, and 2.3 are proved in Section 9, renamed as Theorems 9.1, 9.2, and 9.3, respectively.

3. The proof of Theorem 2.6

The spectrum $\sigma(\mathbf{S}[\mathcal{N}])$ is a closed subset of the unit circle $\mathbb{T}$. The set $\sigma(\mathbf{S}[\mathcal{N}])$ is empty if and only if $\mathbf{S}[\mathcal{N}]$ acts on the trivial space $\{0\}$, in which case $\mathcal{N} = \mathfrak{L}^2(\mathbb{T}, \omega)$. A non-empty closed and countable subset of $\mathbb{T}$ necessarily has isolated points. It remains to prove that isolated points cannot occur in the spectrum $\sigma(\mathbf{S}[\mathcal{N}])$. To this end, let $\lambda_0 \in \sigma(\mathbf{S}[\mathcal{N}])$ be an isolated point. By the Riesz decomposition theorem (see, for example, [24, Section 2.2]), the bilaterally invariant subspace $\mathcal{N}$ can be written as $\mathcal{N} = \mathcal{N}_0 \cap \mathcal{N}_1$, where $\sigma(\mathbf{S}[\mathcal{N}_0]) = \{\lambda_0\}$ and $\sigma(\mathbf{S}[\mathcal{N}_1]) = \sigma(\mathbf{S}[\mathcal{N}]) \backslash \{\lambda_0\}$. We shall prove that $\mathbf{S}[\mathcal{N}_0]$ cannot have one-point spectrum, which does it. In other words, after replacing $\mathcal{N}$ by $\mathcal{N}_0$ and after a rotation of the circle, we may assume that $\mathcal{N}$ has $\sigma(\mathbf{S}[\mathcal{N}]) = \{1\}$. We use the holomorphic functional calculus (again, in fact) to define the operator $\log \mathbf{S}[\mathcal{N}]$, with spectrum $\{0\}$. This permits us to form the expression

$$\mathbf{S}[\mathcal{N}]^\zeta = \exp(\zeta \log \mathbf{S}[\mathcal{N}]), \quad \zeta \in \mathbb{C},$$

which amounts to an entire function of zero exponential type taking values in the space of operators on $\mathfrak{L}^2(\mathbb{T}, \omega)/\mathcal{N}$. We identify the quotient space $\mathfrak{L}^2(\mathbb{T}, \omega)/\mathcal{N}$ with the subspace $\mathfrak{L}^2(\mathbb{T}, \omega) \ominus \mathcal{N}$ of vectors perpendicular to $\mathcal{N}$. Let $\mathbf{P} : \mathfrak{L}^2(\mathbb{T}, \omega) \to \mathfrak{L}^2(\mathbb{T}, \omega) \ominus \mathcal{N}$ stand for the orthogonal projection, and denote by

$$\langle f, g \rangle = \sum_{n \in \mathbb{Z}} \hat{f}(n) \overline{\hat{g}(n)} \Omega(n)$$

the sesquilinear form on the Hilbert space $\mathfrak{L}^2(\mathbb{T}, \omega)$, where for a given element $f \in \mathfrak{L}^2(\mathbb{T}, \omega)$, $\hat{f}(n)$ are the corresponding Laurent coefficients for the formal series expansion

$$f(z) = \sum_{n \in \mathbb{Z}} \hat{f}(n) z^n, \quad z \in \mathbb{T}.$$

We recall the agreed convention that $\Omega(n) = 1/\Omega(-n-1)$ for $n < 0$. The operator $\mathbf{S}[\mathcal{N}]$ is identified with $\mathbf{PS}$. For $f, \phi \in \mathfrak{L}^2(\mathbb{T}, \omega)$, we consider the function

$$E_{f,\phi}(\zeta) = \langle \mathbf{S}[\mathcal{N}]^\zeta \mathbf{P} f, \mathbf{P} \phi \rangle, \quad \zeta \in \mathbb{C},$$

which is entire and of zero exponential type. Using the invariance of $\mathcal{N}$, we see that $\mathbf{PSP} = \mathbf{PS}$; more generally, for integers n, we have

$$\mathbf{S}[\mathcal{N}]^n \mathbf{P} = (\mathbf{PS})^n \mathbf{P} = \mathbf{PS}^n.$$

holds, so that as we plug in $f = 1$ and $\phi \in \mathfrak{L}^2(\mathbb{T}, \omega) \ominus \mathcal{N}$ into the above expression, we obtain, for $n \in \mathbb{Z}$,

$$E_{1,\phi}(n) = \langle \mathbf{S}[\mathcal{N}]^n \mathbf{P} 1, \phi \rangle = \langle \mathbf{PS}^n 1, \phi \rangle = \langle z^n, \phi \rangle = \Omega(n) \overline{\hat{\phi}(n)}.$$

On the other hand, if we instead fix $\phi = 1$ and let $f \in \mathfrak{L}^2(\mathbb{T}, \omega) \ominus \mathcal{N}$ vary, we obtain

$$E_{f,1}(\zeta) = \langle \mathbf{S}[\mathcal{N}]^\zeta f, \mathbf{P}1 \rangle = \langle \mathbf{S}[\mathcal{N}]^\zeta f, 1 \rangle,$$

and hence

$$E_{f,1}(n) = \langle z^n f, 1 \rangle = \Omega(0)\hat{f}(-n), \quad n \in \mathbb{Z}.$$

Now, the entire function $F(\zeta) = E_{f,1}(-\zeta)E_{1,f}(\zeta)$ has zero exponential type, and by the above, it is l^1-summable at the integers:

$$\sum_{n\in\mathbb{Z}} |F(n)| = \sum_{n\in\mathbb{Z}} |E_{f,1}(-n)E_{1,f}(n)|$$
$$= \Omega(0) \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2 \Omega(n) = \Omega(0)||f||^2_{\mathfrak{L}^2(\mathbb{T},\omega)} < +\infty.$$

By the classical Cartwright theorem, the function F is bounded along the real line, and in view of the growth restriction which is a consequence of F having zero exponential type, the Phragmén–Lindelöf principle forces F to be constant. That constant must then be 0, in view of the convergence of the above sum. Then at least one of the entire functions $E_{f,1}$ and $E_{1,f}$ must vanish identically. In either case, $\hat{f}(n) = 0$ for all integers n, that is, $f = 0$. Since f was arbitrary in $\mathfrak{L}^2(\mathbb{T}, \omega) \ominus \mathcal{N}$, we obtain $\mathcal{N} = \mathfrak{L}^2(\mathbb{T}, \omega)$, and hence $\sigma(\mathbf{S}[\mathcal{N}]) = \emptyset$, as desired. $\square$

4. The proof of Theorem 2.4

We start with a function $f \in B^2(\mathbb{D}, \omega)$ as in Theorem 2.2. So, we know that $[f] \neq B^2(\mathbb{D}, \omega)$ and

$$\int_{\mathbb{D}} \int_{\mathbb{D}} \frac{1}{|f(z)|^2} \frac{|f(z) - f(w)|^2}{|z - w|^2} \omega(z)\omega(w)\, dm_{\mathbb{D}}(z)\, dm_{\mathbb{D}}(w) < +\infty. \tag{4.1}$$

Recall that $\mathbf{S}$ is the shift operator on $\mathfrak{L}^2(\mathbb{T}, \omega)$, and consider the induced operator $\mathbf{T}$ on $B^2(\mathbb{D}, \omega)/[f]$. Since $\mathbf{S}$ is a contraction on $B^2(\mathbb{D}, \omega)$, we have $||\mathbf{T}|| \leqslant 1$. The function f has no zeros in the unit disk, hence, the operators $\lambda - \mathbf{T}$ are invertible for $\lambda \in \mathbb{D}$,

$$(\lambda - \mathbf{T})^{-1}(g + [f]) = \frac{g(z) - f(z)g(\lambda)/f(\lambda)}{\lambda - z} + [f], \tag{4.2}$$

the spectrum of $\mathbf{T}$ lies on the unit circle,

$$\lim_{n\to+\infty} ||\mathbf{T}^{-n}||^{1/n} \leqslant 1,$$

and

$$(\lambda - \mathbf{T})^{-1} = -\sum_{n>0} \lambda^{n-1}\mathbf{T}^{-n}, \quad \lambda \in \mathbb{D}. \tag{4.3}$$

Next, we produce a (uniquely defined) continuous linear operator $\mathbf{R} : \mathfrak{L}^2(\mathbb{T}, \omega) \to B^2(\mathbb{D}, \omega)/[f]$ coinciding with the canonical projection $x \mapsto x + [f]$ on $B^2(\mathbb{D}, \omega)$ and such that $\mathbf{T} \circ \mathbf{R} = \mathbf{R} \circ \mathbf{S}$. Then,

$$\mathscr{M} \stackrel{\text{def}}{=} [f] \subset \ker \mathbf{R}.$$

Moreover, $\ker \mathbf{R}$ is a non-trivial bilaterally invariant subspace of $\mathfrak{L}^2(\mathbb{T}, \omega)$. Indeed,

$$\mathbf{R}g = 0 \;\Leftrightarrow\; \mathbf{T}\mathbf{R}g = 0 \;\Leftrightarrow\; \mathbf{R}\mathbf{S}g = 0.$$

By the definition of $\mathbf{R}$, we have $\ker \mathbf{R} \cap B^2(\mathbb{D}, \omega) = [f]$. It follows that f is non-cyclic in $\mathfrak{L}^2(\mathbb{T}, \omega)$ with respect to the bilateral shift. Since $z^n - \mathbf{T}^n(1 + [f]) \subset \mathscr{M}_{\mathbb{T}},\ n \in \mathbb{Z}$, for every $x \in \mathfrak{L}^2(\mathbb{T}, \omega)$, we obtain $x - \mathbf{R}x \subset \mathscr{M}_{\mathbb{T}}$, hence $x \in B^2(\mathbb{D}, \omega) + \mathscr{M}_{\mathbb{T}}$. Thus, $\ker \mathbf{R} = \mathscr{M}_{\mathbb{T}}$ and $\mathfrak{L}^2(\mathbb{T}, \omega) = B^2(\mathbb{D}, \omega) + \mathscr{M}_{\mathbb{T}}$.

The operator $\mathbf{R}$ is already defined on $B^2(\mathbb{D}, \omega)$. Furthermore, we have $\mathbf{R}(z^{-n}) = \mathbf{T}^{-n}(1 + [f]),\ n \in \mathbb{Z}$, and $\mathbf{R}$ extends by linearity to the linear span of $\{z^{-n}\}_{n=1}^{+\infty}$. We are to verify that $\mathbf{R}$ extends continuously to $\mathfrak{L}^2(\mathbb{T}, \omega) \ominus B^2(\mathbb{D}, \omega)$. For every polynomial $q(z) = \sum_{n=1}^{+\infty} a_n z^{-n}$ in the variable z^{-1}, we have

$$\mathbf{R}(q) = \sum_{n=1}^{+\infty} a_n \mathbf{T}^{-n}(1 + [f]),$$

the norm of which we estimate as follows:

$$\begin{aligned}
||\mathbf{R}(q)||_{B^2(\mathbb{D},\omega)/[f]} &\leqslant \sum_{n=1}^{+\infty} |a_n| ||\mathbf{T}^{-n}(1 + [f])||_{B^2(\mathbb{D},\omega)/[f]} \\
&\leqslant \left\{ \sum_{n=1}^{+\infty} \frac{|a_n|^2}{\Omega(n-1)} \right\}^{1/2} \left\{ \sum_{n=1}^{+\infty} ||\mathbf{T}^{-n}(1 + [f])||^2_{B^2(\mathbb{D},\omega)/[f]} \Omega(n-1) \right\}^{1/2} \\
&= ||q||_{\mathfrak{L}^2(\mathbb{T},\omega)} \left\{ \sum_{n=1}^{+\infty} ||\mathbf{T}^{-n}(1 + [f])||^2_{B^2(\mathbb{D},\omega)/[f]} \Omega(n-1) \right\}^{1/2}.
\end{aligned}$$

As a consequence, we have

$$||\mathbf{R}|_{\mathfrak{L}^2(\mathbb{T},\omega) \ominus B^2(\mathbb{D},\omega)}||^2 \leqslant \sum_{n=1}^{+\infty} ||\mathbf{T}^{-n}(1 + [f])||^2_{B^2(\mathbb{D},\omega)/[f]} \Omega(n-1).$$

To estimate the right-hand side of this inequality, we apply the identity

$$\begin{aligned} A &\stackrel{\text{def}}{=} \int_{\mathbb{D}} ||(\lambda - \mathbf{T})^{-1}(1 + [f])||^2_{B^2(\mathbb{D},\omega)/[f]} \omega(\lambda)\, dm_{\mathbb{D}}(\lambda) \\ &= \int_{\mathbb{D}} \left|\left| \sum_{n=1}^{+\infty} \lambda^{n-1}\mathbf{T}^{-n}(1+[f]) \right|\right|^2_{B^2(\mathbb{D},\omega)/[f]} \omega(\lambda)\, dm_{\mathbb{D}}(\lambda) \\ &= \sum_{n=1}^{+\infty} ||\mathbf{T}^{-n}(1+[f])||^2_{B^2(\mathbb{D},\omega)/[f]} \int_{\mathbb{D}} |\lambda|^{2n-2}\omega(\lambda)\, dm_{\mathbb{D}}(\lambda) \\ &= \sum_{n=1}^{+\infty} ||\mathbf{T}^{-n}(1+[f])||^2_{B^2(\mathbb{D},\omega)/[f]} \Omega(n-1), \end{aligned}$$

which follows from (2.4), (4.3), and the fact that for radial ω and for integers $n \neq k$,

$$\int_{\mathbb{D}} \lambda^n \bar{\lambda}^k \omega(\lambda)\, dm_{\mathbb{D}}(\lambda) = 0.$$

Thus, to prove the theorem, we need only to verify that $A < +\infty$. To estimate the norm of $(\lambda - \mathbf{T})^{-1}$ does not seem very promising; fortunately, we do not need this. By (4.2),

$$(\lambda - \mathbf{T})^{-1}(1+[f]) = \frac{1 - f(z)/f(\lambda)}{\lambda - z} + [f],$$

hence we have

$$||(\lambda - \mathbf{T})^{-1}(1+[f])||_{B^2(\mathbb{D},\omega)/[f]} \leqslant \left|\left| \frac{1 - f(z)/f(\lambda)}{\lambda - z} \right|\right|_{B^2(\mathbb{D},\omega)},$$

and finally, we get

$$\begin{aligned} A &\leqslant \int_{\mathbb{D}} \left|\left| \frac{1 - f(z)/f(\lambda)}{\lambda - z} \right|\right|^2_{B^2(\mathbb{D},\omega)} \omega(\lambda)\, dm_{\mathbb{D}}(\lambda) \\ &= \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{1}{|f(\lambda)|^2} \frac{|f(\lambda) - f(z)|^2}{|\lambda - z|^2} \omega(\lambda)\omega(z)\, dm_{\mathbb{D}}(\lambda)\, dm_{\mathbb{D}}(z) < +\infty, \end{aligned}$$

due to inequality (4.1). □

5. A Phragmén–Lindelöf-type estimate for functions in the disk

Here, we will prove that if an analytic function f is bounded in the unit disk, and satisfies a kind of ℓ^p bound on a sequence of points tending to the unit circle along a collection of circles, then we can bound the analytic function in modulus by a given radial function, and the bound is independent of the H^∞-norm of f.

Later on, we will need the *pseudo-hyperbolic metric* for the unit disk, as given by

$$\rho_{\mathbb{D}}(z,w) = \left|\frac{z-w}{1-z\bar{w}}\right|, \quad (z,w)\in\mathbb{D}^2.$$

Fix a constant $0<\kappa<1$, and let $\{r_n\}_n$ be a sequence of numbers in the interval $[\frac{4}{5},1)$ tending to 1 rather quickly. For every $n=1,2,3,\ldots$, let N_n be the integer that satisfies

$$N_n \leqslant \frac{\kappa}{1-r_n} < N_n+1.$$

For each integer k with $0\leqslant k<N_n$, set

$$w_{n,k} = r_n e^{2\pi ik/N_n},$$

and select by some process (which will be explained later on in Section 9) a point $z_{n,k}$ from each disk

$$\mathscr{D}_{n,k} = \{z\in\mathbb{D}\colon |z-w_{n,k}|<(1-r_n)^2\}.$$

Finally, consider a discrete measure μ equal to the sum of point masses of size $1/(n^2N_n)$ at the points $z_{n,k}$, $0\leqslant k<N_n$, $n=1,2,3,\ldots$. In this section, we show that for $0<p<+\infty$, the set of so-called *analytic bounded point evaluations* for $P^p(\mu)$—the closure of the polynomials in $L^p(\mu)$—coincides with $\mathbb{D}$, and, more generally, we supply effective estimates for the constants $C_p(z)$ in the inequality

$$|f(z)|^p \leqslant C_p(z)\sum_{n,k}\frac{|f(z_{n,k})|^p}{n^2N_n}, \quad f\in H^\infty.$$

Since we use only information on a discrete set in $\mathbb{D}$, we cannot apply the standard technique of subharmonic functions. Instead, we make use of the Lagrange interpolation formula; the same method was applied earlier in [21]. Let B_n be the finite Blaschke product

$$B_n(z) = \prod_{k=0}^{N_n-1}\frac{z-z_{n,k}}{1-\bar{z}_{n,k}z};$$

an explicit calculation reveals that for $j=0,1,2,\ldots,N_n-1$, we have

$$(1-|z_{n,j}|^2)|B'(z_{n,j})| = \prod_{k\colon k\neq j}\left|\frac{z_{n,j}-z_{n,k}}{1-\bar{z}_{n,k}z_{n,j}}\right| = \prod_{k\colon k\neq j}\rho_{\mathbb{D}}(z_{n,j},z_{n,k}).$$

Lemma 5.1. *For some positive constant* $c(\kappa)$, *depending only on* κ, *we have*

(a) $|B_n{}'(z_{n,k})| \geqslant c(\kappa) N_n,\ 0 \leqslant k < N_n.$

If $0 < r < 1,\ \varepsilon > 0$ *are fixed, and* r_n *is sufficiently close to* 1, *then*

(b) $|\log|B_n(z)| + \kappa| \leqslant \varepsilon,\ |z| \leqslant r.$

Proof. Let A_n be the finite Blaschke product

$$A_n(z) = \prod_k \frac{z - w_{n,k}}{1 - \bar{w}_{n,k} z} = \frac{z^{N_n} - r_n^{N_n}}{1 - r_n^{N_n} z^{N_n}},$$

which is quite analogous to B_n. In view of our assumptions on the numbers N_n and the finite sequence $\{w_{n,k}\}_k$, we have

$$c_1 N_n \leqslant |A_n{}'(w_{n,j})|, \quad j = 0, 1, 2, \ldots, N_n - 1, \tag{5.1}$$

$$\lim_{n \to +\infty} |A_n(z)| = \lim_{n \to +\infty} r_n^{N_n} = e^{-\kappa}, \quad |z| < 1, \tag{5.2}$$

where $c_1 = c_1(\kappa)$ is a positive constant. The function $\log|B_n/A_n| = \log|B_n| - \log|A_n|$ equals the sum of the functions

$$s_{n,k}(z) = \log \rho_{\mathbb{D}}(z, z_{n,k}) - \log \rho_{\mathbb{D}}(z, w_{n,k})$$

over $k = 0, 1, \ldots N_n - 1$. On the complement in $\mathbb{D}$ of the pseudohyperbolic disk of radius $\frac{1}{2}$ centered at $w_{n,k}$, we have

$$|s_{n,k}(z)| \leqslant c_2(1 - r_n)$$

for some positive constant $c_2 = c_2(\kappa)$, whereas on the circle $r\mathbb{T}$, we have

$$|s_{n,k}(z)| = o(1 - r_n) \quad \text{as } r_n \to 1;$$

along the unit circle $\mathbb{T}$, on the other hand, $s_{n,k} = 0$. Summing up all the terms, we obtain, after an application of the maximum principle, that

$$|\log|B_n(z)| - \log|A_n(z)|| = o(1) \quad \text{as } r_n \to 1 \text{ for } |z| \leqslant r, \tag{5.3}$$

and that for each $j = 0, 1, 2, \ldots, N_n - 1$,

$$|\log|B_n{}'(z_{n,j})| - \log|A_n{}'(w_{n,j})|| < c_3, \tag{5.4}$$

with some positive constant $c_3 = c_3(\kappa)$ independent of the radius r_n. We get both assertions (a) and (b) from the estimates (5.1), (5.2), (5.3), and (5.4). □

Let card be the function that computes the number of points in a given set (it stands for *cardinality*).

Lemma 5.2. *For* $n = 1, 2, 3, \ldots$, *put* $\mathcal{N}_n = \{0, \ldots, N_n - 1\}$, *and take a subset* $\mathcal{N}_n^*$ *of* $\mathcal{N}_n$. *Let*

$$\sigma_n = \frac{\operatorname{card}(\mathcal{N}_n \backslash \mathcal{N}_n^*)}{N_n},$$

be the density of $\mathcal{N}_n^*$ *in* $\mathcal{N}_n$, *and define*

$$B_n^*(z) = \prod_{k \in \mathcal{N}_n^*} \frac{z - z_{n,k}}{1 - \bar{z}_{n,k} z}. \tag{5.5}$$

If r *and* ε, *with* $0 < r < 1$ *and* $0 < \varepsilon < +\infty$ *are fixed, and* r_n *is sufficiently close to* 1, *then*

$$-\varepsilon \leqslant \log |B_n^*(z)| + \kappa \leqslant \frac{3\kappa\sigma_n}{1 - |z|} + \varepsilon, \quad |z| \leqslant r.$$

Proof. The assertion for $\sigma_n = 0$ follows from Lemma 5.1(b). It remains to note that for $|z| \in [0, r]$ and for r_n sufficiently close to 1,

$$\begin{aligned} 0 \leqslant \log |B_n^*(z)| - \log |B_n(z)| &= - \sum_{k \in \mathcal{N}_n \backslash \mathcal{N}_n^*} \log \rho_{\mathbb{D}}(z, z_{n,k}) \\ &\leqslant \log \frac{1}{\rho_{\mathbb{D}}(|z|, r_n)} \operatorname{card}(\mathcal{N}_n \backslash \mathcal{N}_n^*) \leqslant \frac{3\kappa\sigma_n}{1 - |z|}. \end{aligned}$$

The proof is complete. □

Proposition 5.3. *Suppose that the radii* r_n *tend to* 1 *sufficiently rapidly*. *If* f *is a function that is bounded and analytic in the unit disk*, *and if for some* $0 < p < +\infty$,

$$\sum_{k=0}^{N_n - 1} |f(z_{n,k})|^p \leqslant n^2 N_n, \quad n = 1, 2, 3, \ldots, \tag{5.6}$$

then, *for some constant* $c = c(\kappa, p)$ *independent of* f,

$$|f(z)| \leqslant c \exp\left(\frac{1}{1 - |z|}\right), \quad z \in \mathbb{D}.$$

Proof. For each $n = 1, 2, 3, \ldots$, we make the choice

$$\mathcal{N}_n^* = \{k \in \mathcal{N}_n : |f(z_{n,k})| \leqslant n^{4/p}\}.$$

Let the associated finite Blaschke product B_n^* be given by (5.5). If r_n tends to 1 sufficiently rapidly, then by Lemma 5.2 and the limit relationship (5.2), we have

$$-\frac{1}{n^2} \leqslant \log|B_n^*(z)| - N_n \log r_n \leqslant \frac{4\kappa\sigma_n}{1-|z|} + \frac{1}{n^2}, \quad |z| \leqslant r_{n-1}. \tag{5.7}$$

Using (5.6) and a weak-type estimate, we obtain

$$\sigma_n = \frac{\operatorname{card}(\mathcal{N}_n \backslash \mathcal{N}_n^*)}{N_n} \leqslant \frac{1}{n_2}. \tag{5.8}$$

Consider the set

$$\Gamma^* = \{z_{n,k}\colon\ k \in \mathcal{N}_n^*,\ n = 1, 2, 3, \dots\},$$

which is a discrete lattice-like subset of $\mathbb{D}$. In view of (5.7) and (5.8), the infinite product

$$B^*(z) = \prod_{n=1}^{+\infty} (r_n^{-N_n} B_n^*(z)),$$

converges uniformly on compact subsets of $\mathbb{D}$. Note that by (5.2), $r_n^{-N_n} \to e^{\kappa}$ as $n \to +\infty$. If we use the obvious estimate that $|B_n^*| < 1$ on $\mathbb{D}$ for small values of n, and the estimates (5.7) and (5.8) for large values of n, we easily establish that for some positive constant $c_1 = c_1(\kappa)$ that depends only on κ, we have

$$|B^*(z)| \leqslant c_1 \exp\left(\frac{1}{5(1-|z|)}\right), \quad z \in \mathbb{D}. \tag{5.9}$$

Moreover, Lemma 5.1—modified to apply to B_n^* instead of B_n, using that $|B_n| \leqslant |B_n^*|$—shows that if the radii r_n tend to 1 sufficiently rapidly as $n \to +\infty$, then

$$|B^*(z)| \geqslant c_2 e^{\kappa n} \quad \text{for } |z| = 1 - 2(1 - r_n), \tag{5.10}$$

$$|(B^*)'(z_{n,k})| \geqslant c_3 N_n e^{\kappa n} \quad \text{for } k \in \mathcal{N}_n^*,\ n = 1, 2, 3, \dots, \tag{5.11}$$

where $c_2 = c_2(\kappa)$ and $c_3 = c_3(\kappa)$ are two positive constants. By the Cauchy residue theorem, if $0 < r < 1$ and $r\mathbb{T} \cap \Gamma^* = \emptyset$, then

$$-\frac{f(z)}{B^*(z)} + \sum_{z_{n,k} \in \Gamma^* \cap r\mathbb{D}} \frac{f(z_{n,k})}{(B^*)'(z_{n,k})(z - z_{n,k})} = \frac{1}{2\pi i} \int_{r\mathbb{T}} \frac{f(\zeta)}{B^*(\zeta)(z-\zeta)} d\zeta, \quad |z| < r.$$

We use estimate (5.10) on circles $r\mathbb{T}$ and let $r \to 1$, and realize that the residue integral on the right-hand side then tends to 0, as it is given that f is bounded in $\mathbb{D}$.

It follows that

$$\frac{f(z)}{B^*(z)} = \sum_{z_{n,k} \in \Gamma^*} \frac{f(z_{n,k})}{(B^*)'(z_{n,k})(z - z_{n,k})}, \quad z \in \mathbb{D},$$

where it is conceivable that the convergence is conditional. Taking absolute values, we arrive at

$$\left|\frac{f(z)}{B^*(z)}\right| \leqslant \sum_{z_{n,k} \in \Gamma^*} \frac{|f(z_{n,k})|}{|(B^*)'(z_{n,k})(z - z_{n,k})|}, \quad z \in \mathbb{D}.$$

Hence, by (5.11), for some positive constant $c_4 = c_4(\kappa, p)$ independent of z and f, we have

$$\left|\frac{f(z)}{B^*(z)}\right| \leqslant \frac{c_4}{d_{\mathbb{C}}(z, \Gamma^*)},$$

where $d_{\mathbb{C}}$ is the usual Euclidean distance function. If we make a rather crude estimate of the above right-hand side, and multiply by $|B^*|$, then, by (5.9), for some positive constant $c_5 = c_5(\kappa, p)$, we get

$$|f(z)| \leqslant c_5 \exp\left(\frac{1}{4(1 - |z|)}\right),$$

provided that

$$|z| \in [0, 1) \setminus \bigcup_n \left[1 - 2(1 - r_n), 1 - \frac{1}{2}(1 - r_n)\right].$$

An application of the maximum principle in each "complementary" annulus

$$|z| \in \left[1 - 2(1 - r_n), 1 - \frac{1}{2}(1 - r_n)\right]$$

completes the proof. □

6. The construction of harmonic functions: tools from Convex Analysis

In this section we start with the weight ω, pass to an associated function Λ on the positive half-line and then regularize it.

Suppose that a decreasing radial weight ω satisfies (2.1), and put

$$\Theta(s) = \log \frac{1}{\omega(1 - s)}, \quad 0 < s < 1.$$

We want to study the behavior of $\omega(t)$ for t near 1, and hence that of $\Theta(s)$ for s near 0. In order to study that behavior in detail, we make an exponential change of coordinates, and set

$$\Lambda(x) = \log \Theta(e^{-x}) = \log\log \frac{1}{\omega(1-e^{-x})}, \quad 0 \leqslant x < +\infty;$$

the parameter

$$x = \log \frac{1}{1-t}$$

equals approximately the hyperbolic distance in $\mathbb{D}$ from 0 to t. The function $\Lambda(x)$ is increasing in x, and it grows at least exponentially:

$$\lim_{x\to+\infty} e^{-\varepsilon_0 x}\Lambda(x) = +\infty, \tag{6.1}$$

where $\varepsilon_0 > 0$ is the same parameter as in (2.1). We need the following lemma from Convex Analysis.

Lemma 6.1. *Let $\Lambda : [0,+\infty) \to (0,+\infty)$ be a C^1-smooth increasing function satisfying* (6.1) *for some ε_0, $0 < \varepsilon_0 \leqslant 1$. Then there exists another C^1-smooth increasing convex function $\lambda : [0,+\infty) \to (0,+\infty)$, such that*

(a) *λ is a minorant of Λ : $\lambda(x) \leqslant \Lambda(x)$ holds on $[0, +\infty)$,*

(b) *$e^{\varepsilon_0 x} \leqslant \lambda(x)$ holds on some interval $[A, +\infty)$, with $0 \leqslant A < +\infty$,*

(c) *$\lambda'(x) \leqslant \lambda(x)^{3/2}$ holds on $[0, +\infty)$,*

and there is a sequence of numbers $\{x_n\}_n$ tending to $+\infty$, such that

(d) *$\lambda(x_n) = \Lambda(x_n)$, for all n,*

(e) *$\varepsilon_0\lambda(x_n) \leqslant \lambda'(x_n)$, for all n, and*

(f) *$\lambda(x_n) + (x-x_n)\lambda'(x_n) + \frac{1}{4}\varepsilon_0(x-x_n)^2\lambda'(x_n) \leqslant \lambda(x)$ for $x \in [0, +\infty)$.*

Proof. Put $Q = \sqrt{\Lambda}$, and write $q = \sqrt{\lambda}$, where λ is the function we seek. Conditions (a)–(e) then correspond to

(a′) q is a minorant of Q on $[0,+\infty)$,

(b′) $e^{\varepsilon_0 x/2} \leqslant q(x)$ holds on $[A,+\infty)$,

(c′) $q'(x) \leqslant \frac{1}{2}q(x)^2$ on $[0,+\infty)$,

(d′) $q(x_n) = Q(x_n)$ for all n,

(e′) $\frac{1}{2}\varepsilon_0 q(x_n) \leqslant q'(x_n)$ for all n.

We shall find a convex q with the above properties, which means that $\lambda = q^2$ is convex as well. Note that by the assumption on Λ,

$$\lim_{x\to+\infty} e^{-\varepsilon_0 x/2}\, Q(x) = +\infty, \tag{6.2}$$

so that if we forget about property (c′), we can just take q to be equal to the greatest convex minorant q_0 of Q; it is not hard to check that all the properties (a′), (b′), (d′), and (e′) are fulfilled for a sequence $\{x_n\}_n$. To get also (c′), we apply an iterative procedure.

As a consequence of (6.2), we get

$$\lim_{x\to+\infty} e^{-\varepsilon_0 x/2} q_0(x) = +\infty. \tag{6.3}$$

This is so because the function q_0 must touch Q along an unbounded closed set, which we denote by $\mathscr{E}$. Note that q_0 is affine on each open interval in the complement of $\mathscr{E}$. Changing q_0 a little on a small interval with the origin as the left end point, if necessary, we can guarantee that ${q_0}'(0) \leqslant \frac{1}{3} q_0(0)^2$. Set $a_0 = 0$. Our iterative procedure runs as follows. For every $k = 0, 1, 2, 3, \ldots$, we start with a convex increasing minorant q_k of Q such that ${q_k}'(a_k) \leqslant \frac{1}{3} q_k(a_k)^2$, ${q_k}'(x) \leqslant \frac{1}{2} q_k(x)^2$ for $x \leqslant a_k$, and $q_k(x) = q_0(x)$ for $x \geqslant a_k$.

If

$$ {q_k}'(x) \leqslant \frac{1}{2} q_k(x)^2 $$

on the whole interval $[a_k, +\infty)$, then we are done, because we pick $q = q_k$. If it is not so, there exists of course a point $x \in (a_k, +\infty)$ with

$$ {q_k}'(x) > \frac{1}{2} q_k(x)^2. $$

Let $b_k \geqslant a_k$ be the infimum of all such points x. Let (c_k, d_k) be the maximal interval in $(a_k, +\infty)$ such that $b_k \in (c_k, d_k)$ and

$$ {q_k}'(x) > \frac{1}{3} q_k(x)^2, \quad x \in (c_k, d_k). $$

Note that ${q_k}'(c_k) = \frac{1}{3} q_k(c_k)^2$. We claim that c_k belongs to the set $\mathscr{E}$ we defined earlier. In fact, every small interval to the right of c_k contains infinitely many points of $\mathscr{E}$. The reason is that away from $\mathscr{E}$, q_k is affine, and since q_k is increasing, we get

$$ \frac{q_k(c_k)^2}{3} \leqslant \frac{q_k(x)^2}{3} < {q_k}'(x) = {q_k}'(c_k) = \frac{q_k(c_k)^2}{3} $$

if for some x with $c_k < x < d_k$, we have $(c_k, x) \cap \mathscr{E} = \emptyset$; this is impossible. We shall now alter the function q_k to the right of the point c_k. Let f_k solve the initial value problem

$$ {f_k}'(x) = \frac{1}{3} f_k(x)^2, \quad f_k(c_k) = q_k(c_k); $$

we see that f_k explodes in finite time. As a matter of fact, an explicit calculation reveals that

$$f_k(x) = \frac{3}{c_k + \frac{3}{q_k(c_k)} - x}, \quad x \in I_k,$$

where $I_k = [c_k, c_k + \frac{3}{q_k(c_k)})$, and the explosion point is the right end point of the indicated interval. Now, take as q_{k+1} the greatest convex minorant of the function that equals q_k on $[0, +\infty)\backslash I_k$ and equals with $\min\{f_k, q_k\}$ on I_k. The function f_k is convex and increasing on I_k, and to the right of c_k, it initially drops below the convex function q_k, but then after a while, it grows above it again, finally to explode at the right end point of I_k. It follows that q_{k+1} is increasing, $q_{k+1} = q_k$ on the interval $[0, c_k]$, $q_{k+1} = f_k$ on some interval $[c_k, e_k] \subset I_k$, q_{k+1} is affine on some interval $[e_k, a_{k+1}]$, at the right end point of which q_{k+1} touches q_k, and $q_{k+1} = q_k$ on the interval $[a_{k+1}, +\infty)$. Since q_{k+1} is increasing, we have

$$\frac{{q_{k+1}}'(x)}{q_{k+1}(x)^2} = \frac{{q_{k+1}}'(e_k)}{q_{k+1}(x)^2} \leqslant \frac{{q_{k+1}}'(e_k)}{q_{k+1}(e_k)^2} = \frac{{f_k}'(e_k)}{f_k(e_k)^2} = \frac{1}{3}, \quad x \in [e_k, a_{k+1}].$$

Hence,

$$a_{k+1} \geqslant d_k. \tag{6.4}$$

We get that q_{k+1} is a convex increasing minorant of Q, with

$${q_{k+1}}'(a_{k+1}) \leqslant \frac{1}{3} q_{k+1}(a_{k+1})^2,$$

and

$${q_{k+1}}'(x) \leqslant \frac{1}{2} q_{k+1}(x)^2 \text{ for } x \leqslant a_{k+1}, \quad q_{k+1}(x) = q_0(x) \text{ for } x \geqslant a_{k+1}.$$

If our iterative procedure does not stop on a finite step (that is, if ${q_k}'(x) > \frac{1}{2} q_k(x)^2$ on an unbounded subset of the interval $[0, +\infty)$), then we put $q = \lim_{k \to \infty} q_k$. Next we verify the properties (a′)−(e′) for q.

Note that $\lim a_k = \infty$. Indeed, by the definition of b_k, (c_k, d_k), and by property (6.4) of a_k, we have $a_k \leqslant c_k < b_k < d_k \leqslant a_{k+1}$. If a_k were to tend to a finite number a_∞ as $k \to +\infty$, then $c_k \to a_\infty$, $b_k \to a_\infty$, as $k \to +\infty$ as well. By the definitions of these points,

$${q_0}'(c_k) = \frac{1}{3} q_0(c_k)^2, \quad {q_0}'(b_k) = \frac{1}{2} q_0(b_k)^2.$$

In the limit, we obtain a contradiction, which does it.

It follows from our construction that $q = q_k$ on $[0, a_k]$, $k = 1, 2, 3, \ldots$, and that q is a convex minorant of Q. Hence, (a′) and (c′) follow. The graph of function q touches that of q_0 except on the intervals (c_k, a_{k+1}), so that if q does not grow too slowly on those intervals, we have

$$\lim_{x\to+\infty} e^{-\varepsilon_0 x/2} q(x) = +\infty.$$

We now show that the above holds in general. We know that the function $e^{-\varepsilon_0 x/2} q_{k+1}(x)$ is sufficiently big for $x \in \{c_k, a_{k+1}\}$. Also, the function

$$e^{-\varepsilon_0 x/2} f_k(x) = e^{-\varepsilon_0 x/2} q_{k+1}(x), \quad x \in [c_k, e_k],$$

increases on $[c_k, e_k]$ (at least for big k). Since q_k is affine on $[e_k, a_{k+1}]$, we obtain $q(x + e_k) = q_k(e_k) + x q_k{}'(e_k)$, $x \in [0, a_{k+1} - e_k]$. Furthermore, a simple calculation yields that the function

$$t \mapsto e^{-\varepsilon_0 (t+e_k)/2} [q_k(e_k) + t q_k{}'(e_k)]$$

either is monotonic or has just one local maximum on $[0, +\infty)$. As a consequence,

$$e^{-\varepsilon_0 x/2} q_k(x) \geqslant \min\{e^{-\varepsilon_0 x/2} q_k(x) \colon x \in \{c_k, a_{k+1}\}\}, \quad e_k \leqslant x \leqslant a_{k+1},$$

which finishes the proof of (b$'$).

We turn to verifying that there exists a sequence $\{x_n\}_n$ of points in $\mathscr{E}$ tending to $+\infty$ for which

$$q(x_n) = q_0(x_n), \quad \frac{1}{2}\varepsilon_0 q(x_n) < q'(x_n). \tag{6.5}$$

The points c_k are in $\mathscr{E}$, and they may work as x_n, provided that there are infinitely many of them. If there is only a finite supply of c_k, then $q(x) = q_0(x)$ for all sufficiently big x. If we then also cannot find arbitrarily big $x_n \in \mathscr{E}$ satisfying (6.5), then the reason is that $q_0{}'(x) \leqslant \frac{1}{2}\varepsilon_0 q_0(x)$ for all x in $\mathscr{E}$ that are sufficiently large, and since q_0 is affine outside $\mathscr{E}$, we get a contradiction with (6.3). Thus, (d$'$) and (e$'$) follow.

As we made clear before, we pick $\lambda(t) = q(t)^2$. We need only verify property (f). Let x_n be as in (6.5), and take an arbitrary $x \in [0, +\infty)$. We then have

$$q(x_n) + (x - x_n) q'(x_n) \leqslant q(x)$$

and hence

$$\begin{aligned}\lambda(x) &\geqslant q(x_n)^2 + 2(x - x_n) q(x_n) q'(x_n) + (x - x_n)^2 q'(x_n)^2 \\ &= \lambda(x_n) + (x - x_n)\lambda'(x_n) + (x - x_n)^2 \lambda'(x_n) \frac{q'(x_n)}{2q(x_n)} \\ &\geqslant \lambda(x_n) + (x - x_n)\lambda'(x_n) + \frac{1}{4}\varepsilon_0 (x - x_n)^2 \lambda'(x_n).\end{aligned}$$

The proof is complete. $\square$

7. The construction of harmonic functions: building blocks

We start with the weight ω, pass on to the associated function Θ and Λ on the interval (0,1] and the positive half-line, respectively, and then apply Lemma 6.1 to Λ in order to obtain the minorant λ to Λ along with the sequence $\{x_n\}_n$. We recall the relationships

$$\Theta(s) = \log\frac{1}{\omega(1-s)} \quad \text{and} \quad \Lambda(x) = \log\Theta(e^{-x}),$$

and put

$$\theta(s) = \exp\left[\lambda\left(\log\frac{1}{s}\right)\right], \quad 0<s<1. \tag{7.1}$$

Then $\exp(s^{-\varepsilon_0}) \leqslant \theta(s) \leqslant \Theta(s)$, for small $s>0$, and we also have $\theta(e^{-x_n}) = \Theta(e^{-x_n})$; as usual, ε_0 is the positive quantity which appears in (2.1).

For the rest of this section, we fix some sufficiently big x_n from the above-mentioned sequence, and put

$$\delta_n = e^{-x_n}, \quad r_n = 1-\delta_n, \quad \text{and} \quad \gamma_n = e^{-\lambda(x_n)/10}.$$

Furthermore, let

$$\left.\begin{aligned} H_n(z) &= e^{\lambda(x_n)-x_n\lambda'(x_n)}(1-z)^{-\lambda'(x_n)} \\ &= e^{\lambda(x_n)-x_n\lambda'(x_n)}\exp\left[\lambda'(x_n)\log\frac{1}{1-z}\right], \quad z\in\mathbb{D}, \\ h_n &= \operatorname{Re} H_n. \end{aligned}\right\} \tag{7.2}$$

The function H_n is holomorphic on $\mathbb{D}$, and consequently, h_n is real-valued and harmonic. There are sectors about the point 1 of angular opening $\pi/\lambda'(x_n)$ where h_n is large, positive or negative, in an alternating fashion. We shall use the functions h_n as "building blocks", like in [13]. The intention is to construct a function that is alternatingly big and small near a prescribed sequence of points on the unit circle.

For the construction, we need sufficiently effective estimates on the growth and decay of h_n.

In the following lemma, we compare the size of $h_n(z)$ with that of the majorants

$$M(z) = \theta(1-|z|), \quad M_n(z) = (1+\gamma_n)^{-1}[\theta(1-|z|) - 2[\log\theta(1-|z|)]^2];$$

clearly, $M_n \leqslant M$. It is easy to see that with our choice of h_n in formula (7.2), and the clever choice of the point x_n of Lemma 6.1 we have $|h_n| \leqslant M$ throughout $\mathbb{D}$, and $h_n(r_n) = M(r_n)$. We prove that $h_n \leqslant M_n$ holds outside a small disk $\mathscr{D}_n$ of radius $(1-r_n)^2$ centered at r_n, and that $-h_n \leqslant M_n$ in a neighborhood of the unit circle that does not depend on the point x_n.

Lemma 7.1. *In the above setting, we have, for* $z \in \mathbb{D}$,

(a) $|h_n(z)| \leqslant \theta(1-|z|)$,
(b) $(1+\gamma_n)h_n(z) \leqslant \theta(1-|z|) - 2[\log\theta(1-|z|)]^2$ *when* $|z-r_n| > \delta_n^2$, *and*
(c) $(1+\gamma_n)h_n(z) > -\theta(1-|z|) + 2[\log\theta(1-|z|)]^2$ *when* $c(\theta) < |z| < 1$, *where* $c(\theta)$, $0 < c(\theta) < 1$, *only depends on the function* θ.

The next step is to verify the following additional properties of the functions h_n:

- $|h_n|$ is small outside a disk of radius proportional to $1-r_n$ centered at 1,
- $-h_n$ is sufficiently big at a point inside this disk, and
- $(1+\gamma_n)h_n - \Theta(1-|z|)$ is sufficiently big (in the integral sense) inside the above-mentioned disk $\mathscr{D}_n$.

Lemma 7.2. *Under the conditions of Lemma* 7.1,

(a) $|h_n(z)| < \delta_n^3$ *when* $z \in \mathbb{D}$ *has* $|z-1| \geqslant \delta_n \exp(-1+2/\varepsilon_0)$,
(b) $h_n(w) = -\theta(\delta_n)$ *holds for some* $w \in \mathbb{D}$ *with* $1-\delta_n < |w| < \frac{1}{2}\delta_n$, *and*
(c) *we have the integral estimate*

$$1 \leqslant \int_{\mathscr{D}_n} \exp[(1+\gamma_n)h_n(z) - \Theta(1-|z|)]\, dm_{\mathbb{D}}(z).$$

For the proofs of these lemmas, we use the following simple estimates.

Lemma 7.3. *For positive* α, *let* F_α *denote the function*

$$F_\alpha(z) = (1-z)^{-\alpha}, \quad z \in \mathbb{D},$$

where the power is defined by the principal branch of the logarithm. For $0 < \varphi < \pi/6$, *let* $\mathscr{R}_\varphi$ *be the domain*

$$\mathscr{R}_\varphi = \{z \in \mathbb{C}\colon \frac{1}{2} < |z| < 1,\ \varphi < |\arg(1-z)|\},$$

and $\mathscr{L}_\alpha$ *the union of two line segments*

$$\mathscr{L}_\alpha = \left\{z \in \mathbb{D}\colon |\arg(1-z)| = \frac{\pi}{\alpha}\right\}.$$

Then the real part of F_α *has the following properties*:

(a) $\operatorname{Re} F_\alpha(z) = |1-z|^{-\alpha}\cos(\alpha\arg(1-z))$, $z \in \mathbb{D}$,
(b) $\operatorname{Re} F_\alpha(z) \leqslant |F_\alpha(z)| \leqslant F_\alpha(|z|) = (1-|z|)^{-\alpha}$, $z \in \mathbb{D}$,
(c) $\operatorname{Re} F_\alpha(z) = -|1-z|^{-\alpha}$, $z \in \mathscr{L}_\alpha$,
and, if α *is sufficiently large*,
(d) $\operatorname{Re} F_\alpha(z) \leqslant |F_\alpha(z)| \leqslant \exp(-\frac{\alpha\varphi^2}{3})(1-|z|)^{-\alpha}$, $z \in \mathscr{R}_\varphi$.

Proof. Equality (a) follows from the definition of the power function; equality (c) is an immediate consequence of (a). Inequality (b) follows from the triangle inequality.

Finally, for large α, a geometric consideration using the inequality

$$\cos\varphi \leqslant 1 - \frac{\varphi^2}{3}, \quad 0 < \varphi < \pi/6,$$

yields the estimate

$$1 - |z| \leqslant \left(1 - \frac{\varphi^2}{3}\right)|1 - z|, \quad z \in \mathscr{R}_\varphi,$$

and (d) follows. □

Proof of Lemma 7.1. Fix $z \in \mathbb{D}$, put

$$t = \log\frac{1}{1 - |z|},$$

and consider the affine function

$$L_n(t) = \lambda(x_n) - x_n\lambda'(x_n) + t\lambda'(x_n).$$

Then $L_n(x_n) = \lambda(x_n)$ and $L_n{}'(x_n) = \lambda'(x_n)$, so that by the convexity of λ, $L_n(t) \leqslant \lambda(t)$ holds everywhere. Using Lemma 7.3(b), we get

$$\log|h_n(z)| \leqslant \log|H_n(z)| \leqslant \lambda(x_n) - x_n\lambda'(x_n) + t\lambda'(x_n) \leqslant \lambda(t) = \log\theta(1 - |z|).$$

This proves part (a).

In the same fashion, we see that keeping the notation

$$t = \log\frac{1}{1 - |z|},$$

we get

$$|\tilde{h}_n(z)| \leqslant \exp[L_n(t) + \gamma_n] \leqslant \exp[\lambda(t) + \gamma_n], \tag{7.3}$$

where

$$\tilde{h}_n(z) = (1 + \gamma_n)h_n(z), \quad z \in \mathbb{D}.$$

Moreover, by Lemma 6.1(f),

$$\begin{aligned}\theta(1 - |z|) = \exp[\lambda(t)] &\geqslant \exp\left[\lambda(x_n) + (t - x_n)\lambda'(x_n) + \frac{1}{4}\varepsilon_0(t - x_n)^2\lambda'(x_n)\right] \\ &= \exp\left[L_n(t) + \frac{1}{4}\varepsilon_0(t - x_n)^2\lambda'(x_n)\right].\end{aligned} \tag{7.4}$$

It follows that

$$\theta(1-|z|)-|\tilde{h}_n(z)| = \theta(1-|z|)\left(1-\frac{|\tilde{h}_n(z)|}{\theta(1-|z|)}\right)$$
$$\geqslant \left(1-\exp\left[\gamma_n-\frac{1}{4}\varepsilon_0(t-x_n)^2\lambda'(x_n)\right]\right)\exp[\lambda(t)], \qquad (7.5)$$

and that

$$\theta(1-|z|)-|\tilde{h}_n(z)| \geqslant \left(\exp\left[\frac{1}{4}\varepsilon_0(t-x_n)^2\lambda'(x_n)\right]-e^{\gamma_n}\right)\exp[L_n(t)]. \qquad (7.6)$$

If z belongs to the domain under consideration in (b), then

$$||z|-r_n|>\frac{1}{2}\delta_n^2 \quad \text{or} \quad |\arg(1-z)|>\frac{\delta_n}{2}.$$

Furthermore, if $h(z)<0$, then $|\arg(1-z)|>\frac{1}{2}\pi/\lambda'(x_n)$. Write

$$\eta_n=\min\left\{\frac{\delta_n}{2},\frac{\pi}{2\lambda'(x_n)}\right\}.$$

To verify (b) and (c), we need to estimate the expression

$$\theta(1-|z|)-|\tilde{h}_n(z)|,$$

for points $z\in\mathbb{D}$ that satisfy at least one of the following two conditions:

(a) $||z|-r_n|>\delta_n^2/2$ and
(b) $|\arg(1-z)|>\eta_n$.

We first look at case (a). Note that if

$$||z|-r_n|>\frac{\delta_n^2}{2},$$

then

$$|t-x_n|=\left|\log\frac{1-r_n}{1-|z|}\right|>\frac{1}{4}\delta_n=\frac{1}{4}e^{-x_n}.$$

(i) If $t-x_n>\frac{1}{4}e^{-x_n}$, then, for big x_n,

$$\frac{\varepsilon_0}{4}(t-x_n)^2\lambda'(x_n)>e^{-\lambda(x_n)/3}+\gamma_n,$$

and by (7.5),

$$\theta(1-|z|)-|\tilde{h}_n(z)|\geqslant(1-\exp[-e^{-\lambda(x_n)/3}])\exp[\lambda(t)]\geqslant\exp\left(\frac{\lambda(t)}{2}\right). \tag{7.7}$$

(ii) If

$$\frac{1}{4}e^{-x_n}<x_n-t<\frac{\lambda(x_n)}{2\lambda'(x_n)},$$

then for big values of x_n,

$$\frac{1}{4}\varepsilon_0(t-x_n)^2\lambda'(x_n)>\exp\left(-\frac{\lambda(x_n)}{5}\right)+\gamma_n$$

and

$$L_n(t)>\frac{1}{2}\lambda(x_n),$$

so that by (7.6),

$$\theta(1-|z|)-|\tilde{h}_n(z)|\geqslant\exp\left(\frac{\lambda(x_n)}{4}\right). \tag{7.8}$$

Next, suppose that $x_n-t\geqslant\frac{1}{2}\lambda(x_n)/\lambda'(x_n)$. Then, by Lemma 6.1(c),

$$\frac{1}{16}\varepsilon_0[\lambda(x_n)]^{1/2}\leqslant\frac{\varepsilon_0[\lambda(x_n)]^2}{16\lambda'(x_n)}\leqslant\frac{1}{4}\varepsilon_0(t-x_n)^2\lambda'(x_n). \tag{7.9}$$

(iii) Thus, if

$$\frac{\lambda(x_n)}{2\lambda'(x_n)}\leqslant x_n-t<\frac{\lambda(x_n)}{\lambda'(x_n)},$$

then $L_n(t)>0$, and by (7.6),

$$\theta(1-|z|)-|\tilde{h}_n(z)|>\exp\left(\frac{1}{32}\varepsilon_0\lambda(x_n)^{1/2}\right). \tag{7.10}$$

(iv) Finally, if

$$x_n-t\geqslant\frac{\lambda(x_n)}{\lambda'(x_n)},$$

then $L_n(t)\leqslant 0$. By (7.3),

$$|\tilde{h}_n(z)|\leqslant e^{\gamma_n}\leqslant e,$$

and

$$\theta(1-|z|)-|\tilde{h}_n(z)|\geqslant \exp[\lambda(t)]-e. \tag{7.11}$$

As a result of (7.7), (7.8), (7.10) and (7.11), for

$$||z|-r_n|>\frac{\delta_n^2}{2},$$

and for big x_n and t, we obtain

$$\theta(1-|z|)-|\tilde{h}_n(z)|>\exp\left(\frac{1}{32}\varepsilon_0\lambda(t)^{1/2}\right)-e\geqslant 2[\lambda(t)]^2=2[\log\theta(1-|z|)]^2. \tag{7.12}$$

(b) In this case,

$$|\arg(1-z)|>\eta_n=\min\left\{\frac{\delta_n}{2},\frac{\pi}{2\lambda'(x_n)}\right\},$$

and by Lemma 7.3(d), we have

$$|\tilde{h}_n(z)|\leqslant \exp\left(L_n(t)+\gamma_n-\frac{1}{3}\lambda'(x_n)\eta_n^2\right).$$

By Lemma 6.1(c),

$$\gamma_n-\frac{\lambda'(x_n)\eta_n^2}{3}<-\frac{\lambda'(x_n)\eta_n^2}{4}.$$

Hence,

$$|\tilde{h}_n(z)|\leqslant \exp\left(L_n(t)-\frac{1}{4}\lambda'(x_n)\eta_n^2\right). \tag{7.13}$$

Using (7.4), we get, analogously to (7.5) and (7.6), that

$$\begin{aligned}\theta(1-|z|)&-|\tilde{h}_n(z)|\\&\geqslant\left(1-\exp\left[-\frac{1}{4}\lambda'(x_n)\eta_n^2-\frac{1}{4}\varepsilon_0(t-x_n)^2\lambda'(x_n)\right]\right)\exp[\lambda(t)],\end{aligned} \tag{7.14}$$

and

$$\begin{aligned}\theta(1-|z|)&-|\tilde{h}_n(z)|\\&\geqslant\left(\exp\left[\frac{1}{4}\varepsilon_0(t-x_n)^2\lambda'(x_n)\right]-\exp\left[-\frac{1}{4}\lambda'(x_n)\eta_n^2\right]\right)\exp[L_n(t)].\end{aligned} \tag{7.15}$$

(i) If $t > x_n$, then by (7.14) and by Lemma 6.1, parts (c) and (e), we have, for big x_n,

$$\theta(1-|z|) - |\tilde{h}_n(z)| > \exp\left(\frac{\lambda(t)}{2}\right). \tag{7.16}$$

(ii) If

$$0 \leqslant x_n - t < \frac{\lambda(x_n)}{2\lambda'(x_n)},$$

then $L_n(t) > \lambda(x_n)/2$, and by (7.15) and by Lemma 6.1, parts (c) and (e), we have, for big x_n,

$$\theta(1-|z|) - |\tilde{h}_n(z)| > \exp\left(\frac{1}{3}\lambda(x_n)\right). \tag{7.17}$$

If

$$x_n - t \geqslant \frac{\lambda(x_n)}{2\lambda'(x_n)},$$

then we argue as in case (a) using estimate (7.9).

(iii) If

$$\frac{\lambda(x_n)}{2\lambda'(x_n)} \leqslant x_n - t < \frac{\lambda(x_n)}{\lambda'(x_n)},$$

then $L_n(t) > 0$, and by (7.15),

$$\theta(1-|z|) - |\tilde{h}_n(z)| > \exp\left(\frac{1}{32}\varepsilon_0[\lambda(x_n)]^{1/2}\right). \tag{7.18}$$

(iv) Finally, if $x_n - t \geqslant \lambda(x_n)/\lambda'(x_n)$, then $L_n(t) \leqslant 0$. By (7.13), $|\tilde{h}_n(z)| \leqslant 1$, and

$$\theta(1-|z|) - |\tilde{h}_n(z)| > \exp[\lambda(t)] - 1. \tag{7.19}$$

As a result of (7.16)–(7.19), for $|\arg(1-z)| > \eta_n$, and for big x_n and t, we get

$$\theta(1-|z|) - |\tilde{h}_n(z)| > \exp\left[\frac{1}{32}\varepsilon_0[\lambda(t)]^{1/2}\right] - 1 \geqslant 2[\lambda(t)]^2 = 2[\log\theta(1-|z|)]^2. \tag{7.20}$$

The estimates (7.12) and (7.20) imply both (b) and (c) for big values of x_n. □

Proof of Lemma 7.2. To verify (a), we note that if

$$|z-1| \geqslant \delta_n e^{2/\varepsilon_0}, \quad \text{then } \log\frac{1}{|1-z|} < x_n - \frac{2}{\varepsilon_0},$$

and by Lemma 6.1, parts (b) and (e), together with Lemma 7.3(b), we have, for big x_n,

$$\log|h_n(z)| \leqslant \lambda(x_n) - x_n\lambda'(x_n) + x_n\lambda'(x_n) - \left(\frac{2}{\varepsilon_0} - 1\right)\lambda'(x_n)$$
$$= \lambda(x_n) - \left(\frac{2}{\varepsilon_0} - 1\right)\lambda'(x_n) < -3x_n,$$

as desired.

To prove (b), note that if

$$w = 1 - \delta_n \exp\left(i\frac{\pi}{\lambda'(x_n)}\right),$$

then we have $\frac{1}{2}\delta_n < 1 - |w| < \delta_n$ and $w \in \mathscr{L}_{\lambda'(x_n)}$, so that in view of part (c) of Lemma 7.3, we obtain

$$h_n(w) = -\exp[\lambda(x_n) - x_n\lambda'(x_n) + x_n\lambda'(x_n)] = -\theta(\delta_n).$$

Finally, to prove the estimate in part (c), we consider the region

$$\mathscr{R}_n = \{re^{i\theta}\colon\ r_n - \gamma_n^2 < r < r_n,\ |\theta| < \gamma_n^2\}.$$

which is a subset of the domain of integration in (c). Using that for $z \in \mathscr{R}_n$,

$$\delta_n \leqslant |1 - z| \leqslant \delta_n + 2\gamma_n^2,$$

$$|\arg(1 - z)| \leqslant 2\frac{\gamma_n^2}{\delta_n},$$

$$\Theta(1 - |z|) \leqslant \Theta(\delta_n) = \theta(\delta_n),$$

we derive from Lemma 7.3, part (a) and Lemma 6.1, parts (b) and (c), together with the convexity of the function λ, that for $z \in \mathscr{R}_n$,

$$\begin{aligned}\log h_n(z) + \log(1 + \gamma_n) &= \lambda(x_n) - x_n\lambda'(x_n) + \lambda'(x_n)\log\frac{1}{|1 - z|}\\ &\quad + \log(\cos[\lambda'(x_n)\arg(1 - z)]) + \log(1 + \gamma_n)\\ &\geqslant \lambda(x_n) - x_n\lambda'(x_n) + \lambda'(x_n)\log\frac{1}{\delta_n} + \frac{2\gamma_n}{3}\\ &\quad - \frac{4\gamma_n^2\lambda'(x_n)}{\delta_n} - \frac{4\gamma_n^4\lambda'(x_n)^2}{\delta_n^2}\\ &\geqslant \lambda(x_n) - x_n\lambda'(x_n) + x_n\lambda'(x_n) + \frac{\gamma_n}{2}\\ &= \log\Theta(\delta_n) + \frac{\gamma_n}{2}.\end{aligned}$$

As a consequence, it follows that for large x_n,

$$\begin{aligned}
&\int_{\mathscr{D}_n} \exp[(1+\gamma_n)h_n(z) - \Theta(1-|z|)]\, dm_{\mathbb{D}}(z) \\
&\geqslant \int_{\mathscr{R}_n} \exp[(1+\gamma_n)h_n(z) - \Theta(1-|z|)]\, dm_{\mathbb{D}}(z) \\
&\geqslant m_{\mathbb{D}}(\mathscr{R}_n)\exp[\Theta(\delta_n)e^{\gamma_n/2} - \Theta(\delta_n)] \geqslant \gamma_n^4 \exp\left[\frac{1}{2}\gamma_n\Theta(\delta_n)\right] \\
&= \exp\left[-\frac{2}{5}\lambda(x_n)\right] \exp\left[\frac{1}{2}e^{-\lambda(x_n)/10}e^{\lambda(x_n)}\right] \geqslant 1.
\end{aligned}$$

This completes the proof of part (c), and hence that of the whole lemma. □

8. The construction of harmonic functions: estimates

For the proof of Theorem 5.3, we need additional estimates on the values of h and H' at pairs of nearby points in the unit disk. We begin with a simple regularity lemma.

Lemma 8.1. *For positive s close to* 0,

$$\theta(s - [\theta(s)]^{-2}) < \theta(s) + 1.$$

Proof. We start with the inequality

$$e^t \lambda'(t) < e^{\lambda(t)/2},$$

which follows from Lemma 6.1, parts (b) and (c). Using (7.1) and passing to the variable $x = e^{-t}$, we get

$$|\theta'(x)| < \theta(x)^{3/2}.$$

Since θ is monotonically decreasing, to prove the lemma, it suffices to note that if $t < s$ and $\theta(t) = \theta(s) + 1$, then $\theta(s) \leqslant \theta(x) \leqslant \theta(s) + 1$ for $t \leqslant x \leqslant s$, and hence

$$1 = \theta(t) - \theta(s) = -\int_t^s \theta'(x)\, dx < \int_t^s [\theta(x)]^{3/2}\, dx \leqslant (s-t)[\theta(s)+1]^{3/2},$$

so that $s - t > [\theta(s)]^{-2}$. □

In the following lemma, for points $z, w \in \mathbb{D}$ that are sufficiently close to one another, we produce upper estimates for $h_n(w)$ and $2\log|H_n{}'(w)| + h_n(w)$ that depend on the size of $|z|$.

Lemma 8.2. *In the notation of Lemma* 7.1, *take* $w \in \mathbb{D}$ *with* $|w - z| < [\theta(1-|z|)]^{-2}$ *for some* $z \in \mathbb{D}$ *with* $|z|$ *sufficiently close to* 1. *Then*

$$|\theta(1-|z|) - \theta(1-|w|)| < 1. \tag{8.1}$$

(a) *If* $\lambda(t) \leqslant \frac{2}{5}\lambda(x_n)$, *then*

$$(1+\gamma_n)h_n(w) < \theta(1-|z|).$$

(b) *If* $\lambda(t) > \frac{1}{3}\lambda(x_n)$, *then*

$$2|\log|H_n{}'(w)|| + h_n(w) \leqslant \frac{3}{2}\theta(1-|z|).$$

(c) *If* $\lambda(t) > \frac{1}{3}\lambda(x_n)$ *and* $|\arg(1-z)| > \frac{1}{2}\pi/\lambda'(x_n)$, *then*

$$2\log|H_n{}'(w)| + h_n(w) \leqslant \frac{\theta(1-|z|)}{1+\gamma_n}.$$

Proof. The first statement follows immediately from Lemma 8.1. For convenience of notation, put

$$s = \log\frac{1}{1-|w|}.$$

We may rewrite estimate (8.1) as

$$|\exp[\lambda(t)] - \exp[\lambda(s)]| < 1. \tag{8.2}$$

(a) Using that $\lambda(t) \leqslant \frac{2}{5}\lambda(x_n)$ as well as the convexity of λ, we obtain

$$\lambda(x_n) - (x_n - s)\lambda'(x_n) \leqslant \lambda(s) \leqslant \frac{1}{2}\lambda(x_n),$$

and, as a consequence, $x_n - s \geqslant \frac{1}{2}\lambda(x_n)/\lambda'(x_n)$. Applying (7.10) and (7.11) with z replaced by w, we complete the proof.

By the definition of H_n, using that $|1-w| \geqslant 1-|w|$ and that, by the convexity of λ, $\lambda(x_n) + (s-x_n)\lambda'(x_n) \leqslant \lambda(s)$, we obtain:

$$\left.\begin{aligned} H_n{}'(w) &= -\exp[\lambda(x_n) - x_n\lambda'(x_n)]\frac{F_{\lambda'(x_n)}(w)\lambda'(x_n)}{1-w}, \\ \log|H_n{}'(w)| &\leqslant \lambda(x_n) - x_n\lambda'(x_n) + (\lambda'(x_n)+1)s + \log\lambda'(x_n), \\ |\log|H_n{}'(w)|| &\leqslant \lambda(s) + s + \log\lambda'(x_n). \end{aligned}\right\} \tag{8.3}$$

In (b) and (c), we have $\lambda(t) > \frac{1}{3}\lambda(x_n)$, and hence $\lambda(s) > \frac{1}{4}\lambda(x_n)$ because of (8.2). By Lemma 6.1, parts (b) and (c), we obtain

$$s + \log \lambda'(x_n) \leqslant \left(2 + \frac{1}{\varepsilon_0}\right) \log \lambda(s). \tag{8.4}$$

Moreover, as in (7.3), we get

$$h_n(w) \leqslant \exp[\lambda(s)]. \tag{8.5}$$

The assertion in part (b) now follows from (8.2)–(8.5).

(c) We have

$$|\arg(1-z)| > \frac{\pi}{2\lambda'(x_n)} \quad \text{and} \quad |w - z| < e^{-2\lambda(t)}.$$

Since $\lambda(t) > \frac{1}{3}\lambda(x_n)$, we get from Lemma 6.1(c) that

$$e^{-2\lambda(t)} < \frac{e^{-2t}}{\lambda'(x_n)}.$$

A simple geometric argument then shows that

$$|\arg(1-w)| > \frac{\pi}{3\lambda'(x_n)}.$$

By Lemma 7.3(d), and by the convexity of λ, we have

$$\begin{aligned}\log(1+\gamma_n) + \log|h_n(w)| &\leqslant \lambda(x_n) + (s - x_n)\lambda'(x_n) + \gamma_n - \frac{1}{3\lambda'(x_n)} \\ &\leqslant \lambda(s) + \gamma_n - \frac{1}{3\lambda'(x_n)}.\end{aligned}$$

Applying once again Lemma 6.1(c), and using (8.3) and (8.4), we are able to complete the proof. □

In the last technical lemma of this section, for nearby points ξ and z in the unit disk, we estimate the size of the quantities $|g_n{}'(\xi)/g_n(z)|$ and $|g_n(\xi)/g_n(z)|$, where g_n is the zero-free analytic function

$$g_n(z) = \exp\left(\frac{1}{2} H_n(z)\right).$$

Lemma 8.3. *In the notation of Lemma* 7.1, *take* $\xi \in \mathbb{D}$ *with* $|\xi - z| < [\theta(1-|z|)]^{-3}$. *Then*

$$\left|\frac{g_n{}'(\xi)}{g_n(z)}\right| \leqslant \exp\left[\frac{\theta(1-|z|)}{1+\gamma_n} - \lambda(t)\right],$$
$$\left|\frac{g_n(\xi)}{g_n(z)}\right| \leqslant \exp\left[\frac{\theta(1-|z|)}{1+\gamma_n} - \lambda(t)\right].$$

Proof. We prove only the first of these two inequalities; the second one is treated analogously.

If $\lambda(t) \leqslant \frac{1}{3}\lambda(x_n)$, then by Lemma 8.1, for every w with $|w-z| \leqslant e^{-2\lambda(t)}$, we have $\log\theta(1-|w|) < \frac{2}{5}\lambda(x_n)$, and by Lemma 8.2,

$$\sup_{|w-z|\leqslant e^{-2\lambda(t)}} |g_n(w)| \leqslant \exp\left[\frac{\theta(1-|z|)}{2(1+\gamma_n)}\right].$$

Using the Cauchy integral formula, we then get

$$|g_n{}'(\xi)| \leqslant \exp\left[\frac{\theta(1-|z|)}{2(1+\gamma_n)} + 3\lambda(t)\right],$$

By Lemma 7.1(c), we obtain

$$\left|\frac{g_n{}'(\xi)}{g_n(z)}\right| \leqslant \exp\left[\frac{\theta(1-|z|)}{1+\gamma_n} + 3\lambda(t) - [\lambda(t)]^2\right] \leqslant \exp\left[\frac{\theta(1-|z|)}{1+\gamma_n} - \lambda(t)\right].$$

If $\lambda(t) > \frac{1}{3}\lambda(x_n)$, then we consider the following two cases: $|g_n(z)| < 1$ and $|g_n(z)| \geqslant 1$.

If $|g_n(z)| < 1$, then $|\arg(1-z)| \geqslant \frac{1}{2}\pi/\lambda'(x_n)$, and by Lemma 8.2(c),

$$|g_n{}'(\xi)| \leqslant \exp\left[\frac{\theta(1-|z|)}{2(1+\gamma_n)}\right].$$

By Lemma 7.1(c),

$$|g_n(z)| \geqslant \exp\left[[\lambda(t)]^2 - \frac{\theta(1-|z|)}{2(1+\gamma_n)}\right],$$

and we get

$$\left|\frac{g_n{}'(\xi)}{g_n(z)}\right| \leqslant \exp\left[\frac{\theta(1-|z|)}{1+\gamma_n} - \lambda(t)\right].$$

If, on the other hand, $1 \leqslant |g_n(z)|$, then by Lemma 8.2(b),

$$|g_n{}'(\xi)| \leqslant \exp\left[\frac{3}{4}\theta(1-|z|)\right],$$

so that

$$\left|\frac{g_n{}'(\xi)}{g_n(z)}\right| \leqslant \exp\left[\frac{3}{4}\theta(1-|z|)\right].$$

This completes the proof of the lemma. □

9. The construction of invertible non-cyclic functions

Throughout this section, we suppose that ω satisfies (2.1) and $\tilde{\omega}$ is defined by (2.2). Using the technical results of Sections 6–8, in conjunction with Section 5, we produce here non-cyclic functions $F \in B^1(\mathbb{D}, \omega)$ satisfying various additional properties.

Theorem 9.1. *There exists a zero-free function $F \in B^1(\mathbb{D}, \omega)$ such that $1/F$ is in $B^1(\mathbb{D}, \tilde{\omega})$ and $F^{1/p}$ is non-cyclic in $B^p(\mathbb{D}, \omega)$ for each p, $0 < p < +\infty$.*

Theorem 9.2. *There exist functions F_j, $j = 1, 2, 3, \dots,$ in $B^1(\mathbb{D}, \omega)$, satisfying the conditions of Theorem 9.1, and such that for every p with $0 < p < +\infty$ and for every integer d with $1 \leqslant d \leqslant +\infty$, the subspace $[F_1^{1/p}, \dots, F_d^{1/p}]$ has index d in $B^p(\mathbb{D}, \omega)$.*

Theorem 9.3. *There exists a function F satisfying the conditions of Theorem 9.1 and such that for $f = F^{1/2}$ we have*

$$\int_{\mathbb{D}} \int_{\mathbb{D}} \frac{1}{|f(z)|^2} \frac{|f(z) - f(w)|^2}{|z-w|^2} \omega(z)\omega(w)\, dm_{\mathbb{D}}(z)\, dm_{\mathbb{D}}(w) < +\infty.$$

Proof of Theorem 9.1. We are going to construct a harmonic function $V = \log|f|$ as the infinite sum of functions harmonic in (different) neighborhoods of the unit disk.

The setup is as in Section 6. Starting with ω, we obtain

$$\Theta(s) = \log \frac{1}{\omega(1-s)} \quad \text{and} \quad \Lambda(x) = \log \Theta(e^{-x})$$

satisfying (6.1) with some ε_0, $0 < \varepsilon_0 < 1$. Then, applying Lemma 6.1 to Λ, we obtain the minorant λ, and a sequence $\{x_n\}_n$; we also have the function θ defined by

$$\theta(s) = \exp\left[\lambda\left(\log\frac{1}{s}\right)\right].$$

We fix κ to be the quantity

$$\kappa = \exp\left(-1 - \frac{2}{\varepsilon_0}\right).$$

We suppose that $r_n = 1 - e^{-x_n}$ tends to 1 as rapidly as required in Proposition 5.3.

We set $V_1 = 0$ and argue by induction. At step n, we start with a function V_n harmonic in a neighborhood of the unit disk, and find a real parameter η with $0 < \eta < 1$, such that

$$|V_n(z) - V_n(w)| < 1, \quad |z - w| < \eta, \ z, w \in \bar{\mathbb{D}}, \tag{9.1}$$

$$\exp|V_n(z)| < \frac{1}{\eta}, \quad z \in \bar{\mathbb{D}}. \tag{9.2}$$

Fix $\{x_n\}_n$ as a subsequence of the sequence in Lemma 6.1 which grows so fast that

$$e^{-x_n} < \eta\kappa e^{-2n}, \tag{9.3}$$

and put

$$\delta_n = e^{-x_n}, \ r_n = 1 - \delta_n, \quad \text{and} \quad \gamma_n = e^{-\lambda(x_n)/10} = e^{-\Lambda(x_n)/10}.$$

We recall from Section 5 that the integer N_n is such that

$$N_n \leqslant \frac{\kappa}{1 - r_n} < N_n + 1.$$

By (9.2) and (9.3), we get

$$\delta_n^2 \leqslant \frac{\exp[-V_n(z)]}{n^2 N_n} \leqslant 1, \quad z \in \bar{\mathbb{D}}.$$

In Section 7, the harmonic function h_n is constructed; it is given explicitly by formula (7.2). Using Lemma 7.1(a) and Lemma 7.2(c), for every k, $0 \leqslant k < N_n$, we can choose numbers $\gamma_{n,k}$ with $0 \leqslant \gamma_{n,k} \leqslant \gamma_n$, such that

$$\int_{\{z \in \mathbb{D}:\ |z - r_n| < \delta_n^2\}} \exp[\tilde{h}_{n,k}(z) - \Theta(1 - |z|)]\, dm_{\mathbb{D}}(z) = \frac{\exp[-V_n(e^{2\pi ik/N_n})]}{n^2 N_n}, \tag{9.4}$$

where, as in Section 7,

$$\tilde{h}_{n,k}(z) = (1 + \gamma_{n,k}) h_n(z).$$

We consider the following N_n equidistributed points on the unit circle $\mathbb{T}$,

$$\zeta_{n,k} = e^{2\pi ik/N_n},$$

and as in Section 5, we put

$$w_{n,k} = r_n\zeta_{n,k}, \quad 0 \leqslant k < N_n.$$

Next, we consider the harmonic functions

$$U(z) = \sum_{0\leqslant k<N_n} h_{n,k}(z\bar{\zeta}_{n,k}),$$
$$V^0_{n+1} = V_n + U.$$

We need a little patch around each point $\zeta_{n,k}$ on the unit circle,

$$\mathcal{O}_{n,k} = \{z \in \mathbb{D}\colon\ |z-\zeta_{n,k}| < \delta_n e^{2/\varepsilon_0}\}, \quad 0 \leqslant k < N_n.$$

Note that these sets $\mathcal{O}_{n,k}$, with $0 \leqslant k < N_n$, are mutually disjoint. We also need the small disk around the point $w_{n,k}$ given by

$$\mathcal{D}_{n,k} = \{z \in \mathbb{D}\colon\ |z-w_{n,k}| < \delta_n^2\},$$

where we recall that $\delta_n = 1 - r_n$.

By Lemma 7.2(a), for some constant C depending only on the parameter ε_0, we have

$$\left.\begin{aligned} |U(z) - \tilde{h}_{n,k}(e^{-2\pi ik/N_n}z)| &< C(1-r_n), \quad z\in\mathcal{O}_{n,k}, \\ |U(z)| &< C(1-r_n), \quad z \in \mathbb{D}\setminus \bigcup_k \mathcal{O}_{n,k}. \end{aligned}\right\} \tag{9.5}$$

In particular,

$$|V^0_{n+1}(z) - V_n(z)| = |U(z)| < C(1-r_n) \quad \text{for } |z| \leqslant 1 - \delta_n e^{2/\varepsilon_0}. \tag{9.6}$$

Brought together, the relations (9.1), (9.4), and (9.5) give us

$$\int_{\mathcal{D}_{n,k}} \exp[V^0_{n+1}(z) - \Theta(1-|z|)]\, dm_{\mathbb{D}}(z) \asymp \frac{1}{n^2N_n}, \tag{9.7}$$

and by Lemma 7.1(b), we obtain

$$\int_{\{z\in\mathbb{D}:\ 1-|z|<\delta_n \exp(2/\varepsilon_0)\}} \exp[V^0_{n+1}(z) - \Theta(1-|z|)]\, dm_{\mathbb{D}}(z) \asymp \frac{1}{n^2}, \tag{9.8}$$

where we use the notation $a \asymp b$ for the relation $a/c < b < ca$ with some positive constant c depending only on ε_0.

The function $y \mapsto y - [\log y]^2$ increases monotonically as y grows to $+\infty$, whence we conclude that

$$\Theta(1-|z|) - [\log \Theta(1-|z|)]^2 > \theta(1-|z|) - [\log \theta(1-|z|)]^2,$$

and by Lemma 7.1(c), (9.3), and (9.5), we have

$$\int_{\{z\in\mathbb{D}:\ 1-|z|<\delta_n \exp(2/\varepsilon_0)\}} \exp[-V^0_{n+1}(z) - \Theta(1-|z|) + [\log\Theta(1-|z|)]^2]\, dm_{\mathbb{D}}(z) < \frac{1}{n^2}. \tag{9.9}$$

Moreover, by Lemma 7.2(b), for some $w\in\mathbb{D}$ with $\delta_n/2 < 1-|w| < \delta_n$, we have

$$V^0_{n+1}(w) < -\frac{2}{3}\theta(\delta_n). \tag{9.10}$$

Replacing V^0_{n+1} by $V_{n+1}(z) = V_n(z) + U(\tau z)$ with τ sufficiently close to 1, $0<\tau<1$, we get the same properties (9.6)–(9.10) with V_{n+1} harmonic in a neighborhood of the unit disk.

As a consequence of (9.3) and (9.6), the functions V_n converge uniformly on compact subsets of the unit disk to a harmonic function V as $n\to+\infty$. We consider the corresponding analytic function $F = \exp[V + i\widetilde{V}]$, where the tilde indicates the harmonic conjugation operation, normalized so that $\widetilde{V}(0) = 0$.

It follows from (9.6) and (9.8) that

$$\int_{\mathbb{D}} |F(z)| e^{-\Theta(1-|z|)}\, dm_{\mathbb{D}}(z) < +\infty. \tag{9.11}$$

Analogously, by (9.6) and (9.9), we have

$$\int_{\mathbb{D}} |F(z)|^{-1} e^{-\Theta(1-|z|)} \exp([\log\Theta(1-|z|)]^2)\, dm_{\mathbb{D}}(z) < +\infty, \tag{9.12}$$

and by (9.6) and (9.7), we get for each $n = 1, 2, 3, \ldots,$

$$\int_{\mathscr{D}_{n,k}} |F(z)| e^{-\Theta(1-|z|)}\, dm_{\mathbb{D}}(z) \asymp \frac{1}{n^2 N_n}, \quad k = 0, 1, 2, \ldots, N_n - 1. \tag{9.13}$$

By (9.6) and (9.10), for every $n = 1, 2, 3, \ldots,$ there exists a point $\xi_n\in\mathbb{D}$ with

$$\frac{\delta_n}{2} < 1 - |\xi_n| < \delta_n$$

such that

$$|F(\xi_n)| < \exp\left[-\frac{1}{2}\theta(\delta_n)\right]. \tag{9.14}$$

It remains to verify that $F^{1/p}$ is non-cyclic in $B^p(\omega)$ for each p, $0<p<+\infty$. So, suppose that for a sequence of polynomials q_j,

$$\|q_j F^{1/p}\|^p_{B^p(\omega)} = \int_{\mathbb{D}} |q_j(z)|^p |F(z)| e^{-\Theta(1-|z|)}\, dm_{\mathbb{D}}(z) \leqslant 1.$$

The disks $\mathscr{D}_{n,k}$ are disjoint for different indices (n,k), so that the above estimate has the immediate consequence

$$\sum_{n=1}^{+\infty}\sum_{k=0}^{N_n-1}\int_{\mathscr{D}_{n,k}}|q_j(z)|^p|F(z)|e^{-\Theta(1-|z|)}\,dm_{\mathbb{D}}(z)\leqslant 1. \tag{9.15}$$

By the mean value theorem for integrals, there exist points $z_{n,k}\in\mathscr{D}_{n,k}$ (which may depend on the index j, too) such that

$$\int_{\mathscr{D}_{n,k}}|q_j(z)|^p|F(z)|e^{-\Theta(1-|z|)}\,dm_{\mathbb{D}}(z)=|q_j(z_{n,k})|^p\int_{\mathscr{D}_{n,k}}|F(z)|e^{-\Theta(1-|z|)}\,dm_{\mathbb{D}}(z).$$

In view of (9.13), we see that (9.15) leads to

$$\sum_{n=1}^{+\infty}\sum_{k=0}^{N_n-1}\frac{1}{n^2N_n}|q_j(z_{n,k})|^p\leqslant C,$$

for some positive constant C, and by a weak type estimate, we obtain for $n=1,2,3,\dots$ that

$$\sum_{k=0}^{N_n-1}|q_j(z_{n,k})|^p\leqslant Cn^2N_n.$$

We find ourselves in the situation described in Section 5. An application of Proposition 5.3 yields

$$|q_j(z)|\leqslant c\exp\left[\frac{1}{1-|z|}\right],\quad z\in\mathbb{D},$$

for some constant c which is independent of the index j. Inequalities (9.14) show that $q_jF^{1/p}$ cannot converge to a non-zero constant uniformly on compact subsets of $\mathbb{D}$. As a result, $F^{1/p}$ is not cyclic in $B^p(\omega)$. □

Proof of Theorem 9.2. For the sake of simplicity, we consider only the case $p=1$, $d=2$. Arguing as in the Proof of Theorem 9.1, we can produce invertible non-cyclic elements F_1 and F_2 in $B^1(\omega)$, numbers

$$\cdots<r_{1,n}<r_{2,n}<r_{1,n+1}<r_{2,n+1}<\cdots\to 1,\quad n\to+\infty,$$

integers $N_{j,n}$ such that

$$N_{j,n}\leqslant\frac{\kappa}{1-r_{j,n}}<N_{j,n+1},\quad j=1,2,$$

points

$$w_{j,n,k} = r_{j,n} e^{2\pi ik/N_{j,n}}, \quad 0 \leqslant k < N_{j,n},$$

and disks

$$\mathscr{D}_{j,n,k} = \{z \in \mathbb{D}\colon\ |z - w_{j,n,k}| < (1 - r_{j,n})^2\}, \quad k = 0, 1, 2, \dots, N_{j,n} - 1,$$

with the properties that

$$\int_{\mathscr{D}_{j,n,k}} |F_j(z)| e^{-\Theta(1-|z|)}\, dm_{\mathbb{D}}(z) \asymp \frac{1}{n^2 N_{j,n}}, \quad j = 1, 2, \tag{9.16}$$

and

$$|F_j(z)| e^{-\theta(1-|z|)} \leqslant \exp\left[-\frac{1}{1 - |z|}\right], \quad z \in \mathscr{D}_{3-j,n,k},\ j = 1, 2. \tag{9.17}$$

If, for some polynomials q_1 and q_2, we are given that

$$||q_1 F_1 + q_2 F_2||_{B^1(\omega)} \leqslant 1, \tag{9.18}$$

and

$$||q_1||_{H^\infty(\mathbb{D})} + ||q_2||_{H^\infty(\mathbb{D})} \leqslant A, \tag{9.19}$$

then we argue as in the proof of Theorem 9.1. First, for sufficiently big n, $n_0 = n_0(A) \leqslant n < +\infty$, inequalities (9.17)–(9.19) imply that for $j = 1, 2$,

$$\int_{\mathscr{D}_{3-j,n,k}} |q_j(z) F_j(z)| e^{-\Theta(1-|z|)}\, dm_{\mathbb{D}}(z) \leqslant \frac{1}{N_{3-j,n}},$$

$$\sum_{0 \leqslant k < N_{j,n}} \int_{\mathscr{D}_{j,n,k}} |q_j(z) F_j(z)| e^{-\Theta(1-|z|)}\, dm_{\mathbb{D}}(z) \leqslant 2,$$

and hence, by (9.16), for some points $z_{j,n,k}$ in the disks $\mathscr{D}_{j,n,k}$, we have

$$\sum_{0 \leqslant k < N_{j,n}} |q_j(z_{j,n,k})| \leqslant 2n^2 N_{j,n}, \quad n = n_0, n_0 + 1, n_0 + 2, \dots\ . \tag{9.20}$$

We have almost the assumption of Proposition 5.3, only with $p = 1$ and the estimates start from index $n = n_0$. If we analyze the proof of that proposition carefully, we obtain the corresponding estimate of the function q_j,

$$|q_j(z)| \leqslant c n_0^4 \exp\left[\frac{1}{1 - |z|}\right], \quad z \in \mathbb{D}, \tag{9.21}$$

where $c = c(\kappa)$ is a positive constant, independent of the value of n_0. Similarly, the inequalities (9.17), (9.18), and (9.21) imply that the estimates $(j = 1, 2)$

$$\int_{\mathscr{D}_{3-j,n,k}} |q_j(z)F_j(z)|e^{-\Theta(1-|z|)}\, dm_{\mathbb{D}}(z) \leqslant \frac{cn_0^4}{R_{3-j,n}^2} \leqslant \frac{1}{N_{3-j,n}},$$

$$\sum_{0 \leqslant k < N_{j,n}} \int_{\mathscr{D}_{j,n,k}} |q_j(z)F_j(z)|e^{-\Theta(1-|z|)}\, dm_{\mathbb{D}}(z) \leqslant 2,$$

hold for all n such that

$$cn_0^4 \leqslant \min\{N_{1,n}, N_{2,n}\}. \tag{9.22}$$

As a consequence of (9.3), $N_{j,n} \geqslant e^n$, and inequality (9.22) holds for all $n \geqslant n_1$, with n_1 equal to the integer part of $\alpha \log n_0$, for some positive real parameter α; clearly, for sufficiently big n_0, we have $n_1 < n_0$. Arguing as before, we get (9.20) for all $n \geqslant n_1$, and then

$$|q_j(z)| \leqslant cn_1^4 \exp\left(\frac{1}{1-|z|}\right), \quad z \in \mathbb{D}.$$

Continuing in this way, we get (9.20) for all $n \geqslant n_\infty = \lim_{k\to\infty} n_k$, with n_∞ independent of A, and, as a consequence,

$$|q_1(z)| + |q_2(z)| < c \exp\left(\frac{1}{1-|z|}\right), \quad z \in \mathbb{D},$$

for some constant c, which is independent of q_1, q_2, and A. This shows that (see, for instance, [25] for $d = 2$, and [9, Section 3] for the general case)

$$\operatorname{ind}([F_1, F_2]) = 2.$$

The proof is complete. □

Proof of Theorem 9.3. Let F be the function constructed in the proof of Theorem 9.1, and consider $f = F^{1/2}$. Inequalities (9.11) and (9.12) together with the inequality

$$x^6 < \exp[(\log x)^2] \quad \text{for } x > e^6,$$

and the identity

$$\omega(z) = e^{-\Theta(1-|z|)}$$

show that we have

$$\int_{\mathbb{D}} |f(z)|^2 \omega(z)\, dm_{\mathbb{D}}(z) < +\infty,$$

$$\int_{\mathbb{D}} \frac{1}{|f(z)|^2} [\Theta(1-|z|)]^6 \omega(z)\, dm_{\mathbb{D}}(z) < +\infty.$$

Let $\mathscr{E}$ be the set

$$\mathscr{E} = \{(z,w) \in \mathbb{D} \times \mathbb{D}\colon\ |w-z| \leqslant [\theta(1-|z|)]^{-3}\},$$

which is a rather small neighborhood of the diagonal; we then have the estimate

$$\begin{aligned}
&\int\int_{\mathbb{D}^2\setminus\mathscr{E}} \frac{1}{|f(z)|^2} \frac{|f(z)-f(w)|^2}{|z-w|^2} \omega(z)\omega(w)\, dm_{\mathbb{D}}(z)\, dm_{\mathbb{D}}(w) \\
&\leqslant 2 \int\int_{\mathbb{D}^2} \frac{|f(z)|^2 + |f(w)|^2}{|f(z)|^2} [\theta(1-|z|)]^6 \omega(z)\omega(w)\, dm_{\mathbb{D}}(z)\, dm_{\mathbb{D}}(w) \\
&\leqslant C \int\int_{\mathbb{D}^2} \frac{1+|f(z)|^2}{|f(z)|^2} [\theta(1-|z|)]^6 \omega(z)\, dm_{\mathbb{D}}(z) < +\infty.
\end{aligned}$$

By Lemma 8.2,

$$|\theta(1-|z|) - \theta(1-|w|)| < 1 \quad \text{for } (z,w) \in \mathscr{E}. \tag{9.23}$$

Hence, it suffices to verify that for $(z,w) \in \mathscr{E}$, we have

$$\frac{1}{|f(z)|^2} \frac{|f(z)-f(w)|^2}{|z-w|^2} \leqslant M \exp[2\theta(1-|z|)], \tag{9.24}$$

for some positive constant M. This is so because, as a consequence of (9.24), we have the estimate

$$\begin{aligned}
&\int\int_{\mathscr{E}} \frac{1}{|f(z)|^2} \frac{|f(z)-f(w)|^2}{|z-w|^2} \omega(z)\omega(w)\, dm_{\mathbb{D}}(z)\, dm_{\mathbb{D}}(w) \\
&\leqslant M \int\int_{\mathscr{E}} \exp[2\theta(1-|z|)] \omega(z)\omega(w)\, dm_{\mathbb{D}}(z)\, dm_{\mathbb{D}}(w) \\
&= M \int\int_{\mathscr{E}} \exp[2\theta(1-|z|) - \Theta(1-|z|) - \Theta(1-|w|)]\, dm_{\mathbb{D}}(z)\, dm_{\mathbb{D}}(w) \\
&\leqslant M \int\int_{\mathscr{E}} \exp[\theta(1-|z|) - \theta(1-|w|)]\, dm_{\mathbb{D}}(z)\, dm_{\mathbb{D}}(w),
\end{aligned}$$

and the latter is bounded by M, if we use (9.23). We turn to the verification of (9.24). We find that there exists a point $\xi = \xi(z, w) \in \mathbb{D}$ with

$$|\xi - z| \leqslant [\theta(1 - |z|)]^{-3},$$

such that

$$\left|\frac{f(z) - f(w)}{z - w}\right| \leqslant |f'(\xi)|,$$

by expressing the function $f(z) - f(w)$ as a path integral from z to w. It follows that we need only to verify that

$$\frac{|f'(\xi)|^2}{|f(z)|^2} \leqslant M \exp[2\theta(1 - |z|)].$$

Now, we fix ζ and ξ in $\mathbb{D}$, with

$$|\zeta - \xi| \leqslant [\theta(1 - |\zeta|)]^{-3},$$

so that we are in the setting above (only with slightly different variable names). We recall the details of the construction of the function F in the proof of Theorem 9.1. We then find a positive integer $n = n(|\zeta|)$ such that

$$\delta_{n+1} e^{2/\varepsilon_0} = \exp\left(-x_{n+1} + \frac{2}{\varepsilon_0}\right) \leqslant 1 - |\zeta| < \exp\left(-x_n + \frac{2}{\varepsilon_0}\right) = \delta_n e^{2/\varepsilon_0}; \qquad (9.25)$$

this is definitely possible at least if ζ is reasonably close to $\mathbb{T}$. On the other hand, if we start with a positive integer n, we may form the annulus

$$\mathscr{U}_n = \{z \in \mathbb{D}\colon\ \delta_{n+1} e^{2/\varepsilon_0} < 1 - |z| \leqslant \delta_n e^{2/\varepsilon_0}\}, \qquad (9.26)$$

so the above relation (9.25) places the point ζ inside $\mathscr{U}_n$. Let V_n be the harmonic function appearing in the proof of Theorem 9.1, and let $W_n = V_n + i\widetilde{V}_n$ be the analytic function having V_n as real part. We form the exponentiated function

$$F_n(z) = \exp(W_n(z)), \quad z \in \mathbb{D},$$

which is analytic and zero-free in a neighborhood of the closed disk $\overline{\mathbb{D}}$. The construction of the function V_n (and hence that of W_n) involves the choice of the points $x_1, x_2, \ldots, x_{n-1}$. By letting the sequence $\{x_n\}_n$ tend to $+\infty$ as rapidly as need be, we can make sure that

$$\max_{z\in\bar{\mathbb{D}}}\left\{|F_n(z)|,\frac{1}{|F_n(z)|},|F'_n(z)|\right\}\leqslant\theta(e^{-x_n+2/\varepsilon_0})\leqslant\exp\left(\frac{1}{4}\lambda(t)\right), \tag{9.27}$$

where

$$t=\log\frac{1}{1-|\zeta|},$$

and the right-hand inequality in (9.27) holds because of (9.25). The function F is the limit of the functions F_n as $n\to+\infty$, and it is of interest to understand the "tail" function F/F_n. By Lemma 7.2(a) and some elementary estimates of harmonic functions and their gradients, plus the fact that the points x_n approach $+\infty$ very rapidly as $n\to+\infty$, the inequalities (9.25) imply that

$$\max_{z\in\bar{\mathscr{U}}_n}\left\{\left|\frac{F}{F_{n+1}}(z)\right|,\left|\frac{F_{n+1}}{F}(z)\right|,\left|\left(\frac{F}{F_{n+1}}\right)'(z)\right|\right\}\leqslant C \tag{9.28}$$

for some positive constant C independent of n; the notation $\bar{\mathscr{U}}_n$ stands for the closure of $\mathscr{U}_n$. We need to apply the estimate (9.28) to the two points ζ and ξ, and although ξ, strictly speaking, need not belong to $\mathscr{U}_n$, it is not far away from this set, and we can make sure that the estimate (9.28) holds for it, simply because Lemma 7.2(a) applies in a slightly bigger annulus than $\mathscr{U}_n$. Next, we pick the point ζ_{n,k_0}, with $0\leqslant k<N_n$, which is closest to the given point $\zeta\in\mathbb{D}$; after a rotation of the disk, we may assume $\zeta_{n,k_0}=\zeta_{n,0}=1$. Since $|\zeta-\xi|$ is much smaller than $1-|\zeta|$, we obtain

$$\min_{k\neq 0}|\zeta_{n,k}-\zeta|\geqslant\exp\left[-x_n+\frac{2}{\varepsilon_0}\right].$$

Let G_n be the function

$$G_n(z)=\frac{F_{n+1}(z)}{F_n(z)}\exp[-(1+\gamma_{n,0})H_n(z)],$$

where $\gamma_{n,0}$ is determined by (9.4), and the analytic function H_n is defined in (7.2); the real part of H_n equals h_n. Then, again by Lemma 7.2(a),

$$\max_{z\in\{\zeta,\xi\}}\max\left\{|G_n(z)|,\frac{1}{|G_n(z)|},|G_n{}'(z)|\right\}\leqslant C, \tag{9.29}$$

where C is some positive constant which does not depend on n. In conclusion, we write

$$f(z)^2=F(z)=g_n(z)^2F_n(z)\frac{F}{F_{n+1}}(z)G_n(z),$$

where

$$g_n(z) = \exp\left(\frac{1}{2}(1+\gamma_{n,0})H_n(z)\right),$$

and by Lemma 8.3 (note that the function g_n appearing in the lemma is slightly different),

$$\left.\begin{aligned}\frac{|g'_n(\xi)|^2}{|g_n(\zeta)|^2} &\leqslant \exp[2\theta(1-|\zeta|)-2\lambda(t)],\\ \frac{|g_n(\xi)|^2}{|g_n(\zeta)|^2} &\leqslant \exp[2\theta(1-|\zeta|)-2\lambda(t)].\end{aligned}\right\} \tag{9.30}$$

Hence, in view of (9.27)–(9.30), we obtain

$$\frac{|f'(\xi)|^2}{|f(\zeta)|^2} \leqslant Ce^{\lambda(t)}\frac{|g_n(\xi)|^2+|g'_n(\xi)|^2}{|g_n(\zeta)|^2} \leqslant \exp[2\theta(1-|\zeta|)].$$

The proof is complete. □

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