

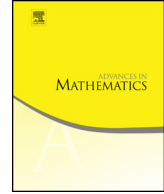


ELSEVIER

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim



Weighted integrability of polyharmonic functions[☆]



Alexander Borichev^a, Haakan Hedenmalm^{b,*}

^a *Laboratoire d'Analyse, Topologie, Probabilités, CMI, Aix-Marseille Université, 39, rue Frédéric Joliot Curie, F-13453 Marseille Cedex 13, France*

^b *Department of Mathematics, KTH Royal Institute of Technology, S-10044 Stockholm, Sweden*

ARTICLE INFO

Article history:

Received 5 November 2013

Accepted 14 July 2014

Available online xxxx

Communicated by Charles Fefferman

In memory of Boris Korenblum

MSC:

primary 31A30

secondary 35J40

Keywords:

Polyharmonic functions

Weighted integrability

Boundary behavior

Cellular decomposition

ABSTRACT

To address the uniqueness issues associated with the Dirichlet problem for the N -harmonic equation on the unit disk \mathbb{D} in the plane, we investigate the L^p integrability of N -harmonic functions with respect to the standard weights $(1 - |z|^2)^\alpha$. The question at hand is the following. If u solves $\Delta^N u = 0$ in \mathbb{D} , where Δ stands for the Laplacian, and

$$\int_{\mathbb{D}} |u(z)|^p (1 - |z|^2)^\alpha dA(z) < +\infty,$$

must then $u(z) \equiv 0$? Here, N is a positive integer, α is real, and $0 < p < +\infty$; dA is the usual area element. The answer will, generally speaking, depend on the triple (N, p, α) . The most interesting case is $0 < p < 1$. For a given N , we find an explicit critical curve $p \mapsto \beta(N, p)$ – a piecewise affine function – such that for $\alpha > \beta(N, p)$ there exist nontrivial functions u with $\Delta^N u = 0$ of the given integrability, while for $\alpha \leq \beta(N, p)$, only $u(z) \equiv 0$ is possible. We also investigate the obstruction to uniqueness for the Dirichlet problem, that is, we study the structure of the functions in $\text{PH}_{N,\alpha}^p(\mathbb{D})$ when this space is nontrivial. We find a new structural decomposition of the polyharmonic functions – the cellular decomposition – which decomposes the polyharmonic

[☆] The research of the first author was partially supported by the ANR grant FRAB. The second author was supported by the Göran Gustafsson Foundation (KVA) and Vetenskapsrådet (VR).

* Corresponding author.

E-mail addresses: borichev@cmi.univ-mrs.fr (A. Borichev), haakanh@math.kth.se (H. Hedenmalm).

weighted L^p space in a canonical fashion. Corresponding to the cellular expansion is a tiling of part of the (p, α) plane into cells.

The above uniqueness for the Dirichlet problem may be considered for any elliptic operator of order $2N$. However, the above-mentioned critical integrability curve will depend rather strongly on the given elliptic operator, even in the constant coefficient case, for $N > 1$.

© 2014 Published by Elsevier Inc.

1. Introduction

1.1. Basic notation

Let

$$\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad dA(z) := dx dy,$$

denote the Laplacian and the area element, respectively. Here, $z = x + iy$ is the standard decomposition into real and imaginary parts. We let \mathbb{C} denote the complex plane, while $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ denote the unit disk and the unit circle, respectively.

1.2. Polyharmonic functions

Given a positive integer N , a function $u : \mathbb{D} \rightarrow \mathbb{C}$ is said to be N -harmonic (alternative terminology: *polyharmonic of degree $N - 1$*) if

$$\Delta^N u = 0$$

holds on \mathbb{D} in the sense of distribution theory. If u is N -harmonic for some N , we say that it is *polyharmonic*. Clearly, 1-harmonic functions are just ordinary harmonic functions. The 2-harmonic functions are of particular interest, and they are usually said to be *biharmonic*. While the Laplacian Δ is associated with a *membrane*, the bilaplacian Δ^2 is associated with a *plate*. There is a sizeable literature related to the bilaplacian, and more generally the N -laplacian Δ^N ; see, e.g., the books [4,10,12,14], and the papers [11, 20,21,23,17,1,15,33,34]. We could also mention that the biharmonic Green function was used in [8] to generalize the factorization of L^2 Bergman space functions found in [18] to the L^p setting (see also [19,3]). Later, it was applied to Hele–Shaw flow on surfaces [27,26,25].

1.3. Uniqueness for the Dirichlet problem

The Dirichlet problem associated with the N -Laplacian is

$$\begin{cases} \Delta^N u = 0 & \text{on } \mathbb{D}, \\ \partial_n^j u = f_j & \text{on } \mathbb{T} \text{ for } j = 0, 1, \dots, N - 1, \end{cases}$$

where ∂_n stands for the (interior) normal derivative. A natural way to interpret this problem is to first construct a function F on \mathbb{D} which encodes the boundary information from the data f_0, \dots, f_{N-1} , and to say that (1.3) asks of u that $\Delta^N u = 0$ and that $u - F$ belongs to a class of functions that decay rapidly to 0 near the boundary \mathbb{T} . Often, this decay is understood in terms of Sobolev spaces (see any book on partial differential equations; for a slightly different approach, see, e.g., [6]). A perhaps simpler requirement is to ask that

$$|u(z) - F(z)| = o((1 - |z|)^{N-1}) \quad \text{as } |z| \rightarrow 1^-, \tag{1.1}$$

and it is easy to see that u is uniquely determined by the differential equation $\Delta^N u = 0$ combined with (1.1). If we focus on uniqueness, then by forming differences we may as well assume $F(z) \equiv 0$. We may then think of (1.1) as

$$(1 - |z|)^{-N+1} |u(z)| \in L^\infty(\mathbb{D}), \tag{1.2}$$

where $L^\infty_0(\mathbb{D})$ stands for the closed subspace of $L^\infty(\mathbb{D})$ consisting of functions with limit 0 along \mathbb{T} . So (1.2) forces an N -harmonic function to vanish identically. What would happen if we replace $L^\infty_0(\mathbb{D})$ by, for instance $L^p(\mathbb{D})$, for some p , $0 < p < +\infty$? We would expect that we ought to change the exponent in the distance to the boundary, but by how much? More precisely, we would like to know for which real (negative) α the implication

$$(1 - |z|)^\alpha |u(z)| \in L^p(\mathbb{D}) \implies u \equiv 0 \tag{1.3}$$

holds for all N -harmonic functions u . Here, we shall obtain the complete answer to this question. We obtain an explicit expression $\beta(N, p)$ such that the implication (1.3) holds if and only if $\alpha \leq \beta(N, p)/p$. When $\alpha > \beta(N, p)/p$, when we have non-uniqueness, we study the source of obstruction to uniqueness (see [Theorems 3.1 and 3.2](#)). We obtain an understanding of those obstructions in terms of a decomposition of N -harmonic functions – which we call the *cellular decomposition* – associated with a nontrivial factorization of the N -Laplacian. This cellular decomposition is of course related to the classical Almansi representation, but the terms are mixed up in a complicated way, optimal for the analysis of the boundary behavior.

1.4. The local uniqueness problem

Let $J \subset \mathbb{T}$ be a proper closed arc with positive length, and let

$$Q(J) := \{z \in \mathbb{D} \setminus \{0\} : z/|z| \in J\} \quad (1.4)$$

be the corresponding angular triangle. For a subset E of \mathbb{C} , we write 1_E for the characteristic function of E , which equals 1 on E and vanishes elsewhere. The local version of the problem (1.3) reads as follows: for which α does the implication

$$1_{Q(J)}(z)(1 - |z|)^\alpha |u(z)| \in L^p(Q(J)) \implies u \equiv 0 \quad (1.5)$$

hold for all N -harmonic functions on \mathbb{D} ? The study of (1.5) can be thought of as a weighted integrability version of Holmgren's uniqueness problem (see, e.g., John's book [30]). We completely resolve this question as well: the implication in (1.5) holds if and only if $\alpha \leq -2N + 1 - \frac{1}{p}$ (see Theorem 8.1). Note that the answer does not depend on the length of the arc J . In addition it is worth observing that $\beta(N, p)/p = -2N + 1 - \frac{1}{p}$ for p in the interval $0 < p \leq 1/(2N)$, so for such small p the global (Dirichlet-type) and local (Holmgren-type) problems have the same uniqueness criteria. For other domains such as the interior of an ellipse the critical range of α will be different. See Sections 8.2 and 8.3 for details.

1.5. Remarks on the setting

It is natural to ask whether our results generalize to other domains and to more general elliptic partial differential operators. After all, our methods are quite specific to the N -Laplacian on circular disks (half-planes would work as well). It is possible to show that already replacing the circle by an ellipse while keeping the bilaplacian Δ^2 (for $N = 2$) changes the problem considered here, so that it obtains a different answer; see Sections 8.2 and 8.3 for details (cf. [22]). We remark here the appearance of a connection with the theory of quadrature domains. In conclusion, we use specific methods because the problem needs them.

2. The Almansi expansion and weighted Lebesgue spaces in the disk

2.1. Some additional notation

For $z_0 \in \mathbb{C}$ and positive real r , let $\mathbb{D}(z_0, r)$ denote the open disk centered at z_0 with radius r ; moreover, we let $\mathbb{T}(z_0, r)$ denote the boundary of $\mathbb{D}(z_0, r)$ (which is a circle). We let $ds(z) = |dz|$ denote arc length measure on curves (usually on circles such as $\mathbb{T} = \mathbb{T}(0, 1)$).

The complex differentiation operators

$$\partial_z := \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \bar{\partial}_z := \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right),$$

will be useful. Here, $z = x + iy$ is the standard decomposition of a complex number into real and imaginary parts. It is easy to check that $\Delta = 4\partial_z\bar{\partial}_z$. If we let

$$\nabla = \nabla_z := \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right)$$

denote the gradient, then we find that

$$|\nabla_z u|^2 = 2(|\partial_z u|^2 + |\bar{\partial}_z u|^2). \tag{2.1}$$

2.2. *The Almansi expansion and the extension of a polyharmonic function*

The classical *Almansi expansion* (or *Almansi representation*) asserts that u is N -harmonic if and only if it is of the form

$$u(z) = u_0(z) + |z|^2 u_1(z) + \dots + |z|^{2N-2} u_{N-1}(z), \tag{2.2}$$

where all the functions u_j are harmonic in \mathbb{D} (see, e.g., Section 32 of [10]). The harmonic functions u_j which appear in the Almansi expansion (2.2) are uniquely determined by the given N -harmonic function u , as is easy to see from the Taylor expansion of u at the origin, by appropriate grouping of the terms. This allows us to define uniquely the *extension operator* \mathbf{E} :

$$\mathbf{E}[u](z, \varrho) = u_0(z) + \varrho^2 u_1(z) + \dots + \varrho^{2N-2} u_{N-1}(z); \tag{2.3}$$

the function $\mathbf{E}[u]$ will be referred to as *the extension* of u . The extension $\mathbf{E}[u](z, \varrho)$ has the following properties:

- (a) it is an even polynomial of degree $2N - 2$ in the variable ϱ ,
- (b) it is harmonic in the variable z , and

$$\mathbf{E}[u](z, |z|) \equiv u(z). \tag{2.4}$$

In fact, the above properties (a)–(b) and (2.4) characterize the extension operator \mathbf{E} .

As for notation, we write $\text{PH}_N(\mathbb{D})$ for the linear space of all N -harmonic function in the unit disk \mathbb{D} .

2.3. The standard weighted Lebesgue spaces

Let $u : \mathbb{D} \rightarrow \mathbb{C}$ be a (Borel) measurable function. Given reals p, α with $0 < p < +\infty$, we consider the Lebesgue space $L^p_\alpha(\mathbb{D})$ of (equivalence classes of) functions u with

$$\|u\|_{p,\alpha}^p := \int_{\mathbb{D}} |u(z)|^p (1 - |z|^2)^\alpha dA(z) < +\infty. \tag{2.5}$$

These spaces are standard in the Bergman space context [24]. For $1 \leq p < +\infty$, they are Banach spaces, and for $0 < p < 1$, they are quasi-Banach spaces. Clearly, we have the inclusion

$$L^p_\alpha(\mathbb{D}) \subset L^p_{\alpha'}(\mathbb{D}) \quad \text{for } \alpha < \alpha'. \tag{2.6}$$

2.4. The L^p -type of a measurable function

We use the standard weighted Lebesgue spaces to define the concept of the L^p -type of a function.

Definition 2.1. ($0 < p < +\infty$) For a Borel measurable function $u : \mathbb{D} \rightarrow \mathbb{C}$, let its L^p -type be the number

$$\beta_p(u) := \inf\{\alpha \in \mathbb{R} : u \in L^p_\alpha(\mathbb{D})\},$$

if the infimum is taken over a non-empty collection. If instead $u \notin L^p_\alpha(\mathbb{D})$ for every $\alpha \in \mathbb{R}$, we write $\beta_p(u) := +\infty$.

The L^p -type measures the boundary growth or decay of the given function u . In particular, it is rather immediate that if u has compact support in \mathbb{D} , then its L^p -type equals $\beta_p(u) = -\infty$ for all p , $0 < p < +\infty$. It is a consequence of Hölder’s inequality that for a fixed u , the function $p \mapsto \beta_p(u)$ is convex (interpreted liberally).

Here, we want to study the p -type in the context of the spaces of N -harmonic functions $\text{PH}_N(\mathbb{D})$. If we think of the elements of the space $\text{PH}_N(\mathbb{D})$ as physical states, we may interpret the L^p -type $\beta_p(f)$ as the “ p -temperature” of the state $f \in \text{PH}_N(\mathbb{D})$. If we fix N and freeze the system – i.e., we consider only states of low “ p -temperature” – we should expect that the degrees of freedom are reduced, and eventually, only the trivial state 0 would remain. The “ p -temperature” at which this transition occurs is the critical “ p -temperature” for the given N .

In mathematical terms, we shall be concerned with the following problem.

Problem 2.2. Given a positive integer n and a p with $0 < p < +\infty$, what is the value of

$$\beta(N, p) := \inf\{\beta_p(f) : f \in \text{PH}_N(\mathbb{D}) \setminus \{0\}\} \tag{2.7}$$

In other words, what is the smallest possible L^p -type of a nontrivial function $f \in \text{PH}_N(\mathbb{D})$? Moreover, is the above infimum attained (i.e., is it a minimum)?

We call the function $p \mapsto \beta(N, p)$ the *critical integrability type curve for the N -harmonic functions*, and the function $(N, p) \mapsto \beta(N, p)$ the *critical integrability type curves for the polyharmonic functions*.

The notion of the critical integrability type is rather parallel to Makarov’s integral means spectrum in the context of bounded univalent functions [32] (see also [28]); there, however, a “sup” is used instead of an “inf” in the formula analogous to (2.7), so Makarov’s integral means spectrum is automatically convex, which is not true about the critical integrability type curve (see remark below).

Remark 2.3. (a) Since we take an infimum over a collection of f , the property that $p \mapsto \beta_p(f)$ is convex does not carry over to $p \mapsto \beta(N, p)$; indeed, examples will show that $p \mapsto \beta(N, p)$ fails to be convex.

(b) If we were to replace the “inf” with a “sup” in the above definition (2.7), we would not obtain an interesting concept, as it is easy to construct a harmonic function f on \mathbb{D} with $\beta_p(f) = +\infty$ for all $p, 0 < p < +\infty$.

2.5. *The Almansi expansion and the boundary decay of polyharmonic functions*

The Almansi expansion (2.2) can be expressed in the following form:

$$u(z) = v_0(z) + (1 - |z|^2)v_1(z) + \dots + (1 - |z|^2)^{N-1}v_{N-1}(z), \tag{2.8}$$

where all the functions v_j are harmonic in \mathbb{D} . In terms of the functions u_j of (2.2), the functions v_j are given as

$$v_j = (-1)^j \sum_{k=j}^{N-1} \binom{k}{j} u_k.$$

In view of (2.8), we might be inclined to believe that the functions u with

$$v_0 = v_1 = \dots = v_{N-2} = 0,$$

that is,

$$u = (1 - |z|^2)^{N-1}v_{N-1}(z), \tag{2.9}$$

should be the smallest near the boundary. To our surprise, we find that this is not true, at least if we understand the question in terms of Problem 2.7 with $0 < p < \frac{1}{3}$. Somehow the functions v_0, \dots, v_{N-1} can cooperate to produce nontrivial functions which decay faster than functions of the type (2.9). We think of this phenomenon as an *entanglement* (see Section 3 for more details).

2.6. *The standard weighted Lebesgue spaces of polyharmonic functions*

We put

$$\text{PH}_{N,\alpha}^p(\mathbb{D}) := \text{PH}_N(\mathbb{D}) \cap L_\alpha^p(\mathbb{D}),$$

and endow it with the norm or quasi-norm structure of $L_\alpha^p(\mathbb{D})$. This is the subspace of $L_\alpha^p(\mathbb{D})$ consisting of N -harmonic functions. It turns out that it is a closed subspace; this is rather nontrivial for $0 < p < 1$, even for $N = 1$. Actually, the proof is based on a property which we will refer to as *Hardy–Littlewood ellipticity of the Laplacian* (cf. [16]; see also [13, pp. 121–123]).

2.7. *The harmonic case ($N = 1$)*

In [2], Aleksandrov studied essentially our [Problem 2.7](#) in the case of $N = 1$ (harmonic functions). To explain the result, we make some elementary calculations. The constant function $U_0(z) \equiv 1$ is harmonic, and

$$U_0 = 1 \in \text{PH}_{1,\alpha}^p(\mathbb{D}) \iff \alpha > -1. \tag{2.10}$$

This shows that

$$\beta_p(U_0) = -1. \tag{2.11}$$

Next, we turn to the Poisson kernel at 1,

$$U_1(z) = P(z, 1) = \frac{1 - |z|^2}{|1 - z|^2}.$$

We shall need the following lemma. The formulation involves the standard Pochhammer symbol notation $(x)_j := x(x + 1) \cdots (x + j - 1)$.

Lemma 2.4. *Let a, b be two real parameters with $b \geq 0$. If we put*

$$I(a, b) := \int_{\mathbb{D}} \frac{(1 - |z|^2)^a}{|1 - z|^{2b}} dA(z),$$

then $I(a, b) = +\infty$ if $a \leq -1$ or if both $b > 0$ and $a \leq 2(b - 1)$. If, on the other hand, $a > -1$ and $b = 0$, then $I(a, b) = \pi/(a + 1)$. Moreover, if $a > -1$, $b > 0$, and $a > 2(b - 1)$, then $I(a, b) < +\infty$, with value

$$I(a, b) = \pi \sum_{j=0}^{+\infty} \frac{[(b)_j]^2}{j!(a + 1)_{j+1}}.$$

Proof. By Taylor expansion and polar coordinates, we find that

$$\begin{aligned}
 I(a, b) &= \int_{\mathbb{D}} |1 - z|^{-2b} (1 - |z|^2)^a dA(z) = \int_{\mathbb{D}} \left| \sum_{j=0}^{+\infty} \frac{(b)_j}{j!} z^j \right|^2 (1 - |z|^2)^a dA(z) \\
 &= \sum_{j=0}^{+\infty} \frac{[(b)_j]^2}{[j!]^2} \int_{\mathbb{D}} |z|^{2j} (1 - |z|^2)^a dA(z) = \pi \sum_{j=0}^{+\infty} \frac{[(b)_j]^2}{[j!]^2} \int_0^1 t^j (1 - t)^a dt.
 \end{aligned}$$

The (Beta) integral on the right-hand side diverges for $a \leq -1$, so that $I(a, b) = +\infty$ then. For $a > -1$, the Beta integral is quickly evaluated, and we obtain that

$$I(a, b) = \pi \sum_{j=0}^{+\infty} \frac{[(b)_j]^2}{j!(a + 1)_{j+1}},$$

and for $b > 0$ the sum on the right-hand side converges if and only if $a > 2(b - 1)$, by the standard approximate formulae for the Gamma function. \square

We see from [Lemma 2.4](#) that

$$U_1 = P(\cdot, 1) \in \text{PH}_{1,\alpha}^p(\mathbb{D}) \iff \alpha > \max\{p - 2, -1 - p\}, \tag{2.12}$$

which implies that its L^p -type is

$$\beta_p(U_1) = \max\{p - 2, -1 - p\}. \tag{2.13}$$

Now, in view of [\(2.11\)](#) and [\(2.13\)](#), the critical integrability type $\beta(1, p)$ satisfies

$$\beta(1, p) \leq \min\{\beta_p(U_0), \beta_p(U_1)\} = \min\{-1, \max\{p - 2, -1 - p\}\}.$$

The profound work of Aleksandrov [\[2\]](#) is mainly concerned with harmonic functions in the unit ball of \mathbb{R}^d and the maximal possible rate of decay of the L^p -integral on concentric spheres $x_1^2 + \dots + x_d^2 = r^2$. In the rather elementary planar case $d = 2$, it gives to the following result (see also Suzuki [\[39\]](#)).

Theorem 2.5. *We have that $\beta(1, p) = \min\{-1, \max\{p - 2, -1 - p\}\}$ for all $p, 0 < p < +\infty$.*

This has the interpretation that the constant function $U_0 = 1$ and the Poisson kernel $U_1 = P(\cdot, 1)$ are jointly extremal for the problem of determining the critical L^p -type for harmonic functions. Indeed, for $0 < p \leq 1$, we have $\beta(1, p) = \beta_p(U_1)$, while for $1 \leq p < +\infty$, we have instead $\beta(1, p) = \beta_p(U_0)$. The function $\beta(1, p)$ is therefore continuous and piecewise affine: $\beta(1, p) = -1 - p$ for $0 < p \leq \frac{1}{2}$, $\beta(1, p) = p - 2$ for $\frac{1}{2} \leq p \leq 1$, and $\beta(1, p) = -1$ for $1 \leq p < +\infty$.

3. Main results

3.1. Characterization of the critical integrability type curve

We let $b_j(p)$ be the function

$$b_{j,N}(p) := \max\{-1 - (j + N - 1)p, -2 + (j - N + 1)p\}, \quad j = 1, \dots, N, \quad (3.1)$$

whose graph is piecewise affine, while for $j = 0$ we put

$$b_{0,N}(p) := -1 - (N - 1)p, \quad (3.2)$$

which is affine. It is easy to check that

$$b_{j,N+1}(p) + p = b_{j,N}(p), \quad j = 0, \dots, N. \quad (3.3)$$

We present our first main theorem.

Theorem 3.1. $(0 < p < +\infty)$ For $N = 1, 2, 3, \dots$ and for real α , we have that

$$\text{PH}_{N,\alpha}^p(\mathbb{D}) = \{0\} \iff \alpha \leq \min_{j:0 \leq j \leq N} b_{j,N}(p).$$

As an immediate consequence, we obtain the explicit evaluation of the critical integrability type curve.

Theorem 3.2 (*The sawtooth theorem*). *The critical integrability type for the polyharmonic functions is given by*

$$\beta(N, p) = \min_{j:0 \leq j \leq N} b_{j,N}(p),$$

for $0 < p < +\infty$ and $N = 1, 2, 3, \dots$

Being the minimum of a finite number of continuous piecewise affine functions, the function $p \mapsto \beta(N, p)$ is then continuous and piecewise affine. It is easy to check that

$$\beta(N, p) = \min_{j:0 \leq j \leq N} b_{j,N}(p) = \min\{b_{0,N}(p), b_{1,N}(p)\} \quad \text{for } \frac{1}{3} \leq p < +\infty, \quad (3.4)$$

and since $b_{0,N}(p)$ and $b_{1,N}(p)$ equal the L^p -type of the N -harmonic functions $z \mapsto (1 - |z|^2)^{N-1}$ and $z \mapsto (1 - |z|^2)^N / |1 - z|^2$, we may interpret this as concrete support for the intuition of Section 2.5 (based on the Almansi expansion) for $\frac{1}{3} \leq p < +\infty$. However, it is again easy to verify that

$$\beta(N, p) = \min_{j:0 \leq j \leq N} b_{j,N}(p) < \min\{b_{0,N}(p), b_{1,N}(p)\} \quad \text{for } 0 < p < \frac{1}{3},$$

so the intuition fails then, and we interpret this as the appearance of entanglement.

Remark 3.3. We draw the graphs of $p \mapsto \beta(N, p)$, for $N = 2, 3$, in Fig. 1 ($N = 2$) and Fig. 2 ($N = 3$), respectively.

We prove Theorem 3.1 in Section 5; the work is based on the property of the N -Laplacian which we call Hardy–Littlewood ellipticity (see Section 4).

3.2. *The structure of polyharmonic functions*

We pass to the study of the structure of the space $\text{PH}_{N,\alpha}^p(\mathbb{D})$ when the space contains nontrivial elements. We fix an integer $N = 2, 3, 4, \dots$, and let $\mathcal{A}_N \subset \mathbb{R}^2$ be the open set

$$\mathcal{A}_N := \left\{ (p, \alpha) : 0 < p < +\infty \text{ and } \min_{j: 0 \leq j \leq N} b_{j,N}(p) < \alpha \right\};$$

then the assertion of Theorem 3.1 is equivalent to the statement

$$(p, \alpha) \in \mathcal{A}_N \iff \text{PH}_{N,\alpha}^p(\mathbb{D}) \neq \{0\}.$$

We will at times refer to \mathcal{A}_N as the *admissible region*. Let us introduce the second order elliptic partial differential operator \mathbf{L}_θ indexed by a real parameter θ ,

$$\mathbf{L}_\theta[u](z) := (1 - |z|^2)\Delta u(z) + 4\theta[z\partial_z u(z) + \bar{z}\bar{\partial}_z u(z)] - 4\theta^2 u(z). \tag{3.5}$$

We note that (when desirable) the complex derivatives can be eliminated by considering polar coordinates:

$$r\partial_r = z\partial_z + \bar{z}\bar{\partial}_z.$$

We remark here that it is possible to represent \mathbf{L}_θ as

$$\mathbf{L}_\theta[u] = (1 - |z|^2)^{2\theta+1} \nabla \cdot \left\{ (1 - |z|^2)^{-2\theta} \nabla u \right\} - 4\theta^2 u,$$

and that \mathbf{L}_θ is somewhat analogous to the partial differential operators considered in the theory of generalized axially symmetric potentials [40] in the context of half-planes. To simplify the notation, we let \mathbf{M} denote the multiplication operator given by

$$\mathbf{M}[v](z) := (1 - |z|^2)v(z).$$

We remark further that the operators \mathbf{L}_θ are related to the operators D_α introduced by Olofsson in [35]: $4\mathbf{M}^{\alpha+1}D_\alpha = \mathbf{L}_{\alpha/2}$. Moreover, we observe that for $j = 0, \dots, N - 1$,

$$\mathbf{M}^j[v] \in L_\alpha^p(\mathbb{D}) \iff v \in L_{\alpha+jp}^p(\mathbb{D}), \tag{3.6}$$

so that

$$\mathbf{M}^j[v] \in \text{PH}_{N,\alpha}^p(\mathbb{D}) \quad \text{and} \quad \Delta^{N-j}v = 0 \quad \iff \quad v \in \text{PH}_{N-j,\alpha+jp}^p(\mathbb{D}). \quad (3.7)$$

The following is our main structure theorem for the N -harmonic functions.

Theorem 3.4 (*The cellular decomposition theorem*). *Let α be real, and let p be positive. Then, for $N = 1, 2, 3, \dots$, every $u \in \text{PH}_{N,\alpha}^p(\mathbb{D})$ has a unique decomposition*

$$u = w_0 + \mathbf{M}[w_1] + \dots + \mathbf{M}^{N-1}[w_{N-1}],$$

where each term $\mathbf{M}^j[w_j]$ is in $\text{PH}_{N,\alpha}^p(\mathbb{D})$, while the functions w_j are $(N - j)$ -harmonic and solve the partial differential equation $\mathbf{L}_{N-j-1}[w_j] = 0$ on \mathbb{D} , for $j = 0, \dots, N - 1$.

Remark 3.5. (a) In the context of [Theorem 3.4](#), the mapping $u \mapsto \mathbf{M}^j[w_j]$ defines an idempotent operator \mathbf{P}_j on $\text{PH}_{N,\alpha}^p(\mathbb{D})$, for $j = 0, \dots, N - 1$. It is clear from the proof of the theorem that each \mathbf{P}_j acts continuously on $\text{PH}_{N,\alpha}^p(\mathbb{D})$, and that $\mathbf{P}_j\mathbf{P}_k = 0$ for $j \neq k$.

(b) The above cellular decomposition theorem should be compared not only with the alternative Almansi representation, but also with a result of Weinstein (see [\[41, p. 251\]](#)).

Though reminiscent of the alternative Almansi expansion [\(2.8\)](#) (compare with Pavlović [\[37\]](#)), the expansion of [Theorem 3.4](#) is different, because here, the functions w_j are not assumed harmonic, instead they solve the partial differential equation $\mathbf{L}_{N-j-1}[w_j] = 0$. We suggest to call the expansion of [Theorem 3.4](#) the *cellular decomposition*, because it relates to the cell structure of the admissible region \mathcal{A}_N (see below). It is a crucial feature of [Theorem 3.4](#) that each term of the decomposition remains in the space $\text{PH}_{N,\alpha}^p(\mathbb{D})$. This means that we may analyze each term separately.

3.3. The structure of polyharmonic functions: the entangled and unentangled regions

We consider the following relatively closed bounded subset of the admissible region \mathcal{A}_N

$$\mathcal{E}_N := \left\{ (p, \alpha) \in \mathcal{A}_N : 0 < p < \frac{1}{3} \text{ and } \alpha \leq -1 - Np \right\}.$$

We will refer to \mathcal{E}_N as the *entangled region*. The complement $\mathcal{N}_N := \mathcal{A}_N \setminus \mathcal{E}_N$ is then open, and we call it the *unentangled region*. These two regions will be further subdivided into smaller units which we refer to as *cells*. In particular, the unentangled region has a *principal unentangled cell*,

$$\mathcal{N}_N^{(1)} := \left\{ (p, \alpha) \in \mathcal{N}_N : \frac{1}{3} < p \text{ and } \alpha \leq \min\{-2 + (3 - N)p, -1 + (2 - N)p\} \right\},$$

which is relatively closed in \mathcal{N}_N .

Proposition 3.6. *We have that $(p, \alpha) \in \mathcal{A}_N$ belongs to the entangled region \mathcal{E}_N if and only if*

$$u = \mathbf{M}^{N-1}[v] \in \text{PH}_{N,\alpha}^p(\mathbb{D}) \quad \text{for harmonic } v \implies u = 0.$$

In other words, the entangled region describes where the space $\text{PH}_{N,\alpha}^p(\mathbb{D})$ contains no nontrivial functions of the form $(1 - |z|^2)^{N-1}v(z)$ with v harmonic. The principal unentangled cell has a similar-looking characterization.

Proposition 3.7. *We have that $(p, \alpha) \in \mathcal{N}_N$ belongs to the principal unentangled cell $\mathcal{N}_N^{(1)}$ if and only if*

$$u \in \text{PH}_{N,\alpha}^p(\mathbb{D}) \implies u = \mathbf{M}^{N-1}[v] \quad \text{for some harmonic } v.$$

Recent work of Olofsson [35] shows the following. We assume that θ is a nonnegative integer for technical reasons; the result may well be true for general real $\theta > -\frac{1}{2}$.

Proposition 3.8. *($0 < p < +\infty$) Fix a nonnegative integer θ and a real parameter α . For a function $v \in L_\alpha^p(\mathbb{D})$, the following are equivalent:*

- (i) *The function v is continuous in \mathbb{D} and solves $\mathbf{L}_\theta[v] = 0$ in \mathbb{D} in the sense of distributions, and*
- (ii) *The function v is of the form*

$$v(z) = \frac{1}{2\pi} \int_{\mathbb{T}} \frac{(1 - |z|^2)^{2\theta+1}}{|1 - z\bar{\xi}|^{2\theta+2}} f(\xi) ds(\xi), \quad z \in \mathbb{D},$$

for some distribution f on \mathbb{T} , where the integral is understood in the sense of distribution theory.

Proposition 3.8 tells us that the functions w_j of Theorem 3.4 may expressed as Poisson-type integrals of their distributional boundary values (compare with the remark below).

Remark 3.9. (a) In the context of Proposition 3.8, the boundary distribution f equals $c(\theta)$ times the limit of of the dilates $v_r(\zeta) = v(r\zeta)$, where $\zeta \in \mathbb{T}$, as $r \rightarrow 1^-$. Here, $c(\theta)$ is the constant

$$c(\theta) := \frac{\Gamma(1 + \theta)^2}{\Gamma(1 + 2\theta)}.$$

- (b) The above theorem of Olofsson should be compared with the work of Huber [29].

3.4. The cells of the admissible region: Entangled and unentangled

To properly analyze all the cells of the admissible region, we should first introduce a modification of the functions $b_{j,N}(p)$ given by (3.1). So, we write, for $j = 1, \dots, N$,

$$a_{j,N}(p) := \min\{b_{j,N}(p), -1 + (j - N)p\} = \begin{cases} -1 - (j + N - 1)p, & 0 < p < 1/(2j), \\ -2 + (j - N + 1)p, & 1/(2j) \leq p < 1, \\ -1 + (j - N)p, & p \geq 1, \end{cases} \tag{3.8}$$

and observe that $a_{j,N}(p) = b_{j,N}(p)$ for $0 < p \leq 1$, while $a_{j,N}(p) = -1 + (j - N)p$ for $p \geq 1$. We check that the graph of $p \mapsto a_{j,N}(p)$ is continuous and piecewise affine. The analogue of (3.3) reads as follows:

$$a_{j,N+1}(p) + p = a_{j,N}(p), \quad j = 1, \dots, N. \tag{3.9}$$

If we draw all the curves $\alpha = a_{j,N}(p)$ within the admissible region \mathcal{A}_N , they slice up the region into pieces we call *cells*. To make this precise, we proceed as follows. For a point $(p, \alpha) \in \mathcal{A}_N$, we put

$$J(p, \alpha) := \{j \in \{0, \dots, N - 1\} : \alpha > a_{N-j,N}(p)\},$$

which defines a function from \mathcal{A}_N to the collection of all subsets of $\{0, \dots, N - 1\}$. The *admissible cells* are the level sets (i.e., the sets of constancy) for this function J . As mentioned above, the admissible cells contained in the entangled region are called *entangled cells*, while those contained in the unentangled region are called *unentangled cells*; all admissible cells belong to one of these categories. We draw the cell decomposition for $N = 2, 3$ in Figs. 1 and 2.

We can now improve upon Theorem 3.4.

Theorem 3.10. *Suppose $(p, \alpha) \in \mathcal{A}_N$. Then every $u \in \text{PH}_{N,\alpha}^p(\mathbb{D})$ has a unique decomposition*

$$u = \sum_{j \in J(p,\alpha)} \mathbf{M}^j[w_j],$$

where each term $\mathbf{M}^j[w_j]$ is in $\text{PH}_{N,\alpha}^p(\mathbb{D})$, while the functions w_j are $(N - j)$ -harmonic and solve the partial differential equation $\mathbf{L}_{N-j-1}[w_j] = 0$ on \mathbb{D} , for $j \in J(p, \alpha)$.

Remark 3.11. (i) The main improvement in Theorem 3.10 is that we may now specify which terms of the cellular decomposition must necessarily vanish. The theorem is sharp, in the sense that each term $\mathbf{M}^j[w_j]$ with $j \in J(p, \alpha)$ is allowed to be nontrivial. This

is easy to see easily by considering Dirac point masses f or uniform density (constant) $f = 1$ in [Proposition 3.8](#) (compare with [Lemma 5.2](#) below). For details, see [Section 7.1](#).

(ii) The cellular decomposition can be likened to the decomposition of a vector with respect to a given basis. [Theorem 3.10](#) tells us which pieces of the “basis” are being used in a given cell. The number of elements of the set $J(p, \alpha) \subset \{0, \dots, N - 1\}$ is like the “dimension” of the subspace spanned by the given vectors. Perhaps “degrees of freedom” would be a better term.

(iii) *An example of entanglement.* Let us try to explain what [Theorem 3.10](#) says in a simple case. Let u be a biharmonic function in \mathbb{D} , so that $N = 2$. By the alternative Almansi representation [\(2.8\)](#), u has a unique representation of the form

$$u(z) = v_0(z) + \mathbf{M}[v_1](z) = v_0(z) + (1 - |z|^2)v_1(z),$$

where v_0, v_1 are harmonic. Suppose that $u \in \text{PH}_{2,\alpha}^p(\mathbb{D})$, where

$$0 < p < \frac{1}{3}, \quad \text{and} \quad -1 - 3p < \alpha \leq -1 - 2p,$$

so that we are in the entangled region (see [Fig. 1](#)). [Theorem 3.10](#) then says that $\mathbf{L}_1[u] = 0$, that is,

$$\begin{aligned} 0 &= \mathbf{L}_1[u] = \mathbf{L}_1[v_0] + \mathbf{L}_1\mathbf{M}[v_1] = \mathbf{L}_1[v_0] + \mathbf{M}\mathbf{L}_0[v_1] - 8v_1 \\ &= 4(z\partial_z v_0 + \bar{z}\bar{\partial}_z v_0) - 4v_0 - 8v_1, \end{aligned}$$

which is equivalent to

$$v_1 = \frac{1}{2}(z\partial_z v_0 + \bar{z}\bar{\partial}_z v_0) - \frac{1}{2}v_0. \tag{3.10}$$

In the above calculation, we used the operator identity [\(6.2\)](#). We conclude that v_1 is completely determined by v_0 . As the harmonic function v_0 is given in terms of its (distributional) boundary “values” on \mathbb{T} , the relation [\(3.10\)](#) expresses how $u = v_0 + \mathbf{M}[v_1]$ is given in terms of its distributional boundary “values”, which coincide with those of v_0 . This is of course in perfect agreement with the Poisson-type representation of Olofsson (see [Proposition 3.8](#)). For general N , related explicit relations hold in every entangled cell; these relations involve a certain finite collection of “free” harmonic functions (like v_0 in [\(3.10\)](#)).

4. The polyharmonic Hardy–Littlewood ellipticity and its applications

4.1. Representation of the extension in terms of the Poisson kernel

If a function $f : \mathbb{D}(0, r) \rightarrow \mathbb{C}$ is harmonic and extends continuously to the boundary, then Poisson’s formula supplies the representation

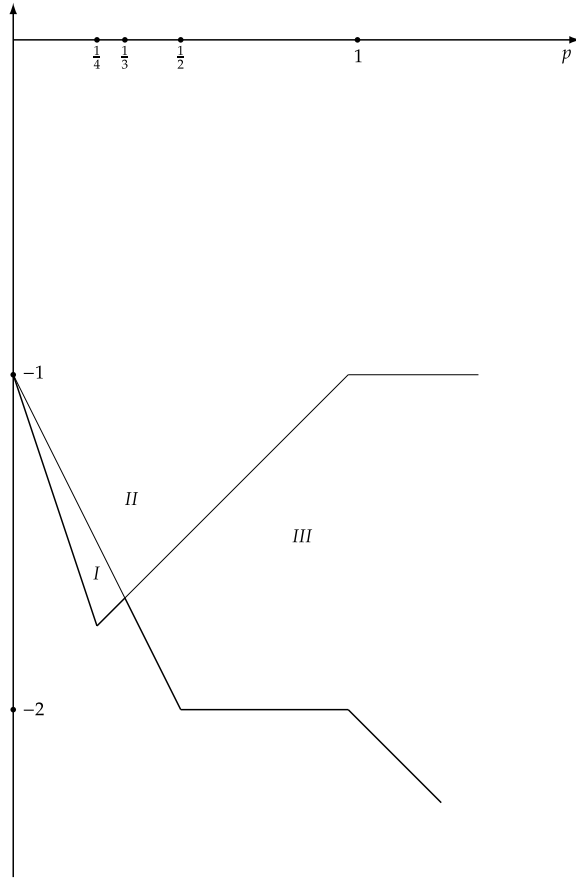


Fig. 1. The graph of $\beta(2, p)$ (bottom “sawtooth” curve, in thick style), plus indication of admissible cells. Here, I is an entangled cell, while II and III are unentangled cells (III is principal).

$$f(z) = \frac{r^2 - |z|^2}{2\pi r} \int_{\mathbb{T}(0,r)} |z - \zeta|^{-2} f(\zeta) ds(\zeta), \quad z \in \mathbb{D}(0, r), \tag{4.1}$$

where we recall that ds stands for arc length measure.

We suppose that $u \in \text{PH}_N(\mathbb{D})$, i.e. that u is N -harmonic in \mathbb{D} , and recall the notation $\mathbf{E}[u](z, \varrho)$ for the extension of u , given by (2.3). With $f(z) = \mathbf{E}[u](z, \varrho)$ and $r = \varrho$, we obtain from (4.1) that

$$\mathbf{E}[u](z, \varrho) = \frac{\varrho^2 - |z|^2}{2\pi \varrho} \int_{\mathbb{T}(0,\varrho)} |z - \zeta|^{-2} u(\zeta) ds(\zeta), \quad z \in \mathbb{D}(0, \varrho), \tag{4.2}$$

for each ϱ with $0 < \varrho < 1$, since $\mathbf{E}[u](\zeta, \varrho) = u(\zeta)$ for $\zeta \in \mathbb{T}(0, \varrho)$ by (2.4). In particular, we obtain in an elementary fashion that

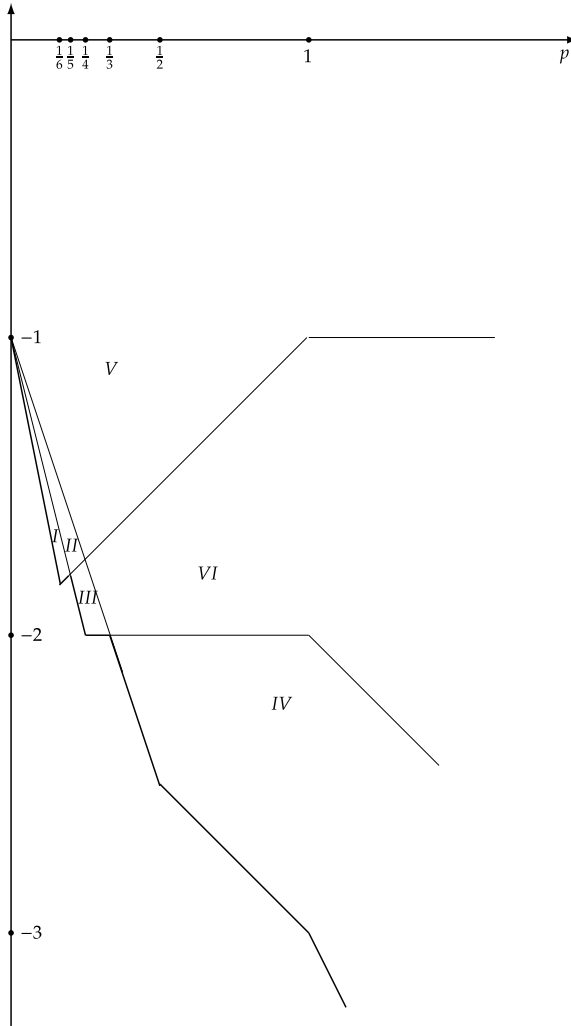


Fig. 2. The graph of $\beta(3, p)$ (bottom “sawtooth” curve, in thick style), plus indication of admissible cells. Here, *I*, *II*, and *III* are entangled cells, while *IV*, *V*, and *VI* are unentangled cells (*IV* is principal).

$$|\mathbf{E}[u](z, \varrho)| \leq \frac{1}{2\pi\varrho} \frac{\varrho + |z|}{\varrho - |z|} \int_{\mathbb{T}(0, \varrho)} |u(\zeta)| ds(\zeta), \quad z \in \mathbb{D}(0, \varrho). \tag{4.3}$$

4.2. Lagrangian interpolation of the extension of a polyharmonic function

We suppose that $u \in \text{PH}_N(\mathbb{D})$, i.e. that u is N -harmonic in \mathbb{D} , and recall the notation $\mathbf{E}[u](z, \varrho)$ for the extension of u , given by (2.3). The polynomial nature of $\varrho \mapsto \mathbf{E}[u](z, \varrho)$ makes it amenable to Lagrangian interpolation. Indeed, if we supply real values ϱ_j , where $j = 1, \dots, N$, with $0 < \varrho_1 < \varrho_2 < \dots < \varrho_N < 1$, then

$$\mathbf{E}[u](z, \varrho) = \sum_{j=1}^N \mathbf{E}[u](z, \varrho_j) L_j(\varrho), \tag{4.4}$$

where $L_j(\varrho)$ is the even interpolation polynomial

$$L_j(\varrho) := \prod_{k:k \neq j} \frac{\varrho^2 - \varrho_k^2}{\varrho_j^2 - \varrho_k^2}, \tag{4.5}$$

here, it is tacitly assumed that the product runs over the set of integers $k \in \{1, \dots, N\}$ (with the exception of j). In particular, with $\varrho = |z|$, (2.4) and (4.4) together give the representation

$$u(z) = \sum_{j=1}^N \mathbf{E}[u](z, \varrho_j) L_j(|z|), \quad z \in \mathbb{D}. \tag{4.6}$$

If we use both (4.2) and (4.6), we see that

$$u(z) = \sum_{j=1}^N L_j(|z|) \frac{\varrho_j^2 - |z|^2}{2\pi\varrho_j} \int_{\mathbb{T}(0, \varrho_j)} |z - \zeta|^{-2} u(\zeta) ds(\zeta), \quad z \in \mathbb{D}(0, \varrho_1). \tag{4.7}$$

We can think about (4.7) as a polyharmonic analogue of the Poisson representation. Let us introduce the even polynomial

$$M_j(\varrho) := L_j(\varrho) \frac{\varrho_j^2 - \varrho^2}{2\varrho_j} = -\frac{\prod_k(\varrho^2 - \varrho_k^2)}{2\varrho_j \prod_{k:k \neq j}(\varrho_j^2 - \varrho_k^2)}, \tag{4.8}$$

which is of degree $2n$ and solves the interpolation problem

$$M_j(\varrho_k) = 0 \quad \text{for all } k, \quad M'_j(\varrho_j) = 1.$$

In terms of the functions M_j , the formula (4.6) simplifies:

$$u(z) = \frac{1}{\pi} \sum_{j=1}^N M_j(|z|) \int_{\mathbb{T}(0, \varrho_j)} |z - \zeta|^{-2} u(\zeta) ds(\zeta), \quad z \in \mathbb{D}(0, \varrho_1). \tag{4.9}$$

We may of course apply the gradient to both sides:

$$\begin{aligned} \nabla u(z) = & \frac{1}{\pi} \sum_{j=1}^N \left\{ [\nabla M_j(|z|)] \int_{\mathbb{T}(0, \varrho_j)} |z - \zeta|^{-2} u(\zeta) ds(\zeta) \right. \\ & \left. + M_j(|z|) \int_{\mathbb{T}(0, \varrho_j)} [\nabla_z |z - \zeta|^{-2}] u(\zeta) ds(\zeta) \right\}, \end{aligned} \tag{4.10}$$

for $z \in \mathbb{D}(0, \varrho_1)$. We write $\varrho_1 = \vartheta$, and let δ denote the quantity

$$\delta := \min_{j,k:j \neq k} |\varrho_j^2 - \varrho_k^2|. \tag{4.11}$$

An elementary calculation then gives that

$$\begin{aligned} |M_j(|z|)| &\leq \delta^{-N+1}(\vartheta - |z|) \quad \text{and} \\ |\nabla M_j(|z|)| &\leq N\delta^{-N+1} \quad \text{for } z \in \mathbb{D}(0, \vartheta), \end{aligned} \tag{4.12}$$

while another gives that

$$\begin{aligned} |z - \zeta|^{-2} &\leq (\vartheta - |z|)^{-2} \quad \text{and} \\ |\nabla_z |z - \zeta|^{-2}| &\leq 2(\vartheta - |z|)^{-3} \quad \text{for } z \in \mathbb{D}(0, \vartheta), \zeta \in \cup_j \mathbb{T}(0, \varrho_j). \end{aligned} \tag{4.13}$$

As we implement these estimates into (4.9) and (4.10), we obtain the following estimates.

Lemma 4.1. *If $0 < \vartheta = \varrho_1 < \dots < \varrho_N < 1$, and δ is given by (4.11), then*

$$|u(z)| \leq \frac{\delta^{-N+1}}{(\vartheta - |z|)\pi} \sum_{j=1}^N \int_{\mathbb{T}(0, \varrho_j)} |u(\zeta)| ds(\zeta), \quad z \in \mathbb{D}(0, \vartheta),$$

and

$$|\nabla u(z)| \leq \frac{(N + 2)\delta^{-N+1}}{(\vartheta - |z|)^2\pi} \sum_{j=1}^N \int_{\mathbb{T}(0, \varrho_j)} |u(\zeta)| ds(\zeta), \quad z \in \mathbb{D}(0, \vartheta).$$

4.3. Hardy–Littlewood ellipticity

The next lemma is an elaboration on a theme going back to Hardy and Littlewood (1931), see Theorem 5 of [16] (see also [13, p. 121]). Consider $N = 1$ (harmonic u), and observe first that for $1 \leq p < +\infty$, the function $|u|^p$ is subharmonic, so that

$$|u(0)|^p \leq \frac{1}{\pi} \int_{\mathbb{D}} |u|^p dA.$$

For $0 < p < 1$, $|u|^p$ need not be subharmonic. However, the above inequality survives nevertheless, if the right hand side is multiplied by a suitable constant. This is an aspect of harmonic functions (and of the Laplacian) which we would like to call *Hardy–Littlewood ellipticity*. This fact was established for harmonic functions in \mathbb{R}^n by Fefferman and Stein [9] (see also Kuran [31]). This Hardy–Littlewood ellipticity survives also in the context of polyharmonic functions.

Here is a 1994 result by Pavlović [36,37] (Lemma 5).

Lemma 4.2. *($0 < p < +\infty$) There exists a positive constant $C_1(N, p)$ depending only on N and p such that for all N -harmonic functions u on the unit disk \mathbb{D} we have*

$$|u(z)|^p \leq C_1(N, p) \int_{\mathbb{D}} |u|^p dA, \quad z \in \mathbb{D} \left(0, \frac{1}{2}\right).$$

Proof. Here we follow the elegant argument by Pavlović. Alternatively, one can develop an argument based on a selection of optimal radii, which shares some features with our previous work [5] (see, e.g., the proof of Theorem 8.4 in [24]).

In the case $1 \leq p < +\infty$, the asserted estimate can be obtained in a rather straightforward fashion based on the integral representation (4.7) (sketch: we let the points ϱ_j vary over short disjoint subintervals of $[\frac{3}{4}, 1[$, and integrate both sides with respect to all the ϱ_j over those short intervals. We then apply Hölder’s inequality).

We now focus on the remaining case $0 < p < 1$. We quickly realize that it is enough to obtain the asserted estimate for a dilate u_r of u , where $u_r(z) = u(rz)$ and $0 < r < 1$. This allows us to suppose that u extends to be N -harmonic in a neighborhood of the closed unit disk. In particular, we may assume that u is bounded in \mathbb{D} .

We pick a $w \in \mathbb{D}$ such that

$$2|u(w)|^p (1 - |w|^2)^2 \geq \max_{z \in \mathbb{D}} [(1 - |z|^2)^2 |u(z)|^p]. \tag{4.14}$$

If we write $\rho := \frac{1}{2}(1 - |w|)$, we quickly realize that (4.14) leads to

$$\max_{\mathbb{D}(w, \varrho)} |u|^p \leq 32|u(w)|^p. \tag{4.15}$$

Then, in view of the already established bound for $p = 1$ (see the first portion of this proof), and (4.14), we have the estimate

$$\begin{aligned} |u(w)|(1 - |w|^2)^2 &\leq c_1(N) \int_{\mathbb{D}(w, \rho)} |u| dA \leq c_1(N) \max_{\mathbb{D}(w, \rho)} |u|^{1-p} \int_{\mathbb{D}(w, \rho)} |u|^p dA \\ &\leq 32^{(1-p)/p} c_1(N) |u(w)|^{1-p} \int_{\mathbb{D}} |u|^p dA. \end{aligned}$$

Here, $c_1(N)$ is a suitable positive constant which only depends on N . It now follows that

$$\begin{aligned} |u(z)|^p &\leq 4(1 - |z|^2)^2 |u(z)|^p \leq 8|u(w)|^p (1 - |w|^2)^2 \\ &\leq 8(32^{(1-p)/p}) c_1(N) \int_{\mathbb{D}} |u|^p dA, \quad z \in \mathbb{D} \left(0, \frac{1}{2}\right), \end{aligned}$$

as required. \square

As a consequence, we obtain the following result of Pavlović:

Corollary 4.3. $(0 < p < +\infty)$ *There exists a positive constant $C_2(N, p)$ depending only on N and p such that for all N -harmonic functions u on the unit disk \mathbb{D} we have*

$$|\nabla u(z)|^p \leq C_2(N, p) \int_{\mathbb{D}} |u|^p dA, \quad z \in \mathbb{D}\left(0, \frac{1}{2}\right).$$

For future use, we formulate the above results in the setting of a general disk $\mathbb{D}(z_0, r)$.

Corollary 4.4. $(0 < p < +\infty)$ *Suppose u is N -harmonic in the disk $\mathbb{D}(z_0, r)$, where $z_0 \in \mathbb{C}$ and the radius r is positive. Then there exist positive constants $C_1(N, p)$ and $C_2(N, p)$ depending only on N and p such that*

$$|u(z)|^p \leq \frac{C_1(N, p)}{r^2} \int_{\mathbb{D}(z_0, r)} |u|^p dA, \quad z \in \mathbb{D}\left(z_0, \frac{1}{2}r\right),$$

and

$$|\nabla u(z)|^p \leq \frac{C_2(N, p)}{r^{2+p}} \int_{\mathbb{D}(z_0, r)} |u|^p dA, \quad z \in \mathbb{D}\left(z_0, \frac{1}{2}r\right).$$

Finally, Pavlović also obtains effective pointwise and integral bounds on u and ∇u given that $u \in \text{PH}_{N, \alpha}^p(\mathbb{D})$.

Corollary 4.5. $(0 < p < +\infty)$ *Suppose u is N -harmonic in \mathbb{D} and that α is a real parameter. Then there exist constants $C_3(N, p, \alpha)$ and $C_4(N, p, \alpha)$ which depend only on N, p, α , such that*

$$|u(z_0)|^p \leq \frac{C_3(N, p, \alpha)}{(1 - |z_0|)^{\alpha+2}} \int_{\mathbb{D}} |u(z)|^p (1 - |z|^2)^\alpha dA(z), \quad z_0 \in \mathbb{D},$$

and

$$|\nabla u(z_0)|^p \leq \frac{C_4(N, p, \alpha)}{(1 - |z_0|)^{\alpha+p+2}} \int_{\mathbb{D}} |u(z)|^p (1 - |z|^2)^\alpha dA(z), \quad z_0 \in \mathbb{D}.$$

Corollary 4.6. $(0 < p < +\infty)$ *Suppose u is N -harmonic in the unit disk \mathbb{D} and that α is a real parameter. Then there exists a constant $C_5(N, p, \alpha)$ which depends only on N, p, α , such that*

$$\int_{\mathbb{D}} |\nabla u|^p (1 - |z|^2)^{\alpha+p} dA(z) \leq C_5(N, p, \alpha) \int_{\mathbb{D}} |u|^p (1 - |z|^2)^\alpha dA(z).$$

Corollary 4.7. ($0 < p < +\infty$) *If $u \in \text{PH}_{N,\alpha}^p(\mathbb{D})$, then $\partial_z u$ and $\bar{\partial}_z u$ are both in $\text{PH}_{N,\alpha+p}^p(\mathbb{D})$.*

Corollary 4.8. ($0 < p < +\infty$) *If $u \in \text{PH}_{N,\alpha}^p(\mathbb{D})$, then $\partial_z^j \bar{\partial}_z^k u \in \text{PH}_{N,\alpha+(j+k)p}^p(\mathbb{D})$ for $j, k = 0, 1, 2, \dots$*

4.4. *Control of the antiderivative*

A particular instance of [Corollary 4.7](#) is when f is holomorphic in \mathbb{D} : If $f \in L_\alpha^p(\mathbb{D})$ then $f' \in L_{\alpha+p}^p(\mathbb{D})$. In the converse direction, we have the following.

Proposition 4.9. ($0 < p < +\infty$) *Suppose f is holomorphic in \mathbb{D} , with $f' \in L_{\alpha+p}^p(\mathbb{D})$ for some real parameter α . Then:*

- (a) *If $\alpha \leq -1 - p$, then f is constant.*
- (b) *If $\alpha > -1$ and $1 \leq p < +\infty$, then $f \in L_\alpha^p(\mathbb{D})$.*
- (c) *If $-1 - p < \alpha \leq -1$ and $1 \leq p < +\infty$, then $f \in L_\beta^p(\mathbb{D})$ for every $\beta > -1$.*
- (d) *If $\frac{1}{2} < p < 1$ and $-1 - p < \alpha \leq p - 2$, then $f \in L_\beta^1(\mathbb{D}) \cap L_\beta^p(\mathbb{D})$ for every $\beta > -1$.*
- (e) *If $0 < p < 1$ and $\alpha > p - 2$, then $f \in L_{-2+(\alpha+2)/p}^1(\mathbb{D})$. Moreover, the inclusion $L_{-2+(\alpha+2)/p}^1(\mathbb{D}) \subset L_{\alpha+1-p+\epsilon}^p(\mathbb{D})$ holds for every $\epsilon > 0$.*

Proof. If $\alpha \leq -1 - p$ we must have $f'(z) \equiv 0$ which makes f constant. This settles part (a). As for parts (b) and (c), this follows from Proposition 1.11 of [\[24\]](#).

We turn to the case $0 < p < 1$. Holomorphic functions are harmonic, so [Corollary 4.5](#) applied to $u = f'$ and $N = 1$ tells us that the function

$$(1 - |z|^2)^{\alpha+p+2} |f'(z)|^p$$

is uniformly bounded in the disk \mathbb{D} . It now follows from this and our assumption $f' \in L_{\alpha+p}^p(\mathbb{D})$ that

$$\int_{\mathbb{D}} |f(z)| (1 - |z|^2)^{\alpha+p+(\alpha+p+2)(1-p)/p} dA(z) < +\infty,$$

which we simplify to $f' \in L_{-1+(\alpha+2)/p}^1(\mathbb{D})$. If $\alpha \leq p - 2$, this gives that $f \in L_\beta^1(\mathbb{D})$ for every $\beta > -1$, by Proposition 1.11 of [\[24\]](#), which settles part (d), if we also use Hölder’s inequality to obtain the L^p statement. If instead $\alpha > p - 2$, then Proposition 1.11 of [\[24\]](#) tells us that $f \in L_{-2+(\alpha+2)/p}^1(\mathbb{D})$, and part (e) follows, except for the inclusion. The inclusion is a simple consequence of Hölder’s inequality. \square

4.5. Control of individual terms in the Almansi expansion

With the help of Proposition 4.9, we may now obtain integral control of the individual terms of the Almansi expansion.

Corollary 4.10. $(0 < p < +\infty)$ *If $u \in \text{PH}_{N,\alpha}^p(\mathbb{D})$ has the Almansi expansion*

$$u(z) = u_0(z) + |z|^2 u_1(z) + \dots + |z|^{2N-2} u_{N-1}(z),$$

where the u_j are harmonic in \mathbb{D} , then

- (a) if $1 \leq p < +\infty$ and $\alpha > -1 - (N - 1)p$, then $u_{N-1} \in L_{\alpha+(N-1)p}^p(\mathbb{D})$, and
- (b) if $0 < p < 1$ and $\alpha \geq -1 - N + p$, then $u_{N-1} \in L_{\alpha+N-p+\epsilon}^p(\mathbb{D})$, for every $\epsilon > 0$.
- (c) if $1 \leq p < +\infty$ and $\alpha \leq -1 - (N - 1)p$, or if $0 < p < 1$ and $\alpha \leq -1 - N + p$, then $u_{N-1} \in L_{-1+\epsilon}^p(\mathbb{D})$, for every $\epsilon > 0$.

Proof. We split $u_j = f_j + \bar{g}_j$, where f_j, g_j are holomorphic, with $g_j(0) = 0$. We calculate that

$$\partial_z \Delta^{N-1} u(z) = (N - 1)! 4^{N-1} \partial_z^N [z^{N-1} f_{N-1}(z)],$$

and see from Corollary 4.8 that $\partial_z \Delta^{N-1} u$ is a holomorphic function in $L_{\alpha+(2N-1)p}^p(\mathbb{D})$. Integrating backwards using Proposition 4.9 shows that if $1 \leq p < +\infty$ and $\alpha > -1 - (N - 1)p$, then $f_{N-1} \in L_{\alpha+(N-1)p}^p(\mathbb{D})$, while if $0 < p < 1$ and $\alpha \geq -1 - N + p$, we instead have $f_{N-1} \in L_{\alpha+N-p+\epsilon}^p(\mathbb{D})$ for every $\epsilon > 0$. If instead $1 \leq p < +\infty$ and $\alpha \leq -1 - (N - 1)p$ or $0 < p < 1$ and $\alpha \leq -1 - N + p$, we find that $f_{N-1} \in L_{-1+\epsilon}^p(\mathbb{D})$ for every $\epsilon > 0$. Analogously, the function g_{N-1} has the same integrability properties, and then $u_{N-1} = f_{N-1} + \bar{g}_{N-1}$ automatically has the asserted properties. The proof is complete. \square

4.6. Divisibility of a polyharmonic functions by a canonical factor

Some N -harmonic functions $u(z)$ are of the form $(1 - |z|^2)\tilde{u}(z)$, where \tilde{u} is $(N - 1)$ -harmonic. The following result offers an instance when this happens.

Proposition 4.11. $(0 < p < +\infty)$ *Suppose $u \in \text{PH}_{N,\alpha}^p(\mathbb{D})$ for some integer $N = 1, 2, 3, \dots$, and a real α with $\alpha \leq \min\{p-2, -1\}$. Then if $N \geq 2$, u has the form $u(z) = (1 - |z|^2)\tilde{u}(z)$, where $\tilde{u} \in \text{PH}_{N-1,\alpha+p}^p(\mathbb{D})$, while if $N = 1$, we have that $u(z) \equiv 0$.*

Proof. We first consider the case $0 < p < 1$. Since then $\alpha \leq p - 2$, we have that $u \in \text{PH}_{N,\alpha}(\mathbb{D}) \subset \text{PH}_{N,p-2}(\mathbb{D})$, because

$$\begin{aligned} \|u\|_{p,p-2}^p &= \int_{\mathbb{D}} |u(z)|^p (1 - |z|^2)^{p-2} dA(z) \\ &\leq \int_{\mathbb{D}} |u(z)|^p (1 - |z|^2)^\alpha dA(z) = \|u\|_{p,\alpha}^p < +\infty, \end{aligned} \tag{4.16}$$

and the pointwise estimate of [Corollary 4.5](#) tells us that

$$[(1 - |z|^2)|u(z)]^p \leq 2^p C_3(N, p, p - 2) \|u\|_{p,p-2}^p, \quad z \in \mathbb{D}. \tag{4.17}$$

We conclude from [\(4.16\)](#) and [\(4.17\)](#) that

$$\int_{\mathbb{D}} \frac{|u(z)|}{1 - |z|^2} dA(z) \leq 2^{1-p} [C_3(N_0, p, p - 2)]^{(1-p)/p} \|u\|_{p,\alpha} < +\infty, \tag{4.18}$$

which by an elementary argument involving polar coordinates implies that

$$\liminf_{\varrho \rightarrow 1^-} \int_{\mathbb{T}(0,\varrho)} |u| ds = 0. \tag{4.19}$$

From the alternative Almansi representation [\(2.8\)](#), we have that

$$u(z) = v_0(z) + (1 - |z|^2)v_1(z) + \dots + (1 - |z|^2)^{N-1} v_{N-1}(z),$$

where the functions v_j are all harmonic in \mathbb{D} . The extension of u can then be written as

$$\mathbf{E}[u](z, \varrho) = v_0(z) + (1 - \varrho^2)v_1(z) + \dots + (1 - \varrho^2)^{N-1} v_{N-1}(z),$$

and by [\(4.3\)](#) combined with [\(4.19\)](#), we find that

$$|v_0(z)| = \lim_{\varrho \rightarrow 1^-} |\mathbf{E}[u](z, \varrho)| \leq \liminf_{\varrho \rightarrow 1^-} \frac{1}{2\pi\varrho} \frac{\varrho + |z|}{\varrho - |z|} \int_{\mathbb{T}(0,\varrho)} |u(\zeta)| ds(\zeta) = 0, \quad z \in \mathbb{D}, \tag{4.20}$$

that is, $v_0(z) \equiv 0$. In case $N = 1$, we are finished. In case $N \geq 2$, we note that by the alternative Almansi representation, this means that the function

$$\tilde{u}(z) := \frac{u(z)}{1 - |z|^2} = v_1(z) + (1 - |z|^2)v_2(z) + \dots + (1 - |z|^2)^{N-2} v_{N-1}(z),$$

is $(N - 1)$ -harmonic. It follows that $\tilde{u} \in \text{PH}_{N-1,\alpha+p}^p(\mathbb{D})$, as claimed.

We finally turn to the case $1 \leq p < +\infty$. Since $\alpha \leq -1$, we have

$$\|u\|_{p,-1-p}^p = \int_{\mathbb{D}} \frac{|u(z)|^p}{1 - |z|^2} dA(z) \leq \int_{\mathbb{D}} |u(z)|^p (1 - |z|^2)^\alpha dA(z) < +\infty.$$

An elementary argument now shows that

$$\liminf_{\varrho \rightarrow 1^-} \int_{\mathbb{T}(0,\varrho)} |u|^p ds = 0$$

so that in particular (recall that $1 \leq p < +\infty$) – by Hölder’s inequality –

$$\liminf_{\varrho \rightarrow 1^-} \int_{\mathbb{T}(0,\varrho)} |u| ds = 0.$$

This is (4.19). We may then use (4.20) to conclude that $v_0(z) \equiv 0$, and if $N = 1$, we are finished. If $N \geq 2$, we obtain instead that $u(z) = (1 - |z|^2)\tilde{u}(z)$ where \tilde{u} is $(N - 1)$ -harmonic. Then $\tilde{u} \in \text{PH}_{N-1,\alpha+p}^p(\mathbb{D})$, as claimed. \square

4.7. A criterion for the triviality of a polyharmonic function

We obtain a sufficient criterion for $\text{PH}_{N,\alpha}^p(\mathbb{D}) = \{0\}$.

Proposition 4.12. *($0 < p < +\infty$) Suppose α is real with $\alpha \leq -1 - (2N - 1)p$, for some integer $N = 1, 2, 3, \dots$. Then $\text{PH}_{N,\alpha}^p(\mathbb{D}) = \{0\}$.*

Proof. As the spaces $\text{PH}_{N,\alpha}^p(\mathbb{D})$ grow with α , it is enough to obtain the result when α is critically big: $\alpha = -1 - (2N - 1)p$.

Step 1. We show that the assertion holds for $N = 1$: $\text{PH}_{1,-1-p}^p(\mathbb{D}) = \{0\}$. To this end, we pick a function $v \in \text{PH}_{1,-1-p}^p(\mathbb{D})$, and observe that since v is harmonic, $\partial_z v$ is holomorphic while $\bar{\partial}_z v$ is conjugate-holomorphic. Moreover, Corollary 4.7 tells us that $\partial_z v, \bar{\partial}_z v \in \text{PH}_{1,-1}^p(\mathbb{D})$. Now $|\partial_z v|^p$ and $|\bar{\partial}_z v|^p$ are both subharmonic in \mathbb{D} , and in particular, their means on the circles $|z| = r$ increase with r , $0 < r < 1$. Using polar coordinates, then, we realize that

$$\|\partial_z v\|_{p,-1}^p + \|\bar{\partial}_z v\|_{p,-1}^p = \int_{\mathbb{D}} (|\partial_z v(z)|^p + |\bar{\partial}_z v(z)|^p) \frac{dA(z)}{1 - |z|^2} < +\infty$$

forces $\partial_z v = 0$ and $\bar{\partial}_z v = 0$, so that v must be constant. As the only constant function in $\text{PH}_{1,-1-p}^p(\mathbb{D})$ is the zero function, we obtain $v = 0$, and the conclusion $\text{PH}_{1,-1-p}^p(\mathbb{D}) = \{0\}$ is immediate.

Step 2. We show that the assertion holds for $N > 1$: $\text{PH}_{1,-1-(2N-1)p}^p(\mathbb{D}) = \{0\}$. We pick a function $u \in \text{PH}_{N,-1-(2N-1)p}^p(\mathbb{D})$ and intend to obtain that $u = 0$. Since u is N -harmonic, the function $v_1 := \Delta^{N-1}u$ is harmonic. Moreover, as

$$v_1(z) = \Delta^{N-1}u(z) = 4^{N-1}\partial_z^{N-1}\bar{\partial}_z^{N-1}u(z)$$

Corollary 4.8 now tells us that $v_1 = \Delta^{N-1}u \in \text{PH}_{1,-1-p}^p(\mathbb{D})$. This case we handled in Step 1, so that we know that $v_1 = 0$. But then the function $v_2 := \Delta^{N-2}u$ is harmonic, and by Corollary 4.8, $v_2 = \Delta^{N-2}u \in \text{PH}_{1,-1-3p}^p(\mathbb{D}) = \{0\}$ because trivially $\text{PH}_{1,-1-3p}^p(\mathbb{D}) \subset \text{PH}_{1,-1-p}^p(\mathbb{D}) = \{0\}$. If $N = 2$, we have arrived at the conclusion that $u = 0$, as needed. If $N > 2$, we continue, and form the successively the functions $v_j := \Delta^{N-j}u$ which belong to $\text{PH}_{1,-1-(2j-1)p}^p(\mathbb{D}) = \{0\}$ for $j = 3, \dots, N$. With $j = N$ we arrive at $u = v_N = 0$, as needed. \square

5. The weighted integrability structure of polyharmonic functions

5.1. A family of polyharmonic kernels

For $N = 1, 2, 3, \dots$ and $j = 0, 1, \dots, N$, let

$$U_{j,N}(z) := \frac{(1 - |z|^2)^{N+j-1}}{|1 - z|^{2j}}, \tag{5.1}$$

so that with

$$U_{j,j}(z) = \frac{(1 - |z|^2)^{2j-1}}{|1 - z|^{2j}},$$

we obtain the relation

$$U_{j,N}(z) = (1 - |z|^2)^{N-j}U_{j,j}(z). \tag{5.2}$$

The functions $U_{j,N}$ are N -harmonic, and as we shall see, they are extremal for the critical integrability type $\beta(N, p)$. The function $U_{1,1}$ is the Poisson kernel for the boundary point at 1, and the function $U_{1,2}$ is known in the context of the bilaplacian as the *harmonic compensator* [20]. We also note that the function $U_{2,2}$ appears implicitly in the biharmonic setting in [1] ($U_{2,2}(z) = 2F(z, 1) - H(z, 1)$ in their notation); cf. [34]. More recently, in [35] the kernels $U_{N,N}$ are shown to solve the Dirichlet problem in the disk \mathbb{D} for a certain (singular) second order elliptic differential operator. It remains to substantiate that $U_{j,N}$ is N -harmonic.

Lemma 5.1. *For $N = 1, 2, 3, \dots$ and $j = 0, 1, \dots, N$, the functions $U_{j,N}$ are all N -harmonic in $\mathbb{C} \setminus \{1\}$.*

Proof. By inspection, the function $U_{0,N}(z) = (1 - |z|^2)^{N-1}$ is N -harmonic. As for the other kernels $U_{j,N}$ with $1 \leq j \leq N$, the identity (5.2) together with the Almansi representation (2.2) shows that it is enough to know that $U_{j,j}$ is j -harmonic. Flipping variables, we just need to show that $U_{N,N}$ is N -harmonic. To this end, we change variables to $\zeta = 1 - z$ and see from the binomial theorem that

$$U_{N,N}(1 - \zeta) = \frac{(\zeta + \bar{\zeta} - \zeta\bar{\zeta})^{2N-1}}{\zeta^N \bar{\zeta}^N} = \sum_{j,k} (-1)^{j+k+1} \frac{(2N-1)!}{j!k!(2N-j-k-1)!} \zeta^{N-k-1} \bar{\zeta}^{N-j-1},$$

where j, k range over integers with $j, k \geq 0$ and $j + k \leq 2N - 1$. It follows that $U_{N,N}$ is N -harmonic for $\zeta \neq 0$, because in every term of the above sum, we have either $0 \leq N - k - 1 \leq N - 1$ or $0 \leq N - j - 1 \leq N - 1$, or both. An application of $\Delta_\zeta^N = 4^N \partial_\zeta^N \bar{\partial}_\zeta^N$ to each term of the finite sum then results in 0, as needed. \square

We recall the definition of the functions $b_{j,N}(p)$ from (3.1) and (3.2).

Lemma 5.2. (0 < p < +∞) For $N = 1, 2, 3, \dots$, $j = 0, \dots, N$, and real α , we have that

$$U_{j,N} \in \text{PH}_{N,\alpha}^p(\mathbb{D}) \iff \alpha > b_{j,N}(p).$$

Proof. To decide when $U_{j,N} \in \text{PH}_{N,\alpha}^p(\mathbb{D})$, we calculate:

$$\|U_{j,N}\|_{p,\alpha}^p = \int_{\mathbb{D}} |U_{j,N}|^p (1 - |z|^2)^\alpha dA = \int_{\mathbb{D}} \frac{(1 - |z|^2)^{(N+j-1)p+\alpha}}{|1 - z|^{2jp}} dA(z). \tag{5.3}$$

By Lemma 2.4, we have the following. For $j = 0$, $U_{j,N} = U_{0,N} \in \text{PH}_{N,\alpha}^p(\mathbb{D})$ if and only if $(N - 1)p + \alpha > -1$. For $1 \leq j \leq N$, however, $U_{j,N} \in \text{PH}_{N,\alpha}^p(\mathbb{D})$ if and only if both $(N + j - 1)p + \alpha > -1$ and $(N + j - 1)p + \alpha > 2(jp - 1)$. After some trivial algebraic manipulations, the assertion of the lemma is now immediate. \square

So, if α meets

$$\alpha > \min_{j:0 \leq j \leq N} b_{j,N}(p),$$

then one of the functions $U_{0,N}, U_{1,N}, \dots, U_{N,N}$ will be in $\text{PH}_{N,\alpha}^p(\mathbb{D})$, so that in particular,

$$\text{PH}_{N,\alpha}^p(\mathbb{D}) \neq \{0\}.$$

It is quite remarkable that this criterion is also necessary for $\text{PH}_{N,\alpha}^p(\mathbb{D}) \neq \{0\}$, as Theorem 3.1 says. The proof will be supplied in Section 7.

6. The structure of polyharmonic functions: the cellular decomposition

6.1. The basic properties of the partial differential operators \mathbf{L}_θ

We recall that \mathbf{L}_θ is the partial differential operator given by (3.5),

$$\mathbf{L}_\theta[u](z) = (1 - |z|^2)\Delta u(z) + 4\theta[z\partial_z u(z) + \bar{z}\bar{\partial}_z u(z)] - 4\theta^2 u(z),$$

and that \mathbf{M} is the operator of multiplication by $1 - |z|^2$. The basic operator identities satisfied by \mathbf{L}_θ are the following:

$$\Delta\mathbf{L}_\theta = \mathbf{L}_{\theta-1}\Delta, \tag{6.1}$$

and

$$\mathbf{L}_\theta\mathbf{M} = \mathbf{M}\mathbf{L}_{\theta-1} - 8\theta\mathbf{I}, \tag{6.2}$$

where \mathbf{I} is the identity operator. To obtain (6.1), we calculate as follows:

$$\begin{aligned} \Delta\mathbf{L}_\theta[u](z) &= \Delta[(1 - |z|^2)\Delta u(z) + 4\theta[z\partial_z u(z) + \bar{z}\bar{\partial}_z u(z)] - 4\theta^2 u(z)] \\ &= (1 - |z|^2)\Delta^2 u(z) - 4z\partial_z \Delta u(z) - 4\bar{z}\bar{\partial}_z \Delta u(z) - 4\Delta u(z) \\ &\quad + 4\theta[2\Delta u(z) + z\partial_z \Delta u(z) + \bar{z}\bar{\partial}_z \Delta u(z)] - 4\theta^2 \Delta u(z) \\ &= (1 - |z|^2)\Delta^2 u(z) + 4(\theta - 1)[z\partial_z \Delta u(z) + \bar{z}\bar{\partial}_z \Delta u(z)] - 4(\theta - 1)^2 \Delta u(z) \\ &= \mathbf{L}_{\theta-1}\Delta u(z). \end{aligned}$$

To instead obtain (6.2), we calculate somewhat analogously:

$$\begin{aligned} \mathbf{L}_\theta\mathbf{M}[u](z) &= (1 - |z|^2)\Delta[(1 - |z|^2)u(z)] + 4\theta[z\partial_z((1 - |z|^2)u(z)) \\ &\quad + \bar{z}\bar{\partial}_z((1 - |z|^2)u(z))] - 4\theta^2(1 - |z|^2)u(z) \\ &= (1 - |z|^2)[(1 - |z|^2)\Delta u(z) - 4z\partial_z u(z) - 4\bar{z}\bar{\partial}_z u(z) - 4u(z)] \\ &\quad + 4\theta[-2|z|^2 u(z) + (1 - |z|^2)(z\partial_z u(z) + \bar{z}\bar{\partial}_z u(z))] - 4\theta^2(1 - |z|^2)u(z) \\ &= (1 - |z|^2)^2 \Delta u(z) + 4(\theta - 1)(1 - |z|^2)[z\partial_z u(z) + \bar{z}\bar{\partial}_z u(z)] \\ &\quad - 4(\theta - 1)^2(1 - |z|^2)u(z) - 8\theta u(z) = (1 - |z|^2)\mathbf{L}_{\theta-1}[u](z) - 8\theta u(z). \end{aligned}$$

By iteration of the operator identity (6.1), we obtain more generally that

$$\Delta^j \mathbf{L}_\theta = \mathbf{L}_{\theta-j} \Delta^j, \quad j = 0, 1, 2, \dots \tag{6.3}$$

If we instead iterate (6.2), we find the following operator identity:

$$\mathbf{L}_\theta \mathbf{M}^j = \mathbf{M}^j \mathbf{L}_{\theta-j} + 4j(j - 1 - 2\theta)\mathbf{M}^{j-1}, \quad j = 0, 1, 2, \dots \tag{6.4}$$

Proposition 6.1. *We have the following factorization:*

$$\mathbf{L}_0 \mathbf{L}_1 \cdots \mathbf{L}_{n-1} = \mathbf{M}^n \Delta^n, \quad n = 1, 2, 3, \dots$$

Proof. We argue by induction. Since $\mathbf{L}_0 = \mathbf{M}\Delta$, the assertion holds trivially for $n = 1$. Next, suppose we have established the identity for $n = n_0 \geq 1$, that is,

$$\mathbf{L}_0 \mathbf{L}_1 \cdots \mathbf{L}_{n_0-1} = \mathbf{M}^{n_0} \Delta^{n_0}$$

holds. We then see that

$$\mathbf{L}_0 \mathbf{L}_1 \cdots \mathbf{L}_{n_0} = \mathbf{M}^{n_0} \Delta^{n_0} \mathbf{L}_{n_0} = \mathbf{M}^{n_0} \mathbf{L}_0 \Delta^{n_0} = \mathbf{M}^{n_0+1} \Delta^{n_0+1},$$

where we used the iterated operator identity (6.3) and that $\mathbf{L}_0 = \mathbf{M}\Delta$. The proof is complete. \square

Corollary 6.2. *Fix a positive integer n . If v solves $\mathbf{L}_{n-1}[v] = 0$ in \mathbb{D} , then v is n -harmonic in \mathbb{D} .*

Corollary 6.3. *Fix a positive integer n . If v is n -harmonic in \mathbb{D} , then $\mathbf{L}_{n-1}[u]$ is $(n - 1)$ -harmonic. If $n = 1$, this should be interpreted as $\mathbf{L}_0[u] = 0$.*

6.2. Mapping properties of \mathbf{L}_θ

We need to understand what space \mathbf{L}_θ maps $\text{PH}_{N,\alpha}^p(\mathbb{D})$ into.

Proposition 6.4. *($0 < p < +\infty$) Suppose $u \in \text{PH}_{N,\alpha}^p(\mathbb{D})$, for some integer $N = 1, 2, 3, \dots$ and a real parameter α . Then, for θ real, we have $\mathbf{L}_\theta[u] \in \text{PH}_{N,\alpha+p}^p(\mathbb{D})$.*

Proof. In view of Corollary 4.8, we have $\partial_z u, \bar{\partial}_z u \in \text{PH}_{N,\alpha+p}^p(\mathbb{D})$ and

$$\Delta u = 4\partial_z \bar{\partial}_z u \in \text{PH}_{N-1,\alpha+2p}^p(\mathbb{D}).$$

Here, we used that Δu is $(N - 1)$ -harmonic, which is immediate because u is N -harmonic. We remark that as a consequence, the function $\mathbf{M}\Delta u(z) = (1 - |z|^2)\Delta u(z)$ is N -harmonic, and then $\mathbf{M}\Delta u \in \text{PH}_{N,\alpha+p}^p(\mathbb{D})$. The assertion now follows. \square

6.3. The work of Olofsson

Given a function v on \mathbb{D} , for $0 < r < 1$, we let v_r denote its *dilate*, the function on \mathbb{T} given by $v_r(\zeta) := v(r\zeta)$. Recently, Olofsson obtained the following result [35], which is somewhat analogous to the classical theory of axially symmetric potentials due to Weinstein et al. (see, e.g., [40]).

Proposition 6.5. ($0 < p < +\infty$, $-\frac{1}{2} < \theta < +\infty$) For a continuous function $v : \mathbb{D} \rightarrow \mathbb{C}$, the following are equivalent:

- (i) The function v solves $\mathbf{L}_\theta[v] = 0$ on \mathbb{D} , and its dilates v_r converge to a distribution as $r \rightarrow 1^-$, and
- (ii) The function v is of the form

$$v(z) = \frac{1}{2\pi} \int_{\mathbb{T}} \frac{(1 - |z|^2)^{2\theta+1}}{|1 - z\bar{\xi}|^{2\theta+2}} f(\xi) d_s(\xi), \quad z \in \mathbb{D},$$

for some distribution f on \mathbb{T} , where the integral is understood in the sense of distribution theory.

We now indicate how to derive Proposition 3.8 from the above Proposition 6.5.

Proof of Proposition 3.8. We realize that Proposition 3.8 follows from Proposition 6.5 once we know that adding the following requirements in the setting of Proposition 6.5(i) forces the function v to have convergent dilates v_r as $r \rightarrow 1^-$ in the sense of distribution theory: (1) $\theta = n - 1$ is a nonnegative integer, and (2) $v \in L^p_\alpha(\mathbb{D})$. First, we observe that since $\mathbf{L}_{n-1}[v] = \mathbf{L}_\theta[v] = 0$ is assumed to hold, we know from Corollary 6.2 that v is n -harmonic, so that $v \in \text{PH}^p_{n,\alpha}(\mathbb{D})$. Next, by suitably iterating Corollary 4.10 and by combining the result with the pointwise bound of Corollary 4.5, we realize that all the harmonic functions u_j in the Almansi representation of $v \in \text{PH}^p_{n,\alpha}(\mathbb{D})$,

$$u(z) = u_0(z) + |z|^2 u_1(z) + \dots + |z|^{2n-2} u_{n-1}(z),$$

have the growth bound

$$|u_j(z)| = O((1 - |z|)^{-A}) \quad \text{as } |z| \rightarrow 1^-,$$

for some big positive constant A (which may depend on n, p, α). By a standard result in distribution theory, this means that each u_j is the Poisson integral of a distribution f_j on \mathbb{T} , and that the dilates $u_{j,r}(\zeta) := u_j(r\zeta)$ (for $\zeta \in \mathbb{T}$) converge to a nonzero constant multiple of f_j in the sense of distribution theory as $r \rightarrow 1^-$. Then the dilates of u_r of u converge in the sense of distribution theory, as needed. \square

6.4. The proof of the cellular decomposition theorem

We turn to the proof of Theorem 3.4.

Proof of Theorem 3.4. UNIQUENESS. We first treat the uniqueness part. We have the equation

$$w_0 + \mathbf{M}[w_1] + \cdots + \mathbf{M}^{N-1}[w_{N-1}] = 0, \tag{6.5}$$

where the functions w_j are $(N - j)$ harmonic and solve the partial differential equation $\mathbf{L}_{N-j-1}[w_j] = 0$ on \mathbb{D} . We need to show that all the functions w_j vanish, where $j = 0, \dots, N - 1$. We resort to an induction argument. Clearly, when $N = 1$, Eq. (6.5) just says $w_0 = 0$, as needed. Next, in the induction step we suppose that uniqueness holds for $N = N_0$, and intend to demonstrate that it must then also hold for $N = N_0 + 1$. Eq. (6.5) with $N = N_0 + 1$ reads

$$\sum_{j=0}^{N_0} \mathbf{M}^j[w_j] = 0,$$

where the functions w_j solve $\mathbf{L}_{N_0-j}[w_j] = 0$. We now apply the operator \mathbf{L}_{N_0} to both sides, and rewrite the equation using (6.4):

$$\sum_{j=0}^{N_0} \{ \mathbf{M}^j \mathbf{L}_{N_0-j}[w_j] + 4j(j - 2N_0 - 1) \mathbf{M}^{j-1}[w_j] \} = 0.$$

If we use the given information that the functions w_j solve $\mathbf{L}_{N_0-j}[w_j] = 0$, the above equation simplifies pleasantly:

$$\sum_{j=0}^{N_0-1} (j + 1)(j - 2N_0) \mathbf{M}^j[w_{j+1}] = 0.$$

By introducing the functions $\tilde{w}_j := (j + 1)(j - 2N_0)w_{j+1}$, the equation simplifies further:

$$\sum_{j=0}^{N_0-1} \mathbf{M}^j[\tilde{w}_j] = 0.$$

We observe that $\mathbf{L}_{N_0-j-1}[\tilde{w}_j] = 0$, and we are in the setting of $N = N_0$, and by the induction hypothesis, we have that $\tilde{w}_j = 0$ for all $j = 0, \dots, N_0 - 1$. As a consequence, $w_j = 0$ for all $j = 1, \dots, N_0$, and $w_0 = 0$ is then immediate from (6.5). This completes the uniqueness part of the proof.

EXISTENCE. Next, we turn to the existence part. We argue by induction in N . We first observe that the assertion is trivial for $N = 1$. In the induction step, we assume that the assertion of the theorem holds for $N = N_0 \geq 1$, and attempt to show that it must then also hold for $N = N_0 + 1$. The function u is now $(N_0 + 1)$ -harmonic, and we form the associated function $\mathbf{L}_{N_0}[u]$, which is then N_0 -harmonic, by Corollary 6.3. Since $u \in \text{PH}_{N_0+1,\alpha}^p(\mathbb{D})$, Proposition 6.4 gives that $\mathbf{L}_{N_0}[u] \in \text{PH}_{N_0,\alpha+p}^p(\mathbb{D})$. By the induction hypothesis, then, we know that

$$\mathbf{L}_{N_0}[u] = \sum_{j=0}^{N_0-1} \mathbf{M}^j[h_j],$$

where h_j are $(N_0 - j)$ -harmonic and solve $\mathbf{L}_{N_0-j-1}[h_j] = 0$. Moreover, again by the induction hypothesis, we have $\mathbf{M}^j[h_j] \in \text{PH}_{N_0, \alpha+p}^p(\mathbb{D})$, so that by (3.7), $h_j \in \text{PH}_{N_0-j, \alpha+(1+j)p}^p(\mathbb{D})$. We form the associated function

$$H := \frac{1}{4} \sum_{j=0}^{N_0-1} \frac{1}{(j+1)(2N_0-j)} \mathbf{M}^{j+1}[h_j],$$

and observe that $H \in \text{PH}_{N_0+1, \alpha}^p(\mathbb{D})$. So the sum $u + H$ is in $\text{PH}_{N_0+1, \alpha}^p(\mathbb{D})$, and we calculate that

$$\begin{aligned} \mathbf{L}_{N_0}[u + H] &= \mathbf{L}_{N_0}[u] + \mathbf{L}_{N_0}[H] = \sum_{j=0}^{N_0-1} \left\{ \mathbf{M}^j[h_j] + \frac{1}{4(j+1)(2N_0-j)} \mathbf{L}_{N_0} \mathbf{M}^{j+1}[h_j] \right\} \\ &= \sum_{j=0}^{N_0-1} \left\{ \mathbf{M}^j[h_j] + \frac{1}{4(j+1)(2N_0-j)} (\mathbf{M}^{j+1} \mathbf{L}_{N_0-j-1}[h_j] \right. \\ &\quad \left. - 4(j+1)(2N_0-j) \mathbf{M}^j[h_j]) \right\} = 0, \end{aligned}$$

where we used the operator identity (6.4) and that $\mathbf{L}_{N_0-j-1}[h_j] = 0$. So, with $w_0 := u + H$ and

$$w_j := -\frac{1}{4j(2N_0-j+1)} h_{j-1}, \quad j = 1, \dots, N_0,$$

we see that

$$u = \sum_{j=0}^{N_0} \mathbf{M}^j[w_j],$$

where w_j is $(N_0 + 1 - j)$ -harmonic with $\mathbf{L}_{N_0-j}[w_j] = 0$, for $j = 0, \dots, N_0$. Moreover, the given integrability properties of the functions h_j lead to $w_j \in \text{PH}_{N_0+1-j, \alpha+jp}^p(\mathbb{D})$ and $\mathbf{M}^j[w_j] \in \text{PH}_{N_0+1, \alpha}^p(\mathbb{D})$. The existence of the asserted expansion has now been obtained. The proof is complete. \square

7. Applications of the cellular decomposition

7.1. Triviality of individual terms in the cellular decomposition

We now analyze each term in the cellular decomposition separately. We recall the definition of the functions $a_{j,N}(p)$ from Section 3.

Proposition 7.1. $(0 < p < +\infty)$ Fix integers $N = 1, 2, 3, \dots$ and $j = 0, \dots, N - 1$. Suppose $u \in \text{PH}_{N,\alpha}^p(\mathbb{D})$ is of the form $u = \mathbf{M}^j[w]$, where w is $(N - j)$ -harmonic and $\mathbf{L}_{N-j-1}[w] = 0$. Then if $\alpha \leq a_{N-j,N}(p)$, we have that $u = 0$.

Proof. By (3.7), the function w is in $\text{PH}_{N-j,\alpha+jp}^p(\mathbb{D})$. From (3.9), we know that

$$\alpha + jp \leq a_{N-j,N}(p) + jp = a_{N-j,N-j}(p),$$

so if we introduce $N' := N - j$ and $\alpha' := \alpha + jp$, it will be sufficient to show the following: If $w \in \text{PH}_{N',\alpha'}^p(\mathbb{D})$ solves $\mathbf{L}_{N'-1}[w] = 0$, and $\alpha' \leq a_{N',N'}(p)$, then $w = 0$. We recall from (3.1) and (3.8) that

$$a_{N',N'}(p) = \min\{b_{N',N'}(p), -1\}, \quad b_{N',N'}(p) = \min\{-1 - (2N' - 1)p, -2 + p\},$$

so that

$$a_{N',N'}(p) = b_{N',N'}(p) = -1 - (2N' - 1)p \quad \text{for } 0 < p \leq \frac{1}{2N'}.$$

The assertion $w = 0$ is immediate from Proposition 4.12 in case $0 < p \leq 1/(2N')$. Next, we consider the remaining interval $1/(2N') < p < +\infty$. Since

$$a_{N',N'}(p) = \min\{p - 2, -1\} \quad \text{for } \frac{1}{2N'} < p < +\infty,$$

the assumption $\alpha \leq a_{N',N'}(p)$ entails in view of Proposition 4.11 that $w = \mathbf{M}[\tilde{w}]$, where \tilde{w} is $(N' - 1)$ -harmonic, which means that $\tilde{w} = 0$ if $N' = 1$. In view of Theorem 3.4, the function $\tilde{w} \in \text{PH}_{N'-1,\alpha+p}^p(\mathbb{D})$ then has a unique expansion

$$\tilde{w} = \sum_{j=0}^{N'-2} \mathbf{M}^j[g_j],$$

where $\mathbf{M}^j[g_j] \in \text{PH}_{N'-1,\alpha+p}^p(\mathbb{D})$ and g_j is $(N - j - 1)$ -harmonic with $\mathbf{L}_{N'-j-2}[g_j] = 0$. This means that $w = \mathbf{M}[\tilde{w}]$ has the expansion

$$w = \mathbf{M}[\tilde{w}] = \sum_{j=1}^{N'-1} \mathbf{M}^j[g_{j-1}] = \sum_{j=1}^{N'-1} \mathbf{M}^j[\tilde{g}_j], \tag{7.1}$$

where the terms are all in $\text{PH}_{N',\alpha}^p(\mathbb{D})$, and $\mathbf{L}_{N'-j-1}[\tilde{g}_j] = 0$; here we have introduced $\tilde{g}_j := g_{j-1}$. From the uniqueness of the cellular decomposition in Theorem 3.4, we realize that (7.1) is only possible if $w = 0$. The proof is complete. \square

Proposition 7.2. $(0 < p < +\infty)$ Fix integers $N = 1, 2, 3, \dots$ and $j = 0, \dots, N - 1$. Suppose α is real with $\alpha > a_{N-j,N}(p)$. Then there exists a nontrivial $u \in \text{PH}_{N,\alpha}^p(\mathbb{D})$ of the form $u = \mathbf{M}^j[w]$, where w is $(N - j)$ -harmonic with $\mathbf{L}_{N-j-1}[w] = 0$.

Proof. For $0 < p \leq 1$, we can use the function $u = U_{N-j,N} = \mathbf{M}^j[U_{N-j,N-j}]$, as given by (5.1), so that $w = U_{N-j,N-j}$. By Proposition 6.5, the function $w = U_{N-j,N-j}$ solves $\mathbf{L}_{N-j-1}[w] = 0$, by Lemma 5.2, u is in $\text{PH}_{N,\alpha}^p(\mathbb{D})$ if and only if $\alpha > b_{N-j,N}(p)$, and we have $a_{N-j,N}(p) = b_{N-j,N}(p)$ for $0 < p \leq 1$. For $1 \leq p < +\infty$, we need to consider instead the function $u = \mathbf{M}^j[w]$, where

$$w(z) = (1 - |z|^2)^{2N-2j-1} \int_{\mathbb{T}} |1 - z\bar{\xi}|^{-2N+2j} \frac{ds(\xi)}{2\pi}, \quad z \in \mathbb{D}, \tag{7.2}$$

which solves $\mathbf{L}_{N-j-1}[w] = 0$, by a second application of Proposition 6.5. The function w given by (7.2) is bounded in the disk \mathbb{D} , so that $u = \mathbf{M}^j[w]$ is in $\text{PH}_{N,\alpha}^p(\mathbb{D})$ for

$$\alpha > -1 - pj = a_{N-j,N}(p), \quad 1 \leq p < +\infty.$$

The proof is complete. \square

7.2. Characterization of the admissible region

We are now ready to supply the proof of Theorem 3.1.

Proof of Theorem 3.1. We first observe that

$$\min_{j:1 \leq j \leq N} a_{j,N}(p) = \min_{j:0 \leq j \leq N} b_{j,N}(p).$$

After that, we understand that the assertion is a consequence of the cellular decomposition of Theorem 3.4 combined with Propositions 7.1 and 7.2. \square

7.3. The entangled region

It remains to supply the proof of Proposition 3.6.

Proof of Proposition 3.6. In terms of the cellular decomposition of Theorem 3.4, it is a matter of deciding for which (p, α) the function w_{N-1} must equal 0. This is easy to do using Propositions 7.1 and 7.2. \square

7.4. The principal unentangled cell

It remains to supply the proof of Proposition 3.7.

Proof of Proposition 3.7. In terms of the cellular decomposition of Theorem 3.4, it is a matter of deciding for which (p, α) the functions w_j , with $j = 0, \dots, N - 2$, must all equal 0. This is easy to do using Propositions 7.1 and 7.2. \square

7.5. *The cellular decomposition for the general admissible cell*

It remains to supply the proof of [Theorem 3.10](#).

Proof of Theorem 3.10. The criteria of the theorem check which terms actually occur in the cellular decomposition of [Theorem 3.4](#), in accordance with [Propositions 7.1 and 7.2](#). □

8. **The local weighted integrability problem and the classical theorem of Holmgren**

8.1. *Uniqueness for the local weighted integrability criterion*

We now prove the local uniqueness theorem alluded to in [Section 1.4](#). We recall the definition of the “pie” $Q(J)$ for a given arc J of the circle \mathbb{T} . Let $L^p_\alpha(\mathbb{D})$ denote the weighted Lebesgue space supplied with the (quasi-)norm

$$\|u\|_{L^p_\alpha(\mathbb{D})}^p = \int_{\mathbb{D}} |u(z)|^p (1 - |z|^2)^\alpha dA(z) < +\infty.$$

Theorem 8.1. $(0 < p < +\infty)$ *Let J be an arc of \mathbb{T} of positive length $< 2\pi$. Then the implication*

$$1_{Q(J)}u \in L^p_\alpha(\mathbb{D}) \implies u \equiv 0$$

holds for all N -harmonic functions on \mathbb{D} if and only if $\alpha \leq -(2N - 1)p - 1$.

Proof. By considering a rotation of the function $U_{N,N}$ we realize that the condition is necessary for the implication to be valid.

In the other direction, we build on the fact that the argument which gives [Corollaries 4.6 and 4.7](#) may be essentially localized. This gives us the conclusion that

$$1_{Q(J')} \partial_z^j \bar{\partial}_z^k u \in L^p_{\alpha+(j+k)p}(\mathbb{D})$$

for any arc J' contained in J whose two endpoints are different than those of J . The rest of the argument mimics the proof of [Proposition 4.12](#), except that we need to know that a holomorphic function f cannot have

$$\int_{Q(J')} \frac{|f(z)|^p}{1 - |z|} dA(z) < +\infty$$

unless f vanishes identically. This can be obtained in a straight-forward fashion, since the integrability requirement entails that

$$\liminf_{r \rightarrow 1^-} \int_{J'} |f(r\zeta)|^p ds(\zeta) = 0,$$

and the pointwise estimates which localize [Corollary 4.5](#) yield that $|f(z)| = O((1 - |z|)^{-1/p})$ holds near J'' for any slightly smaller arc inside J' . Since $\log |f|$ is subharmonic, an elementary estimate of harmonic measure can now be used to deduce that $f(z) \equiv 0$. \square

8.2. Remarks on other domains than the disk and constant-coefficient elliptic operators

In this subsection, we will need the concept of a Schwarz function; for details, we refer to the monographs [\[7,38\]](#).

For simplicity, we will focus on the biharmonic case $N = 2$. Suppose that the disk \mathbb{D} is replaced by a bounded simply connected Jordan domain Ω with, say, C^∞ -smooth boundary. Let $\Gamma_\Omega(z, w)$ denote the Green function for Δ^2 on Ω with vanishing Dirichlet boundary data. Using Green’s formula, we may represent any biharmonic function which is C^2 -smooth up to the boundary in terms of integrals with respect to a continuous density in w along $\partial\Omega$ of the two Poisson-type kernels

$$H_1(z, w) := \Delta_w \Gamma_\Omega(z, w), \quad H_2(z) := \partial_{\mathfrak{n}(w)} \Delta_w \Gamma_\Omega(z, w),$$

for $w \in \partial\Omega$ and $z \in \Omega$. For the disk \mathbb{D} it happens that a certain nontrivial linear combination of H_1, H_2 is $O(\text{dist}(z, \partial\mathbb{D})^3)$ except near the singularity w , which is the maximal degree of flatness permitted by Holmgren’s uniqueness theorem (see the book of John [\[30\]](#)). Indeed, with $w = 1$, this is the function $U_{2,2}(z)$ appearing in [Section 5](#). We do not expect such degeneracy for more general domains. For instance, when Ω is a non-circular ellipse it is impossible to find a nontrivial biharmonic function which decays like $O(\text{dist}(z, \partial\Omega)^3)$ even along an arc of the boundary $\partial\Omega$. Indeed, we have the following result (cf. [\[22\]](#)).

Theorem 8.2. *Suppose Ω is a Jordan domain in the complex plane \mathbb{C} , and that $I \subset \partial\Omega$ is a nontrivial arc (so that it contains more than one point), which is real-analytically smooth. Let $S(z)$ be the local Schwarz function, that is, the function which is holomorphic near I with $S(z) = \bar{z}$ on I . Let F be a biharmonic function F in Ω , which extends C^2 -smoothly to $\Omega \cup I$, whose partial derivatives of order at most 2 extend to vanish on I . If F does not vanish identically on Ω , then the local Schwarz function $S(z)$ extends meromorphically to Ω .*

Proof. In the indicated setting, it is well-known that the local Schwarz function exists as a function holomorphic in a neighborhood of I with $S(z) = \bar{z}$ on I . We form the function $H(z) := \Delta F(z)$, which is harmonic in Ω and extends continuously to $\Omega \cup I$, with $H|_I = 0$. By Schwarzian reflection, the function H extends harmonically across I , in such a manner that in a neighborhood of the arc I , $H(z) = -H(\bar{S}(z))$. We may now

think of F as the solution to the Cauchy problem $\Delta F = H$ and $F|_I = \partial_n F|_I = 0$ (here, ∂_n denotes the exterior normal derivative). The Cauchy–Kovalevskaya theorem tells us that there exists a real-analytic solution to this Cauchy problem in a neighborhood of I , and by Holmgren’s theorem, the solution must be unique (see, e.g., the book of John [30]). In conclusion, F extends real-analytically across I .

In a second step, we form the function $G := \partial_z^2 F$, which is then bianalytic in Ω , as

$$16\bar{\partial}_z^2 \partial_z^2 F(z) = \Delta^2 F(z) = 0$$

in Ω , while $G(z) = 0$ holds for $z \in I$. By an Almansi-type expansion, G has the form $G(z) = G_0(z) + \bar{z}G_1(z)$, where G_0, G_1 are holomorphic in Ω and continuous in $\Omega \cup I$. Actually, by the previous argument, we know more: G_0, G_1 both extend holomorphically across I . Since $G(z) = 0$ on I , we conclude that $G_0(z) = -\bar{z}G_1(z) = -S(z)G_1(z)$ on I , so that by the uniqueness theorem for holomorphic functions, $G_0(z) = -S(z)G_1(z)$ holds on a neighborhood of I . Next, unless $G(z) \equiv 0$, this means that the Schwarz function $S(z)$ has a meromorphic extension to all of Ω given by $-G_0(z)/G_1(z)$. This argument does not apply if $G(z) \equiv 0$. In this remaining case, we realize that \bar{F} is a nontrivial bianalytic function, which vanishes on I . So the same argument applied to \bar{F} in place of G gives us a local Schwarz function $S(z)$ which extends meromorphically to Ω . \square

The following corollary should be compared with the classical uniqueness theorem of Holmgren (cf. [22]).

Corollary 8.3. *Let Ω be the domain interior to an ellipse, which is not a circle. Moreover, let I be a nontrivial arc of $\partial\Omega$. If F is a biharmonic function in Ω which extends to a C^2 -smooth function on $\Omega \cup I$, whose partial derivatives of order at most 2 vanish along I , then $F(z) \equiv 0$.*

Proof. It is well-known that the Schwarz function for the ellipse develops a branch cut along the segment between the focal points (cf. [7,38]), so it cannot in particular be meromorphic in Ω . So, in view of Theorem 8.2, we must have $F(z) \equiv 0$, as claimed. \square

Remark 8.4. If we apply a suitable affine transformation of the plane, the ellipse turns into \mathbb{D} , but the Laplacian changes to a related constant-coefficient elliptic operator of order 2. The above example of the ellipse strongly suggests that the results of this paper do not carry over to the more general setting of elliptic constant-coefficient operators of order $2N$ on the disk \mathbb{D} , even when we have the N -th power of a given elliptic constant-coefficient operator of order 2 (for $N > 1$). Indeed, for $N = 2$, this follows from the results presented below in Section 8.3, since already the local uniqueness problem finds a different solution.

8.3. The weighted integrability of biharmonic functions on an ellipse

It is a natural question to ask what happens with the local uniqueness in [Theorem 8.1](#) when the circle is replaced by an ellipse. Let Ω be the interior to an ellipse, which is assumed not to be a circle. We let $L^p_\alpha(\Omega)$ consist of the functions u with

$$\int_{\Omega} |u(z)|^p [\text{dist}(z, \partial\Omega)]^\alpha \, dA(z) < +\infty.$$

For an arc I of the ellipse $\partial\Omega$, we let $Q(I)$ denote the corresponding “pie”, whose boundary consists of two linear rays from the center of the ellipse to the endpoints of I , together with the arc I .

Theorem 8.5. *Let Ω be the domain interior to an ellipse, which is not circular. Moreover, let $I \subset \partial\Omega$ be a nontrivial boundary arc, such that the complementary arc $\partial\Omega \setminus I$ is also nontrivial. Then the implication*

$$1_{Q(I)}u \in L^p_\alpha(\Omega) \implies u \equiv 0$$

holds for all biharmonic functions u on Ω if and only if $\alpha \leq -2p - 1$.

Proof. By solving the Dirichlet problem for the bilaplacian with nontrivial smooth boundary data (but with data that vanish on I), we may obtain a nontrivial biharmonic function u on Ω which decays like $u(z) = O(\text{dist}(z, \partial\Omega)^2)$ near the arc I . This function u will have $1_{Q(I)}u \in L^p_\alpha(\Omega)$ for every $\alpha > -2p - 1$. This settles the “only if” part.

We now turn to the “if” part. So we assume $\alpha \leq -1 - 2p$ and try to show that the asserted implication holds. As in the proof of [Theorem 8.1](#), we observe that we may obtain from the technique in [Corollaries 4.6 and 4.7](#) that

$$1_{Q(I')} \partial_z^j \bar{\partial}_z^k u \in L^p_{\alpha+(j+k)p}(\Omega),$$

for any arc I' contained in I whose endpoints are different. We form the function $G(z) := \partial_z^2 u(z)$, which is bianalytic in Ω with $1_{Q(I')}G \in L^p_{\alpha+2p}(\Omega)$. The function G is bianalytic, so that $G(z) = G_1(z) + \bar{z}G_2(z)$, where G_1, G_2 are holomorphic. We know also that $G_2(z) = \bar{\partial}_z \partial_z^2 u(z)$ has $1_{Q(I')}G_2 \in L^p_{\alpha+3p}(\Omega)$. We let $S(z)$ be the Schwarz function for the boundary ellipse $\partial\Omega$, which has the property that $\bar{z} - S(z) = O(\text{dist}(z, \partial\Omega))$. We conclude that if I' is made smaller so that the “pie” $Q(I')$ avoids the foci of the ellipse, the function $1_{Q(I')}(z)(S(z) - \bar{z})G_2(z)$ is in $L^p_{\alpha+2p}(\Omega)$. Next, since $G(z) = G_1(z) + \bar{z}G_2(z)$ has $1_{Q(I')}G \in L^p_{\alpha+2p}(\Omega)$, we conclude that $1_{Q(I')}(z)\{G_1(z) + S(z)G_2(z)\}$ is in $L^p_{\alpha+2p}(\Omega) \subset L^p_{-1}(\Omega)$. The implied integral decay of the holomorphic function $G_1 + SG_2$ is too strong near the arc I' , which leads to the conclusion that $G_1(z) + S(z)G_2(z) \equiv 0$ (an argument can be based on harmonic measure estimates; compare with the proof of [Theorem 8.1](#)).

We are left with $\partial_z^2 u(z) = G_1(z) + \bar{z}G_2(z) = (\bar{z} - S(z))G_2(z)$ which is possible only if $G_2(z) \equiv 0$ as a result of the branch cuts for $S(z)$ at the foci of the ellipse. Finally, if $G_2(z) \equiv 0$ then $G(z) \equiv 0$ and \bar{u} is bianalytic. Arguing with \bar{u} in place of G settles the “if” part. The proof of the theorem is complete. \square

Remark 8.6. We see from a comparison of [Theorems 8.1 and 8.5](#) that circles and non-circular ellipses behave quite differently.

Open problem 8.7. It remains an open problem whether the critical integrability type – defined in an analogous fashion – for biharmonic functions ($N = 2$) on a bounded simply connected quadrature domain Ω is the same as for the disk or not (compare with [Theorem 3.2](#)). Even the analogue of the local theorem ([Theorem 8.1](#)) appears unknown in the context of quadrature domains.

9. Concluding remarks

9.1. Boundary effects

It would be interesting to find out necessary and sufficient conditions on a distribution f on \mathbb{T} in order to have that the potential

$$U_{N,N}[f](z) = (1 - |z|^2)^{2N-1} \int_{\mathbb{T}} |1 - z\bar{\xi}|^{-2N} f(\xi) \frac{ds(\xi)}{2\pi},$$

belongs to, say, $\text{PH}_{N,\alpha}^p(\mathbb{D})$ for $0 < p < 1/(2N)$ and $\alpha > b_{N,N}(p)$ close to $b_{N,N}(p)$. It is obvious that a sum of point masses at points of \mathbb{T} with coefficients from ℓ^p gives rise to such a distribution f . However, it is easy to see that the functions $U_{N,N}(\gamma z)$ depend continuously on $\gamma \in \mathbb{T}$ in the space $\text{PH}_{N,\alpha}^p(\mathbb{D})$, so the correct answer is not the trivial one (sums of point masses at points of \mathbb{T} with coefficients from ℓ^p).

9.2. Sharpening of some results

It is possible to sharpen the assertion of [Theorem 3.1](#) to arrive at the following results (cf. Section 4 of [\[2\]](#)).

- (i) Suppose $0 < p < +\infty$, and that u is N -harmonic in \mathbb{D} . If

$$\liminf_{r \rightarrow 1^-} (1 - r)^{\beta(N,p)} \int_{\mathbb{D} \setminus \mathbb{D}(0,r)} |u|^p \, dA = 0,$$

then $u(z) \equiv 0$.

(ii) If $0 < p < \frac{1}{2N-1}$, and if

$$\liminf_{r \rightarrow 1^-} \frac{\int_{\mathbb{D} \setminus \mathbb{D}(0,r)} |u|^p dA}{\int_{\mathbb{D} \setminus \mathbb{D}(0,r)} |U_{N,N}|^p dA} = 0,$$

then $u(z) \equiv 0$.

Here, it is useful to know that

$$\int_{\mathbb{D} \setminus \mathbb{D}(0,r)} |U_{N,N}(z)|^p dA(z) \asymp \begin{cases} (1-r)^{2-p}, & \text{if } p > \frac{1}{2N}, \\ (1-r)^{2-p} \log \frac{1}{1-r}, & \text{if } p = \frac{1}{2N}, \\ (1-r)^{1+(2n-1)p}, & \text{if } 0 < p < \frac{1}{2N}. \end{cases}$$

9.3. Remarks on the weighted integrability of polyanalytic functions

In the polyanalytic case the corresponding critical exponent is $\beta = -1 - (N - 1)p$, which may be interpreted as saying that no entanglement takes place. To be more precise, let $PA_{N,\alpha}^p(\mathbb{D})$ denote the subspace of $L_\alpha^p(\mathbb{D})$ consisting of N -analytic functions, which by definition solve the partial differential equation $\bar{\partial}_z^N f = 0$ on \mathbb{D} . Then

$$PA_{N,\alpha}^p(\mathbb{D}) = \{0\} \iff \alpha \leq -1 - (N - 1)p.$$

We explain the necessary argument for $N = 2$. Then $f \in PA_{2,\alpha}^p(\mathbb{D})$ decomposes into $f(z) = f_1(z) + \bar{z}f_2(z)$ with holomorphic f_1, f_2 , so that

$$zf(z) = zf_1(z) + |z|^2 f_2(z)$$

is biharmonic. We form the extension of zf , $\mathbf{E}[zf](z, \varrho) = zf_1(z) + \varrho^2 f_2(z)$, so that $\mathbf{E}[zf](z, \varrho) = zf(z)$ for $z \in \mathbb{T}(0, \varrho)$. Now, if

$$\int_{\mathbb{D}} |zf(z)|^p (1 - |z|^2)^{-1-p} dA(z) < +\infty,$$

an elementary argument shows that

$$\liminf_{\varrho \rightarrow 1^-} (1 - \varrho^2)^{-p} \int_{\mathbb{T}(0,\varrho)} |\mathbf{E}[zf](\zeta, \varrho)|^p ds(\zeta) = \liminf_{\varrho \rightarrow 1^-} (1 - \varrho^2)^{-p} \int_{\mathbb{T}(0,\varrho)} |\zeta f(\zeta)|^p ds(\zeta) = 0.$$

As the function $|\mathbf{E}[zf](\cdot, \varrho)|^p$ is subharmonic, we have that

$$|\mathbf{E}[zf](z, \varrho)|^p \leq \frac{1}{2\pi\varrho} \frac{\varrho + |z|}{\varrho - |z|} \int_{\mathbb{T}(0,\varrho)} |F_\varrho(\zeta)|^p ds(\zeta), \quad z \in \mathbb{D}(0, \varrho),$$

and a combination of the above tells us that $\mathbf{E}[zf](z, 1) = zf_1(z) + f_2(z) \equiv 0$. We arrive at $zf(z) = z(1 - |z|^2)f_1(z)$, and the problem reduces to $N = 1$, $\alpha \leq -1$, which is trivial.

Acknowledgments

We are grateful to Miroslav Pavlović for pointing out to us the results contained in his papers [36,37]. We also thank Elias Stein for making us aware of the theory of axially symmetric potentials, and Alexandru Ionescu for suggesting the local uniqueness problem. Finally, we thank Dmitry Khavinson for valuable comments.

References

- [1] A. Abkar, H. Hedenmalm, A Riesz representation formula for super-biharmonic functions, *Ann. Acad. Sci. Fenn. Math.* 26 (2001) 305–324.
- [2] A.B. Aleksandrov, On boundary decay in the mean of harmonic functions, *St. Petersburg Math. J.* 7 (1996) 507–542.
- [3] A. Aleman, S. Richter, C. Sundberg, Beurling’s theorem for the Bergman space, *Acta Math.* 177 (2) (1996) 275–310.
- [4] N. Aronszajn, T.M. Creese, L.J. Lipkin, *Polyharmonic Functions*, notes taken by Eberhard Gerlach, *Oxford Math. Monogr.*, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1983.
- [5] A. Borichev, H. Hedenmalm, Harmonic functions of maximal growth: invertibility and cyclicity in Bergman spaces, *J. Amer. Math. Soc.* 10 (4) (1997) 761–796.
- [6] B.E.J. Dahlberg, On the radial boundary values of subharmonic functions, *Math. Scand.* 40 (2) (1977) 301–317.
- [7] P.J. Davis, *The Schwarz Functions and Its Applications*, *Carus Math. Monogr.*, vol. 17, The Mathematical Association of America, Buffalo, NY, 1974.
- [8] P. Duren, D. Khavinson, H.S. Shapiro, C. Sundberg, Contractive zero-divisors in Bergman spaces, *Pacific J. Math.* 157 (1) (1993) 37–56.
- [9] C. Fefferman, E.M. Stein, H^p spaces of several variables, *Acta Math.* 129 (3–4) (1972) 137–193.
- [10] F.D. Gakhov, *Boundary Value Problems*, Pergamon Press, Addison–Wesley Publishing Co., Inc., Oxford, New York, Paris, Reading, MA, London, 1966.
- [11] P.R. Garabedian, A partial differential equation arising in conformal mapping, *Pacific J. Math.* 1 (1951) 485–524.
- [12] P.R. Garabedian, *Partial Differential Equations*, Second edition, Chelsea Publishing Co., New York, 1986.
- [13] J.B. Garnett, *Bounded Analytic Functions*, *Pure Appl. Math.*, vol. 96, Academic Press, Inc., New York, London, 1981.
- [14] F. Gazzola, H.-C. Grunau, G. Sweers, *Polyharmonic Boundary Value Problems. Positivity Preserving and Nonlinear Higher Order Elliptic Equations in Bounded Domains*, *Lecture Notes in Math.*, vol. 1991, Springer-Verlag, Berlin, 2010.
- [15] H.-C. Grunau, F. Robert, Positivity and almost positivity of biharmonic Green’s functions under Dirichlet boundary conditions, *Arch. Ration. Mech. Anal.* 195 (3) (2010) 865–898.
- [16] G.H. Hardy, J.E. Littlewood, Some properties of conjugate functions, *J. Reine Angew. Math.* 167 (1931) 405–423.
- [17] W.K. Hayman, B. Korenblum, Representation and uniqueness theorems for polyharmonic functions, *J. Anal. Math.* 60 (1993) 113–133.
- [18] H. Hedenmalm, A factorization theorem for square area-integrable analytic functions, *J. Reine Angew. Math.* 422 (1991) 45–68.
- [19] H. Hedenmalm, A factoring theorem for the Bergman space, *Bull. Lond. Math. Soc.* 26 (2) (1994) 113–126.
- [20] H. Hedenmalm, A computation of Green functions for the weighted biharmonic operators $\Delta|z|^{-2\alpha}\Delta$, with $\alpha > 1$, *Duke Math. J.* 75 (1) (1994) 51–78.
- [21] H. Hedenmalm, The Beurling operator for the hyperbolic plane, *Ann. Acad. Sci. Fenn. Math.* 37 (2012) 3–18.

- [22] H. Hedenmalm, On the uniqueness theorem of Holmgren, arXiv:1312.4348.
- [23] H. Hedenmalm, S. Jakobsson, S. Shimorin, A biharmonic maximum principle for hyperbolic surfaces, *J. Reine Angew. Math.* 550 (2002) 25–75.
- [24] H. Hedenmalm, B. Korenblum, K. Zhu, *Theory of Bergman spaces*, Grad. Texts in Math., vol. 199, Springer-Verlag, New York, 2000.
- [25] H. Hedenmalm, A. Olofsson, Hele–Shaw flow on weakly hyperbolic surfaces, *Indiana Univ. Math. J.* 54 (4) (2005) 1161–1180.
- [26] H. Hedenmalm, Y. Perdomo-G., Mean value surfaces with prescribed curvature form, *J. Math. Pures Appl.* (9) 83 (9) (2004) 1075–1107.
- [27] H. Hedenmalm, S. Shimorin, Hele–Shaw flow on hyperbolic surfaces, *J. Math. Pures Appl.* (9) 81 (3) (2002) 187–222.
- [28] H. Hedenmalm, S. Shimorin, Weighted Bergman spaces and the integral means spectrum of conformal mappings, *Duke Math. J.* 127 (2) (2005) 341–393.
- [29] A. Huber, On the uniqueness of generalized axially symmetric potentials, *Ann. of Math.* (2) 60 (1954) 351–358.
- [30] F. John, *Partial Differential Equations*, Fourth edition, Appl. Math. Sci., vol. 1, Springer-Verlag, New York, 1982.
- [31] Ū. Kuran, Subharmonic behaviour of $|h|^p$ ($p > 0$, h harmonic), *J. Lond. Math. Soc.* (2) 8 (1974) 217–268.
- [32] N.G. Makarov, Fine structure of harmonic measure, *St. Petersburg Math. J.* 10 (1999) 217–268.
- [33] S. Mayboroda, V. Maz’ya, Boundedness of the gradient of a solution and Wiener test of order one for the biharmonic equation, *Invent. Math.* 175 (2) (2009) 287–334.
- [34] A. Olofsson, A computation of Poisson kernels for some standard weighted biharmonic operators in the unit disc, *Complex Var. Elliptic Equ.* 53 (6) (2008) 545–564.
- [35] A. Olofsson, Differential operators for a scale of Poisson type kernels in the unit disc, *J. Anal. Math.* (2014), in press.
- [36] M. Pavlović, On subharmonic behaviour and oscillation of functions on balls in \mathbb{R}^n , *Publ. Inst. Math. (Beograd) (N.S.)* 55 (69) (1994) 18–22.
- [37] M. Pavlović, Decompositions of L^p and Hardy spaces of polyharmonic functions, *J. Math. Anal. Appl.* 216 (2) (1997) 499–509.
- [38] H.S. Shapiro, *The Schwarz function and its generalization to higher dimensions*, Univ. Arkansas Lect. Notes Math. Sci., vol. 9, Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1992.
- [39] N. Suzuki, Nonintegrability of harmonic functions in a domain, *Jpn. J. Math.* 16 (1990) 269–278.
- [40] A. Weinstein, Generalized axially symmetric potential theory, *Bull. Amer. Math. Soc.* 59 (1953) 20–38.
- [41] A. Weinstein, On a class of partial differential equations of even order, *Ann. Mat. Pura Appl.* (4) 39 (1955) 245–254.