BOUNDARY PROPERTIES OF GREEN FUNCTIONS IN THE PLANE

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Introduction

Ω – bounded simply connected domain in $\mathbf{C}$ which contains 0.

$\varphi : \mathbf{D} \to \Omega$ – the conformal mapping with $\varphi(0) = 0$, $\varphi'(0) > 0$.

$G_\Omega(z, w)$ is the Green function for Ω $(z, w \in \Omega)$.

We write $G_\Omega(z) = G_\Omega(z, 0)$.

Wirtinger derivatives:

$$\partial_z = \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\right), \qquad \bar{\partial}_z = \frac{1}{2}\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right).$$

Multiplicative counterparts:

$$\partial_z^\times = z\partial_z, \qquad \bar{\partial}_z^\times = \bar{z}\bar{\partial}_z.$$

PROBLEM. Compare, for complex τ and real α,

$$\left|\left[\partial_z^\times G_\Omega(z)\right]^\tau\right| \quad \text{with} \quad |G_\Omega(z)|^{-\alpha}.$$

More precisely, when do we have

$$(1) \qquad \int_\Omega \left|\left[\partial_z^\times G_\Omega(z)\right]^\tau\right| |G_\Omega(z)|^\alpha \, \mathrm{d}A(z) < +\infty?$$

We denote by $A_\Omega(\tau)$ the "best possible" α for a given τ.

MAIN THEOREM. We have

$$A_\Omega(\tau) \leq -\mathrm{Re}\,\tau + \left[\frac{9e^2}{2} + o(1)\right] |\tau|^2 \log \frac{1}{|\tau|}$$

as $|\tau| \to 0$. The $o(1)$ term is independent of the choice of the bounded simply connected domain Ω.

If $A_\Omega(\tau) + A_\Omega(-\tau) \leq 0$, our scheme of comparing the quantities in (1) in terms of L^1 integrals is very successful. It is therefore natural to view the quadratic-logarithmic remainder term in the Main Theorem as the amount by which the L^1 comparison might fail.

In terms of φ,

$$G_\Omega(\varphi(z)) = \log(|z|^2), \qquad z \in \mathbf{D},$$

and we get

$$\int_\Omega \left|[\partial_z^\times G(z)]^\tau\right| |G(z)|^\alpha \,\mathrm{d}A(z)$$

$$= \int_\mathbf{D} \left| \left[\frac{z\varphi'(z)}{\varphi(z)}\right]^{-\tau} \right| \left\{\log \frac{1}{|z|^2}\right\}^\alpha |\varphi'(z)|^2 \mathrm{d}A(z).$$

Integral means spectra. Let $B_\varphi(\tau)$ be "the best" β such that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \left| \left[\frac{re^{i\theta}\varphi'(re^{i\theta})}{\varphi(re^{i\theta})}\right]^\tau \right| \mathrm{d}\theta = O\left(\frac{1}{(1-r)^\beta}\right)$$

as $r \to 1^-$. It is possible to show that

$$B_\varphi(\tau) = A_\Omega(2-\tau) + 1$$

for all complex τ. the *universal integral means spectrum* for the class of bounded univalent functions S_b is the function $B_b(\tau)$, obtained by taking the sup of $B_\varphi(\tau)$ over all φ. As a consequence of the Main Theorem, we get

$$B_b(2-\tau) \le 1 - \mathrm{Re}\tau + \left[\frac{9e^2}{2} + o(1)\right] |\tau|^2 \log \frac{1}{|\tau|}$$

as $|\tau| \to 0$. For *real* τ, P. W. Jones and N. G. Makarov obtained a smaller error term:

$$B_b(2-\tau) \le 1 - \tau + O(\tau^2), \qquad \mathbf{R} \ni \tau \to 0.$$

The Grunsky identity and generalizations

We need the *Cauchy transform* $\mathcal{C}_\Omega$,

$$\mathcal{C}_\Omega[f](z) = \int_\Omega \frac{f(w)}{w-z}\, \mathrm{d}A(w),$$

and the *Beurling transform*

$$\mathcal{B}_\Omega[f](z) = \partial_z \mathcal{C}_\Omega[f](z) = \mathrm{pv}\int_\Omega \frac{f(w)}{(w-z)^2}\, \mathrm{d}A(w).$$

It is clear that in the sense of distribution theory,

$$\bar{\partial}_z \mathcal{C}_\Omega[f](z) = -f(z), \qquad z \in \Omega,$$

and

$$\partial_z \mathcal{C}_\Omega[f](z) = \mathcal{B}_\Omega f(z).$$

It is well-known that for $\Omega = \mathbf{C}$, $\mathcal{B}_\mathbf{C}$ is a *unitary transformation* $L^2(\mathbf{C}) \to L^2(\mathbf{C})$. In general, $\mathcal{B}_\Omega$ is a contraction $L^2(\Omega) \to L^2(\Omega)$.

We connect two functions f and g, on Ω and $\mathbf{D}$, respectively, via

$$g(z) = \bar{\varphi}'(z)\, f \circ \varphi(z),$$

and define the integral operator

$$\begin{aligned} \mathcal{C}_\varphi[g](z) =& (\mathcal{C}_\Omega[f]) \circ \varphi(z) \\ =& \int_\mathbf{D} \frac{\varphi'(w)}{\varphi(w) - \varphi(z)}\, g(w)\, \mathrm{d}A(w), \qquad z \in \mathbf{D}; \end{aligned}$$

$\mathcal{C}_\varphi$ is then a contraction $L^2(\mathbf{D}) \to W^{1,2}(\mathbf{D})/\mathbf{C}$.

It is known that $\mathcal{B}_\mathbf{C}$ is bounded $L^p(\mathbf{C}) \to L^p(\mathbf{C})$, for all p with $1 < p < +\infty$. Let $K(p)$ be a positive constant such that

(2) $$\|\mathcal{B}_\mathbf{C} f\|_{L^p(\mathbf{C})} \le K(p)\, \|f\|_{L^p(\mathbf{C})}, \qquad f \in L^p(\mathbf{C}).$$

The optimal constant $K(p)$ in (2) is not known; however, we may choose, e. g., $K(p) = 2(p^* - 1)$, where $p^* = \max\{p, p'\}$, and $p' = p/(p-1)$ is the dual exponent (one expects $K(p) = p^* - 1$ is the optimal choice). For $0 \le \theta \le 2$, we introduce the *θ-skewed Beurling transform*, as defined by

$$\mathcal{B}^\theta_\varphi[f] = \operatorname{pv} \int_{\mathbf{D}} \frac{\varphi'(z)^\theta \varphi'(w)^{2-\theta}}{(\varphi(z) - \varphi(w))^2} f(w) \, \mathrm{d}A(w).$$

It follows from (2) that

$$\|\mathcal{B}^{2/p}_\varphi f\|_{L^p(\mathbf{D})} \le K(p) \, \|f\|_{L^p(\mathbf{D})}, \qquad f \in L^p(\mathbf{D}),$$

for all p with $1 < p < +\infty$. In the symmetric case $\theta = 1$, we write $\mathcal{B}_\varphi$ in place of $\mathcal{B}^1_\varphi$. We note that $\mathcal{B}_\varphi$ is a contraction on $L^2(\mathbf{D})$.

BASIC IDENTITY. We have the identity

$$\log \frac{z(\varphi(z) - \varphi(\zeta))}{(z - \zeta)\varphi(z)} + \log(1 - \bar{z}\zeta) = \int_{\mathbf{D}} \frac{\varphi'(w)}{\varphi(w) - \varphi(z)} \frac{\zeta}{1 - \bar{w}\zeta} \, \mathrm{d}A(w).$$

GRUNSKY IDENTITY (integral form). We have

$$\frac{\varphi'(z)\varphi'(\zeta)}{(\varphi(z) - \varphi(\zeta))^2} - \frac{1}{(z - \zeta)^2} = \int_{\mathbf{D}} \frac{\varphi'(z)\varphi'(w)}{(\varphi(w) - \varphi(z))^2} \frac{1}{(1 - \bar{w}\zeta)^2} \, \mathrm{d}A(w).$$

Let $\mathcal{P}$ be the Bergman projection operator

$$\mathcal{P}[f](z) = \int_{\mathbf{D}} \frac{f(w)}{(1-\bar{w}z)^2}\, \mathrm{d}A(w),$$

which is the orthogonal projection $L^2(\mathbf{D}) \to A^2(\mathbf{D})$. Let $\mathcal{B} = \mathcal{B}_{\mathbf{D}}$.

GRUNSKY IDENTITY (operator form).

$$\mathcal{B}_\varphi - \mathcal{B} = \mathcal{P}\mathcal{B}_\varphi = \mathcal{B}_\varphi \bar{\mathcal{P}} = \mathcal{P}\mathcal{B}_\varphi \bar{\mathcal{P}}.$$

The strong Grunsky inequality is equivalent to the statement that $\mathcal{B}_\varphi - \mathcal{B}$ is a contraction on $L^2(\mathbf{D})$, which immediately follows from the above.

Let $\mathcal{D}$ denote the operator

$$\mathcal{D}[f](z) = \int_{\mathbf{D}} \frac{f(w)}{(w-z)(1-\bar{w}z)}\, \mathrm{d}A(w),$$

and $\mathcal{M}_F$ the operator of multiplication by F.

SKEWED GRUNSKY IDENTITY. $(0 < \theta < 2)$

$$\mathcal{B}_\varphi^\theta - \mathcal{B} + (\theta - 1)\mathcal{D}\mathcal{M}_{1-|z|^2}\mathcal{M}_{\varphi''/\varphi'} = \mathcal{P}\mathcal{B}_\varphi^\theta.$$

The skewed Grunsky identity is suitable for the space $L^p(\mathbf{D})$, provided $\theta = 2/p$.

VARIANT OF BASIC IDENTITY. We have

$$\log \frac{z(\varphi(z)-\varphi(\zeta))}{(z-\zeta)\varphi(z)}$$
$$-\zeta(1-|\zeta|^2)\left[\frac{\varphi'(\zeta)}{\varphi(\zeta)-\varphi(z)}-\frac{1}{\zeta-z}\right]$$
$$+\log\left(1-\bar{z}\zeta\right)+\bar{z}\zeta\,\frac{1-|\zeta|^2}{1-\bar{z}\zeta}$$
$$=\zeta^2\int_{\mathbf{D}}\frac{\varphi'(w)}{\varphi(w)-\varphi(z)}\,\frac{\bar{\zeta}-\bar{w}}{(1-\bar{w}\zeta)^2}\,\mathrm{d}A(w).$$

LEMMA.

$$(1-|\zeta|^2)\left|\frac{\varphi'(\zeta)}{\varphi(\zeta)-\varphi(z)}-\frac{1}{\zeta-z}\right|\le C.$$

COROLLARY.

$$\log\frac{z\varphi'(z)}{\varphi(z)}+\log\left(1-|z|^2\right)$$
$$=z^2\int_{\mathbf{D}}\frac{\varphi'(w)}{\varphi(w)-\varphi(z)}\,\frac{\bar{z}-\bar{w}}{(1-\bar{w}z)^2}\,\mathrm{d}A(w)+O(1).$$

Marcinkiewicz-Zygmund integrals

Suppose $0<\kappa<1$. Let $\delta(w)$ be the Euclidean distance from w to $\mathbf{C}\setminus\Omega$. Pick a real parameter γ, confined to the interval $0 < \gamma < 1$. The Marcinkiewicz-Zygmund integral is defined by the formula

$$I_\kappa(z) = \int_\Omega \min\left\{ \frac{\delta(w)^\kappa}{|z-w|^{2+\kappa}}, \frac{\gamma^{-2-\kappa}}{\delta(w)^2} \right\} \mathrm{d}A(w).$$

Zygmund showed (essentially) in 1969 that

$$\left\| e^{\lambda I_\kappa} - 1 \right\|_{L^1(\mathbf{C})} \le \frac{\kappa\,|\Omega|_A}{\kappa - 9e|\lambda|\gamma^{-\kappa}(2+\kappa)} - |\Omega|_A$$

for complex λ with

$$|\lambda| < \frac{\kappa\,\gamma^\kappa}{9e(2+\kappa)}.$$

Uniform Sobolev imbedding

We work with

$$\text{(3)}\quad \widetilde{\mathcal{C}}_\varphi[f](z) = \int_{\mathbf{D}} \frac{\varphi'(w)}{\varphi(w)-\varphi(z)} \, \frac{\bar z - \bar w}{1-\bar w z} f(w)\,\mathrm{d}A(w).$$

For $0 < \kappa < 1$, we consider the Lebesgue space

$$X_\kappa(\mathbf{D}) = L^p(\mathbf{D}, \mu),$$

where

$$p = \frac{2+\kappa}{1+\kappa}, \qquad \mathrm{d}\mu(z) = (1-|z|^2)^{-\kappa/(1+\kappa)}\,\mathrm{d}A(z).$$

By Hölder's inequality, we get

$$\begin{aligned} &\left|\widetilde{\mathcal{C}}_\varphi[f](z)\right| \\ &\qquad\le\left\{\int_{\mathbf{D}}\left|\frac{(w-z)\varphi'(w)}{(1-\bar{w}z)(\varphi(w)-\varphi(z))}\right|^{2+\kappa}\right. \\ &\qquad\qquad\left.\times(1-|w|^2)^\kappa\,\mathrm{d}A(w)\right\}^{1/(2+\kappa)} \\ &\qquad\qquad\times\|f\|_{X_\kappa(\mathbf{D})}. \end{aligned}$$

The function

$$\begin{aligned} &J_\kappa[\varphi](z) \\ &\quad=\int_{\mathbf{D}}\left|\frac{(w-z)\varphi'(w)}{(1-\bar{w}z)(\varphi(w)-\varphi(z))}\right|^{2+\kappa}(1-|w|^2)^\kappa\mathrm{d}A(w) \end{aligned}$$

is essentially the familiar Marcinkiewicz-Zygmund integral:

$$J_k[\varphi](z)\le 4^\kappa\,I_\kappa(\varphi(z))+O(1),\qquad z\in\mathbf{D}.$$

We get:

UNIFORM SOBOLEV IMBEDDING. For complex λ with

$$|\lambda|<\frac{\kappa 4^{-\kappa}}{9e(2+\kappa)},$$

we have

$$\int_{\mathbf{D}} \exp\left\{ |\lambda| \sup_{f\in \mathrm{ball}(X_\kappa(\mathbf{D}))} \left| \widetilde{\mathcal{C}}_\varphi\left[f\right](z)\right|^{2+\kappa} \right\} \times |\varphi'(z)|^2 \, \mathrm{d}A(z) < +\infty.$$

The proof of the main theorem

By the Corollary on p. 7,

$$\log \frac{z\varphi'(z)}{\varphi(z)} + \log(1-|z|^2) = z^2 \widetilde{\mathcal{C}}_\varphi[g_z](z) + O(1),$$

where

$$g_z(w) = \frac{1}{1-\bar{w}z}.$$

We plan to apply the Uniform Sobolev Imbedding to the function $f = f_z = g_z/\|g_z\|_{X_\kappa(\mathbf{D})}$. We get

$$\|g_z\|^{2+\kappa}_{X_\kappa(\mathbf{D})} \sim \left[\frac{\Gamma\left(\frac{1-\kappa}{1+\kappa}\right)}{\Gamma\left(\frac{1}{1+\kappa}\right)^2} \log \frac{1}{1-|z|^2} \right]^{1+\kappa}.$$

Let Λ be such that

$$\Lambda > \left[\frac{\Gamma\left(\frac{1-\kappa}{1+\kappa}\right)}{\Gamma\left(\frac{1}{1+\kappa}\right)^2} \right]^{1+\kappa}.$$

We now find that for

$$|\lambda| < \frac{\kappa 4^{-\kappa}}{9e(2+\kappa)},$$

$$\int_{\mathbf{D}} \exp\left\{ \frac{|\lambda|}{\Lambda} \left| 1 - \frac{\log \frac{z\,\varphi'(z)}{\varphi(z)}}{\log \frac{1}{1-|z|^2}} \right|^{2+\kappa} \log \frac{1}{1-|z|^2} \right\}$$

$$\times |\varphi'(z)|^2 \mathrm{d}A(z) < +\infty.$$

It remains to apply a *linear approximation argument.* We apply the convexity estimate (a, b complex)

$$\begin{aligned} |a|^{2+\kappa} = |\bar{a}|^{2+\kappa} & \\ \geq & |b|^{2+\kappa} - (2+\kappa)|b|^{\kappa} \mathrm{Re}\big[b(\bar{b} - a)\big] \\ & = |b|^{2+\kappa} + (2+\kappa)|b|^{\kappa}\big[\mathrm{Re} b - |b|^2\big] \\ & - (2+\kappa)|b|^{\kappa} \mathrm{Re}\big[b(1-a)\big], \end{aligned}$$

to

$$a = 1 - \log \frac{z\varphi'(z)}{\varphi(z)} \log \frac{1}{1-|z|^2},$$

and obtain

$$\left| 1 - \frac{\log \frac{z\,\varphi'(z)}{\varphi(z)}}{\log \frac{1}{1-|z|^2}} \right|^{2+\kappa} \log \frac{1}{1-|z|^2}$$

$$\geq \Big[|b|^{2+\kappa} + (2+\kappa)|b|^{\kappa}\big[\mathrm{Re} b - |b|^2\big]\Big] \log \frac{1}{1-|z|^2}$$

$$- (2+\kappa)|b|^{\kappa} \mathrm{Re}\left[b \log \frac{z\varphi'(z)}{\varphi(z)}\right]$$

for any $b \in \mathbf{C}$. We now insert this estimate into the estimate we got from the Uniform Sobolev Imbedding, and find that

$$\int_{\mathbf{D}} \exp\left\{\frac{|\lambda|}{\Lambda}\left[|b|^{2+\kappa} + (2+\kappa)|b|^{\kappa}\left[\mathrm{Re} b - |b|^2\right]\right] \times \log\frac{1}{1-|z|^2} - \frac{|\lambda|}{\Lambda}(2+\kappa)|b|^{\kappa}\mathrm{Re}\left[b\log\frac{z\varphi'(z)}{\varphi(z)}\right]\right\} \times |\varphi'(z)|^2 \mathrm{d}A(z) < +\infty.$$

Next, we assume $b \neq 0$, and put $\tau = \Lambda^{-1}|\lambda|(2+\kappa)\,|b|^{\kappa}b$. Note also that

$$\exp\left\{-\frac{|\lambda|}{\Lambda}(2+\kappa)\,|b|^{\kappa}\,\mathrm{Re}\left[b\log\frac{z\varphi'(z)}{\varphi(z)}\right]\right\} = \left|\left[\frac{z\varphi'(z)}{\varphi(z)}\right]^{-\tau}\right|.$$

We now get that (in view of the restrictions on λ, Λ)

$$\int_{\mathbf{D}} \left|\left[\frac{z\varphi'(z)}{\varphi(z)}\right]^{-\tau}\right| (1-|z|^2)^{-\mathrm{Re}\tau + R(\tau)} \times |\varphi'(z)|^2 \mathrm{d}A(z) < +\infty$$

holds so long as $R(\tau)$ satisfies

$$R(\tau) > R_0(\tau) :=$$

$$\inf_{0<\kappa<1} \left(\frac{9e4^{\kappa}}{\kappa} \right)^{1/(1+\kappa)} \frac{(1+\kappa)\Gamma\left(\frac{1-\kappa}{1+\kappa}\right)}{(2+\kappa)\Gamma\left(\frac{1}{1+\kappa}\right)^2} |\tau|^{(2+\kappa)/(1+\kappa)}.$$

The choice (for small $|\tau|$)

$$\kappa = \frac{1}{\log \frac{1}{|\tau|}},$$

yields the asserted asymptotics.