Berezin Transform in Polynomial Bergman Spaces

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Abstract

Fix a smooth weight function Q in the plane, subject to a growth condition from below. Let $K_{m,n}$ denote the reproducing kernel for the Hilbert space of analytic polynomials of degree at most $n-1$ of finite L^2-norm with respect to the measure $\mathrm{e}^{-mQ}\,\mathrm{d}A$. Here $\mathrm{d}A$ is normalized area measure, and m is a positive real scaling parameter. The (polynomial) Berezin measure $\mathrm{d}B_{m,n}^{\langle z_0\rangle}(z) = K_{m,n}(z_0,z_0)^{-1}|K_{m,n}(z,z_0)|^2\mathrm{e}^{-mQ(z)}\,\mathrm{d}A(z)$ for the point z_0 is a probability measure that defines the (polynomial) Berezin transform

$$\mathrm{B}_{m,n} f(z_0) = \int_{\mathbb{C}} f\,\mathrm{d}B_{m,n}^{\langle z_0\rangle}$$

for continuous $f \in L^\infty(\mathbb{C})$. We analyze the semiclassical limit of the Berezin measure (and transform) as $m \to +\infty$ while $n = m\tau + \mathrm{o}(1)$, where τ is fixed, positive, and real. We find that the Berezin measure for z_0 converges weak-star to the unit point mass at the point z_0 provided that $\Delta Q(z_0) > 0$ and that z_0 is contained in the interior of a compact set $\mathscr{S}_\tau$, defined as the coincidence set for an obstacle problem. As a refinement, we show that the appropriate local blowup of the Berezin measure converges to the standardized Gaussian measure in the plane. For points $z_0 \in \mathbb{C}\setminus\mathscr{S}_\tau$, the Berezin measure cannot converge to the point mass at z_0. In the model case $Q(z) = |z|^2$, when $\mathscr{S}_\tau$ is a closed disk, we find that the Berezin measure instead converges to harmonic measure at z_0 relative to $\mathbb{C}\setminus\mathscr{S}_\tau$.

Our results have applications to the study of the eigenvalues of random normal matrices. The auxiliary results include weighted L^2-estimates for the equation $\bar\partial u = f$ when f is a suitable test function and the solution u is restricted by a polynomial growth bound at ∞. © 2010 Wiley Periodicals, Inc.

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1 Introduction

1.1 General Introduction to Berezin Quantization

In a version of quantum theory, a Bargmann-Fock-type space of polynomials plays the role of the quantized system, while the corresponding weighted L^2-space is the classical analogue. It is therefore natural to study the asymptotics of the quantized system as we approach the semiclassical limit. A particularly useful object is the *reproducing kernel* of the Bargmann-Fock-type space.

To make matters more concrete, let μ be a finite positive Borel measure on $\mathbb{C}$, and $L^2(\mathbb{C};\mu)$ the usual L^2-space with inner product

$$\langle f, g\rangle_{L^2(\mathbb{C};\mu)} = \int_{\mathbb{C}} f(z)\overline{g(z)}\mathrm{d}\mu(z).$$

The subspace of $L^2(\mathbb{C};\mu)$ consisting of entire functions is the Bergman space $A^2(\mathbb{C};\mu)$.

We assume that the μ is carried by infinitely many points so that any given polynomial corresponds to a unique element of $L^2(\mathbb{C};\mu)$, and write

$$J_\mu = \sup\left\{ j \in \mathbb{Z} : \int_{\mathbb{C}} |z|^{2(j-1)}\mathrm{d}\mu(z) < +\infty \right\}.$$

Because μ is finite, we are ensured that $1 \le J_\mu \le +\infty$.

Let $P_n = P_n(\mathbb{C})$ be the linear (i.e., $\mathbb{C}$-linear) space of analytic polynomials of degree at most $n-1$, and write

$$A^2_{\mu,n} = L^2(\mathbb{C};\mu) \cap P_n(\mathbb{C}) \subset L^2(\mathbb{C};\mu).$$

If we put $n' = \min\{n, J_\mu\}$, we see that $A^2_{\mu,n}$ equals $P_{n'}(\mathbb{C})$ in the sense of sets, with the norm inherited from $L^2(\mathbb{C};\mu)$. Hence $A^2_{\mu,n} = A^2_{\mu,n'}$ for all n, and there is no loss of generality in assuming that $n = n'$ in what follows.

Let $e_1, \ldots, e_{n'}$ be an orthonormal basis for $A^2_{\mu,n}$. The reproducing kernel $K_{\mu,n}$ for the space $A^2_{\mu,n}$ is the function

$$K_{\mu,n}(z,w) = \sum_{j=1}^{n'} e_j(z)\overline{e_j(w)}.$$

Then $K_{\mu,n}$ is independent of the choice of an orthonormal basis for $A^2_{\mu,n}$, and it is characterized by the properties that for $z_0 \in \mathbb{C}$, the function $z \mapsto K_{\mu,n}(z, z_0)$ is in $A^2_{\mu,n}$ and

$$\begin{aligned} u(z_0) &= \langle u, K_{\mu,n}(\,\cdot\,, z_0)\rangle_{L^2(\mathbb{C};\mu)} \\ &= \int_{\mathbb{C}} u(z)\overline{K_{\mu,n}(z,z_0)}\mathrm{d}\mu(z), \quad u \in A^2_{\mu,n}. \end{aligned} \tag{1.1}$$

For a given complex number z_0 we now consider the measure

$$\mathrm{d}B_{\mu,n}^{\langle z_0\rangle}(z) = \frac{|K_{\mu,n}(z, z_0)|^2}{K_{\mu,n}(z_0, z_0)}\,\mathrm{d}\mu(z), \tag{1.2}$$

which we may call the *Berezin measure* associated with μ, n, and z_0. The reproducing property (1.1) applied to $u = K_{\mu,n}(\cdot\,, z_0)$ implies that $B_{\mu,n}^{\langle z_0\rangle}$ is a probability measure. In classical physics, $B_{\mu,n}^{\langle z_0\rangle}$ corresponds to a point mass at z_0, so our interest focuses on how closely this measure approximates the point mass.

1.2 Class of Weights

By a *weight* we mean a (Lebesgue) measurable function $\phi : \mathbb{C} \to \mathbb{R}$ such that the measure

$$\mathrm{d}\mu_\phi(z) = \mathrm{e}^{-\phi(z)}\mathrm{d}A(z)$$

is a finite positive measure on $\mathbb{C}$, where $\mathrm{d}A(z) = \mathrm{d}x\,\mathrm{d}y/\pi$ with $z = x + \mathrm{i}y$.

We write L^2_ϕ for the space $L^2(\mathbb{C}; \mu_\phi)$ and A^2_ϕ for $A^2(\mathbb{C}; \mu_\phi)$, and the norm of an element $u \in L^2_\phi$ will be denoted by

$$\|u\|^2_\phi = \int_{\mathbb{C}} |u|^2 \mathrm{e}^{-\phi}\,\mathrm{d}A.$$

For a positive integer n, we frequently write

$$L^2_{\phi,n} = \{u \in L^2_\phi : u(z) = \mathrm{O}(|z|^{n-1}) \text{ when } |z| \to +\infty\} \tag{1.3}$$

and

$$A^2_{\phi,n} = L^2_{\phi,n} \cap A^2_\phi = L^2_{\phi,n} \cap P_n. \tag{1.4}$$

We observe that $L^2_{\phi,n}$ is usually nonclosed in L^2_ϕ, while the finite-dimensional space $A^2_{\phi,n}$ is always closed in L^2_ϕ.

1.3 Weights Considered

Let $Q : \mathbb{C} \to \mathbb{R}$ be a given locally bounded, Lebesgue-measurable function that satisfies a growth condition of the form

$$Q(z) \geq \rho \log |z|^2, \quad |z| \geq C, \tag{1.5}$$

for some positive numbers ρ and C. For a positive number m, we now consider weights ϕ_m of the form $\phi_m = mQ$. By abuse of notation, we will in this context sometimes refer to Q as the weight. To simplify the notation, we set

$$H_{m,n} = A^2_{mQ,n} = L^2_{mQ} \cap P_n,$$

and denote by $K_{m,n}$ the reproducing kernel for $H_{m,n}$. For a given point z_0, the corresponding Berezin measure $B_{m,n}^{\langle z_0\rangle}$ is defined accordingly; cf. (1.2).

Adding a real constant to Q means that the inner product in $H_{m,n}$ is only changed by a positive multiplicative constant, and $K_{m,n}$ gets multiplied by the inverse of that number. Hence the corresponding Berezin measures are unchanged, and we may, for instance, without loss of generality assume that $Q \geq 1$ on $\mathbb{C}$ when this is convenient.

1.4 Limits Considered; the Parameter τ

We think of Q as being fixed while the parameters m and n vary, and also fix a number τ satisfying $0 < \tau < \rho$. To avoid bulky notation, it is customary to reduce the number of parameters to one by regarding $n = n(m)$ as a function of $m > 0$. We will adopt this convention and study the behavior of the measures $B_{m,n}^{\langle z_0 \rangle}$ as $m \to +\infty$ while $n = m\tau + \mathrm{o}(1)$.

1.5 Word on Notation

For real x, we write $]x[$ for the largest integer that is strictly smaller than x. We frequently write

$$\partial_z = \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathrm{i}\,\frac{\partial}{\partial y}\right), \quad \bar{\partial}_z = \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \mathrm{i}\,\frac{\partial}{\partial y}\right), \quad z = x + \mathrm{i}y,$$

and use $\Delta = \Delta_z$ to denote the normalized Laplacian

$$\Delta_z = \partial_z \bar{\partial}_z = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

We write $\mathbb{D}(z_0, r)$ for the open disk $\{z \in \mathbb{C} : |z - z_0| < r\}$, and we simplify the notation to $\mathbb{D}$ when $z_0 = 0$ and $r = 1$. The boundary of $\mathbb{D}(z_0, r)$ is denoted $\mathbb{T}(z_0, r)$. When S is a subset of $\mathbb{C}$, we write S° for the interior of S, and $\overline{S}$ for its closure; the support of a continuous function f is denoted by supp f. The symbol $A \subset B$ means that A is a subset of B, while $A \Subset B$ means that A is a precompact subset of B. We write $\mathrm{dist}_{\mathbb{C}}(A, B)$ for the Euclidean distance between A and B. The symbol "a.e." is short for "dA-almost everywhere," and "p.m." is short for "probability measure." Finally, we use the shorthand L^2 to denote the unweighted space $L^2_0 = L^2(\mathbb{C}; \mathrm{d}A)$.

1.6 Applications to Random Normal Matrices

One of the main motivations for this paper comes from the fact that some of our results have applications in random matrix theory. We will return to this topic in a subsequent article [1]. The relevant results for these applications are primarily Theorem 2.3 and Corollary 8.2.

2 Main Results

2.1 Berezin Quantization

Fix a nonnegative weight Q and two positive numbers τ and ρ, $\tau < \rho$, such that the growth condition (1.5) is satisfied and form the probability measure

$$\mathrm{d}B_{m,n}^{\langle z_0 \rangle} = \mathrm{B}_{m,n}^{\langle z_0 \rangle} \mathrm{d}A \quad \text{where } \mathrm{B}_{m,n}^{\langle z_0 \rangle}(z) = \frac{|K_{m,n}(z, z_0)|^2}{K_{m,n}(z_0, z_0)}\,\mathrm{e}^{-mQ(z)}.$$

We shall refer to the function $\mathrm{B}_{m,n}^{\langle z_0\rangle}$ as the *Berezin kernel* associated with m, n, and z_0. The *Berezin transform* of a function $f \in L^\infty(\mathbb{C})$ is the function given by

$$\mathrm{B}_{m,n} f(z_0) = \int_{\mathbb{C}} f(z)\, \mathrm{d}B_{m,n}^{\langle z_0\rangle}(z), \quad z_0 \in \mathbb{C}.$$

For background material on the Berezin transform, we refer to [11] and F. A. Berezin's papers [2, 3], as well as the books [10, 15]. Let us now assume that $Q \in \mathscr{C}^2(\mathbb{C})$. In the first instance, we ask for which z_0 do we have the convergence

$$B_{m,n}^{\langle z_0\rangle} \to \delta_{z_0} \quad \text{as } m \to +\infty \text{ while } n = m\tau + \mathrm{o}(1) \tag{2.1}$$

in the weak-star sense of measures. Here we use $\mathscr{C}_b(\mathbb{C})$, the Banach space of all bounded continuous functions on $\mathbb{C}$, as the predual. In terms of the Berezin transform, we are asking whether

$$\mathrm{B}_{m,n} f(z_0) \to f(z_0) \quad \text{as } m \to +\infty \text{ while } n = m\tau + \mathrm{o}(1) \tag{2.2}$$

for all $f \in \mathscr{C}_b(\mathbb{C})$.

Let $\mathscr{N}_+$ and $\mathscr{N}_{+,0}$ be the sets of points defined by

$$\mathscr{N}_+ = \{z \in \mathbb{C} : \Delta Q(z) > 0\}, \quad \mathscr{N}_{+,0} = \{z \in \mathbb{C} : \Delta Q(z) \geq 0\}.$$

We shall find that there exists a compact set $\mathscr{S}_\tau$ contained in $\mathscr{N}_{+,0}$ such that (2.2) holds for all z_0 in the interior of $\mathscr{S}_\tau \cap \mathscr{N}_+$, while (2.2) fails whenever $z_0 \in \mathbb{C} \setminus \mathscr{S}_\tau$.

2.2 Ingredients from Potential Theory

To define $\mathscr{S}_\tau$, we first need to introduce some notions from weighted potential theory; cf. [16, 21]. It is here advantageous to slightly relax the regularity assumption on Q and assume that Q is in the class $\mathscr{C}^{1,1}(\mathbb{C})$ consisting of all $\mathscr{C}^1$-smooth functions with Lipschitzian gradients. We will make frequent use of the simple fact that the distributional Laplacian ΔF of a function $F \in \mathscr{C}^{1,1}(\mathbb{C})$ makes sense almost everywhere and is of class $L^\infty_{\mathrm{loc}}(\mathbb{C})$. After all, $F \in \mathscr{C}^{1,1}(\mathbb{C})$ means that the first-order partial derivatives of F are locally Lip 1, and therefore the second-order partial derivatives are locally bounded.

Let $\mathrm{Subh}_\tau(\mathbb{C})$ denote the set of all subharmonic functions $f : \mathbb{C} \to \mathbb{R}$ that satisfy a growth bound of the form

$$f(z) \leq \tau \log |z|^2 + \mathrm{O}(1) \quad \text{as } |z| \to +\infty.$$

The *equilibrium potential* corresponding to the weight Q and the parameter τ is defined by

$$\widehat{Q}_\tau(z) = \sup\{f(z) : f \in \mathrm{Subh}_\tau(\mathbb{C}) \text{ and } f \leq Q \text{ on } \mathbb{C}\}.$$

Clearly, $\widehat{Q}_\tau$ is subharmonic on $\mathbb{C}$. What is perhaps not so obvious is that $\widehat{Q}_\tau$ is of class $\mathscr{C}^{1,1}(\mathbb{C})$, a fact that is well-known in the context of variational inequalities of partial differential equations (associated with the work of Kinderlehrer, Stampacchia, and Caffarelli; see [16] for some details, and [19, chap. 1] for the main ingredient). We now define $\mathscr{S}_\tau$ as the coincidence set

$$\mathscr{S}_\tau = \{z \in \mathbb{C} : Q(z) = \widehat{Q}_\tau(z)\}. \tag{2.3}$$

It is easy to see that the set $\mathscr{S}_\tau$ increases with τ (with respect to containment). Moreover, $\widehat{Q}_\tau$ is harmonic in $\mathbb{C} \setminus \mathscr{S}_\tau$.

We collect our observations in a lemma and add a little concerning the growth of $\widehat{Q}_\tau$.

LEMMA 2.1 *The function $\widehat{Q}_\tau$ is of class $\mathscr{C}^{1,1}(\mathbb{C})$, and $\mathscr{S}_\tau$ is compact. Furthermore, $\widehat{Q}_\tau$ is harmonic in $\mathbb{C} \setminus \mathscr{S}_\tau$, and there is a number C such that $\widehat{Q}_\tau(z) \le \tau \log_+ |z|^2 + C$ for all $z \in \mathbb{C}$.*

PROOF: This is [16, prop. 4.2, p. 10] and [21, theorem I.4.7, p. 54]. □

Given the smoothness of Q and $\widehat{Q}_\tau$, one can show that

$$\Delta Q(z) = \Delta \widehat{Q}_\tau(z), \quad z \in \mathscr{S}_\tau,$$

in the almost-everywhere sense. In particular, as $\widehat{Q}_\tau$ is subharmonic, we have

$$\mathscr{S}_\tau \subset \mathscr{N}_{+,0} = \{z \in \mathbb{C} : \Delta Q(z) \ge 0\}.$$

Constructing subharmonic minorants of critical growth is easy; for C large enough, the function $z \mapsto \tau \log_+ \left(|z|^2/C\right)$ is a minorant of Q of class $\mathrm{Subh}_\tau(\mathbb{C})$. It follows that

$$\widehat{Q}_\tau(z) = \tau \log |z|^2 + \mathrm{O}(1) \quad \text{as } |z| \to +\infty. \tag{2.4}$$

2.3 Technical Tools: Expansion Formula for the Polynomial Bergman Kernel

Many results in this paper rely on a suitable expansion formula for $K_{m,n}(z,w)$ when z and w are both close to some point $z_0 \in \mathscr{N}_+$, and m and n are large. In the presentation, we shall assume that $Q \in \mathscr{C}^2(\mathbb{C})$ is *real analytic* in $\mathscr{S}_\tau^\circ$. A general expansion formula was recently obtained by Berman [6, p. 9] based on the methods of [7]. Here we will only need a special case of that result. We first need some notation.

If Q is real analytic in a neighborhood of a point z_0, we may introduce the function $\psi = \psi_Q$ given by

$$\psi(z,w) = \sum_{j,k=0}^{+\infty} \frac{1}{j!\,k!}(z - z_0)^j (w - \bar{z}_0)^k \partial^j \bar\partial^k Q(z_0),$$

which is holomorphic in (z,w) when z is close to z_0 and w is close to $\bar{z}_0$. By Taylor's formula, we have

$$\psi(z,\bar{z}) = Q(z)$$

for all z close to z_0. By varying z_0, we find that $\psi(z,w)$ extends to a holomorphic function in a neighborhood of the conjugate diagonal $w = \bar{z}$ provided that Q

is real analytic throughout $\mathbb{C}$. *We shall assume that Q is real analytic in $\mathscr{S}_\tau^\circ$.* Then $\psi(z,w)$ is well-defined and holomorphic in a neighborhood of the conjugate diagonal $w=\bar{z}$, where $z \in \mathscr{S}_\tau^\circ$. It is clear that $\psi(z,\bar{z}) = Q(z)$ holds for all $z \in \mathscr{S}_\tau^\circ$.

DEFINITION 2.2 Let b_0 and b_1 denote the functions

$$b_0(z,w) = \partial_z \partial_w \psi(z,w), \quad b_1(z,w) = \frac{1}{2}\,\partial_w \frac{\partial_z b_0(z,w)}{b_0(z,w)}. \tag{2.5}$$

The function $b_1(z,w)$ is well-defined where there is no division by 0; in particular, this is so in some neighborhood of the conjugate diagonal $w=\bar{z}$, where $z \in \mathscr{S}_\tau^\circ \cap \mathscr{N}_+$. Along the conjugate diagonal $w=\bar{z}$, we have

$$b_0(z,\bar{z}) = \Delta Q(z), \quad b_1(z,\bar{z}) = \frac{1}{2}\Delta \log \Delta Q(z), \quad z \in \mathscr{S}_\tau^\circ \cap \mathscr{N}_+.$$

We define the *first-order approximating Bergman kernel* $K_m^1(z,w)$:

$$K_m^1(z,w) = (m b_0(z,\bar{w}) + b_1(z,\bar{w}))\, \mathrm{e}^{m\psi(z,\bar{w})}.$$

We have the following theorem.

THEOREM 2.3 *Assume that $Q \in \mathscr{C}^2(\mathbb{C})$ is real analytic in $\mathscr{S}_\tau^\circ$, and suppose $\mathscr{K}$ is a compact subset of $\mathscr{S}_\tau^\circ \cap \mathscr{N}_+$. Fix a point $z_0 \in \mathscr{K}$ and a real number $M > 0$. Then there exists a number m_0 depending only on Q, τ, and M and positive numbers C and ε depending only on Q, τ, M, and $\mathscr{K}$ such that*

$$|K_{m,n}(z,w) - K_m^1(z,w)|\mathrm{e}^{-\frac{1}{2}m(Q(z)+Q(w))} \le \frac{C}{m}, \quad z_0 \in \mathscr{K}, \; z,w \in \mathbb{D}(z_0,\varepsilon),$$

for all $m \ge m_0$ and $n \ge m\tau - M$. In particular, by restricting to the diagonal, we get

$$\left| K_{m,n}(z,z)\mathrm{e}^{-mQ(z)} - \left(m\Delta Q(z) + \frac{1}{2}\Delta \log \Delta Q(z) \right) \right| \le \frac{C}{m}, \quad z \in \mathscr{K},$$

when $m \ge m_0$ and $n \ge m\tau - M$.

PROOF: See Section 6. See also Remark 2.5 below. □

Remark 2.4. Since $\psi(z,w)$, $b_0(z,w)$, and $b_1(z,w)$ are all real-valued along the conjugate diagonal $w=\bar{z}$, a suitable version of the reflection principle implies that K_m^1 is Hermitian, that is,

$$\overline{K_m^1(z,w)} = K_m^1(w,z).$$

Remark 2.5. We should point out that Theorem 2.3 has a long history; analogous expansions are well-known for (weighted) Bergman kernels of several complex variables (SCV); see, for instance, [6, 7, 9, 12, 13, 18], and the references therein. Moreover, as was mentioned above, Theorem 2.3 is a slight modification of a more general SCV result stated by Berman [6, theorem 3.8] (see also [7]). In the proof, we make frequent use of the ideas and techniques developed in [4, 6, 7] and in the book [22].

2.4 Extensions and Possible Generalizations of the Expansion Formula

Theorem 2.3 is a special case of a general asymptotic expansion of Bergman kernels; cf. Berman [6, theorem 3.8].

Given a positive integer k, a k^{th}-*order approximating Bergman kernel* is an expression of the form

$$K_m^k(z,w) = \big(mb_0(z,\bar{w}) + b_1(z,\bar{w}) + \ldots + m^{-k+1}b_k(z,\bar{w})\big)\mathrm{e}^{m\psi(z,\bar{w})}$$

for z and w in a neighborhood of the conjugate diagonal $w=\bar{z}$, where the coefficient functions $b_0,\ldots,b_k$ are certain holomorphic functions. It is possible to find $b_0,\ldots,b_k$ such that

$$\big|K_{m,n}(z,w) - K_m^k(z,w)\big|\mathrm{e}^{-\frac{1}{2}m(Q(z)+Q(w))} \le Cm^{-k}, \quad z,w \in \mathbb{D}(z_0,\varepsilon),$$

for m large and $n \ge m\tau - 1$ provided that $z_0 \in \mathscr{S}_\tau^\circ \cap \mathscr{N}_+$ is fixed. In principle, the $b_0,\ldots,b_k$ are determined from a recursion formula involving partial differential equations of increasing order; cf. Berman et al. [7, eq. (2.20), p. 209]. However, the analysis required for calculating higher-order coefficients b_j, $j \ge 2$, is quite involved, and the first-order expansion is sufficient for many practical purposes; cf. [1]. This is why here we prefer a more direct approach.

The real analyticity assumption on Q in Theorem 2.3 is of course excessive. An analogous theorem holds under weaker assumptions, e.g., $\mathscr{C}^\infty$-smoothness. In this case, the functions b_0, b_1, and ψ are defined using almost-holomorphic extensions (cf., e.g., [9] for a relevant discussion). The modifications are essentially outlined in [7, sec. 2.6].

2.5 Asymptotic Behavior of the Berezin Measure

We begin with the following basic observation:

PROPOSITION 2.6 *Let $Q \in \mathscr{C}^{1,1}(\mathbb{C})$ and $z_0 \in \mathbb{C}$ be an arbitrary point. Then, for every open neighborhood $\mathscr{D}$ of $\mathscr{S}_\tau$,*

$$B_{m,n}^{\langle z_0\rangle}(\mathscr{D}) \to 1 \quad \textit{as } m \to +\infty \textit{ while } n \le m\tau + 1.$$

PROOF: See Section 3. □

It follows from Proposition 2.6 that for each fixed $z_0 \in \mathbb{C}$ and each open neighborhood $\mathscr{D}$ of $\mathscr{S}_\tau$, $B_{m,n}^{\langle z_0\rangle}(\mathbb{C}\setminus\mathscr{D}) \to 0$ as $m \to +\infty$ while $n \le m\tau+1$. Hence if $z_0 \in \mathbb{C}\setminus\mathscr{S}_\tau$, then (2.1) fails, as the measures $B_{m,n}^{\langle z_0\rangle}$ remain essentially confined to (neighborhoods of) $\mathscr{S}_\tau$. When the point z_0 is in $\mathscr{S}_\tau \cap \mathscr{N}_+$, this confinement is no longer an obstacle to (2.1). Indeed, for interior points z_0 in $\mathscr{S}_\tau \cap \mathscr{N}_+$, we have the following.

THEOREM 2.7 *Assume that $Q \in \mathscr{C}^2(\mathbb{C})$, and let $z_0 \in \mathscr{S}_\tau^\circ \cap \mathscr{N}_+$. Then, for any fixed real number $M \ge 0$, the measures $B_{m,n}^{\langle z_0\rangle}$ converge to δ_{z_0} in the weak-star sense as $m \to +\infty$ and $n \ge m\tau - M$.*

PROOF: See Section 7. See also Remark 2.13. □

2.6 Gaussian Convergence of the Berezin Measure

Fix a point $z_0 \in \mathscr{S}_\tau^\circ \cap \mathscr{N}_+$. It is convenient to introduce the *rescaled* Berezin measure $\widehat{B}_{m,n}^{\langle z_0 \rangle}$ by

$$\text{(2.6)} \quad \mathrm{d}\widehat{B}_{m,n}^{\langle z_0 \rangle} = \widehat{\mathrm{B}}_{m,n}^{\langle z_0 \rangle} \mathrm{d}A$$

$$\text{where } \widehat{\mathrm{B}}_{m,n}^{\langle z_0 \rangle}(z) = \frac{1}{m\Delta Q(z_0)} \mathrm{B}_{m,n}^{\langle z_0 \rangle}\left(z_0 + \frac{z}{\sqrt{m\Delta Q(z_0)}}\right).$$

We denote the standard Gaussian p.m. on $\mathbb{C}$ by

$$\mathrm{d}P(z) = \mathrm{e}^{-|z|^2}\,\mathrm{d}A(z).$$

We have the following central limit theorem, which gives much more precise information than Theorem 2.7. The proof is based on the expansion formula of Theorem 2.3.

THEOREM 2.8 *Assume that $Q \in \mathscr{C}^2(\mathbb{C})$ is real analytic in $\mathscr{S}_\tau^\circ$. Fix a compact subset $\mathscr{K} \Subset \mathscr{S}_\tau^\circ \cap \mathscr{N}_+$, a point $z_0 \in \mathscr{K}$, and a number $M \geq 0$. We then have*

$$\text{(2.7)} \quad \int_{\mathbb{C}} \left|\widehat{\mathrm{B}}_{m,n}^{\langle z_0 \rangle}(z) - \mathrm{e}^{-|z|^2}\right| \mathrm{d}A(z) \to 0 \quad \text{as } m \to +\infty \text{ while } n \geq m\tau - M,$$

where the convergence is uniform over $z_0 \in \mathscr{K}$. Equivalently, $\widehat{B}_{m,n}^{\langle z_0 \rangle} \to P$ in norm as $m \to +\infty$ and $n \geq m\tau - M$.

PROOF: See Section 7. □

Remark 2.9. Scaling asymptotics of correlation kernels to model Gaussian-type models have been studied in a different but related context by Bleher, Shiffman, Zelditch, and others. See, e.g., [8].

2.7 Bargmann-Fock Case and Harmonic Measure

When $z_0 \in \mathbb{C} \setminus (\mathscr{S}_\tau^\circ \cap \mathscr{N}_+)$, Proposition 2.6 states that the measures $B_{m,n}^{\langle z_0 \rangle}$ tend to concentrate on $\mathscr{S}_\tau$ as $m \to +\infty$ while $n = m\tau + \mathrm{o}(1)$. However, our general results do not suggest what the asymptotic behavior of the p.m. $B_{m,n}^{\langle z_0 \rangle}$ might be.

We specialize to the Bargmann-Fock weight $Q(z) = |z|^2$, in which case we obviously have $\mathscr{N}_+ = \mathbb{C}$. It is convenient to introduce the truncated exponentials

$$E_k(z) = \sum_{j=0}^{k} \frac{z^j}{j!}.$$

We check that

$$K_{m,n}(z,w) = mE_{n-1}(mz\bar{w})$$

and that

$$\hat{Q}_\tau(z) = \begin{cases} |z|^2 & \text{for } |z| \leq \sqrt{\tau}, \\ \tau + \tau \log |z|^2/\tau & \text{for } |z| > \sqrt{\tau}, \end{cases}$$

so that $\mathscr{S}_\tau = \overline{\mathbb{D}}(0, \sqrt{\tau})$. We infer that

$$\mathrm{d}B_{m,n}^{\langle z_0 \rangle}(z) = m \, \frac{|E_{n-1}(mz\bar{z}_0)|^2}{E_{n-1}(m|z_0|^2)} \, \mathrm{e}^{-m|z|^2} \, \mathrm{d}A(z). \tag{2.8}$$

Let $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$ denote the extended plane.

THEOREM 2.10 *Let $Q(z) = |z|^2$ and $z_0 \in \mathbb{C} \setminus \mathscr{S}_\tau$. Then, as $m \to +\infty$ and $n/m \to \tau$, the measures $B_{m,n}^{\langle z_0 \rangle}$ converge to harmonic measure at z_0 with respect to $\mathbb{C}^* \setminus \mathscr{S}_\tau$.*

PROOF: See Section 9. □

Remark 2.11. The above theorem is a special case of theorem 8.7 in [1].

2.8 Technical Tools: Estimates for the $\bar{\partial}$-Equation

An essential ingredient in our analysis is an estimate of the norm-minimal solution $u_{0,n}$ in the space $L^2_{mQ,n}$ to the equation $\bar{\partial} u = f$, where f is a suitable bounded measurable function with supp $f \subset \mathscr{S}_\tau$. What makes the problem slightly sophisticated is the growth bound associated with $L^2_{mQ,n}$. In Section 4 we show how to modify the standard approach to get the estimates we need. Compare Berman's compactification approach [6] involving sections of line bundles in the SCV setting.

2.9 Long-Range Damping Bound on the Berezin Density

Theorem 2.7 suggests that if z_0 is a given point in the interior of $\mathscr{S}_\tau \cap \mathscr{N}_+$, m large, and $n \geq m\tau - M$, then the density $\mathrm{B}_{m,n}^{\langle z_0 \rangle}(z)$ should be big near z_0 and decay rapidly away from z_0. A concrete decay bound is offered by the following theorem. The assumption $Q \geq 1$ is technical and may be removed at the expense of altering the expression for the positive constant λ_0. The proof is mainly based on the weighted L^2-estimates for the $\bar{\partial}$-equation obtained in Section 4.

THEOREM 2.12 *Assume that $Q \in \mathscr{C}^2(\mathbb{C})$ with $Q \geq 1$. Let $\mathscr{K}$ be a compact subset of $\mathscr{S}_\tau^\circ \cap \mathscr{N}_+$, and let $\hat{\mathscr{K}}$ be the closure of the union*

$$\bigcup_{z_0 \in \mathscr{K}} \mathbb{D}(z_0, r(z_0)) \quad \text{where } r(z_0) = \frac{1}{2} \operatorname{dist}_{\mathbb{C}}(z_0, \mathbb{C} \setminus (\mathscr{S}_\tau^\circ \cap \mathscr{N}_+)),$$

which is automatically a compact subset of $\mathscr{S}_\tau^\circ \cap \mathscr{N}_+$. We also put

$$\alpha_{\hat{\mathscr{K}}} = \inf\{\Delta Q(z) : z \in \hat{\mathscr{K}}\} > 0.$$

Then, given a positive real M, we have, for all positive integers n confined to $m\tau - M \leq n \leq m\tau + 1$, the estimate

$$\mathrm{B}_{m,n}^{\langle z_0 \rangle}(z) \leq C m \mathrm{e}^{-\lambda_0 \sqrt{m} \min\{r(z_0), |z - z_0|\}} \mathrm{e}^{-m(Q(z) - \hat{Q}_\tau(z))}, \quad z_0 \in \mathscr{K}, \ z \in \mathbb{C},$$

where $\lambda_0 = 2^{-3/2} \sqrt{\alpha_{\hat{\mathscr{K}}}} \mathrm{e}^{-(1/2)M\tau^{-1} q_\tau}$ with $q_\tau = \sup_{\mathscr{S}_\tau} Q$, and the constant C depends only on $\mathscr{K}$, Q, M, and τ.

PROOF: See Section 8. □

Remark 2.13. The proof of Theorem 2.12 is quite different from that supplied for Theorem 2.7. Hence Theorem 2.12 offers an alternative means to obtain Theorem 2.7 in the case of interest when $n = m\tau + \mathrm{o}(1)$.

Remark 2.14. Theorem 2.12 was generalized (independently) to the SCV context by Berman [5].

3 Preparatory Lemmas

3.1 Preliminaries

In this section, we introduce the cutoff functions needed repeatedly throughout this paper. We also introduce the technique of subharmonicity estimates, which will allow us to turn integral bounds into pointwise bounds. An application of the maximum principle allows us to obtain Proposition 2.6. As for the results obtained here, it is enough to suppose that Q is $\mathscr{C}^{1,1}$-smooth (and satisfies the growth assumption (1.5)). In particular, we have $\Delta Q \in L^\infty_{\mathrm{loc}}(\mathbb{C})$.

3.2 Cutoff Functions

Given reals $\delta > 0$, $r > 0$, and $C > 1$, there exists $\chi \in \mathscr{C}_0^\infty(\mathbb{C})$ such that $\chi = 1$ on $\mathbb{D}(0, \delta)$, $\chi = 0$ outside $\mathbb{D}(0, \delta(1 + r))$, and $\chi \le 1$ and $|\bar\partial\chi|^2 \le C r^{-2}\delta^{-2}\chi$ on $\mathbb{C}$, where C is an absolute constant. In fact, any $C > 1$ will do. It is then easy to check that $\|\bar\partial\chi\|^2_{L^2} \le C(1 + 2/r)$. A Lipschitzian χ is given by the radial expression

$$\chi(z) = \left(\frac{1+r}{r} - \frac{|z|}{r\delta}\right)^2, \qquad \delta \le |z| \le \delta(1+r),$$

and a smooth χ is obtained by a suitable regularization of this solution. Later on, we will often use the values $\delta = 3\varepsilon/2$ and $\delta(1 + r) = 2\varepsilon$, where $\varepsilon > 0$ is given. We may then arrange that $\|\bar\partial\chi\|_{L^2} \le 3$.

3.3 Subharmonicity Estimates

We start by giving two simple lemmas, which are based on the submean property of subharmonic functions.

LEMMA 3.1 *Let ϕ be a weight of class $\mathscr{C}^{1,1}(\mathbb{C})$ and put $A = \operatorname{ess\,sup}\{\Delta\phi(\zeta) : \zeta \in \mathbb{D}\}$. Then if u is bounded and holomorphic in $\mathbb{D}$, we have that*

$$|u(0)|^2 \mathrm{e}^{-\phi(0)} \le \int_{\mathbb{D}} |u(\zeta)|^2 \mathrm{e}^{-\phi(\zeta)} \mathrm{e}^{A|\zeta|^2}\, \mathrm{d}A(\zeta) \le \mathrm{e}^A \int_{\mathbb{D}} |u(\zeta)|^2 \mathrm{e}^{-\phi(\zeta)} \mathrm{d}A(\zeta).$$

PROOF: Consider the function $F(\zeta) = |u(\zeta)|^2 \mathrm{e}^{-\phi(\zeta) + A|\zeta|^2}$, which satisfies

$$\Delta \log F = \Delta \log |u|^2 - \Delta\phi + A \ge 0 \quad \text{a.e. on } \mathbb{D},$$

making F logarithmically subharmonic. But then F is subharmonic itself, and so

$$F(0) \le \int_{\mathbb{D}} F(z)\mathrm{d}A(z).$$

The assertion of the lemma is immediate from this. □

LEMMA 3.2 *Let $Q \in \mathscr{C}^{1,1}(\mathbb{C})$ and fix $\delta > 0$ and $z \in \mathbb{C}$ and put*

$$A = A_{z,\delta,m} = \operatorname{ess\,sup}\{\Delta Q(\zeta) : \zeta \in \mathbb{D}(z, m^{-1/2}\delta)\}.$$

Also, let u be holomorphic and bounded in $\mathbb{D}(z, m^{-1/2}\delta)$. Then

$$|u(z)|^2 e^{-mQ(z)} \le \frac{m\mathrm{e}^{A\delta^2}}{\delta^2} \int_{\mathbb{D}(z,m^{-1/2}\delta)} |u(\zeta)|^2 e^{-mQ(\zeta)} \mathrm{d}A(\zeta).$$

PROOF: The assertion follows if we make the change of variables $\zeta = z + m^{-1/2}\delta\xi$ where $\zeta \in \mathbb{D}(z, m^{-1/2}\delta)$ and $\xi \in \mathbb{D}$, and apply Lemma 3.1 with the weight $\phi(\xi) = mQ(\zeta)$. □

We note the following consequence of the subharmonicity estimates. We will need it in later sections.

LEMMA 3.3 *Let $\mathscr{K}$ be a compact subset of $\mathbb{C}$ and δ a given positive number. We put*

$$\mathscr{K}_{m^{-1/2}\delta} = \{z \in \mathbb{C} : \operatorname{dist}_{\mathbb{C}}(z, \mathscr{K}) \le m^{-1/2}\delta\},$$
$$A = \operatorname{ess\,sup}\{\Delta Q(z) : z \in \mathscr{K}_{m^{-1/2}\delta}\}.$$

Then, for all positive real m and all integers $n \ge 1$,

$$|K_{m,n}(z, w)|^2 \mathrm{e}^{-mQ(z)} \le \frac{m\mathrm{e}^{A\delta^2}}{\delta^2} \int_{\mathbb{D}(z,m^{-1/2}\delta)} |K_{m,n}(\zeta, w)|^2 \mathrm{e}^{-mQ(\zeta)} \mathrm{d}A(\zeta)$$

for all $z \in \mathscr{K}$ and $w \in \mathbb{C}$.

PROOF: Apply Lemma 3.2 to $u(\zeta) = K_{m,n}(\zeta, w)$. □

3.4 A Weak Maximum Principle for Weighted Polynomials

Maximum principles for weighted polynomials have a long history; see, e.g., [21, chap. III]. The following simple lemma will suffice for our present purposes; it is a consequence of [21, theorem III.2.1]. (We recall that $\mathscr{S}_\tau$ is the coincidence set given by (2.3).)

LEMMA 3.4 *Suppose that a polynomial u of degree at most $n-1$ satisfies*

$$|u(z)|^2 \mathrm{e}^{-mQ(z)} \le 1, \quad z \in \mathscr{S}_{\tau(m,n)},$$

where $\tau(m,n) = (n-1)/m < \rho$. Then

$$|u(z)|^2 \mathrm{e}^{-m\hat{Q}_{\tau(m,n)}(z)} \le 1, \quad z \in \mathbb{C}.$$

PROOF: Put $q(z) = m^{-1} \log |u(z)|^2$. Then $q \in \operatorname{Subh}_{\tau(m,n)}(\mathbb{C})$ and $q \le Q$ on $\mathscr{S}_{\tau(m,n)}$. Hence, by the maximum principle, $q \le \widehat{Q}_{\tau(m,n)}$ on $\mathbb{C}$, as desired. □

LEMMA 3.5 *Let $u \in H_{m,n}$, and suppose that $m \ge 1$ and $n \le m\tau + 1$. Then*

$$|u(z)|^2 \le m\mathrm{e}^A \|u\|_{mQ}^2 \mathrm{e}^{m\widehat{Q}_\tau(z)},$$

where A denotes the essential supremum of ΔQ over the set

$$\{z \in \mathbb{C} : \operatorname{dist}_{\mathbb{C}}(z, \mathscr{S}_\tau) \le 1\}.$$

PROOF: The assertion that $n \le m\tau + 1$ is equivalent to $\tau(m,n) \le \tau$ where $\tau(m,n) = (n-1)/m$. Thus $\mathscr{S}_{\tau(m,n)} \subset \mathscr{S}_\tau$ and $\widehat{Q}_{\tau(m,n)} \le \widehat{Q}_\tau$. An application of Lemma 3.2 with $\delta = 1$ now gives

$$\begin{aligned} |u(z)|^2 \mathrm{e}^{-mQ(z)} &\le m\mathrm{e}^A \int_{\mathbb{D}(z,m^{-1/2})} |u(\zeta)|^2 \mathrm{e}^{-mQ(\zeta)} \mathrm{d}A(\zeta) \\ &\le m\mathrm{e}^A \int_{\mathbb{C}} |u(\zeta)|^2 \mathrm{e}^{-mQ(\zeta)} \mathrm{d}A(\zeta), \quad z \in \mathscr{S}_\tau. \end{aligned}$$

As a consequence, the same estimate holds on $\mathscr{S}_{\tau(m,n)}$. We can therefore apply Lemma 3.4, which yields that

$$|u(z)|^2 \le m\mathrm{e}^A \|u\|_{mQ}^2 \mathrm{e}^{m\widehat{Q}_{\tau(m,n)}(z)}, \quad z \in \mathbb{C}.$$

The desired assertion is immediate, as $\widehat{Q}_{\tau(m,n)} \le \widehat{Q}_\tau$. □

PROOF OF PROPOSITION 2.6: Fix two points z and z_0 in $\mathbb{C}$. We apply Lemma 3.5 to the polynomial

$$u(\zeta) = \frac{K_{m,n}(\zeta, z_0)}{\sqrt{K_{m,n}(z_0, z_0)}},$$

which is of class $H_{m,n}$ and satisfies $\|u\|_{mQ} = 1$. It follows that $|u(z)|^2 \le m\mathrm{e}^A \mathrm{e}^{\widehat{Q}_\tau(z)}$, so that

$$\begin{aligned} \text{Б}_{m,n}^{\langle z_0 \rangle}(z) &= |u(z)|^2 \mathrm{e}^{-mQ(z)} \\ &\le m\mathrm{e}^A \mathrm{e}^{m(\widehat{Q}_\tau(z) - Q(z))}, \quad z \in \mathbb{C},\ n \le m\tau + 1. \end{aligned} \tag{3.1}$$

Next, let $\mathscr{D}$ be an open neighborhood of $\mathscr{S}_\tau$. Since $Q > \widehat{Q}_\tau$ on $\mathbb{C} \setminus \mathscr{S}_\tau$, the continuity of the functions involved coupled with the growth conditions (2.4) and (1.5) shows that $Q - \widehat{Q}_\tau$ is bounded from below by a positive number on $\mathbb{C} \setminus \mathscr{D}$. Moreover, $Q - \widehat{Q}_\tau$ increases sufficiently rapidly near infinity that it follows that

$$B_{m,n}^{\langle z_0 \rangle}(\mathbb{C} \setminus \mathscr{D}) = \int_{\mathbb{C} \setminus \mathscr{D}} \text{Б}_{m,n}^{\langle z_0 \rangle} \, \mathrm{d}A \to 0$$

as $m \to +\infty$ and $n \le m\tau + 1$. Since $B_{m,n}^{\langle z_0 \rangle}$ is a p.m. on $\mathbb{C}$, the assertion of Proposition 2.6 is immediate. □

3.5 Estimate for the One-Point Function

Our proof of Proposition 2.6 implies a useful estimate for the one-point function $z \mapsto K_{m,n}(z,z)\mathrm{e}^{-mQ(z)}$. The following result is implicit in [6].

PROPOSITION 3.6 *Suppose that $n \le m\tau + 1$. Then there exists a number C depending only on Q and τ such that*

$$K_{m,n}(z,z)\mathrm{e}^{-mQ(z)} \le Cm\,\mathrm{e}^{-m(Q(z)-\widehat{Q}_\tau(z))}, \quad z \in \mathbb{C}, \tag{3.2}$$

and

$$|K_{m,n}(z,w)|^2\mathrm{e}^{-m(Q(z)+Q(w))} \le Cm^2\mathrm{e}^{-m(Q(z)-\widehat{Q}_\tau(z))}\mathrm{e}^{-m(Q(w)-\widehat{Q}_\tau(w))}, \quad z,w \in \mathbb{C}. \tag{3.3}$$

In particular, $|K_{m,n}(z,w)|^2\mathrm{e}^{-m(Q(z)+Q(w))} \le Cm^2$ for all $z,w \in \mathbb{C}$.

PROOF: The function $\mathrm{B}_{m,n}^{\langle z_0 \rangle}(z)$ in the diagonal case $z = z_0$ reduces to

$$\mathrm{B}_{m,n}^{\langle z \rangle}(z) = K_{m,n}(z,z)\,\mathrm{e}^{-mQ(z)}.$$

Thus estimate (3.1) implies

$$K_{m,n}(z,z)\mathrm{e}^{-mQ(z)} \le m\mathrm{e}^{A}\mathrm{e}^{m(\widehat{Q}_\tau(z)-Q(z))},$$

which proves (3.2). In order to get (3.3) from (3.2), it is enough to apply the Cauchy-Schwarz inequality. □

4 Weighted Estimates for the $\bar\partial$-Equation with Growth Constraint

4.1 General Introduction: Bergman Projection and the $\bar\partial$-Equation

Let ϕ be a weight on $\mathbb{C}$ (cf. Section 1.3). We assume throughout that ϕ is of class $\mathscr{C}^{1,1}(\mathbb{C})$ (so that $\Delta\phi \in L^\infty_{\mathrm{loc}}(\mathbb{C})$) and that $\int_{\mathbb{C}} \mathrm{e}^{-\phi}\,\mathrm{d}A < +\infty$, so that L^2_ϕ contains the constant functions. We fix a positive integer n and recall the definition of the "truncated" spaces $L^2_{\phi,n}$ and $A^2_{\phi,n}$ (see (1.3) and (1.4)).

Let $K_{\phi,n}$ denote the reproducing kernel for the space $A^2_{\phi,n}$, and let $\mathrm{P}_{\phi,n} : L^2_\phi \to A^2_{\phi,n}$ be the orthogonal projection,

$$\mathrm{P}_{\phi,n}u(z) = \int_{\mathbb{C}} u(z)K_{\phi,n}(z,w)\mathrm{e}^{-\phi(w)}\mathrm{d}A(w), \quad u \in L^2_\phi. \tag{4.1}$$

For f in the class $\mathscr{C}_0(\mathbb{C})$ (continuous functions with compact support), we consider the Cauchy transform $u = \mathrm{C}f$ defined by

$$\mathrm{C}f(z) = \int_{\mathbb{C}} \frac{f(w)}{z-w}\,\mathrm{d}A(w),$$

which solves the $\bar\partial$-equation

$$\bar\partial u = f. \tag{4.2}$$

Moreover, $u = \mathrm{C} f$ is bounded and is therefore of class $L^2_{\phi,n}$ for any $n \geq 1$. Hence the function

$$u_{0,n}(z) = u(z) - \mathrm{P}_{\phi,n} u(z) \tag{4.3}$$

solves (4.2) and is of class $L^2_{\phi,n}$. It is easy to check that $u_{0,n}$, as defined by (4.3), is the unique norm-minimal solution to (4.2) in $L^2_{\phi,n}$ whenever $u \in L^2_{\phi,n}$ is a solution to (4.2). In a sense, the study of the orthogonal projection $\mathrm{P}_{\phi,n} u$ is therefore equivalent to the study of the $L^2_{\phi,n}$-minimal solution $u_{0,n}$ to (4.2).

It is useful to observe that $u_{0,n}$ is characterized among the solutions of class $L^2_{\phi,n}$ to (4.2) by the condition $u_{0,n} \perp A^2_{\phi,n}$, that is,

$$\int_{\mathbb{C}} u_{0,n}(z)\overline{h(z)} \mathrm{e}^{-\phi(z)} \mathrm{d}A(z) = 0 \quad \text{for all } h \in A^2_{\phi,n}. \tag{4.4}$$

Our principal result in this section, Theorem 4.1, states that a variant of the (one-dimensional) weighted L^2-estimates of Hörmander apply to $u_{0,n}$. As for further developments, the important results are, however, Corollaries 4.5 and 4.4. The reader may consider glancing at those results and skipping to the next section on a first read.

4.2 Hörmander's Weighted L^2-Estimates

The L^2-estimates of Hörmander in the most elementary, one-dimensional form applies only to weights ϕ that are strictly subharmonic in the entire plane $\mathbb{C}$. The result states that u_0, the L^2_ϕ-minimal solution to (4.2) (where $f \in \mathscr{C}_0(\mathbb{C})$) satisfies

$$\int_{\mathbb{C}} |u_0|^2 \mathrm{e}^{-\phi} \, \mathrm{d}A \leq \int_{\mathbb{C}} |f|^2 \frac{\mathrm{e}^{-\phi}}{\Delta\phi} \, \mathrm{d}A, \tag{4.5}$$

provided that ϕ is $\mathscr{C}^2$-smooth on $\mathbb{C}$. See [17, eq. (4.2.6), p. 250] (this is essentially just Green's formula).

It is important to observe that the estimate (4.5) remains valid for strictly subharmonic weights ϕ in the larger class $\mathscr{C}^{1,1}(\mathbb{C})$ (that ϕ is strictly subharmonic here means that $\Delta\phi > 0$ a.e. on $\mathbb{C}$). The proof in [17, sec. 4.2] goes through without changes in this more general case.

4.3 Weighted $L^2_{\phi,n}$-Estimates

If $f \in L^p(\mathscr{X})$ for some Borel set $\mathscr{X} \subset \mathbb{C}$, it is tacitly extended to all of $\mathbb{C}$ by declaring it to vanish on $\mathbb{C} \setminus \mathscr{X}$.

We have the following theorem, where we consider two weights ϕ and $\widehat{\phi}$ with various properties.

THEOREM 4.1 *Let $\mathscr{T}$ be a compact subset of $\mathbb{C}$, while ϕ, $\hat{\phi}$, and ρ are three real-valued functions of class $\mathscr{C}^{1,1}(\mathbb{C})$ and n is a positive integer. We assume that*:

(i) $L^2_{\hat{\phi}}$ *contains the function* $z \mapsto (1+|z|)^{-1}$, *and that there are real constants* a, b *with* $0 \le a, b < +\infty$, *such that*

$$\hat{\phi} \le \phi + a \quad \text{on } \mathbb{C} \quad \text{and} \quad \phi \le \hat{\phi} + b \quad \text{on } \mathscr{T},$$

(ii) $A^2_{\hat{\phi}} \subset A^2_{\phi,n}$,

(iii) $\Delta(\hat{\phi} + \rho) > 0$ *holds a.e. on* $\mathbb{C}$,

(iv) ρ *is locally constant on* $\mathbb{C} \setminus \mathscr{T}$, *and*

(v) *there exists a number* κ, $0 < \kappa < 1$, *such that*

$$\text{(v-a)} \qquad \frac{|\bar{\partial}\rho|^2}{\Delta(\hat{\phi} + \rho)} \le \frac{\kappa^2}{e^{a+b}} \quad \text{a.e. on } \mathscr{T}.$$

If $f \in L^\infty(\mathscr{T})$, *then* $u_{0,n}$, *the* $L^2_{\phi,n}$*-minimal solution to* $\bar{\partial}u = f$, *satisfies*

$$\int_{\mathbb{C}} |u_{0,n}|^2 e^{\rho-\phi}\, dA \le \frac{e^{a+b}}{(1-\kappa)^2} \int_{\mathscr{T}} |f|^2 \frac{e^{\rho-\phi}}{\Delta(\hat{\phi} + \rho)}\, dA.$$

Before we supply the proof, we state a basic lemma.

LEMMA 4.2 *Suppose that* $f \in L^\infty(\mathbb{C})$ *has compact support as a distribution. Then* (4.2) *has a solution* u *in* $L^2_{\hat{\phi}}$. *Moreover,* v_0, *the* $L^2_{\hat{\phi}}$*-minimal solution to* (4.2), *is of class* $L^2_{\phi,n}$.

PROOF: As f is bounded with compact support, the Cauchy transform $\mathrm{C}f$ is bounded and decays like $|\mathrm{C}f(z)| = \mathrm{O}(|z|^{-1})$ as $|z| \to +\infty$. By assumption (i), the Cauchy transform $\mathrm{C}f$ is in $L^2_{\hat{\phi}}$. Thus the $L^2_{\hat{\phi}}$-minimal solution v_0 to (4.2) exists, and it is necessarily of the form $v_0 = \mathrm{C}f + g$ with some $g \in A^2_{\hat{\phi}}$. In view of property (ii), we know that $g \in A^2_{\phi,n}$. The assertion is now immediate. □

PROOF OF THEOREM 4.1: The assertion is trivial unless

$$\int_{\mathscr{T}} |f|^2 \frac{e^{\rho-\phi}}{\Delta(\hat{\phi} + \rho)}\, dA < +\infty,$$

which will be assumed in the rest of the proof. In view of (4.4), the condition that $u_{0,n}$ is $L^2_{\phi,n}$-minimal may be expressed as

$$\int_{\mathbb{C}} u_{0,n} e^{\rho} \bar{h} e^{-(\phi+\rho)} dA = 0 \quad \text{for all } h \in A^2_{\phi,n}.$$

The latter relation means that the function $w_0 = u_{0,n}\mathrm{e}^{\rho}$ minimizes the norm in $L^2_{\phi+\rho}$ over elements of the (nonclosed) subspace

$$\mathrm{e}^{\rho}L^2_{\phi,n} = \{w : w = \mathrm{e}^{\rho}h \text{ where } h \in L^2_{\phi,n}\} \subset L^2_{\phi+\rho},$$

which solve the (modified) $\bar\partial$-equation

$$\bar\partial w = \bar\partial(u_{0,n}\mathrm{e}^{\rho}) = f\mathrm{e}^{\rho} + u_{0,n}\bar\partial(\mathrm{e}^{\rho}). \tag{4.6}$$

Since ρ is locally constant outside the compact set $\mathscr{T}$ and bounded on $\mathscr{T}$, it is automatically bounded throughout $\mathbb{C}$. It follows that we have

$$L^2_{\phi+\rho,n} = \mathrm{e}^{\rho}\, L^2_{\phi,n} = L^2_{\phi,n} \quad \text{(as sets)},$$

and we conclude that w_0 is the norm-minimal solution to (4.6) in $L^2_{\phi+\rho,n}$.

Let v_0 denote the $L^2_{\hat\phi}$-minimal solution to $\bar\partial u = f$ (see Lemma 4.2). We form the continuous function $g = \bar\partial((u_{0,n} - v_0)\mathrm{e}^{\rho}) = (u_{0,n} - v_0)\,\bar\partial(\mathrm{e}^{\rho})$, whose support is contained in $\mathscr{T}$, and consider the $\bar\partial$-equation

$$\bar\partial\xi = g = (u_{0,n} - v_0)\,\bar\partial(\mathrm{e}^{\rho}). \tag{4.7}$$

The assertion of Lemma 4.2 remains valid if $\hat\phi$ is replaced by $\hat\phi + \rho$; it follows that (4.7) has a solution $\xi \in L^2_{\hat\phi+\rho}$, and moreover, if ξ_0 denotes the norm-minimal solution to (4.7) in $L^2_{\hat\phi+\rho}$, we have $\xi_0 \in L^2_{\phi+\rho,n}$.

We next continue by forming the function

$$w_1 = v_0\mathrm{e}^{\rho} + \xi_0.$$

It is clear that $w_1 \in L^2_{\phi+\rho,n}$, and a calculation yields that

$$\bar\partial w_1 = f\mathrm{e}^{\rho} + v_0\,\bar\partial(\mathrm{e}^{\rho}) + (u_{0,n} - v_0)\,\bar\partial(\mathrm{e}^{\rho}) = \bar\partial(u_{0,n}\mathrm{e}^{\rho}) = \bar\partial w_0. \tag{4.8}$$

Since w_0 is norm-minimal in $L^2_{\phi+\rho,n}$ over functions w such that $\bar\partial w = \bar\partial w_0$, we must have

$$\int_{\mathbb{C}} |w_0|^2 e^{-(\phi+\rho)}\mathrm{d}A \le \int_{\mathbb{C}} |w_1|^2\mathrm{e}^{-(\phi+\rho)}\mathrm{d}A \le \mathrm{e}^{a}\int_{\mathbb{C}} |w_1|^2\mathrm{e}^{-(\hat\phi+\rho)}\mathrm{d}A, \tag{4.9}$$

where we have used the condition $\hat\phi \le \phi + a$ to deduce the second inequality. Moreover, since ξ_0 is norm-minimal in $L^2_{\hat\phi+\rho}$ among solutions to (4.7), we have

$$\int_{\mathbb{C}} w_1\bar h\mathrm{e}^{-(\hat\phi+\rho)}\mathrm{d}A = \int_{\mathbb{C}} v_0\bar h\mathrm{e}^{-\hat\phi}\,\mathrm{d}A + \int_{\mathbb{C}} \xi_0\bar h\mathrm{e}^{-(\hat\phi+\rho)}\mathrm{d}A = 0$$

for all $h \in A^2_{\hat\phi+\rho}$, so the function w_1 is in fact the norm-minimal solution to a $\bar\partial$-equation in $L^2_{\hat\phi+\rho}$. The $\bar\partial$-equation satisfied by w_1 is (see (4.8))

$$\bar\partial w_1 = f\mathrm{e}^{\rho} + u_{0,n}\bar\partial(\mathrm{e}^{\rho}) = f\mathrm{e}^{\rho} + u_{0,n}\mathrm{e}^{\rho}\bar\partial\rho.$$

By the Hörmander estimate (4.5) applied to the weight $\widehat{\phi} + \rho$, we obtain that

$$\begin{aligned}\int_{\mathbb{C}} |w_1|^2 \mathrm{e}^{-(\widehat{\phi}+\rho)} \mathrm{d}A &\le \int_{\mathbb{C}} \left| f \mathrm{e}^{\rho} + u_{0,n} \mathrm{e}^{\rho} \bar{\partial} \rho \right|^2 \frac{\mathrm{e}^{-(\widehat{\phi}+\rho)}}{\Delta(\widehat{\phi}+\rho)} \mathrm{d}A \\ &= \int_{\mathbb{C}} \left| f + u_{0,n} \bar{\partial}\rho \right|^2 \frac{\mathrm{e}^{\rho-\widehat{\phi}}}{\Delta(\widehat{\phi}+\rho)} \mathrm{d}A. \end{aligned} \tag{4.10}$$

Since f and $\bar{\partial}\rho$ are supported in $\mathscr{T}$, and since $\mathrm{e}^{-\widehat{\phi}} \le \mathrm{e}^{b}\mathrm{e}^{-\phi}$ there (see condition (i)), it is seen that the right-hand side in (4.10) may be estimated by

$$\mathrm{e}^{b} \int_{\mathbb{C}} |f + u_{0,n} \bar{\partial}\rho|^2 \frac{\mathrm{e}^{\rho-\phi}}{\Delta(\widehat{\phi}+\rho)} \mathrm{d}A. \tag{4.11}$$

For positive t, we now use the inequality

$$|a+b|^2 \le (1+t)|a|^2 + (1+t^{-1})|b|^2$$

and condition (v-a) to conclude that the integral in (4.11) is dominated by

$$\begin{aligned} &(1+t) \int_{\mathbb{C}} |f|^2 \frac{\mathrm{e}^{\rho-\phi}}{\Delta(\widehat{\phi}+\rho)} \mathrm{d}A + (1+t^{-1}) \int_{\mathbb{C}} |u_{0,n}|^2 \frac{|\bar{\partial}\rho|^2}{\Delta(\widehat{\phi}+\rho)} \mathrm{e}^{\rho-\phi} \mathrm{d}A \\ &\le (1+t) \int_{\mathbb{C}} |f|^2 \frac{\mathrm{e}^{\rho-\phi}}{\Delta(\widehat{\phi}+\rho)} \mathrm{d}A + (1+t^{-1}) \frac{\kappa^2}{\mathrm{e}^{a+b}} \int_{\mathbb{C}} |u_{0,n}|^2 \mathrm{e}^{\rho-\phi} \mathrm{d}A. \end{aligned} \tag{4.12}$$

Tracing back through (4.9), (4.10), (4.11), and (4.12), we get

$$\begin{aligned} \int_{\mathbb{C}} |u_{0,n}|^2 \mathrm{e}^{\rho-\phi} \mathrm{d}A &\le \mathrm{e}^{a+b}(1+t) \int_{\mathbb{C}} |f|^2 \frac{\mathrm{e}^{\rho-\phi}}{\Delta\widehat{\phi} + \Delta\rho} \mathrm{d}A \\ &\quad + (1+t^{-1})\kappa^2 \int_{\mathbb{C}} |u_{0,n}|^2 \mathrm{e}^{\rho-\phi} \mathrm{d}A, \end{aligned}$$

which we write as

$$(1-(1+t^{-1})\kappa^2) \int_{\mathbb{C}} |u_{0,n}|^2 \mathrm{e}^{\rho-\phi} \mathrm{d}A \le \mathrm{e}^{a+b}(1+t) \int_{\mathbb{C}} |f|^2 \frac{\mathrm{e}^{\rho-\phi}}{\Delta(\widehat{\phi}+\rho)} \mathrm{d}A.$$

The desired estimate now follows if we pick $t = \kappa/(1-\kappa)$ and recall that f is supported on $\mathscr{T}$. □

4.4 Implementation Scheme

We now fix $Q \in \mathscr{C}^{1,1}(\mathbb{C})$ satisfying the growth assumption (1.5) with a fixed $\rho > 0$. Adding a constant to Q does not change the problem, and so we may assume that $Q \geq 1$ on $\mathbb{C}$. Let us put

$$q_\tau = \sup_{z \in \mathscr{S}_\tau} \{Q(z)\}, \quad l_\tau = \sup_{z \in \mathscr{S}_\tau} \{\log(1 + |z|^2)\}. \tag{4.13}$$

We next fix a positive number $\tau < \rho$ and two positive numbers M_0 and M_1 such that

$$M_1 \log(1 + |z|^2) \leq M_0 \widehat{Q}_\tau(z), \quad z \in \mathbb{C}. \tag{4.14}$$

This is possible since $\widehat{Q}_\tau \geq 1$ and $\widehat{Q}_\tau(z) = \tau \log |z|^2 + \mathrm{O}(1)$ when $|z| \to +\infty$ (see (2.4)). In particular, it follows that $M_1 \leq M_0 \tau$.

We now put

$$\phi_m = mQ \quad \text{and} \quad \widehat{\phi}_{m,M_0,M_1}(z) = (m - M_0)\widehat{Q}_\tau(z) + M_1 \log(1 + |z|^2). \tag{4.15}$$

Note that $\widehat{\phi}_{m,M_0,M_1}$ is strictly subharmonic on $\mathbb{C}$ whenever $m \geq M_0$ with

$$\Delta\widehat{\phi}_{m,M_0,M_1}(z) = (m - M_0)\Delta\widehat{Q}_\tau(z) + M_1(1+|z|^2)^{-2} \geq M_1(1+|z|^2)^{-2} \tag{4.16}$$

and that (4.14) implies that (recall that $\widehat{Q}_\tau \leq Q$)

$$\widehat{\phi}_{m,M_0,M_1} \leq \phi_m \quad \text{on } \mathbb{C}. \tag{4.17}$$

In the other direction, we easily see that

$$\phi_m \leq \widehat{\phi}_{m,M_0,M_1} + M_0 q_\tau \quad \text{on } \mathscr{S}_\tau. \tag{4.18}$$

Near the point at infinity, we instead have

$$\phi_m(z) - \widehat{\phi}_{m,M_0,M_1}(z) \geq m(\rho - \tau) \log |z|^2 + \mathrm{O}(1) \quad \text{as } |z| \to +\infty,$$

given the growth assumption on Q and the asymptotics

$$\widehat{\phi}_{m,M_0,M_1} = ((m - M_0)\tau + M_1) \log |z|^2 + \mathrm{O}(1) \quad \text{as } |z| \to +\infty, \tag{4.19}$$

which follows from (2.4). Note also that

$$A^2_{\phi_m,n} = A^2_{mQ} \cap \mathscr{P}_n = H_{m,n}.$$

We now check conditions (i) and (ii) of Theorem 4.1. We recall that $]x[$ denotes the largest integer that is strictly smaller than x.

LEMMA 4.3 *In view of* (4.17) *and* (4.18), *condition* (i) *of Theorem* 4.1 *holds for* $\phi = \phi_m$, $\widehat{\phi} = \widehat{\phi}_{m,M_0,M_1}$, *provided the compact set* $\mathscr{T}$ *is contained in* $\mathscr{S}_\tau$, *and the constants* a *and* b *are given by* $a = 0$ *and* $b = M_0 q_\tau$, *while* m *is kept* $m \geq M_0$. *On the other hand, condition* (ii) *of Theorem* 4.1, *which requires that* $A^2_{\widehat{\phi}_{m,M_0,M_1}} \subset H_{m,n}$, *holds if* $n \geq](m - M_0)\tau + M_1[$.

PROOF: As for condition (i), it remains to check that the function $(1+|z|)^{-1}$ is in $L^2_{\hat{\phi}_{m,M_0,M_1}}$ for $m \geq M_0$. This follows from the asymptotics (4.19), since

$$(m - M_0)\tau + M_1 > 0$$

and

$$\int_{\mathbb{C}} (1+|z|^2)^{-r} \mathrm{d}A(z) < +\infty \iff r > 1. \tag{4.20}$$

As for condition (ii), we note that since $\hat{\phi}_{m,M_0,M_1} \leq \phi_m$, we have the containment $A^2_{\hat{\phi}_{m,M_0,M_1}} \subset A^2_{\phi_m}$, and it remains only to check that $A^2_{\hat{\phi}_{m,M_0,M_1}} \subset \mathscr{P}_n$ provided that $n \geq \,](m-M_0)\tau + M_1[$. But this is immediate from (4.19) and (4.20). □

We first apply Theorem 4.1 in a rather simple fashion; cf. [6, p. 10].

COROLLARY 4.4 *Suppose $Q \in \mathscr{C}^{1,1}(\mathbb{C})$, with $Q \geq 1$ on $\mathbb{C}$. If $f \in L^\infty(\mathscr{S}_\tau)$, then the function $u_{0,n}$, which is the $L^2_{mQ,n}$-minimal solution to $\bar\partial u = f$, satisfies*

$$\|u_{0,n}\|^2_{mQ} \leq \frac{\mathrm{e}^{M_0 q_\tau + 2l_\tau}}{M_1} \|f\|^2_{mQ},$$

provided that $m \geq M_0$ and $n \geq \,](m-M_0)\tau + M_1[$.

PROOF: Given Lemma 4.3, the estimate follows if we put $\mathscr{T} = \mathscr{S}_\tau$ and $\varrho_m = 0$ in Theorem 4.1, while observing that κ may be chosen as close to 0 as we like. □

We next apply Theorem 4.1 in a more challenging context. The result will be required to obtain off-diagonal damping estimates for the kernel $K_{m,n}(z,w)$.

COROLLARY 4.5 *Suppose $Q \in \mathscr{C}^{1,1}(\mathbb{C})$ with $Q \geq 1$ on $\mathbb{C}$. Let $\mathscr{T}$ be a compact subset of $\mathscr{S}_\tau$ such that*

$$\Delta Q \geq \alpha > 0 \quad \text{a.e. on } \mathscr{T},$$

where α is a constant. Also, suppose there is a real-valued function $\rho_m \in \mathscr{C}^{1,1}(\mathbb{C})$ with

$$\bar\partial \rho_m = 0 \quad \text{on } \mathbb{C} \setminus \mathscr{T}, \tag{i}$$

such that

$$\Delta \rho_m \geq -\frac{m\alpha}{2} + M_0 \alpha \quad \text{a.e. on } \mathscr{T}, \tag{ii}$$

while

$$|\bar\partial \rho_m|^2 \leq \frac{m\alpha\kappa^2}{2\mathrm{e}^{M_0 q_\tau}} \quad \text{a.e. on } \mathscr{T} \tag{iii}$$

for some constant κ, $0 < \kappa < 1$. *Then, if* $m \geq M_0$ *and* $n \geq \,](m - M_0)\tau + M_1[$, *and if* $f \in L^\infty(\mathscr{T})$, *we have that* $u_{0,n}$, *the* $L^2_{mQ,n}$*-minimal solution to* $\bar\partial u = f$, *satisfies*

$$\int_{\mathbb{C}} |u_{0,n}|^2 \mathrm{e}^{\rho_m - mQ}\,\mathrm{d}A \leq \frac{2\mathrm{e}^{M_0 q_\tau}}{(1-\kappa)^2(m\alpha + 2M_1\mathrm{e}^{-2l_\tau})} \int_{\mathscr{T}} |f|^2 \mathrm{e}^{\rho_m - mQ}\,\mathrm{d}A.$$

PROOF: The function ρ_m being real-valued, assumption (i) entails that ρ_m is locally constant in $\mathbb{C} \setminus \mathscr{T}$. Since

$$\Delta(\hat\phi_{m,M_0,M_1} + \rho_m) = (m - M_0)\Delta\hat{Q}_\tau + \Delta\rho_m + M_1(1 + |z|^2)^{-2}, \tag{4.21}$$

and since $\Delta\hat{Q}_\tau = \Delta Q$ a.e. on $\mathscr{S}_\tau$, we get from the assumptions on $\mathscr{T}$ and ρ_m that

$$\Delta(\hat\phi_{m,M_0,M_1} + \rho_m) \geq \frac{m\alpha}{2} + M_1\mathrm{e}^{-2l_\tau} \quad \text{a.e. on } \mathscr{T}. \tag{4.22}$$

Moreover, the assumption that $\bar\partial\rho_m = 0$ on $\mathbb{C} \setminus \mathscr{T}$ implies that $\Delta\rho_m = 0$ on $\mathbb{C} \setminus \mathscr{T}$ as well, and so, in view of (4.21), we have

$$\Delta(\hat\phi_{m,M_0,M_1} + \rho_m) > 0 \quad \text{a.e. on } \mathbb{C} \setminus \mathscr{T}.$$

In view of (iii) and (4.22), we have

$$\frac{|\bar\partial\rho_m|^2}{\Delta(\hat\phi_{m,M_0,M_1} + \rho_m)} \leq \frac{\kappa^2}{\mathrm{e}^{M_0 q_\tau}},$$

which means that condition (v) of Theorem 4.1 is fulfilled with the parameter choices $a = 0$ and $b = M_0 q_\tau$ of Lemma 4.3. We have already verified that conditions (i)–(iv) of Theorem 4.1 are satisfied (conditions (i) and (ii) are dealt with in Lemma 4.3). The assertion of the corollary is now an immediate consequence of Theorem 4.1. □

5 Approximate Bergman Projections

5.1 Preliminaries

In this section, we state and prove a result (Theorem 5.2 below) that we later use to obtain the asymptotic expansion of Theorem 2.3.

5.2 Earlier Work

Our approach is based on a paper by Berman, Berndtsson, and Sjöstrand [7] with some slight modifications. Perhaps it is of value to explain how weighted Bergman kernel expansions work explicitly in the elementary $\mathbb{C}^1$ case. For this reason, our exposition is rather detailed.

5.3 Local Integral Operator

We recall the ansatz

$$K^1_m(z,w) = (mb_0(z,\bar{w}) + b_1(z,\bar{w}))\mathrm{e}^{m\psi(z,\bar{w})},$$

where the functions b_0 and b_1 are as in Section 2.3. Associated with the kernel, we define the local integral operator

$$\mathrm{I}^{1,\chi}_m u(z) = \int_{\mathbb{D}(z_0,2\varepsilon)} u(w)\chi(w)K^1_m(z,w)\mathrm{e}^{-mQ(w)}\mathrm{d}A(w), \tag{5.1}$$

where χ is a fixed, smooth cutoff function, with the following properties: $\chi = 1$ on $\mathbb{D}(z_0, \frac{3}{2}\varepsilon)$, $\chi = 0$ on $\mathbb{C} \setminus \mathbb{D}(z_0, 2\varepsilon)$, $0 \le \chi \le 1$ throughout $\mathbb{C}$, and $\|\chi\|_{L^2} \le 3$. The point $z_0 \in \mathscr{N}_+$ is assumed fixed. Moreover, the parameter ε is assumed small but positive, and independent of the parameter m. We specify how small ε should be as we go along.

If we put

$$B^1_m(z,\bar{w}) = b_0(z,\bar{w}) + m^{-1}b_1(z,\bar{w}),$$

we may write

$$\mathrm{I}^{1,\chi}_m u(z) = m\int_{\mathbb{D}(z_0,2\varepsilon)} u(w)\chi(w)B^1_m(z,\bar{w})\,\mathrm{e}^{m(\psi(z,\bar{w})-Q(w))}\mathrm{d}A(w). \tag{5.2}$$

In order for this expression to make sense, we assume that ε is so small that

$$\psi(z,\bar{w}) = \sum_{k=0}^{+\infty}\frac{1}{k!}(\bar{w}-\bar{z})^k\bar{\partial}^k Q(z)$$

makes sense as a convergent series for all $z, w \in \mathbb{D}(z_0, 3\varepsilon)$.

5.4 Phase Function

The function

$$\phi(z,w) = \frac{\psi(z,\bar{w}) - Q(w)}{z-w} = \frac{\psi(z,\bar{w}) - \psi(w,\bar{w})}{z-w} \tag{5.3}$$

will be called the *phase function*. It is clear that $\phi(z,w)$ is holomorphic in z and real analytically smooth in w across $z = w$, and we observe that

$$\begin{aligned}\bar{\partial}_w \mathrm{e}^{m(z-w)\phi(z,w)} &= m(z-w)\bar{\partial}_w\phi(z,w)\mathrm{e}^{m(z-w)\phi(z,w)}\\ &= m(z-w)\bar{\partial}_w\phi(z,w)\mathrm{e}^{m(\psi(z,\bar{w})-Q(w))}.\end{aligned} \tag{5.4}$$

As a consequence, (5.2) may be written in the form

$$\mathrm{I}^{1,\chi}_m u(z) = \int_{\mathbb{D}(z_0,2\varepsilon)} \frac{u(w)\chi(w)}{z-w}\,\frac{B^1_m(z,\bar{w})}{\bar{\partial}_w\phi(z,w)}\,\bar{\partial}_w \mathrm{e}^{m(z-w)\phi(z,w)}\mathrm{d}A(w). \tag{5.5}$$

We note that

$$\bar\partial_w \phi(z,w)\big|_{w=z} = \Delta Q(z). \tag{5.6}$$

5.5 Diagonal Taylor Expansion

By Taylor's formula,

$$\begin{aligned}\psi(z,\bar w) - Q(w) &= Q(z) + (\bar w - \bar z)\bar\partial Q(z) + \frac{1}{2}(\bar w - \bar z)^2\bar\partial^2 Q(z)\\ &\quad - \Big\{Q(z) + (\bar w - \bar z)\bar\partial Q(z) + (w-z)\partial Q(z)\\ &\quad + \frac{1}{2}(\bar w - \bar z)^2\bar\partial^2 Q(z) + \frac{1}{2}(w-z)^2\partial^2 Q(z)\\ &\quad + |w-z|^2\Delta Q(z)\Big\} + \mathrm{O}(|w-z|^3)\\ &= -(w-z)\partial Q(z) - \frac{1}{2}(w-z)^2\partial^2 Q(z)\\ &\quad - |w-z|^2\Delta Q(z) + \mathrm{O}(|w-z|^3),\end{aligned}$$

which in terms of the quadratic expression

$$L_z(w) = (w-z)\partial Q(z) + \frac{1}{2}(w-z)^2\partial^2 Q(z)$$

becomes

$$\psi(z,\bar w) - Q(w) = -L_z(w) - |w-z|^2\Delta Q(z) + \mathrm{O}(|w-z|^3)$$

for $z, w \in \mathbb{D}(z_0, 2\varepsilon)$.

An analogous calculation shows that

$$2\psi(z,\bar w) - Q(z) - Q(w) = -2\mathrm{i}\,\mathrm{Im}\,L_z(w) - |w-z|^2\Delta Q(z) + \mathrm{O}(|w-z|^3),$$

and, if we take real parts, we obtain

$$2\,\mathrm{Re}\,\psi(z,\bar w) - Q(z) - Q(w) = -|w-z|^2\Delta Q(z) + \mathrm{O}(|w-z|^3). \tag{5.7}$$

In several instances we only need a rough estimate of this type:

$$2\,\mathrm{Re}\,\psi(z,\bar w) - Q(z) - Q(w) \le -\frac{1}{2}|w-z|^2\Delta Q(z_0), \quad z, w \in \mathbb{D}(z_0, 3\varepsilon). \tag{5.8}$$

This entails a restriction on the size of $\varepsilon > 0$. We note that this requirement contains

$$\Delta Q(z) \ge \frac{1}{2}\Delta Q(z_0) > 0, \quad z \in \mathbb{D}(z_0, 3\varepsilon). \tag{5.9}$$

We recall that $\Delta Q(z_0) > 0$ holds as $z_0 \in \mathscr{N}_+$.

5.6 Differential Operator and Negligible Amplitudes

Let M_{z-w} denote the operator of multiplication by $z-w$:

$$\mathrm{M}_{z-w} f(z,w) = (z-w) f(z,w),$$

and consider the (partial) differential operator

$$\nabla_m = \frac{1}{\bar\partial_w \phi(z,w)} \bar\partial_w + m\mathrm{M}_{z-w}. \tag{5.10}$$

For smooth functions $X(z,w)$, we see that

$$\bar\partial_w \{\mathrm{e}^{m(z-w)\phi(z,w)} X(z,w)\} = \bar\partial_w \phi(z,w) \mathrm{e}^{m(z-w)\phi(z,w)} \nabla_m X(z,w). \tag{5.11}$$

Functions of the form $\nabla_m X$ are called *negligible amplitudes* because, in an integral expression like (5.12) below, the term involving $\nabla_m X$ decreases exponentially in m.

We turn to the main technical result of this section.

PROPOSITION 5.1 *Fix a point $z_0 \in \mathscr{N}_+$. We then have that*

$$\begin{aligned}\frac{B_m^1(z,\bar w)}{\bar\partial_w \phi(z,w)} &= \frac{b_0(z,\bar w)}{\bar\partial_w \phi(z,w)} + m^{-1} \frac{b_1(z,\bar w)}{\bar\partial_w \phi(z,w)} \\ &= 1 + m^{-1} \nabla_m X_m(z,w) + m^{-2} Y(z,w),\end{aligned}$$

where $X_m = X^0 + m^{-1} X^1$, and the functions X^0, X^1, and Y are all holomorphic in z and real analytically smooth in w for $z, w \in \mathbb{D}(z_0, 3\varepsilon)$ provided $\varepsilon > 0$ is small enough. Moreover, if $\mathscr{K}$ is a compact subset of $\mathscr{N}_+$, an $\varepsilon > 0$ can be found that works for all $z_0 \in \mathscr{K}$ simultaneously.

We supply the proof at the end of this section.

5.7 Application to Local Integral Operator

We apply Proposition 5.1 to (5.5) (in the second and third terms on the right-hand side, we use relation (5.4) to expand $\bar\partial_w \mathrm{e}^{m(z-w)\phi(z,w)}$, and recall that $(z-w)\phi(z,w) = \psi(z,\bar w) - Q(w))$ to obtain

$$\begin{aligned}\mathrm{I}_m^{1,\chi} u(z) = &\int_{\mathbb{D}(z_0,2\varepsilon)} \frac{u(w)\chi(w)}{z-w} \bar\partial_w \mathrm{e}^{m(z-w)\phi(z,w)} \mathrm{d}A(w) \\ &+ \int_{\mathbb{D}(z_0,2\varepsilon)} u(w)\chi(w) \nabla_m X_m(z,w) \bar\partial_w \phi(z,w) \mathrm{e}^{m(\psi(z,\bar w) - Q(w))} \mathrm{d}A(w) \\ &+ m^{-1} \int_{\mathbb{D}(z_0,2\varepsilon)} u(w)\chi(w) Y(z,w) \bar\partial_w \phi(z,w) \mathrm{e}^{m(\psi(z,\bar w) - Q(w))} \mathrm{d}A(w).\end{aligned} \tag{5.12}$$

Let C_1 be a positive constant such that

$$|Y(z,w)\bar\partial_w \phi(z,w)| \le C_1, \quad z, w \in \mathbb{D}(z_0, 2\varepsilon).$$

Then, for $z \in \mathbb{D}(z_0, 2\varepsilon)$, we have, in view of (5.8) and the Cauchy-Schwarz inequality,

$$\begin{aligned}
&\left| \int_{\mathbb{D}(z_0,2\varepsilon)} u(w)\chi(w)\, Y(z,w)\, \bar\partial_w \phi(z,w)\, \mathrm{e}^{m(\psi(z,\bar w)-Q(w))} \mathrm{d}A(w) \right| \\
&\quad \le C_1 \int_{\mathbb{D}(z_0,2\varepsilon)} |u(w)|\, \mathrm{e}^{m(\operatorname{Re}\psi(z,\bar w)-Q(w))} \mathrm{d}A(w) \\
&\quad \le C_1 \mathrm{e}^{\frac12 mQ(z)} \int_{\mathbb{D}(z_0,2\varepsilon)} |u(w)|\, \mathrm{e}^{-\frac12 mQ(w)} \mathrm{e}^{-\frac14 \Delta Q(z_0)|w-z|^2}\, \mathrm{d}A(w) \\
&\quad \le C_1 \mathrm{e}^{\frac12 mQ(z)} \|u\mathrm{e}^{-\frac12 mQ}\|_{L^2(\mathbb{D}(z_0,2\varepsilon))} \left\{ \int_{\mathbb{C}} \mathrm{e}^{-\frac12 \Delta Q(z_0)|w-z|^2}\, \mathrm{d}A(w) \right\}^{1/2} \\
&\quad = C_1' m^{-1/2}\, \mathrm{e}^{\frac12 mQ(z)} \|u\mathrm{e}^{-\frac12 mQ}\|_{L^2(\mathbb{D}(z_0,2\varepsilon))},
\end{aligned}$$

where $C_1' = 2^{1/2}[\Delta Q(z_0)]^{-1/2} C_1$.

Next, by (5.11), we have

$$\begin{aligned}
&\int_{\mathbb{D}(z_0,2\varepsilon)} u(w)\chi(w)\, \nabla_m X_m(z,w)\, \bar\partial_w \phi(z,w)\, \mathrm{e}^{m(\psi(z,\bar w)-Q(w))} \mathrm{d}A(w) = \\
&\qquad\qquad \int_{\mathbb{D}(z_0,2\varepsilon)} u(w)\chi(w)\, \bar\partial_w \{\mathrm{e}^{m(\psi(z,\bar w)-Q(w))} X_m(z,w)\} \mathrm{d}A(w),
\end{aligned}$$

and if u is holomorphic in, say, $\mathbb{D}(z_0, 3\varepsilon)$, we can use Green's formula to conclude that

$$\begin{aligned}
&\int_{\mathbb{D}(z_0,2\varepsilon)} u(w)\chi(w)\, \bar\partial_w \{\mathrm{e}^{m(\psi(z,\bar w)-Q(w))} X_m(z,w)\} \mathrm{d}A(w) = \\
&\qquad\qquad - \int_{\mathbb{D}(z_0,2\varepsilon)} u(w) \bar\partial \chi(w)\, X_m(z,w)\, \mathrm{e}^{m(\psi(z,\bar w)-Q(w))} \mathrm{d}A(w).
\end{aligned}$$

We now confine the parameter z to the disk $\mathbb{D}(z_0, \varepsilon)$ and observe that then

$$|w-z| \ge \frac{\varepsilon}{2}, \quad w \in \mathbb{D}(z_0, 2\varepsilon) \setminus \mathbb{D}(z_0, \tfrac32 \varepsilon),$$

so that by (5.8),

$$\operatorname{Re}\psi(z,\bar w) \le \frac12 Q(z) + \frac12 Q(w) - \frac{\varepsilon^2}{16} \Delta Q(z_0), \qquad w \in \mathbb{D}(z_0, 2\varepsilon) \setminus \mathbb{D}(z_0, \tfrac32 \varepsilon). \tag{5.13}$$

Moreover, we recall that m should be big, at least $m \geq 1$, so that there exists a positive constant C_2 with

$$|X_m(z,w)| \leq C_2, \quad z \in \mathbb{D}(z_0,\varepsilon),\ w \in \mathbb{D}(z_0, 2\varepsilon),$$

for small enough ε. Recalling that $\bar\partial\chi = 0$ on $\mathbb{D}(z_0, \frac{3}{2}\varepsilon)$, we find that

$$\begin{aligned}\Bigg|\int_{\mathbb{D}(z_0,2\varepsilon)} & u(w)\bar\partial\chi(w)\, X_m(z,w)\, \mathrm{e}^{m(\psi(z,\bar w)-Q(w))}\mathrm{d}A(w)\Bigg| \\ &\leq C_2 \int_{\mathbb{D}(z_0,2\varepsilon)} |u(w)\bar\partial\chi(w)|\mathrm{e}^{m(\operatorname{Re}\psi(z,\bar w)-Q(w))}\mathrm{d}A(w) \\ &\leq C_2 \mathrm{e}^{\frac12 mQ(z)}\mathrm{e}^{-\frac{1}{16}m\varepsilon^2\Delta Q(z_0)}\|u\mathrm{e}^{-\frac12 mQ}\|_{L^2(\mathbb{D}(z_0,2\varepsilon))}\,\|\bar\partial\chi\|_{L^2}.\end{aligned}$$

Since χ was assumed to have $\|\bar\partial\chi\|_{L^2} \leq 3$, we obtain in summary that

$$\int_{\mathbb{D}(z_0,2\varepsilon)} u(w)\chi(w)\, \nabla_m X_m(z,w)\, \bar\partial_w\phi(z,w)\, \mathrm{e}^{m(\psi(z,\bar w)-Q(w))}\mathrm{d}A(w) \leq \\ 3C_2\mathrm{e}^{\frac12 mQ(z)}\mathrm{e}^{-\frac{1}{16}m\varepsilon^2\Delta Q(z_0)}\|u\mathrm{e}^{-\frac12 mQ}\|_{L^2(\mathbb{D}(z_0,2\varepsilon))}.$$

We finally turn to the first term on the right-hand side of (5.12) and use Green's formula to get

$$\int_{\mathbb{D}(z_0,2\varepsilon)} \frac{u(w)\chi(w)}{z-w}\,\bar\partial_w \mathrm{e}^{m(z-w)\phi(z,w)}\mathrm{d}A(w) = \\ u(z) - \int_{\mathbb{D}(z_0,2\varepsilon)} \frac{u(w)\bar\partial\chi(w)}{z-w}\,\mathrm{e}^{m(\psi(z,\bar w)-Q(w))}\mathrm{d}A(w),$$

where again $z \in \mathbb{D}(z_0,\varepsilon)$ and u is assumed holomorphic on $\mathbb{D}(z_0, 3\varepsilon)$.

To estimate the rightmost term, we again recall that $\bar\partial\chi = 0$ on $\mathbb{D}(z_0, \frac32\varepsilon)$ and find that

$$\begin{aligned}\int_{\mathbb{D}(z_0,2\varepsilon)} & \left|\frac{u(w)\bar\partial\chi(w)}{z-w}\,\mathrm{e}^{m(\psi(z,\bar w)-Q(w))}\right|\mathrm{d}A(w) \\ &\leq \frac{2}{\varepsilon}\int_{\mathbb{D}(z_0,2\varepsilon)} |u(w)\bar\partial\chi(w)|\mathrm{e}^{m(\operatorname{Re}\psi(z,\bar w)-Q(w))}\mathrm{d}A(w) \\ &\leq \frac{2}{\varepsilon}\,\mathrm{e}^{\frac12 mQ(z)}\mathrm{e}^{-\frac{1}{16}m\varepsilon^2\Delta Q(z_0)} \int_{\mathbb{D}(z_0,2\varepsilon)} |u(w)|\mathrm{e}^{-\frac12 mQ(w)}|\bar\partial\chi(w)|\,\mathrm{d}A(w) \leq\end{aligned}$$

$$\leq \frac{2}{\varepsilon}\,\mathrm{e}^{\frac{1}{2}mQ(z)}\mathrm{e}^{-\frac{1}{16}m\varepsilon^2\Delta Q(z_0)}\|u\mathrm{e}^{-\frac{1}{2}mQ}\|_{L^2(\mathbb{D}(z_0,2\varepsilon))}\,\|\bar\partial\chi\|_{L^2}$$
$$\leq \frac{6}{\varepsilon}\,\mathrm{e}^{\frac{1}{2}mQ(z)}\mathrm{e}^{-\frac{1}{16}m\varepsilon^2\Delta Q(z_0)}\|u\mathrm{e}^{-\frac{1}{2}mQ}\|_{L^2(\mathbb{D}(z_0,2\varepsilon))},\quad z_0\in\mathbb{D}(z_0,\varepsilon),$$

where we have again used (5.13), the Cauchy-Schwarz inequality, and the estimate $\|\bar\partial\chi\|_{L^2}\leq 3$. We gather our observations in a theorem.

THEOREM 5.2 *Fix a compact subset $\mathscr{K}$ of $\mathscr{N}_+$. Then there exists an $\varepsilon>0$ that depends only on Q and $\mathscr{K}$ such that the following is true: If $z_0\in\mathscr{K}$ and if u is holomorphic in $\mathbb{D}(z_0,3\varepsilon)$, we have*

$$|\mathrm{I}_m^{1,\chi}u(z)-u(z)|\leq Cm^{-3/2}\mathrm{e}^{\frac{1}{2}mQ(z)}\|u\mathrm{e}^{-\frac{1}{2}mQ}\|_{L^2(\mathbb{D}(z_0,2\varepsilon)},\quad z_0\in\mathbb{D}(z_0,\varepsilon),$$

for some positive constant C that depends only on Q, $\mathscr{K}$, and ε.

PROOF: The above sequence of estimates shows that for $z\in\mathbb{D}(z_0,\varepsilon)$,

$$\left|\mathrm{I}_m^{1,\chi}u(z)-u(z)\right|\leq$$
$$\left\{C_1'm^{-3/2}+(3C_2+6\varepsilon^{-1})\mathrm{e}^{-\frac{1}{16}m\varepsilon^2\Delta Q(z_0)}\right\}\mathrm{e}^{\frac{1}{2}mQ(z)}\|u\mathrm{e}^{-\frac{1}{2}mQ}\|_{L^2(\mathbb{D}(z_0,2\varepsilon)},$$

whence the assertion is immediate from the observation that the constants C_1' and C_2 are uniformly bounded for $z_0\in\mathscr{K}$, and that $\varepsilon>0$ can be picked to work simultaneously for all $z_0\in\mathscr{K}$. □

Remark 5.3. The above theorem is the reason that we call the local integral operator $\mathrm{I}_m^{1,\chi}$ an approximate Bergman projection.

5.8 Lift of the Phase Function

As before, we carry out our local analysis near a point $z_0\in\mathscr{N}_+$. The points z and w are usually confined to the neighborhood $\mathbb{D}(z_0,3\varepsilon)$ of the point z_0, where $\varepsilon>0$ is as small as needed. We will want to replace any previous occurrence of $\bar w$ by an independent variable ζ. An instance of this is the function

$$\theta(z,w,\zeta)=\frac{\psi(z,\zeta)-\psi(w,\zeta)}{z-w},\quad z,w,\zeta\in\mathbb{D}(z_0,3\varepsilon),\tag{5.14}$$

which extends the phase function $\phi(z,w)$. Indeed, $\theta(z,w\zeta)$ is holomorphic in all three variables

$$(z,w,\zeta)\in\mathbb{D}(z_0,3\varepsilon)\times\mathbb{D}(z_0,3\varepsilon)\times\mathbb{D}(\bar z_0,3\varepsilon),$$

and it follows from (5.14) that

$$\theta(z,w,\bar w)=\phi(z,w),\quad z,w\in\mathbb{D}(z_0,3\varepsilon).\tag{5.15}$$

By the definition of θ (see (5.14)) and (5.9), we have

$$\partial_\zeta\theta(z,w,\zeta)\big|_{w=z,\zeta=\bar z}=\Delta Q(z)\geq\frac{1}{2}\Delta Q(z_0)>0,\quad z\in\mathbb{D}(z_0,3\varepsilon),$$

so that if we let $\varepsilon>0$ be a little smaller, we are guaranteed that

$$\partial_\zeta\theta(z,w,\zeta)\neq 0,\quad (z,w,\zeta)\in\mathbb{D}(z_0,3\varepsilon)\times\mathbb{D}(z_0,3\varepsilon)\times\mathbb{D}(\bar z_0,3\varepsilon).$$

To simplify the notation, we frequently write θ in place of $\theta(z, w, \zeta)$.

5.9 Some Differential Operators and Change of Variables

We introduce the commuting differential operators

$$\{\partial_\theta\} = \frac{1}{\partial_\zeta \theta}\,\partial_\zeta, \quad \{\partial_w\} = \partial_w - \frac{\partial_w \theta}{\partial_\zeta \theta}\partial_\zeta, \tag{5.16}$$

where the curly brackets are used to differentiate these differential operators from the usual partial derivatives, e.g., ∂_w. We shall adopt the convention of writing, e.g., $\{\partial_\theta \partial_w\}$ for the product $\{d_\theta\}\{\partial_w\}$. The differential operators $\{\partial_\theta\}$ and $\{\partial_w\}$ come from the (local) change of variables

$$(z, w, \zeta) \mapsto (z, w, \theta) \quad \text{where } \theta = \theta(z, w, \zeta).$$

If we put $\zeta = \bar{w}$, we get from (5.16) that

$$\{\partial_\theta\}\big|_{\zeta=\bar{w}} = \frac{1}{\bar{\partial}_w \phi(z, w)}\,\bar{\partial}_w, \tag{5.17}$$

and we recognize the expression on the right-hand side as a component of the differential operator ∇_m.

5.10 Lift of Operator ∇_m

If we put

$$\nabla'_m = \{\partial_\theta\} + m\mathrm{M}_{z-w}, \tag{5.18}$$

we have in view of (5.17) defined a lift of ∇_m to the context of the three complex variables (z, w, θ). We shall suppress the distinction between ∇_m and ∇'_m and write ∇_m in place of ∇'_m.

5.11 Negligible Amplitude Equations

We recall that in view of Proposition 5.1, we are to solve the equation

$$A_m = 1 + m^{-1}\nabla_m X_m + m^{-2} Y, \tag{5.19}$$

where

$$A_m = A^0 + m^{-1}A^1, \quad A^0 = \frac{b_0(z,\zeta)}{\partial_\zeta \theta(z, w, \zeta)}, \quad A^1 = \frac{b_1(z,\zeta)}{\partial_\zeta \theta(z, w, \zeta)}. \tag{5.20}$$

More generally, we may consider the "negligible amplitude" equation

$$A_m = 1 + m^{-1}\nabla_m X_m + m^{-k-1} Y_m, \tag{5.21}$$

where

$$A_m = \sum_{j=0}^{k} m^{-j} A^j, \quad X_m = \sum_{j=0}^{+\infty} m^{-j} X^j, \quad Y_m = \sum_{j=0}^{+\infty} m^{-j} Y^j.$$

The key to studying equations of this type is to introduce the generalized differential operator

$$\mathrm{e}^{m^{-1}\{\partial_\theta \partial_w\}} = \sum_{j=0}^{+\infty} \frac{1}{j!m^j} \{\partial_\theta \partial_w\}^j$$

because of the identity

$$\mathrm{e}^{m^{-1}\{\partial_\theta \partial_w\}} \nabla_m = m \mathrm{M}_{z-w} \mathrm{e}^{m^{-1}\{\partial_\theta \partial_w\}}.$$

As we apply this operator $\mathrm{e}^{m^{-1}\{\partial_\theta \partial_w\}}$ to both sides of the equation, we find that

$$\begin{aligned} \mathrm{e}^{m^{-1}\{\partial_\theta \partial_w\}} A_m &= 1 + m^{-1} \mathrm{e}^{m^{-1}\{\partial_\theta \partial_w\}} \nabla_m X_m + m^{-k-1} \mathrm{e}^{m^{-1}\{\partial_\theta \partial_w\}} Y_m \\ &= 1 + \mathrm{M}_{z-w} \mathrm{e}^{m^{-1}\{\partial_\theta \partial_w\}} X_m + m^{-k-1} \mathrm{e}^{m^{-1}\{\partial_\theta \partial_w\}} Y_m, \end{aligned}$$

so that by restricting to the diagonal $w = z$, we get

$$\begin{aligned} \mathrm{e}^{m^{-1}\{\partial_\theta \partial_w\}} A_m \Big|_{w=z} &= \mathrm{e}^{m^{-1}\{\partial_\theta \partial_w\}} \sum_{j=0}^{k} m^{-j} A^j \Big|_{w=z} \\ &= 1 + m^{-k-1} \mathrm{e}^{m^{-1}\{\partial_\theta \partial_w\}} Y_m \Big|_{w=z}. \end{aligned}$$

This gives $A^0|_{w=z} = 1$, and also

$$A^l \Big|_{w=z} = - \sum_{j=1}^{l} \frac{1}{j!} \{\partial_\theta \partial_w\}^j A^{l-j} \Big|_{w=z}, \quad l = 1, \dots, k.$$

These equations are *compatibility conditions* that must be fulfilled in order for (5.21) to have a solution. In particular, when $k = 1$, we get

$$A^0 \Big|_{w=z} = 1, \quad A^1 \Big|_{w=z} = -\{\partial_\theta \partial_w\} A^0 \Big|_{w=z}. \tag{5.22}$$

To simplify the notation, we put

$$b_0 = b_0(z, \zeta), \quad b_1 = b_1(z, \zeta),$$

and see that (5.20) expresses that

$$A^0 = \frac{b_0}{\partial_\zeta \theta}, \quad A^1 = \frac{b_1}{b_0} A^0,$$

so that equation (5.22) may be written as

$$A^0 \Big|_{w=z} = \frac{b_0}{\partial_\zeta \theta} \Big|_{w=z} = 1, \quad A^1 \Big|_{w=z} = \frac{b_1}{b_0} = -\{\partial_\theta \partial_w\} A^0 \Big|_{w=z}. \tag{5.23}$$

We proceed to check that (5.23) holds. It is an easy consequence of (5.14) that $\partial_\zeta \theta|_{w=z} = \partial_z \partial_\zeta \psi = b_0$, and so the first part of (5.23) is clearly true. In fact, a Taylor series argument gives more precise information:

$$\frac{b_0}{\partial_\zeta \theta} = 1 - \frac{1}{2}(w - z) \frac{\partial_z b_0}{b_0} + \mathrm{O}(|w - z|^2). \tag{5.24}$$

An application of the operator $\{\partial_w\}$ to both sides of the above equality gives

$$\{\partial_w\}\frac{b_0}{\partial_\zeta\theta} = -\frac{1}{2}\frac{\partial_z b_0}{b_0} + \mathrm{O}(|w-z|), \tag{5.25}$$

and if we also apply $\{\partial_\theta\}$, the result is

$$\{\partial_\theta\partial_w\}\frac{b_0}{\partial_\zeta\theta} = -\frac{1}{2\partial_\zeta\theta}\,\partial_\zeta\,\frac{\partial_z b_0}{b_0} + \mathrm{O}(|w-z|) = -\frac{1}{2b_0}\,\partial_\zeta\,\frac{\partial_z b_0}{b_0} + \mathrm{O}(|w-z|),$$

and so

$$\{\partial_\theta\partial_w\}A^0\big|_{w=z} = \{\partial_\theta\partial_w\}\frac{b_0}{\partial_\zeta\theta}\bigg|_{w=z} = -\frac{1}{2b_0}\partial_\zeta\frac{\partial_z b_0}{b_0}.$$

A comparison with (2.5), which defines b_1, now shows that the second part of (5.23) is also true.

5.12 Solution to the Negligible Amplitude Equation

We put

$$Z^0 = \frac{1-A^0}{z-w}, \quad z \neq w, \tag{5.26}$$

which extends holomorphically across $w = z$:

$$Z^0\big|_{w=z} = \{\partial_w\}A^0\big|_{w=z}.$$

In view of (5.22), the relation

$$A^1 + \{\partial_\theta\partial_w\}A^0 = (z-w)Z^1$$

defines a function Z^1 that is holomorphic when z and w are close to z_0, and ζ is close to $\bar{z}_0$. We finally put

$$X^0 = -Z^0, \quad X^1 = Z^1 + \{\partial_\theta\partial_w\}Z^0, \quad Y = -\{\partial_\theta\}Z^1 - \{\partial_\theta^2\partial_w\}Z^0.$$

Then X^0, X^1, and Y are all holomorphic when z and w are close to z_0, and ζ is close to $\bar{z}_0$. We rewrite these expressions in terms of A^0 and A^1:

$$X^0 = \frac{A^0-1}{z-w}, \quad X^1 = \frac{A^1}{z-w} - \frac{\{\partial_\theta\}A^0}{(z-w)^2}, \quad Y = -\frac{\{\partial_\theta\}A^1}{z-w} + \frac{\{\partial_\theta^2\}A^0}{(z-w)^2}.$$

It follows that

$$\nabla_m X^0 = \{\partial_\theta\}X^0 + m(z-w)X^0 = \frac{\{\partial_\theta\}A^0}{z-w} + m(A^0-1)$$

and

$$\begin{aligned}\nabla_m X^1 &= \{\partial_\theta\}X^1 + m(z-w)X^1\\ &= \frac{\{\partial_\theta\}A^1}{z-w} - \frac{\{\partial_\theta^2\}A^0}{(z-w)^2} + m\left(A^1 - \frac{\{\partial_\theta\}A^0}{z-w}\right)\end{aligned}$$

so that

$$\nabla_m X_m = \nabla_m(X^0 + m^{-1}X^1)$$
$$= m(A^0 - 1) + A^1 + m^{-1}\left(\frac{\{\partial_\theta\}A^1}{z-w} - \frac{\{\partial_\theta^2\}A^0}{(z-w)^2}\right).$$

By adding up the pieces, we find that

$$1 + m^{-1}\nabla_m X_m + m^{-2}Y = A^0 + m^{-1}A^1, \tag{5.27}$$

which constitutes the solution to the negligible amplitude equation (5.19) for triplets $(z, w, \zeta) \in \mathbb{D}(z_0, 3\varepsilon) \times \mathbb{D}(z_0, 3\varepsilon) \times \mathbb{D}(\bar{z}_0, 3\varepsilon)$.

PROOF OF PROPOSITION 5.1: If we put $\zeta = \bar{w}$ in (5.27) and recall the definition (5.20) of A^0 and A^1, the assertion is immediate. □

6 Proof of Theorem 2.3

6.1 Preliminaries

In this section, we prove Theorem 2.3 by following the approach outlined in [7]. The proof is obtained by combining Lemma 3.2, Theorem 5.2, and Corollary 4.4. To facilitate the presentation, we divide the proof into several steps.

6.2 Setting

We first note that adding a constant to Q means that $K_{m,n}$ is changed by a multiplicative constant only and hence we can (and will) assume that $Q \geq 1$ on $\mathbb{C}$.

We fix a point $z_0 \in \mathscr{K}$, and let $\varepsilon > 0$ be as needed for Theorem 5.2. The cutoff function χ is as in Section 5: it has $\chi = 1$ on $\mathbb{D}(z_0, \varepsilon)$, $\chi = 0$ on $\mathbb{C} \setminus \mathbb{D}(z_0, \frac{3}{2}\varepsilon)$, while $0 \leq \chi \leq 1$ everywhere else. By the definition of the approximate Bergman kernel $K_m^1(z, w)$ back in Section 2.3, we have

$$|K_m^1(z,w)|\mathrm{e}^{-\frac{1}{2}m(Q(z)+Q(w))} = |mb_0(z,\bar{w}) + b_1(z,\bar{w})|\mathrm{e}^{\frac{1}{2}m(2\operatorname{Re}\psi(z,\bar{w})-Q(z)-Q(w))},$$

and if $m \geq 1$, we have

$$|mb_0(z,\bar{w}) + b_1(z,\bar{w})|^2 \leq C_1 m, \quad z, w \in \mathbb{D}(z_0, 2\varepsilon),$$

where C_1 is a suitable constant (which depends only on Q, $\mathscr{K}$, and ε), and by (5.8), we have

$$\mathrm{e}^{\frac{1}{2}m(2\operatorname{Re}\psi(z,\bar{w})-Q(z)-Q(w))} \leq \mathrm{e}^{-\frac{1}{4}m|z-w|^2\Delta Q(z_0)}.$$

A combination of the above leads to

$$|K_m^1(z,w)|\mathrm{e}^{-\frac{1}{2}m(Q(z)+Q(w))} \leq C_1 m\mathrm{e}^{-\frac{1}{4}m|z-w|^2\Delta Q(z_0)}, \quad z, w \in \mathbb{D}(z_0, 2\varepsilon). \tag{6.1}$$

6.3 Integral Kernel Techniques

Let $\mathrm{P}_{m,n} : L^2_{mQ} \to H_{m,n}$ denote the orthogonal projection, that is,

$$\mathrm{P}_{m,n} u(z) = \int_{\mathbb{C}} u(w) K_{m,n}(z,w) \mathrm{e}^{-mQ(w)} \mathrm{d}A(w).$$

We introduce the functions

$$K_{m,n;z}(\zeta) = K_{m,n}(\zeta, z) \quad \text{and} \quad K^{1,\chi}_{m;w}(\zeta) = \chi(\zeta) K^1_m(\zeta, w), \tag{6.2}$$

with the tacit understanding that $K^{1,\chi}_w(\zeta) = 0$ whenever $\zeta \in \mathbb{C} \setminus \mathbb{D}(z_0, 2\varepsilon)$. We also need the difference

$$R^\chi_w(\zeta) = K^{1,\chi}_{m;w}(\zeta) - \mathrm{P}_{m,n} K^{1,\chi}_{m;w}(\zeta), \quad \zeta \in \mathbb{C}. \tag{6.3}$$

From the definition (5.1) of the local integral operator $\mathrm{I}^{1,\chi}_m$, we have

$$\begin{aligned}\mathrm{I}^{1,\chi}_m K_{m,n;z}(\zeta) &= \int_{\mathbb{D}(z_0,2\varepsilon)} K_{m,n;z}(\xi)\chi(\xi) K^1_m(\zeta,\xi) \mathrm{e}^{-mQ(\xi)} \mathrm{d}A(\xi) \\ &= \int_{\mathbb{D}(z_0,2\varepsilon)} K_{m,n}(\xi,z)\chi(\xi) K^1_m(\zeta,\xi) \mathrm{e}^{-mQ(\xi)} \mathrm{d}A(\xi),\end{aligned}$$

so that as we take complex conjugates, we find that

$$\begin{aligned}\overline{\mathrm{I}^{1,\chi}_m K_{m,n;z}(w)} &= \int_{\mathbb{C}} K^1_m(\xi, w)\chi(\xi) K_{m,n}(z,\xi) \mathrm{e}^{-mQ(\xi)} \mathrm{d}A(\xi) \\ &= \mathrm{P}_{m,n} K^{1,\chi}_{m;w}(z).\end{aligned} \tag{6.4}$$

Next, by Theorem 5.2, there is a constant C_2 (which depends only on Q, $\mathscr{K}$, and ε) such that

$$|\mathrm{I}^{1,\chi}_m K_{m,n;z}(w) - K_{m,n;z}(w)| \le C_2 m^{-3/2} \mathrm{e}^{\frac{1}{2}mQ(w)} \|K_{m,n;z} \mathrm{e}^{-\frac{1}{2}mQ}\|_{L^2(\mathbb{D}(z_0,2\varepsilon))},$$

and in view of (6.4) we may rewrite this as

$$|\mathrm{P}_{m,n} K^{1,\chi}_{m;w}(z) - K_{m,n}(z,w)| \le C_2 m^{-3/2} \mathrm{e}^{\frac{1}{2}mQ(w)} \|K_{m,n;z} \mathrm{e}^{-\frac{1}{2}mQ}\|_{L^2(\mathbb{D}(z_0,2\varepsilon))}. \tag{6.5}$$

We note that

$$\begin{aligned}&\|K_{m,n;z} \mathrm{e}^{-\frac{1}{2}mQ}\|_{L^2(\mathbb{D}(z_0,2\varepsilon))} \\ &\quad \le \|K_{m,n;z} \mathrm{e}^{-\frac{1}{2}mQ}\|_{L^2(\mathbb{C})} \\ &\quad = \|K_{m,n;z}\|_{mQ} = K_{m,n}(z,z)^{\frac{1}{2}} \le C_3 m^{\frac{1}{2}} \mathrm{e}^{\frac{1}{2}mQ(z)},\end{aligned}$$

with a number C_3 that depends only on Q and τ (see Proposition 3.6), so that by (6.5),

$$|\mathrm{P}_{m,n} K^{1,\chi}_{m;w}(z) - K_{m,n}(z,w)| \le C_4 m^{-1} \mathrm{e}^{\frac{1}{2}m(Q(z)+Q(w))}, \tag{6.6}$$

where $C_4 = C_2 C_3$.

6.4 Application of Weighted $\bar\partial$-Methods

The function R^χ_w given by (6.3) is the $L^2_{mQ,n}$-minimal norm solution to the $\bar\partial$-problem

$$\bar\partial_\zeta R^\chi_w(\zeta) = \bar\partial_\zeta K^{1,\chi}_{m,w}(\zeta) = K^1_{m;w}(\zeta)\bar\partial\chi(\zeta),$$

with the understanding that the rightmost expression vanishes on $\mathbb{C} \setminus \mathbb{D}(z_0, 2\varepsilon)$. Here we use the notation $K^1_{m;w}(\zeta) = K^1_m(\zeta, w)$. We now add the requirement on ε that

$$\mathbb{D}(z_0, 2\varepsilon) \subset \mathscr{S}_\tau,$$

so that $\bar\partial\chi$ vanishes off $\mathscr{S}_\tau$. By Corollary 4.4, then,

$$\|R^\chi_w\|_{mQ} \le \frac{\mathrm{e}^{\frac{1}{2}M_0 q_\tau + l_\tau}}{{M_1}^{1/2}} \left\| K^1_{m;w}\bar\partial\chi \right\|_{mQ},$$

provided that $m \ge M_0$ and $n \ge](m - M_0)\tau + M_1[$. We recall that the positive reals M_0 and M_1 are constrained by (4.14), and that q_τ and l_τ are given by (4.13).

Next, we recall that $\|\bar\partial\chi\|_{L^2} \le 3$, so that

$$\begin{aligned}\left\| K^1_{m;w}\bar\partial\chi \right\|_{mQ} &= \left\| K^1_{m;w}\bar\partial\chi \mathrm{e}^{-\frac{1}{2}mQ} \right\|_{L^2} \\ &\le \|\bar\partial\chi\|_{L^2} \left\| K^1_{m;w}\mathrm{e}^{-\frac{1}{2}mQ} \right\|_{L^\infty(\mathbb{D}(z_0,2\varepsilon)\setminus\mathbb{D}(z_0,\frac{3}{2}\varepsilon))} \\ &\le 3\left\| K^1_{m;w}\mathrm{e}^{-\frac{1}{2}mQ} \right\|_{L^\infty(\mathbb{D}(z_0,2\varepsilon)\setminus\mathbb{D}(z_0,\frac{3}{2}\varepsilon))}.\end{aligned}$$

A combination of the above gives

$$\|R^\chi_w\|_{mQ} \le \frac{3\mathrm{e}^{\frac{1}{2}M_0 q_\tau + l_\tau}}{{M_1}^{1/2}} \left\| K^1_{m;w}\mathrm{e}^{-\frac{1}{2}mQ} \right\|_{L^\infty(\mathbb{D}(z_0,2\varepsilon)\setminus\mathbb{D}(z_0,\frac{3}{2}\varepsilon))}. \tag{6.7}$$

In view of (6.1), we have

$$\left\| K^1_{m;w}\mathrm{e}^{-\frac{1}{2}mQ} \right\|_{L^\infty(\mathbb{D}(z_0,2\varepsilon)\setminus\mathbb{D}(z_0,\frac{3}{2}\varepsilon))} \le C_1 m \mathrm{e}^{-\frac{1}{16}m\varepsilon^2\Delta Q(z_0)} \mathrm{e}^{\frac{1}{2}mQ(w)}, \qquad w \in \mathbb{D}(z_0, \varepsilon),$$

so that (6.7) entails that

$$\|R^\chi_w\|_{mQ} \le \frac{3C_1\mathrm{e}^{\frac{1}{2}M_0 q_\tau + l_\tau}}{{M_1}^{1/2}} m \mathrm{e}^{-\frac{1}{16}m\varepsilon^2\Delta Q(z_0)} \mathrm{e}^{\frac{1}{2}mQ(w)}, \quad w \in \mathbb{D}(z_0, \varepsilon). \tag{6.8}$$

Since R^χ_w is holomorphic in the disk $\mathbb{D}(z_0, \frac{3}{2}\varepsilon)$, we may use Lemma 3.2 to get a pointwise estimate in place of an integral one. If we assume ε is so small that $\Delta Q \le 2\Delta Q(z_0)$ on $\mathbb{D}(z_0, \varepsilon)$, we find that

$$\begin{aligned}|R^\chi_w(z)|\mathrm{e}^{-\frac{1}{2}mQ(z)} &\le \frac{2\sqrt{m}\,\mathrm{e}^{\frac{1}{4}\varepsilon^2\Delta Q(z_0)}}{\varepsilon}\left\| R^\chi_w\,\mathrm{e}^{-\frac{1}{2}mQ}\right\|_{L^2(\mathbb{D}(z,\frac{1}{2}m^{-1/2}\varepsilon))} \\ &\le \frac{2\sqrt{m}\,\mathrm{e}^{\frac{1}{4}\varepsilon^2\Delta Q(z_0)}}{\varepsilon}\|R^\chi_w\|_{mQ}, \quad z, w \in \mathbb{D}(z_0, \varepsilon),\end{aligned} \tag{6.9}$$

where $m \ge 1$ is assumed. We get from a combination of (6.8) and (6.9) that

$$|R^\chi_w(z)| \le C_5\, m^{\frac{3}{2}}\mathrm{e}^{-\frac{1}{16}m\varepsilon^2\Delta Q(z_0)}\mathrm{e}^{\frac{1}{2}m(Q(z)+Q(w))}, \quad z, w \in \mathbb{D}(z_0, \varepsilon), \tag{6.10}$$

where C_5 is the constant

$$C_5 = \frac{6C_1\mathrm{e}^{\frac{1}{2}M_0 q_\tau + l_\tau + \frac{1}{4}\varepsilon^2\Delta Q(z_0)}}{{M_1}^{1/2}\varepsilon}.$$

CONCLUSION OF THE PROOF OF THEOREM 2.3: We have assumed that $m \ge m_0$ and $n \ge m\tau - M$ for some large positive real m_0. We need to find M_0 and M_1 such that (4.14) holds, while

$$n \ge m\tau - M \quad \Longrightarrow \quad n \ge \,](m - M_0)\tau + M_1[$$

for all $m \ge m_0$. We should also make sure $m_0 \ge \max\{1, M_0\}$. For the implication to hold, it is enough to require that

$$m\tau - M \ge (m - M_0)\tau + M_1,$$

which amounts to

$$M_0\tau - M_1 \ge M.$$

By choosing, e.g., $M_1 = 1$ and M_0 appropriately big, we can make sure the above inequality is fulfilled, while (4.14) holds. In conclusion, if we recall the definition (6.3) of R^χ_w, we get by combining (6.6) with (6.10) the estimate

$$\begin{aligned}&|K_{m,n}(z, w) - K^1_m(z, w)| \\ &\quad = |K_{m,n}(z, w) - \mathrm{P}_{m,n} K^{1,\chi}_{m;w}(z) - R^\chi_w(z)| \\ &\quad \le |K_{m,n}(z, w) - \mathrm{P}_{m,n} K^{1,\chi}_{m;w}(z)| + |R^\chi_w(z)| \\ &\quad \le \big(C_4 m^{-1} + C_5 m^{\frac{3}{2}}\mathrm{e}^{-\frac{1}{16}m\varepsilon^2\Delta Q(z_0)}\big)\mathrm{e}^{\frac{1}{2}m(Q(z)+Q(w))}, \quad z, w \in \mathbb{D}(z_0, \varepsilon).\end{aligned}$$

Finally, we note that

$$\inf_{z_0\in\mathscr{K}} \Delta Q(z_0) > 0,$$

and that ε is to be chosen independently of $z_0 \in \mathscr{K}$, so that the expression

$$m^{\frac{3}{2}}\mathrm{e}^{-\frac{1}{16}m\varepsilon^2\Delta Q(z_0)}$$

is dominated by m^{-1} for large m (we assume that $m \ge m_0$). The constants C_4 and C_5 may also be chosen independently of $z_0 \in \mathscr{K}$. This completes the proof. □

7 Berezin Quantization and Gaussian Convergence

7.1 Preliminaries

In this section we use the expansion formula for $K_{m,n}$ (Theorem 2.3) to prove Theorems 2.7 and 2.8. We fix a compact subset $\mathscr{K} \Subset \mathscr{S}_1^\circ \cap \mathscr{N}_+$, a positive real ε with the properties listed in Theorem 2.3, and write, for $m \geq 3$,

$$\delta_m = \varepsilon\, m^{-1/2} \log m.$$

7.2 Application of Expansion Formula for Polynomial Bergman Kernel

By Theorem 2.3, we have for each $z_0 \in \mathscr{K} \Subset \mathscr{S}_\tau^\circ \cap \mathscr{N}_+$,

$$\begin{aligned} &K_{m,n}(z, z_0)\mathrm{e}^{-\frac{1}{2}m(Q(z)+Q(z_0))} \\ &\quad = m(b_0(z, \bar{z}_0) + m^{-1} b_1(z, \bar{z}_0))\, \mathrm{e}^{\frac{1}{2}m(2\operatorname{Re}\psi(z,\bar{z}_0) - Q(z) - Q(z_0))} \\ &\quad\quad + \mathrm{O}(m^{-1}), \quad z \in \mathbb{D}(z_0, \varepsilon), \end{aligned} \tag{7.1}$$

when $m \to +\infty$ while $n \geq m\tau - M$ for some fixed positive real M. The O expression is uniform in $z_0 \in \mathscr{K}$. In particular, with $z = z_0$, we have

$$K_{m,n}(z_0, z_0)\mathrm{e}^{-mQ(z_0)} = m\Delta Q(z_0) + \frac{1}{2}\Delta \log \Delta Q(z_0) + \mathrm{O}(m^{-1}), \quad z \in \mathbb{D}(z_0, \varepsilon). \tag{7.2}$$

Moreover, by (5.7) with z in place of w, and z_0 in place of z, we get

$$2\operatorname{Re}\psi(z, \bar{z}_0) - Q(z) - Q(z_0) = -|z - z_0|^2 \Delta Q(z_0) + R_{z_0}(z),$$

where R_{z_0} is a function with

$$|R_{z_0}(z)| \leq C_1 |z - z_0|^3, \quad z \in \mathbb{D}(z_0, 2\varepsilon),$$

for some constant C_1 independent of $z_0 \in \mathscr{K}$. By Taylor's formula,

$$\begin{aligned} b_0(z, \bar{z}_0) &= b_0(z_0, \bar{z}_0) + \mathrm{O}(|z - z_0|) \\ &= \Delta Q(z_0) + \mathrm{O}(|z - z_0|), \quad z \in \mathbb{D}(z_0, \varepsilon), \end{aligned}$$

so that

$$b_0(z, \bar{z}_0) = \Delta Q(z_0) + \mathrm{O}(m^{-1/2} \log m), \quad z \in \mathbb{D}(z_0, \delta_m), \tag{7.3}$$

since—by the way we defined δ_m—we have $\delta_m \leq \varepsilon$ for $m \geq 3$. From the bound on R_{z_0} we get that

$$m|R_{z_0}(z)| \leq C_1 m \delta_m^3 \leq C_1 \varepsilon^3 m^{-1/2} \log^3 m, \quad z \in \mathbb{D}(z_0, \delta_n),\ z_0 \in \mathscr{K},$$

which gives

$$\mathrm{e}^{mR_{z_0}(z)} = 1 + \mathrm{O}(m^{-1/2} \log^3 m), \quad z \in \mathbb{D}(z_0, \delta_n),\ z_0 \in \mathscr{K}. \tag{7.4}$$

As we implement this into (7.1), while using (7.3) and that $b_1(z,\bar z_0)$ is bounded for $z \in \mathbb{D}(z_0,\varepsilon)$, we find that

$$K_{m,n}(z,z_0)\mathrm{e}^{-\frac{1}{2}m(Q(z)+Q(z_0))} = m(\Delta Q(z_0)+\mathrm{O}(m^{-\frac{1}{2}}\log^3 m))\mathrm{e}^{-\frac{1}{2}m|z-z_0|^2\Delta Q(z_0)} + \mathrm{O}(m^{-1}) \tag{7.5}$$

for $z \in \mathbb{D}(z_0,\delta_n)$ as $m \to +\infty$ while $n \geq m\tau - M$. Forming the square of the modulus of each side gives us

$$\begin{aligned} &|K_{m,n}(z,z_0)|^2\mathrm{e}^{-m(Q(z)+Q(z_0))} \\ &\quad= m^2((\Delta Q(z_0))^2+\mathrm{O}(m^{-\frac{1}{2}}\log^3 m))\mathrm{e}^{-m\Delta Q(z_0)|z-z_0|^2} \\ &\qquad+\mathrm{O}(\mathrm{e}^{-\frac{1}{2}m\Delta Q(z_0)|z-z_0|^2})+\mathrm{O}(m^{-2}), \end{aligned} \tag{7.6}$$

so that in view of (7.2), we obtain

$$\begin{aligned} &Б^{\langle z_0\rangle}_{m,n}(z) \\ &\quad= \frac{|K_{m,n}(z,z_0)|^2}{K_{m,n}(z_0,z_0)}\mathrm{e}^{-mQ(z)} \\ &\quad= m(\Delta Q(z_0)+\mathrm{O}(m^{-\frac{1}{2}}\log^3 m))\mathrm{e}^{-m\Delta Q(z_0)|z-z_0|^2}+\mathrm{O}(m^{-1}) \end{aligned} \tag{7.7}$$

for $z \in \mathbb{D}(z_0,\delta_n)$ as $m \to +\infty$ while $n \geq m\tau - M$.

PROOF OF THEOREM 2.7: Our next step is to calculate the integral

$$\begin{aligned} \int_{\mathbb{D}(z_0,\delta_m)} \mathrm{e}^{-m\Delta Q(z_0)|z-z_0|^2}\,\mathrm{d}A(z) &= \frac{1}{m\Delta Q(z_0)}(1-\mathrm{e}^{-\varepsilon^2\Delta Q(z_0)\log^2 m}) \\ &= \frac{1}{m\Delta Q(z_0)}(1+\mathrm{O}(m^{-1})), \end{aligned} \tag{7.8}$$

which allows us to conclude from (7.7) that

$$B^{\langle z_0\rangle}(\mathbb{D}(z_0,\delta_m)) = \int_{\mathbb{D}(z_0,\delta_m)} Б^{\langle z_0\rangle}_{m,n}(z)\mathrm{d}A(z) = 1+\mathrm{O}(m^{-1/2}\log^3 m).$$

Since the Berezin measure $B^{\langle z_0\rangle}$ is a p.m., we get

$$B^{\langle z_0\rangle}(\mathbb{C}\setminus\mathbb{D}(z_0,\delta_m)) = \int_{\mathbb{C}\setminus\mathbb{D}(z_0,\delta_m)} Б^{\langle z_0\rangle}_{m,n}(z)\mathrm{d}A(z) = \mathrm{O}(m^{-1/2}\log^3 m), \tag{7.9}$$

which tends to 0 as $m \to +\infty$. In particular, since $\delta_m \to 0$ as $m \to +\infty$, the mass of $B^{\langle z_0\rangle}$ concentrates to the point z_0, which concludes the proof. □

PROOF OF THEOREM 2.8: By the linear change of variables

$$z = z_0 + \frac{\zeta}{\sqrt{m\Delta Q(z_0)}},$$

we have

$$\int_{\mathbb{C}} \left|\widehat{Б}^{\langle z_0\rangle}_{m,n}(\zeta) - \mathrm{e}^{-|\zeta|^2}\right| \mathrm{d}A(\zeta) = \int_{\mathbb{C}} \left|Б^{\langle z_0\rangle}_{m,n}(z) - m\Delta Q(z_0)\mathrm{e}^{-m\Delta Q(z_0)|z-z_0|^2}\right| \mathrm{d}A(z). \tag{7.10}$$

If we use (7.8), we get from (7.7) that

$$\int_{\mathbb{D}(z_0,\delta_m)} \left|Б^{\langle z_0\rangle}_{m,n}(z) - m\Delta Q(z_0)\mathrm{e}^{-m\Delta Q(z_0)|z-z_0|^2}\right| \mathrm{d}A(z) = \mathrm{O}(m^{-1/2}\log^3 m), \tag{7.11}$$

while (7.9) and

$$m\Delta Q(z_0) \int_{\mathbb{C}\setminus\mathbb{D}(z_0,\delta_m)} \mathrm{e}^{-m\Delta Q(z_0)|z-z_0|^2}\,\mathrm{d}A(z) = \mathrm{e}^{-\varepsilon^2\Delta Q(z_0)\log^2 m} = \mathrm{O}(m^{-1})$$

show that

$$\int_{\mathbb{C}\setminus\mathbb{D}(z_0,\delta_m)} \left|Б^{\langle z_0\rangle}_{m,n}(z) - m\Delta Q(z_0)\mathrm{e}^{-m\Delta Q(z_0)|z-z_0|^2}\right| \mathrm{d}A(z) = \mathrm{O}(m^{-1/2}\log^3 m). \tag{7.12}$$

We add up (7.11) and (7.12):

$$\int_{\mathbb{C}} \left|Б^{\langle z_0\rangle}_{m,n}(z) - m\Delta Q(z_0)\mathrm{e}^{-m\Delta Q(z_0)|z-z_0|^2}\right| \mathrm{d}A(z) = \mathrm{O}(m^{-1/2}\log^3 m).$$

The assertion is immediate, since the right-hand side approaches 0 uniformly in $z_0 \in \mathscr{K}$ when $m \to +\infty$. □

8 Off-Diagonal Damping

8.1 Background

Theorem 8.1 is the main result of this section. It supplies an off-diagonal damping bound of the polynomial Bergman kernel $K_{m,n}$. Off-diagonal damping techniques are well-known for the Bergman kernel, under the assumption that the weight Q is globally strictly subharmonic. See, e.g., Lindholm's paper [20, prop. 9, pp. 404–407], for background on the damping technique. In our situation, two non-standard elements arise: (1) we consider polynomial Bergman kernels, and (2) we do not assume the weight Q to be globally strictly subharmonic. Our proof is based on the weighted L^2-estimate provided by Corollary 4.5.

8.2 Some Notation

In this section we prove Theorem 2.12. We shall obtain that theorem from Corollary 8.2 below, which is derived from Theorem 8.1. The estimate of Theorem 8.1 and Corollary 8.2 is of independent interest and has applications to random matrix theory; see [1]. It will be convenient to define the set

$$\mathscr{S}_{\tau,1} = \{\zeta \in \mathbb{C} : \operatorname{dist}_{\mathbb{C}}(\zeta, \mathscr{S}_\tau) \le 1\}.$$

We begin by picking a point $z_0 \in \mathscr{S}_\tau^\circ \cap \mathscr{N}_+$ and a radius $r_0 > 0$ such that

$$\overline{\mathbb{D}}(z_0, r_0) \subset \mathscr{S}_\tau \cap \mathscr{N}_+. \tag{8.1}$$

Let α and A be two positive constants such that

$$\Delta Q \begin{cases} \ge \alpha & \text{on } \overline{\mathbb{D}}(z_0, r_0), \\ \le A & \text{on } \mathscr{S}_{\tau,1}. \end{cases} \tag{8.2}$$

8.3 Application of Weighted Estimates for the $\bar\partial$-Equation with Growth Constraint

Suppose we have found a real-valued function ρ_m of class $\mathscr{C}^{1,1}(\mathbb{C})$ with these conditions:

(1) ρ_m is constant on $\mathbb{C} \setminus \overline{\mathbb{D}}(z_0, r_0)$,
(2) $\Delta \rho_m \ge -\frac{1}{2} m\alpha + M_0 \alpha$ a.e. on $\overline{\mathbb{D}}(z_0, r_0)$,
(3) $|\bar\partial \rho_m|^2 \le \frac{1}{8} m\alpha \mathrm{e}^{-M_0 q_\tau}$ a.e. on $\overline{\mathbb{D}}(z_0, r_0)$,
where q_τ and l_τ are given by (4.13), and the general setting with M_0 and M_1 is as in the implementation scheme of Section 4 (so we of course assume $Q \ge 1$). This means that the conditions on ρ_m of Corollary 4.5 are fulfilled with $\kappa = \frac{1}{2}$ and $\mathscr{T} = \overline{\mathbb{D}}(z_0, r_0)$. Corollary 4.5 then asserts that if $f \in L^\infty(\mathbb{D}(z_0, r_0))$, and if $m \ge M_0$ and $n \ge](m - M_0)\tau + M_1[$, then $u_{0,n}$, the $L^2_{mQ,n}$-minimal solution to $\bar\partial u = f$, satisfies

$$\int_{\mathbb{C}} |u_{0,n}|^2 \mathrm{e}^{\rho_m - mQ} \, \mathrm{d}A \le \frac{8\mathrm{e}^{M_0 q_\tau}}{m\alpha + 2M_1 \mathrm{e}^{-2l_\tau}} \int_{\mathbb{D}(z_0,r_0)} |f|^2 \mathrm{e}^{\rho_m - mQ} \, \mathrm{d}A. \tag{8.3}$$

We make one more assumption on ρ_m, namely,

(4) $\rho_m = 0$ on $\mathbb{D}(z_0, m^{-1/2})$.

Then we get immediately from (8.3) that

$$\int_{\mathbb{D}(z_0,m^{-1/2})} |u_{0,n}|^2 \mathrm{e}^{-mQ} \, \mathrm{d}A \le \frac{8\mathrm{e}^{M_0 q_\tau}}{m\alpha + 2M_1 \mathrm{e}^{-2l_\tau}} \int_{\mathbb{D}(z_0,r_0)} |f|^2 \mathrm{e}^{\rho_m - mQ} \, \mathrm{d}A. \tag{8.4}$$

An application of Lemma 3.2 with $\delta = 1$ shows that for $m \geq 1$,

$$|u_{0,n}(z_0)|^2 e^{-mQ(z_0)} \leq m e^A \int_{\mathbb{D}(z_0, m^{-1/2})} |u_{0,n}|^2 e^{-mQ} \, dA,$$

and if we combine this with (8.4), the result is

$$|u_{0,n}(z_0)|^2 e^{-mQ(z_0)} \leq \frac{8m e^{A+M_0 q_\tau}}{m\alpha + 2M_1 e^{-2l_\tau}} \int_{\mathbb{D}(z_0,r_0)} |f|^2 e^{\rho_m - mQ} \, dA. \tag{8.5}$$

8.4 Juggling with Test Functions

We now require that f be of the form $f = g\bar{\partial}\chi_0$, where g is a polynomial in $H_{m,n}$, and χ_0 is a smooth real-valued function with $0 \leq \chi_0 \leq 1$ everywhere, and, more specifically,

$$\chi_0 = \begin{cases} 0 & \text{on } \mathbb{D}(z_0, r_1), \\ 1 & \text{on } \mathbb{C} \setminus \mathbb{D}(z_0, 2r_1), \end{cases} \tag{8.6}$$

where $0 < r_1 \leq \frac{1}{2} r_0$. It is possible to require in addition that

$$|\bar{\partial}\chi_0|^2 \leq 2r_1^{-2}\chi_0; \tag{8.7}$$

cf. the subsection on cutoff functions in Section 3. One solution of the equation $\bar{\partial}u = g\bar{\partial}\chi_0$ is given by $u = g\chi_0$, so the difference $g\chi_0 - u_{0,n}$ is entire. In fact, from Section 4.1, we have that

$$g\chi_0 - u_{0,n} = \mathrm{P}_{m,n}[g\chi_0],$$

where $\mathrm{P}_{m,n} : L^2_{mQ} \to H_{m,n}$ is the orthogonal projection. As χ_0 vanishes at z_0, we see that

$$u_{0,n}(z_0) = -\mathrm{P}_{m,n}[g\chi_0](z_0),$$

so that (8.5) with $f = g\chi_0$ expresses that

$$|\mathrm{P}_{m,n}[g\chi_0](z_0)|^2 e^{-mQ(z_0)} \leq \frac{8m e^{A+M_0 q_\tau}}{m\alpha + 2M_1 e^{-2l_\tau}} \int_{\mathbb{D}(z_0,r_0)} |g|^2 |\bar{\partial}\chi_0|^2 e^{\rho_m - mQ} \, dA. \tag{8.8}$$

If we also use estimate (8.7), we get in place of (8.8) the estimate

$$\begin{aligned} |\mathrm{P}_{m,n}[g\chi_0](z_0)|^2 e^{-mQ(z_0)} &\leq \frac{16m r_1^{-2} e^{A+M_0 q_\tau}}{m\alpha + 2M_1 e^{-2l_\tau}} \int_{\mathbb{D}(z_0,r_0)} |g|^2 \chi_0 e^{\rho_m - mQ} \, dA \\ &\leq \frac{16m r_1^{-2} e^{A+M_0 q_\tau}}{m\alpha + 2M_1 e^{-2l_\tau}} \int_{\mathbb{C}} |g|^2 \chi_0 e^{\rho_m - mQ} \, dA. \end{aligned} \tag{8.9}$$

This suggests that we should introduce the norm

$$\|g\|^2_{\chi_0,mQ} = \int_{\mathbb{C}} |g|^2 \chi_0 \mathrm{e}^{-mQ}\, \mathrm{d}A,$$

with the associated inner product. The Hilbert space of polynomials of degree $\le n-1$ supplied with this new norm will be denoted by $H_{\chi_0,m,n}$. We write $\mathrm{ball}(H_{\chi_0,m,n})$ for the closed unit ball of $H_{\chi_0,m,n}$. Next, we recall that $\mathrm{P}_{m,n}$ is expressed in terms of the reproducing kernel $K_{m,n}$,

$$\mathrm{P}_{m,n}[F](\zeta) = \int_{\mathbb{C}} K_{m,n}(\zeta,\xi)\, F(\xi)\mathrm{e}^{-mQ(\xi)}\, \mathrm{d}A(\xi) = \langle F, K_{m,n}(\,\cdot\,,\zeta)\rangle_{mQ}.$$

This allows us to make the following calculation:

$$\begin{aligned}
&\sup\{|\mathrm{P}_{m,n}[g\chi_0](z_0)|^2 : g\in \mathrm{ball}(H_{\chi_0,m,n})\}\\
&\quad= \sup\{|\langle g\chi_0, K_{m,n}(\,\cdot\,,z_0)\rangle_{mQ}|^2 : g\in \mathrm{ball}(H_{\chi_0,m,n})\}\\
&\quad= \sup\{|\langle g, K_{m,n}(\,\cdot\,,z_0)\rangle_{\chi_0,mQ}|^2 : g\in \mathrm{ball}(H_{\chi_0,m,n})\}\\
&\quad= \|K_{m,n}(\,\cdot\,,z_0)\|^2_{\chi_0,m,n} = \int_{\mathbb{C}} |K_{m,n}(\zeta,z_0)|^2\chi_0(\zeta)\mathrm{e}^{-mQ(\zeta)}\mathrm{d}A(\zeta).
\end{aligned} \tag{8.10}$$

We realize from (8.10) that our estimate (8.9) entails that

$$\begin{aligned}
&\mathrm{e}^{-mQ(z_0)}\int_{\mathbb{C}} |K_{m,n}(\zeta,z_0)|^2\chi_0(\zeta)\mathrm{e}^{-mQ(\zeta)}\mathrm{d}A(\zeta) \le\\
&\qquad\qquad \frac{16 m r_1^{-2}\mathrm{e}^{A+M_0 q_\tau}}{m\alpha + 2M_1\mathrm{e}^{-2l_\tau}}\sup\{\mathrm{e}^{\rho_m(\zeta)} : \zeta\in\mathbb{C}\setminus\mathbb{D}(z_0,r_1)\}
\end{aligned} \tag{8.11}$$

as χ_0 vanishes on $\mathbb{D}(z_0,r_1)$.

We need to turn this integral estimate into a pointwise one. We pick a point $z_1 \in \mathscr{S}_\tau \setminus \mathbb{D}(z_0, 2r_1 + m^{-1/2})$ and note that $\mathbb{D}(z_1, m^{-1/2}) \subset \mathscr{S}_{\tau,1}$ for $m \ge 1$. In addition, $\chi_0 = 1$ on $\mathbb{D}(z_1, m^{-1/2})$, and so, by Lemma 3.2, with $\delta = 1$,

$$\begin{aligned}
&|K_{m,n}(z_1,z_0)|^2\mathrm{e}^{-mQ(z_1)}\\
&\quad\le m\mathrm{e}^{A}\int_{\mathbb{D}(z_1,m^{-1/2})} |K_{m,n}(\zeta,z_0)|^2\mathrm{e}^{-mQ(\zeta)}\mathrm{d}A(\zeta)\\
&\quad\le m\mathrm{e}^{A}\int_{\mathbb{C}} |K_{m,n}(\zeta,z_0)|^2\chi_0(\zeta)\mathrm{e}^{-mQ(\zeta)}\mathrm{d}A(\zeta),
\end{aligned} \tag{8.12}$$

where it is tacitly assumed that $m \geq 1$. A combination of (8.11) and (8.12) leads to

(8.13) $$|K_{m,n}(z_1, z_0)|^2 \mathrm{e}^{-m(Q(z_0)+Q(z_1))} \leq \frac{16m^2 r_1^{-2} \mathrm{e}^{2A+M_0 q_\tau}}{m\alpha + 2M_1 \mathrm{e}^{-2l_\tau}} \sup\{\mathrm{e}^{\rho_m(\zeta)} : \zeta \in \mathbb{C} \setminus \mathbb{D}(z_0, r_1)\},$$

If we assume $r_1 \geq \frac{1}{2} m^{-1/2}$ and recall that $M_1 > 0$, we may relax (8.13) to

(8.14) $$|K_{m,n}(z_1, z_0)|^2 \mathrm{e}^{-m(Q(z_0)+Q(z_1))} \leq 64m^2 \alpha^{-1} \mathrm{e}^{2A+M_0 q_\tau} \sup\{\mathrm{e}^{\rho_m(\zeta)} : \zeta \in \mathbb{C} \setminus \mathbb{D}(z_0, r_1)\}.$$

Our hope is that ρ_m can be made rather big negative on $\mathbb{C} \setminus \mathbb{D}(z_0, r_1)$, for then estimate (8.14) supplies strong decay off the diagonal.

8.5 Construction of ρ_m

We now construct a function ρ_m that fulfills conditions (1)–(4) above, under the assumption

(8.15) $$m \geq \max\left\{4M_0, \frac{(\sqrt{2} + \sqrt{\alpha})^2}{\alpha r_0^2}\right\}.$$

We look for a radial function of the form

$$\rho_m(\zeta) = -\sqrt{m}\sigma_m(|\zeta - z_0|),$$

where the function $\sigma_m : [0, +\infty) \to [0, +\infty)$ is of class $\mathscr{C}^{1,1}$. By condition (1), we should have

(8.16) $$\sigma_m'(t) = 0, \quad r_0 < t < +\infty.$$

Also, condition (4) requires that

(8.17) $$\sigma_m(t) = 0, \quad 0 \leq t \leq m^{-1/2}.$$

Condition (3) is fulfilled if

(8.18) $$0 \leq \sigma_m'(t) \leq 2^{-1/2} \sqrt{\alpha}\, \mathrm{e}^{-\frac{1}{2} M_0 q_\tau}, \quad 0 < t < r_0.$$

As for condition (2), we first check that

$$\Delta\rho_m(\zeta) = -\frac{\sqrt{m}}{4}\left\{\sigma_m''(|\zeta - z_0|) + \frac{\sigma_m'(|\zeta - z_0|)}{|\zeta - z_0|}\right\},$$

so that condition (2) is equivalent to having

(8.19) $$\sigma_m''(t) + \frac{\sigma_m'(t)}{t} \leq 2\alpha m^{1/2} - 4\alpha M_0 m^{-1/2}, \quad 0 < t < r_0.$$

By (8.15), $m \geq 4M_0$, and we see that (8.19) is implied by the slightly simpler-looking condition

(8.20) $$\sigma_m''(t) + \frac{\sigma_m'(t)}{t} \leq \alpha m^{1/2}, \quad 0 < t < r_0.$$

We define $\sigma_m(t)$ in the following manner: We introduce points t_1, t_2, t_3, and t_4 with

$$m^{-1/2} = t_1 < t_2 < t_3 < t_4 = r_0,$$

and we let

$$\sigma_m''(t) = a_m \mathbb{1}_{[t_1,t_2]}(t) - b_m \mathbb{1}_{[t_3,t_4]}(t),$$

where the positive real constants a_m and b_m remain to be determined. Here $\mathbb{1}_X$ denotes the characteristic function of the set X. Condition (8.16) then requires that

$$a_m(t_2 - t_1) - b_m(t_4 - t_3) = 0.$$

In this problem the interval $[t_3, t_4]$ should be really short, and the number b_m correspondingly large. As for a_m, we pick

$$a_m = \frac{1}{2}\alpha m^{1/2},$$

because then

$$\sigma_m''(t) + \frac{\sigma_m'(t)}{t} = a_m + a_m \frac{t - t_1}{t} \le 2a_m = \alpha m^{1/2}, \quad t_1 < t < t_2,$$

and (8.20) holds at least on the interval $[t_1, t_2]$. In the remaining interval $[t_2, t_4]$, we have

$$\sigma_m''(t) + \frac{\sigma_m'(t)}{t} \le \frac{\sigma_m'(t)}{t} \le a_m \frac{t_2 - t_1}{t} \le a_m = \frac{1}{2}\alpha m^{1/2}, \quad t_2 < t < t_4,$$

and so (8.20) holds throughout.

It remains to arrange that (8.18) holds as well. The maximum value of $\sigma_m'(t)$ is attained at $t = t_2$, so that (8.18) holds once

$$\sigma_m'(t_2) = a_m(t_2 - t_1) = \frac{1}{2}\alpha m^{\frac{1}{2}}(t_2 - t_1) \le 2^{-\frac{1}{2}}\sqrt{\alpha}\, \mathrm{e}^{-\frac{1}{2}M_0 q_\tau}. \tag{8.21}$$

If we put

$$t_2 = (1 + 2^{\frac{1}{2}}\alpha^{-\frac{1}{2}}\mathrm{e}^{-\frac{1}{2}M_0 q_\tau})m^{-\frac{1}{2}},$$

then (8.21) is valid with equality. Condition (8.15) guarantees that $t_2 < t_4 = r_0$. The choice of t_3 is essentially trivial; we should just pick it very close to $t_4 = r_0$. We consider the function

$$\begin{aligned}
\tilde{\sigma}_m(t) &= 2^{-\frac{1}{2}}\sqrt{\alpha}\,\mathrm{e}^{-\frac{1}{2}M_0 q_\tau} \max\{0, \min\{t - t_2, t_4 - t_2\}\} \\
&= \max\bigl\{0, 2^{-\frac{1}{2}}\sqrt{\alpha}\,\mathrm{e}^{-\frac{1}{2}M_0 q_\tau} \min\{t, r_0\} \\
&\qquad - m^{-\frac{1}{2}}(2^{-\frac{1}{2}}\sqrt{\alpha}\,\mathrm{e}^{-\frac{1}{2}M_0 q_\tau} + \mathrm{e}^{-M_0 q_\tau})\bigr\} \\
&= \max\bigl\{0, 2\lambda_0 \min\{t, r_0\} - m^{-\frac{1}{2}}(2\lambda_0 + \lambda_1)\bigr\},
\end{aligned}$$

where

$$\lambda_0 = 2^{-\frac{3}{2}}\sqrt{\alpha}\,\mathrm{e}^{-\frac{1}{2}M_0 q_\tau}, \quad \lambda_1 = \mathrm{e}^{-M_0 q_\tau},$$

and note that the above construction produces a function $\sigma_m(t)$ with

$$\tilde{\sigma}_m(t) \le \sigma_m(t), \quad 0 \le t < +\infty. \tag{8.22}$$

8.6 Implementation of Estimates

We return to (8.14), with the given choice of ρ_m:

$$\begin{aligned}|K_{m,n}(z_1,z_0)|^2 e^{-m(Q(z_0)+Q(z_1))} &\le 64m^2\alpha^{-1}e^{2A+M_0q_\tau}e^{-\sqrt{m}\sigma_m(r_1)} \\ &\le 64m^2\alpha^{-1}e^{2A+M_0q_\tau}e^{-\sqrt{m}\,\tilde{\sigma}_m(r_1)},\end{aligned} \tag{8.23}$$

where we use (8.22) and recall that it is assumed that $r_1 \ge \frac{1}{2}m^{-1/2}$. We now turn things around, and try to pick first z_1 and only later r_1. If we begin with

$$|z_1 - z_0| \ge 2m^{-\frac{1}{2}},$$

we may put

$$r_1 = \frac{1}{2}\min\{r_0, |z_1 - z_0| - m^{-\frac{1}{2}}\},$$

and it is an easy matter to obtain the estimate

$$e^{-\sqrt{m}\,\tilde{\sigma}_m(r_1)} \le e^{3\lambda_0+\lambda_1}e^{-\lambda_0\sqrt{m}\min\{r_0,|z_0-z_1|\}}.$$

If we suppose that

$$r_0 \ge m^{-\frac{1}{2}},$$

which is a consequence of (8.15), then

$$r_1 \ge \frac{1}{2}m^{-\frac{1}{2}}$$

as needed, and we may derive from (8.23) that

$$|K_{m,n}(z_1,z_0)|^2 e^{-m(Q(z_0)+Q(z_1))} \le 64m^2\alpha^{-1}e^{2A+M_0q_\tau+3\lambda_0+\lambda_1}e^{-\lambda_0\sqrt{m}\min\{r_0,|z_1-z_0|\}}. \tag{8.24}$$

It remains to pick r_0 and α so that both (8.1) and (8.2) are fulfilled. One way—which we shall follow—is to assume Q is $\mathscr{C}^2$-smooth and choose

$$r_0 = \frac{1}{2}\operatorname{dist}_{\mathbb{C}}(z_0, \mathbb{C}\setminus(\mathscr{S}_\tau^\circ \cap \mathscr{N}_+)),$$

and let α be the biggest possible value such that (8.2) is met. We recall that the assumptions leading up to (8.24) are as follows: the parameters M_0 and M_1 are as in Section 4.4, the parameter m is assumed so big that (8.15) holds, $n \ge \,](m - M_0)\tau + M_1[$, and $|z_1 - z_0| \ge 2m^{-1/2}$.

THEOREM 8.1 *Assume that* $Q \in \mathscr{C}^2(\mathbb{C})$ *and that* $Q \ge 1$. *Fix a point* $z_0 \in \mathscr{S}_\tau^\circ \cap \mathscr{N}_+$ *and a real parameter* $M > 0$. *Finally, put*

$$r_0 = \frac{1}{2}\operatorname{dist}_{\mathbb{C}}(z_0, \mathbb{C}\setminus(\mathscr{S}_\tau^\circ \cap \mathscr{N}_+)), \quad \alpha = \inf_{\mathbb{D}(z_0,r_0)} \Delta Q, \quad A = \sup_{\mathscr{S}_{\tau,1}} \Delta Q.$$

Then there exists a positive constant C such that for all $m \geq 1$ and all positive integers n with $n \geq m\tau - M$,

$$|K_{m,n}(z_1, z_0)|^2 e^{-m(Q(z_0)+Q(z_1))} \leq C m^2 e^{-\lambda_0 \sqrt{m} \min\{r_0, |z_1 - z_0|\}}, \quad z_1 \in \mathscr{S}_\tau, \tag{8.25}$$

where $\lambda_0 = 2^{-3/2}\sqrt{\alpha}\, e^{-\frac{1}{2}M\tau^{-1}q_\tau}$ and q_τ is given by (4.13). *The constant C only depends on the parameters α, A, q_τ, and $M\tau^{-1}$.*

PROOF: We may consider in the context of Section 4.4 that $M_1 \to 0$ while $M_0 \to M\tau^{-1}$. In other words, whenever M_1 appears, we replace it with 0, and when M_0 appears, we replace it by $M\tau^{-1}$. It is important that (4.14) no longer restrains M_0. By (8.24), the asserted estimate (8.25) holds for any C with

$$C \geq 64\alpha^{-1} e^{2A + M\tau^{-1}q_\tau + 3\lambda_0 + \lambda_1}$$

provided that (8.15) holds and $|z_1 - z_0| \geq 2m^{-1/2}$. We note that by Lemma 3.2, one can show that (cf. Proposition 3.6)

$$|K_{m,n}(z_1, z_0)|^2 e^{-m(Q(z_0)+Q(z_1))} \leq e^{2A} m^2, \quad z_1 \in \mathscr{S}_\tau.$$

In case $|z_1 - z_0| < 2m^{-1/2}$, we have

$$e^{-\lambda_0 \sqrt{m} \min\{r_0, |z_1 - z_0|\}} \geq e^{-2\lambda_0},$$

so that (8.25) holds whenever

$$C \geq e^{2A + 2\lambda_0}.$$

Next, we observe that the opposite to (8.15) splits into two possibilities,

$$m < 4M\tau^{-1} \quad \text{and} \quad m^{\frac{1}{2}} < \frac{\sqrt{2} + \sqrt{\alpha}}{\sqrt{\alpha}\, r_0}.$$

If $m < 4M\tau^{-1}$, then

$$e^{-\lambda_0 \sqrt{m} \min\{r_0, |z_1 - z_0|\}} \geq e^{-2\lambda_0 r_0 \sqrt{M\tau^{-1}}},$$

so that (8.25) holds whenever

$$C \geq e^{2A + 2\lambda_0 r_0 \sqrt{M\tau^{-1}}}.$$

If

$$m^{\frac{1}{2}} < \frac{\sqrt{2} + \sqrt{\alpha}}{\sqrt{\alpha}\, r_0},$$

then

$$e^{-\lambda_0 \sqrt{m} \min\{r_0, |z_1 - z_0|\}} \geq e^{-\lambda_0 (1 + 2^{\frac{1}{2}} \alpha^{-\frac{1}{2}})},$$

so that (8.25) holds whenever

$$C \geq e^{2A + \lambda_0 (1 + 2^{\frac{1}{2}} \alpha^{-\frac{1}{2}})}.$$

In conclusion, it follows that (8.25) holds generally, with

$$C = \mathrm{e}^{2A} \max\left\{64\alpha^{-1}\mathrm{e}^{M\tau^{-1}q_\tau+3\lambda_0+\lambda_1}, \mathrm{e}^{2\lambda_0}, \mathrm{e}^{\lambda_0 r_0\sqrt{M\tau^{-1}}}, \mathrm{e}^{\lambda_0(1+2^{\frac{1}{2}}\alpha^{-\frac{1}{2}})}\right\}.$$

The proof is complete. □

8.7 Application of the Maximum Principle

The maximum principle allows us to extend the estimate of Theorem 8.1 to general $z_1 \in \mathbb{C}$.

COROLLARY 8.2 *Assume $Q \in \mathscr{C}^2(\mathbb{C})$ and that $Q \geq 1$. Fix a point $z_0 \in \mathscr{S}_\tau^\circ \cap \mathscr{N}_+$ and a real parameter $M > 0$. Let r_0, α, and A be as in Theorem* 8.1. *Then, if the positive integer n is confined to $m\tau - M \leq n \leq m\tau + 1$, we have, for all $z_1 \in \mathbb{C}$,*

$$|K_{m,n}(z_0, z_1)|^2 \mathrm{e}^{-m(Q(z_0)+Q(z_1))} \leq \\ C m^2 \mathrm{e}^{-\lambda_0\sqrt{m}\min\{r_0,|z_0-z_1|\}} \mathrm{e}^{-m(Q(z_1)-\widehat{Q}_\tau(z_1))},$$

where λ_0 and the constant C are as in Theorem 8.1.

PROOF: By Theorem 8.1, it is enough to obtain the claimed estimate when $z_1 \in \mathbb{C} \setminus \mathscr{S}_\tau$, since $\widehat{Q}_\tau = Q$ holds on $\mathscr{S}_\tau$. To this end, we consider the function

$$f(\zeta) = \log|K_{m,n}(\zeta, z_0)|^2 - m\widehat{Q}_\tau(\zeta).$$

Since $\widehat{Q}_\tau$ is harmonic on $\mathbb{C} \setminus \mathscr{S}_\tau$, f is subharmonic there. Moreover, since $n - 1 \leq m\tau$ and since the polynomial $K_{m,n}(\,\cdot\,, z_0)$ is of degree $\leq n-1$, we have that

$$\begin{aligned}\log|K_{m,n}(\zeta, z_0)|^2 &\leq (n-1)\log|\zeta|^2 + \mathrm{O}(1) \\ &\leq m\tau\log|\zeta|^2 + \mathrm{O}(1) \quad \text{as } |\zeta| \to +\infty,\end{aligned}$$

while the relation (2.4) says that $\widehat{Q}_\tau(\zeta) = \tau\log|\zeta|^2 + \mathrm{O}(1)$ when $|\zeta| \to +\infty$. Hence f is bounded above in the plane $\mathbb{C}$ (the bound might depend on m, though). Moreover, it is clear that f is harmonic in a punctured neighborhood of infinity, so that f has a representation

$$f(\zeta) = h(\zeta) - c\log|\zeta|$$

for all large enough $|\zeta|$, where c is a nonnegative number and h is harmonic at infinity (see, e.g., [21, cor. 0.3.7, p. 12]). In particular, f extends to a subharmonic function on $\mathbb{C}^* \setminus \mathscr{S}_\tau$. By Theorem 8.1, we have

$$f(\zeta) \leq \log(Cm^2) + mQ(z_0) - \lambda_0\sqrt{m}r_0 \quad \text{when } \zeta \in \partial\mathscr{S}_\tau,$$

since it is clear that $r_0 < |\zeta - z_0|$ for $\zeta \in \partial\mathscr{S}_\tau$. The maximum principle extends this estimate to all $\zeta \in \mathbb{C}^* \setminus \mathscr{S}_\tau$. The conclusion is that

$$|K_{m,n}(z_0, z_1)|^2 \mathrm{e}^{-m\widehat{Q}_\tau(z_1)-mQ(z_0)} \leq Cm^2\mathrm{e}^{-\lambda_0 r_0\sqrt{m}} \quad \text{when } z_1 \in \mathbb{C} \setminus \mathscr{S}_\tau,$$

and the claimed estimate follows. □

PROOF OF THEOREM 2.12: Let $\mathscr{K}$ be a compact subset of $\mathscr{S}_\tau^\circ \cap \mathscr{N}_+$, and pick $M > 0$. By Theorem 2.3 we have that $K_{m,n}(z_0, z_0)\mathrm{e}^{-mQ(z_0)} = m\Delta Q(z_0) + \mathrm{O}(1)$ as $m \to +\infty$ and $n \geq m\tau - M$, where the O(1) is uniform in $z_0 \in \mathscr{K}$. It follows that

$$\begin{aligned} \mathrm{B}_{m,n}^{\langle z_0 \rangle}(z) &= \frac{|K_{m,n}(z, z_0)|^2}{K_{m,n}(z_0, z_0)} \mathrm{e}^{-mQ(z)} \\ &= \frac{|K_{m,n}(z, z_0)|^2}{m\Delta Q(z_0) + \mathrm{O}(1)} \mathrm{e}^{-m(Q(z)+Q(z_0))}, \quad z \in \mathbb{C}, \end{aligned} \tag{8.26}$$

as $m \to +\infty$ and $n \geq m\tau - M$. Since ΔQ is bounded from below by a positive number on $\mathscr{K}$, the right-hand side in (8.26) can be estimated by

$$C m^{-1} |K_{m,n}(z, z_0)|^2 \mathrm{e}^{-m(Q(z)+Q(z_0))}, \quad z \in \mathbb{C},$$

where C depends on the lower bound of ΔQ on $\mathscr{K}$. The assertion now follows from Corollary 8.2. □

9 Bargmann-Fock Case and Harmonic Measure

9.1 Preliminaries

In this section, we prove Theorem 2.10. We therefore put $Q(z) = |z|^2$. Recall that in this case $\mathscr{S}_\tau = \overline{\mathbb{D}}(0, \sqrt{\tau})$, and (see (2.8))

$$\mathrm{d}B_{m,n}^{\langle z_0 \rangle}(z) = m \frac{|E_{n-1}(mz\bar{z}_0)|^2}{E_{n-1}(m|z_0|^2)} \mathrm{e}^{-m|z|^2} \, \mathrm{d}A(z) \quad \text{where } E_k(z) = \sum_{j=0}^{k} \frac{z^j}{j!}. \tag{9.1}$$

9.2 Berezin Transform on Polynomials

We first consider the action of the Berezin transform on polynomials.

PROPOSITION 9.1 *Fix a complex number $z_0 \neq 0$ and a positive integer d, and let n be an integer, $n \geq d + 1$. Then, for all analytic polynomials u of degree at most d, we have*

$$\mathrm{pv} \int_{\mathbb{C}} u(z^{-1}) \mathrm{d}B_{m,n}^{\langle z_0 \rangle} \to u(z_0^{-1}) \quad \text{as } m \to +\infty, \tag{9.2}$$

uniformly in n, $n \geq d + 1$.

PROOF: It is sufficient to prove the statement for $u(z) = z^j$ with $j \leq d$. The left-hand side in (9.2) can then be written

$$\mathrm{pv} \int_{\mathbb{C}} z^{-j} \, \mathrm{d}B_{m,n}^{\langle z_0 \rangle} = \frac{m b_{m,n}^j(z_0)}{E_{n-1}(m|z_0|^2)},$$

where we have put

$$b_{m,n}^{j}(z_0) = \mathrm{pv}\int_{\mathbb{C}} z^{-j}\left|\sum_{k=0}^{n-1}\frac{(mz_0\bar{z})^k}{k!}\right|^2 \mathrm{e}^{-m|z|^2}\,\mathrm{d}A(z).$$

Expanding the square yields

$$b_{m,n}^{j}(z_0) = \sum_{k,l=0}^{n-1}\frac{m^{k+l}{z_0}^k\bar{z}_0^l}{k!l!}\,\mathrm{pv}\int_{\mathbb{C}} \bar{z}^k z^{l-j}\mathrm{e}^{-m|z|^2}\mathrm{d}A(z).$$

Clearly only those k, l for which $k = l - j$ give a nonzero contribution to the sum, and therefore,

$$\begin{aligned} b_{m,n}^{j}(z_0) &= {z_0}^{-j}\sum_{l=j}^{n-1}\frac{m^{2l-j}|z_0|^{2l}}{(l-j)!l!}\int_{\mathbb{C}} |z|^{2(l-j)}\mathrm{e}^{-m|z|^2}\mathrm{d}A(z) \\ &= \frac{1}{m{z_0}^j}\sum_{l=j}^{n-1}\frac{m^l|z_0|^{2l}}{l!}. \end{aligned}$$

It follows that

$$b_{m,n}^{j}(z_0) = \frac{1}{m{z_0}^j}\sum_{l=j}^{n-1}\frac{(m|z_0|^2)^l}{l!} = \frac{1}{m{z_0}^j}\big(E_{n-1}(m|z_0|^2) - E_{j-1}(m|z_0|^2)\big),$$

and so

$$\frac{mb_{m,n}^{j}(z_0)}{E_{n-1}(m|z_0|^2)} = \frac{1}{{z_0}^j}\left(1 - \frac{E_{j-1}(m|z_0|^2)}{E_{n-1}(m|z_0|^2)}\right).$$

Finally, since $j \le d < n$,

$$\frac{E_{j-1}(m|z_0|^2)}{E_{n-1}(m|z_0|^2)} \le \frac{E_{d-1}(m|z_0|^2)}{E_{d}(m|z_0|^2)} \to 0 \quad \text{as } m \to +\infty.$$

□

PROPOSITION 9.2 *Fix* $z_0 \in \mathbb{C} \setminus \overline{\mathbb{D}}(0, \sqrt{\tau})$*, and let* u *be an analytic polynomial. Then, for any fixed* r *with* $0 < r < \sqrt{\tau}$*, we have*

$$\mathrm{pv}\int_{\mathbb{D}(0,r)} u(z^{-1})\mathrm{d}B_{m,n}^{\langle z_0\rangle}(z) \to 0 \quad \text{as } m \to +\infty \text{ while } n = m\tau + \mathrm{o}(m).$$

PROOF: Put, for $\nu = 0, 1, 2, \ldots$,

$$b_{m,n}^{\nu}(z_0, r) = \mathrm{pv}\int_{\mathbb{D}(0,r)} z^{-\nu}\,\mathrm{d}B_{m,n}^{\langle z_0\rangle}.$$

A straightforward calculation based on (9.1) leads to

$$b^{\nu}_{m,n}(z_0,r)=\frac{1}{z_0^{\nu}E_{n-1}(m|z_0|^2)}\sum_{j=\nu}^{n-1}\frac{(m|z_0|^2)^j}{j!(j-\nu)!}\int_0^{mr^2}s^{j-\nu}\mathrm{e}^{-s}\,\mathrm{d}s. \tag{9.3}$$

We suppose n is greater than ν by at least two units, so that we may pick an integer k with $\nu<k<n$ and split the sum (9.3) accordingly:

$$\begin{aligned}b^{\nu}_{m,n}(z_0,r)=\frac{1}{z_0^{\nu}E_{n-1}(m|z_0|^2)}\Bigg\{&\sum_{j=\nu}^{k-1}\frac{(m|z_0|^2)^j}{j!(j-\nu)!}\int_0^{mr^2}s^{j-\nu}\mathrm{e}^{-s}\,\mathrm{d}s\\&+\sum_{j=k}^{n-1}\frac{(m|z_0|^2)^j}{j!(j-\nu)!}\int_0^{mr^2}s^{j-\nu}\mathrm{e}^{-s}\,\mathrm{d}s\Bigg\}.\end{aligned} \tag{9.4}$$

We estimate the first term trivially as follows:

$$\begin{aligned}\sum_{j=\nu}^{k-1}\frac{(m|z_0|^2)^j}{j!(j-\nu)!}\int_0^{mr^2}s^{j-\nu}\mathrm{e}^{-s}\,\mathrm{d}s&\le\sum_{j=\nu}^{k-1}\frac{(m|z_0|^2)^j}{j!(j-\nu)!}\int_0^{+\infty}s^{j-\nu}\mathrm{e}^{-s}\,\mathrm{d}s\\&=\sum_{j=0}^{k-1}\frac{(m|z_0|^2)^j}{j!}=E_{k-1}(m|z_0|^2).\end{aligned} \tag{9.5}$$

As for the second term, we use the fact that the function $s\mapsto s^{j-\nu}\mathrm{e}^{-s}$ is increasing on the interval $[0,j-\nu]$ to say that

$$\int_0^{mr^2}s^{j-\nu}\mathrm{e}^{-s}\,\mathrm{d}s\le(mr^2)^{j-\nu+1}\mathrm{e}^{-mr^2}$$

provided that $j\ge mr^2+\nu$. It follows that if $k\ge mr^2+\nu$, then

$$\sum_{j=k}^{n-1}\frac{(m|z_0|^2)^j}{j!(j-\nu)!}\int_0^{mr^2}s^{j-\nu}\mathrm{e}^{-s}\,\mathrm{d}s\le(mr^2)^{1-\nu}\mathrm{e}^{-mr^2}\sum_{j=k}^{n-1}\frac{(mr|z_0|)^{2j}}{j!(j-\nu)!}.$$

By Stirling's formula, $j!\ge\sqrt{2\pi}\,j^{j+1/2}\mathrm{e}^{-j}$, so that

$$\frac{(mr|z_0|)^{2j}}{j!}\,\mathrm{e}^{-mr^2}\le\frac{1}{\sqrt{2\pi j}}\,m^j|z_0|^{2j}\left(\frac{mr^2}{j}\,\mathrm{e}^{1-\frac{mr^2}{j}}\right)^j.$$

Since the function $x\mapsto x\mathrm{e}^{1-x}$ is increasing on the interval $[0,1]$, it yields that

$$\frac{(mr|z_0|)^{2j}}{j!}\,\mathrm{e}^{-mr^2}\le\frac{1}{\sqrt{2\pi j}}\,m^j|z_0|^{2j}\left(\frac{mr^2}{k}\,\mathrm{e}^{1-\frac{mr^2}{k}}\right)^j,\quad mr^2+\nu\le k\le j.$$

We write

$$c_{k,m}=\frac{mr^2}{k}\,\mathrm{e}^{1-\frac{mr^2}{k}}\le1,\quad mr^2+\nu\le k, \tag{9.6}$$

and conclude that

$$\begin{aligned}
&\sum_{j=k}^{n-1} \frac{(m|z_0|^2)^j}{j!(j-\nu)!} \int_0^{mr^2} s^{j-\nu} \mathrm{e}^{-s}\, \mathrm{d}s \\
&\quad \le (mr^2)^{1-\nu} \sum_{j=k}^{n-1} \frac{(m|z_0|^2 c_{k,m})^j}{(j-\nu)!\sqrt{2\pi j}} \\
&\quad \le \frac{(mr^2)^{1-\nu}}{\sqrt{2\pi}} (c_{k,m})^k \sum_{j=k}^{n-1} \frac{(m|z_0|^2)^j}{(j-\nu)!} \\
&\quad \le \frac{mr^2}{\sqrt{2\pi}} \left(\frac{|z_0|}{r}\right)^{2\nu} (c_{k,m})^k E_{n-\nu-1}(m|z_0|^2).
\end{aligned} \tag{9.7}$$

Now, a combination of (9.5) and (9.7) applied to (9.4) yields

$$\begin{aligned}
&|z_0^\nu b_{m,n}^\nu(z_0, r)| \\
&\quad \le \frac{E_{k-1}(m|z_0|^2)}{E_{n-1}(m|z_0|^2)} + \frac{mr^2}{\sqrt{2\pi}} \left(\frac{|z_0|}{r}\right)^{2\nu} (c_{k,m})^k \frac{E_{n-\nu-1}(m|z_0|^2)}{E_{n-1}(m|z_0|^2)} \\
&\quad \le \frac{E_{k-1}(m|z_0|^2)}{E_{n-1}(m|z_0|^2)} + \frac{mr^2}{\sqrt{2\pi}} \left(\frac{|z_0|}{r}\right)^{2\nu} (c_{k,m})^k.
\end{aligned} \tag{9.8}$$

We would like to show that each of the terms on the right-hand side of (9.8) can be made small by choosing k cleverly. As for the first term, we appeal to a theorem of Szegö [23, Hilfssatz 1, p. 54], which states that

$$E_l(lx) = \frac{1}{\sqrt{2\pi l}} (\mathrm{e}x)^l \frac{x}{x-1} (1 + \varepsilon_l(x)), \quad x > 1,$$

where $\varepsilon_l(x) \to 0$ uniformly on compact subintervals of $(1, +\infty)$ as $l \to +\infty$. It follows that

$$\frac{E_{k-1}(m|z_0|^2)}{E_{n-1}(m|z_0|^2)} = \sqrt{\frac{n-1}{k-1}} \frac{m|z_0|^2 - n + 1}{m|z_0|^2 - k + 1} \left(\frac{\mathrm{e}m|z_0|^2}{k-1}\right)^{k-1} \left(\frac{\mathrm{e}m|z_0|^2}{n-1}\right)^{1-n} \frac{1 + \varepsilon_{k-1}(\frac{m|z_0|^2}{k-1})}{1 + \varepsilon_{n-1}(\frac{m|z_0|^2}{n-1})}.$$

Finally, we decide to pick k such that

$$\frac{k}{m} \to \beta$$

as $k, m \to +\infty$, where $r^2 < \beta < \tau$. We observe that with this choice of k, the above epsilons tend to 0 as $k, m, n \to +\infty$. The function

$$y \mapsto \left(\frac{e}{y}\right)^y, \quad 0 < y \le 1,$$

is strictly increasing, so that with

$$y_1 = \frac{k-1}{m|z_0|^2} \approx \frac{\beta}{|z_0|^2}, \quad y_2 = \frac{n-1}{m|z_0|^2} \approx \frac{\tau}{|z_0|^2},$$

we have

$$\frac{(\mathrm{e}/y_1)^{y_1}}{(\mathrm{e}/y_2)^{y_2}} \le \theta < 1,$$

where at least for large k, m, and n, the number θ may be taken to be independent of k, m, and n. It follows that

$$\left(\frac{\mathrm{e}m|z_0|^2}{k-1}\right)^{k-1} \left(\frac{\mathrm{e}m|z_0|^2}{n-1}\right)^{1-n} \le \theta^{-m|z_0|^2},$$

so that

$$\frac{E_{k-1}(m|z_0|^2)}{E_{n-1}(m|z_0|^2)} \le (1 + \mathrm{o}(1)) \sqrt{\frac{\tau}{\beta}} \, \frac{|z_0|^2 - \tau}{|z_0|^2 - \beta} \, \theta^{-m|z_0|^2} \to 0$$

exponentially quickly as $k, m, n \to +\infty$.

Finally, as for the second term, we observe that the numbers $c_{k,m}$ defined by (9.6) have the property that

$$c_{k,m} \to \frac{r^2}{\beta} \, \mathrm{e}^{1-\frac{r^2}{\beta}} < 1,$$

as $k, m, n \to +\infty$ in the given fashion. In particular, the second term converges exponentially quickly to 0. The proof is complete. □

COROLLARY 9.3 *Let $z_0 \in \mathbb{C} \setminus \overline{\mathbb{D}}(0, \sqrt{\tau})$, and let $\mathscr{D}$ be an open set in $\mathbb{C}$ that contains the circle $\mathbb{T}(0, \sqrt{\tau})$. Further, let u be an analytic polynomial. Then*

$$\int_{\mathscr{D}} u(z^{-1}) \mathrm{d}B^{\langle z_0 \rangle}_{m,n}(z) \to u(z_0^{-1}) \quad \textit{as } m \to +\infty \textit{ while } n = m\tau + \mathrm{o}(m).$$

PROOF: This follows from Propositions 9.1, 9.2, and 2.6. □

PROOF OF THEOREM 2.10: Let $\mathscr{C}_h(\mathbb{C}; \mathbb{C} \setminus \mathscr{S}_\tau)$ denote the class of bounded continuous functions $\mathbb{C} \to \mathbb{C}$ that are harmonic on $\mathbb{C} \setminus \mathscr{S}_\tau$. For a function $f \in \mathscr{C}_b(\mathbb{C})$, we write $\tilde{f}$ for the unique function of class $\mathscr{C}_h(\mathbb{C}; \mathscr{S}_\tau)$ that coincides with f on $\mathbb{D}(0, \sqrt{\tau})$. We must show that

$$\int_{\mathbb{C}} f(z) \mathrm{d}B^{\langle z_0 \rangle}_{m,n}(z) \to \tilde{f}(z_0)$$

as $m \to +\infty$ and $n = m\tau + \mathrm{o}(m)$. See, e.g., [14, p. 90]. By convolving with the Féjer kernel, we find that f may be uniformly approximated on a neighborhood $\mathscr{D}$ of $\mathbb{T}(0, \sqrt{\tau})$ by functions of the form $u(z^{-1})$ with u a harmonic polynomial. We may therefore without loss of generality suppose that f itself is of this form, i.e., $f(z) = u(z^{-1})$ when $z \in \mathscr{D}$. Thus $f(z) = \tilde{f}(z) = u(z^{-1})$ on $\mathscr{D}$. By Proposition 2.6,

$$\int_{\mathbb{C}} (f(z) - \tilde{f}(z)) \mathrm{d}B_{m,n}^{\langle z_0 \rangle}(z) \to 0$$

as $m \to +\infty$ while $n = m\tau + \mathrm{o}(m)$. Moreover, Corollary 9.3 gives that

$$\int_{\mathbb{C}} \tilde{f}(z) \mathrm{d}B_{m,n}^{\langle z_0 \rangle}(z) = \int_{\mathbb{C}} u(z^{-1}) \mathrm{d}B_{m,n}^{\langle z_0 \rangle}(z) \to u(z_0^{-1}) = \tilde{f}(z_0)$$

as $m \to +\infty$ while $n = m\tau + \mathrm{o}(m)$. □

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