Heisenberg uniqueness pairs and the Klein-Gordon equation

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Heisenberg uniqueness pairs (HUPs)

A Heisenberg uniqueness pair (HUP) is a pair (Γ, Λ), where Γ is a curve in the plane and Λ is a set in the plane, with the following property: any finite Borel measure μ in the plane supported on Γ, which is absolutely continuous with respect to arc length, and whose Fourier transform $\widehat{\mu}$ vanishes on Λ, must automatically be the zero measure.

Two parallel lines

Put

$$\Gamma = \mathbb{R} \times \{0, 1\}.$$

Let $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ be the projection to the first coordinate. Then

$$(\Gamma, \Lambda) \quad \Longrightarrow \quad \pi_1(\Lambda) \subset \mathbb{R} \text{ dense.}$$

Let

$$\mathrm{d}\mu(x) = f(x_1)\mathrm{d}x_1\mathrm{d}\delta_0(x_2) + g(x_1)\mathrm{d}x_1\mathrm{d}\delta_1(x_2),$$

where $f, g \in L^1(\mathbb{R})$ (δ_y is Dirac), so:

$$\widehat{\mu}(\xi) = \widehat{f}(\xi_1) + \mathrm{e}^{\pi \mathrm{i} \xi_2}\widehat{g}(\xi_1).$$

Two parallel lines (2)

We split

$$\pi_1(\Lambda) = \pi_1^a(\Lambda) \cup \pi_1^b(\Lambda),$$

where the two sets are disjoint: $t \in \pi_1^a(\Lambda)$ if there exist two lifted points $\xi = (\xi_1, \xi_2)$ and $\eta = (\eta_1, \eta_2)$ in Λ, with $\xi_1 = \eta_1 = t$ and $\xi_2 - \eta_2 \notin 2\mathbb{Z}$. For $t \in \pi_1^b(\Lambda)$, put $\chi(t) = \mathrm{e}^{\pi\mathrm{i}\xi_2}$ where $(\xi_1, \xi_2) \in \Lambda$, $\xi_1 = t$. Let $\pi_1^c(\Lambda)$ consist of those points $t_0 \in \pi_1^b(\Lambda)$ where $\chi : \pi_1^b(\Lambda) \to \mathbb{C}$ is locally the Fourier transform of an $L^1(\mathbb{R})$ function around t_0.

THEOREM. (Γ, Λ) is a Heisenberg uniqueness pair iff $\pi_1^a(\Lambda) \cup (\pi_1^b(\Lambda) \setminus \pi_1^c(\Lambda))$ is dense in $\mathbb{R}$.

Three parallel lines

The case of three parallel lines has been treated by D. Blasi-B. in successful Swedish-Catalan interaction!

The hyperbola

Let $\Gamma : x_1 x_2 = \epsilon$, with $\epsilon \neq 0$. Let Λ be the lattice-cross

$$\Lambda = (\alpha\mathbb{Z} \times \{0\}) \cup (\{0\} \times \beta\mathbb{Z}),$$

where $\alpha, \beta > 0$ are positive reals.

THEOREM. (Γ, Λ) is a Heisenberg uniqueness pair iff $\alpha\beta|\epsilon| \leq 1$.

One remark

We note that $\hat{\mu}$ solves the Klein-Gordon equation, which is of hyperbolic type. It is therefore unexpected to get such a small uniqeness set as Λ.

Ideas for the proof

The proof is based on the Birkhoff Ergodic Theorem, and a theorem of Schweiger on the uniqueness of the absolutely continuous invariant measure for the map $\mapsto 1/x$ modulo 2 on the symmetric interval $[-1, 1]$.