ISRAEL MATHEMATICAL CONFERENCE PROCEEDINGS Vol. 15, 2001

CURIOUS PROPERTIES OF CANONICAL DIVISORS IN WEIGHTED BERGMAN SPACES

by ALEXANDRU ALEMAN,

Fachbereich Mathematik, Fernuniversität Hagen, D-58084 Hagen, Germany
Current address: Department of Mathematics, Lund University, Box 118, S-22100 Lund, Sweden
E-mail address: aleman@maths.lth.se

HÅKAN HEDENMALM,

Department of Mathematics, Lund University, Box 118, S-221 00 Lund, Sweden
E-mail address: haakan@maths.lth.se

STEFAN RICHTER

Department of Mathematics, The University of Tennessee, Knoxville, TN 37996, USA
E-mail address: richter@math.utk.edu

and CARL SUNDBERG

Department of Mathematics, The University of Tennessee, Knoxville, TN 37996, USA
E-mail address: sundberg@math.utk.edu

1. Introduction.

Let ω be a nonnegative Borel measurable function in the open unit disk $\mathbb{D}$ such that

$$h(0) = \int_{\mathbb{D}} h(z)\omega(z)dS(z), \qquad h \in L_h^\infty(\mathbb{D}), \tag{1.1}$$

where dS stands for area measure in the complex plane, normalized so that $\mathbb{D}$ has mass 1, and $L_h^\infty(\mathbb{D})$ is the space of bounded complex-valued harmonic functions on $\mathbb{D}$. We associate with ω the Hilbert space $L^2(\omega) = L^2(\mathbb{D}, \omega)$ of Borel measurable functions f on $\mathbb{D}$, with

$$\|f\|_{L^2(\omega)} = \left(\int_{\mathbb{D}} |f(z)|^2\omega(z)dS(z)\right)^{1/2} < +\infty.$$

Consider the subspaces $L_a^2(\omega)$ and $L_h^2(\omega)$ of $L^2(\omega)$, consisting of those functions that can be altered on null sets so as to be analytic and harmonic on $\mathbb{D}$, respectively. If two harmonic functions on $\mathbb{D}$ are given, which coincide as elements of $L^2(\omega)$, then the harmonic functions coincide. This allows us to think of $L_a^2(\omega)$ and $L_h^2(\omega)$ as spaces of analytic and harmonic functions, respectively. These linear subspaces of

$L^2(\omega)$ need not be closed. For instance, if ω is supported on a disk centered at the origin of radius less than 1, they certainly are not. However, if ω is locally bounded away from 0 on $\mathbb{D}\setminus K$, for some compact subset K of $\mathbb{D}$, then they are, in which case we refer to them as weighted analytic and harmonic Bergman spaces, with weight ω. We sometimes drop the word "analytic" and talk about weighted Bergman spaces. When the weight is the constant function $\omega = 1$, we talk about the analytic and harmonic Bergman spaces, and write L^2_a and L^2_h, dropping the weight from the notation. In the sequel, we shall make the assumption that ω is such that $L^2_a(\omega)$ and $L^2_h(\mathbb{D})$ are closed subspaces of $L^2(\omega)$.

Condition (1.1) implies that the decomposition of a harmonic polynomial f into a sum of an analytic polynomial g and an antianalytic polynomial h, where $h(0) = 0$, is actually orthogonal: g and h are perpendicular in $L^2_h(\omega)$. If the harmonic polynomials are dense in $L^2_h(\omega)$, this extends to a direct sum decomposition $L^2_h(\omega) = L^2_a(\omega) \oplus L^2_{\bar{a},0}(\omega)$, where $L^2_{\bar{a},0}(\omega)$ stands for the closed subspace of $L^2_h(\omega)$ consisting of antianalytic functions that vanish at the origin. In the sequel, we shall assume that ω is such that the harmonic polynomials are dense in $L^2_h(\omega)$.

2. Basic concepts.

The reproducing kernel function in any of the spaces $L^2_h(\omega)$, $L^2_a(\omega)$, and $L^2_{\bar{a},0}(\omega)$, is obtained by taking an orthonormal basis $e_1, e_2, \dots$, and forming

$$k(z,\zeta) = \sum_{n=1}^{\infty} e_n(z)\bar{e}_n(\zeta), \qquad (z,\zeta) \in \mathbb{D}\times\mathbb{D}.$$

To distinguish the kernels for the various spaces, we use subscripts and superscripts: $k^\omega_h(z,\zeta)$, $k^\omega_a(z,\zeta)$, and $k^\omega_{\bar{a},0}(z,\zeta)$, with obvious interpretations. It should be pointed out that the sum is independent of the particular orthonormal basis. One way to see this is to use the reproducing property of the kernel functions. For instance, $k^\omega_h(z,\zeta)$ has the property that

$$f(z) = \int_{\mathbb{D}} f(\zeta) k^\omega_h(z,\zeta)\omega(\zeta)dS(\zeta), \qquad z \in \mathbb{D},$$

for all $f \in L^2_h(\omega)$. The direct decomposition of $L^2_h(\omega)$ has its counterpart for the kernel functions,

$$k^\omega_h(z,\zeta) = k^\omega_a(z,\zeta) + k^\omega_{\bar{a},0}(z,\zeta), \qquad (z,\zeta) \in \mathbb{D}\times\mathbb{D}.$$

One checks that $k^\omega_{\bar{a},0}(z,\zeta) = \bar{k}^\omega_a(z,\zeta) - 1$, so that the above identity reduces to

$$k^\omega_h(z,\zeta) = 2\,\mathrm{Re}\, k^\omega_a(z,\zeta) - 1, \qquad (z,\zeta) \in \mathbb{D}\times\mathbb{D}. \tag{1.2}$$

We shall work with weights ω such that the kernel $k^\omega_a(z,\zeta)$ extends continuously to $\bar{\mathbb{D}}^2 \setminus \delta(\mathbb{T})$, where $\delta(\mathbb{T}) = \{(z,z) : z \in \mathbb{T}\}$ is the diagonal on $\mathbb{T}^2$. If, for instance, ω is C^∞-smooth and bounded away from 0 near $\mathbb{T}$, this is so.

3. General properties of one-point extremal functions.

We assume the weight ω is C^∞ smooth near $\mathbb{T}$ in $\overline{\mathbb{D}}$ and bounded away from 0 there as well. Then the kernel $k_a^\omega(z,\zeta)$ extends to a C^∞ smooth function on $\overline{\mathbb{D}}^2 \setminus \delta(\mathbb{T})$, and the norm of $k_a^\omega(\cdot,\zeta)$ in $L_a^2(\omega)$ is $\asymp (1-|\zeta|^2)^{-2}$, since the space is just the usual L_a^2 Bergman space with a different equivalent norm. If Λ, $\Lambda \subset \mathbb{D}\setminus\{0\}$, is a subsequence of a zero sequence (counting multiplicities) for $L_a^2(\omega)$, the *extremal function* for Λ in $L_a^2(\omega)$, denoted G_Λ, is the function in $L_a^2(\omega)$ that vanishes (counting multiplicities) on Λ, and has biggest positive value at the origin among all such functions of norm 1. If Λ only contains one point λ, then we write G_λ in place of G_Λ. A calculation reveals that

$$G_\lambda(z) = \big(1 - \|k_a^\omega(\cdot,\lambda)\|^{-2}\big)^{-1/2}\big(1 - \|k_a^\omega(\cdot,\lambda)\|^{-2}k_a^\omega(z,\lambda)\big), \qquad z \in \mathbb{D}. \tag{3.1}$$

Let us agree to say that the one-point zero extremal functions G_λ for $L_a^2(\omega)$ are asymptotically expansive multipliers if

$$\liminf_{|\lambda|\to 1}(1-|\lambda|^2)^{-2}\big(\|G_\lambda f\|_{L^2(\omega)} - \|f\|_{L^2(\omega)}\big) \geq 0.$$

For instance, if they are expansive for all λ in an annulus $r < |\lambda| < 1$, then they are asymptotically expansive.

The extremal functions for subzero sequences (a subzero sequence is a subsequence of a zero sequence) form a subclass among the $L_a^2(\omega)$-inner functions G, which are described by their property that

$$h(0) = \int_{\mathbb{D}} h(z)|G(z)|^2\omega(z)dS(z), \qquad h \in L_h^\infty(\mathbb{D}). \tag{3.2}$$

Note the similarity between (3.2) and (1.1). It is standard to express these conditions as saying that ωdS and $|G|^2\omega dS$ are representing measures for the origin.

THEOREM 3.1. *Suppose that the one-point zero extremal functions in $L_a^2(\omega)$ are asymptotically expansive multipliers on $L_a^2(\omega)$. Then if G is $L_a^2(\omega)$-inner and continuous on $\overline{\mathbb{D}}$, $|G| \geq 1$ on $\mathbb{T}$.*

PROOF. It follows from (3.1) that

$$\big(\|k_a^\omega(\cdot,\lambda)\|^2 - 1\big)\big(|G_\lambda(z)|^2 - 1\big) = \|k_a^\omega(\cdot,\lambda)\|^{-2}\big|k_a^\omega(z,\lambda)\big|^2 - 2\,\mathrm{Re}\,k_a^\omega(z,\lambda) + 1. \tag{3.3}$$

Since both $|G|^2 - 1$ and $|G_\lambda|^2 - 1$ annihilate the bounded harmonic functions with respect to the inner product of $L^2(\omega)$, one sees that

$$\begin{aligned}\int_{\mathbb{D}} \big(|G(z)|^2 - 1\big)\,|G_\lambda|^2\omega dS &= \int_{\mathbb{D}} \big(|G(z)|^2 - 1\big)\big(|G_\lambda|^2 - 1\big)\omega dS \\ &= \int_{\mathbb{D}} |G(z)|^2\big(|G_\lambda|^2 - 1\big)\omega dS,\end{aligned}$$

whence, in view of (3.3),

$$\big\|k_a^\omega(\cdot,\lambda)\big\|^{-2}\int_{\mathbb{D}}\big(|G(z)|^2-1\big)\big|k_a^\omega(z,\lambda)\big|^2\omega(z)dS(z)$$
$$=\big(\|k_a^\omega(\cdot,\lambda)\|^2-1\big)\int_{\mathbb{D}}|G(z)|^2\big(|G_\lambda|^2-1\big)\omega dS.$$

As λ approaches some point $\lambda_0\in\mathbb{T}$, the left hand side tends to $|G(\lambda_0)|^2-1$, and the right hand side has a limes inferior that is nonnegative, by the condition of asymptotic expansion. The assertion of the theorem follows. □

It has been known for some time that an expansive multiplier that extends continuously to the closed unit disk necessarily has boundary values of modulus ≥ 1. The following offers a converse to that statement for one-point zero set extremal functions. For an analytic function f in $\mathbb{D}$, $Z(f)$ denotes its zero sequence, counting multiplicities.

THEOREM 3.2. *Suppose that for some $\lambda\in\mathbb{D}\setminus\{0\}$, $|G_\lambda(z)|\geq 1$ for all $z\in\mathbb{T}$. If also $Z(G_\lambda)=\{\lambda\}$, then G_λ is an expansive multiplier.*

PROOF. For $\lambda\in\mathbb{D}\setminus\{0\}$, let L_λ be the operator

$$L_\lambda f(z)=\frac{f(z)-f(\lambda)}{G_\lambda(z)},\qquad z\in\mathbb{D}\setminus\{\lambda\},$$

which takes $L_a^2(\omega)$ into itself. Note that since G_λ is bounded and ω is so regular, (3.2) extends to all $h\in L_h^2$. The functions $|f|^2$ and $|f-f(\lambda)|^2=|G_\lambda L_\lambda f|^2$ differ by a function in L_h^2, and so

$$0\leq\int_{\mathbb{D}}\big(|G_\lambda(z)|^2-1\big)^2|L_\lambda f(z)|^2\omega(z)dS(z)$$
$$=\int_{\mathbb{D}}\big(|G_\lambda(z)|^2-1\big)\big(|f(z)-f(\lambda)|^2-|L_\lambda f(z)|^2\big)\omega(z)dS(z)$$
$$\int_{\mathbb{D}}\big(|G_\lambda(z)|^2-1\big)\big(|f(z)|-|L_\lambda f(z)|^2\big)\omega(z)dS(z)$$
$$=\big(\|G_\lambda f\|^2_{L^2(\omega)}-\|f\|^2_{L^2(\omega)}\big)-\big(\|G_\lambda L_\lambda f\|^2_{L^2(\omega)}-\|L_\lambda f\|^2_{L^2(\omega)}\big).$$

If we apply the above formula to L_λ^j in place of f, and then sum over $j=1,\dots,n-1$, the right hand side becomes a telescoping sum, and the result is

$$0\leq\int_{\mathbb{D}}\big(|G_\lambda(z)|^2-1\big)^2\sum_{j=1}^{n-1}|L_\lambda^j f(z)|^2\omega(z)dS(z)\tag{3.4}$$
$$=\big(\|G_\lambda f\|^2_{L^2(\omega)}-\|f\|^2_{L^2(\omega)}\big)-\big(\|G_\lambda L_\lambda^n f\|^2_{L^2(\omega)}-\|L_\lambda^n f\|^2_{L^2(\omega)}\big).$$

As we let n tend to infinity, the following is obtained.

PROPOSITION 3.3. *If $f \in L^2_a(\omega)$ is such that $L^n_\lambda f$ tends to 0 in norm as $n \to +\infty$, then $\|G_\lambda f\|_{L^2(\omega)} \ge \|f\|_{L^2(\omega)}$. Indeed,*

$$\|G_\lambda f\|^2_{L^2(\omega)} = \|f\|^2_{L^2(\omega)} + \int_{\mathbb{D}} \left(|G_\lambda(z)|^2 - 1\right)^2 \sum_{j=1}^{\infty} |L^j_\lambda f(z)|^2 \omega(z) dS(z).$$

REMARK. If $f = p(G_\lambda)$, where p is a polynomial, then for sufficiently large n, $L^n_\lambda f = 0$. If functions of this type are dense in $L^2_a(\omega)$, then G_λ is an expansive multiplier on $L^2_a(\omega)$. This fact was shown by Sergeĭ Shimorin in [**3**]. Actually, our result grew out of an effort to generalize Shimorin's method.

We continue the proof of Theorem 3.2. In view of Proposition 3.3, it suffices to show that the assumption that G_λ is greater than or equal to 1 in modulus on $\mathbb{T}$ entails that $L^n_\lambda f$ tends to 0 in norm as $n \to +\infty$, for every $f \in L^2_a(\omega)$. An approximation argument shows that we need only do this for polynomials f. Let H^2_λ be the usual Hardy space H^2 on the unit disk, endowed with the norm that makes the mapping $f \mapsto f \circ \varphi_\lambda$ an isometry $H^2 \to H^2_\lambda$; here, $\varphi_\lambda(z) = (\lambda - z)/(1 - \bar{\lambda} z)$ is the standard Möbius automorphism. This renormalization has the virtue that $f - f(\lambda)$ is an orthogonal projection of f, and hence has smaller norm than f in H^2_λ. Division by G_λ then reduces the norm even more, and hence $\|L_\lambda f\|_{H^2_\lambda} \le \|f\|_{H^2_\lambda}$. It follows that $\|L^n_\lambda f\|_{H^2_\lambda} \le \|f\|_{H^2_\lambda}$ for all $n = 1, 2, 3, \ldots$. The functions $L^n_\lambda f$ clearly form a normal family on $\mathbb{D}$, so that we can select a normal limit $g \in H^2_\lambda$. This normal limit has $\|L_\lambda g\|_{H^2_\lambda} = \|g\|_{H^2_\lambda}$, which is only possible if both the projection $g \mapsto g - g(\lambda)$ and the division of $g - g(\lambda)$ by G_λ are isometric. Unless g is the 0 function, the division is isometric only if $|G_\lambda(z)| = 1$ everywhere on $\mathbb{T}$. By the maximum principle, and the fact that G_λ is nonconstant, we obtain $|G_\lambda(z)| < 1$ throughout $\mathbb{D}$. This, however, makes (3.2) impossible for $G = G_\lambda$ and $h = 1$. The only remaining logical alternative is that $g = 0$. But then every normal limit of $L^n_\lambda f$ is 0, whence $L^n_\lambda f$ tends to 0 as $n \to +\infty$, uniformly on compact subsets of $\mathbb{D}$. Since the functions $L^n_\lambda f$ were known to be bounded in H^2_λ, an elementary argument now shows that $L^n_\lambda f \to 0$ in the norm of L^2_a, which is equivalent to that of $L^2_a(\omega)$. The proof is complete. □

4. A refinement of Shimorin's theorem on compositions.

The following result is a slight sharpening of a theorem of Shimorin [**3**]. The space $L^p_a(\mathbb{C}, \mu)$ denotes the closure of the polynomials in the space $L^p(\mathbb{C}, \mu)$, and $\|\cdot\|_{L^p(\mu)}$ stands for the norm in the latter space. Similarly, $L^\infty_a(\mathbb{C}, \mu)$ is the weak-star closure of the polynomials in $L^\infty(\mathbb{C}, \mu)$.

THEOREM 4.1. *Let p be a positive even integer, and let μ be a compactly supported Borel probability measure in $\mathbb{C}$ such that*

$$q(0) = \int_{\mathbb{C}} q(z)\, d\mu(z)$$

holds for all polynomials q. Suppose $\varphi \in L^\infty_a(\mathbb{C},\mu)$ satisfies

$$q(0) = \int_{\mathbb{C}} q(z)\,|\varphi(z)|^p\,d\mu(z)$$

for all polynomials q. Then

$$\|q\circ\varphi\|_{L^p(\mu)} \le \|(q\circ\varphi)\varphi\|_{L^p(\mu)},$$

again for all polynomials q.

PROOF. Let $\mathcal{P}_a$ stand for the linear space of all polynomials, and let $\mathcal{P}_h$ stand for the space of harmonic polynomials (that is, all functions of type $q_1(z)+\bar{q}_2(z)$, with $q_1, q_2 \in \mathcal{P}_a$). It is clear from the hypotheses and the standard approximation of bounded harmonic functions in $\mathbb{D}$ by harmonic polynomials that

$$\int_{\mathbb{C}} u(z)\big(|\varphi(z)|^p - 1\big)\,d\mu(z) = 0, \qquad u \in L^\infty_h(\mathbb{D}). \tag{4.1}$$

For $f \in \mathcal{P}_a$, we define

$$Tf(z) = \frac{f(z)-f(0)}{z},$$

which is another polynomial; the mapping T is known as the backward shift. Since $zTf(z) = f(z) - f(0)$, we see that

$$\Delta|zTf(z)|^2 = \Delta|f(z)|^2. \tag{4.2}$$

We write $p = 2N$, where N is a positive integer, and let $q \in \mathcal{P}_a$. Also, let $g(z) = q(z)^N$, and observe that as g is a polynomial, the sum

$$\sum_{n=1}^{\infty} |T^n g(z)|^2$$

is actually finite. Repeated applications of (4.2) show that

$$\Delta\left((1-|z|^2)\sum_{n=1}^{\infty}|T^n g(z)|^2\right) = -\Delta|g(z)|^2,$$

whence

$$u_g(z) = |g(z)|^2 + (1-|z|^2)\sum_{n=1}^{\infty}|T^n g(z)|^2$$

is harmonic; in fact, it is in $\mathcal{P}_h$. We now see that, in view of (4.1),

$$\begin{aligned}\|(q\circ\varphi)\varphi\|_{L^p(\mu)} - \|q\circ\varphi\|_{L^p(\mu)} &= \int_{\mathbb{C}} |q(\varphi(z))|^p \big(|\varphi(z)|^p - 1\big)\, d\mu(z)\\ = \int_{\mathbb{C}} |g(\varphi(z))|^2 \big(|\varphi(z)|^{2N} - 1\big)\, d\mu(z) &= \int_{\mathbb{C}} |g(\varphi(z))|^2 \big(|\varphi(z)|^{2N} - 1\big)\, d\mu(z)\\ &= \int_{\mathbb{C}} \Big(|g(\varphi(z))|^2 - u_g(\varphi(z))\Big)\big(|\varphi(z)|^{2N} - 1\big)\, d\mu(z)\\ &= \int_{\mathbb{C}} \sum_{n=1}^{\infty} |(T^n g)(\varphi(z))|^2 \big(|\varphi(z)|^2 - 1\big)\big(|\varphi(z)|^{2N} - 1\big)\, d\mu(z) \geq 0,\end{aligned}$$

where the last relation holds simply because the integrand is greater than or equal to 0. The proof is complete. □

We now show that the assumption in Theorem 4.1 that p is an even integer cannot be avoided.

THEOREM 4.2. *Let p be a positive real number, other than an even integer. Then there is a Borel probability measure supported in the closed unit disk $\overline{\mathbb{D}}$, with the following properties:*

(1) $u(0) = \int_{\overline{\mathbb{D}}} u\, d\mu$ *for all* $u \in \mathcal{P}_h$,
(2) $u(0) = \int_{\overline{\mathbb{D}}} u(z)\, |2z|^p\, d\mu$ *for all* $u \in \mathcal{P}_h$,
(3) $\|(z-\frac{1}{2})(2z)\|_{L^p(\mu)} < \|(z-\frac{1}{2})\|_{L^p(\mu)}$.

PROOF. Consider the real Banach space $C(\overline{\mathbb{D}})$, consisting of the real-valued continuous functions on $\overline{\mathbb{D}}$ with the supremum norm. The dual space of $C(\overline{\mathbb{D}})$ is identified with the space of real-valued Borel measures on $\overline{\mathbb{D}}$, and we shall show the existence of the measure μ by showing that a suitable continuous linear functional on $C(\overline{\mathbb{D}})$ exists. Let L be the linear subspace of $C(\overline{\mathbb{D}})$ consisting of the functions of the form

$$u(z) + \big(|2z|^p - 1\big)\, v(z), \qquad u, v \in \mathcal{P}_h, \tag{4.3}$$

and let $\Lambda : L \to \mathbb{R}$ be the linear functional on L taking the functions of the form (4.3) to $u(0)$.

Since

$$\begin{aligned}|u(0)| \leq \sup\big\{|u(z)| : |z| = \tfrac{1}{2}\big\} &= \sup\big\{\big|u(z) + (|2z|^p - 1)v(z)\big| : |z| = \tfrac{1}{2}\big\}\\ &\leq \sup\big\{\big|u(z) + (|2z|^p - 1)v(z)\big| : |z| \leq 1\big\},\end{aligned}$$

and $\Lambda(1) = 1$, we see that the norm of Λ as a linear functional from the subspace L of $C(\overline{\mathbb{D}})$ to $\mathbb{R}$ equals 1. We wish to extend Λ in a norm-preserving way to L_1, the subspace of $C(\overline{\mathbb{D}})$ spanned by L and the function

$$F(z) = \big(|2z|^p - 1\big)\, |z - \tfrac{1}{2}|^p.$$

Suppose $\Lambda_1 : L_1 \to \mathbb{R}$ is defined by

$$\Lambda_1(f + \lambda F) = \Lambda(f) + \lambda\alpha, \qquad f \in L,\ \lambda \in \mathbb{R},$$

where $\alpha \in \mathbb{R}$. The functional Λ_1 is obviously a linear extension of Λ, and by the proof of the Hahn-Banach theorem (see [2]), the norm of Λ_1 will be 1 provided that

$$\sup\left\{\Lambda(f) - \|f - F\|_{C(\overline{\mathbb{D}})} : f \in L\right\} \le \alpha \le \inf\left\{\Lambda(f) + \|f - F\|_{C(\overline{\mathbb{D}})} : f \in L\right\};$$

it is part of the Hahn-Banach theorem that the left hand side is less than or equal to the right hand side, so that a permissible α can be found. We shall choose α to be the quantity on the left, that is,

(4.4)

$$\alpha = \sup\left\{u(0) - \sup\left\{\left|u(z) + (|2z|^p - 1)v(z) - (|2z|^p - 1)|z - \tfrac{1}{2}|^p\right| : |z| \le 1\right\} : u, v \in \mathcal{P}_h\right\}.$$

Again by the Hahn-Banach theorem, there is a norm-preserving extension of Λ_1 to a bounded linear functional on $C(\overline{\mathbb{D}})$. Identifying this functional with a real-valued Borel measure μ on $\overline{\mathbb{D}}$, we see that it has the properties

$$u(0) = \int_{\overline{\mathbb{D}}} \big(u(z) + (|2z|^p - 1)v(z)\big)\, d\mu(z), \qquad u, v \in \mathcal{P}_h, \tag{4.5}$$

$$\int_{\overline{\mathbb{D}}} (|2z|^p - 1)|z - \tfrac{1}{2}|^p \, d\mu(z) = \alpha, \tag{4.6}$$

$$\|\mu\| = 1. \tag{4.7}$$

By (4.5), μ has properties (1) and (2). As we combine (4.5) with (4.7), we see that

$$\Lambda(1) = \int_{\overline{\mathbb{D}}} d\mu = \int_{\overline{\mathbb{D}}} d|\mu| = \|\mu\| = 1,$$

so that μ is a probability measure. We shall complete the proof by demonstrating that $\alpha < 0$, thus showing that property (3) is satisfied.

Suppose, on the contrary, that $\alpha \ge 0$. From (4.4), we see that there must exist $u_n, v_n \in \mathcal{P}_h$ for $n = 1, 2, 3, \ldots$, such that

$$\sup_{z \in \overline{\mathbb{D}}} \left| u_n(z) + \big(|2z|^p - 1\big)\, v_n(z) - \big(|2z|^p - 1\big)\, |z - \tfrac{1}{2}|^p \right| < u_n(0) + \frac{1}{n}.$$

It follows from this that

$$u_n(z) - u_n(0) + \big(|2z|^p - 1\big)\, v_n(z) - \big(|2z|^p - 1\big)\, |z - \tfrac{1}{2}|^p < \frac{1}{n}, \qquad z \in \overline{\mathbb{D}}. \tag{4.8}$$

Plugging in $z = 0$ into this gives

$$v_n(0) > 2^{-p} - \frac{1}{n}.$$

We now set

$$f_n(z) = u_n(z) - u_n(0) + (2^p - 1)v_n(z),$$
$$g_n(z) = u_n(z) - u_n(0) + ((\tfrac{3}{2})^p - 1)v_n(z).$$

From (4.8) we can see that

$$u_n(z) - u_n(0) + \big(|2z|^p - 1\big)\, v_n(z) < \big(|2z|^p - 1\big)\big(|z| + \tfrac{1}{2}\big)^p + \frac{1}{n}, \qquad \tfrac{1}{2} \le |z| \le 1.$$

Plugging $|z| = 1$ and $|z| = \frac{3}{4}$ into this yields

$$f_n(z) < (2^p - 1)(\tfrac{3}{2})^p + \frac{1}{n}, \qquad |z| = 1, \tag{4.10}$$

and

$$g_n(z) < ((\tfrac{3}{2})^p - 1)(\tfrac{5}{4})^p + \frac{1}{n}, \qquad |z| = \tfrac{3}{4}. \tag{4.11}$$

By (4.9), we see that

$$f_n(0) > (2^p - 1)(2^{-p} - n^{-1}), \tag{4.12}$$

$$g_n(0) > \big((\tfrac{3}{2})^p - 1\big)(2^{-p} - n^{-1}). \tag{4.13}$$

The functions f_n and g_n are harmonic polynomials. It follows from (4.10)–(4.13) that $\{f_n\}$ and $\{g_n\}_n$ form normal families of harmonic functions in $|z| < \frac{3}{4}$, and hence there are subsequences $\{f_{n_j}\}$ and $\{g_{n_j}\}$ that converge uniformly on compact subsets of $|z| < \frac{3}{4}$. It follows that the corresponding subsequences of $\{u_n\}$ and $\{v_n\}$, which may be represented by

$$u_{n_j}(z) - u_{n_j}(0) = \big(2^p - (\tfrac{3}{2})^p\big)^{-1}\big((2^p - 1)g_{n_j}(z) - ((\tfrac{3}{2})^p - 1)f_{n_j}(z)\big),$$

$$v_{n_j}(z) = \big(2^p - (\tfrac{3}{2})^p\big)^{-1}\big(f_{n_j}(z) - g_{n_j}(z)\big),$$

converge uniformly on compact subsets of $|z| < \frac{3}{4}$. Say $u_{n_j}(z) - u_{n_j}(0) \to U(z)$ and $u_{n_j}(z) \to V(z)$ as $j \to +\infty$, where $U(z)$ and $V(z)$ are harmonic in $|z| < \frac{3}{4}$. It follows from (4.8) that

$$U(z) + \big(|2z|^p - 1\big)V(z) - \big(|2z|^p - 1\big)|z - \tfrac{1}{2}|^p \le 0, \qquad |z| < \tfrac{3}{4}. \tag{4.14}$$

Hence $U(z) \le 0$ for $|z| = \frac{1}{2}$. Since U is harmonic in $|z| < \frac{3}{4}$ and $U(0) = 0$, we conclude that $U(z) \equiv 0$. The relation (4.14) now reads

$$\big(|2z|^p - 1\big)\big(V(z) - |z - \tfrac{1}{2}|^p\big) \le 0, \qquad |z| < \tfrac{3}{4}. \tag{4.15}$$

However, for $|z| < \frac{1}{2}$, (4.15) says that

$$V(z) - |z - \tfrac{1}{2}|^p \ge 0,$$

and for $|z| > \frac{1}{2}$, it says that

$$V(z) - |z - \tfrac{1}{2}|^p \le 0.$$

It follows by continuity that

$$V(z) = |z - \tfrac{1}{2}|^p, \qquad |z| = \tfrac{1}{2}.$$

This is a contradiction, since the function $|z-\frac{1}{2}|^p$ fails to be infinitely differentiable at $z=\frac{1}{2}$. □

References

1. H. Hedenmalm, *A factorization theorem for square area-integrable analytic functions*, J. Reine Angew. Math. **422** (1991), 45–68.
2. W. Rudin, *Real and Complex Analysis*, 3rd edition, McGraw-Hill, 1987.
3. S. Shimorin, *Single point extremal functions in weighted Bergman spaces. Nonlinear boundary-value problems and some questions of function theory*, J. Math. Sci. **80** (1996), 2349–2356.