BRANCH POINT AREA METHODS IN CONFORMAL MAPPING

By

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Abstract. The classical estimate of Bieberbach that $|a_2| \leq 2$ for a given univalent function $\varphi(z) = z + a_2 z^2 + \cdots$ in the class S leads to the best possible pointwise estimates of the ratio $\varphi''(z)/\varphi'(z)$ for $\varphi \in S$, first obtained by Kœbe and Bieberbach. For the corresponding class Σ of univalent functions in the exterior disk, Goluzin found in 1943 by variational methods the corresponding best possible pointwise estimates of $\psi''(z)/\psi'(z)$ for $\psi \in \Sigma$. It was perhaps surprising that this time, the expressions involve elliptic integrals. Here, we obtain an area-type theorem which has Goluzin's pointwise estimate as a corollary. This shows that Goluzin's estimate, like the Kœbe–Bieberbach estimate, is firmly rooted in area-based methods. The appearance of elliptic integrals finds a natural explanation: they arise because a certain associated covering surface of the Riemann sphere is a torus.

1 Introduction

Area methods. Area methods play an important role in the theory of conformal mappings. The original Grönwall area theorem states that if ψ belongs to the class Σ, with series expansion

$$\psi(z) = z + \sum_{n=0}^{+\infty} b_n\, z^{-n},$$

then

$$\frac{1}{\pi}\int_{\mathbb{D}_e} \left|\psi'(z) - 1\right|^2 \mathrm{d}A(z) = \sum_{n=0}^{+\infty} n\,|b_n|^2 \leq 1. \tag{1.1}$$

Here, $\mathrm{d}A(z) = \mathrm{d}x\mathrm{d}y$ is ordinary area measure in the plane. Recall that $\psi \in \Sigma$ means that ψ is a conformal mapping from the exterior disk

$$\mathbb{D}_e = \left\{z \in \mathbb{C} \cup \{\infty\} : 1 < |z| \leq +\infty\right\}$$

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to some domain on the Riemann sphere $\mathbb{S} = \mathbb{C}_\infty$, normalized so that $\psi(\infty) = \infty$ and $\psi'(\infty) = 1$. In particular, (1.1) implies that $|b_1| \le 1$. After an inversion of the plane plus a square root transformation, it follows that for φ in the class $\mathcal{S}$ of conformal mappings of the unit disk $\mathbb{D}$ into $\mathbb{C}$ with $\varphi(0) = 0$ and $\varphi'(0) = 1$, we have the estimate $|\varphi''(0)| \le 4$. The Mœbius automorphisms of the unit disk allow us to move the point at the origin to an arbitrary point in $\mathbb{D}$; this results in the Kœbe–Bieberbach estimate

$$\left|\frac{\varphi''(z)}{\varphi'(z)} - \frac{2\bar{z}}{1-|z|^2}\right| \le \frac{4}{1-|z|^2}, \quad z \in \mathbb{D}. \tag{1.2}$$

This estimate is best possible in the sense that if we consider, for a given $z_0 \in \mathbb{D}$, the set of points

$$\left\{\frac{\varphi''(z_0)}{\varphi'(z_0)} : \varphi \in \mathcal{S}\right\},$$

we obtain a closed circular disk of radius $4/(1-|z_0|^2)$ centered at $2\bar{z}_0/(1-|z|^2)$.

Goluzin's inequality. For the class Σ, Goluzin [5], [6, p. 132] found in 1943 the estimate analogous to (1.2) using variational methods. Given $\psi \in \Sigma$, it reads:

$$\left|\frac{\psi''(z)}{\psi'(z)} + \frac{4|z|^2-2}{z(|z|^2-1)} - \frac{4\bar{z}}{|z|^2-1}\frac{E(1/|z|)}{K(1/|z|)}\right| \le \frac{4|z|}{|z|^2-1}\left(1 - \frac{E(1/|z|)}{K(1/|z|)}\right), \tag{1.3}$$

for $z \in \mathbb{D}_e$. Here, E and K are the elliptic integrals

$$E(\lambda) = \int_0^1 \sqrt{\frac{1-\lambda^2 t^2}{1-t^2}}\,\mathrm{d}t, \quad \lambda \in \mathbb{D}, \tag{1.4}$$

and

$$K(\lambda) = \int_0^1 \frac{\mathrm{d}t}{\sqrt{(1-\lambda^2 t^2)(1-t^2)}}, \quad \lambda \in \mathbb{D}. \tag{1.5}$$

Like (1.2), the estimate (1.3) is best possible. However, the derivation of (1.3) which Goluzin employs is quite different from the above-mentioned classical derivation of (1.2) in terms of area estimates. Here, *we find the area-type estimate needed to derive* (1.3). Basically, we introduce a square root slit in $\mathbb{S}$ between the point at infinity and a given point $\psi(z_0)$ for $z_0 \in \mathbb{D}_e$, and apply Stokes' theorem to the resulting compact covering surface over the Riemann sphere. The application of Stokes' theorem involves the use of the Green function for the part of the covering surface which covers $\psi(\mathbb{D}_e)$; in terms of the coordinates of $\mathbb{D}_e$, this Green function results from inserting a square root slit in $\mathbb{D}_e$ between infinity and z_0. This latter surface is conformally equivalent to an annulus. From the area-type method point of view which is hinted at above and described in detail in the following sections, the Green function for the annulus, which is expressible in terms of elliptic integrals, is the reason why elliptic integrals appear in (1.3). On the other hand, the reason why elliptic integrals appear in (1.3) in the context of Goluzin's proof is that the extremal function is admissible for a quadratic differential, and hence is given by the integral of a square root of a rational function.

In Section 2 below, we explain how the general area method, due to Nehari [7, 8], applies for compact Riemann surfaces. In the actual implementation of the area method, however, Nehari weakens the estimate in order to get a result he can compute using reproducing properties. We do not need to weaken the estimate at the corresponding point, possibly because we are happy to work with area integrals in place of curve integrals. We explain this in greater detail toward the end of Section 2.

As informative background material for the reader, we mention the paper of Bergman and Schiffer [2], where the Grunsky inequalities (a general version of the area method) are explained from the perspective of Bergman kernels.

2 The area-type inequality

An application of Stokes' theorem. Let S be a compact Riemann surface. Later we consider the special case when S is a (branched) covering surface of the Riemann sphere $\mathbb{S} = \mathbb{C}_\infty$. The Sobolev space $W^{1,2}(\mathrm{S})$ consists of those locally summable functions $f : \mathrm{S} \to \mathbb{C}$ for which the first-order differential $\omega_f = \mathrm{d}f$ is an element of the Hilbert space of 1-forms $L^2_1(\mathrm{S})$ (see [11, Ch. 7, pp. 181–182]). We recall the standard definition of the norm in $L^2_1(\mathrm{S})$:

$$\|\omega\|^2_{L^2} = \int_{\mathrm{S}} \omega \wedge {}^*\bar{\omega}.$$

Here, we use the standard Hodge notation

$$\omega = u\,\mathrm{d}z + v\,\mathrm{d}\bar{z}, \qquad {}^*\omega = -\mathrm{i}u\,\mathrm{d}z + \mathrm{i}v\,\mathrm{d}\bar{z},$$

where z is any local complex parameter. The space $W^{1,2}(\mathrm{S})$ is supplied with the semi-norm

$$\|f\|^2_{W^{1,2}} = \|\mathrm{d}f\|^2_{L^2}.$$

We consider the space $W^{1,2}(\mathrm{S})$ as taken modulo the constant functions; that is, any constant function is to be thought of as the zero function. This is done with the intention of making the above semi-norm a norm on $W^{1,2}(\mathrm{S})$. In terms of a local complex parameter z, the differential $\omega_f = \mathrm{d}f$ may be written as

$$\omega_f = \partial_z f\,\mathrm{d}z + \bar{\partial}_z f\,\mathrm{d}\bar{z}.$$

This is the local form of the global decomposition

$$\omega_f = \omega_{f,1} + \omega_{f,2},$$

where in terms of local coordinates $\omega_{f,1} = \partial_z f \, \mathrm{d}z$, $\omega_{f,2} = \bar{\partial}_z f \, \mathrm{d}\bar{z}$ (see [4, Ch. 1, pp. 62–63 and Ch. 2, p. 153]).

The function $f \in W^{1,2}(\mathbf{S})$ generates the second-order differentials

$$\Lambda_{f,1} = \omega_{f,1} \wedge \bar{\omega}_{f,1}, \quad \Lambda_{f,2} = -\omega_{f,2} \wedge \bar{\omega}_{f,2},$$

which have the form

$$\Lambda_{f,1} = |\partial_z f|^2 \, \mathrm{d}z \wedge \mathrm{d}\bar{z}, \qquad \Lambda_{f,2} = |\bar{\partial}_z f|^2 \, \mathrm{d}z \wedge \mathrm{d}\bar{z}, \tag{2.1}$$

in a local complex parameter z. Note that

$$\|f\|^2_{W^{1,2}} = \mathrm{i} \int_{\mathbf{S}} \Lambda_{f,1} + \mathrm{i} \int_{\mathbf{S}} \Lambda_{f,2}.$$

The next result is a consequence of Stokes' theorem.

Proposition 2.1. *For $f \in W^{1,2}(\mathbf{S})$, both integrals $\int_{\mathbf{S}} \Lambda_{f,1}$ and $\int_{\mathbf{S}} \Lambda_{f,2}$ are finite, and*

$$\int_{\mathbf{S}} \Lambda_{f,1} = \int_{\mathbf{S}} \Lambda_{f,2}. \tag{2.2}$$

Proof. Assume that $f \in C^2(\mathbf{S})$, and consider the integral

$$\int_{\mathbf{S}} \mathrm{d}(f \, \mathrm{d}\bar{f}).$$

Simple calculations give us

$$\mathrm{d}(f \, \mathrm{d}\bar{f}) = \left(|\partial_z f|^2 - |\bar{\partial}_z f|^2\right) \mathrm{d}z \wedge \mathrm{d}\bar{z}$$

in a local complex parameter z. This means that

$$\int_{\mathbf{S}} \mathrm{d}(f \, \mathrm{d}\bar{f}) = \int_{\mathbf{S}} \Lambda_{f,1} - \int_{\mathbf{S}} \Lambda_{f,2}. \tag{2.3}$$

By Theorem 6-4 [11, Ch. 6, p. 167], we have

$$\int_{\mathbf{S}} \mathrm{d}(f \, \mathrm{d}\bar{f}) = 0.$$

In view of (2.3), we obtain

$$\int_{\mathbf{S}} \Lambda_{f,1} = \int_{\mathbf{S}} \Lambda_{f,2}.$$

The general case $f \in W^{1,2}(\mathbf{S})$ follows by an approximation argument. □

Note that Proposition 2.1 claims that for the exact first-order differential form $\omega = \omega_f$,

$$\int_{\mathbb{S}} \omega \wedge \bar{\omega} = 0;$$

of course, this is not true for an arbitrary 1-form.

Solution of Laplace's equation on a subdomain. We consider a nontrivial finitely connected subdomain Ω of the compact Riemann surface $\mathbb{S}$ (nontrivial means that $\Omega \neq \emptyset, \mathbb{S}$) and a meromorphic function R on $\mathbb{S}$, the poles of which are all contained in Ω. The poles of R are denoted by $p_1, \dots, p_N$, and m_j is the order of the pole p_j, for $j = 1, \dots, N$.

Proposition 2.2. *There exists a function* $Q : \mathbb{S} \to \mathbb{S}$ *with the following properties:*

(Q1) Q *equals zero on* $\mathbb{S} \setminus \Omega$;
(Q2) Q *is harmonic on* $\Omega \setminus \{p_1, \dots, p_N\}$;
(Q3) *the function* $P = R - Q$ *is of Hölder class* Lip $\frac{1}{2}$ *on* $\mathbb{S}$, *and it belongs to the Sobolev space* $W^{1,2}(\mathbb{S})$.

Proof. As a matter of convenience, we assume in the first part of this proof that the domain Ω has real-analytic boundary. For Ω, considered as a Riemann surface, we introduce its *conjugate surface* Ω^* (see [11, Ch. 8, p. 217, Problem 1]). Let Ω^* be another copy of Ω and $* : \Omega \to \Omega^*$ be the identity mapping, $p^* = *(p)$. We also use the same notation $*$ for the inverse mapping, $* = *^{-1}$, so that $p^{**} = p$. The complex structures of Ω and Ω^* are different, however: if $z = \Phi(p)$ is a local complex parameter about some point $p_0 \in \Omega$, with $\Phi(p_0) = 0$, we pick $\bar{z} = \bar{\Phi}(p) = \Phi^*(p^*)$ as a local complex parameter about p_0^*, where the latter relation is used to define the function Φ^*. Out of Ω and Ω^*, we form the *Schottky double*

$$\widehat{\Omega} = \Omega \cup \Omega^* \cup \partial\Omega$$

by identifying conjugate boundary points $p \in \partial\Omega$ and $p^* \in \partial\Omega^*$. As a local complex parameter near the identified boundary points $p_0 = p_0^* \in \partial\Omega$, we pick

$$z = \begin{cases} \Phi(p), & p \in \Omega \cup \partial\Omega, \\ \bar{\Phi}(p^*), & p \in \Omega^*, \end{cases}$$

where $z = \Phi(p)$ is defined on some neighborhood $V \subset \mathbb{S}$ around p_0 and maps $V \cap \Omega$ onto a region in the upper half-plane $\operatorname{Im} z > 0$, with $\Phi(p_0) = 0$ in such a way that the connected segment of $\partial\Omega \cap V$ containing p_0 is mapped onto a segment of the real axis (see [11, Ch. 8, p. 217, Problem 2]). Thus we endow $\widehat{\Omega}$ with the structure of a compact Riemann surface. By Corollary 8-1 in [11, Ch. 8, p. 211], for every point p_j, there exist functions g_j and g_j^* such that

- g_j is harmonic in $\widehat{\mathbf{\Omega}} \setminus \{p_j\}$, and g_j^* is harmonic in $\widehat{\mathbf{\Omega}} \setminus \{p_j^*\}$;
- g_j has at the point p_j the same singularity as R, while g_j^* has at the point p_j^* the same singularity as $-R \circ *$.

Now put

$$Q_j(p) = \frac{1}{2}\Big\{g_j(p) + g_j^*(p) - g_j(p^*) - g_j^*(p^*)\Big\}.$$

The function Q_j has the following properties, for $j = 1, \dots, N$:

(1) it is harmonic in $\mathbf{\Omega} \setminus \{p_j\}$;
(2) the function $R - Q_j$ is regular at the point p_j;
(3) Q_j is continuous in $\bar{\mathbf{\Omega}} \setminus \{p_j\}$, and $Q_j(p) = 0$ for $p \in \partial\mathbf{\Omega}$.

Next, define the function Q by

$$Q(p) = \begin{cases} \sum\limits_{j=1}^{N} Q_j(p), & p \in \mathbf{\Omega}, \\ 0, & p \in \mathbf{S} \setminus \mathbf{\Omega}, \end{cases}$$

and introduce the associated function P, given by

$$P(p) = R(p) - Q(p).$$

The properties of Q imply that P coincides with R on the compact set $\mathbf{S} \setminus \mathbf{\Omega}$ and that P extends harmonically across the set $\{p_1, \dots, p_N\}$. Moreover, in view of the real-analyticity of the boundary $\partial\mathbf{\Omega}$, it follows that the function Q is Lipschitz-continuous near $\partial\mathbf{\Omega}$, making P Lipschitz-continuous on all of S. Hence $P \in W^{1,2}(\mathbf{S})$.

All the above considerations are valid under the assumption that $\mathbf{\Omega}$ has real-analytic boundary. In the general case, we may approximate $\mathbf{\Omega}$ by an increasing sequence of domains $\mathbf{\Omega}_n$ with real-analytic boundaries. For each such domain $\mathbf{\Omega}_n$, we construct the function Q_n according to the above scheme. We then appeal to a well-known result of Beurling [3, p. 53], which implies the uniform boundedness of the local Lip $\frac{1}{2}$-norms of Q_n (away from the poles $\{p_1, \dots, p_N\}$ of R). Thus, the sequence $\{Q_n\}$ converges in a weak sense to some function Q, defined on $\mathbf{\Omega}$. We set $P = R - Q$ with this limit function Q. The functions P and Q satisfy all required conditions, with one possible exception: we need to show that $P \in W^{1,2}(\mathbf{S})$. However, this is an obvious consequence of the fact that the function P solves the Dirichlet problem on $\mathbf{\Omega}$ with boundary values equal to R, and the solution to the Dirichlet problem minimizes the Dirichlet integral over $\mathbf{\Omega}$. The $W^{1,2}$ $(\mathbf{S})$-(semi-)norm of P is the sum of its Dirichlet integral over $\mathbf{\Omega}$ and the Dirichlet integral of R over $\mathbf{S} \setminus \mathbf{\Omega}$, both of which are finite. It follows that P belongs to $W^{1,2}(\mathbf{S})$. □

The area-theorem type inequality. We want to apply (2.2) to $P = R - Q$. For this function, we have, by (2.1),

$$\Lambda_{P,1} = \left|\partial_z(R-Q)\right|^2 \mathrm{d}z \wedge \mathrm{d}\bar{z}, \quad \Lambda_{P,2} = \left|\bar{\partial}_z(R-Q)\right|^2 \mathrm{d}z \wedge \mathrm{d}\bar{z},$$

where z is a local complex parameter.

Note that the area element $\mathrm{d}A(z)$ is $\frac{\mathrm{i}}{2}\,\mathrm{d}z \wedge \mathrm{d}\bar{z}$. We have

$$\frac{\mathrm{i}}{2}\int_{\mathbf{S}} \Lambda_{P,1} \geq \frac{\mathrm{i}}{2}\int_{\mathbf{\Omega}} \Lambda_{P,1} \quad \text{and} \quad \int_{\mathbf{S}} \Lambda_{P,2} = \int_{\mathbf{\Omega}} \Lambda_{P,2}.$$

Combining these relations with (2.2) applied to the function P, we obtain

$$\int_{\mathbf{\Omega}} \frac{\mathrm{i}}{2}\Lambda_{P,1} \leq \int_{\mathbf{\Omega}} \frac{\mathrm{i}}{2}\Lambda_{P,2}, \tag{2.4}$$

where, in terms of local coordinates,

$$\frac{\mathrm{i}}{2}\Lambda_{P,1} = \left|\partial_z R - \partial_z Q\right|^2 \mathrm{d}A(z), \quad \frac{\mathrm{i}}{2}\Lambda_{P,2} = \left|\bar{\partial}_z Q\right|^2 \mathrm{d}A(z). \tag{2.5}$$

Note that equality holds in (2.4) precisely when the complement $\mathbf{S} \setminus \mathbf{\Omega}$ has zero area.

In the next section, we consider more concrete choice of $\mathbf{S}$, $\mathbf{\Omega}$ and P, to derive from (2.4) area theorem type estimates for univalent functions.

Comparison with Nehari's results. As mentioned in the introduction, all the results of this section are from Nehari's papers [7, 8], except for the Lip $\frac{1}{2}$ estimate, which follows from the work of Beurling. Nevertheless, in some of the stated theorems, Nehari chooses to make assertions that are weaker than necessary. For instance, in Theorem III [7], the stated inequality (23) is equivalent to

$$0 \leq \langle S, S \rangle_{\mathbf{S}\setminus\mathbf{\Omega}} + \langle p - S, p - S \rangle_{\mathbf{\Omega}},$$

while our inequality (2.4)–(2.5) amounts to

$$0 \leq \langle S, S \rangle_{\mathbf{S}\setminus\mathbf{\Omega}},$$

which is a stronger assertion. The notation here is Nehari's; in our context, S is the real part of R, while p is the real part of Q. The inner product is the standard Dirichlet form: for a subdomain $\mathbf{D}$ of $\mathbf{S}$, and a function f on $\mathbf{D}$,

$$\langle f, f \rangle_{\mathbf{D}} = \int_{\mathbf{D}} \left|\nabla f(z)\right|^2 \mathrm{d}A(z),$$

if $\mathbf{D}$ has a global coordinate chart. If $\partial\mathbf{D}$ is smooth, then the Dirichlet form over the complement $\mathbf{S}\setminus\mathbf{D}$ is taken to mean the same as the Dirichlet form over the interior of $\mathbf{S}\setminus\mathbf{D}$.

3 Applications

The torus subdomain. For a covering surface $\mathbf{S}$ of the Riemann sphere $\mathbb{S}$, we denote by π the projection mapping of $\mathbf{S}$ onto $\mathbb{S}$.

Let $\mathbf{S}$ be the image of $\mathbb{S}$ under the mapping $z \mapsto z^2$. Thought of as a covering surface of $\mathbb{S}$, $\mathbf{S}$ is a two-sheeted covering, with associated projection $\pi : \mathbf{S} \to \mathbb{S}$. The covering has two branch points in $\mathbf{S}$, which we call $\mathbf{0}$ and $\boldsymbol{\infty}$. They project to the points 0 and ∞ : $\pi(\mathbf{0}) = 0$ and $\pi(\boldsymbol{\infty}) = \infty$.

We now describe a concrete domain $\boldsymbol{\Omega}$. Let $\varphi(w)$ be a univalent function, defined in the unit disk $\mathbb{D}$, which maps into $\mathbb{S}$, such that for some real parameter x_0, $0 < x_0 < 1$, we have

$$\varphi(x_0) = 0, \quad \varphi(-x_0) = \infty, \quad \varphi'(x_0) = 1.$$

We put $\Omega = \varphi(\mathbb{D})$ and note that Ω contains the points 0 and ∞. We use the notation ϕ for the inverse function to φ :

$$\phi = \varphi^{-1} : \Omega \to \mathbb{D}.$$

Denote by $\boldsymbol{\Omega}$ the lifting of Ω to $\mathbf{S}$, so that $\pi(\boldsymbol{\Omega}) = \Omega$. To get $\boldsymbol{\Omega}$, we should first cut Ω from 0 to ∞, then take two copies of such cut Ω, and attach them crosswise along the cuts.

The preimage of the cut from 0 to ∞ in Ω is a cut from x_0 to $-x_0$ in the unit disk $\mathbb{D}$. Attaching crosswise along these preimage cuts two replicas of "cut $\mathbb{D}$", we get a two-sheeted covering surface $\mathbf{D}$, which is conformally equivalent to $\boldsymbol{\Omega}$. The surface $\mathbf{D}$ has two branch points, which project to the points x_0 and $-x_0$ of the unit disk. We use the notation π also for the projection $\mathbf{D} \to \mathbb{D}$.

We need to define an analytic self-mapping $\mathbf{D} \to \mathbf{D}$. It is the correspondence $p \mapsto p'$ between the points p and p' belonging to the different sheets of $\mathbf{D}$. Namely, the point p with the projection $\pi(p) = z$, $z \in \mathbb{D} \setminus \{x_0, -x_0\}$, is mapped to another point $p' \in \mathbf{D}$ with the same projection $\pi(p') = z$. For p such that $\pi(p) = \pm x_0$, we put $p' = p$. We call p' *the mirror point* to the point p.

We now define the mapping $\boldsymbol{\varphi} : \mathbf{D} \to \boldsymbol{\Omega}$ to be the lifting of φ to $\mathbf{D}$. By definition, it maps the point $p \in \mathbf{D}$ to the point $\mathbf{p} \in \boldsymbol{\Omega}$ with projection $\varphi(\pi(p))$; to determine the lifting uniquely, we should specify that the top sheet of $\mathbf{D}$ is to be mapped onto the top sheet of $\mathbf{\Omega}$, and the same for the bottom sheets. We also need

$$\phi = \varphi^{-1} : \mathbf{\Omega} \to \mathbf{D},$$

which is the lifting of $\phi : \Omega \to \mathbb{D}$.

Further, denote by $\mathbf{p}'$ the point $\varphi(p')$ for $\mathbf{p} = \varphi(p) \in \mathbf{\Omega}$. We get an analytic self-mapping $\mathbf{\Omega} \to \mathbf{\Omega}$, which takes any point $\mathbf{p} \in \mathbf{\Omega} \setminus \{\mathbf{0}, \infty\}$ to the other point $\mathbf{p}' \in \mathbf{\Omega} \setminus \{\mathbf{0}, \infty\}$ with the same projection: $\pi(\mathbf{p}) = \pi(\mathbf{p}')$; each of the points $\mathbf{0}$, ∞, is taken to itself. As in the case of the points p, $p' \in \mathbf{D}$, we call $\mathbf{p}'$ *the mirror point* to the point $\mathbf{p}$.

Next, we introduce a meromorphic function $R(\mathbf{p})$, $\mathbf{p} \in \mathbf{S}$, which has a simple pole at the branch point $\mathbf{0}$ and has no other poles. Note that any meromorphic function f on our surface $\mathbf{S}$ can be expressed in terms of the global coordinates of $\mathbb{S} = \mathbb{C} \cup \{\infty\}$ as

$$f(z) = f_1(z) + \sqrt{z} f_2(z),$$

where f_1 and f_2 are meromorphic functions on $\mathbb{S}$, and $\sqrt{z}$ means the algebraic square root of z. *We define the function R to be the above f with the choices $f_1(z) = 0$, $f_2(z) = 1/z$.*

Our next project is to construct the function Q, which satisfies the conditions (Q1)–(Q3) of Proposition 2.2 for this given R. To this end, as a first step, we consider the Green function $G_{\mathbf{\Omega}}(\mathbf{p}, \mathbf{q})$ of the domain $\mathbf{\Omega}$. For fixed $\mathbf{q} \in \mathbf{\Omega}$, the function $\mathbf{p} \mapsto G_{\mathbf{\Omega}}(\mathbf{p}, \mathbf{q})$ is harmonic on $\mathbf{\Omega} \setminus \{\mathbf{q}\}$, vanishes on the boundary $\partial\mathbf{\Omega}$, and has the logarithmic singularity $-\log|z| + O(1)$ in terms of local coordinates around $\mathbf{p} = \mathbf{q}$. The function ϕ maps $\mathbf{\Omega}$ onto $\mathbf{D}$ conformally. It follows (see [10, Ch. 6, §2, pp. 201–202]) that

$$G_{\mathbf{\Omega}}(\mathbf{p}, \mathbf{q}) = G_{\mathbf{D}}(\phi(p), \phi(q)), \quad p, q \in \mathbf{D},\ \mathbf{p} = \phi(p),\ \mathbf{q} = \phi(q).$$

For $p, q \in \mathbf{D}$, define

$$\begin{aligned} G^{\mathrm{alt}}_{\mathbf{D}}(p, q) &= G_{\mathbf{D}}(p, q) - G_{\mathbf{D}}(p', q), \\ G^{\mathrm{alt}}_{\mathbf{\Omega}}(\mathbf{p}, \mathbf{q}) &= G^{\mathrm{alt}}_{\mathbf{D}}(\phi(p), \phi(q)), \quad \mathbf{p} = \phi(p),\ \mathbf{q} = \phi(q). \end{aligned}$$

It follows from the above definitions that

$$\begin{aligned} G^{\mathrm{alt}}_{\mathbf{\Omega}}(\mathbf{p}', \mathbf{q}) &= -G^{\mathrm{alt}}_{\mathbf{\Omega}}(\mathbf{p}, \mathbf{q}), \quad \mathbf{p},\ \mathbf{q} \in \mathbf{\Omega}, \\ G^{\mathrm{alt}}_{\mathbf{D}}(p', q) &= -G^{\mathrm{alt}}_{\mathbf{D}}(p, q), \quad p,\ q \in \mathbf{D}. \end{aligned} \tag{3.1}$$

The functions R and $R_{\mathbf{D}} = R \circ \varphi$ have the same property:

$$\begin{aligned} R(\mathbf{p}') &= -R(\mathbf{p}), \quad \mathbf{p},\ \mathbf{p}' \in \mathbf{S}, \\ R_{\mathbf{D}}(p') &= -R_{\mathbf{D}}(p), \quad p,\ p' \in \mathbf{D}. \end{aligned}$$

For our further considerations, we need yet another covering surface of $\mathbb{S}$. To obtain it, we first supply $\mathbb{S}$ with two cuts. One of the cuts is made in $\mathbb{D}$ from $-x_0$ to x_0 as we did earlier for the description of $\mathbf{D}$. The second cut goes from $-1/x_0$ to $1/x_0$. Also, this cut is to be obtained from the first one by reflection in the unit circle: $z \mapsto 1/\bar{z}$. Attaching two copies of such cut Riemann spheres crosswise along the corresponding (same) cuts, we obtain a compact surface, which we denote by $\mathbf{\Pi}$. It is a two-sheeted covering surface of $\mathbb{S}$ with four branch points. In terms of conformal equivalence, $\mathbf{\Pi}$ is a torus. As the second cut from $-1/x_0$ to $1/x_0$ falls outside the unit disk $\mathbb{D}$, we may think of the surface $\mathbf{D}$ as a subdomain of $\mathbf{\Pi}$.

For a moment, let us fix an arbitrary $q \in \mathbf{D}$. In addition to (3.1), $G_{\mathbf{D}}^{\mathrm{alt}}(p,q)$ has the following properties.

(1) $G_{\mathbf{D}}^{\mathrm{alt}}(p,q) = 0$, for $p \in \partial\mathbf{D}$.
(2) the function $p \mapsto G_{\mathbf{D}}^{\mathrm{alt}}(p,q)$ has the logarithmic singularity $-\log|z| + O(1)$ in terms of local coordinates around the point $p = q$ and the logarithmic singularity $\log|z| + O(1)$ in terms of local coordinates around the point $p = q'$, where q, q' are mirror points to each other (so that $q \neq q'$ and $\pi(q) = \pi(q') \in \mathbb{D}$).
(3) $G_{\mathbf{D}}^{\mathrm{alt}}(p,q)$ is harmonic on $\bar{\mathbf{D}} \setminus \{q\}$.

Next, we describe a self-mapping $\mathbf{\Pi} \to \mathbf{\Pi}$, *reflection in* $\partial\mathbf{D}$. Namely, this mapping takes the point p with the projection $\pi(p) = z$ to the point p^* with the projection $\pi(p^*) = 1/\bar{z}$. The choice of p^* from the two different points of $\mathbf{\Pi}$ with the same projection $1/\bar{z}$ is defined by the following requirements: our mapping must be continuous on $\mathbf{\Pi}$, and $p^* = p$ for all $p \in \partial\mathbf{D}$. The function $G_{\mathbf{D}}^{\mathrm{alt}}$ may be extended harmonically across the boundary $\partial\mathbf{D}$. Indeed, by the Schwarz reflection principle, for any fixed $q \in \mathbf{D}$, we define $G_{\mathbf{D}}^{\mathrm{alt}}(p,q)$ on the complement of $\mathbf{D}$ by

$$G_{\mathbf{D}}^{\mathrm{alt}}(p,q) = -G_{\mathbf{D}}^{\mathrm{alt}}(p^*,q), \quad p \in \mathbf{\Pi} \setminus \mathbf{D} \setminus \{q^*, (q')^*\}.$$

The extended function $p \mapsto G_{\mathbf{D}}^{\mathrm{alt}}(p,q)$ is harmonic on $\mathbf{\Pi} \setminus \{q, q', q^*, (q')^*\}$. It has the singularity $-\log|z| + O(1)$ in terms of local coordinates around the points $p = q$ and $p = (q')^*$, and the singularity $\log|z| + O(1)$ in terms of local coordinates around the points $p = q'$ and $p = q^*$.

Remark 3.1. The reason why we consider the Green functions G_Ω, $G_{\mathbf{D}}$ and the functions G_Ω^{alt}, $G_{\mathbf{D}}^{\mathrm{alt}}$, is explained by the following observation. Let Ω be a subdomain of $\mathbb{S}$ with analytic boundary such that Ω contains 0 and $w \in \Omega \iff -w \in \Omega$. Let

$$G_\Omega^{\mathrm{alt}}(w,\lambda) = G_\Omega(w,\lambda) - G_\Omega(-w,\lambda),$$

where G_Ω is the Green function of Ω. This function can be represented as

$$G_\Omega^{\mathrm{alt}}(z,\lambda) = -\frac{1}{2}\log|w-\lambda|^2 + \frac{1}{2}\log|w+\lambda|^2 + H(w,\lambda),$$

where $H(w,\lambda)$ is an odd harmonic function of the variable w. We observe that the function Q defined by

$$Q(w) = \partial_\lambda G_\Omega^{\mathrm{alt}}(w,\lambda)|_{\lambda=0} = \frac{1}{w} + \partial_\lambda H(w,\lambda)|_{\lambda=0}$$

is harmonic on $\bar{\Omega}\setminus\{0\}$, with a simple pole at the point $w=0$, and it vanishes on $\partial\Omega$. So, in the special case $\mathbf{S}=\mathbb{S}$, $R(z)=1/z$, we obtain the function Q of Proposition 2.2 from the function G_Ω^{alt} in the above manner.

We seek the required function $Q:\mathbf{S}\to\mathbb{S}$ for the given $R:\mathbf{S}\to\mathbb{S}$ in an analogous fashion. The theory of elliptic functions (or integrals) is needed to obtain the explicit form of $G_{\mathbf{D}}^{\mathrm{alt}}$.

Elliptic functions and the Green function for the torus subdomain. We recall some definitions and facts from the elliptic functions theory (see [1, Ch. V, VI]).

Let k be a real parameter, $0<k<1$. We introduce the following notation:

$$\begin{array}{llll} k'=\sqrt{1-k^2}, & l=\frac{1-k'}{1+k'}, & l'=\sqrt{1-l^2}, & M=\frac{1}{1+k'}, \\ K=K(k), & K'=K(k'), & L=K(l), & L'=K(l'), \end{array} \tag{3.2}$$

where the function $K(\lambda)$ is defined by (1.5). The values L, L', K, K' are connected by Landen's transformation (see [1, Ch. VI]), namely,

$$K=2ML,\quad K'=ML'. \tag{3.3}$$

Let $h=e^{-\pi K'/K}$. One of Jacobi's theta-functions $\vartheta_0(u)$ is defined by

$$\vartheta_0(u) = 1-2h\cos 2\pi u + 2h^4\cos 4\pi u - 2h^9\cos 6\pi u + \cdots,\quad u\in\mathbb{C}.$$

We also recall the definitions of the following Jacobi elliptic functions:

$$\theta_0(z)=\vartheta_0\left(\frac{z}{2K}\right),\qquad Z(z)=\frac{\theta_0'(z)}{\theta_0(z)}, \tag{3.4}$$

$$\operatorname{sn}(z;k)=\frac{\mathrm{i}e^{-\frac{\pi\mathrm{i}}{4K}(2z+\mathrm{i}K')}}{\sqrt{k'}}\,\frac{\theta_0(z-\mathrm{i}K')}{\theta_0(z)},\qquad \operatorname{dn}(z;k)=\sqrt{k'}\,\frac{\theta_0(z-K)}{\theta_0(z)},$$

$$\operatorname{cn}(z;k)=-\mathrm{i}e^{-\frac{\pi\mathrm{i}}{4K}(2z+\mathrm{i}K')}\sqrt{\frac{k'}{k}}\,\frac{\theta_0(z-K-\mathrm{i}K')}{\theta_0(z)}.$$

Let $\kappa = 2x_0/(1+x_0^2)$ for some real x_0, $0 < x_0 < 1$. The point x_0 is the same one we used to define the surfaces $\mathbf{D}$, $\boldsymbol{\Omega}$, $\boldsymbol{\Pi}$. In our further considerations, *we use the functions* $\theta_0(z)$, $Z(z)$, *which are defined with the parameter* $k = \kappa$. In addition, we require $\operatorname{sn}(z;\kappa)$, $\operatorname{cn}(z;\kappa)$, $\operatorname{dn}(z;\kappa)$, as well as $\operatorname{sn}(z;x_0^2)$, $\operatorname{cn}(z;x_0^2)$, $\operatorname{dn}(z;x_0^2)$; the argument x_0^2 appears because for $k=\kappa$, we have $l = x_0^2$.

The function $\theta_0(z)$ is entire and has simple zeros at the points

$$z_{m,n} = \mathrm{i}K' + 2mK + 2\mathrm{i}nK', \qquad \text{for } m,n \in \mathbb{Z};$$

likewise, $Z(z)$ is a meromorphic function with the simple poles at the points $z_{m,n}$, for $m,n \in \mathbb{Z}$. In addition, the functions θ_0 and Z are "almost" double-periodic:

$$\begin{aligned} \theta_0(z+2K) &= \theta_0(z), &\qquad \theta_0(z+2\mathrm{i}K') &= -h^{-1}e^{-\pi \mathrm{i} z/K}\theta_0(z),\\ Z(z+2K) &= Z(z), &\qquad Z(z+2\mathrm{i}K') &= Z(z) - \pi\mathrm{i}/K. \end{aligned}$$

We consider the rectangle

$$\mathcal{D} = \big\{ z \in \mathbb{C} :\ -2L < \operatorname{Re} z < 2L,\ -L' < \operatorname{Im} z < L' \big\},$$

and the analytic function

$$\boldsymbol{\sigma}(z) = x_0 \operatorname{sn}(z+L; x_0^2).$$

Let us introduce the following subrectangles of $\mathcal{D}$:

$$\begin{aligned} \mathcal{D}^- &= \big\{ z \in \mathbb{C} :\ -2L < \operatorname{Re} z < 0,\ -L' < \operatorname{Im} z < L' \big\},\\ \mathcal{D}^+ &= \big\{ z \in \mathbb{C} :\ 0 < \operatorname{Re} z < 2L,\ -L' < \operatorname{Im} z < L' \big\},\\ \mathcal{D}_1^- &= \big\{ z \in \mathbb{C} :\ -2L < \operatorname{Re} z < 0,\ -L' < \operatorname{Im} z < 0 \big\}\\ \mathcal{D}_0^- &= \big\{ z \in \mathbb{C} :\ -2L < \operatorname{Re} z < 0,\ 0 < \operatorname{Im} z < L' \big\},\\ \mathcal{D}_1^+ &= \big\{ z \in \mathbb{C} :\ 0 < \operatorname{Re} z < 2L,\ -L' < \operatorname{Im} z < 0 \big\}\\ \mathcal{D}_0^+ &= \big\{ z \in \mathbb{C} :\ 0 < \operatorname{Re} z < 2L,\ 0 < \operatorname{Im} z < L' \big\}. \end{aligned}$$

The function $\boldsymbol{\sigma}$ maps each of the rectangles $\mathcal{D}^-$ and $\mathcal{D}^+$ conformally onto the slit sphere

$$\mathbb{S} \setminus \Big( ]-\infty; -x_0] \cup [x_0; +\infty[\,\cup \{\infty\} \Big).$$

It is also known that $w = \boldsymbol{\sigma}(z)$ maps the closed rectangle $\bar{\mathcal{D}}$, with both pairs of opposite sides identified, conformally onto $\boldsymbol{\Pi}$ (see [1, Ch. VIII] or [9, Ch. VI, pp. 280–285]).

The inverse function of the restriction of $w = \boldsymbol{\sigma}(z)$ to $\mathcal{D}^-$ is given by the elliptic integral

$$z = \boldsymbol{\tau}(w) = \int_0^{w/x_0} \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-x_0^4 t^2)}} - L.$$

As a conformal mapping, $z = \tau(w)$ sends the upper half-plane

$$\mathbb{C}_+ = \{w \in \mathbb{C} : \operatorname{Im} w > 0\}$$

onto the rectangle $\mathcal{D}_0^-$ and the lower half-plane

$$\mathbb{C}_- = \{w \in \mathbb{C} : \operatorname{Im} w < 0\}$$

onto the rectangle $\mathcal{D}_1^-$ in such a way that

$$\tau(x_0) = 0, \qquad \tau(-x_0) = -2L, \qquad \tau(0) = -L,$$

$$\lim_{\mathbb{C}_+ \ni w \to 1/x_0} \tau(w) = \mathrm{i}L', \qquad \lim_{\mathbb{C}_- \ni w \to 1/x_0} \tau(w) = -\mathrm{i}L',$$

$$\lim_{\mathbb{C}_+ \ni w \to -1/x_0} \tau(w) = -2L + \mathrm{i}L', \qquad \lim_{\mathbb{C}_- \ni w \to -1/x_0} \tau(w) = -2L - \mathrm{i}L',$$

$$\lim_{\mathbb{C}_+ \ni w \to \infty} \tau(w) = -L + \mathrm{i}L', \qquad \lim_{\mathbb{C}_- \ni w \to \infty} \tau(w) = -L - \mathrm{i}L'.$$

The function $z = \tau(w)$ extends to an analytic function on

$$\mathbb{C}_+ \cup \mathbb{C}_- \cup \,]-x_0, x_0[.$$

Its restriction to the upper half plane $\mathbb{C}_+$ has an analytic continuation across the remaining segments

$$\mathbb{R} \cup \{\infty\} \setminus [-1/x_0, 1/x_0], \quad]-1/x_0, -x_0[, \quad]x_0, 1/x_0[,$$

and so does its restriction to the lower half plane $\mathbb{C}_-$. If we look carefully at these extensions, we find that the mapping $z = \tau(w)$ lifts to a conformal mapping $\mathbf{\Pi} \to \mathbb{C}/\mathbf{\Gamma}$, where $\mathbf{\Gamma}$ is the additive group generated by the elements $4L$ and $2\mathrm{i}L'$. We let $\mathcal{D}_{\text{fund}}$ denote the set $\mathcal{D}$ with the left vertical and the lower horizontal sides of this rectangle adjoined; then $\mathcal{D}_{\text{fund}}$ is a fundamental domain for $\mathbb{C}/\mathbf{\Gamma}$.

We need to understand the operations $p \mapsto p'$ and $p \mapsto p^*$ on $\mathbf{\Pi}$ in terms of this identification of $\mathbf{\Pi}$ with $\mathbb{C}/\mathbf{\Gamma}$. It is easy to see that the mirror mapping $p \mapsto p'$ corresponds to $z \mapsto -z$ on $\mathbb{C}/\mathbf{\Gamma}$. Also, the mapping $p \mapsto p^*$ of reflection in $\partial\mathbf{D}$ corresponds to $z \mapsto z^*$, where z^* is the reflected point in the line $\frac{\mathrm{i}}{2}L' + \mathbb{R}$ (modulo $\mathbf{\Gamma}$). This latter fact is perhaps not entirely obvious. To see that it is nevertheless so, pick a point $z \in \mathcal{D}_{\text{fund}}$. We have

$$\sigma(z) = w, \qquad \operatorname{sn}(z + L; x_0^2) = w/x_0.$$

Using the relation (see [1, table XII])

$$\operatorname{sn}(u + \mathrm{i}L'; x_0^2) = 1/\big(x_0^2 \operatorname{sn}(u; x_0^2)\big),$$

we find that

$$\operatorname{sn}(\bar{z}+L+\mathrm{i}L';x_0^2)=1/\big(x_0^2\operatorname{sn}(\bar{z}+L;x_0^2)\big)=1/(x_0\bar{w}),$$

so that

$$x_0\operatorname{sn}((\bar{z}+\mathrm{i}L')+L;x_0^2)=w^*.$$

We realize that

$$z^*=\bar{z}+\mathrm{i}L',$$

which is the formula expressing reflection in the line $\frac{\mathrm{i}}{2}L'+\mathbb{R}$.

Finally, we obtain a description of the image $\boldsymbol{\tau}(\mathbf{D})$: *it is the rectangle*

$$\mathcal{D}_{\mathbf{D}}=\big\{z\in\mathbb{C}:\ -2L\le\operatorname{Re}z<2L,\ \ |\operatorname{Im}z|<L'/2\big\}.$$

The image of $\partial\mathbf{D}$ consists of the two horizontal line segments

$$\gamma_\pm=\big\{-2L\le\operatorname{Re}z<2L,\ \operatorname{Im}z=\pm L'/2\big\}.$$

For $(z,\zeta)\in\mathbb{C}\times\mathbb{C}$, we define the function $G(z,\zeta)$ by

$$G(z,\zeta)=-\frac{1}{2}\log\left|\frac{\theta_0(Mz-M\zeta+\mathrm{i}K')\theta_0(M\bar{z}+M\zeta)}{\theta_0(Mz+M\zeta-\mathrm{i}K')\theta_0(M\bar{z}-M\zeta)}\right|^2-\frac{\pi M}{K}\left[\frac{2M}{K'}\operatorname{Im}\zeta-1\right]\operatorname{Im}z,$$

where the function θ_0 is given by (3.4); here, we think of log as taking values in $[-\infty;+\infty]$.

From the properties of the function θ_0, and (3.3), we see easily that $G(z,\zeta)$ has the following properties:

1° $G(z,\zeta)=G(\zeta,z)$;
2° the function $z\mapsto G(z,\zeta)$ is periodic with respect to the group $\boldsymbol{\Gamma}$, making it a function on $\mathbb{C}/\boldsymbol{\Gamma}$;
3° for a fixed $\zeta\in\mathcal{D}_{\mathbf{D}}$, the function $z\mapsto G(z,\zeta)$ is harmonic in the variable z in the domain $\mathcal{D}_{\mathbf{D}}\setminus\{\zeta,-\zeta\}$, it has the logarithmic singularities $\log|z-\zeta|+O(1)$ near $z=\zeta$ and $-\log|z+\zeta|+O(1)$ near $z=-\zeta$;
4° $G(z,\zeta)=0$ for $z\in\gamma_+\cup\gamma_-$;
5° $G(-z,\zeta)=-G(z,\zeta)$.

The property 2° means that $G(\boldsymbol{\tau}(p),\boldsymbol{\tau}(q))$ is a function on $\boldsymbol{\Pi}\times\boldsymbol{\Pi}$. From the above properties of G, it also follows that $G(\boldsymbol{\tau}(p),\boldsymbol{\tau}(q))$ coincides with the previously considered function $G_{\mathbf{D}}^{\mathrm{alt}}(p,q)$:

$$G_{\mathbf{D}}^{\mathrm{alt}}(\boldsymbol{\sigma}(z),\boldsymbol{\sigma}(\zeta))\equiv G(z,\zeta),\quad (z,\zeta)\in\mathbb{C}/\boldsymbol{\Gamma}\times\mathbb{C}/\boldsymbol{\Gamma}.\tag{3.5}$$

We denote by $\boldsymbol{\mathcal{D}}_{\mathrm{D}}$ the subdomain of the torus $\mathbb{C}/\boldsymbol{\Gamma}$ whose restriction to the fundamental domain $\mathcal{D}_{\mathrm{fund}}$ is the subrectangle $\mathcal{D}_{\mathrm{D}}$ and by $G_{\boldsymbol{\mathcal{D}}_{\mathrm{D}}}(z,\zeta)$ the Green function of this subdomain. Then, the relation (3.5) is equivalent to

$$G(z,\zeta) = G^{\mathrm{alt}}_{\boldsymbol{\mathcal{D}}_{\mathrm{D}}}(z,\zeta) = G_{\boldsymbol{\mathcal{D}}_{\mathrm{D}}}(z,\zeta) - G_{\boldsymbol{\mathcal{D}}_{\mathrm{D}}}(-z,\zeta), \quad (z,\zeta) \in \boldsymbol{\mathcal{D}}_{\mathrm{D}} \times \boldsymbol{\mathcal{D}}_{\mathrm{D}}.$$

Let us consider the function

$$(3.6)\ Q_{\mathrm{D}}(z) = \partial_\zeta G(z,\zeta)|_{\zeta=0} = MZ(Mz+\mathrm{i}K') - MZ(M\bar{z}) + \frac{\pi\mathrm{i}M}{KK'}\mathrm{Im}\,(Mz) + \frac{\pi\mathrm{i}M}{2K},$$

where Z is Jacobi Z-function (see (3.4)). The above properties of $G(z,\zeta)$ imply that $Q_{\mathrm{D}}(z)$ has the following properties:

(1) it is a periodic function with respect to $\boldsymbol{\Gamma}$, so that $Q_{\mathrm{D}}(z)$ is a function on $\mathbb{C}/\boldsymbol{\Gamma}$;
(2) Q_{D} is harmonic on $\mathcal{D}_{\mathrm{D}}\setminus\{0\}$, and has the singularity $1/z + O(1)$ at the point 0;
(3) $Q_{\mathrm{D}}(z) = 0$ for $z \in \gamma_+ \cup \gamma_-$;
(4) $Q_{\mathrm{D}}(-z) = -Q_{\mathrm{D}}(z)$, for $z \in \mathbb{C}$.

Put

$$Q_1(\mathbf{p}) \equiv \big(Q_{\mathrm{D}} \circ \boldsymbol{\tau} \circ \boldsymbol{\phi}\big)(\mathbf{p}), \quad \mathbf{p} \in \boldsymbol{\Omega}.$$

This function satisfies the conditions (Q1) and (Q2) of Proposition 2.2. Also, it has the singularity

$$1/\boldsymbol{\tau}(\boldsymbol{\phi}(z^2)) + O(1) \sim b/z + O(1)$$

at the point $\mathbf{0} \in \boldsymbol{\Omega}$; here,

(3.7)
$$\begin{aligned} b = \lim_{z\to 0} \frac{z}{\boldsymbol{\tau}(\boldsymbol{\phi}(z^2))} &= \lim_{w\to x_0} \frac{\sqrt{\varphi(w)}}{\boldsymbol{\tau}(w)} \\ &= \lim_{w\to x_0} \frac{\sqrt{w - x_0}}{\boldsymbol{\tau}(w)} = -\lim_{w\to x_0} \frac{(\sqrt{w-x_0})'_w}{\boldsymbol{\tau}'(w)} = \frac{\mathrm{i}}{\sqrt{2}}\sqrt{x_0(1-x_0^4)}. \end{aligned}$$

In view of the above, it follows that

$$Q(\mathbf{p}) = \frac{1}{b}\,Q_1(\mathbf{p}) = \frac{1}{b}\,\big(Q_{\mathrm{D}} \circ \boldsymbol{\tau} \circ \boldsymbol{\phi}\big)(\mathbf{p})$$

is exactly the function we are looking for.

The area-theorem type inequality for univalent function on $\mathbb{D}$. We now write down the inequality (2.4) for the function

$$P(z) = (R \circ \boldsymbol{\varphi} \circ \boldsymbol{\sigma})(z) - \frac{1}{b}\,Q_{\mathrm{D}}(z), \quad z \in \mathcal{D}_{\mathrm{D}}.$$

Recalling the definition of the function R, we see that

$$(R \circ \varphi \circ \boldsymbol{\sigma})(z) = 1/\sqrt{\varphi(\boldsymbol{\sigma}(z))},$$

where $\sqrt{u}$ means the algebraic square root of u. Then, for our choice of P, (2.4) assumes the form

$$\int_{\mathcal{D}_{\mathrm{D}}} \left| \frac{\varphi'(\boldsymbol{\sigma}(z))\boldsymbol{\sigma}'(z)}{2\left[\varphi(\boldsymbol{\sigma}(z))\right]^{3/2}} + \frac{1}{b}\,\partial_z Q_{\mathrm{D}}(z)\right|^2 \mathrm{d}A(z) \leq \frac{1}{|b|^2}\int_{\mathcal{D}_{\mathrm{D}}} \left|\bar{\partial}_z Q_{\mathrm{D}}(z)\right|^2 \mathrm{d}A(z); \tag{3.8}$$

here, as usual, $\mathrm{d}A(z)$ is the area element, and the constant b is as in (3.7).

We intend to simplify the inequality (3.8). First, we evaluate the right-hand side of (3.8). Recall that

$$Q_{\mathrm{D}}(z) = \partial_\zeta G(z,\zeta)\big|_{\zeta=0} = \partial_\zeta \left\{ G_{\mathcal{D}_{\mathrm{D}}}(z,\zeta) - G_{\mathcal{D}_{\mathrm{D}}}(-z,\zeta)\right\}\Big|_{\zeta=0},$$

so that

$$\bar{\partial}_z Q_{\mathrm{D}}(z) = \left\{ \bar{\partial}_z \partial_\zeta G_{\mathcal{D}_{\mathrm{D}}}(z,\zeta) + \bar{\partial}_z \partial_\zeta G_{\mathcal{D}_{\mathrm{D}}}(-z,\zeta)\right\}\Big|_{\zeta=0}.$$

The kernel

$$K_{\mathcal{D}_{\mathrm{D}}}(z,\zeta) = -\frac{2}{\pi}\,\partial_z \bar{\partial}_\zeta\, G_{\mathcal{D}_{\mathrm{D}}}(z,\zeta), \quad z \neq \zeta,$$

has the following reproducing property: *for any analytic function $f \in L^2(\mathcal{D}_{\mathrm{D}})$,*

$$f(\zeta) = \int_{\mathcal{D}_{\mathrm{D}}} f(z) \bar{K}_{\mathcal{D}_{\mathrm{D}}}(z,\zeta)\,\mathrm{d}A(z), \quad \zeta \in \mathcal{D}_{\mathrm{D}}.$$

In particular, taking into account that the function $z \mapsto K_{\mathcal{D}_{\mathrm{D}}}(z,\zeta)$ is analytic and bounded near the point $z = \zeta$, we have

$$\int_{\mathcal{D}_{\mathrm{D}}} |K_{\mathcal{D}_{\mathrm{D}}}(z,\zeta)|^2 \mathrm{d}A(z) = K_{\mathcal{D}_{\mathrm{D}}}(\zeta,\zeta).$$

From the above, it follows that

$$\begin{aligned} \int_{\mathcal{D}_{\mathrm{D}}} \left|\bar{\partial}_z Q_{\mathrm{D}}(z)\right|^2 \mathrm{d}A(z) &= \frac{\pi^2}{4}\int_{\mathcal{D}_{\mathrm{D}}} \left|\bar{K}_{\mathcal{D}_{\mathrm{D}}}(z,0) + \bar{K}_{\mathcal{D}_{\mathrm{D}}}(-z,0)\right|^2 \mathrm{d}A(z) \\ &= \pi^2 K_{\mathcal{D}_{\mathrm{D}}}(0,0) = -2\pi\,\partial_z\bar{\partial}_\zeta\, G_{\mathcal{D}_{\mathrm{D}}}(0,0) \\ &= -\pi\,\overline{\bar{\partial}_z\, Q_{\mathrm{D}}(0)} = -\pi\,\partial_z\,\bar{Q}_{\mathrm{D}}(0). \end{aligned} \tag{3.9}$$

The calculations of the right-hand side of (3.8) can be completed by using the following facts from the elliptic functions theory (see [1, Ch.V]):

$$Z'(u) = \left[\mathrm{dn}\,(u;\kappa)\right]^2 - E/K, \tag{3.10}$$

$$\mathrm{dn}\,(0;\kappa) = 1,$$

$$EK' + E'K - KK' = \pi/2, \tag{3.11}$$

where $E = E(\kappa)$, $E' = E(\kappa')$, $K = K(\kappa)$, and $K' = K(\kappa')$ (see equations (1.4), (1.5), and (3.2)). In view of (3.6), we have

$$\bar{\partial}_z Q_{\mathbf{D}}(z) = M^2 \left(-\left[\mathrm{dn}\,(M\bar{z};\kappa)\right]^2 + \frac{E}{K} - \frac{\pi}{2KK'} \right), \tag{3.12}$$

so that

$$\bar{\partial}_z Q_{\mathbf{D}}(0) = -M^2 E'/K',$$

which is a real number. We get, by (3.9),

$$\frac{1}{|b|^2} \int_{\mathcal{D}_{\mathbf{D}}} \left|\bar{\partial}_z Q_{\mathbf{D}}(z)\right|^2 \mathrm{d}A(z) = \frac{\pi M^2 E'}{|b|^2 K'} = \frac{\pi(1+x_0^2)E'}{2x_0(1-x_0^2)K'}. \tag{3.13}$$

Finally, the inequality (3.8) becomes

$$\int_{\mathcal{D}_{\mathbf{D}}} \left| -\frac{\varphi'(\boldsymbol{\sigma}(z))\boldsymbol{\sigma}'(z)}{2\left[\varphi(\boldsymbol{\sigma}(z))\right]^{3/2}} - \frac{1}{b}\bar{\partial}_z Q_{\mathbf{D}}(z) \right|^2 \mathrm{d}A(z) \le \frac{\pi M^2 E'}{|b|^2 K'} = \frac{\pi(1+x_0^2)E'}{2x_0(1-x_0^2)K'}. \tag{3.14}$$

A pointwise estimate. Put

$$\boldsymbol{\Psi}(z) = -\frac{\varphi'(\boldsymbol{\sigma}(z))\boldsymbol{\sigma}'(z)}{2\left[\varphi(\boldsymbol{\sigma}(z))\right]^{3/2}} - \frac{1}{b}\,\bar{\partial}_z Q_{\mathbf{D}}(z), \qquad z \in \mathcal{D}_{\mathbf{D}};$$

the inequality (3.14) now takes form

$$\int_{\mathcal{D}_{\mathbf{D}}} |\boldsymbol{\Psi}(z)|^2 \,\mathrm{d}A(z) \le \frac{\pi M^2 E'}{|b|^2 K'}. \tag{3.15}$$

By (3.13), (3.15), and the reproducing property of the function

$$-\frac{1}{\pi}\overline{\bar{\partial}_z Q_{\mathbf{D}}(z)} = \frac{1}{2}\left(K_{\mathcal{D}_{\mathbf{D}}}(z,0) + K_{\mathcal{D}_{\mathbf{D}}}(-z,0)\right),$$

we have by the Cauchy–Schwarz inequality

$$\begin{aligned} |\boldsymbol{\Psi}(0)|^2 &= \left| \int_{\mathcal{D}_{\mathbf{D}}} \boldsymbol{\Psi}(z) \left[-\frac{1}{\pi}\,\bar{\partial}_z Q_{\mathbf{D}}(z) \right] \mathrm{d}A(z) \right|^2 \\ &\le \frac{1}{\pi^2} \int_{\mathcal{D}_{\mathbf{D}}} |\boldsymbol{\Psi}(z)|^2 \mathrm{d}A(z) \int_{\mathcal{D}_{\mathbf{D}}} \left|\partial_z \bar{Q}_{\mathbf{D}}(z)\right|^2 \mathrm{d}A(z) \le \frac{M^4}{|b|^2} \left(\frac{E'}{K'}\right)^2, \end{aligned}$$

whence

$$|\boldsymbol{\Psi}(0)| \le \frac{M^2}{|b|} \cdot \frac{E'}{K'}. \tag{3.16}$$

Below, we demonstrate that this inequality is equivalent to the estimate (1.3) of Goluzin for the class Σ.

Rewriting the area-type inequality in the coordinates of the unit disk. We first rewrite the left-hand side of (3.14) as an integral over $\mathbf{D}$ rather than $\mathcal{D}_{\mathbf{D}}$:

$$\begin{aligned}&\int_{\mathcal{D}_{\mathbf{D}}}\left|-\frac{\varphi'(\boldsymbol{\sigma}(z))\boldsymbol{\sigma}'(z)}{2\big[\varphi(\boldsymbol{\sigma}(z))\big]^{3/2}}-\frac{1}{b}\partial_z Q_{\mathbf{D}}(z)\right|^2\mathrm{d}A(z)\\ &\quad=\int_{\mathbf{D}}\left|-\frac{\varphi'(w)}{2\big[\varphi(w)\big]^{3/2}}-\frac{1}{b}\,\partial_z Q_{\mathbf{D}}(z)\Big|_{z=\boldsymbol{\tau}(w)}\boldsymbol{\tau}'(w)\right|^2\mathrm{d}A(w);\end{aligned}\tag{3.17}$$

here, the area measure $\mathrm{d}A$ is implicitly lifted from $\mathbb{D}$ to $\mathbf{D}$. From (3.10), (3.11) and the following relations between Jacobi elliptic functions ([1, table XII]),

$$\mathrm{dn}\,(u+\mathrm{i}K';\kappa)=-\mathrm{i}\frac{\mathrm{cn}\,(u;\kappa)}{\mathrm{sn}\,(u;\kappa)},$$
$$\big[\mathrm{sn}\,(u;\kappa)\big]^2+\big[\mathrm{cn}\,(u;\kappa)\big]^2=1,$$

we obtain, in view of (3.6),

$$\partial_z Q_{\mathbf{D}}(z)=M^2\left[-\frac{1}{\big[\mathrm{sn}\,(Mz;\kappa)\big]^2}+\frac{E'}{K'}\right].$$

Note that the expression

$$\big[\mathrm{sn}\,(Mz;\kappa)\big]^2\Big|_{z=\boldsymbol{\tau}(w)}$$

can be simplified by using Landen's transformation of Jacobi functions ([1, Ch.VI]). This transformation allows us to express $\big[\mathrm{sn}\,(Mz;\kappa)\big]^2$ as a function of the expression

$$\xi(z)=\frac{\mathrm{cn}\,(z;x_0^2)}{\mathrm{dn}\,(z;x_0^2)}.$$

We have

$$\mathrm{cn}\,(z;x_0^2)=\frac{1-(1+\kappa')\big[\mathrm{sn}\,(Mz;\kappa)\big]^2}{\mathrm{dn}\,(Mz;\kappa)},\quad \mathrm{dn}\,(z;x_0^2)=\frac{1-(1-\kappa')\big[\mathrm{sn}\,(Mz;\kappa)\big]^2}{\mathrm{dn}\,(Mz;\kappa)}.$$

From these formulas, we find that

$$\big[\mathrm{sn}\,(Mz;\kappa)\big]^2=\frac{1-\xi(z)}{1+\kappa'-(1-\kappa')\xi(z)}.$$

Further, taking into account the relation

$$\mathrm{sn}\,(z+L;x_0^2)=\frac{\mathrm{cn}\,(z;x_0^2)}{\mathrm{dn}\,(z;x_0^2)},$$

we conclude that $w=\xi(z)$ is the inverse function to

$$w \mapsto \int_0^w \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-x_0^4t^2)}}.$$

From the above, we obtain

$$\big[\mathrm{sn}\,(Mz;\kappa)\big]^2\big|_{z=\tau(w)} = \frac{1-w/x_0}{1+\kappa'-(1-\kappa')w/x_0} = \frac{(1+x_0^2)(w-x_0)}{2x_0(x_0w-1)}.$$

Finally, we arrive at

$$\partial_z Q_{\mathbf{D}}(z)\Big|_{z=\tau(w)} = \frac{x_0(1+x_0^2)}{2}\frac{1-x_0w}{w-x_0} + \frac{(1+x_0^2)^2E'}{4K'}, \qquad w\in\mathbf{D}.$$

Note that the above expression is a well-defined function on $\mathbb{D}$. Substituting the last expression as well as

$$\tau'(w) = \frac{1}{\mathrm{i}\sqrt{(w^2-x_0^2)(1-x_0^2w^2)}}$$

into the right-hand side integral of (3.17), we get

$$\begin{aligned}
&\int_{\mathbf{D}}\left|-\frac{\varphi'(w)}{2\big[\varphi(w)\big]^{3/2}} - \frac{1}{b}\partial_z Q_{\mathbf{D}}(z)\Big|_{z=\tau(w)}\tau'(w)\right|^2 \mathrm{d}A(w)\\
&= \int_{\mathbf{D}}\left|\frac{\varphi'(w)}{2\big[\varphi(w)\big]^{3/2}} - \frac{(1+x_0^2)\sqrt{x_0}}{\sqrt{2(1-x_0^4)}}\sqrt{\frac{1-x_0w}{1+x_0w}}\frac{1}{\sqrt{w+x_0}}\frac{1}{(w-x_0)^{3/2}}\right.\\
&\qquad\left. - \frac{(1+x_0^2)^2}{2\sqrt{2x_0(1-x_0^4)}}\frac{E'}{K'}\frac{1}{\sqrt{(w+x_0)(1-x_0^2w^2)}}\frac{1}{(w-x_0)^{1/2}}\right|^2 \mathrm{d}A(w).
\end{aligned}$$

As we multiply by $\sqrt{w^2-x_0^2}$ inside the absolute value signs of the integral and divide by $|w^2-x_0^2|$ outside them, which permits us to integrate over $\mathbb{D}$ instead of over the covering surface $\mathbf{D}$, we realize that we have derived the following from (3.14).

Proposition 3.2. *Let $\varphi:\mathbb{D}\to\mathbb{S}$ be a univalent function with the property that for some real x_0, $0<x_0<1$, we have $\varphi(x_0)=0$, $\varphi(-x_0)=\infty$, and $\varphi'(x_0)=1$. Then*

$$\begin{aligned}
\text{(3.18)}\quad &\int_{\mathbb{D}}\left|\frac{\varphi'(w)\sqrt{w^2-x_0^2}}{\big[\varphi(w)\big]^{3/2}} - \frac{(1+x_0^2)\sqrt{2x_0}}{\sqrt{(1-x_0^4)}}\sqrt{\frac{1-x_0w}{1+x_0w}}\frac{1}{w-x_0}\right.\\
&\qquad\left. - \frac{E'}{K'}\frac{(1+x_0^2)^2}{\sqrt{2x_0(1-x_0^4)}}\frac{1}{\sqrt{1-x_0^2w^2}}\right|^2\frac{\mathrm{d}A(w)}{|w^2-x_0^2|} \le \frac{\pi E'}{K'}\frac{1+x_0^2}{x_0(1-x_0^2)},
\end{aligned}$$

where $E' = E\big((1-x_0^2)/(1+x_0^2)\big)$, $K' = K\big((1-x_0^2)/(1+x_0^2)\big)$ *and the functions* $E(\lambda)$, $K(\lambda)$ *are defined by* (1.4) *and* (1.5). *Equality is attained in* (3.18) *if and only if* φ *is a full mapping.*

The corresponding estimates the class Σ. Let $\psi(z) = z + b_0 + b_1 z^{-1} + \cdots$ be an element of the class Σ. Fix a point $\zeta \in \mathbb{D}_e \setminus \{\infty\}$. Then

$$x_0 = \frac{1-\sqrt{1-|\zeta|^{-2}}}{1+\sqrt{1-|\zeta|^{-2}}} \tag{3.19}$$

satisfies $0 < x_0 < 1$, and we have the inverse relation

$$|\zeta| = \frac{1+x_0^2}{2x_0}.$$

The mapping

$$\eta(z) = \frac{|\zeta| - x_0\bar{\zeta}z}{\bar{\zeta}z - x_0|\zeta|}$$

maps $\mathbb{D}_e$ onto $\mathbb{D}$ conformally and takes ∞ to $-x_0$, while ζ is mapped to x_0. The inverse mapping is

$$\eta^{-1}(w) = \frac{\zeta}{|\zeta|}\,\frac{1+x_0 w}{w+x_0}.$$

Consider the related function

$$\varphi(w) = \frac{|\zeta|}{\zeta}\,\frac{(1+x_0^2)^2}{1-x_0^2}\,\frac{\psi(\eta^{-1}(w)) - \psi(\zeta)}{\zeta^2\,\psi'(\zeta)}, \tag{3.20}$$

which is univalent on $\mathbb{D}$ with $\varphi(-x_0) = \infty$, $\varphi(x_0) = 0$, $\varphi'(x_0) = 1$.

Substituting (3.20) into (3.18) and making the change of variable $w = \eta(z)$, we obtain, after some simplification, the corresponding inequality for ψ. We write it down in the following form.

Theorem 3.3 (The area-type estimate). *Fix a point* $\zeta \in \mathbb{D}_e \setminus \{\infty\}$. *Then, for any* $\psi \in \Sigma$,

$$\begin{aligned}\int_{\mathbb{D}_e}\Bigg|\left(\frac{\psi'(\zeta)(z-\zeta)}{\psi(z)-\psi(\zeta)}\right)^{1/2}\frac{\psi'(z)}{\psi(z)-\psi(\zeta)} - \left(\frac{1-(\bar{\zeta}z)^{-1}}{1-|\zeta|^{-2}}\right)^{1/2}\frac{1}{z-\zeta}\\ +\frac{E'}{K'}\,\frac{1}{\big[(1-|\zeta|^{-2})(1-(\bar{\zeta}z)^{-1})\big]^{1/2}\,z}\Bigg|^2\frac{\mathrm{d}A(z)}{|z-\zeta|} \le \frac{2\pi E'}{K'}\,\frac{|\zeta|}{|\zeta|^2-1},\end{aligned} \tag{3.21}$$

where $E' = E\big(\sqrt{1-|\zeta|^{-2}}\big)$, $K' = K\big(\sqrt{1-|\zeta|^{-2}}\big)$ *and the functions* $E(\lambda)$, $K(\lambda)$ *are defined by* (1.4) *and* (1.5). *Equality holds if and only if* ψ *is a full mapping.*

The derivation of Goluzin's inequality from the area-type estimate. Put

$$\Psi(z,\zeta) = \left(\frac{\psi'(\zeta)(z-\zeta)}{\psi(z)-\psi(\zeta)}\right)^{1/2} \frac{\psi'(z)}{\psi(z)-\psi(\zeta)} - \left(\frac{1-(\bar\zeta z)^{-1}}{1-|\zeta|^{-2}}\right)^{1/2} \frac{1}{z-\zeta}$$
$$+ \frac{E'}{K'} \frac{1}{\left[(1-|\zeta|^{-2})(1-(\bar\zeta z)^{-1})\right]^{1/2} z}.$$

As we recall how the inequality (3.15) containing the function Ψ is transformed into (3.21) involving the analogous function Ψ, we find that

$$|\Psi(\zeta,\zeta)| = \frac{x_0^{3/2}\sqrt{1-x_0^4}}{\sqrt{2}} \frac{|\zeta|^2}{|\zeta|^2-1} |\Psi(0)|.$$

In view of (3.16), we then have

$$|\Psi(\zeta,\zeta)| \le \frac{M^2 E'}{\sqrt{2}\,|b|\,K'} x_0^{3/2}\sqrt{1-x_0^4}\, \frac{|\zeta|^2}{|\zeta|^2-1},$$

where x_0 is given in terms of $|\zeta|$ by (3.19). Substituting the expressions for the constants M and b (see (3.2) and (3.7)) and simplifying further, we obtain the estimate

$$|\Psi(\zeta,\zeta)| \le \frac{E'}{K'} \frac{|\zeta|}{|\zeta|^2-1}. \tag{3.22}$$

On the other hand, a direct calculation yields

$$\Psi(\zeta,\zeta) = \frac{\psi''(\zeta)}{4\psi'(\zeta)} - \frac{1}{2\zeta} - \frac{2-|\zeta|^2}{2\,(|\zeta|^2-1)\,\zeta} + \frac{E'}{K'} \frac{|\zeta|^2}{(|\zeta|^2-1)\,\zeta}.$$

The inequality (3.22) thus takes the form

$$\left|\frac{\zeta\,\psi''(\zeta)}{\psi'(\zeta)} - 2 + \frac{2(|\zeta|^2-2)}{|\zeta|^2-1} + \frac{4E'}{K'} \frac{|\zeta|^2}{(|\zeta|^2-1)}\right| \le \frac{E'}{K'} \frac{4|\zeta|^2}{|\zeta|^2-1}. \tag{3.23}$$

From (3.11), we have

$$\frac{E'}{K'} = 1 - \frac{E}{K} + \frac{\pi}{2KK'},$$

which, together with (3.23), leads to

$$\left|\frac{\zeta\,\psi''(\zeta)}{\psi'(\zeta)} + \frac{4|\zeta|^2-2}{|\zeta|^2-1} - \frac{4|\zeta|^2}{|\zeta|^2-1} \frac{E\,(1/|\zeta|)}{K\,(1/|\zeta|)}\right| \le \frac{4|\zeta|^2}{|\zeta|^2-1}\left(1 - \frac{E\,(1/|\zeta|)}{K\,(1/|\zeta|)}\right)$$

after some simplification; here, the functions $E(\lambda)$, $K(\lambda)$ are defined by (1.4) and (1.5). This is the classical inequality due to Goluzin (see [5], [6, Ch. IV, §3, p. 132]), and if we divide by ζ inside the absolute value signs, we arrive at (1.3).

Remark 3.4. To find the extremal $\psi \in \Sigma$ which gives equality in Goluzin's inequality (1.3) at a given point $z \in \mathbb{D}_e$, we should just check when we have equality in the Cauchy–Schwarz inequality leading up to (3.16). Of course, the result of this exercise of course agrees with Goluzin's findings.

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