Алгебра и анализ
Том 12 (2000), вып. 1

St. Petersburg Math. J.
Vol. 12 (2001), No. 1

INVARIANT SUBSPACES IN QUASI-BANACH SPACES OF ANALYTIC FUNCTIONS

A. ABKAR AND H. HEDENMALM

ABSTRACT. Let X be a quasi-Banach space of analytic functions on a finitely connected bounded domain Ω on the complex plane. We prove a theorem that reduces the study of the hyperinvariant subspaces of X to that of the hyperinvariant subspaces of X_1, where X_1 is a quasi-Banach space of analytic functions on a domain Ω_1 obtained from Ω by adding some of the bounded connectivity components of $\mathbb{C} \setminus \Omega$. In particular, the lattice structure (incident to the hyperinvariant subspaces) of a quasi-Banach space X of analytic functions on the annulus $\{ z \in \mathbb{C} : \rho < |z| < 1 \}$, $0 < \rho < 1$, is understood in terms of the lattice structure of the space X_1, the counterpart of X for the unit disk.

§0. INTRODUCTION

A *topological vector space* over the complex field $\mathbb{C}$ is a vector space equipped with a Hausdorff topology with respect to which the vector space operations, addition and multiplication by scalars, are continuous. A subset E of a topological vector space X is said to be *bounded* if for every neighborhood U of the origin in X there is a positive number t such that $E \subset tU$. A topological vector space X is said to be *locally bounded* if the origin in X has a bounded neighborhood. In this case, the family $\{\frac{1}{n}U\}_{n=1}^{\infty}$ forms a countable basis of neighborhoods of the origin in X, which is equivalent to saying that the topological vector space X is metrizable. We recall that a topological vector space X is *metrizable* if there exists an invariant metric on X that induces the topology (see [11, p. 163] or [15, pp. 17–18]). Note that in any topological vector space the notions of a *Cauchy sequence*, *convergence*, and *completeness* can be defined without reference to any metric (see [15, p. 20]). Now, suppose that X is a topological vector space whose topology τ is compatible with an invariant metric d (both τ and d have the same open sets). It is well known that a sequence $\{x_n\}_{n=1}^{\infty}$ in X is a Cauchy sequence with respect to τ if and only if it is a Cauchy sequence with respect to d [15, p. 20]. By a *complete locally bounded space* we mean a locally bounded space complete with respect to its original vector topology, or equivalently, with respect to an invariant metric compatible with the original topology.

A topological vector space X is called an *F-space* if its topology is induced by a complete invariant metric. In particular, any complete locally bounded space is an F-space. It is well known [11, p. 163] that the topology of an F-space is induced by an F-norm; an F-norm is a nonnegative real-valued function $\|\cdot\|$ satisfying the following conditions (x and y are arbitrary elements of X):

(F1) $\|x\| \geq 0$ always, and $\|x\| = 0$ if and only if $x = 0$;
(F2) $\|\lambda x\| \leq \|x\|$ for every complex number λ with $|\lambda| \leq 1$;
(F3) $\|x + y\| \leq \|x\| + \|y\|$;

2000 *Mathematics Subject Classification.* Primary 47A15, 30H05.
Key words and phrases. Locally bounded spaces of analytic functions, invariant subspace, multiplier, index, spectrum, linear operator, holomorphic functional calculus.

©2001 American Mathematical Society

(F4) $\|\lambda x_n\| \to 0$ if $\|x_n\| \to 0$, where λ is a fixed complex number and $\{x_n\}_{n=1}^{\infty}$ is a sequence of elements of X;

(F5) $\|\lambda_n x\| \to 0$ if $\lambda_n \to 0$, where $\{\lambda_n\}_{n=1}^{\infty}$ is a sequence of complex numbers and x is a fixed element of X.

An F-norm $\|\cdot\|$ in X is said to be *p-homogeneous*, $0 < p \le 1$, if $\|\lambda x\| = |\lambda|^p \|x\|$ for every $x \in X$ and every complex number λ. Sometimes, a p-homogeneous F-norm is called a *p-norm*. The following characterization of a locally bounded space is known as the Aoki–Rolewicz theorem: a topological vector space X is locally bounded if and only if its topology is induced by a p-norm for some p with $0 < p \le 1$ (see [11, p. 161] or [14, p. 95]).

Let X be a topological vector space. By definition, a *subspace* of X is a closed linear subset of X. If Y is a subspace of X, we can form the quotient space X/Y. It is clear that if Y is closed, then the space X/Y, endowed with the quotient topology, has the structure of a topological vector space (see [15, p. 29]). We also note that if X is an F-space or a locally bounded space, then so is the quotient space [15, Theorem 1.41]. In this paper, we shall work with complete locally bounded spaces (of analytic functions). We shall use the term *quasi-Banach space* to refer to a complete locally bounded space; this terminology emphasizes the completeness and the relationship with the Banach spaces.

In this paper, we investigate the dependence of the lattice of multiplier invariant subspaces on the connectivity of the underlying domain. Our main result shows that, in order to characterize all multiplier invariant subspaces of index one of a Bergman space $L^p_a(\Omega)$, $0 < p < 1$, on a finitely connected smoothly bordered domain Ω, we only need to characterize the multiplier invariant subspaces of the Bergman spaces $L^p_a(\Omega')$, $0 < p < 1$, where Ω' is a smoothly bordered domain with fewer holes. In particular, if Ω is the annulus $A_\rho = \{ z \in \mathbb{C} : \rho < |z| < 1 \}$, $0 < \rho < 1$, then its associated domain Ω' becomes the unit disk, which in turn simplifies the problem significantly.

A (planar) *domain* is an open and connected subset of the complex plane. In the sequel we shall consider a finitely connected bounded domain Ω. Let Ω_1 be a domain obtained from Ω by adding some of the bounded connectivity components of $\mathbb{C} \setminus \Omega$. More precisely, if $\mathbb{C} \setminus \Omega$ is the union of two nonempty closed sets K_1 and K_2 such that $K_1 \cap K_2 = \varnothing$, K_1 is unbounded, and K_2 is compact, we set $\Omega_1 = \mathbb{C} \setminus K_1$ and $\Omega_2 = \mathbb{C} \setminus K_2$. Then

(a) $\Omega = \Omega_1 \cap \Omega_2$, where Ω_1 and Ω_2 are two planar domains;

(b) Ω_1 is bounded, Ω_2 is unbounded, and $\overline{\Omega_2} \ne \mathbb{C}$;

(c) $\Omega_1 \cup \Omega_2 = \mathbb{C}$.

Observe that such a decomposition of Ω is by no means unique. That is, given a finitely connected bounded domain Ω, we may have different choices of Ω_1 and Ω_2. However, if Ω is a domain with only one hole—such as an annulus—there is precisely one choice of such a decomposition of Ω into Ω_1 and Ω_2.

Let X denote a quasi-Banach space of analytic functions on Ω (see §1 for the definition), and let X_1 be the subspace of X consisting of the functions in X that extend analytically to Ω_1. In fact, $X_1 = X \cap \mathcal{O}(\Omega_1)$, where $\mathcal{O}(\Omega_1)$ denotes the Fréchet space of all functions holomorphic in Ω_1. The maximum principle shows that X_1 is a closed subspace of X. Moreover, equipped with the norm of X, the space X_1 is a quasi-Banach space of analytic functions on Ω_1 (see §4). We need the following basic concepts.

Let T be a bounded operator on X. A closed subspace J of X is said to be *invariant for the operator* T if $TJ \subset J$. We say that J is *invariant* if $zJ \subset J$, i.e., J is invariant for the operator $M_z \colon f \to zf$ defined on X. By a *multiplier* in X we mean a holomorphic function φ on Ω such that $\varphi f \in X$ for every $f \in X$. The collection of all multipliers in X is denoted by $M(X)$. It turns out that $M(X)$ has the structure of a quasi-Banach algebra (see §2 for the definition). A closed subspace J of X is said to be *multiplier invariant* if $\varphi J \subset J$ for every $\varphi \in M(X)$. An invariant subspace of X is said to be *hyperinvariant* for the operator M_z if it is invariant for any operator commuting with M_z. The multiplier invariant subspaces turn out to be precisely the hyperinvariant subspaces for the operator M_z (see §1).

We wish to compare the multiplier invariant subspaces of X with those of X_1. Assuming that I is a multiplier invariant subspace of X_1 subject to some conditions (to be fixed later), we define

$$I \cdot M(X) = \operatorname{span}\{\, f\varphi : f \in I,\ \varphi \in M(X)\,\},$$

where by the span of a set we mean the collection of all finite linear combinations of the elements of the set. The closure $\Lambda(I)$ of $I \cdot M(X)$ in X is a multiplier invariant subspace of X; in fact $\Lambda(I)$ is the smallest multiplier invariant subspace of X containing I. Conversely, if we start with a given multiplier invariant subspace of X, say J, which again satisfies some conditions, then $J \cap X_1$ is a multiplier invariant subspace of X_1 provided that $M(X_1) \subset M(X)$. To ascertain that the above operations result in a one-to-one correspondence between the classes of multiplier invariant subspaces of X and of X_1, we need to check the following:

$$I = \Lambda(I) \cap X_1, \qquad J = \Lambda(J \cap X_1).$$

Under suitable conditions, it turns out that this is indeed the case, so that the mapping $I \mapsto \Lambda(I)$ is injective; the inverse is given by $J \mapsto J \cap X_1$.

The results of this paper generalize those of the paper [1] where Banach spaces of analytic functions were treated.

§1. Quasi-Banach spaces of analytic functions

In this section we formally define a quasi-Banach space of analytic functions.

Definition 1.1. We say that X is a quasi-Banach space of analytic functions on Ω if X is a linear subspace of $\mathcal{O}(\Omega)$ such that

(a) X is a quasi-Banach space;

(b) the injection mapping $X \hookrightarrow \mathcal{O}(\Omega)$ is continuous.

It should be noted that a quasi-Banach space of analytic functions is not merely a linear space of analytic functions equipped with a p-homogeneous F-norm (or a p-norm) that makes it a quasi-Banach space, as the name suggests. It must also have the substantial continuity property (b). From the extended form of the Banach–Steinhaus theorem for F-spaces (see [14, p. 39]) it follows that condition (b) is equivalent to the continuity of all the point evaluation functionals $X \to \mathbb{C}$ corresponding to the points of Ω.

We use the notation $\mathbb{C}_\infty$ for the Riemann sphere $\mathbb{C} \cup \{\infty\}$. In this paper we shall consider quasi-Banach spaces of analytic functions on Ω that satisfy the following two natural axioms (we assume that Ω is not dense in $\mathbb{C}$):

(A1) *If $f \in X$, then $rf \in X$ for any rational function r with poles in $\mathbb{C}_\infty \setminus \overline{\Omega}$, where the closure is taken in $\mathbb{C}_\infty$.*

(A2) *If $f \in X$, $\lambda \in \Omega$, $f(\lambda) = 0$, then there exists $g \in X$ such that $f = (z - \lambda)g$.*

From (A1) we conclude that any rational function r with poles off $\overline{\Omega}$ belongs to the multiplier space $M(X)$. For a bounded domain Ω, consider the operator $M_z\colon X \to X$ that sends $f \in X$ to zf. The closed graph theorem for F-spaces and axiom (A1) imply that the operator M_z is continuous. The spectrum of this operator is defined as the set of all $\lambda \in \mathbb{C}$ for which the operator $\lambda - M_z$ is not invertible. The spectrum of M_z, denoted by $\sigma = \sigma(M_z)$, is a compact subset of the plane. Suppose that $X \neq \{0\}$, and let

$\lambda \in \Omega$; axiom (A2) implies that there is a function $f \in X$ such that $f(\lambda) \neq 0$. Hence, the operator $M_z - \lambda$ cannot be surjective. Consequently, $M_z - \lambda$ is not invertible, whence $\lambda \in \sigma(M_z)$. We see that $\overline{\Omega} \subset \sigma(M_z)$. If $\overline{\Omega}$ contains the point at infinity, M_z is not a bounded operator, but we may still talk of the spectrum of this operator, meaning the set of all complex numbers λ for which $(\lambda - z)^{-1} \notin M(X)$. This set can be shown to be closed, for $X \neq \{0\}$. Similarly, for a fixed $\varphi \in M(X)$ we introduce the operator $M_\varphi \colon X \to X$ that sends any $f \in X$ to φf. This operator is also bounded, and we can make $M(X)$ a quasi-Banach algebra (see §2 for the definition) by defining $\|\varphi\| = \|M_\varphi\|$. This definition implies that

$$\|\varphi f\|_X \leq \|\varphi\|_{M(X)} \cdot \|f\|_X, \quad \varphi \in M(X), \ f \in X.$$

As was mentioned in the Introduction, the multiplier invariant subspaces of X coincide with the hyperinvariant subspaces for M_z. This is a well-known result in the case where X is a Banach space of analytic functions satisfying axioms (A1) and (A2) (see [13, Proposition 2.4]). More generally, the following statement is true.

Proposition 1.2. *Let $X \neq \{0\}$ be a quasi-Banach space of analytic functions on Ω satisfying axioms* (A1) *and* (A2). *Then an invariant subspace J of X is multiplier invariant if and only if J is hyperinvariant for the operator M_z.*

Proof. It suffices to check that for an operator T on X we have $TM_z = M_zT$ if and only if $T = M_\varphi$ with some $\varphi \in M(X)$. The "if" part of this assertion is clear; in order to prove the "only if" part, we assume that T is an operator on X commuting with M_z. It follows that for every $\lambda \in \Omega$ the operator T commutes with the operator $M_z - \lambda \colon f \mapsto (z-\lambda)f$, $f \in X$. Fixing $f \in X$ and $\lambda \in \Omega$, we use axiom (A2) and the assumption $X \neq \{0\}$ to conclude that there is a function $h \in X$ satisfying $h(\lambda) \neq 0$ and

$$f = \frac{f(\lambda)}{h(\lambda)} h + (z - \lambda) g \quad \text{for some } g \in X.$$

Consequently,

$$Tf = \frac{f(\lambda)}{h(\lambda)} Th + (z - \lambda) Tg,$$

whence

$$(Tf)(\lambda) = \frac{f(\lambda)}{h(\lambda)} Th(\lambda) = f(\lambda) \frac{Th(\lambda)}{h(\lambda)}. \tag{1.1}$$

Freeing up the variable λ, we can view (1.1) as an identity valid for all $\lambda \in \Omega$ with $h(\lambda) \neq 0$. The function $\varphi(z) = Th(z)/h(z)$ is meromorphic in Ω and, by (1.1), is independent of the choice of h. Taking different $h \in X$, we see that φ is holomorphic on the entire Ω, because the functions in X have no common zeros in Ω. In other words, $\varphi \in \mathcal{O}(\Omega)$. By (1.1), for every $\lambda \in \Omega$ we have $Tf(\lambda) = f(\lambda)\varphi(\lambda)$, which means that $T = M_\varphi$ with $\varphi \in M(X)$. □

We note that if the constant function 1 belongs to X, then every multiplier is in X, i.e., $M(X) \subset X$; moreover, $M(X) \subset H^\infty(\Omega) \cap X$, where $H^\infty(\Omega)$ stands for the algebra of all bounded analytic functions on Ω.

Examples. We turn to some examples of quasi-Banach spaces of analytic functions satisfying axioms (A1) and (A2), namely, the Hardy spaces $H^p(\Omega)$ for $0 < p < 1$, and the Bergman spaces $L^p_a(\Omega)$ for $0 < p < 1$ (see [5, 9]). The remaining part of this section is devoted to the definitions of the Hardy and the Bergman spaces in a general domain Ω.

Let $0 < p < +\infty$; a holomorphic function f on Ω is in $H^p(\Omega)$ if the subharmonic function $|f(z)|^p$ has a harmonic majorant on Ω, i.e., there exists a harmonic function $v(z)$ such that $|f(z)|^p \le v(z)$ for every $z \in \Omega$. It follows that every $f \in H^p(\Omega)$ has a unique least harmonic majorant; this means that there is a unique harmonic function u_f such that $|f(z)|^p \le u_f(z)$, $z \in \Omega$, and $u_f(z) \le v(z)$, $z \in \Omega$, for any harmonic majorant v of $|f(z)|^p$ (see [6, p. 51]. Fixing a point $z_0 \in \Omega$, we put

$$\|f\|_{H^p(\Omega)} = \big(u_f(z_0)\big)^{1/p}, \quad 0 < p < +\infty.$$

This defines a norm on $H^p(\Omega)$ if $1 \le p < +\infty$; moreover, the resulting topology does not depend on the particular choice of $z_0 \in \Omega$ (see [6, p. 52]). The space $H^p(\Omega)$ with $1 \le p < +\infty$ is a Banach space (see [6, p. 54]), and for $0 < p < 1$ it is a quasi-Banach space, equipped with the p-norm equal to the pth power of the above norm expression.

For $0 < p < +\infty$, we define $L^p_a(\Omega)$, the Bergman space on a domain Ω, to be the space of all holomorphic functions f on Ω such that

$$\|f\|_{L^p_a(\Omega)} = \left(\int_\Omega |f(z)|^p dA(z)\right)^{1/p} < +\infty,$$

where $dA(z)$ denotes the area measure on Ω. More generally, let $w\colon \Omega \to \mathbb{C}$ be a continuous and strictly positive function, and let $0 < p < +\infty$; the space $L^p_a(\Omega, w)$, the weighted Bergman space on Ω, consists of all holomorphic functions f on Ω such that

$$\|f\|_{L^p_a(\Omega,w)} = \left(\int_\Omega |f(z)|^p w(z) dA(z)\right)^{1/p} < +\infty.$$

For $1 \le p < +\infty$, this is a norm making $L^p_a(\Omega, w)$ a Banach space. In contrast, for $0 < p < 1$, the weighted Bergman spaces are quasi-Banach spaces; the pth power of the above norm expression determines a p-norm.

§2. The holomorphic functional calculus

Classically, one considers a Banach space X and an operator T in the algebra $\mathcal{L}(X)$ of bounded operators on X (the multiplication operation in $\mathcal{L}(X)$ is the usual composition of operators). We denote by $\sigma(T)$ the spectrum of T, i.e., the set of all complex numbers λ for which $\lambda - T$ is not invertible. Consider a complex-valued function f analytic in a neighborhood of $\sigma(T)$, say U. Let V be a domain with smooth boundary such that $\sigma(T) \subset V \subset \overline{V} \subset U$. With f we associate the operator-valued integral

$$f(T) = \frac{1}{2\pi i}\int_{\partial V} f(\lambda)(\lambda - T)^{-1} d\lambda, \tag{2.1}$$

where the boundary ∂V of V is oriented in the positive direction. The mapping $f \mapsto f(T)$, known as the holomorphic functional calculus, has a number of interesting properties (see [4, 15, 16]); for instance, it maps the constant function 1 to the identity operator in $\mathcal{L}(X)$, and the coordinate function $\lambda \mapsto \lambda$ to T. The holomorphic functional calculus is multiplicative, i.e., if f and g are two functions holomorphic in a common neighborhood of $\sigma(T)$, then $(fg)(T) = f(T)g(T)$. Moreover, the holomorphic functional calculus has the following continuity property: if $\{f_n\}_n$ is a sequence of functions analytic in a fixed neighborhood of $\sigma(T)$ and if $f_n \to f$ locally uniformly on this neighborhood, then $f_n(T) \to f(T)$ in the norm of $\mathcal{L}(X)$.

We wish to apply the holomorphic functional calculus in a more general setting where X is a quasi-Banach space, equipped with a p-norm, $0 < p \leq 1$. Consider the space $\mathcal{L}(X)$ of bounded linear operators $X \to X$; a mapping T from an F-space X to another F-space Y is said to be bounded if it maps bounded sets to bounded sets. It is well known that for metrizable X, the boundedness of an operator $T \in \mathcal{L}(X)$ is equivalent to the continuity of T (see [14, p. 37] or [15, p. 23]). We endow $\mathcal{L}(X)$ with the topology of bounded convergence; this is the topology induced by the invariant metric

$$\|T\| = \sup\{ \|Tx\| : \|x\| \leq 1 \}, \quad T \in \mathcal{L}(X)$$

(it is easy to check that this is a p-norm with the same p, $0 < p \leq 1$, as for X). With this p-norm, $\mathcal{L}(X)$ is a quasi-Banach space. Moreover, for T and S in $\mathcal{L}(X)$ we have

$$\|TS\| \leq \|T\|\|S\|.$$

In fact, the space $\mathcal{L}(X)$ is a *quasi-Banach algebra* in the sense that it satisfies all axioms of a Banach algebra with the only exception that the norm is replaced by a p-norm, $0 < p \leq 1$ [16]. These algebras were studied extensively by W. Żelazko (see [16, 17]).

A topological vector space Y is said to be *locally convex* provided that the origin in Y has a basis of convex neighborhoods. Note that the locally convex spaces and the locally bounded spaces are two natural generalizations of the normed spaces. By the Kolmogorov theorem, a topological vector space Y is a normed space if and only if Y is both locally bounded and locally convex (see [15, Theorem 1.39]).

Integration theory for functions defined on subsets of the complex plane and taking values in an F-space Y works rather well for locally convex spaces; both the Bochner–Lebesgue integral and the Riemann integral of such functions can be defined (see [15, pp. 73, 85] and also the references cited in [15] on page 375). If we do not make any convexity assumption about the topology of Y, the Bochner–Lebesgue integral of measurable Y-valued functions, which is based on approximation of a function by piecewise constant functions, cannot be defined in general. The problem is that in such a space there is a sequence of piecewise constant functions converging uniformly to zero and such that the sequence of integrals does not converge to zero (see [14, p. 123]). However, the Riemann integration theory, based on the limit of "Riemann sums" in the norm of Y, can be developed in the setting of F-spaces, as observed by B. Gramsch [7], and independently by D. Przeworska-Rolewicz and S. Rolewicz [12]. However, if Y is not locally convex, then, by a theorem of Mazur–Orlicz, there exists a continuous Y-valued function $f(t)$, $0 \leq t \leq 1$, that fails to be Riemann integrable (see [14, p. 121]). It is important to note that the Y-valued analytic functions (to be defined below) are always Riemann integrable.

Let $u(z)$ be a function defined on some domain Ω and taking values in an F-space X. We say that u is *analytic* in Ω if for every $z_0 \in \Omega$ there exists a neighborhood $V(z_0)$ of z_0 such that in $V(z_0)$ the function $u(z)$ can be represented as a convergent power series $u(z) = \sum_{n=0}^{\infty}(z - z_0)^n u_n$, $u_n \in X$. It is known (see [3, 14]) that if X is a quasi-Banach space and if the above series is convergent at some point z_1, then it is convergent for all z with $|z - z_0| < |z_1 - z_0|$. This is not so for general F-spaces [18].

By a theorem of Gramsch and Przeworska-Rolewicz, for any smooth curve Γ contained in Ω the Riemann integral $\int_\Gamma u(z)dz$ of an analytic function $u(z)$ exists, and moreover, the Cauchy integral formula and the Liouville theorem are valid in this setting (for the details, see [14, p. 124]).

Let Y be a commutative Banach algebra with unit. It is well known that the spectrum of every element of Y is a nonempty compact subset of the complex plane. Passing to quasi-Banach algebras, such as the operator algebra $\mathcal{L}(X)$ for a quasi-Banach space X, we need to make sure that the spectrum is well defined, nonempty, and compact.

In what follows we shall see that the quasi-Banach spaces share almost all features of the Banach spaces. We proceed with some definitions.

Let X be a quasi-Banach space, and let $T \in \mathcal{L}(X)$. In accordance with [14, p. 176], on its domain of definition the function $(\lambda - T)^{-1}$ is analytic in λ (here, analyticity is understood in the sense indicated above). By the *spectrum* $\sigma(T)$ of T we mean the set of all complex numbers λ for which $\lambda - T$ is not invertible. The set $\mathbb{C} \setminus \sigma(T)$ is an open subset of the complex plane; moreover, for every λ with $|\lambda| > \|T\|$ the operator $\lambda - T$ is invertible (see [14, p. 175]). This implies that the spectrum of T is both bounded and closed; hence, it is compact. Finally, we note that the spectrum of T is nonempty provided that the quasi-Banach space X is nontrivial, $X \neq \{0\}$. This follows from the Liouville theorem (see [16, p. 75 and the comments on p. 110]).

Let X be a quasi-Banach space. For $T \in \mathcal{L}(X)$ we can define an operator-valued function $f(T)$ by (2.1), since this formula makes sense. Observe that now "all is well" by the said above. If f is an analytic function defined on a domain U containing the spectrum $\sigma(T)$, and if ∂V is an oriented closed smooth curve containing $\sigma(T)$ inside the domain surrounded by ∂V, then the integral on the right-hand side of (2.1) exists; furthermore, the spectral mapping theorem holds true, i.e., $\sigma(f(T)) = f(\sigma(T))$ (see [14, p. 182]). Finally, we mention that the mapping $f \mapsto f(T)$ has the same properties as in the classical case (see either [2] for a detailed account, or [16, Theorem 16.4] and the comments on page 110 of [16]).

Now we introduce the operator of multiplication by φ on the quotient space X/J, where J is a multiplier invariant subspace of X and φ is some fixed element of $M(X)$. This is the operator $M_\varphi[X/J]\colon X/J \to X/J$ given by

$$f + J \mapsto \varphi f + J.$$

This operator is continuous. In the special case where $\varphi = z$, the spectrum of $M_z[X/J]$, the set of all complex numbers λ for which $\lambda - M_z[X/J]$ is not invertible, is denoted by $\sigma_J = \sigma(M_z[X/J])$. We recall that for a closed subset E in the complex plane the notation $\mathcal{O}(E)$ is used to indicate the space of all functions analytic in a neighborhood of E. It is rather easy to use the holomorphic functional calculus in order to deduce that $\mathcal{O}(\overline{\Omega}) \subset M(X)$ and that if J is an invariant subspace of X and $g \in \mathcal{O}(\overline{\Omega})$, then $g(M_z[X/J]) = M_g[X/J]$, where the operator on the left is understood in the sense of the holomorphic functional calculus.

§3. Stability of the index

In the following lemma we establish a stability property for the codimension of $(z-\lambda)J$ in J, where J is an invariant subspace of X. We note that for a Banach space X this property follows from the perturbation theory of semi-Fredholm operators (see [10]).

Lemma 3.1. *Let X be a quasi-Banach space of analytic functions on a domain Ω satisfying axioms* (A1) *and* (A2), *and let J be an invariant subspace of X. Then the function $\lambda \mapsto \dim\bigl(J/(z-\lambda)J\bigr)$ is constant on Ω.*

Proof. We denote $J_\lambda = (z - \lambda)J$ and make the following two observations.

Observation 1:

$$\begin{aligned}\dim(J/J_\lambda) &= \sup\bigl\{\dim\bigl(L/(L \cap J_\lambda)\bigr) : L \subset J,\ \dim L < +\infty\bigr\}\\ &= \sup\bigl\{\dim L : L \subset J,\ \dim L < +\infty,\ L \cap J_\lambda = \{0\}\bigr\}.\end{aligned}$$

The second equality is clear. Recalling the algebraic relations $L/(L\cap J_\lambda) \cong (L+J_\lambda)/J_\lambda \subset J/J_\lambda$, and the fact that $L \cong L/(L\cap J_\lambda)$ if $L\cap J_\lambda = \{0\}$, we conclude that $\dim(J/J_\lambda)$ cannot be less than either of the quantities involved in the suprema. As for equality, first we consider the case where $\dim(J/J_\lambda) = n < +\infty$ and choose a basis $\{h_j + J_\lambda\}_{j=1}^n$ of J/J_λ, where $h_1, \dots, h_n \in J$. Then we set $L = \operatorname{span}\{h_1,\dots,h_n\} \subset J$, and note that $\dim L = n$. We show that $L\cap J_\lambda = \{0\}$. To this end, let $\varphi \in L\cap J_\lambda$, so that $\varphi = \sum_{j=1}^n c_j h_j$, where c_j are some constants. Then $0 + J_\lambda = \varphi + J_\lambda = \sum_{j=1}^n c_j(h_j + J_\lambda)$, which implies that $c_j = 0$ for all $1 \le j \le n$. The conclusion $L\cap J_\lambda = \{0\}$ follows, whence $\dim\bigl(L/(L\cap J_\lambda)\bigr) = \dim L = n$. Thus, we have proved the above relations under the assumption that $\dim(J/J_\lambda) < +\infty$. If $\dim(J/J_\lambda) = +\infty$, then for any positive integer N we can find N linearly independent vectors in J/J_λ. Proceeding as above, we construct an N-dimensional subspace $L \subset J$ such that $L\cap J_\lambda = \{0\}$. The quotient space $L/(L\cap J_\lambda)$ has dimension N, and since N can be taken arbitrarily large, the above suprema are equal to $+\infty$.

Observation 2: Let L be a subspace of J of finite dimension, and let λ_0 be a point in Ω. If $L\cap J_{\lambda_0} = \{0\}$, then $L \cap J_\lambda = \{0\}$ for every λ close to λ_0.

Suppose this is not true. Then we can find a sequence $\{\lambda_k\}_k \subset \Omega$ such that $\lambda_k \to \lambda_0$ as $k \to +\infty$ and $L\cap J_{\lambda_k} \neq \{0\}$. We choose a sequence $\{f_k\}$ in $L\cap J_{\lambda_k}$ such that $\|f_k\| = 1$. Since L is finite-dimensional, a subsequence of $\{f_k\}$ converges in norm to some $f \in L\cap J_{\lambda_0}$ with $\|f\| = 1$. We keep the same notation $\{f_k\}_k$ for this subsequence. For each f_k we have $f_k = (z-\lambda_k)g_k$ with some $g_k \in J$. Now we consider the operator $M_z - \lambda$ restricted to J; this operator is bounded from below. By the uniform boundedness principle, the operators $M_z - \lambda$ are uniformly bounded from below on each compact subset of Ω. It follows that there is a constant $\varepsilon > 0$, independent of k, such that $\varepsilon\|g_k\| \le \|(z-\lambda_k)g_k\| = \|f_k\| = 1$. This implies that

$$\begin{aligned}\varepsilon\|g_k - g_j\| &\le \|(z-\lambda_k)(g_k - g_j)\| = \|f_k - (z-\lambda_k)g_j\| = \|f_k - f_j + (\lambda_k - \lambda_j)g_j\| \\ &\le \|f_k - f_j\| + \varepsilon^{-1}|\lambda_k - \lambda_j|^p,\end{aligned}$$

where p is the exponent of homogeneity of the space X, $0 < p \le 1$. This shows that $\{g_k\}_k$ is a Cauchy sequence in J, so that g_k converges to some $g \in J$. Finally, $f = (z-\lambda_0)g \in L\cap J_{\lambda_0}$, which contradicts the assumption $L\cap J_{\lambda_0} = \{0\}$, because f has norm 1.

Now we turn to the proof of the lemma. Let λ_0 be a point in Ω, and let $\dim(J/J_{\lambda_0}) = n$, where $0 < n \le +\infty$. Let L be a finite-dimensional subspace of J with $L\cap J_{\lambda_0} = \{0\}$. By the second observation, we have $L\cap J_\lambda = \{0\}$ for every λ close to λ_0. Now, we apply the first observation to this L to obtain $\dim(J/J_\lambda) \ge \dim L$. But $\dim L$ can be n if n is finite, and can be any large integer if $n = +\infty$; therefore,

$$\dim\bigl(J/(z-\lambda)J\bigr) = \dim(J/J_\lambda) \ge n \quad \text{if } n < +\infty. \tag{3.1}$$

If $n = +\infty$, then for any integer N there is a neighborhood $U_N(\lambda_0)$ of λ_0 such that $\dim(J/J_\lambda) \ge N$ for every $\lambda \in U_N(\lambda_0)$.

We turn to the reverse inequality in the case where n is finite. Without loss of generality, we may assume that $0 \in \Omega$ and that $\lambda_0 = 0$. By assumption, J/zJ is generated by n linearly independent vectors $\{h_j + zJ\}_{j=1}^n$. This means that any coset $f + zJ$ can be written as a linear combination of these vectors; moreover, if $\sum_{j=1}^n c_j h_j \in zJ$ for some $c_1,\dots,c_n \in \mathbb{C}$, then $c_j = 0$ for all $1 \le j \le n$. Let $\{\phi_j\}_{j=1}^n$ be the basis functionals corresponding to the n-dimensional quotient space J/zJ and dual to the vectors $\{h_j + zJ\}_{j=1}^n$, so that each coset $f + zJ \in J/zJ$ has the representation

$$f + zJ = \sum_{j=1}^n \langle f + zJ, \phi_j\rangle (h_j + zJ). \tag{3.2}$$

Then for any $f \in J$ we have $f = P_1 f + P_2 f$, where $P_1 f \in zJ$ and $P_2 f \in \operatorname{span}\{h_1, h_2, \dots, h_n\}$. More precisely, the projections P_1 and P_2 are defined by the formulas

$$P_1 f = f - \sum_{j=1}^{n} \langle f + zJ, \phi_j \rangle h_j, \qquad P_2 f = \sum_{j=1}^{n} \langle f + zJ, \phi_j \rangle h_j.$$

For $|\lambda|$ sufficiently small, we define $Q\colon J \to J$ by the relation

$$Qf = \frac{P_1 f}{z} = \frac{f - \sum_{j=1}^{n} \langle f + zJ, \phi_j \rangle h_j}{z}.$$

Then $zQf = f - \sum_{j=1}^{n} \langle f + zJ, \phi_j \rangle h_j$, whence

$$(z - \lambda) Qf = f - \sum_{j=1}^{n} \langle f + zJ, \phi_j \rangle h_j - \lambda Qf.$$

Since

$$Q^2 f = Q(Qf) = \frac{Qf - \sum_{j=1}^{n} \langle Qf + zJ, \phi_j \rangle h_j}{z},$$

we get

$$(z - \lambda)(Qf + \lambda Q^2 f) = f - \sum_{j=1}^{n} \langle f + zJ, \phi_j \rangle h_j - \lambda \sum_{j=1}^{n} \langle Qf + zJ, \phi_j \rangle h_j - \lambda^2 Q^2 f.$$

Proceeding inductively, for $|\lambda|$ small, in the limit we obtain

$$\begin{aligned} &(z - \lambda)\big(Qf + \lambda Q^2 f + \lambda^2 Q^3 f + \cdots\big) \\ &= f - \left(\sum_{j=1}^{n} \langle f + zJ, \phi_j \rangle h_j + \lambda \sum_{j=1}^{n} \langle Qf + zJ, \phi_j \rangle h_j \right. \\ &\qquad \left. + \lambda^2 \sum_{j=1}^{n} \langle Q^2 f + zJ, \phi_j \rangle h_j + \cdots \right). \end{aligned} \tag{3.3}$$

We know that $Q = M_z^{-1} P_1$ is a bounded operator. If $|\lambda|$ is smaller than $\|Q\|^{-1}$, then the operator $1 - \lambda Q$ is invertible, and its norm is at most $(1 - |\lambda|^p \|Q\|)^{-1}$, where $0 < p \le 1$. Therefore, the left-hand side of (3.3) becomes $(z - \lambda) Q_\lambda f$, where

$$Q_\lambda = Q(1 - \lambda Q)^{-1} = Q + \lambda Q^2 + \lambda^2 Q^3 + \cdots.$$

From (3.3) it follows that

$$f = (z - \lambda) Q_\lambda f + \sum_{j=1}^{n} \langle (1 - \lambda Q)^{-1} f + zJ, \phi_j \rangle h_j.$$

Thus, $f - \sum_{j=1}^{n} c_j^\lambda h_j \in J_\lambda$, where $c_j^\lambda = \langle (1 - \lambda Q)^{-1} f + zJ, \phi_j \rangle$. Consequently,

$$f + J_\lambda = \sum_{j=1}^{n} c_j^\lambda (h_j + J_\lambda). \tag{3.4}$$

In other words, $P_1^\lambda f = f - \sum_{j=1}^n c_j^\lambda h_j$ is a projection $J \to J_\lambda$. Setting $P_2^\lambda f = \sum_{j=1}^n c_j^\lambda h_j$, we obtain $f = P_1^\lambda f + P_2^\lambda f$. This argument shows that

$$\dim(J/J_\lambda) \le \dim(J/J_{\lambda_0}) = n. \tag{3.5}$$

From (3.1) and (3.5) it follows that if $n < +\infty$, then $\dim(J/J_\lambda) = \dim(J/J_{\lambda_0}) = n$ for each λ close to λ_0. This means that for each integer $n < +\infty$ the set $G_n = \{\lambda \in \Omega : \dim(J/J_\lambda) = n\}$ is open in Ω. In other words, the function $\lambda \mapsto \dim(J/J_\lambda)$ is locally constant, *a fortiori* continuous, provided that $\lambda \in G_n$ for some $n < +\infty$. Also, the statement after (3.1) implies that the function $\lambda \mapsto \dim(J/J_\lambda)$ is continuous at every $\lambda \in G_{+\infty} = \{\lambda \in \Omega : \dim(J/J_\lambda) = +\infty\}$. Since the continuous function $\lambda \mapsto \dim(J/J_\lambda)$, $\lambda \in \Omega$, takes its values in the discrete set $\{0, 1, 2, \ldots, +\infty\}$, this function must be constant on the connected set Ω. □

In accordance with the above lemma, it makes sense to define the index of an invariant subspace J as the dimension of $J/(z-\lambda)J$. In the case where X is a Banach space of analytic functions, S. Richter gave a nice characterization of the invariant subspaces of index one in [13, Lemma 3.1]. Fortunately, the result remains true in a more general case, as the following lemma shows. Since the argument is the same, we omit the proof. We introduce the notation

$$Z(J) = \{\lambda \in \Omega : f(\lambda) = 0,\ f \in J\}$$

for the common zero set of an invariant subspace J in X.

Lemma 3.2 (Richter). *Let X be a quasi-Banach space of analytic functions satisfying axioms* (A1) *and* (A2), *let J be an invariant subspace of X, and let $\lambda \in \Omega \setminus Z(J)$. The following statements are equivalent*:

(a) $\operatorname{index}(J) = 1$;
(b) *if $f \in J$ and $f(\lambda) = 0$, then there is a function $h \in J$ such that $f = (z-\lambda)h$*;
(c) *if $(z-\lambda)h \in J$ for some $h \in X$, then $h \in J$.*

We have already defined the spectrum σ_J of the operator $M_z[X/J]$. Another notion related to σ_J is the *weak spectrum*, by which we mean the collection of all complex numbers λ such that the operator $\lambda - M_z[X/J]$ is not onto (see [8]). The weak spectrum of $M_z[X/J]$ is denoted by σ'_J. It is clear that $\sigma'_J \subset \sigma_J \subset \overline{\Omega}$. The following proposition tells us more about the relationship between these two spectra and the index.

Proposition 3.3. *Let X be a nontrivial quasi-Banach space of analytic functions on Ω satisfying axioms* (A1) *and* (A2). *Let J be an invariant subspace of X. Then*:

(a) $\sigma'_J \cap \Omega = Z(J)$;
(b) *if* $\operatorname{index}(J) = 1$, *then* $\sigma'_J \cap \Omega = \sigma_J \cap \Omega$;
(c) *if* $\operatorname{index}(J) > 1$, *then* $\sigma_J = \overline{\Omega}$;
(d) *if $\sigma_J \subset \Omega$, then $J = \{pf : f \in X\}$, where p is a polynomial with zeros in Ω.*

Proof. For the proof of statements (c) and (d) we may argue, with little modifications, as in the proofs of Propositions 2.3 and 2.5 in [1]. In order to prove (a), we let $\lambda \in Z(J)$ and suppose that $\lambda - M_z[X/J]$ is onto. Axiom (A2) guarantees the existence of a function $f \in X$ with $f(\lambda) \neq 0$. Therefore, there is a function $h \in X$ such that $(\lambda - M_z[X/J])(h + J) = f + J$. This implies that $(\lambda - z)h - f \in J$. On the other hand, since $\lambda \in Z(J)$, we have $f(\lambda) = 0$. This contradiction shows that the operator $\lambda - M_z[X/J]$ is not onto, whence $Z(J) \subset \sigma'_J \cap \Omega$. To verify the reverse inclusion, assume that $\lambda \in \Omega \setminus Z(J)$. Then we can find a function $h \in J$ with $h(\lambda) = 1$. Let $g \in X$ be arbitrary, and let $f = (g - g(\lambda)h)/(\lambda - z)$; axiom (A2) shows that $f \in X$. Thus,

$(\lambda - z)f - g = -g(\lambda)h \in J$. This means that $\big(\lambda - M_z[X/J]\big)(f + J) = g + J$. Therefore, the operator $\lambda - M_z[X/J]$ is onto or $\lambda \notin \sigma'_J$, which completes the proof of (a). As for part (b), from the definition it follows that $\sigma'_J \cap \Omega \subset \sigma_J \cap \Omega$ for any invariant subspace J. Suppose that J has index one, and let $\lambda \in \Omega \setminus \sigma'_J$, so that the operator $\lambda - M_z[X/J]$ is onto; we claim that it is also one-to-one. To see this, we let $(\lambda - M_z[X/J])(f+J) = 0+J$, which is equivalent to saying that $(\lambda - z)f \in J$. By Lemma 3.2, we have $f \in J$, which proves our claim. Thus, the operator $\lambda - M_z[X/J]$ is invertible, or $\lambda \in \Omega \setminus \sigma_J$; this completes the proof of (b). □

§4. A TRANSFER THEOREM

We recall that X is a quasi-Banach space of analytic functions on Ω satisfying axioms (A1) and (A2). Moreover, we assume that $1 \in X$. Let X_1 denote the subspace of X consisting of the functions that extend analytically to Ω_1, i.e., $X_1 = X \cap \mathcal{O}(\Omega_1)$. The fact that X_1 is closed in X follows from the maximum principle. Indeed, let K be a compact subset of Ω_1. The representation of Ω shows that there exists an open subset U of Ω_1 containing K and such that its boundary ∂U lies in Ω. For any $f \in X_1$ and $z \in K$, we have

$$\sup_{z\in K} |f(z)| \le \sup_{z\in\partial U} |f(z)| \le C\|f\|_X,$$

where $C = C(\partial U)$ is a constant depending on the set ∂U. It follows that the evaluation functionals corresponding to the points of Ω_1 are continuous, so that X_1 is closed in X. This argument shows that X_1 is a quasi-Banach space of analytic functions on Ω_1. Similarly, we set $X_2 = X \cap \mathcal{O}(\Omega_2)$ and $X_2^0 = X \cap \mathcal{O}_0(\Omega_2)$, where $\mathcal{O}_0(\Omega_2)$ is the subspace of $\mathcal{O}(\Omega_2)$ consisting of the functions that tend to 0 as $|z| \to \infty$. It is clear that $X_1 \oplus X_2^0 \subset X$, and that X_1 and X_2^0 satisfy axiom (A1) on the corresponding domains. We assume that X can be written as $X = X_1 \oplus X_2^0$, and that X_1 and X_2^0 satisfy axiom (A2) as well. Also, we make similar assumption about the multiplier space $M(X)$; more precisely, we assume that $M(X) = M(X_1) \oplus M_0(X_2)$, where $M_0(X_2) = M(X_2) \cap \mathcal{O}_0(\Omega_2)$.

Definition 4.1. Let X be a quasi-Banach space of analytic functions, and let Y be a quasi-Banach space. A bounded linear mapping $L\colon X \to Y$ is said to be a *multiplier module homomorphism* if there is a continuous homomorphism $L_M\colon M(X) \to \mathcal{L}(Y)$ such that $L(\varphi f) = L_M(\varphi)L(f)$ for every $\varphi \in M(X)$ and every $f \in X$.

This definition deserves some comments. Concerning the term *multiplier module homomorphism*, it should be mentioned that any quasi-Banach space X has the structure of an $M(X)$-module. This module structure is induced by the mapping $\big(M(X), X\big) \to X$ that sends each couple (φ, f) to the element $\varphi f \in X$. For this reason, any linear mapping on X preserving this structure is named a *multiplier module homomorphism*. It is easy to check that if a linear mapping L as above is onto, then the associated homomorphism L_M is unique. It turns out that the kernel of any multiplier module homomorphism L is a multiplier invariant subspace of X. If X is replaced by a quotient space X/J (here J is a multiplier invariant subspace of X), then we require that

$$L\big(\varphi(f + J)\big) = L_M(\varphi)L(f + J), \quad f \in X, \ \ \varphi \in M(X).$$

In this case L is called a *quotient multiplier module homomorphism*. A surjective and injective (quotient) multiplier module homomorphism will be called a *(quotient) multiplier module isomorphism* if its inverse is also a (quotient) multiplier module homomorphism. Since, largely, in this paper we are concerned with quotient multiplier module isomorphisms, it is worth while to make this concept more transparent. Let Y be a quasi-Banach space of analytic functions and I a multiplier invariant subspace in Y. Assume that the quotient multiplier module homomorphism $L\colon X/J \to Y/I$ is both one-to-one and onto. From the definition we know that there is a continuous homomorphism $L_M\colon M(X) \to \mathcal{L}(Y/I)$ satisfying the above condition. Let $K = L^{-1}\colon Y/I \to X/J$; this operator is also continuous. If we can find a continuous homomorphism $K_M\colon M(Y) \to \mathcal{L}(X/J)$ such that $K\big(\psi(f+I)\big) = K_M(\psi)K(f+I)$ for every $\psi \in M(Y)$ and every $f \in I$, then L will be a quotient multiplier module isomorphism.

Let I be a proper multiplier invariant subspace of X_1, and, similarly, let J be a multiplier invariant subspace of X. We recall that σ_I denotes the spectrum of the operator $M_z[X_1/I]$. Let $\mathcal{O}(\sigma_I)$ denote the algebra of all functions h analytic in some neighborhood U_h of the compact set σ_I, and let D_h be a compact subset of U_h such that its interior contains σ_I. We define an operator $H_I\colon \mathcal{O}(\sigma_I) \to \mathcal{L}(X_1/I)$ by the rule

$$h \mapsto \frac{1}{2\pi i}\int_{\partial D_h} h(\lambda)\big(\lambda - M_z[X_1/I]\big)^{-1}d\lambda. \tag{4.1}$$

The boundary of D_h is assumed to consist of finitely many piecewise smooth curves, and integration is taken in the positive direction. The Cauchy theorem (see §2) shows that the above integral does not depend on the particular choice of U_h. The mapping H_I, known as the holomorphic functional calculus, has a number of nice properties. It maps the constant function 1 and the coordinate function z to the identity operator on X_1/I and to the operator $M_z[X_1/I]$, respectively. Also, H_I is multiplicative in the sense that $H_I(fg) = H_I(f)H_I(g)$ for any f and g in $\mathcal{O}(\sigma_I)$. Finally, H_I has the following continuity property: if U is a fixed neighborhood of σ_I and if $f_n \to f$ locally uniformly in U, then $H_I(f_n)$ tends to $H_I(f)$ in the norm of $\mathcal{L}(X_1/I)$. Therefore, H_I is a continuous homomorphism between two quasi-Banach algebras.

By our assumptions on X, we have $1 \in X_1$, so that the coset $1+I$ belongs to X_1/I. Now, we define a mapping $P_I\colon \mathcal{L}(X_1/I) \to X_1/I$ by the rule

$$S \mapsto S(1+I).$$

Then the composition $P_I \circ H_I$ acts from $\mathcal{O}(\sigma_I)$ into X_1/I and sends the constant function 1 and the coordinate function z to $1+I$ and to $z+I$, respectively. We know that $\sigma_I \subset \overline{\Omega}_2$. Assume that σ_I does not touch the boundary of Ω_2, i.e., $\sigma_I \subset \Omega_2$; then $X_2^0 \subset \mathcal{O}_0(\Omega_2) \subset \mathcal{O}(\sigma_I)$, allowing us to introduce the following important multiplier module homomorphism: we define $L_I\colon X = X_1 \oplus X_2^0 \to X_1/I$ by the formula

$$L_I(f) = \begin{cases} f+I & \text{if } f \in X_1, \\ P_I\big(H_I(f)\big) & \text{if } f \in X_2^0. \end{cases} \tag{4.2}$$

In [1, Proposition 3.3] it was proved that, in the Banach space case, L_I is indeed a multiplier module homomorphism. Here we sketch a proof in the general setting of quasi-Banach spaces.

Lemma 4.2. *Let I be a multiplier invariant subspace of X_1 such that $\sigma_I \subset \Omega_2$. Then $L_I\colon X \to X_1/I$ is a multiplier module homomorphism.*

Sketch of the proof. First, we assume that $\mathcal{O}_0(\overline{\Omega}_2)$ is dense in $M_0(X_2)$ and in X_2^0; eventually we shall lift this restriction (see [1, Lemma 3.2]). With L_I we associate the following continuous homomorphism $L_{I,M}$. We write $\varphi \in M(X)$ as $\varphi_1 + \varphi_2$, where $\varphi_1 \in M(X_1)$ and $\varphi_2 \in M_0(X_2)$. Then we define $L_{I,M}\colon M(X) \to \mathcal{L}(X_1/I)$ by the formula

$$L_{I,M}(\varphi) = M_{\varphi_1}[X_1/I] + H_I(\varphi_2), \tag{4.3}$$

where H_I is the holomorphic functional calculus defined by (4.1). To prove that $L_{I,M}$ is a homomorphism, first we verify that

$$L_{I,M}\left(\frac{\varphi_1}{\lambda - z}\right) = L_{I,M}(\varphi_1)L_{I,M}\left(\frac{1}{\lambda - z}\right),$$

where $\varphi_1 \in M(X_1)$ and $\lambda \in \mathbb{C} \setminus \overline{\Omega}_2$. Since for such λ the finite linear combinations of the functions $z \mapsto (\lambda - z)^{-1}$ are dense in $\mathcal{O}_0(\overline{\Omega}_2)$, we may replace $(\lambda - z)^{-1}$ by any $\varphi_2 \in M_0(X_2)$. Consequently, $L_{I,M}$ is a homomorphism. Finally, it can be shown that

$$L_I(\varphi f) = L_{I,M}(\varphi)L_I(f), \quad \varphi \in M(X), \quad f \in X,$$

which means that L_I is a multiplier module homomorphism. □

In the sequel, we shall see that the kernel of L_I is a multiplier invariant subspace of X. It turns out that, in fact, this is the desired counterpart of the multiplier invariant subspace I of X_1 with $\sigma_I \subset \Omega_2$. Denoting $\ker L_I = J$, we prove a statement which makes the relationship between I and J transparent.

Proposition 4.3. *Let I be a multiplier invariant subspace of X_1 with $\sigma_I \subset \Omega_2$. If $J = \ker L_I$, then*

(a) *J is a multiplier invariant subspace in X;*
(b) $\operatorname{clos}(I \cdot M(X)) \subset J$;
(c) $J \cap X_1 = I$;
(d) $\sigma_I = \sigma_J$.

Proof. The proof is essentially the same as that of Proposition 3.4 in [1]. □

Let V_1 and V_2 denote two finitely connected smoothly bordered domains satisfying the following conditions:

(a) $\mathbb{C} \setminus \Omega_2 \subset V_1 \subset\subset \Omega_1$;
(b) $\mathbb{C} \setminus \Omega_1 \subset V_2 \subset \overline{V}_2 \subset \Omega_2$;
(c) $\overline{V}_1 \cap \overline{V}_2 = \varnothing$.

Definition 4.4. Let X be a quasi-Banach space of analytic functions on Ω satisfying axioms (A1) and (A2). We say that X satisfies the *shrinking domain condition* with respect to the couple (V_1, V_2) if there exist two quasi-Banach spaces of analytic functions $X^{(1)}$ and $X^{(2)}$ satisfying axioms (A1) and (A2) on $\Omega_1 \cap V_2$ and $\Omega_2 \cap V_1$, respectively, and such that

(a) $X^{(1)} \cap \mathcal{O}(\Omega_1) = X_1$, and $X^{(2)} \cap \mathcal{O}_0(\Omega_2) = X_2^0$;
(b) $X^{(1)} = X_1 \oplus X_{2,0}^{(1)}$, where $X_{2,0}^{(1)} = X^{(1)} \cap \mathcal{O}_0(V_2)$;
(c) $X^{(2)} = X_1^{(2)} \oplus X_2^0$, where $X_1^{(2)} = X^{(2)} \cap \mathcal{O}(V_1)$;
(d) $M(X^{(1)}) = M(X_1) \oplus M_0(X_2^{(1)})$, where $M_0(X_2^{(1)}) = M(X_2^{(1)}) \cap \mathcal{O}_0(V_2)$, and $M(X^{(2)}) = M(X_1^{(2)}) \oplus M_0(X_2)$.

At first glance, it seems that too many conditions are involved here. However, it should be realized that our typical example $X = L_a^p(\Omega)$, $0 < p < 1$, the Bergman space, indeed satisfies the shrinking domain condition. In this case $X_1 = L_a^p(\Omega_1)$ and $X_2 = L_a^p(\Omega_2)$ (here, we use area measure on the Riemann sphere rather than on the complex plane to define the Bergman space). Now, we may regard $L_a^p(\Omega_1 \cap V_2)$ and $L_a^p(\Omega_2 \cap V_1)$ as our $X^{(1)}$ and $X^{(2)}$, respectively. These spaces split into direct sums, and the same is true for their multiplier spaces, i.e., the spaces of bounded analytic functions on the corresponding domains. It follows (see §2) that

$$X_1 \subset \mathcal{O}(\Omega_1) \subset \mathcal{O}(\overline{V}_1) \subset \mathcal{O}(\overline{\Omega_2 \cap V_1}) \subset X^{(2)},$$

whence $X \subset X^{(2)}$. In a similar way we prove that $X \subset X^{(1)}$.

Assuming that X satisfies the shrinking domain condition, we can prove a factorization theorem for the functions in X, at the same time keeping some control on the zeros of the factors.

Lemma 4.5. *Suppose that X satisfies the shrinking domain condition with respect to the couple (V_1, V_2). If $f \in X$ is not identically zero, then for every $\overline{V}_2 \subset E \subset \mathbb{C} \setminus \overline{V}_1$ there exist $f_1 \in X_1$ and $f_2 \in X_2$ such that*
(a) $f = f_1 \cdot f_2$,
(b) $Z_{\Omega_1}(f_1) = Z_\Omega(f) \cap E$,
(c) $Z_{\Omega_2}(f_2) = Z_\Omega(f) \setminus E$.

Proof. The proof follows the same line of arguments as in [1, Lemma 3.7]. □

The next lemma provides, in fact, the spectral decomposition of operators with disjoint spectra, which was known for some time for Banach spaces. In what follows, we use this classical argument in the case where X is a quasi-Banach space of analytic functions.

Lemma 4.6. *Let J be a multiplier invariant subspace of X of index 1, and let E be as in the preceding lemma. Then in X there exist two multiplier invariant subspaces J_1 and J_2 such that*
(a) $J = J_1 \cap J_2$,
(b) $\sigma_{J_1} = \sigma_J \cap E$,
(c) $\sigma_{J_2} = \sigma_J \setminus E$.

Proof. First, we assume that J is an invariant subspace of X. Let T denote the operator $M_z[X/J]$. Since index$(J) = 1$, from Proposition 3.3(a,b) it follows that the spectrum $\sigma_J = \sigma(T)$ is discrete in Ω. Putting $\sigma_1 = \sigma_J \cap E$ and $\sigma_2 = \sigma_J \setminus E$, we let e_1 be a function in $\mathcal{O}(\sigma_J)$ such that e_1 is equal to 1 near σ_1 and to 0 near σ_2. We consider the projection $Q_1 = e_1(T)$, where the operator on the right is defined via the holomorphic functional calculus (see §2). Similarly, we can introduce the projection Q_2. It is easy to check that $Q_1 Q_2 = Q_2 Q_1 = 0$ and that $Q_1 + Q_2$ is equal to identity. This shows that $X/J = \operatorname{im} Q_1 \oplus \operatorname{im} Q_2$, where im stands for the image of X/J under the corresponding projection. It follows that for $k = 1, 2$ the subspace $\operatorname{im} Q_k$ is an invariant subspace of X/J with the property that the spectrum of T restricted to $\operatorname{im} Q_k$ coincides with σ_k. We define an invariant subspace $J_2 \subset X$ by $J_2 = \{ f \in X : f + J \in \operatorname{im} Q_1 \} \supset J$. Observe that, in fact, we have $\operatorname{im} Q_1 = J_2/J$. Now, we let $f \in J_1 \cap J_2$; then $f + J$ belongs to $\operatorname{im} Q_1 \cap \operatorname{im} Q_2$, whence $f + J = 0 + J$ or $f \in J$, and (a) follows. To prove (b) , we denote by T_1 and T_2 the operators of multiplication by z on X/J_1 and on J_2/J, respectively. Since, in fact, T_2 is the restriction of T to the quotient space $J_2/J = \operatorname{im} Q_1$, we see that $\sigma(T_2) = \sigma_1$. On the other hand, the mapping $\Psi\colon J_2/J \to X/J_1$ that sends $f + J$ to $f + J_1$ is an isomorphism, so that $\sigma(T_1) = \sigma(T_2) = \sigma_1$. Part (c) can be proved in a similar way. So far, we have proved the lemma in the case where J is an invariant subspace of X. Now we assume that J is a multiplier invariant subspace of X and that $\varphi \in M(X)$. We know that φ is analytic in Ω, and since $\varphi(\lambda)e_1(\lambda) = e_1(\lambda)\varphi(\lambda)$, we see that $M_\varphi[X/J]Q_1 = Q_1 M_\varphi[X/J]$. Let $f \in J_1$; then $f + J \in \operatorname{im} Q_2 = \ker Q_1$ and

$$Q_1(\varphi f + J) = Q_1 M_\varphi[X/J](f + J) = M_\varphi[X/J]Q_1(f + J) = 0,$$

meaning that $\varphi f \in J_1$. Similarly, we can prove that J_2 is also a multiplier invariant subspace. □

We have prepared the ground for the following proposition about factorization of functions in J in terms of functions in $J_1 \cap X_1$ and in $J_2 \cap X_2$.

Proposition 4.7. *Suppose that $X = X_1 \oplus X_2^0$ satisfies the shrinking domain condition with respect to (V_1, V_2), and that J is a multiplier invariant subspace of X of index 1. If $\mathcal{O}(\overline{\Omega}_1)$ and $\mathcal{O}(\overline{\Omega}_2)$ are dense in X_1 and X_2, respectively, then for every $f \in J$ we have $f = f_1 \cdot f_2$, where $f_1 \in J_1 \cap X_1$ and $f_2 \in J_2 \cap X_2$.*

Proof. Let f_1 and f_2 be as in Lemma 4.5. First, we want to show that $f_1 \in J_1$ and $f_2 \in J_2$. Let g_k be a sequence in $\mathcal{O}(\overline{\Omega}_2)$ such that $g_k \to f_2$ in X_2. Since $X_2 = X^{(2)} \cap \mathcal{O}(\Omega_2)$, it follows that $g_k \to f_2$ in $\mathcal{O}(\Omega_2)$, and consequently, in $\mathcal{O}(\sigma_{J_1})$. Using the continuity of the holomorphic functional calculus, we see that $g_k(M_z[X/J_1]) \to f_2(M_z[X/J_1])$ as $k \to \infty$. In accordance with the arguments of §2, we have

$$M_{g_k}[X/J_1](f_1 + J_1) \to f_2\big(M_z[X/J_1]\big)(f_1 + J_1) \quad \text{as } k \to \infty.$$

Suppose for the moment that $g_k f_1 \to f$ in the quasi-Banach space X, so that

$$f_2\big(M_z[X/J_1]\big)(f_1 + J_1) = \big(\lim_{k\to\infty} g_k f_1\big) + J_1 = f + J_1 = 0 + J_1.$$

Since, by Lemma 4.5, the function f_2 has no zeros near σ_{J_1}, we have

$$f_1 + J_1 = (1/f_2)\big(M_z[X/J_1]\big)(0 + J_1) = 0 + J_1,$$

whence $f_1 \in J_1$. Likewise, $f_2 \in J_2$. To complete the proof, we must show that $g_k f_1 \to f$ in X. To this end, we observe that, combined with the closed graph theorem for F-spaces, Definition 4.4 implies that

$$\|f\|_X \asymp \|f\|_{X^{(1)}} + \|f\|_{X^{(2)}},$$

where $\asymp$ means equivalence of p-norms. From the definition of the norm in the space of multipliers we get $\|f\|_{X^{(1)}} \le \|f_2\|_{M(X^{(1)})} \cdot \|f_1\|_{X^{(1)}}$ and $\|f\|_{X^{(2)}} \le \|f_1\|_{M(X^{(2)})} \cdot \|f_2\|_{X^{(2)}}$. Since

$$\|f_1(g_k - f_2)\|_X \asymp \|f_1(g_k - f_2)\|_{X^{(1)}} + \|f_1(g_k - f_2)\|_{X^{(2)}},$$

we see that for some constant C,

$$\|f_1(g_k - f_2)\|_X \le C\big(\|f_1\|_{X^{(1)}} \cdot \|g_k - f_2\|_{M(X^{(1)})} + \|g_k - f_2\|_{X^{(2)}} \cdot \|f_1\|_{M(X^{(2)})}\big).$$

From §2 we know that $\mathcal{O}(\Omega_2) \subset M(X^{(1)})$, and $g_k \to f_2$ in $\mathcal{O}(\Omega_2)$, by assumption; then g_k tends to f_2 in $M(X^{(1)})$. Since $\|g_k - f_2\|_{X_2} \to 0$, it follows that $g_k - f_2$ tends to zero in $X^{(2)}$ as $k \to \infty$. Consequently, $\|f_1(g_k - f_2)\|_X \to 0$ as $k \to \infty$, concluding the proof of the proposition. □

Now we are ready to prove the main result of the paper. We recall that for a multiplier invariant subspace I of X_1 we denote by $\Lambda(I)$ the closure in X of $I \cdot M(X)$. Finally, let

$$\mathfrak{S}_1 = \big\{\, I : I \text{ is a multiplier invariant subspace of } X_1 \text{ with } \sigma_I \subset \Omega_2 \,\big\},$$

and let $\mathfrak{S}$ be the class of all multiplier invariant subspaces $J \subset X$ with $\sigma_J \subset \Omega_2$.

Theorem 4.8. *Let $X = X_1 \oplus X_2^0$ be a quasi-Banach space of analytic functions on Ω satisfying the shrinking domain condition with respect to the couple (V_1, V_2). If $\mathcal{O}(\overline{\Omega}_1)$ and $\mathcal{O}(\overline{\Omega}_2)$ are dense in X_1 and X_2, respectively, then:*

(a) *The mapping $I \mapsto \Lambda(I)$ is a bijection from $\mathfrak{S}_1$ onto $\mathfrak{S}$. Furthermore, $\sigma_I = \sigma_{\Lambda(I)}$ and the inverse mapping is given by $J \mapsto J \cap X_1$.*

(b) *For every* $I \in \mathfrak{S}_1$ *the mapping* $L_I\colon X \to X_1/I$ *is a surjective module homomorphism that is canonical on* X_1 *and has kernel* $\Lambda(I)$.

(c) *For every* $I \in \mathfrak{S}_1$ *the quotient quasi-Banach spaces* X_1/I *and* $X/\Lambda(I)$ *are canonically quotient multiplier module isomorphic.*

Proof. First we prove (b). By Lemma 4.2, L_I is a surjective multiplier module homomorphism canonical on X_1. It remains to show that $\ker L_I = \Lambda(I)$. The inclusion $\Lambda(I) \subset \ker L_I$ was already proved in Proposition 4.3(b). Putting $\ker L_I = J$, we want to verify that $J \subset \Lambda(I)$. We have $\sigma_J = \sigma_I \subset \Omega_2$ by Proposition 4.3(d), so that Proposition 3.3(c) implies that $\operatorname{index}(J) = 1$. Applying Proposition 4.7 to each $f \in J$, we obtain $f = f_1 \cdot f_2$ with $f_1 \in J_1 \cap X_1$, and $f_2 \in J_2 \cap X_2$, where J_1 and J_2 are as in Proposition 4.7. We may assume that $J_1 = J$ and $J_2 = X_2$. Since $\sigma_J \subset \Omega \cup \partial\Omega_1$, we see that in the situation of Lemma 4.6 σ_{J_2} consists of finitely many points in Ω; hence, we may use Proposition 3.3(d) to deduce that $J_2 = pX$ for some polynomial p with zeros in σ_{J_2}. Therefore, $f_2 = pg$ for some $g \in X$. Since, by construction, p divides any element of J_2, we have $g \in X_2$. Thus, $f = (f_1 \cdot p) \cdot g$, where $f_1 \cdot p \in J_1 \cap J_2 = J$ and $g \in X_2$. But $f_1 \cdot p \in X_1$, so that $f_1 \cdot p \in J \cap X_1 = I$ by Proposition 4.3(c). Finally, $J = (J \cap X_1) \cdot X_2 = I \cdot X_2$. Now, suppose that $f \in J$ can be written as $f = f_1 \cdot f_2$, where $f_1 \in J \cap X_1 = I$ and $f_2 \in X_2$. By our assumption, there exists a sequence $\{f_{2,k}\}_k$ of functions in $\mathcal{O}(\overline{\Omega}_2)$ that approximate f_2 in X_2. As in the proof of the preceding proposition, we have

$$\|f_1(f_2 - f_{2,k})\|_X \asymp \|f_1(f_2 - f_{2,k})\|_{X^{(1)}} + \|f_1(f_2 - f_{2,k})\|_{X^{(2)}},$$
$$\|f_1(f_2 - f_{2,k})\|_X \le C\big(\|f_1\|_{X^{(1)}} \cdot \|f_2 - f_{2,k}\|_{M(X^{(1)})} + \|f_2 - f_{2,k}\|_{X^{(2)}} \cdot \|f_1\|_{M(X^{(2)})}\big),$$

where C is some constant. Since $f_{2,k} \to f_2$ in $\mathcal{O}(\Omega_2)$, we obtain $\|f_2 - f_{2,k}\|_{M(X^{(1)})} \to 0$, because $M(X^{(1)}) \supset \mathcal{O}(\Omega_2)$ by §2. Also, $\|f_2 - f_{2,k}\|_{X^{(2)}} \to 0$ by assumption, whence

$$\|f - f_1 f_{2,k}\|_X = \|f_1(f_2 - f_{2,k})\|_X \to 0 \quad \text{as } k \to \infty.$$

Finally, observing that

$$f_1 f_{2,k} \in (J \cap X_1) \cdot M(X_2) = I \cdot M(X_2) \subset I \cdot M(X),$$

we obtain $f \in \Lambda(I)$, which completes the proof of part (b).

In order to prove part (c), we consider $I \in \mathfrak{S}_1$ and the surjective multiplier module homomorphism L_I that induces L_I^* defined by (4.4). Since $L_I(f) = f + I$ for $f \in X_1$, it follows that $(L_I^*)^{-1}\colon X_1/I \to X/\Lambda(I)$ coincides with the canonical homomorphism $X_1/I \to X/\Lambda(I)$. Observe that for $f \in X_1$ we have $L_I^*\big(f + \Lambda(I)\big) = L_I(f)$ if and only if $(L_I^*)^{-1}\big(L_I(f)\big) = f + \Lambda(I)$, or $(L_I^*)^{-1}(f + I) = f + \Lambda(I)$. Thus, for any $\varphi \in M(X_1)$ and any $f \in X_1$ we can write

$$(L_I^*)^{-1}(\varphi f + I) = \varphi f + \Lambda(I) = M_\varphi[X/\Lambda(I)]\big(f + \Lambda(I)\big),$$

which shows that $(L_I^*)^{-1}$ is a quotient multiplier module homomorphism. Finally, for every $f \in X$ and every $\varphi \in M(X)$ we have

$$L_I^*(\varphi f + \Lambda(I)) = L_I(\varphi f) = L_{I,M}(\varphi) L_I(f) = L_{I,M}(\varphi) L_I^*(f + \Lambda(I)),$$

where $L_{I,M}$ is as in (4.3). Thus, L_I^* is a quotient multiplier module isomorphism (in accordance with the comments following Definition 4.1).

Now we prove (a). The coincidence of the two spectra in question is the content of Proposition 4.3(d). Also, in part (b) above we proved that for every $I \in \mathfrak{S}_1$ we have $\ker L_I = \Lambda(I)$; hence, Proposition 4.3(c) yields $\Lambda(I) \cap X_1 = I$. This means that the mapping $I \mapsto \Lambda(I)$ is injective. It remains to prove surjectivity. Let $J \in \mathfrak{S}$. As in part (b) above, we may assume that $J = (J \cap X_1) \cdot X_2$. Thus, any $f \in J$ can be written as $f = f_1 \cdot f_2$, where $f_1 \in J \cap X_1$ and $f_2 \in X_2$. As before, first we approximate $f_2 \in X_2$ by functions in $\mathcal{O}(\overline{\Omega}_2)$ and then use the same technique to deduce that

$$J \subset \operatorname{clos}\bigl\{(J \cap X_1) \cdot M(X_2)\bigr\} \subset \operatorname{clos}\bigl\{(J \cap X_1) \cdot M(X)\bigr\}.$$

Since the reverse inclusion is clear, we obtain

$$J = \operatorname{clos}\bigl\{(J \cap X_1) \cdot M(X)\bigr\} = \Lambda(J \cap X_1),$$

completing the proof of the theorem. □

Remark 4.9. Let Ω be the annulus $A_\rho = \{\, z \in \mathbb{C} : \rho < |z| < 1 \,\}$, so that Ω_1 is the unit disk and Ω_2 is the exterior disk $\{\, z \in \mathbb{C} : |z| > \rho \}$. Let $\mathfrak{S}_2$ denote the class of all multiplier invariant subspaces I of X_2 satisfying the condition $\sigma_I = \sigma(M_z[X_2/I]) \subset \Omega_1$. Using a suitable Möbius transformation, we can interchange the roles of X_1 and X_2. Then Theorem 4.8 supplies a one-to-one correspondence between $\mathfrak{S}_2$ and $\widehat{\mathfrak{S}}$, where the latter class consists of the multiplier invariant subspaces $J \subset X$ with $\sigma_J \subset \Omega_1$. As was observed in Lemma 4.6, any multiplier invariant subspace $J \subset X$ of index 1 splits: $J = J_1 \cap J_2$ with $J_1 \in \mathfrak{S}$ and $J_2 \in \widehat{\mathfrak{S}}$. By Theorem 4.8, J_1 corresponds to $I_1 = J_1 \cap X_1$, and J_2 corresponds to $I_2 = J_2 \cap X_2$. Consequently, to describe J we only need to understand I_1 and I_2. Moreover, by Proposition 4.7, any $f \in J$ can be written as the product of $f_1 \in I_1$ and $f_2 \in I_2$. Therefore, I_1 and I_2 take care of the boundary behavior of the elements of J near $\partial\Omega_1$ and $\partial\Omega_2$, respectively. In this way, the structure of any multiplier invariant subspace $J \subset X$ of index 1 can be described in terms of the structure of the multiplier invariant subspaces of X_1 and X_2.

Acknowledgement. We thank professor Stefan Richter, University of Tennessee, for his assistance in the proof of Lemma 3.1. We also thank professor Wiesław Żelazko, Polish Academy of Sciences, for helping us with references regarding locally bounded spaces.

References

1. A. Abkar and H. Hedenmalm, *Invariant subspaces on multiply connected domains*, Publ. Mat. **42** (1998), 521–557.
2. G. R. Allan, H. G. Dales, and J. P. McClure, *Pseudo-Banach algebras*, Studia Math. **40** (1971), 55–69.
3. L. Arnold, *Convergence in probability of random power series and a related problem in linear topological spaces*, Statist. Lab. Publ., Michigan State Univ., East Lansing, Michigan, 1966.
4. N. Dunford and J. T. Schwartz, *Linear operators.* I. *General theory*, Interscience, New York, 1958.
5. P. L. Duren, B. W. Romberg, and A. L. Shields, *Linear functionals on H^p spaces with* $0 < p < 1$, J. Reine Angew. Math. **238** (1969), 32–60.
6. S. D. Fisher, *Function theory on planar domains*, Wiley, New York, 1983.
7. B. Gramsch, *Integration und holomorphe Funktionen in lokalbeschränkten Räumen*, Math. Ann. **162** (1965), 190–210.
8. H. Hedenmalm, *Spectral properties of invariant subspaces in the Bergman space*, J. Funct. Anal. **116** (1993), 441–448.
9. H. Hedenmalm, S. Richter, and K. Seip, *Interpolating sequences and invariant subspaces of given index in the Bergman spaces*, J. Reine Angew. Math. **477** (1996), 13–30.
10. T. Kato, *Perturbation theory for linear operators*, Grundlehren Math. Wiss., Bd. 132, Springer-Verlag, New York, 1966.

11. G. Köthe, *Topological vector spaces.* I, Springer-Verlag, New York, 1969.
12. D. Przeworska-Rolewicz and S. Rolewicz, *On integrals of functions with values in a complete linear metric space*, Studia Math. **26** (1966), 121–131.
13. S. Richter, *Invariant subspaces in Banach spaces of analytic functions*, Trans. Amer. Math. Soc. **304** (1987), 585–616.
14. S. Rolewicz, *Metric linear spaces*, Math. Appl. (East European Ser.), vol. 20, PWN-Polish Scientific Publishers, Warsaw; Reidel, Dordrecht–Boston.
15. W. Rudin, *Functional analysis*, McGraw-Hill, New York, 1973.
16. W. Żelazko, *Banach algebras*, Elsevier Publishing Co., Amsterdam; PWN-Polish Scientific Publishers, Warsaw, 1973.
17. ———, *Metric generalizations of Banach algebras*, Dissertationes Math. (Rozprawy Mat.) **47** (1965), 70 pp.
18. ———, *A power series with a finite domain of convergence*, Comment. Math. Prace Mat. **15** (1971), 115–117.

MATHEMATICS DEPARTMENT, LUND UNIVERSITY, BOX 118, 22100 LUND, SWEDEN
E-mail address: ali@maths.lth.se

MATHEMATICS DEPARTMENT, LUND UNIVERSITY, BOX 118, 22100 LUND, SWEDEN
E-mail address: haakan@maths.lth.se

Received 16/NOV/1998

Originally published in English