FOURIER UNIQUENESS IN $\mathbb{R}^4$

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Abstract. We show an interrelation between the uniqueness aspect of the recent Fourier interpolation formula of Radchenko and Viazovska and the Heisenberg uniqueness for the Klein-Gordon equation and the lattice-cross of critical density, studied by Hedenmalm and Montes-Rodríguez. This has been known since 2017.

1. Introduction

1.1. **Basic notation in the plane.** We write $\mathbb{Z}$ for the integers, $\mathbb{Z}_+$ for the positive integers, $\mathbb{R}$ for the real line, and $\mathbb{C}$ for the complex plane. We write $\mathbb{H}$ for the upper half-plane $\{\tau\in\mathbb{C}:\ \mathrm{Im}\,\tau>0\}$. Moreover, we let $\langle\cdot,\cdot\rangle_d$ denote the Euclidean inner product of $\mathbb{R}^d$.

1.2. **The Fourier transform of radial functions.** For a function $f\in L^1(\mathbb{R}^d)$, we consider its Fourier transform (with $x=(x_1,\ldots,x_d)$ and $y=(y_1,\ldots,y_d)$)

$$\hat f(y):=\int_{\mathbb{R}^d}\mathrm{e}^{-\mathrm{i}2\pi\langle x,y\rangle_d}f(x)\mathrm{dvol}_d(x),\quad \mathrm{dvol}_d(x):=\mathrm{d}x_1\cdots\mathrm{d}x_d.$$

If f is radial, then $\hat f$ is radial too. A particular example of a radial function is the Gaussian

$$G_\tau(x):=\mathrm{e}^{\mathrm{i}\pi\tau|x|^2}, \tag{1.2.1}$$

which decays nicely provided that $\mathrm{Im}\,\tau>0$, that is, when $\tau\in\mathbb{H}$. The Fourier transform of a Gaussian is another Gaussian, in this case

$$\hat G_\tau(y):=\Big(\frac{\tau}{\mathrm{i}}\Big)^{-d/2}\mathrm{e}^{-\mathrm{i}\pi|y|^2/\tau}=\Big(\frac{\tau}{\mathrm{i}}\Big)^{-d/2}G_{-1/\tau}(y),$$

Here, it is important that $\tau\mapsto-1/\tau$ preserves hyperbolic space $\mathbb{H}$. In the sense of distribution theory, the above relationship extends to boundary points $\tau\in\mathbb{R}$ as well. We now consider the relationship

$$\Phi(x):=\int_{\mathbb{R}}G_\tau(x)\phi(\tau)\mathrm{d}\tau=\int_{\mathbb{R}}\mathrm{e}^{\mathrm{i}\pi\tau|x|^2}\phi(\tau)\mathrm{d}\tau,\qquad x\in\mathbb{R}^d. \tag{1.2.2}$$

In terms of the Fourier transform, the relationship reads

$$\Phi(x)=\hat\phi_1\Big(-\frac{|x|^2}{2}\Big),$$

where the subscript signifies that we are dealing with the Fourier transform on $\mathbb{R}^1$. This tells us that Φ is radial, but pretty arbitrary, if, say, $\phi\in L^1(\mathbb{R})$. In view of the functional identity (1.2.1), the Fourier transform of the radial function Φ equals

$$\hat\Phi(y):=\int_{\mathbb{R}}\hat G_\tau(y)\phi(\tau)\mathrm{d}\tau=\int_{\mathbb{R}}\Big(\frac{\tau}{\mathrm{i}}\Big)^{-d/2}G_{-1/\tau}(y)\phi(\tau)\mathrm{d}\tau=\int_{\mathbb{R}}\Big(\frac{\tau}{\mathrm{i}}\Big)^{-d/2}\mathrm{e}^{-\mathrm{i}\pi|y|^2/\tau}\phi(\tau)\mathrm{d}\tau. \tag{1.2.3}$$

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We now rewrite the relationships (1.2.2) and (1.2.3) using integration by parts. If ϕ is a tempered test function, integration by parts applied to (1.2.2) gives that

$$\Phi(x)=\frac{\mathrm{i}}{\pi|x|^2}\int_{\mathbb{R}}\mathrm{e}^{\mathrm{i}\pi\tau|x|^2}\phi'(\tau)\mathrm{d}\tau,\qquad x\in\mathbb{R}^d\setminus\{0\}. \tag{1.2.4}$$

A similar application of integration by parts to (1.2.3) gives that

$$\hat{\Phi}(y)=\frac{\mathrm{i}}{\pi|y|^2}\int_{\mathbb{R}}\Big(\frac{\tau}{\mathrm{i}}\Big)^{(4-d)/2}\phi(\tau)\partial_\tau\mathrm{e}^{-\mathrm{i}\pi|y|^2/\tau}\mathrm{d}\tau=\frac{1}{\mathrm{i}\pi|y|^2}\int_{\mathbb{R}}\partial_\tau\Big\{\Big(\frac{\tau}{\mathrm{i}}\Big)^{(4-d)/2}\phi(\tau)\Big\}\mathrm{e}^{-\mathrm{i}\pi|y|^2/\tau}\mathrm{d}\tau, \tag{1.2.5}$$

where $y\in\mathbb{R}^d\setminus\{0\}$, and we need to be a little careful around $\tau=0$ unless $d\in\{0,2,4\}$. We now *restrict to* $d:=4$, so that (1.2.5) simplifies to

$$\hat{\Phi}(y)=\frac{1}{\mathrm{i}\pi|y|^2}\int_{\mathbb{R}}\phi'(\tau)\mathrm{e}^{-\mathrm{i}\pi|y|^2/\tau}\mathrm{d}\tau,\qquad y\in\mathbb{R}^4\setminus\{0\}. \tag{1.2.6}$$

As for the test function ϕ, we could think of the relations (1.2.4) and (1.2.6) as the fundamental relationship in place of (1.2.2) and (1.2.3). This allows us to place conditions on the derivative ϕ' in place of ϕ. For our considerations, we need one more piece of information:

$$\int_{\mathbb{R}}\phi'(\tau)\mathrm{d}\tau=0, \tag{1.2.7}$$

which is obvious for tempered test functions ϕ.

2. Main results

2.1. Fourier uniqueness meets Heisenberg uniqueness and the Klein-Gordon equation. Let $C_0(\mathbb{R})$ denote the space of continuous functions on $\mathbb{R}$ with limit value 0 at infinity. Moreover, let $H^1_+(\mathbb{R})$ denote the Hardy space of the upper half-plane. It may be defined as the subspace of functions in $L^1(\mathbb{R})$ with Poisson harmonic extension to $\mathbb{H}$ which is holomorphic.

Theorem 2.1.1. *Let Φ be given by ϕ in the above fashion, where $\phi\in C_0(\mathbb{R})$ with $\phi'\in L^1(\mathbb{R})$ and $d=4$. If $\Phi(x)=\hat{\Phi}(y)=0$ whenever $x,y\in\mathbb{R}^4$ meet $|x|^2,|y|^2\in\mathbb{Z}_+$, then $\phi\in H^1_+(\mathbb{R})$ and as a consequence, $\Phi(x)\equiv0$ on $\mathbb{R}^4\setminus\{0\}$.*

Proof. In view of the assumptions that $\phi\in C_0(\mathbb{R})$ and $\phi'\in L^1(\mathbb{R})$, it follows that (1.2.7) holds, so that $\phi'\in L^1(\mathbb{R})$ annihilates as a functional the constant function 1. Moreover, by (1.2.4) and (1.2.6), $\phi'\in L^1(\mathbb{R})$ annihilates the subspace of $L^\infty(\mathbb{R})$ spanned by the functions $\mathrm{e}^{\mathrm{i}\pi m\tau}$ and $\mathrm{e}^{-\mathrm{i}\pi n/\tau}$ as well, where $m,n\in\mathbb{Z}_+$ and τ is the real variable. By Theorem 1.8.2 in [4], which relies on technology developed in [5] and is motivated by [3], it follows that $\phi'\in H^1_+(\mathbb{R})$. Finally, in view of the standard Fourier analysis characterization of $H^1_+(\mathbb{R})$, it follows from this and (1.2.4) that $\Phi=0$ on $\mathbb{R}^4\setminus\{0\}$. ◻

Remark 2.1.2. (a) The method used in [3], [4], and [5], as well as in [1] is based on ideas from dynamical systems and ergodic theory, We use the strongest result that is available to us. However, we could use the results in e.g. [3] to derive a similar assertion under stronger assumptions on the function ϕ.
(b) The above theorem is a four-dimensional analogue of the uniqueness part of the Fourier interpolation formula found by Radchenko and Viazovska [6]. That work in its turn was motivated by Fourier interpolation formulæ associated with optimizing the Cohn-Elkies method for sphere packing [7], [2].

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