Fourier uniqueness in even dimensions

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In recent work, methods from the theory of modular forms were used to obtain Fourier uniqueness results in several key dimensions ($d = 1, 8, 24$), in which a function could be uniquely reconstructed from the values of it and its Fourier transform on a discrete set, with the striking application of resolving the sphere packing problem in dimensions $d = 8$ and $d = 24$. In this short note, we present an alternative approach to such results, viable in even dimensions, based instead on the uniqueness theory for the Klein–Gordon equation. Since the existing method for the Klein–Gordon uniqueness theory is based on the study of iterations of Gauss-type maps, this suggests a connection between the latter and methods involving modular forms. The derivation of Fourier uniqueness from the Klein–Gordon theory supplies conditions on the given test function for Fourier interpolation, which are hoped to be optimal or close to optimal.

Fourier transform | Fourier uniqueness | Heisenberg uniqueness pairs | Klein–Gordon equation

1. Introduction

1.1. Basic Notation in the Plane. We write $\mathbb{Z}$ for the integers, $\mathbb{Z}_+$ for the positive integers, $\mathbb{R}$ for the real line, and $\mathbb{C}$ for the complex plane. We write $\mathbb{H}$ for the upper half-plane $\{\tau \in \mathbb{C} : \mathrm{Im}\tau > 0\}$. Moreover, we let $\langle \cdot, \cdot \rangle_d$ denote the Euclidean inner product of $\mathbb{R}^d$.

1.2. Motivation. Oscillatory processes are governed by hyperbolic equations, and little is known about interpolation problems for such equations. About 10 y ago, Hedenmalm and Montes-Rodríguez (1) considered bounded solutions of the Klein–Gordon equation $\partial_x \partial_y u + u = 0$ of the form

$$u_\rho(x, y) = \int_{\mathbb{R}} \exp(\mathrm{i}xt + \mathrm{i}y/t)\, \rho(t) \mathrm{d}t, \quad \rho \in L^1(\mathbb{R})$$

and examined the Goursat problem for such solutions, where instead of prescribing their values on the two characteristics $x = 0$ and $y = 0$, they assumed that these values are known only along a discrete subset of the characteristics consisting of the equidistant points $(an, 0)$ and $(0, bn)$, where $n \in \mathbb{Z}$ and $a, b > 0$ are fixed. They obtained that the values of u_ρ at those points determine the function u_ρ uniquely if and only if $ab \le \pi^2$. The most delicate case is when $ab = \pi^2$, and then we may take $a = b = \pi$ without loss of generality. An essential property of the solutions u_ρ is that the boundary values in any spacelike quarter-plane are connected across the boundary half-lines by a Hankel (or Fourier–Bessel) transform. In particular, the quadrant $[0, +\infty) \times (-\infty, 0]$ is spacelike, and the boundary values of $u = u_\rho$ relate according to

$$u(0, -y) = u(0, 0) - \int_0^{+\infty} \mathrm{J}_1(y, t) u(t, 0) \mathrm{d}t, \qquad y \in [0, +\infty).$$

Here we write $\mathrm{J}_\nu(x, y) := (x/y)^{\nu/2} J_\nu(2\sqrt{xy})$, where J_ν denotes the standard Bessel function, so that this function

$$\mathrm{J}_\nu(x, y) = \sum_{m=0}^{+\infty} \frac{(-1)^m}{m!\Gamma(m + \nu + 1)} x^{m+\nu} y^m$$

becomes well defined and solves the Klein–Gordon equation for $x > 0$ and $\nu > -1$. We associate to this extended Bessel function the corresponding Hankel operator $\mathbf{J}_\nu f(x) = \int_0^{+\infty} \mathrm{J}_\nu(t, x) f(t) \mathrm{d}t$. Later, the result on the discretized Goursat problem was extended to give local uniqueness on spacelike quarter-planes, from data along discrete sequences along the boundary half-lines (2, 3). This

Significance

We show an interrelation between the uniqueness aspect of the recent Fourier interpolation formula of D.R. and M.V. and the lattice-cross uniqueness set for the Klein–Gordon equation studied by H.H. and A.M.-R. With appropriate modifications, the approach applies in any even dimension ≥ 4 and is based on a sophisticated analysis of the iterates of a Gauss-type map.

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suggests the following general problem. Let $\mathfrak{F}$ be a linear space of continuous functions on the half-line $[0,+\infty)$ with the property that if $f\in\mathfrak{F}$, then $\mathbf{J}_\nu f$ is well defined and continuous as well. For a given parameter value ν and $a, b>0$ as above, we would like to have the implication

$$(\forall n=0,1,2,\ldots:\ f(bn)=\mathbf{J}_\nu f(an)=0)\implies f=0$$

for as wide as possible space of functions $f\in\mathfrak{F}$. Moreover, since it is well known that the Fourier transform of radial functions in $\mathbb{R}^d$ may be expressed in terms of the Hankel transform with $\nu=\frac{d}{2}-1$, $d\geq 1$,

$$\int_{\mathbb{R}^d}\mathrm{e}^{-2\pi\mathrm{i}\langle x,y\rangle_d}f(\pi|x|^2)\,\mathrm{dvol}_d(x)=\mathbf{J}_{\frac{d}{2}-1}f(\pi|y|^2),\qquad \mathrm{dvol}_d(x):=\mathrm{d}x_1\cdots\mathrm{d}x_d,\qquad |x|^2=\langle x,x\rangle_d$$

this question can be recast in terms of the uniqueness of a radial function in terms of its values and those of its Fourier transform along a sequence that is a fixed positive multiple of the square roots of nonnegative integers. This program was carried out for radial test functions in the Schwartz class in dimension $d=1$ by Radchenko and Viazovska with explicit interpolation formulæ (4), and a variant with double zeros along the sequences was successful in dimensions $d=8$ and $d=24$ (5, 6). Recently, Stoller extended the Fourier uniqueness problem to nonradial test functions with the function and its Fourier transform given along spherical shells, with explicit interpolation formulæ (7) in all dimensions $d\geq 2$. The goal of the present paper is to show that the uniqueness property holds for a substantially wider class of radial functions in even-dimensional $\mathbb{R}^d$, without further explicit use of Bessel functions.

1.3. The Fourier Transform of Radial Functions. For a function $f\in L^1(\mathbb{R}^d)$. we consider its Fourier transform (with $x=(x_1,\ldots,x_d)$ and $y=(y_1,\ldots,y_d)$)

$$\hat{f}(y):=\int_{\mathbb{R}^d}\mathrm{e}^{-2\pi\mathrm{i}\langle x,y\rangle_d}f(x)\mathrm{dvol}_d(x),$$

If f is radial, then $\hat{f}$ is radial too. A particular example of a radial function is the Gaussian

$$G_\tau(x):=\mathrm{e}^{\mathrm{i}\pi\tau|x|^2}, \tag{1}$$

which decays nicely provided that $\mathrm{Im}\tau>0$, that is, when $\tau\in\mathbb{H}$. The Fourier transform of a Gaussian is another Gaussian, in this case

$$\hat{G}_\tau(y):=\left(\frac{\tau}{\mathrm{i}}\right)^{-d/2}\mathrm{e}^{-\mathrm{i}\pi|y|^2/\tau}=\left(\frac{\tau}{\mathrm{i}}\right)^{-d/2}G_{-1/\tau}(y). \tag{2}$$

Here it is important that $\tau\mapsto -1/\tau$ preserves hyperbolic space $\mathbb{H}$. The Fourier transform extends to tempered distributions, and in this sense, the above relationship [**2**] extends to boundary points $\tau\in\mathbb{R}$ as well. We now consider the relationship

$$\Phi(x):=\int_{\mathbb{R}}G_\tau(x)\phi(\tau)\mathrm{d}\tau=\int_{\mathbb{R}}\mathrm{e}^{\mathrm{i}\pi\tau|x|^2}\phi(\tau)\mathrm{d}\tau,\qquad x\in\mathbb{R}^d. \tag{3}$$

In terms of the Fourier transform, the relationship reads

$$\Phi(x)=\hat{\phi}_1\left(-\frac{|x|^2}{2}\right),$$

where the subscript signifies that we are dealing with the Fourier transform on $\mathbb{R}^1$. This tells us that Φ is radial, but pretty arbitrary, if, say, $\phi\in L^1(\mathbb{R})$. If, e.g., ϕ ranges over the Schwartz test functions on $\mathbb{R}$, then Φ ranges over the radial Schwartz test functions on $\mathbb{R}^d$ [this is a consequence of the work of Hassler Whitney on the structure of smooth even functions (8)]. In view of the functional identity [**1**], the Fourier transform of the radial function Φ equals

$$\hat{\Phi}(y):=\int_{\mathbb{R}}\hat{G}_\tau(y)\phi(\tau)\mathrm{d}\tau=\int_{\mathbb{R}}\left(\frac{\tau}{\mathrm{i}}\right)^{-d/2}G_{-1/\tau}(y)\phi(\tau)\mathrm{d}\tau=\int_{\mathbb{R}}\left(\frac{\tau}{\mathrm{i}}\right)^{-d/2}\mathrm{e}^{-\mathrm{i}\pi|y|^2/\tau}\phi(\tau)\mathrm{d}\tau. \tag{4}$$

The function $\hat{\Phi}$ may be thought of as a Gauss–Schrödinger transform of ϕ, given that the integral kernel $\hat{G}_\tau$ is the fundamental solution of a Schrödinger equation without potential. We now rewrite the relationships [**3**] and [**4**] using integration by parts. If ϕ is a tempered test function, integration by parts applied to [**3**] gives that

$$\Phi(x)=\frac{\mathrm{i}}{\pi|x|^2}\int_{\mathbb{R}}\mathrm{e}^{\mathrm{i}\pi\tau|x|^2}\phi'(\tau)\mathrm{d}\tau,\qquad x\in\mathbb{R}^d\setminus\{0\}. \tag{5}$$

A similar application of integration by parts to [**4**] gives that

$$\hat{\Phi}(y)=\frac{\mathrm{i}}{\pi|y|^2}\int_{\mathbb{R}}\left(\frac{\tau}{\mathrm{i}}\right)^{(4-d)/2}\phi(\tau)\partial_\tau\mathrm{e}^{-\mathrm{i}\pi|y|^2/\tau}\mathrm{d}\tau=\frac{1}{\mathrm{i}\pi|y|^2}\int_{\mathbb{R}}\partial_\tau\left\{\left(\frac{\tau}{\mathrm{i}}\right)^{(4-d)/2}\phi(\tau)\right\}\mathrm{e}^{-\mathrm{i}\pi|y|^2/\tau}\mathrm{d}\tau, \tag{6}$$

where $y\in\mathbb{R}^d\setminus\{0\}$, and we need to be a little careful around $\tau=0$ unless $d\in\{0,2,4\}$. For $d=4$, [**6**] simplifies to

$$\hat{\Phi}(y)=\frac{1}{\mathrm{i}\pi|y|^2}\int_{\mathbb{R}}\phi'(\tau)\mathrm{e}^{-\mathrm{i}\pi|y|^2/\tau}\mathrm{d}\tau,\qquad y\in\mathbb{R}^4\setminus\{0\}. \tag{7}$$

As for the test function ϕ, we could think of the relations [**5**] and [**7**] as the fundamental relationship in place of [**3**] and [**4**]. This allows us to place conditions on the derivative ϕ' in place of ϕ. For our considerations, we need one more piece of information:

$$\int_{\mathbb{R}} \phi'(\tau)\mathrm{d}\tau = 0, \tag{8}$$

which is obvious for, e.g., Schwartz test functions ϕ.

2. Main Results

2.1. The Setup in Dimension 4. We focus on $\mathbb{R}^4$ only and consider for $\psi \in L^1(\mathbb{R})$ the associated function

$$\Psi(x) = \frac{\mathrm{i}}{\pi|x|^2}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\pi\tau|x|^2}\psi(\tau)\mathrm{d}\tau, \qquad x \in \mathbb{R}^4 \setminus \{0\}. \tag{9}$$

This is the same as relation [**5**], only ψ replaces ϕ', while Ψ replaces Φ. For real τ, let $H_{\tau,4}$ denote the function

$$H_{\tau,4}(x) := \frac{\mathrm{e}^{\mathrm{i}\pi|x|^2\tau}}{|x|^2}, \qquad x \in \mathbb{R}^4 \setminus \{0\}, \tag{10}$$

which is locally integrable and decays at infinity. As such, it is a tempered distribution, and its Fourier transform equals

$$\hat{H}_{\tau,4}(y) = \frac{1-\mathrm{e}^{-\mathrm{i}\pi|y|^2/\tau}}{|y|^2} = \frac{1}{|y|^2} - H_{-1/\tau,4}(y). \tag{11}$$

This is the integrated version of the Fourier transformation law for Gaussians [**2**] in dimension $d = 4$. Indeed, if we differentiate with respect to τ in [**11**], we recover [**2**]. In other words, differentiation with respect to τ gives us that $\hat{H}_{\tau,4} + H_{-1/\tau,4}$ is independent of τ. By letting τ tend to 0, the identification with the Newton kernel as in [**11**] follows from the Riemann–Lebesgue lemma. In view of [**11**], the Fourier transform of the function Ψ given by [**9**] is in the sense of distribution theory

$$\hat{\Psi}(y) = \frac{\mathrm{i}}{\pi}\int_{\mathbb{R}} \hat{H}_{\tau,4}(y)\psi(\tau)\mathrm{d}\tau = \frac{\mathrm{i}}{\pi|y|^2}\int_{\mathbb{R}}\psi(\tau)\mathrm{d}\tau - \frac{\mathrm{i}}{\pi|y|^2}\int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\pi|y|^2/\tau}\psi(\tau)\mathrm{d}\tau, \qquad y \in \mathbb{R}^4 \setminus \{0\}. \tag{12}$$

This formula extends [**1.7**].

2.2. Fourier Uniqueness Meets Heisenberg Uniqueness and the Klein–Gordon Equation. In ref. 1, in the context of the Klein–Gordon equation in $1+1$ dimensions, Hedenmalm and Montes-Rodríguez found discrete uniqueness sets along characteristic directions, based on ideas from dynamical systems and ergodic theory. We apply the approach in refs. 1–3 and 9 to obtain a uniqueness result for the pair ψ, Ψ connected by [**9**]. Let $H^1_+(\mathbb{R})$ denote the Hardy space of the upper half-plane. It may be defined as the subspace of functions in $L^1(\mathbb{R})$ with Poisson harmonic extension to $\mathbb{H}$ which is holomorphic.

Theorem 1. *Let $\psi \in L^1(\mathbb{R})$ and Ψ be as above. If $\Psi(x) = \hat{\Psi}(y) = 0$ holds for all $x, y \in \mathbb{Z}^4 \setminus \{0\}$, and if $\Psi(x) = \mathrm{o}(|x|^{-2})$ as $|x| \to 0$, then $\psi \in H^1_+(\mathbb{R})$ and, as a consequence, $\Psi(x) \equiv 0$ on $\mathbb{R}^4 \setminus \{0\}$.*

Proof: In view of the assumption that $\Psi(x) = \mathrm{o}(|x|^{-2})$ as $|x| \to 0$, it follows from [**9**] that $\psi \in L^1(\mathbb{R})$ annihilates the constant function 1. Moreover, by the Lagrange four squares theorem, each positive integer may be written as $|x|^2$ for some $x \in \mathbb{Z}^4 \setminus \{0\}$. Consequently, we see from [**5**] and [**7**] that ψ also annihilates the subspace of $L^\infty(\mathbb{R})$ spanned by the functions $\mathrm{e}^{\mathrm{i}\pi m\tau}$ and $\mathrm{e}^{-\mathrm{i}\pi n/\tau}$, where $m, n \in \mathbb{Z}_+$ and τ denotes the real variable. By theorem 1.8.2 in ref. 2, which relies on methods developed in ref. 3 and is motivated by ref. 1, we may conclude that $\psi \in H^1_+(\mathbb{R})$. Finally, in view of the standard Fourier analysis characterization of $H^1_+(\mathbb{R})$, it follows from this and [**5**] that $\Psi = 0$ on $\mathbb{R}^4 \setminus \{0\}$. This finishes the proof of the theorem.

We return to the initial setup with ϕ and Φ and think of $\phi' = \psi$ and $\Phi = \Psi$. In terms of notation, let $C_0(\mathbb{R})$ denote the space of continuous functions on $\mathbb{R}$ with limit value 0 at infinity. Then the condition at the origin in *Theorem 1* may be replaced by $\phi \in C_0(\mathbb{R})$.

Corollary 2. *Let Φ be given by [**5**], where $\phi \in C_0(\mathbb{R})$ with $\phi' \in L^1(\mathbb{R})$ and $d = 4$. If $\Phi(x) = \hat{\Phi}(y) = 0$ for all $x, y \in \mathbb{Z}^4 \setminus \{0\}$, then $\phi' \in H^1_+(\mathbb{R})$ and, as a consequence, $\Phi(x) \equiv 0$ on $\mathbb{R}^4 \setminus \{0\}$.*

Remark. The above theorem is a four-dimensional analogue of the uniqueness part of the Fourier interpolation formula found by Radchenko and Viazovska (4). That work is based on a method invented by Viazovska to realize the Cohn–Elkies upper bound for sphere packing (5, 6).

3. Modifications in Higher Even Dimensions

3.1. The Higher-Dimensional Kernel and Its Fourier Transform. We consider even dimensions $d \geq 4$ only and write $d = 2d_0$ with $d_0 \geq 2$. We are interested in the kernel

$$H_{\tau,d}(x) := \frac{\mathrm{e}^{\mathrm{i}\pi|x|^2\tau}}{|x|^{d-2}}, \qquad x \in \mathbb{R}^d \setminus \{0\}. \tag{13}$$

Its Fourier transform as a tempered distribution is given by

$$\hat{H}_{\tau,d}(y) = \frac{(\mathrm{i}\tau)^{d_0-2}}{|y|^{d-2}}\left\{\sum_{j=0}^{d_0-2}\frac{1}{j!}(-\mathrm{i}\pi|y|^2/\tau)^j - \mathrm{e}^{-\mathrm{i}\pi|y|^2/\tau}\right\}, \qquad y \in \mathbb{R}^d \setminus \{0\}.$$

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For fixed y, the function $\tau \mapsto \hat{H}_{\tau,d}(y)$ is a bounded holomorphic function in $\mathbb{H}$. Moreover, $(\tau, y) \mapsto \hat{H}_{\tau,d}(y)$ solves a Schrödinger equation without potential and initial datum $\hat{H}_{0,d}(y) = c_d |y|^{-2}$. Here $c_d = \pi^{d_0-2}/(d_0-2)!$ is the volume of the ball in dimension $d-4$.

3.2. The Fourier Uniqueness Theorem in Higher Even Dimensions. Next, for $\psi \in L^1(\mathbb{R})$, we let

$$\Psi(x) = \left(\frac{\mathrm{i}}{\pi}\right)^{d_0-1} \int_{\mathbb{R}} H_{\tau,d}(x)\psi(\tau)\mathrm{d}\tau = \left(\frac{\mathrm{i}}{\pi|x|^2}\right)^{d_0-1} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\pi\tau|x|^2}\psi(\tau)\mathrm{d}\tau, \qquad x \in \mathbb{R}^d \setminus \{0\}. \qquad [14]$$

If, e.g., $\psi = \phi^{(d_0-1)}$ for a function $\phi \in C_0(\mathbb{R})$ with $\phi^{(j)} \in C_0(\mathbb{R})$ for $j \leq d_0 - 2$, we may make sense of [**14**] as arising from [**3**] by successive integration by parts, with $\Phi = \Psi$. The Fourier transform of Ψ is given by

$$\hat{\Psi}(y) = \left(\frac{\mathrm{i}}{\pi}\right)^{d_0-1} \int_{\mathbb{R}} \hat{H}_{\tau,d}(x)\psi(\tau)\mathrm{d}\tau, \qquad y \in \mathbb{R}^d \setminus \{0\}.$$

We may now formulate the analogue of *Theorem 1* which applies in arbitrary even dimension $d \geq 4$.

Theorem 2. *Suppose the dimension $d \geq 4$ is even, and let $\psi \in L^1(\mathbb{R})$ and Ψ be as above. If $\Psi(x) = \hat{\Psi}(y) = 0$ holds for all $x, y \in \mathbb{Z}^d \setminus \{0\}$, and if $\Psi(x) = \mathrm{o}(|x|^{-d+2})$ as $|x| \to 0$, then $\psi \in H^1_+(\mathbb{R})$, and as a consequence, $\Psi(x) \equiv 0$ on $\mathbb{R}^d \setminus \{0\}$.*

The proof of this theorem for $d \geq 6$ is based on an extension of the methods developed in refs. 2, 3 and will be presented elsewhere. Here we only mention that the method relies on a sophisticated analysis of the iterates of the weighted transfer operators

$$\mathbf{T}_{d_0} f(t) = \sum_{k \in \mathbb{Z} \setminus \{0\}} (2k - t)^{-d_0} f\left(\frac{1}{2k - t}\right)$$

for the even Gauss map $x \mapsto -1/x - 2\lfloor (x-1)/(2x) \rfloor$ on the punctured interval $[-1, 1) \setminus \{0\}$, which requires a new blend of ideas involving, e.g., methods of totally positive matrices. The needed results about the iterates of the weighted transfer operator cannot be obtained by application of the standard methods of ergodic theory. Here $\lfloor x \rfloor$ denotes the usual integer part of $x \in \mathbb{R}$. As a side remark, we mention that higher-dimensional uniqueness results for radial functions lead to Fourier uniqueness results along concentric shells for nonradial functions; see Stoller (7) for details.

Data Availability. There are no data underlying this work.

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