The Bergman projection

We let $\mathbb{D}$ be the open unit disk.

Let $P$ denote the Bergman projection

$$Pf(z) := \int_{\mathbb{D}} \frac{f(w)}{(1 - z\bar{w})^2} \, dA(w), \quad z \in \mathbb{D},$$

which is well-defined if $f \in L^1(\mathbb{D})$ ($dA$ is normalized area measure). It is well-known that $P$ maps $L^p(\mathbb{D}) \to L^p(\mathbb{D})$ for $1 < p < +\infty$, and that

$$P : L^2(\mathbb{D}) \to L^2(\mathbb{D})$$

is a norm contraction. We write $\langle \cdot, \cdot \rangle_{\mathbb{D}}$ for the sesquilinear form

$$\langle f, g \rangle_{\mathbb{D}} := \int_{\mathbb{D}} f(z)\bar{g}(z) \, dA(z),$$

which is well-defined if $f\bar{g} \in L^1(\mathbb{D})$. We shall be concerned with the space $PL^\infty(\mathbb{D})$, supplied with the canonical norm

$$\|f\|_{PL^\infty(\mathbb{D})} := \inf \{ \|\mu\|_{L^\infty(\mathbb{D})} : \mu \in L^\infty(\mathbb{D}) \text{ and } f = P\mu \}.$$
The Bloch space

It is well-known that as a space, $\mathbf{PL}^\infty(\mathbb{D}) = \mathcal{B}(\mathbb{D})$, the Bloch space. This seems to have been observed first in a 1976 paper by Coifman, Rochberg, Weiss [CRW]. We recall that the Bloch space consists of all holomorphic $f : \mathbb{D} \to \mathbb{C}$ subject to the seminorm boundedness condition

$$\|f\|_{\mathcal{B}(\mathbb{D})} := \sup_{z \in \mathbb{D}} (1 - |z|^2)|f'(z)| < +\infty.$$ 

Indeed, if $f = \mathbf{P}\mu$, where $\|\mu\|_{L^\infty(\mathbb{D})} \leq 1$, then

$$\begin{align*}
(1 - |z|^2)|(\mathbf{P}\mu)'(z)| &= 2(1 - |z|^2)\left| \int_{\mathbb{D}} \frac{\bar{w}\mu(w)}{(1 - z\bar{w})^3} dA(w) \right| \\
&\leq 2(1 - |z|^2) \int_{\mathbb{D}} \frac{|w|}{|1 - z\bar{w}|^3} dA(w) = 2(1 - |z|^2) \sum_{j=0}^{+\infty} \frac{[(\frac{3}{2})j]^2}{(j!)^2(j + \frac{3}{2})}|z|^{2j} \leq \frac{8}{\pi},
\end{align*}$$

where the main loss of information is in the application of the triangle inequality.
On the other hand, if $f \in \mathcal{B}(\mathbb{D})$ with $f(0) = f'(0) = 0$, and we put

$$
\mu_f(w) := (1 - |w|^2) \frac{f'(w)}{\bar{w}}, \quad w \in \mathbb{D},
$$

then $\mu_f \in L^\infty(\mathbb{D})$, and

$$
|\mu_f(w)| = (1 - |w|^2) \left| \frac{f'(w)}{w} \right| \leq (1 + o(1)) \|f\|_{\mathcal{B}(\mathbb{D})} \quad \text{as} \quad |w| \to 1,
$$

while

$$
P_{\mu_f}(z) = \int_{\mathbb{D}} \frac{(1 - |w|^2)f'(w)}{\bar{w}(1 - z\bar{w})^2} dA(w) = f(z) - f(0) = f(z), \quad z \in \mathbb{D}.
$$

so in terms of boundary effects there would appear to be a gap of the size $8/\pi$ between the two norms. Perälä [Per] has shown that the bound $8/\pi$ in (1) is best possible.
The Bloch space as a dual space

It is known that with respect to $\langle \cdot, \cdot \rangle_{\mathbb{D}}$, $\mathcal{B}(\mathbb{D})$ can be identified with the dual space of the Bergman space $A^1(\mathbb{D})$ in much the same way that $\text{BMOA}(\mathbb{D})$ is the dual space of $H^1(\mathbb{D})$ with respect to the dual action on the circle $\mathbb{T}$:

$$\langle f, g \rangle_{\mathbb{T}} := \int_{\mathbb{T}} f(z) \overline{g}(z) ds(z),$$

where $ds(z) := |dz|/(2\pi)$ is normalized arc length measure. Indeed, the norm induced on $\mathcal{B}(\mathbb{D})$ by $A^1(\mathbb{D})$ is that of $PL^{\infty}(\mathbb{D})$:

**PROPOSITION 1**

For $f \in \mathcal{B}(\mathbb{D})$, we have that

$$\|f\|_{PL^{\infty}(\mathbb{D})} = \sup \{ |\langle f, g \rangle_{\mathbb{D}}| : g \in A^2(\mathbb{D}), \|g\|_{A^1(\mathbb{D})} \leq 1 \}.$$
Remarks on Proposition 1

The assertion is a consequence of the Hahn-Banach theorem. Indeed, a bounded linear functional on $A^1(\mathbb{D})$ is lifted to a bounded linear functional on $L^1(\mathbb{D})$ with the same norm. Such functionals then correspond to elements of $L^\infty(\mathbb{D})$.

This is used to see that with respect to $\langle \cdot, \cdot \rangle_\mathbb{D}$, the dual space to $A^1(\mathbb{D})$ is $PL^\infty(\mathbb{D})$, isometrically and isomorphically.
Duality and dilations

For a function $f$, we let $f_r(z) := f(rz)$ denote its dilate. We shall need the following identities.

**PROPOSITION 2**
Suppose $g, h \in L^1(\mathbb{D})$ are both harmonic. Then, for $0 < r < 1$, we have

$$\langle g_r, h \rangle_D = \langle g, h_r \rangle_D$$

In other words, the dilation $D_r f(z) = f(rz)$ is self-adjoint with respect to the duality $\langle \cdot, \cdot \rangle_D$.

**PROPOSITION 3**
Suppose $g = P\mu$, where $\mu \in L^\infty(\mathbb{D})$, and that $h \in L^\infty(\mathbb{D})$ is harmonic. Then

$$\langle zg_r, h \rangle_T = \langle g_r, \partial h \rangle_D = \langle g, (\partial h)_r \rangle_D = \langle P\mu, (\partial h)_r \rangle_D = \langle \mu, (\partial h)_r \rangle_D.$$
Remarks on Proposition 3

The first equality follows from Green’s formula. The second step uses that the dilation is self-adjoint, which is easy to check using Taylor series expansions. The third equality expresses that $g = P \mu$, while the fourth uses that $P$ is self-adjoint and preserves the holomorphic functions.
A basic estimate

**Corollary 4**

Suppose $g = P\mu$, where $\mu \in L^\infty(\mathbb{D})$, and that $h \in H^\infty(\mathbb{D})$. Then

$$|\langle zg_r, h \rangle_T| \leq \|\mu\|_{L^\infty(\mathbb{D})} \|(h')_r\|_{A^1(\mathbb{D})} = \frac{\|\mu\|_{L^\infty(\mathbb{D})}}{r^2} \int_{\mathbb{D}(0, r)} |h'| dA.$$

We now resort to the Cauchy-Schwarz inequality:

In the setting of Corollary 4, we have that

$$|\langle zg_r, h \rangle_T| \leq \frac{\|\mu\|_{L^\infty(\mathbb{D})}}{r^2} \int_{\mathbb{D}(0, r)} |h'| dA$$

$$\leq \frac{\|\mu\|_{L^\infty(\mathbb{D})}}{r^2} \left( \int_{\mathbb{D}(0, r)} |h'(z)|^2 (1 - |z|^2) dA(z) \right)^{1/2} \left( \int_{\mathbb{D}(0, r)} \frac{dA(z)}{1 - |z|^2} \right)^{1/2}$$

$$\leq \frac{\|\mu\|_{L^\infty(\mathbb{D})}}{r^2} \left( \int_{\mathbb{D}} |h'(z)|^2 (1 - |z|^2) dA(z) \right)^{1/2} \left( \log \frac{1}{1 - r^2} \right)^{1/2}.$$

(2)
Application of the Littlewood-Paley identity

If \( h(0) = 0 \), the Littlewood-Paley identity asserts that

\[
\|h\|_{L^2(T)}^2 = \int_{\mathbb{D}} |h'(z)|^2 \log \frac{1}{|z|^2} \, dA(z) \geq \int_{\mathbb{D}} |h'(z)|^2 (1 - |z|^2) \, dA(z)
\]

by the elementary inequality

\[
1 - |z|^2 \leq \log \frac{1}{|z|^2}.
\]

It now follows from (2) that

\[
|\langle zg_r, h \rangle_T| \leq \frac{\|\mu\|_{L^\infty(D)}}{r^2} \|h\|_{L^2(T)} \left( \log \frac{1}{1 - r^2} \right)^{1/2},
\]

and with \( h = zg_r \) we obtain

\[
\|g_r\|_{L^2(T)} \leq \frac{\|\mu\|_{L^\infty(D)}}{r^2} \left( \log \frac{1}{1 - r^2} \right)^{1/2}.
\]  \hspace{1cm} (3)
The asymptotic variance of a Bloch function

Definition
For a given function $g$ in the Bloch space, its asymptotic variance is the quantity

$$\sigma(g)^2 := \limsup_{r \to 1^-} \frac{\|gr\|^2_{L^2(\mathbb{T})}}{\log \frac{1}{1-r^2}}.$$ 

The estimate (3) shows that for $g = P\mu$,

$$\sigma(g)^2 \leq \|\mu\|^2_{L^\infty(\mathbb{D})}.$$ 

Definition
The universal asymptotic variance for $PL^\infty(\mathbb{D})$ is the quantity

$$\Sigma^2 := \sup_{\|\mu\|_{L^\infty(\mathbb{D})} \leq 1} \sigma(P\mu)^2.$$ 

From (3) it is clear that $\Sigma^2 \leq 1$. This estimate was obtained by other means in [AIPP].
Why is \( \Sigma^2 \) of interest?

\( \Sigma^2 \) is related to quasiconformal mappings of with small \( k \sim 0 \):

\[
|\bar{\partial}\varphi| \leq k|\partial\varphi|.
\]

Let \( S_k \) denote the class of \( \varphi \) which are conformal on \( \mathbb{D} \) and have a global \( k \)-quasiconformal extension. For \( t \in \mathbb{C} \), we define

\[
\beta_\varphi(t) := \limsup_{r \to 1^-} \frac{\log \int_T |(\varphi'_r)^t| ds}{\log \log \frac{1}{1-r}}
\]

and put (quasiconformal integral means spectrum)

\[
B(k, t) := \sup_{\varphi \in S_k} \beta_\varphi(t).
\]

**THEOREM (Ivrii)**

For \( k, t \sim 0 \), we have that

\[
B(k, t) \sim \frac{\Sigma^2}{4} k^2 |t|^2.
\]

See [Ivr].
Let $D(k)$ denote the maximal Hausdorff (or Minkowski) dimension of a $k$-quasicircle (the image of the unit circle under a $k$-quasiconformal mapping). It turns out that it is also related to $\Sigma^2$:

$$D(k) = 1 + \Sigma^2 k^2 + O(k^{8/3}) \quad \text{as} \quad k \to 0.$$ 

See [Ivr]. It is an open question whether the error bound can be improved to $O(k^3)$. 
The bound $\Sigma^2 < 1$

Let

$$\rho_{\alpha, \beta}(\mathbb{H}) := \liminf_{r \to 1^-} \inf f \frac{\int_{\mathbb{D}(0, r)} ((1 - |z|^2)^\alpha |f(z)|^\beta - 1)^2 \frac{dA(z)}{1 - |z|^2}}{\int_{\mathbb{D}(0, r)} \frac{dA(z)}{1 - |z|^2}}.$$ 

where the infimum is over all polynomials $f$.

**THEOREM**

We have that $\Sigma^2 = 1 - \rho_{1,1}(\mathbb{H})$. Moreover, $\rho_{1,1}(\mathbb{H}) \geq 2.3 \times 10^{-8}$ so that in particular $\Sigma^2 < 1$. 
A Hilbert space lemma

The following lemma is very helpful.

**LEMMA**

Let $\mathcal{H}$ be a real-linear Hilbert space, and suppose $u, v \in \mathcal{H}$ and $0 \leq \theta \leq 1$. TFAE:

(a) $\forall c \in \mathbb{R}: \|u - cv\|_\mathcal{H} \geq \theta \|u\|_\mathcal{H}$,

(b) $\forall c \in \mathbb{R}: \|u - cv\|_\mathcal{H} \geq |c|\theta \|v\|_\mathcal{H}$,

(c) $|\langle u, v \rangle_\mathcal{H}| \leq (1 - \theta^2)^{1/2} \|u\|_\mathcal{H} \|v\|_\mathcal{H}$. 
Alternative definition of $\Sigma^2$

It is natural to also consider the possibility of changing the order of the operations in the definition of $\Sigma^2$:

$$\tilde{\Sigma}^2 := \limsup_{r \to 1^-} \sup_{\|\mu\|_{L^\infty(D)} \leq 1} \frac{\|g_r\|^2_{L^2(T)}}{\log \frac{1}{1-r^2}}, \quad g = P\mu.$$  

Then clearly $\tilde{\Sigma}^2 \geq \Sigma^2$, and one might guess that $\tilde{\Sigma}^2 = \Sigma^2$. In terms of the dilation operator $D_rf(z) = f(rz)$, it is a matter of definition that

$$\tilde{\Sigma}^2 := \limsup_{r \to 1^-} \frac{\|D_r\|^2_{PL^\infty(D) \to L^2(T)}}{\log \frac{1}{1-r^2}}.$$  

Since $D_r$ is self-adjoint by Proposition 2, and the $L^2(T)$ norm is the same as the norm on $H^2(D)$ norm on holomorphic functions, we must also have

$$\tilde{\Sigma}^2 = \limsup_{r \to 1^-} \frac{\|D_r\|^2_{(H^2(D))^* \to A^1(D)}}{\log \frac{1}{1-r^2}},$$  

where $(H^2(D))^*$ is the dual with respect to $\langle \cdot, \cdot \rangle_D$. 
The identity $\Sigma^2 = 1 - \rho_{1,1}(\mathbb{H})$

Up to inessential contributions, the space $(H^2(\mathbb{D}))^*$ can be identified with the Hilbert space $A^2_1(\mathbb{D})$ of holomorphic functions $f$ with finite norm

$$\|f\|^2_{A^2_1(\mathbb{D})} := \int_{\mathbb{D}} |f(z)|^2 (1 - |z|^2) dA(z) < +\infty.$$ 

It follows that

$$\tilde{\Sigma}^2 = \limsup_{r \to 1^-} \sup_f \frac{\|f_r\|^2_{A^1_1(\mathbb{D})}}{\|f\|^2_{A^2_1(\mathbb{D})} \log \frac{1}{1-r^2}},$$

and Lemma ?? now shows that $\tilde{\Sigma}^2 = 1 - \rho_{1,1}(\mathbb{H})$. Actually, a recent theorem of Wennman is needed as well. Finally, an argument involving the construction of a single quasioptimizing $\mu$ out of a sequence of optimizers shows that $\tilde{\Sigma}^2 = \Sigma^2$. 
The theorem $\rho_{1,1}(\mathbb{H}) \geq 2.3 \times 10^{-8}$

Basically, the argument begins with a local estimate which obtains a universal lower bound for

$$\int_{D(0, \frac{1}{2})} ((1 - |z|^2)|f(z)| - 1)^2 \frac{dA(z)}{1 - |z|^2}.$$ 

This estimate is the moved around by a Möbius mapping to other disks and after suitable integration the desired result is obtained.
Analogous planar densities

We consider the asymptotic density

$$\rho_\beta(\mathbb{C}) := \liminf_{R \to +\infty} R^{-2} \inf_f \int_{\mathbb{D}(0, R)} \left( e^{-|z|^2} |f(z)|^\beta - 1 \right)^2 dA(z),$$

where again $f$ runs over all polynomials. Here, $\beta > 0$ is assumed. Again, one shows that $\rho_\beta(\mathbb{C}) > 0$, so the density is nontrivial. This is a (local) limit case of the hyperbolic density, since

$$\left( 1 - \frac{|z|^2}{\alpha} \right)^\alpha \to e^{-|z|^2}$$

as $\alpha \to +\infty$. 
Abrikosov’s work on superconductivity à la Nier

In view of Lemma ??,

\[
\frac{1}{1 - \rho_\beta(\mathbb{C})} = \liminf_{R \to +\infty} \frac{R^2 \int_{\mathbb{D}(0,R)} |f(z)|^{2\beta} e^{-2|z|^2} \, dA(z)}{\left( \int_{\mathbb{D}(0,R)} |f(z)|^{\beta} e^{-|z|^2} \, dA(z) \right)^2}.
\]

This density appears in the work of Aftalion, Blanc, and Nier [ABN] for \( \beta = 2 \) in the context of Abrikosov’s analysis ([Abr], 1957) of type II superconductors.
The equilateral triangular lattice (or honeycomb) appears naturally in the physical context ($\beta = 2$). The associated zeros are located inside the grayish dots.
What about compact surfaces?

Let $\mathcal{S}$ be a compact Riemann surface. How to formulate a zero packing problem? Let $U(z, w) = U_{\mathcal{S}}(z, w)$ be a real-valued function with

$$\Delta^S U(\cdot, w))dA_S = \frac{1}{2} \delta_w - \frac{1}{2a(S)}dA_S,$$

where $dA_S$ is the area measure on $\mathcal{S}$ (with constant Gaussian curvature!) and $\Delta^S$ the Laplace-Beltrami operator, both normalized. This means that for $z \sim w$, using chart coordinates,

$$U(\cdot, w)) = \log |z - w| + O(1).$$

We call $U(\cdot, w)$ a *logarithmic monopole* (but they are known as Green functions in the literature). These functions are a substitute for polynomials!
Zero packing density for a compact surface

DEFINITION
We put

$$\rho_{n,\beta}(S) := \inf_{b, z_1, \ldots, z_n} \frac{1}{a(S)} \int_S \left( b e^{\beta U(\cdot, z_1) + \cdots + U(\cdot, z_n)} - 1 \right)^2 dA_S,$$

where the infimum is over all positive reals $b$ and all points $z_1, \ldots, z_n$ on the surface $S$.

PROPOSITION
The function $\beta \mapsto \rho_{n,\beta}(S)$ is monotonically increasing.
Relationship with zero packing on the universal cover

Suppose the compact surface \( S \) has genus \( \geq 2 \). Then the universal covering surface is the hyperbolic plane, which can be modelled by the disk \( \mathbb{D} \). Then \( S \cong \mathbb{D}/\Gamma \), where \( \Gamma \) is a Fuchsian group.

**THEOREM**

Suppose \( f : \mathbb{D} \to \mathbb{C} \) is holomorphic, and that the function 
\( z \mapsto (1 - |z|^2)^\alpha |f(z)|^\beta \) is \( \Gamma \)-periodic, where \( 2\alpha a(\mathbb{D}/\Gamma) = n\beta \), where \( n \geq 1 \) is an integer. Then

\[
\lim_{r \to 1^-} \frac{1}{\log \frac{1}{1-r^2}} \int_{\mathbb{D}(0,r)} ((1 - |z|^2)^\alpha |f(z)|^\beta - 1)^2 \frac{dA(z)}{1 - |z|^2} = \frac{1}{a(\mathbb{D}/\Gamma)} \int_{\mathbb{D}/\Gamma} ((1 - |z|^2)^\alpha |f(z)|^\beta - 1)^2 dA_H(z).
\]

The proof uses the ergodicity of geodesic flow for negatively curved surfaces (Hopf).


[Ivr] Ivrii, O., Quasicircles of dimension $1 + k^2$ do not exist. arXiv: 1511.07240

