The Klein-Gordon equation, the Hilbert transform, and dynamics of Gauss-type maps

Joint work with Alfonso Montes-Rodríguez

June 1-4, 2015

Motivation

Model uniqueness theorems

- Suppose that u is a differentiable function on $\mathbb{C}$ and

$$\bar{\partial}_z u(z) = 0, \qquad z \in \mathbb{C}.$$

  If $u(1/n) = 0$ for $n = 1, 2, 3, \ldots$, then $u(z) \equiv 0$.

- The Blaschke condition for the Hardy space (Münz-Szasz).
- Carlson's Theorem for entire functions of exponential type.

Motivation

Model uniqueness theorems

- Suppose that u is a differentiable function on $\mathbb{C}$ and

$$\bar{\partial}_z u(z) = 0, \qquad z \in \mathbb{C}.$$

If $u(1/n) = 0$ for $n = 1, 2, 3, \ldots$, then $u(z) \equiv 0$.

- The Blaschke condition for the Hardy space (Münz-Szasz).
- Carlson's Theorem for entire functions of exponential type.

Motivation

Model uniqueness theorems

- Suppose that u is a differentiable function on $\mathbb{C}$ and

$$\bar{\partial}_z u(z) = 0, \qquad z \in \mathbb{C}.$$

  If $u(1/n) = 0$ for $n = 1, 2, 3, \ldots$, then $u(z) \equiv 0$.

- The Blaschke condition for the Hardy space (Münz-Szasz).
- Carlson's Theorem for entire functions of exponential type.

Motivation

Model uniqueness theorems

- Suppose that u is a differentiable function on $\mathbb{C}$ and

$$\bar{\partial}_z u(z) = 0, \qquad z \in \mathbb{C}.$$

If $u(1/n) = 0$ for $n = 1, 2, 3, \ldots$, then $u(z) \equiv 0$.

- The Blaschke condition for the Hardy space (Münz-Szasz).
- Carlson's Theorem for entire functions of exponential type.

Heisenberg uniqueness pairs

Let μ be a finite complex Borel measure on $\mathbb{R}^2$

Its Fourier transform is

$$\widehat{\mu}(\xi) = \int_{\mathbb{R}^2} e^{i\pi\langle x,\xi\rangle} d\mu(x).$$

- Γ a curve in $\mathbb{R}^2$ s. t. $\operatorname{supp}(\mu) \subset \Gamma$, with μ absolutely continuous with respect to the arc length.
- Let $\Lambda \subset \mathbb{R}^2$.
- (Γ, Λ) is said to be a *Heisenberg uniqueness pair* whenever

$$\widehat{\mu}\big|_{\Lambda} \equiv 0 \Rightarrow \mu \equiv 0.$$

Heisenberg uniqueness pairs

Let μ be a finite complex Borel measure on $\mathbb{R}^2$

Its Fourier transform is

$$\widehat{\mu}(\xi) = \int_{\mathbb{R}^2} e^{i\pi\langle x,\xi\rangle} d\mu(x).$$

- Γ a curve in $\mathbb{R}^2$ s. t. $\operatorname{supp}(\mu) \subset \Gamma$, with μ absolutely continuous with respect to the arc length.
- Let $\Lambda \subset \mathbb{R}^2$.
- (Γ, Λ) is said to be a *Heisenberg uniqueness pair* whenever

$$\widehat{\mu}\big|_{\Lambda} \equiv 0 \Rightarrow \mu \equiv 0.$$

Heisenberg uniqueness pairs

Let μ be a finite complex Borel measure on $\mathbb{R}^2$

Its Fourier transform is

$$\widehat{\mu}(\xi) = \int_{\mathbb{R}^2} e^{i\pi\langle x,\xi\rangle} d\mu(x).$$

- Γ a curve in $\mathbb{R}^2$ s. t. $\operatorname{supp}(\mu) \subset \Gamma$, with μ absolutely continuous with respect to the arc length.
- Let $\Lambda \subset \mathbb{R}^2$.
- (Γ, Λ) is said to be a *Heisenberg uniqueness pair* whenever

$$\widehat{\mu}\big|_{\Lambda} \equiv 0 \Rightarrow \mu \equiv 0.$$

Heisenberg uniqueness pairs

Let μ be a finite complex Borel measure on $\mathbb{R}^2$

Its Fourier transform is

$$\widehat{\mu}(\xi) = \int_{\mathbb{R}^2} e^{i\pi\langle x,\xi\rangle} d\mu(x).$$

- Γ a curve in $\mathbb{R}^2$ s. t. $\operatorname{supp}(\mu) \subset \Gamma$, with μ absolutely continuous with respect to the arc length.
- Let $\Lambda \subset \mathbb{R}^2$.
- (Γ, Λ) is said to be a *Heisenberg uniqueness pair* whenever

$$\widehat{\mu}\big|_{\Lambda} \equiv 0 \Rightarrow \mu \equiv 0.$$

Heisenberg uniqueness pairs

Let μ be a finite complex Borel measure on $\mathbb{R}^2$

Its Fourier transform is

$$\widehat{\mu}(\xi) = \int_{\mathbb{R}^2} e^{i\pi\langle x,\xi\rangle} d\mu(x).$$

- Γ a curve in $\mathbb{R}^2$ s. t. $\operatorname{supp}(\mu) \subset \Gamma$, with μ absolutely continuous with respect to the arc length.
- Let $\Lambda \subset \mathbb{R}^2$.
- (Γ, Λ) is said to be a *Heisenberg uniqueness pair* whenever

$$\widehat{\mu}\big|_{\Lambda} \equiv 0 \Rightarrow \mu \equiv 0.$$

Heisenberg uniqueness pairs

Let μ be a finite complex Borel measure on $\mathbb{R}^2$

Its Fourier transform is

$$\widehat{\mu}(\xi) = \int_{\mathbb{R}^2} e^{i\pi\langle x,\xi\rangle} d\mu(x).$$

- Γ a curve in $\mathbb{R}^2$ s. t. $\operatorname{supp}(\mu) \subset \Gamma$, with μ absolutely continuous with respect to the arc length.
- Let $\Lambda \subset \mathbb{R}^2$.
- (Γ, Λ) is said to be a *Heisenberg uniqueness pair* whenever

$$\widehat{\mu}\big|_{\Lambda} \equiv 0 \Rightarrow \mu \equiv 0.$$

Properties

Dual reformulation

(Γ, Λ) is a HUP $\Leftrightarrow$ the functions

$$e_\xi(x) = e^{i\pi\langle x,\xi\rangle}, \quad \xi \in \Lambda,$$

span a weak* dense subspace in $L^\infty(\Gamma)$.

Properties

Dual reformulation

(Γ, Λ) is a HUP $\Leftrightarrow$ the functions

$$e_\xi(x) = e^{i\pi\langle x,\xi\rangle}, \quad \xi \in \Lambda,$$

span a weak* dense subspace in $L^\infty(\Gamma)$.

Properties

Dual reformulation

(Γ, Λ) is a HUP $\Leftrightarrow$ the functions

$$e_\xi(x) = e^{i\pi\langle x,\xi\rangle}, \quad \xi \in \Lambda,$$

span a weak* dense subspace in $L^\infty(\Gamma)$.

Properties

Dual reformulation

(Γ, Λ) is a HUP $\Leftrightarrow$ the functions

$$e_\xi(x) = e^{i\pi\langle x,\xi\rangle}, \quad \xi \in \Lambda,$$

span a weak* dense subspace in $L^\infty(\Gamma)$.

Algebraic curves and PDEs

Let p be a polynomial in two variables

- $$p\left(\frac{\partial_1}{\pi i}, \frac{\partial_2}{\pi i}\right)\widehat{\mu}(\xi) = \int_{\mathbb{R}^2} e^{i\pi\langle x,\xi\rangle} p(x_1, x_2)\, d\mu(x).$$
- If p is real valued and Γ is the zero locus

$$p(x_1, x_2) = 0,$$

then $\widehat{\mu}$ solves the PDE

$$p\left(\frac{\partial_1}{\pi i}, \frac{\partial_2}{\pi i}\right)\widehat{\mu}(\xi) = 0.$$

Algebraic curves and PDEs

Let p be a polynomial in two variables

- $$p\left(\frac{\partial_1}{\pi i}, \frac{\partial_2}{\pi i}\right)\widehat{\mu}(\xi) = \int_{\mathbb{R}^2} e^{i\pi\langle x,\xi\rangle} p(x_1, x_2)\, d\mu(x).$$

- If p is real valued and Γ is the zero locus

$$p(x_1, x_2) = 0,$$

then $\widehat{\mu}$ solves the PDE

$$p\left(\frac{\partial_1}{\pi i}, \frac{\partial_2}{\pi i}\right)\widehat{\mu}(\xi) = 0.$$

Algebraic curves and PDEs

Let p be a polynomial in two variables

- $$p\left(\frac{\partial_1}{\pi i}, \frac{\partial_2}{\pi i}\right)\widehat{\mu}(\xi) = \int_{\mathbb{R}^2} e^{i\pi\langle x,\xi\rangle} p(x_1, x_2)\, d\mu(x).$$

- If p is real valued and Γ is the zero locus

$$p(x_1, x_2) = 0,$$

then $\widehat{\mu}$ solves the PDE

$$p\left(\frac{\partial_1}{\pi i}, \frac{\partial_2}{\pi i}\right)\widehat{\mu}(\xi) = 0.$$

Algebraic curves and PDEs

Let p be a polynomial in two variables

- $$p\left(\frac{\partial_1}{\pi i}, \frac{\partial_2}{\pi i}\right)\widehat{\mu}(\xi) = \int_{\mathbb{R}^2} e^{i\pi\langle x,\xi\rangle} p(x_1, x_2)\, d\mu(x).$$

- If p is real valued and Γ is the zero locus

  $$p(x_1, x_2) = 0,$$

  then $\widehat{\mu}$ solves the PDE

  $$p\left(\frac{\partial_1}{\pi i}, \frac{\partial_2}{\pi i}\right)\widehat{\mu}(\xi) = 0.$$

Simple algebraic curves Γ: the conic sections

Examples

- The line.
- Two parallel lines.
- The ellipse (Sjölin and Lev).
- The cross.
- The parabola (Sjölin).

Simple algebraic curves Γ: the conic sections

Examples

- The line.
- Two parallel lines.
- The ellipse (Sjölin and Lev).
- The cross.
- The parabola (Sjölin).

Simple algebraic curves Γ: the conic sections

Examples

- The line.
- Two parallel lines.
- The ellipse (Sjölin and Lev).
- The cross.
- The parabola (Sjölin).

Simple algebraic curves Γ: the conic sections

Examples

- The line.
- Two parallel lines.
- The ellipse (Sjölin and Lev).
- The cross.
- The parabola (Sjölin).

Simple algebraic curves Γ: the conic sections

Examples

- The line.
- Two parallel lines.
- The ellipse (Sjölin and Lev).
- The cross.
- The parabola (Sjölin).

Simple algebraic curves Γ: the conic sections

Examples

- The line.
- Two parallel lines.
- The ellipse (Sjölin and Lev).
- The cross.
- The parabola (Sjölin).

Simple algebraic curves Γ: the conic sections

Examples

- The line.
- Two parallel lines.
- The ellipse (Sjölin and Lev).
- The cross.
- The parabola (Sjölin).

Simple algeraic Γ: the conic sections

The hyperbola and the Klein-Gordon equation

- $\Gamma : x_1 x_2 = \frac{M^2}{4\pi}$, $M > 0$. The Klein-Gordon equation:

$$\partial_{\xi_1}\partial_{\xi_2} u(\xi_1, \xi_2) + \frac{M^2}{4} u(\xi_1, \xi_2) = 0.$$

  Here, $M > 0$ is the mass of a particle.
- Consider $\Lambda := (\alpha\mathbb{Z} \times \{0\}) \cup (\{0\} \times \beta\mathbb{Z})$, where $\alpha, \beta > 0$.

(H & Montes-Rodríguez) Then (Γ, Λ) is a Heisenberg uniqueness pair iff $\alpha\beta M^2 \leq 4\pi^2$.

Simple algeraic Γ: the conic sections

The hyperbola and the Klein-Gordon equation

- $\Gamma : x_1 x_2 = \frac{M^2}{4\pi}$, $M > 0$. The Klein-Gordon equation:

$$\partial_{\xi_1}\partial_{\xi_2} u(\xi_1, \xi_2) + \frac{M^2}{4} u(\xi_1, \xi_2) = 0.$$

  Here, $M > 0$ is the mass of a particle.
- Consider $\Lambda := (\alpha\mathbb{Z} \times \{0\}) \cup (\{0\} \times \beta\mathbb{Z})$, where $\alpha, \beta > 0$.

(H & Montes-Rodríguez) Then (Γ, Λ) is a Heisenberg uniqueness pair iff $\alpha\beta M^2 \leq 4\pi^2$.

Simple algeraic Γ: the conic sections

The hyperbola and the Klein-Gordon equation

- $\Gamma: x_1 x_2 = \frac{M^2}{4\pi}$, $M > 0$. The Klein-Gordon equation:

$$\partial_{\xi_1}\partial_{\xi_2} u(\xi_1, \xi_2) + \frac{M^2}{4} u(\xi_1, \xi_2) = 0.$$

  Here, $M > 0$ is the mass of a particle.

- Consider $\Lambda := (\alpha\mathbb{Z} \times \{0\}) \cup (\{0\} \times \beta\mathbb{Z})$, where $\alpha, \beta > 0$.

(H & Montes-Rodríguez) Then (Γ, Λ) is a Heisenberg uniqueness pair iff $\alpha\beta M^2 \leq 4\pi^2$.

Simple algeraic Γ: the conic sections

The hyperbola and the Klein-Gordon equation

- $\Gamma : x_1 x_2 = \frac{M^2}{4\pi}$, $M > 0$. The Klein-Gordon equation:

$$\partial_{\xi_1}\partial_{\xi_2} u(\xi_1, \xi_2) + \frac{M^2}{4} u(\xi_1, \xi_2) = 0.$$

  Here, $M > 0$ is the mass of a particle.
- Consider $\Lambda := (\alpha\mathbb{Z} \times \{0\}) \cup (\{0\} \times \beta\mathbb{Z})$, where $\alpha, \beta > 0$.

(H & Montes-Rodríguez) Then (Γ, Λ) is a Heisenberg uniqueness pair iff $\alpha\beta M^2 \leq 4\pi^2$.

Simple algeraic Γ: the conic sections

The hyperbola and the Klein-Gordon equation

- $\Gamma: x_1 x_2 = \frac{M^2}{4\pi}$, $M > 0$. The Klein-Gordon equation:

$$\partial_{\xi_1}\partial_{\xi_2} u(\xi_1, \xi_2) + \frac{M^2}{4} u(\xi_1, \xi_2) = 0.$$

  Here, $M > 0$ is the mass of a particle.

- Consider $\Lambda := (\alpha\mathbb{Z} \times \{0\}) \cup (\{0\} \times \beta\mathbb{Z})$, where $\alpha, \beta > 0$.

(H & Montes-Rodríguez) Then (Γ, Λ) is a Heisenberg uniqueness pair iff $\alpha\beta M^2 \leq 4\pi^2$.

Extension of the trigonometric system

The trigonometric system & the periodic functions

- The trigonometric system $\{e_n(x) = e^{i\pi nx}\}_{n\in\mathbb{Z}}$.
- Consider $e_n^\beta(x) = e_n(\beta/x) = e^{i\pi\beta n/x}$ with $n \in \mathbb{Z}$.
- Normalize so that $\alpha := 1$ and $M := 2\pi$.

THEOREM. The sets $\{e_n\}_{n\in\mathbb{Z}}$ and $\{e_n^\beta\}_{n\in\mathbb{Z}}$ together form a weak-star spanning system of $L^\infty(\mathbb{R}) \Leftrightarrow 0 < \beta \leq 1$.

Extension of the trigonometric system

The trigonometric system & the periodic functions

- The trigonometric system $\{e_n(x) = e^{i\pi nx}\}_{n\in\mathbb{Z}}$.
- Consider $e_n^{\beta}(x) = e_n(\beta/x) = e^{i\pi\beta n/x}$ with $n \in \mathbb{Z}$.
- Normalize so that $\alpha := 1$ and $M := 2\pi$.

THEOREM. The sets $\{e_n\}_{n\in\mathbb{Z}}$ and $\{e_n^{\beta}\}_{n\in\mathbb{Z}}$ together form a weak-star spanning system of $L^{\infty}(\mathbb{R}) \Leftrightarrow 0 < \beta \leq 1$.

Extension of the trigonometric system

The trigonometric system & the periodic functions

- The trigonometric system $\{e_n(x) = e^{i\pi nx}\}_{n\in\mathbb{Z}}$.
- Consider $e_n^\beta(x) = e_n(\beta/x) = e^{i\pi\beta n/x}$ with $n \in \mathbb{Z}$.
- Normalize so that $\alpha := 1$ and $M := 2\pi$.

THEOREM. The sets $\{e_n\}_{n\in\mathbb{Z}}$ and $\{e_n^\beta\}_{n\in\mathbb{Z}}$ together form a weak-star spanning system of $L^\infty(\mathbb{R}) \Leftrightarrow 0 < \beta \leq 1$.

Necessity of $0 < \beta \leq 1$.

Harmonic extension to $\mathbb{C}_+$

- $e_n(z) = e^{\pi inz}$ if $n \geq 0$ and $e_n(z) = e^{\pi ni\bar{z}}$ if $n < 0$.
- $e_n^\beta(z) = e^{\pi i\beta n/\bar{z}}$ if $n \geq 0$ and $e_n^\beta(z) = e^{\pi i\beta n/z}$ if $n < 0$.

Necessity of $0 < \beta \leq 1$.

Harmonic extension to $\mathbb{C}_+$

- $e_n(z) = e^{\pi inz}$ if $n \geq 0$ and $e_n(z) = e^{\pi ni\bar{z}}$ if $n < 0$.
- $e_n^\beta(z) = e^{\pi i\beta n/\bar{z}}$ if $n \geq 0$ and $e_n^\beta(z) = e^{\pi i\beta n/z}$ if $n < 0$.

Necessity of $0 < \beta \leq 1$

The point separation condition

- If $f \in L^\infty(\mathbb{R})$, then the harmonic extension is
$$f(z_0) = \frac{1}{\pi}\int_{-\infty}^{\infty} P(t, z_0) f(t)\, dt, \qquad z_0 \in \mathbb{C}_+,$$
where $P(t, z_0) := \frac{\Im z_0}{\pi}|t - z_0|^{-2}$ is the Poisson kernel.
- The bounded harmonic functions on $\mathbb{C}_+$ separate points.
- When there are $z_1, z_2 \in \mathbb{C}_+$ with $z_1 \neq z_2$ such that $\forall n$: $e_n(z_1) = e_n(z_2)$ and $e_n^\beta(z_1) = e_n^\beta(z_2)$, it is impossible to span.
- For $1 < \beta < +\infty$, we can take
$$z_1 = 1 + i\sqrt{\beta^2 - 1} \quad \text{and} \quad z_2 = -1 + i\sqrt{\beta^2 - 1}.$$

Necessity of $0 < \beta \leq 1$

The point separation condition

- If $f \in L^\infty(\mathbb{R})$, then the harmonic extension is

$$f(z_0) = \frac{1}{\pi}\int_{-\infty}^{\infty} P(t, z_0) f(t)\, dt, \qquad z_0 \in \mathbb{C}_+,$$

  where $P(t, z_0) := \frac{\Im z_0}{\pi}|t - z_0|^{-2}$ is the Poisson kernel.
- The bounded harmonic functions on $\mathbb{C}_+$ separate points.
- When there are $z_1, z_2 \in \mathbb{C}_+$ with $z_1 \neq z_2$ such that $\forall n$: $e_n(z_1) = e_n(z_2)$ and $e_n^\beta(z_1) = e_n^\beta(z_2)$, it is impossible to span.
- For $1 < \beta < +\infty$, we can take

$$z_1 = 1 + i\sqrt{\beta^2 - 1} \quad \text{and} \quad z_2 = -1 + i\sqrt{\beta^2 - 1}.$$

Necessity of $0 < \beta \leq 1$

The point separation condition

- If $f \in L^\infty(\mathbb{R})$, then the harmonic extension is

$$f(z_0) = \frac{1}{\pi}\int_{-\infty}^{\infty} P(t, z_0) f(t)\, dt, \qquad z_0 \in \mathbb{C}_+,$$

where $P(t, z_0) := \frac{\Im z_0}{\pi}|t - z_0|^{-2}$ is the Poisson kernel.
- The bounded harmonic functions on $\mathbb{C}_+$ separate points.
- When there are $z_1, z_2 \in \mathbb{C}_+$ with $z_1 \neq z_2$ such that $\forall n$: $e_n(z_1) = e_n(z_2)$ and $e_n^\beta(z_1) = e_n^\beta(z_2)$, it is impossible to span.
- For $1 < \beta < +\infty$, we can take

$$z_1 = 1 + i\sqrt{\beta^2 - 1} \quad \text{and} \quad z_2 = -1 + i\sqrt{\beta^2 - 1}.$$

Necessity of $0 < \beta \leq 1$

The point separation condition

- If $f \in L^\infty(\mathbb{R})$, then the harmonic extension is

$$f(z_0) = \frac{1}{\pi}\int_{-\infty}^{\infty} P(t, z_0) f(t)\, dt, \qquad z_0 \in \mathbb{C}_+,$$

  where $P(t, z_0) := \frac{\Im z_0}{\pi}|t - z_0|^{-2}$ is the Poisson kernel.
- The bounded harmonic functions on $\mathbb{C}_+$ separate points.
- When there are $z_1, z_2 \in \mathbb{C}_+$ with $z_1 \neq z_2$ such that $\forall n$: $e_n(z_1) = e_n(z_2)$ and $e_n^\beta(z_1) = e_n^\beta(z_2)$, it is impossible to span.
- For $1 < \beta < +\infty$, we can take

$$z_1 = 1 + i\sqrt{\beta^2 - 1} \quad \text{and} \quad z_2 = -1 + i\sqrt{\beta^2 - 1}.$$

Necessity of $0 < \beta \leq 1$

The point separation condition

- If $f \in L^\infty(\mathbb{R})$, then the harmonic extension is

$$f(z_0) = \frac{1}{\pi}\int_{-\infty}^{\infty} P(t, z_0) f(t)\, dt, \qquad z_0 \in \mathbb{C}_+,$$

  where $P(t, z_0) := \frac{\Im z_0}{\pi}|t - z_0|^{-2}$ is the Poisson kernel.
- The bounded harmonic functions on $\mathbb{C}_+$ separate points.
- When there are $z_1, z_2 \in \mathbb{C}_+$ with $z_1 \neq z_2$ such that $\forall n$: $e_n(z_1) = e_n(z_2)$ and $e_n^\beta(z_1) = e_n^\beta(z_2)$, it is impossible to span.
- For $1 < \beta < +\infty$, we can take

$$z_1 = 1 + i\sqrt{\beta^2 - 1} \quad \text{and} \qquad z_2 = -1 + i\sqrt{\beta^2 - 1}.$$

Periodic and inverted periodic functions

Weak-star closures

- The weak-star closure of $\operatorname{span}\{e_n(x)\}_{n\in\mathbb{Z}}$ is $L^\infty_2(\mathbb{R})$ (2-periodic).
- The weak-star closure of $\operatorname{span}\{e^\beta_n(x)\}_{n\in\mathbb{Z}}$ is $L^\infty_{(\beta)}(\mathbb{R})$, which consists of bounded functions such that

$$x \mapsto f(-\beta/x) \quad \text{is 2-periodic.}$$

Periodic and inverted periodic functions

Weak-star closures

- The weak-star closure of $\operatorname{span}\{e_n(x)\}_{n\in\mathbb{Z}}$ is $L_2^\infty(\mathbb{R})$ (2-periodic).
- The weak-star closure of $\operatorname{span}\{e_n^\beta(x)\}_{n\in\mathbb{Z}}$ is $L_{(\beta)}^\infty(\mathbb{R})$, which consists of bounded functions such that

$$x \mapsto f(-\beta/x) \quad \text{is 2-periodic.}$$

Periodic and inverted periodic functions

Weak-star closures

- The weak-star closure of $\operatorname{span}\{e_n(x)\}_{n\in\mathbb{Z}}$ is $L_2^\infty(\mathbb{R})$ (2-periodic).
- The weak-star closure of $\operatorname{span}\{e_n^\beta(x)\}_{n\in\mathbb{Z}}$ is $L_{(\beta)}^\infty(\mathbb{R})$, which consists of bounded functions such that

$$x \mapsto f(-\beta/x) \quad \text{is 2-periodic.}$$

Periodic and inverted periodic functions

Weak-star closures

- The weak-star closure of $\operatorname{span}\{e_n(x)\}_{n\in\mathbb{Z}}$ is $L^\infty_2(\mathbb{R})$ (2-periodic).
- The weak-star closure of $\operatorname{span}\{e_n^\beta(x)\}_{n\in\mathbb{Z}}$ is $L^\infty_{(\beta)}(\mathbb{R})$, which consists of bounded functions such that

$$x \mapsto f(-\beta/x) \quad \text{is 2-periodic.}$$

An involution and periodization

Some operators

Consider the involution operator defined by

$$\mathbf{J}_\beta f(x) := \frac{\beta}{x^2} f\left(-\frac{\beta}{x}\right).$$

Its adjoint is

$$\mathbf{J}_\beta^\star f(x) := f\left(-\frac{\beta}{x}\right).$$

Also consider the operator

$$\mathbf{\Pi}_2 f(x) := \sum_{j\in\mathbb{Z}} f(x+2j),$$

which *periodizes*.

Duality

Recall that $L^\infty_2(\mathbb{R})$ consists of the 2-periodic bounded functions.

PROPOSITION.

A function $f \in L^1(\mathbb{R})$ annihilates $L^\infty_2(\mathbb{R})$ iff $\mathbf{\Pi}_2 f = 0$.

COROLLARY.

A function $f \in L^1(\mathbb{R})$ annihilates $L^\infty_{(\beta)}(\mathbb{R}) = \mathbf{J}^\star_\beta L^\infty_2(\mathbb{R})$ iff $\mathbf{\Pi}_2 \mathbf{J}_\beta f = 0$.

DUAL THEOREM.

Suppose $f \in L^1(\mathbb{R})$. Then if $0 < \beta \le 1$,

$$\mathbf{\Pi}_2 f = \mathbf{\Pi} \mathbf{J}_\beta f = 0 \quad \Longleftrightarrow \quad f = 0.$$

Duality

Recall that $L_2^\infty(\mathbb{R})$ consists of the 2-periodic bounded functions.

PROPOSITION.

A function $f \in L^1(\mathbb{R})$ annihilates $L_2^\infty(\mathbb{R})$ iff $\mathbf{\Pi}_2 f = 0$.

COROLLARY.

A function $f \in L^1(\mathbb{R})$ annihilates $L_{(\beta)}^\infty(\mathbb{R}) = \mathbf{J}_\beta^\star L_2^\infty(\mathbb{R})$ iff $\mathbf{\Pi}_2 \mathbf{J}_\beta f = 0$.

DUAL THEOREM.

Suppose $f \in L^1(\mathbb{R})$. Then if $0 < \beta \leq 1$,

$$\mathbf{\Pi}_2 f = \mathbf{\Pi} \mathbf{J}_\beta f = 0 \quad \Longleftrightarrow \quad f = 0.$$

Proof approach

We split $\mathbf{\Pi}_2 = \mathbf{I} + \mathbf{\Sigma}_2$, where

$$\mathbf{\Sigma}_2 f(x) = \sum_{k \in \mathbb{Z}^\times} f(x + 2k).$$

(i) The condition $\mathbf{\Pi}_2 f = 0$ reads $f = -\mathbf{\Sigma}_2 f$.
(ii) The condition $\mathbf{\Pi}_2 \mathbf{J}_\beta f = 0$ reads $\mathbf{J}_\beta f = -\mathbf{\Sigma}_2 \mathbf{J}_\beta f$.
(iii) $\mathbf{J}_\beta f = -\mathbf{\Sigma}_2 \mathbf{J}_\beta f$ can be written $f = -\mathbf{J}_\beta \mathbf{\Sigma}_2 \mathbf{J}_\beta f$.
(iv) We may combine conditions (i) and (ii) in two different ways:
$f = \mathbf{\Sigma}_2 \mathbf{J}_\beta \mathbf{\Sigma}_2 \mathbf{J}_\beta f$ and $f = \mathbf{J}_\beta \mathbf{\Sigma}_2 \mathbf{J}_\beta \mathbf{\Sigma}_2 f$.

Proof approach

We split $\mathbf{\Pi}_2 = \mathbf{I} + \mathbf{\Sigma}_2$, where

$$\mathbf{\Sigma}_2 f(x) = \sum_{k \in \mathbb{Z}^\times} f(x + 2k).$$

(i) The condition $\mathbf{\Pi}_2 f = 0$ reads $f = -\mathbf{\Sigma}_2 f$.
(ii) The condition $\mathbf{\Pi}_2 \mathbf{J}_\beta f = 0$ reads $\mathbf{J}_\beta f = -\mathbf{\Sigma}_2 \mathbf{J}_\beta f$.
(iii) $\mathbf{J}_\beta f = -\mathbf{\Sigma}_2 \mathbf{J}_\beta f$ can be written $f = -\mathbf{J}_\beta \mathbf{\Sigma}_2 \mathbf{J}_\beta f$.
(iv) We may combine conditions (i) and (ii) in two different ways:
$f = \mathbf{\Sigma}_2 \mathbf{J}_\beta \mathbf{\Sigma}_2 \mathbf{J}_\beta f$ and $f = \mathbf{J}_\beta \mathbf{\Sigma}_2 \mathbf{J}_\beta \mathbf{\Sigma}_2 f$.

Transfer operators and dynamics

Let $I_1 := (-1, 1)$ the the symmetric unit interval. Let $\mathbf{T}_\beta$ be defined by

$$\mathbf{T}_\beta f(x) := \sum_{j \in \mathbb{Z}^\times} \frac{\beta}{(x+2j)^2} f\left(-\frac{\beta}{x+2j}\right), \qquad x \in I_1,$$

and $\mathbf{V}_\beta$ by

$$\mathbf{V}_\beta f(x) := \frac{\beta}{x^2} \sum_{j \in \mathbb{Z}^\times} f\left(-\frac{\beta}{x} + 2j\right), \qquad x \in \mathbb{R} \setminus I_1.$$

Here, $\mathbb{Z}^\times := \mathbb{Z} \setminus \{0\}$. These are *transfer operators* for $\beta = 1$, and for $0 < \beta < 1$ *subtransfer operators*. The transformation on I_1 corresponding to $\mathbf{T}_\beta$ is $\tau_\beta(x) := \{-\beta/x\}_2 \pmod{2\mathbb{Z}}$.

The connection with Dynamics

For $f \in L^1(\mathbb{R})$, we have that

$$\mathbf{\Sigma}_2 \mathbf{J}_\beta f(x) = \mathbf{T}_\beta f(x), \qquad x \in I_1,$$

and

$$\mathbf{J}_\beta \mathbf{\Sigma}_2 f(x) = \mathbf{V}_\beta f(x), \qquad x \in \mathbb{R} \setminus I_1.$$

N.B. $\mathbf{T}_\beta f$ uses only $f|_{I_1}$ while $\mathbf{V}_\beta f$ uses only $f|_{\mathbb{R}\setminus I_1}$.

PROPOSITION.

$(0 < \beta \leq 1)$ Suppose $f \in L^1(\mathbb{R})$ has $\mathbf{\Pi}_2 f = 0$ and $\mathbf{\Pi}_2 \mathbf{J}_\beta f = 0$. Put $f_0 := f_{I_1}$ and $f_1 := f_{\mathbb{R}\setminus I_1}$. Then $f_0 = \mathbf{T}_\beta^2 f_0$, $f_1 = \mathbf{V}_\beta^2 f_1$, and $f_1 = -\mathbf{J}_\beta \mathbf{T}_\beta f_0$ on $\mathbb{R} \setminus I_1$.

How to solve the problem

OBSERVATION.

($0 < \beta \leq 1$) Suppose we can show that the fixed point equation $f_0 = \mathbf{T}_\beta^2 f_0$ with $f_0 \in L^1(I_1)$ has only the trivial solution $f_0 = 0$. Then the following equivalence holds for $f \in L^1(\mathbb{R})$:

$$\mathbf{\Pi}_2 f = \mathbf{\Pi J}_\beta f = 0 \quad \Longleftrightarrow \quad f = 0.$$

N.B.

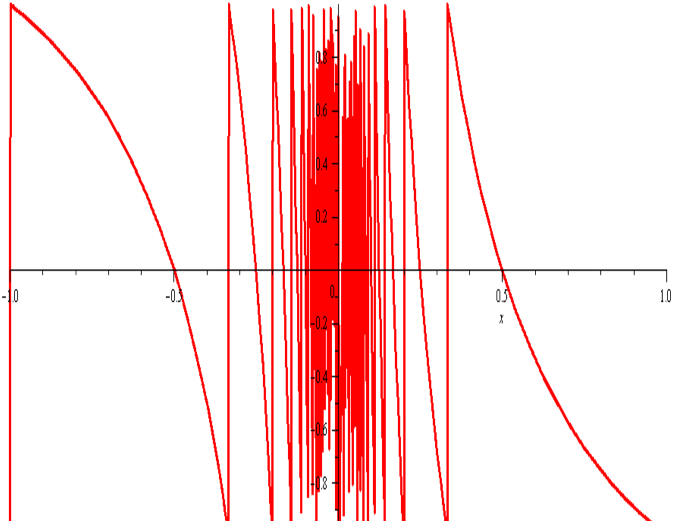
($0 < \beta \leq 1$) If $\mathbf{T}_\beta^2 f_0 = f_0$ has a nontrivial solution $f_0 \in L^1(I_1)$, then $\mathbf{T}_\beta g_0 = \lambda g_0$ has a nontrivial solution $g_0 \in L^1(I_1)$ either for $\lambda = 1$ or $\lambda = -1$.

How to solve the problem II

Put $\tau_\beta = \{-\beta/x\}_2 \pmod{2\mathbb{Z}}$

- $\mathbf{T}_\beta : L^1(I_1) \to L^1(I_1)$ is a norm contraction that preserves the positive cone, for $0 < \beta \leq 1$.
- For $0 < \beta < 1$, τ_β has an *attractor*: $I_1 \setminus I_\beta$, where $I_\beta := (-\beta, \beta)$. After a number of iterations, almost all points are eventually attracted to the attractor. This means that $\mathbf{T}_\beta^n g_0 \to 0$ as $n \to +\infty$ for every given $g_0 \in L^1(I_1)$.
- For $\beta = 1$, τ_1 is ergodic with invariant measure $(1-x^2)^{-1}dx$, which has *infinite mass*. So the only solution to $\mathbf{T}_1 g_0 = \lambda g_0$ with $|\lambda| = 1$ and $g_0 \in L^1(I_1)$ is the trivial one: $g_0 = 0$.

The graph of the Gauss-type map $\tau_1(x) = \{-1/x\}_2$



The transfer operator for $\tau_\beta(x) = \{-\beta/x\}_2$

$0 < \beta \leq 1$

- We put

$$\mathcal{T}_\beta f(x) = 1_{I_1}(x) \sum_{j\in\mathbb{Z}} \frac{1}{(x+2j)^2} f\left(-\frac{1}{x+2j}\right), \qquad x \in I_1.$$

- $\mathcal{T}_\beta f$ is positive for positive f.
- $\mathcal{T}_\beta$ is a norm contraction on $L^1(I_1)$, and preserves the L^1-norms of positive functions, that is,

$$\|\mathcal{T}_\beta f\|_{L^1(I_1)} = \|f\|_{L^1(I_1)}.$$

The transfer operator for $\tau_\beta(x) = \{-\beta/x\}_2$

$0 < \beta \leq 1$

- We put

$$\mathcal{T}_\beta f(x) = 1_{I_1}(x) \sum_{j \in \mathbb{Z}} \frac{1}{(x+2j)^2} f\left(-\frac{1}{x+2j}\right), \qquad x \in I_1.$$

- $\mathcal{T}_\beta f$ is positive for positive f.
- $\mathcal{T}_\beta$ is a norm contraction on $L^1(I_1)$, and preserves the L^1-norms of positive functions, that is,

$$\|\mathcal{T}_\beta f\|_{L^1(I_1)} = \|f\|_{L^1(I_1)}.$$

THE TRANSFER OPERATOR FOR $\tau_\beta(x) = \{-\beta/x\}_2$

$0 < \beta \leq 1$

- We put

$$\mathcal{T}_\beta f(x) = 1_{I_1}(x) \sum_{j \in \mathbb{Z}} \frac{1}{(x+2j)^2} f\left(-\frac{1}{x+2j}\right), \qquad x \in I_1.$$

- $\mathcal{T}_\beta f$ is positive for positive f.
- $\mathcal{T}_\beta$ is a norm contraction on $L^1(I_1)$, and preserves the L^1-norms of positive functions, that is,

$$\|\mathcal{T}_\beta f\|_{L^1(I_1)} = \|f\|_{L^1(I_1)}.$$

THE TRANSFER OPERATOR FOR $\tau_\beta(x) = \{-\beta/x\}_2$

$0 < \beta \leq 1$

- We put

$$\mathcal{T}_\beta f(x) = 1_{I_1}(x) \sum_{j \in \mathbb{Z}} \frac{1}{(x+2j)^2} f\left(-\frac{1}{x+2j}\right), \qquad x \in I_1.$$

- $\mathcal{T}_\beta f$ is positive for positive f.
- $\mathcal{T}_\beta$ is a norm contraction on $L^1(I_1)$, and preserves the L^1-norms of positive functions, that is,

$$\|\mathcal{T}_\beta f\|_{L^1(I_1)} = \|f\|_{L^1(I_1)}.$$

The connection between the transfer and the subtransfer operators

$0 < \beta \leq 1$

For $f \in L^1(I_1)$, we have that

$$\mathbf{T}_\beta f = \mathcal{T}_\beta(1_{I_\beta} f).$$

So $\mathbf{T}_\beta f$ uses only the restriction of f to the subinterval $I_\beta = (-\beta, \beta)$, which means that it throws away the attractor!

Consequences for the Hardy spaces

Let H^p, $1 \leq p \leq +\infty$, be the standard Hardy space on the disk

- Consider the algebra $\mathcal{A}$ generated by the inner functions

$$\exp\left(\pi\frac{z+1}{z-1}\right) \quad \text{and} \quad \exp\left(\pi\frac{z-1}{z+1}\right).$$

  Then $\mathcal{A}$ is weak-star dense in BMOA, and in particular, norm dense in each Hardy space H^p, $1 \leq p < +\infty$.

- For $p = +\infty$, one may ask whether the algebra generated by the inner functions above is weak-star dense in H^∞ or not.

  Why this question? The lattice of $\mathcal{A}$-invariant subspaces coincides with the lattice of z-invariant subspaces if and only if $\mathcal{A}$ is weak-star dense in H^∞.

Consequences for the Hardy spaces

Let H^p, $1 \leq p \leq +\infty$, be the standard Hardy space on the disk

- Consider the algebra $\mathcal{A}$ generated by the inner functions

$$\exp\left(\pi \frac{z+1}{z-1}\right) \quad \text{and} \quad \exp\left(\pi \frac{z-1}{z+1}\right).$$

  Then $\mathcal{A}$ is weak-star dense in BMOA, and in particular, norm dense in each Hardy space H^p, $1 \leq p < +\infty$.

- For $p = +\infty$, one may ask whether the algebra generated by the inner functions above is weak-star dense in H^∞ or not.

  Why this question? The lattice of $\mathcal{A}$-invariant subspaces coincides with the lattice of z-invariant subspaces if and only if $\mathcal{A}$ is weak-star dense in H^∞.

Consequences for the Hardy spaces

Let H^p, $1 \leq p \leq +\infty$, be the standard Hardy space on the disk

- Consider the algebra $\mathcal{A}$ generated by the inner functions

$$\exp\left(\pi\frac{z+1}{z-1}\right) \quad \text{and} \quad \exp\left(\pi\frac{z-1}{z+1}\right).$$

  Then $\mathcal{A}$ is weak-star dense in BMOA, and in particular, norm dense in each Hardy space H^p, $1 \leq p < +\infty$.

- For $p = +\infty$, one may ask whether the algebra generated by the inner functions above is weak-star dense in H^∞ or not.

  Why this question? The lattice of $\mathcal{A}$-invariant subspaces coincides with the lattice of z-invariant subspaces if and only if $\mathcal{A}$ is weak-star dense in H^∞.

Consequences for the Hardy spaces

Let H^p, $1 \leq p \leq +\infty$, be the standard Hardy space on the disk

- Consider the algebra $\mathcal{A}$ generated by the inner functions

$$\exp\left(\pi\frac{z+1}{z-1}\right) \quad \text{and} \quad \exp\left(\pi\frac{z-1}{z+1}\right).$$

  Then $\mathcal{A}$ is weak-star dense in BMOA, and in particular, norm dense in each Hardy space H^p, $1 \leq p < +\infty$.

- For $p = +\infty$, one may ask whether the algebra generated by the inner functions above is weak-star dense in H^∞ or not.

  Why this question? The lattice of $\mathcal{A}$-invariant subspaces coincides with the lattice of z-invariant subspaces if and only if $\mathcal{A}$ is weak-star dense in H^∞.

The holomorphic version of the problem

We replace $L^\infty(\mathbb{R})$ by $H^\infty(\mathbb{C}_+)$

- Let $H^\infty(\mathbb{C}_+)$ be the subspace of $L^\infty(\mathbb{R})$ whose harmonic extensions to the upper half plane $\mathbb{C}_+$ are analytic on $\mathbb{C}_+$.
- Let $H_2^\infty(\mathbb{C}_+)$ be the weak-star closed subspace of $L_2^\infty(\mathbb{R})$, whose harmonic extensions are analytic on $\mathbb{C}_+$.
- Let $H_{(\beta)}^\infty(\mathbb{C}_+)$ be the weak-star closed subspace of $L_{(\beta)}^\infty(\mathbb{R})$, whose harmonic extensions are analytic on $\mathbb{C}_+$.
- **QUESTION:** Is $H_2^\infty(\mathbb{C}_+) + H_{(\beta)}^\infty(\mathbb{C}_+)$ weak-star dense in $H^\infty(\mathbb{C}_+)$? **N.B.** It is weak-star dense in $BMOA(\mathbb{C}_+)$!

The holomorphic version of the problem

We replace $L^\infty(\mathbb{R})$ by $H^\infty(\mathbb{C}_+)$

- Let $H^\infty(\mathbb{C}_+)$ be the subspace of $L^\infty(\mathbb{R})$ whose harmonic extensions to the upper half plane $\mathbb{C}_+$ are analytic on $\mathbb{C}_+$.
- Let $H_2^\infty(\mathbb{C}_+)$ be the weak-star closed subspace of $L_2^\infty(\mathbb{R})$, whose harmonic extensions are analytic on $\mathbb{C}_+$.
- Let $H_{(\beta)}^\infty(\mathbb{C}_+)$ be the weak-star closed subspace of $L_{(\beta)}^\infty(\mathbb{R})$, whose harmonic extensions are analytic on $\mathbb{C}_+$.
- **QUESTION:** Is $H_2^\infty(\mathbb{C}_+) + H_{(\beta)}^\infty(\mathbb{C}_+)$ weak-star dense in $H^\infty(\mathbb{C}_+)$? **N.B.** It is weak-star dense in $BMOA(\mathbb{C}_+)$!

The holomorphic version of the problem

We replace $L^\infty(\mathbb{R})$ by $H^\infty(\mathbb{C}_+)$

- Let $H^\infty(\mathbb{C}_+)$ be the subspace of $L^\infty(\mathbb{R})$ whose harmonic extensions to the upper half plane $\mathbb{C}_+$ are analytic on $\mathbb{C}_+$.
- Let $H^\infty_2(\mathbb{C}_+)$ be the weak-star closed subspace of $L^\infty_2(\mathbb{R})$, whose harmonic extensions are analytic on $\mathbb{C}_+$.
- Let $H^\infty_{(\beta)}(\mathbb{C}_+)$ be the weak-star closed subspace of $L^\infty_{(\beta)}(\mathbb{R})$, whose harmonic extensions are analytic on $\mathbb{C}_+$.
- **QUESTION:** Is $H^\infty_2(\mathbb{C}_+) + H^\infty_{(\beta)}(\mathbb{C}_+)$ weak-star dense in $H^\infty(\mathbb{C}_+)$? **N.B.** It is weak-star dense in $BMOA(\mathbb{C}_+)$!

The holomorphic version of the problem

We replace $L^\infty(\mathbb{R})$ by $H^\infty(\mathbb{C}_+)$

- Let $H^\infty(\mathbb{C}_+)$ be the subspace of $L^\infty(\mathbb{R})$ whose harmonic extensions to the upper half plane $\mathbb{C}_+$ are analytic on $\mathbb{C}_+$.
- Let $H_2^\infty(\mathbb{C}_+)$ be the weak-star closed subspace of $L_2^\infty(\mathbb{R})$, whose harmonic extensions are analytic on $\mathbb{C}_+$.
- Let $H_{(\beta)}^\infty(\mathbb{C}_+)$ be the weak-star closed subspace of $L_{(\beta)}^\infty(\mathbb{R})$, whose harmonic extensions are analytic on $\mathbb{C}_+$.
- **QUESTION:** Is $H_2^\infty(\mathbb{C}_+) + H_{(\beta)}^\infty(\mathbb{C}_+)$ weak-star dense in $H^\infty(\mathbb{C}_+)$? **N.B.** It is weak-star dense in $BMOA(\mathbb{C}_+)$!

The holomorphic version of the problem

We replace $L^\infty(\mathbb{R})$ by $H^\infty(\mathbb{C}_+)$

- Let $H^\infty(\mathbb{C}_+)$ be the subspace of $L^\infty(\mathbb{R})$ whose harmonic extensions to the upper half plane $\mathbb{C}_+$ are analytic on $\mathbb{C}_+$.
- Let $H_2^\infty(\mathbb{C}_+)$ be the weak-star closed subspace of $L_2^\infty(\mathbb{R})$, whose harmonic extensions are analytic on $\mathbb{C}_+$.
- Let $H_{(\beta)}^\infty(\mathbb{C}_+)$ be the weak-star closed subspace of $L_{(\beta)}^\infty(\mathbb{R})$, whose harmonic extensions are analytic on $\mathbb{C}_+$.
- **QUESTION:** Is $H_2^\infty(\mathbb{C}_+) + H_{(\beta)}^\infty(\mathbb{C}_+)$ weak-star dense in $H^\infty(\mathbb{C}_+)$? **N.B.** It is weak-star dense in $BMOA(\mathbb{C}_+)$!

Consequences for the Klein-Gordon equation

The Zariski closure, I

- Consider
$$\Lambda^{++} = (\alpha\mathbb{Z}_{+,0} \times \{0\}) \cup (\{0\} \times \beta\mathbb{Z}_{+})$$
and
$$\Lambda^{+-} = (\alpha\mathbb{Z}_{+,0} \times \{0\}) \cup (\{0\} \times \beta\mathbb{Z}_{-}),$$
where $\alpha, \beta > 0$ are positive and $\mathbb{Z}_{+,0} = \{0, 1, 2, \ldots\}$. The sets Λ^{+-} and Λ^{--} are defined analogously.
- Suppose that $u(\xi_1, \xi_2)$ solves the Klein-Gordon equation
$$\partial_1\partial_2 u(\xi_1, \xi_2) + \frac{M^2}{4} u(\xi_1, \xi_2) = 0.$$

Consequences for the Klein-Gordon equation

The Zariski closure, I

- Consider

$$\Lambda^{++} = (\alpha\mathbb{Z}_{+,0} \times \{0\}) \cup (\{0\} \times \beta\mathbb{Z}_{+})$$

and

$$\Lambda^{+-} = (\alpha\mathbb{Z}_{+,0} \times \{0\}) \cup (\{0\} \times \beta\mathbb{Z}_{-}),$$

where $\alpha, \beta > 0$ are positive and $\mathbb{Z}_{+,0} = \{0, 1, 2, \ldots\}$. The sets Λ^{+-} and Λ^{--} are defined analogously.

- Suppose that $u(\xi_1, \xi_2)$ solves the Klein-Gordon equation

$$\partial_1\partial_2 u(\xi_1, \xi_2) + \frac{M^2}{4} u(\xi_1, \xi_2) = 0.$$

The Zariski closure, I

- Consider

$$\Lambda^{++} = (\alpha\mathbb{Z}_{+,0} \times \{0\}) \cup (\{0\} \times \beta\mathbb{Z}_{+})$$

and

$$\Lambda^{+-} = (\alpha\mathbb{Z}_{+,0} \times \{0\}) \cup (\{0\} \times \beta\mathbb{Z}_{-}),$$

where $\alpha, \beta > 0$ are positive and $\mathbb{Z}_{+,0} = \{0, 1, 2, \ldots\}$. The sets Λ^{+-} and Λ^{--} are defined analogously.

- Suppose that $u(\xi_1, \xi_2)$ solves the Klein-Gordon equation

$$\partial_1\partial_2 u(\xi_1, \xi_2) + \frac{M^2}{4} u(\xi_1, \xi_2) = 0.$$

The Zariski closure, I

- Consider

$$\Lambda^{++} = (\alpha\mathbb{Z}_{+,0} \times \{0\}) \cup (\{0\} \times \beta\mathbb{Z}_{+})$$

and

$$\Lambda^{+-} = (\alpha\mathbb{Z}_{+,0} \times \{0\}) \cup (\{0\} \times \beta\mathbb{Z}_{-}),$$

where $\alpha, \beta > 0$ are positive and $\mathbb{Z}_{+,0} = \{0, 1, 2, \ldots\}$. The sets Λ^{+-} and Λ^{--} are defined analogously.

- Suppose that $u(\xi_1, \xi_2)$ solves the Klein-Gordon equation

$$\partial_1\partial_2 u(\xi_1, \xi_2) + \frac{M^2}{4} u(\xi_1, \xi_2) = 0.$$

Applications to the Klein-Gordon equation

The Zariski closure, II

- Suppose also that $u = \widehat{\mu}$ where μ is a finite Borel measure, absolutely continuous w.r.t. 1-dim Hausdorff measure. We then have the implication

$$u|_{\Lambda^{+-}} = 0 \Longrightarrow u|_{\mathbb{R}_+ \times \mathbb{R}_-} = 0$$

  if and only if $\alpha\beta M^2 \leq 4\pi^2$.

- The corresponding assertion holds for Λ^{-+} as well. The quarter-planes $\mathbb{R}_+ \times \mathbb{R}_-$ and $\mathbb{R}_- \times \mathbb{R}_+$ are both *space-like*. *There is no analogue for $\Lambda^{++}, \Lambda^{--}$ (time-like).*

Applications to the Klein-Gordon equation

The Zariski closure, II

- Suppose also that $u = \widehat{\mu}$ where μ is a finite Borel measure, absolutely continuous w.r.t. 1-dim Hausdorff measure. We then have the implication

$$u|_{\Lambda^{+-}} = 0 \Longrightarrow u|_{\mathbb{R}_+ \times \mathbb{R}_-} = 0$$

  if and only if $\alpha\beta M^2 \leq 4\pi^2$.

- The corresponding assertion holds for Λ^{-+} as well. The quarter-planes $\mathbb{R}_+ \times \mathbb{R}_-$ and $\mathbb{R}_- \times \mathbb{R}_+$ are both *space-like*. *There is no analogue for $\Lambda^{++}, \Lambda^{--}$ (time-like).*

The Zariski closure, II

- Suppose also that $u = \widehat{\mu}$ where μ is a finite Borel measure, absolutely continuous w.r.t. 1-dim Hausdorff measure. We then have the implication

$$u|_{\Lambda^{+-}} = 0 \Longrightarrow u|_{\mathbb{R}_+ \times \mathbb{R}_-} = 0$$

if and only if $\alpha\beta M^2 \leq 4\pi^2$.

- The corresponding assertion holds for Λ^{-+} as well. The quarter-planes $\mathbb{R}_+ \times \mathbb{R}_-$ and $\mathbb{R}_- \times \mathbb{R}_+$ are both *space-like. There is no analogue for $\Lambda^{++}, \Lambda^{--}$ (time-like).*

The Zariski closure, II

- Suppose also that $u = \widehat{\mu}$ where μ is a finite Borel measure, absolutely continuous w.r.t. 1-dim Hausdorff measure. We then have the implication

$$u|_{\Lambda^{+-}} = 0 \Longrightarrow u|_{\mathbb{R}_+ \times \mathbb{R}_-} = 0$$

if and only if $\alpha\beta M^2 \leq 4\pi^2$.

- The corresponding assertion holds for Λ^{-+} as well. The quarter-planes $\mathbb{R}_+ \times \mathbb{R}_-$ and $\mathbb{R}_- \times \mathbb{R}_+$ are both *space-like*. *There is no analogue for* $\Lambda^{++}, \Lambda^{--}$ *(time-like)*.

The holomorphic exponentials

The predual of $H^\infty(\mathbb{C}_+)$

- In what follows we focus on the critical case $\alpha\beta M^2 = 4\pi$.
- We normalize: $M = 2\pi$ and $\alpha = \beta = 1$.
- The predual of $H^\infty(\mathbb{C}_+)$ is identified with $L^1(\mathbb{R})/H^1(\mathbb{C}_+)$.
- We need to look for $f \in L^1(\mathbb{R})/H^1(\mathbb{C}_+)$ such that $\forall n = 0, 1, 2 \ldots$

$$\int_{\mathbb{R}} f(x) e^{in\pi x}\, dx = 0 = \int_{\mathbb{R}} f(x) e^{-in\pi/x}\, dx.$$

  In particular, $f \perp 1$, which means that f has mean 0.
- Let $L^1_0(\mathbb{R})$ denote the $L^1(\mathbb{R})$ functions that satisfy

$$\int_{\mathbb{R}} f(x) dx = 0.$$

The holomorphic exponentials

The predual of $H^\infty(\mathbb{C}_+)$

- In what follows we focus on the critical case $\alpha\beta M^2 = 4\pi$.
- We normalize: $M = 2\pi$ and $\alpha = \beta = 1$.
- The predual of $H^\infty(\mathbb{C}_+)$ is identified with $L^1(\mathbb{R})/H^1(\mathbb{C}_+)$.
- We need to look for $f \in L^1(\mathbb{R})/H^1(\mathbb{C}_+)$ such that $\forall n = 0, 1, 2 \ldots$

$$\int_{\mathbb{R}} f(x)e^{in\pi x}\,dx = 0 = \int_{\mathbb{R}} f(x)e^{-in\pi/x}\,dx.$$

  In particular, $f \perp 1$, which means that f has mean 0.
- Let $L^1_0(\mathbb{R})$ denote the $L^1(\mathbb{R})$ functions that satisfy

$$\int_{\mathbb{R}} f(x)dx = 0.$$

The holomorphic exponentials

The predual of $H^\infty(\mathbb{C}_+)$

- In what follows we focus on the critical case $\alpha\beta M^2 = 4\pi$.
- We normalize: $M = 2\pi$ and $\alpha = \beta = 1$.
- The predual of $H^\infty(\mathbb{C}_+)$ is identified with $L^1(\mathbb{R})/H^1(\mathbb{C}_+)$.
- We need to look for $f \in L^1(\mathbb{R})/H^1(\mathbb{C}_+)$ such that $\forall n = 0, 1, 2 \ldots$

$$\int_{\mathbb{R}} f(x)e^{in\pi x}\, dx = 0 = \int_{\mathbb{R}} f(x)e^{-in\pi/x}\, dx.$$

  In particular, $f \perp 1$, which means that f has mean 0.
- Let $L^1_0(\mathbb{R})$ denote the $L^1(\mathbb{R})$ functions that satisfy

$$\int_{\mathbb{R}} f(x)dx = 0.$$

The holomorphic exponentials

The predual of $H^\infty(\mathbb{C}_+)$

- In what follows we focus on the critical case $\alpha\beta M^2 = 4\pi$.
- We normalize: $M = 2\pi$ and $\alpha = \beta = 1$.
- The predual of $H^\infty(\mathbb{C}_+)$ is identified with $L^1(\mathbb{R})/H^1(\mathbb{C}_+)$.
- We need to look for $f \in L^1(\mathbb{R})/H^1(\mathbb{C}_+)$ such that $\forall n = 0, 1, 2 \ldots$

$$\int_{\mathbb{R}} f(x) e^{in\pi x}\, dx = 0 = \int_{\mathbb{R}} f(x) e^{-in\pi/x}\, dx.$$

  In particular, $f \perp 1$, which means that f has mean 0.
- Let $L^1_0(\mathbb{R})$ denote the $L^1(\mathbb{R})$ functions that satisfy

$$\int_{\mathbb{R}} f(x) dx = 0.$$

The holomorphic exponentials

The predual of $H^\infty(\mathbb{C}_+)$

- In what follows we focus on the critical case $\alpha\beta M^2 = 4\pi$.
- We normalize: $M = 2\pi$ and $\alpha = \beta = 1$.
- The predual of $H^\infty(\mathbb{C}_+)$ is identified with $L^1(\mathbb{R})/H^1(\mathbb{C}_+)$.
- We need to look for $f \in L^1(\mathbb{R})/H^1(\mathbb{C}_+)$ such that $\forall n = 0, 1, 2 \ldots$

$$\int_{\mathbb{R}} f(x) e^{in\pi x}\, dx = 0 = \int_{\mathbb{R}} f(x) e^{-in\pi/x}\, dx.$$

  In particular, $f \perp 1$, which means that f has mean 0.
- Let $L^1_0(\mathbb{R})$ denote the $L^1(\mathbb{R})$ functions that satisfy

$$\int_{\mathbb{R}} f(x) dx = 0.$$

The holomorphic exponentials

The predual of $H^\infty(\mathbb{C}_+)$

- In what follows we focus on the critical case $\alpha\beta M^2 = 4\pi$.
- We normalize: $M = 2\pi$ and $\alpha = \beta = 1$.
- The predual of $H^\infty(\mathbb{C}_+)$ is identified with $L^1(\mathbb{R})/H^1(\mathbb{C}_+)$.
- We need to look for $f \in L^1(\mathbb{R})/H^1(\mathbb{C}_+)$ such that $\forall n = 0, 1, 2 \ldots$

$$\int_{\mathbb{R}} f(x)e^{in\pi x}\,dx = 0 = \int_{\mathbb{R}} f(x)e^{-in\pi/x}\,dx.$$

  In particular, $f \perp 1$, which means that f has mean 0.
- Let $L^1_0(\mathbb{R})$ denote the $L^1(\mathbb{R})$ functions that satisfy

$$\int_{\mathbb{R}} f(x)dx = 0.$$

An equivalent statement

Back to the operators

- So, we need to check that if $f \in L^1_0(\mathbb{R})$ has

$$\mathbf{\Pi}_2 f(x) = \sum_{j\in\mathbb{Z}} f(x+2j) \in H^1_2(\mathbb{C}_+),$$

and $$\mathbf{\Pi}_2 \mathbf{J}_1 f(x) = \sum_{j\in\mathbb{Z}} \frac{1}{(x+2j)^2} f\left(-\frac{1}{x+2j}\right) \in H^1_2(\mathbb{C}_+).$$

then $f \in H^1(\mathbb{C}_+)$. Here, $H^1_2(\mathbb{C}_+) := H^1(\mathbb{C}_+/2\mathbb{Z})$.

Back to the operators

- So, we need to check that if $f \in L^1_0(\mathbb{R})$ has

$$\boldsymbol{\Pi}_2 f(x) = \sum_{j\in\mathbb{Z}} f(x+2j) \in H^1_2(\mathbb{C}_+),$$

and

$$\boldsymbol{\Pi}_2 \mathbf{J}_1 f(x) = \sum_{j\in\mathbb{Z}} \frac{1}{(x+2j)^2} f\left(-\frac{1}{x+2j}\right) \in H^1_2(\mathbb{C}_+).$$

then $f \in H^1(\mathbb{C}_+)$. Here, $H^1_2(\mathbb{C}_+) := H^1(\mathbb{C}_+/2\mathbb{Z})$.

An equivalent statement

Back to the operators

- So, we need to check that if $f \in L^1_0(\mathbb{R})$ has

$$\mathbf{\Pi}_2 f(x) = \sum_{j\in\mathbb{Z}} f(x+2j) \in H^1_2(\mathbb{C}_+),$$

$$\text{and} \quad \mathbf{\Pi}_2 \mathbf{J}_1 f(x) = \sum_{j\in\mathbb{Z}} \frac{1}{(x+2j)^2} f\left(-\frac{1}{x+2j}\right) \in H^1_2(\mathbb{C}_+).$$

then $f \in H^1(\mathbb{C}_+)$. Here, $H^1_2(\mathbb{C}_+) := H^1(\mathbb{C}_+/2\mathbb{Z})$.

An equivalent statement

Back to the operators

- So, we need to check that if $f \in L^1_0(\mathbb{R})$ has

$$\mathbf{\Pi}_2 f(x) = \sum_{j\in\mathbb{Z}} f(x+2j) \in H^1_2(\mathbb{C}_+),$$

$$\text{and} \quad \mathbf{\Pi}_2 \mathbf{J}_1 f(x) = \sum_{j\in\mathbb{Z}} \frac{1}{(x+2j)^2} f\left(-\frac{1}{x+2j}\right) \in H^1_2(\mathbb{C}_+).$$

then $f \in H^1(\mathbb{C}_+)$. Here, $H^1_2(\mathbb{C}_+) := H^1(\mathbb{C}_+/2\mathbb{Z})$.

An equivalent statement

Back to the operators

- So, we need to check that if $f \in L^1_0(\mathbb{R})$ has

$$\mathbf{\Pi}_2 f(x) = \sum_{j\in\mathbb{Z}} f(x+2j) \in H^1_2(\mathbb{C}_+),$$

$$\text{and} \quad \mathbf{\Pi}_2 \mathbf{J}_1 f(x) = \sum_{j\in\mathbb{Z}} \frac{1}{(x+2j)^2} f\left(-\frac{1}{x+2j}\right) \in H^1_2(\mathbb{C}_+).$$

then $f \in H^1(\mathbb{C}_+)$. Here, $H^1_2(\mathbb{C}_+) := H^1(\mathbb{C}_+/2\mathbb{Z})$.

The Hilbert transform

Some properties

- For $f \in L^1(\mathbb{R})$, its Hilbert transform is

$$(\mathbf{H}f)(x) = \text{pv}\frac{1}{\pi}\int_{\mathbb{R}} \frac{f(t)}{x-t}dt.$$

- The Hilbert transform commutes with translations.
- Indeed, $\mathbf{H} = -i\mathbf{F}\mathbf{M}_{\mathrm{sgn}}\mathbf{F}$, where $\mathbf{F}$ is the Fourier transform and sgn is the sign function. Thus $\mathbf{H}^2 = -\mathbf{I}$,
- The Hilbert transform also commutes with $\mathbf{J}_1$, where $(\mathbf{J}_1 f)(x) = x^{-2}f(-1/x)$, but only on $L^1_0(\mathbb{R})$.

The Hilbert transform

Some properties

- For $f \in L^1(\mathbb{R})$, its Hilbert transform is

$$(\mathbf{H}f)(x) = \mathrm{pv}\frac{1}{\pi}\int_{\mathbb{R}} \frac{f(t)}{x-t}dt.$$

- The Hilbert transform commutes with translations.
- Indeed, $\mathbf{H} = -i\mathbf{F}\mathbf{M}_{\mathrm{sgn}}\mathbf{F}$, where $\mathbf{F}$ is the Fourier transform and sgn is the sign function. Thus $\mathbf{H}^2 = -\mathbf{I}$,
- The Hilbert transform also commutes with $\mathbf{J}_1$, where $(\mathbf{J}_1 f)(x) = x^{-2}f(-1/x)$, but only on $L^1_0(\mathbb{R})$.

The Hilbert transform

Some properties

- For $f \in L^1(\mathbb{R})$, its Hilbert transform is

$$(\mathbf{H}f)(x) = \mathrm{pv}\frac{1}{\pi}\int_{\mathbb{R}} \frac{f(t)}{x-t}dt.$$

- The Hilbert transform commutes with translations.
- Indeed, $\mathbf{H} = -i\mathbf{F}\mathbf{M}_{\mathrm{sgn}}\mathbf{F}$, where $\mathbf{F}$ is the Fourier transform and sgn is the sign function. Thus $\mathbf{H}^2 = -\mathbf{I}$,
- The Hilbert transform also commutes with $\mathbf{J}_1$, where $(\mathbf{J}_1 f)(x) = x^{-2}f(-1/x)$, but only on $L^1_0(\mathbb{R})$.

The Hilbert transform

Some properties

- For $f \in L^1(\mathbb{R})$, its Hilbert transform is

$$(\mathbf{H}f)(x) = \text{pv}\frac{1}{\pi}\int_{\mathbb{R}} \frac{f(t)}{x-t}dt.$$

- The Hilbert transform commutes with translations.
- Indeed, $\mathbf{H} = -i\mathbf{F}\mathbf{M}_{\mathrm{sgn}}\mathbf{F}$, where $\mathbf{F}$ is the Fourier transform and sgn is the sign function. Thus $\mathbf{H}^2 = -\mathbf{I}$,
- The Hilbert transform also commutes with $\mathbf{J}_1$, where $(\mathbf{J}_1 f)(x) = x^{-2} f(-1/x)$, but only on $L^1_0(\mathbb{R})$.

The Hilbert transform

Some properties

- For $f \in L^1(\mathbb{R})$, its Hilbert transform is

$$(\mathbf{H}f)(x) = \text{pv}\frac{1}{\pi}\int_{\mathbb{R}} \frac{f(t)}{x-t}dt.$$

- The Hilbert transform commutes with translations.
- Indeed, $\mathbf{H} = -i\mathbf{F}\mathbf{M}_{\mathrm{sgn}}\mathbf{F}$, where $\mathbf{F}$ is the Fourier transform and sgn is the sign function. Thus $\mathbf{H}^2 = -\mathbf{I}$,
- The Hilbert transform also commutes with $\mathbf{J}_1$, where $(\mathbf{J}_1 f)(x) = x^{-2}f(-1/x)$, but only on $L_0^1(\mathbb{R})$.

The space $\mathcal{L}(\mathbb{R})$

Definition.

We write $\mathcal{L}(\mathbb{R}) := L^1(\mathbb{R}) + \mathbf{H}L^1_0(\mathbb{R})$, interpreted as a space of distributions.

There exists a *valeur au point* map $\mathrm{pv} : \mathcal{L}(\mathbb{R}) \to L^{1,\infty}(\mathbb{R})$, which assigns point value a.e., and is 1-to-1. This allows us to think of $\mathcal{L}(\mathbb{R})$ as distributions as well as functions in $L^{1,\infty}(\mathbb{R})$ – the weak L^1 space.

Interpretation of the space $\mathcal{L}(\mathbb{R})$

Physical intepretation: pure states

The space $L^1(\mathbb{R})$ arises from the (L^1-smooth) linear combinations of pure states $\delta_y(x)$ – point masses (particles). In a similar way, the space arises from (L^1-smooth) linear combinations of two kinds of pure states:
(i) $\delta_y(x)$ – point masses (particles), and
(ii) $\mathbf{H}\delta_y(x) = \frac{1}{\pi(x-y)}$ – pseudoparticles.

An equivalent reformulation

$\mathcal{L}_0(\mathbb{R}) := L^1_0(\mathbb{R}) + \mathbf{H}L^1_0(\mathbb{R})$

Using the projection conjugate-holomorphic projection $\mathbf{P}_- := \frac{1}{2}(\mathbf{I} - i\mathbf{H})$, it is possible to see that the implication (for $f \in L^1(\mathbb{R})$)

$$\mathbf{\Pi}_2 f \in H^1_2(\mathbb{C}_+),\ \ \mathbf{\Pi}_2 \mathbf{J}_1 f \in H^1_2(\mathbb{C}_+) \implies f \in H^1(\mathbb{C}_+)$$

is equivalent to the following: For $u \in \mathcal{L}_0(\mathbb{R})$, it holds that

$$\mathbf{\Pi}_2 u = 0,\ \ \mathbf{\Pi}_2 \mathbf{J}_1 u = 0 \implies u = 0.$$

A set of solutions in $L^{1,\infty}(\mathbb{R})$ (weak L^1)

The problem is now in the same form as the previous theorem, only the space $\mathcal{L}_0(\mathbb{R})$ is much bigger than $L^1_0(\mathbb{R})$. How dangerous is this?

$\mathcal{L}_0(\mathbb{R}) = L^1_0(\mathbb{R}) + \mathbf{H}L^1_0(\mathbb{R}) \subset L^{1,\infty}(\mathbb{R})$

- Take $x_1 = j + \sqrt{j^2-1}$ and $x_2 = \sqrt{j^2-1} - j$, $j = 2, 3, \ldots$.
- We have $e^{i\pi n(x_1 - x_2)} = 1$ and $e^{i\pi n(1/x_1 - 1/x_2)} = 1$ for each $n \in \mathbb{Z}$. In addition, $x_1 x_2 = -1$.
- Consider $\mu = \delta_{x_1} - \delta_{x_2}$, and put

$$f(x) = (\mathbf{H}\mu)(x) = \frac{\pi^{-1}}{x - x_1} - \frac{\pi^{-1}}{x - x_2}.$$

  Then $f \in L^{1,\infty}(\mathbb{R})$ is a nontrivial solution to

$$\mathbf{\Pi}_2 f = 0, \quad \mathbf{\Pi}_2 \mathbf{J}_1 f = 0.$$

A set of solutions in $L^{1,\infty}(\mathbb{R})$ (weak L^1)

The problem is now in the same form as the previous theorem, only the space $\mathcal{L}_0(\mathbb{R})$ is much bigger than $L^1_0(\mathbb{R})$. How dangerous is this?

$\mathcal{L}_0(\mathbb{R}) = L^1_0(\mathbb{R}) + \mathbf{H}L^1_0(\mathbb{R}) \subset L^{1,\infty}(\mathbb{R})$

- Take $x_1 = j + \sqrt{j^2-1}$ and $x_2 = \sqrt{j^2-1} - j$, $j = 2, 3, \ldots$.
- We have $e^{i\pi n(x_1 - x_2)} = 1$ and $e^{i\pi n(1/x_1 - 1/x_2)} = 1$ for each $n \in \mathbb{Z}$. In addition, $x_1 x_2 = -1$.
- Consider $\mu = \delta_{x_1} - \delta_{x_2}$, and put

$$f(x) = (\mathbf{H}\mu)(x) = \frac{\pi^{-1}}{x - x_1} - \frac{\pi^{-1}}{x - x_2}.$$

Then $f \in L^{1,\infty}(\mathbb{R})$ is a nontrivial solution to

$$\mathbf{\Pi}_2 f = 0, \quad \mathbf{\Pi}_2 \mathbf{J}_1 f = 0.$$

A set of solutions in $L^{1,\infty}(\mathbb{R})$ (weak L^1)

The problem is now in the same form as the previous theorem, only the space $\mathcal{L}_0(\mathbb{R})$ is much bigger than $L^1_0(\mathbb{R})$. How dangerous is this?

$\mathcal{L}_0(\mathbb{R}) = L^1_0(\mathbb{R}) + \mathbf{H}L^1_0(\mathbb{R}) \subset L^{1,\infty}(\mathbb{R})$

- Take $x_1 = j + \sqrt{j^2-1}$ and $x_2 = \sqrt{j^2-1} - j$, $j = 2, 3, \ldots$.
- We have $e^{i\pi n(x_1-x_2)} = 1$ and $e^{i\pi n(1/x_1 - 1/x_2)} = 1$ for each $n \in \mathbb{Z}$. In addition, $x_1 x_2 = -1$.
- Consider $\mu = \delta_{x_1} - \delta_{x_2}$, and put

$$f(x) = (\mathbf{H}\mu)(x) = \frac{\pi^{-1}}{x - x_1} - \frac{\pi^{-1}}{x - x_2}.$$

Then $f \in L^{1,\infty}(\mathbb{R})$ is a nontrivial solution to

$$\mathbf{\Pi}_2 f = 0, \quad \mathbf{\Pi}_2 \mathbf{J}_1 f = 0.$$

A set of solutions in $L^{1,\infty}(\mathbb{R})$ (weak L^1)

The problem is now in the same form as the previous theorem, only the space $\mathcal{L}_0(\mathbb{R})$ is much bigger than $L^1_0(\mathbb{R})$. How dangerous is this?

$\mathcal{L}_0(\mathbb{R}) = L^1_0(\mathbb{R}) + \mathbf{H} L^1_0(\mathbb{R}) \subset L^{1,\infty}(\mathbb{R})$

- Take $x_1 = j + \sqrt{j^2 - 1}$ and $x_2 = \sqrt{j^2 - 1} - j$, $j = 2, 3, \ldots$.
- We have $e^{i\pi n(x_1 - x_2)} = 1$ and $e^{i\pi n(1/x_1 - 1/x_2)} = 1$ for each $n \in \mathbb{Z}$. In addition, $x_1 x_2 = -1$.
- Consider $\mu = \delta_{x_1} - \delta_{x_2}$, and put

$$f(x) = (\mathbf{H}\mu)(x) = \frac{\pi^{-1}}{x - x_1} - \frac{\pi^{-1}}{x - x_2}.$$

Then $f \in L^{1,\infty}(\mathbb{R})$ is a nontrivial solution to

$$\mathbf{\Pi}_2 f = 0, \quad \mathbf{\Pi}_2 \mathbf{J}_1 f = 0.$$

A set of solutions in $L^{1,\infty}(\mathbb{R})$ (weak L^1)

The problem is now in the same form as the previous theorem, only the space $\mathcal{L}_0(\mathbb{R})$ is much bigger than $L^1_0(\mathbb{R})$. How dangerous is this?

$\mathcal{L}_0(\mathbb{R}) = L^1_0(\mathbb{R}) + \mathbf{H}L^1_0(\mathbb{R}) \subset L^{1,\infty}(\mathbb{R})$

- Take $x_1 = j + \sqrt{j^2 - 1}$ and $x_2 = \sqrt{j^2 - 1} - j$, $j = 2, 3, \ldots$.
- We have $e^{i\pi n(x_1 - x_2)} = 1$ and $e^{i\pi n(1/x_1 - 1/x_2)} = 1$ for each $n \in \mathbb{Z}$. In addition, $x_1 x_2 = -1$.
- Consider $\mu = \delta_{x_1} - \delta_{x_2}$, and put

$$f(x) = (\mathbf{H}\mu)(x) = \frac{\pi^{-1}}{x - x_1} - \frac{\pi^{-1}}{x - x_2}.$$

Then $f \in L^{1,\infty}(\mathbb{R})$ is a nontrivial solution to

$$\mathbf{\Pi}_2 f = 0, \quad \mathbf{\Pi}_2 \mathbf{J}_1 f = 0.$$

The transfer operator in the distributional setting

The space $\mathcal{L}(I_1)$

If u is a distribution on I_1, we write $u \in \mathcal{L}(I_1)$ if there is a $U \in \mathcal{L}(\mathbb{R})$ whose restrivtion to I_1 equals u.

The transfer operator $\mathbf{T}_1$ extends continuously $\mathcal{L}(I_1) \to \mathcal{L}(I_1)$ – this requires some work. However, it is not a contraction!

Basic observation.

As before, our problem reduces to showing that $\mathbf{T}_1 u = \lambda u$ for $\lambda = \pm 1$ and $u \in \mathcal{L}(I_1)$ has only the solution $u = 0$.

The transfer operator in the distributional setting

The space $\mathcal{L}(I_1)$

If u is a distribution on I_1, we write $u \in \mathcal{L}(I_1)$ if there is a $U \in \mathcal{L}(\mathbb{R})$ whose restrivtion to I_1 equals u.

The transfer operator $\mathbf{T}_1$ extends continuously $\mathcal{L}(I_1) \to \mathcal{L}(I_1)$ – this requires some work. However, it is not a contraction!

Basic observation.

As before, our problem reduces to showing that $\mathbf{T}_1 u = \lambda u$ for $\lambda = \pm 1$ and $u \in \mathcal{L}(I_1)$ has only the solution $u = 0$.

The main theorem

Asymptotic stability in $L^{1,\infty}$ (Lyapunov energy)

THEOREM. If $u \in \mathcal{L}(I_1)$ is odd, then $1_{I_\eta} \mathrm{vp}\, \mathbf{T}_1^n u \to 0$ in $L^{1,\infty}(I_1)$ as $n \to +\infty$, for each fixed $0 < \eta < 1$.

REMARK.

(i) This shows the asymptotic stability of the $\mathbf{T}_1$-orbits of odd distributions, which is a delicate matter as $\mathbf{T}_1$ *is not a norm contraction*.

(ii) What about even distributions? There is a *trick* which allows us to focus exclusively on the odd distributions in $\mathcal{L}(I_1)$ and still be able to prove that $\mathbf{T}u = \lambda u$ for $\lambda = \pm 1$ implies that $u = 0$ for $u \in \mathcal{L}(I_1)$.

The main theorem

Asymptotic stability in $L^{1,\infty}$ (Lyapunov energy)

THEOREM. If $u \in \mathcal{L}(I_1)$ is odd, then $1_{I_\eta} \mathrm{vp}\, \mathbf{T}_1^n u \to 0$ in $L^{1,\infty}(I_1)$ as $n \to +\infty$, for each fixed $0 < \eta < 1$.

REMARK.

(i) This shows the asymptotic stability of the $\mathbf{T}_1$-orbits of odd distributions, which is a delicate matter as $\mathbf{T}_1$ *is not a norm contraction*.
(ii) What about even distributions? There is a *trick* which allows us to focus exclusively on the odd distributions in $\mathcal{L}(I_1)$ and still be able to prove that $\mathbf{T}u = \lambda u$ for $\lambda = \pm 1$ implies that $u = 0$ for $u \in \mathcal{L}(I_1)$.

Underlying ideas

The commutator of the Hilbert transform and the transfer operator

The commutator (for $n = 1, 2, 3, \ldots$ and $f \in L^1(I_1)$)

$$[\mathbf{T}_1^n, \mathbf{H}]f = \mathbf{T}_1^n \mathbf{H} f - \mathbf{H} \mathbf{T}_1^n f$$

is basic to our analysis. Our main problem is after all that the Hilbert transform has *long tails* (non-local).

Underlying ideas

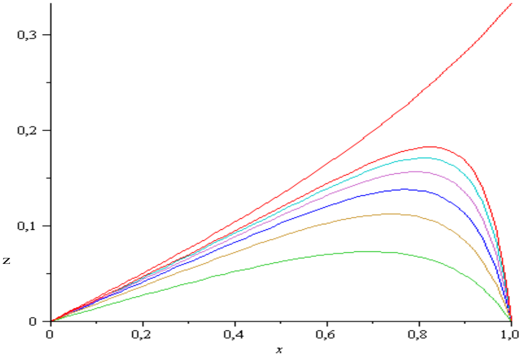
The commutator of the Hilbert transform and the transfer operator

The commutator (for $n = 1, 2, 3, \ldots$ and $f \in L^1(I_1)$)

$$[\mathbf{T}_1^n, \mathbf{H}]f = \mathbf{T}_1^n \mathbf{H} f - \mathbf{H}\mathbf{T}_1^n f$$

is basic to our analysis. Our main problem is after all that the Hilbert transform has *long tails* (non-local).

A dynamical splitting of the odd part of the Hilbert kernel.



Some ingredients of the proof

The splitting of the Hilbert kernel

Let
$$Q(t,x) := \frac{t}{1+tx}$$
be the Hilbert kernel. It odd part w. r. t. x is
$$Q^{II}(t,x) := \frac{t^2 x}{1-t^2x^2}.$$
Let
$$q^{II}(t,x) := (\mathbf{I} - \mathbf{T}_1)Q^{II}(t,\cdot)(x)$$
be the corresponding *dynamically reduced kernel*.

Some ingredients of the proof

The splitting of the Hilbert kernel

Let
$$Q(t,x) := \frac{t}{1+tx}$$
be the Hilbert kernel. It odd part w. r. t. x is
$$Q^{II}(t,x) := \frac{t^2x}{1-t^2x^2}.$$
Let
$$q^{II}(t,x) := (\mathbf{I}-\mathbf{T}_1)Q^{II}(t,\cdot)(x)$$
be the corresponding *dynamically reduced kernel*.

The splitting using Neumann series

Neumann series

We have that

$$Q^{II}(t,x) = \sum_{j=0}^{+\infty} \mathbf{T}_1^j q^{II}(t,\cdot)(x).$$

LEMMA

(Uniform control of summands) We have that

$$0 < \mathbf{T}_1^j q^{II}(t,\cdot)(x) < \mathbf{T}_1^j q^{II}(1,\cdot)(x), \qquad 0 < x < 1,$$

for fixed $t \in I_1$.

Further remarks on the proof

T

he proof of the dynamical decomposition lemma involves:

- Hurwitz zeta function estimate.
- Application of the theory of totally positivite matrices.

Bibliography

HM1. Hedenmalm, H., Montes-Rodriguez, A., *Heisenberg uniqueness pairs and the Klein-Gordon equation*. Ann. of Math. 173 (2011), 1507-1527.

HM2. Hedenmalm, H., Montes-Rodriguez, A., *Perron-Frobenius operators and the Klein-Gordon equation*. J. Eur. Math. Soc. 16 (2014), 31-66.

HM3. Hedenmalm, H., Montes-Rodriguez, A., *The Klein-Gordon equation, the Hilbert transform, and dynamics of Gauss-type maps*. Submitted.