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A critical topology for L^p-Carleman classes with $0 < p < 1$

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Abstract In this paper, we explore a sharp phase transition phenomenon which occurs for L^p-Carleman classes with exponents $0 < p < 1$. These classes are defined as for the standard Carleman classes, only the L^∞-bounds are replaced by corresponding L^p-bounds. We study the quasinorms

$$\|u\|_{p,\mathcal{M}} = \sup_{n \geq 0} \frac{\|u^{(n)}\|_p}{M_n},$$

for some weight sequence $\mathcal{M} = \{M_n\}_n$ of positive real numbers, and consider as the corresponding L^p-Carleman space the completion of a given collection of smooth test functions. To mirror the classical definition, we add the feature of dilatation invariance as well, and consider a larger soft-topology space, the L^p-Carleman class. A particular degenerate instance is when $M_n = 1$ for $0 \leq n \leq k$ and $M_n = +\infty$ for $n > k$. This would give the L^p-Sobolev spaces, which were analyzed by Peetre, following an initial insight by Douady. Peetre found that these L^p-Sobolev spaces are highly degenerate for $0 < p < 1$. Indeed, the canonical map $W^{k,p} \to L^p$ fails to be injective, and there is even an isomorphism

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$$W^{k,p} \cong L^p \oplus L^p \oplus \cdots \oplus L^p,$$

corresponding to the canonical map $f \mapsto (f, f', \ldots, f^{(k)})$ acting on the test functions. This means that e.g. the function and its derivative lose contact with each other (they "disconnect"). Here, we analyze this degeneracy for the more general L^p-Carleman classes defined by a weight sequence $\mathcal{M}$. If $\mathcal{M}$ has some regularity properties, and if the given collection of test functions is what we call (p, θ)-*tame*, then we find that there is a sharp boundary, defined in terms of the weight $\mathcal{M}$: on the one side, we get Douady–Peetre's phenomenon of "disconnexion", while on the other, the completion of the test functions consists of C^∞-smooth functions and the canonical map $f \mapsto (f, f', f'', \ldots)$ is correspondingly well-behaved in the completion. We also look at the more standard second phase transition, between non-quasianalyticity and quasianalyticity, in the L^p setting, with $0 < p < 1$.

1 Introduction

1.1 The Sobolev spaces for $0 < p < 1$

We first survey the properties of the Sobolev spaces with exponents in the range $0 < p < 1$. These were considered by Peetre [18] after Adrien Douady suggested that they behave very differently from the standard $1 \le p \le +\infty$ case. For an exponent p with $0 < p < 1$, and an integer $k \ge 0$, *we define the Sobolev space* $W^{k,p}(\mathbb{R})$ *on the real line* $\mathbb{R}$ *to be the abstract completion of the space* C_0^k *of compactly supported k times continuously differentiable functions*, *with respect to the quasinorm*

$$\|f\|_{k,p} = \left(\|f\|_p^p + \|f'\|_p^p + \cdots \|f^{(k)}\|_p^p\right)^{1/p}. \tag{1.1}$$

Here, as a matter of notation, $\|\cdot\|_p$ denotes the quasinorm of L^p. Defined in this manner, $W^{k,p}$ becomes a quasi-Banach space whose elements consist of equivalence classes of Cauchy sequences of test functions. At first glance, this definition is very natural, and other approaches to define the same object in the classical case $p \ge 1$ do not generalize. For instance, we cannot use the usual notion of weak derivatives to define these spaces, as functions in L^p need not be locally L^1 for $0 < p < 1$.

Some observations suggest that things are not the way we might expect them to be. For instance, as a consequence of the failure of local integrability, for a given point $a \in \mathbb{R}$, the formula for the primitive

$$F(x) = \int_a^x f(t)\,\mathrm{d}t$$

cannot be expected to make sense. A related difficulty is the invisibility of Dirac "point masses" in the quasinorm of L^p. Indeed, if $0 < p < 1$ and $u_\epsilon := \epsilon^{-1} 1_{[0,\epsilon]}$, then as $\epsilon \to 0^+$, we have the convergence $u_\epsilon \to 0$ in L^p while $u_\epsilon \to \delta_0$ in the sense of distribution theory. Here, we use standard notational conventions: δ_0 denotes the unit point mass at 0, and 1_E is the characteristic function of the subset E, which equals 1 on E and vanishes off E.

1.2 Independence of the derivatives in Sobolev spaces

For $1 \le p \le +\infty$, we think of the Sobolev space $W^{k,p}$ as a subspace of L^p, consisting of functions of a specified degree of smoothness. As such, the identity mapping $\boldsymbol{\alpha}\colon W^{k,p} \to L^p$ defines a canonical injection. Douady observed that we cannot have this picture in mind when $0 < p < 1$, as the corresponding canonical map $\boldsymbol{\alpha}$ fails to be injective. Peetre built on Douady's observation and showed that this uncoupling (or disconnexion) between the derivatives goes even deeper. In fact, the standard map $f \mapsto (f, f', \ldots, f^{(k)})$ on test functions f defines a topological isomorphism of the completion $W^{k,p}$ onto the direct sum of $k+1$ copies of L^p. For convenience, we state the theorem here.

Theorem 1.1 [18] *Let $0 < p < 1$ and $k = 1, 2, 3, \ldots$. Then $W^{k,p}$ is isometrically isomorphic to $k+1$ copies of L^p:*

$$W^{k,p} \cong L^p \oplus L^p \oplus \cdots \oplus L^p. \tag{1.2}$$

This decoupling occurs as a result of the availability of approximate point masses which are barely visible in the quasinorm. As a consequence, if we define Sobolev spaces as completions with respect to the Sobolev quasinorm, we obtain highly pathological (and rather useless) objects.

1.3 A bootstrap argument to control the L^∞-norm in terms the L^p-quasinorms of the higher derivatives

From the Douady–Peetre analysis of the Sobolev spaces $W^{k,p}$ for exponents $0 < p < 1$, we might be inclined to believe that for such small p, we always run into pathology. However, there is in fact an argument of bootstrap type which can save the situation if we simultaneously control *the derivatives of all orders*. To explain the bootstrap argument, we take a (weight) sequence $\mathcal{M} = \{M_k\}_{k=0}^{+\infty}$ of positive reals, and define the quasinorm

$$\|f\|_{p,\mathcal{M}} = \sup_{k\ge 0} \frac{\|f^{(k)}\|_p}{M_k}, \quad f \in \mathscr{S}, \tag{1.3}$$

on some appropriate linear space $\mathscr{S} \subset C^\infty(\mathbb{R})$ of test functions that we choose to begin with. In the analogous setting of periodic functions (i.e., with the unit circle $\mathbb{T} \cong \mathbb{R}/\mathbb{Z}$ in place of $\mathbb{R}$), it would be natural to work with the linear span of the periodic complex exponentials as $\mathscr{S}$. In the present setting of the line $\mathbb{R}$, there is no such canonical choice. Short of a natural explicit linear space of functions, we ask instead that $\mathscr{S}$ should satisfy a property.

Definition 1.1 ($0 < p \leq 1$, $\theta \in \mathbb{R}$) We say that $f \in C^\infty(\mathbb{R})$ is (p, θ)-*tame* if $f^{(n)} \in L^\infty(\mathbb{R})$ for each $n = 0, 1, 2, \ldots,$ and

$$\limsup_{n \to +\infty} (1-p)^n \log \|f^{(n)}\|_\infty \leq \theta. \tag{1.4}$$

Moreover, for subsets $\mathscr{S} \subset C^\infty(\mathbb{R})$, we say that $\mathscr{S}$ is (p, θ)-*tame* if every element $f \in \mathscr{S}$ is (p, θ)-tame.

Remark 1.1 (a) For $\theta < 0$, only the constant functions $f = 0$ are (p, θ)-tame, unless if $p = 1$, in which case no (p, θ)-tame function exists (the limsup vanishes and the right-hand side $\theta < 0$). The claim that only the constant functions are (p, θ)-tame for $0 < p < 1$ and $\theta < 0$ may be obtained using an argument involving entire functions, Liouville's theorem, and an application of the Phragmen–Lindelöf principle. For this reason, in the sequel, *we shall restrict our attention to* $\theta \geq 0$.

(b) Loosely speaking, for $\theta \geq 0$, the requirement (1.4) asks that the L^∞-norms of the higher order derivatives do not grow too wildly. We note that for p close to one, (p, θ)-tameness is a very weak requirement; indeed, at the endpoint value $p = 1$, it is void.

A natural suitable choice of a (p, θ)-tame collection of test functions might be the following Hermite class:

$$\mathscr{S}^{\mathrm{Her}} := \left\{ f : f(x) = \mathrm{e}^{-x^2} q(x), \text{ where } q \text{ is a polynomial} \right\}.$$

Indeed, a rather elementary argument shows that (1.4) holds with $\theta = 0$ for $\mathscr{S} = \mathscr{S}^{\mathrm{Her}}$ (for the details, see Lemma 4.1 below). However, it might be the case that not all $f \in \mathscr{S}^{\mathrm{Her}}$ have finite quasinorm $\|f\|_{p,\mathcal{M}} < +\infty$ (this depends on the choice of weight sequence $\mathcal{M}$). In that case, we would then replace $\mathscr{S}^{\mathrm{Her}}$ by its linear subspace

$$\mathscr{S}^{\mathrm{Her}}_{p,\mathcal{M}} := \left\{ f \in \mathscr{S}^{\mathrm{Her}} : \|f\|_{p,\mathcal{M}} < +\infty \right\},$$

and hope that this collection of test functions is not too small.

The bootstrap argument. We proceed with the bootstrap argument. We assume that our parameters are confined to the intervals $0 < p \leq 1$ and $0 \leq \theta < +\infty$. Moreover, we assume that the collection of test functions $\mathscr{S}$ is (p, θ)-tame and that $\|f\|_{p,\mathcal{M}} < +\infty$ holds for each $f \in \mathscr{S}$. We pick a normalized element $f \in \mathscr{S}$ with $\|f\|_{p,\mathcal{M}} = 1$. Since $1 = p + (1-p)$, it follows from the fundamental theorem of Calculus that for $x \leq y$,

$$\begin{aligned} |f(y) - f(x)| = \left| \int_x^y f'(t)\mathrm{d}t \right| \leq \int_x^y |f'(t)|\mathrm{d}t \leq \|f'\|_\infty^{1-p} \int_x^y |f'(t)|^p \mathrm{d}t \\ \leq \|f'\|_\infty^{1-p} \|f'\|_p^p. \end{aligned} \tag{1.5}$$

As $f \in L^p(\mathbb{R})$, the function f must assume values arbitrarily close to 0 on rather big subsets of $\mathbb{R}$. By taking the limit of such y in (1.5), we arrive at

$$|f(x)| \leq \|f'\|_\infty^{1-p} \|f'\|_p^p, \quad x \in \mathbb{R},$$

which gives that

$$\|f\|_\infty \leq \|f'\|_\infty^{1-p} \|f'\|_p^p. \tag{1.6}$$

By iteration of the estimate (1.6), we obtain that for $n = 1, 2, 3, \ldots,$

$$\begin{aligned} \|f\|_\infty &\leq \|f^{(n)}\|_\infty^{(1-p)^n} \|f^{(n)}\|_p^{p(1-p)^{n-1}} \cdots \|f'\|_p^p \\ &= \|f^{(n)}\|_\infty^{(1-p)^n} \prod_{j=1}^n \|f^{(j)}\|_p^{(1-p)^{j-1}p}. \end{aligned} \tag{1.7}$$

As it is given that $\|f\|_{p,\mathcal{M}} = 1$, we have that $\|f^{(j)}\|_p \leq M_j$, which we readily implement into (1.7):

$$\|f\|_\infty \leq \|f^{(n)}\|_\infty^{(1-p)^n} \prod_{j=1}^n M_j^{(1-p)^{j-1}p} \quad \text{if } \|f\|_{p,\mathcal{M}} = 1. \tag{1.8}$$

Finally, we let n approach infinity, so that in view of the (p, θ)-tameness assumption (1.4) and homogeneity, we obtain that

$$\|f\|_\infty \leq e^\theta \|f\|_{p,\mathcal{M}} \limsup_{n\to+\infty} \prod_{j=1}^n M_j^{(1-p)^{j-1}p}, \quad f \in \mathscr{S}. \tag{1.9}$$

The estimate (1.9) tells us that under the requirement

$$\kappa(p, \mathcal{M}) := \limsup_{n\to+\infty} \sum_{j=1}^n (1-p)^j \log M_j < +\infty, \tag{1.10}$$

we may control the sup-norm of a test function $f \in \mathscr{S}$ in terms of its quasinorm $\|f\|_{p,\mathcal{M}}$. We will refer to the quantity $\kappa(p, \mathcal{M})$ as the *p-characteristic of the sequence* $\mathcal{M}$. It follows that the Douady–Peetre disconnexion phenomenon does not occur if we simultaneously control all the higher derivatives under (1.10) (provided the test function space $\mathscr{S}$ is (p, θ)-tame). But the condition (1.10) achieves more. To see this, we first observe that as the inequality (1.7) applies to an arbitrary smooth function f, and in particular to a derivative $f^{(k)}$, for $k = 0, 1, 2, \ldots$:

$$\|f^{(k)}\|_\infty \leq \|f^{(n+k)}\|_\infty^{(1-p)^n} \prod_{j=1}^n \|f^{(j+k)}\|_p^{(1-p)^{j-1}p}, \quad n = 1, 2, 3, \ldots. \tag{1.11}$$

Next, we let n tend to infinity in (1.11) and use homogeneity and (p, θ)-tameness as in (1.9), and arrive at

$$\|f^{(k)}\|_\infty \le \|f\|_{p,\mathcal{M}} \limsup_{n\to+\infty} \|f^{(k+n)}\|_\infty^{(1-p)^n} \limsup_{n\to+\infty} \prod_{j=1}^{n} M_{j+k}^{(1-p)^{j-1}p}, \quad f \in \mathscr{S}. \tag{1.12}$$

Moreover, since

$$\sum_{j=1}^{n}(1-p)^j \log M_{j+k} = (1-p)^{-k} \sum_{j=1}^{n+k}(1-p)^j \log M_j - (1-p)^{-k} \sum_{j=1}^{k}(1-p)^j \log M_j,$$

and

$$(1-p)^n \log \|f^{(k+n)}\|_\infty = (1-p)^{-k}(1-p)^{n+k} \log \|f^{(k+n)}\|_\infty$$

it follows from (1.12) and (1.10) that

$$\|f^{(k)}\|_\infty \le \|f\|_{p,\mathcal{M}} \, \mathrm{e}^{\theta(1-p)^{-k}+p(1-p)^{-k-1}\kappa(p,\mathcal{M})} \prod_{j=1}^{k} M_j^{-(1-p)^{j-k-1}p}, \quad f \in \mathscr{S}, \; k = 0, 1, 2, \ldots. \tag{1.13}$$

It is immediate from (1.13) that under the summability condition (1.10), we may in fact control the sup-norm of all the higher order derivatives, which guarantees that the elements of the completion of the test function class $\mathscr{S}$ under the quasinorm $\|\cdot\|_{p,\mathcal{M}}$ consists of C^∞ functions, and the failure of the Douady–Peetre disconnexion phenomenon is complete. We refer to the argument leading up to (1.13) as a "bootstrap" because we were able to rid ourselves of the sup-norm control on the right-hand side by diminishing its contribution in the preceding estimate and taking the limit.

Remark 1.2 The above argument is inspired by an argument which goes back to work of Hardy and Littlewood on Hardy spaces of harmonic functions. The phenomenon is coined *Hardy–Littlewood ellipticity* in [7]. To explain the background, we recall that for a function $u(z)$ harmonic in the unit disk $\mathbb{D}$, the function $z \mapsto |u(z)|^p$ is subharmonic if $p \ge 1$. As such, it enjoys the mean value estimate

$$|u(0)|^p \le \frac{1}{\pi} \int_{\mathbb{D}} |u(z)|^p \mathrm{d}A(z).$$

A remarkable fact is that this inequality survives even for $0 < p < 1$ (with a different constant) even though subharmonicity fails. See, e.g., [7, Lemma 4.2], [13, Lemma 3.7], and the original work of Hardy and Littlewood [14]. The similarity with the bootstrap argument used here is striking if we compare with e.g. [13, Lemma 3.7].

1.4 The L^p-Carleman spaces and classes

The Carleman class $\mathscr{C}_{\mathcal{M}}$ associated with the weight sequence $\mathcal{M}$ is defined as the linear subspace of $f \in C^\infty(\mathbb{R})$ for which

$$\frac{\|f^{(k)}\|_\infty}{M_k} \le CA^k,$$

for some positive constant $A = A_f$ (which may depend on f). The theory for Carleman classes was developed in order to understand for which classes of functions the (formal) Taylor series at a point uniquely determines the function. Denjoy [12] provided an answer under regularity assumptions on the weight sequence, and Carleman [8–10] proved what has since become known as the Denjoy–Carleman Theorem: if $\mathcal{N}$ denotes the largest logarithmically convex minorant of $\mathcal{M}$, then $\mathscr{C}_{\mathcal{M}}$ has this uniqueness property if and only if

$$\sum_{j\ge 0} \frac{N_j}{N_{j+1}} = +\infty.$$

Bang later found a simplified proof of this result [1]. Bang also found numerous other remarkable results for the Carleman classes. To name one, he proved that the Bang degree $\mathfrak{n}_f$ of $f \in \mathscr{C}_{\mathcal{M}}([0,1])$, defined as the maximal integer $N \ge 0$ such that

$$\sum_{\log\|f\|_\infty^{-1} < j \le N} \frac{M_{j-1}}{M_j} < \mathrm{e},$$

is an upper bound for the number of zeros of f on the interval [0, 1]. For an account of several of the interesting results in [1] as well as in Bang's thesis [2], together with some further developments in the theory of quasianalytic functions, we refer to the work of Borichev, Nazarov, Sodin, and Volberg [6, 17].

The present work is devoted to the study of the analogous classes defined in terms of L^p-norms, mainly for $0 < p < 1$. In view of the preceding subsection, it is natural to select the biggest possible collection of test functions so that the bootstrap argument has a chance to apply under the assumption (1.10):

$$\mathscr{S}^{\circledast}_{p,\theta,\mathcal{M}} := \left\{ f \in C^\infty(\mathbb{R}) : \|f\|_{p,\mathcal{M}} < +\infty \text{ and } \limsup_{n\to+\infty} (1-p)^n \log \|f^{(n)}\|_\infty \le \theta \right\}.$$

Here, we keep our standing assumptions that $0 < p < 1$ and $\theta \ge 0$ (many assertion will hold also at the endpoint value $p = 1$). Then $\mathscr{S}^{\circledast}_{p,\theta,\mathcal{M}}$ automatically meets the asymptotic growth condition (1.4), but for (p, θ)-tameness to hold we also need for each individual derivative to be bounded. However, the condition

$$\limsup_{n\to+\infty}(1-p)^n \log \|f^{(n)}\|_\infty \le \theta$$

guarantees that $f^{(n)} \in L^\infty$ at least for big positive integers n, say for $n \ge n_0$. But then, since $\|f\|_{p,\mathcal{M}} < +\infty$ and in particular $f^{(k)} \in L^p$ for all $k = 0, 1, 2, \ldots$, the estimate (1.6) gives that $f^{(n-1)} \in L^\infty$ as well. Proceeding iteratively, we find that $f^{(n)} \in L^\infty$ for all $n = 0, 1, 2, \ldots$ if $f \in \mathscr{S}^{\circledast}_{p,\theta,\mathcal{M}}$. This means that all the estimates of the preceding subsection are sound for $f \in \mathscr{S}^{\circledast}_{p,\theta,\mathcal{M}}$. Of course, for very degenerate weight sequences $\mathcal{M}$, it might unfortunately happen that $\mathscr{S}^{\circledast}_{p,\theta,\mathcal{M}} = \{0\}$. We proceed to define the L^p-Carleman spaces.

Definition 1.2 $(0 < p < 1)$ The L^p-Carleman space $W^{p,\theta}_{\mathcal{M}}$ is the completion of the test function class $\mathscr{S}^{\circledast}_{p,\theta,\mathcal{M}}$ with respect to the quasinorm $\|\cdot\|_{p,\mathcal{M}}$.

In the standard textbook presentations, the Carleman classes are defined for a regular weight sequence $\mathcal{M}$ in the same way, only the L^p quasinorm is replaced by the L^∞ norm, and the class is to be minimal given the following two requirements: (i) the space is contained in the class, and (ii) the class is invariant with respect to dilatation [10,15,16]. It is easy to see that claiming that $f \in \mathscr{C}_{\mathcal{M}}$ is the same as saying that $f(t) = g(at)$ for some positive real a, where

$$\frac{\|g^{(k)}\|_\infty}{M_k} \le C, \quad k = 0, 1, 2, \ldots,$$

for some constant C, which we understand as the requirement $\|g\|_{\infty,\mathcal{M}} < +\infty$ (with exponent $p = +\infty$). This allows us to extend the notion of the Carleman classes for exponents $0 < p < 1$ as follows.

Definition 1.3 $(0 < p < 1)$ The L^p-Carleman class $\mathscr{C}^{p,\theta}_{\mathcal{M}}$ consists of all the dilates $f_a(t) := f(at)$ (with $a > 0$) of functions $f \in W^{p,\theta}_{\mathcal{M}}$.

Here, to avoid unnecessary abstraction, we need to understand that each element in $W^{p,\theta}_{\mathcal{M}}$ gives rise to an element of the Cartesian product space $L^p \times L^p \times \cdots$ via the lift of the map $f \mapsto (f, f', f'', \ldots)$ initially defined on test functions $f \in \mathscr{S}^{\circledast}_{p,\mathcal{M}}$ (for more details, see the next subsection). Moreover, it is easy to see that $f \in W^{p,\theta}_{\mathcal{M}}$ is uniquely determined by the corresponding element in $L^p \times L^p \times \cdots$. It is important to note that in the Cartesian product space, the dilatation operation is well-defined, so that the L^p-Carleman class $\mathscr{C}^{p,\theta}_{\mathcal{M}}$ can be understood as a submanifold of $L^p \times L^p \times \cdots$ in the above sense. Moreover, $\mathscr{C}^{p,\theta}_{\mathcal{M}}$ is actually a linear subspace, as the quasinorm criterion for being in the class is (analogously)

$$\frac{\|f^{(k)}\|_p}{M_k} \le CA^k,$$

for some positive constants C and A, and this kind of bound is closed under linear combination.

1.5 Classes of weight sequences

From this point onward, we will restrict attention to positive, logarithmically convex weight sequences, i.e. sequences $\mathcal{M} = \{M_j\}_j$ of positive numbers such that the function $j \mapsto \log M_j$ is convex.

We will consider the following notions of regularity for weight sequences.

Definition 1.4 A logarithmically convex sequence $\mathcal{M} = \{M_n\}_{n=0}^{+\infty}$ with infinite p-characteristic $\kappa(p, \mathcal{M}) = +\infty$ is called *p-regular* if one of the following conditions (i)–(ii) holds:

- (i) $\liminf_{n\to+\infty}(1-p)^n \log M_n > 0$, or
- (ii) $\log M_n = \mathrm{o}((1-p)^{-n})$ and $n \log n \le (\delta + \mathrm{o}(1)) \log M_n$ as $n \to +\infty$, for some δ with $0 < \delta < 1$.

In view of the Denjoy–Carleman theorem, either of the conditions (i) or (ii) implies that the standard Carleman class with the weight sequence $\mathcal{M}$ is non-quasianalytic. Using this observation, it can be shown that the class of test functions $\mathscr{S}^{\circledast}_{p,\theta,\mathcal{M}}$ contains nontrivial compactly supported functions.

It is a simple observation that p-regular sequences are stable under the process of shifts and under replacing M_n for a finite number of indices, as long as log-convexity is kept.

Remark 1.3 (a) We note in the context of Definition 1.4 that if $\mathcal{M}$ grows so fast that $\kappa(p, \mathcal{M}) = +\infty$ holds, then in particular the asymptotic estimate $\log M_n = \mathrm{O}(q^{-n})$ fails as $n \to +\infty$ for any given q with $1 - p < q < 1$. The second condition in (ii), which says that $n \log n \le (\delta + \mathrm{o}(1)) \log M_n$, is a rather mild lower bound on $\log M_n$ compared with this exponential growth along subsequences.

(b) For logarithmically convex sequences $\mathcal{M}$, the sum

$$\kappa(p, \mathcal{M}) = \sum_{j=0}^{+\infty} (1-p)^j \log M_j \tag{1.14}$$

is actually convergent to an extended real number in $\mathbb{R} \cup \{+\infty\}$.

We conclude with a notion of regularity that applies in the regime with finite p-characteristic.

Definition 1.5 A logarithmically convex weight sequence $\mathcal{M}$ for which $\kappa(p, \mathcal{M}) < +\infty$ is said to be *decay-regular* if $M_n/M_{n+1} \geq \epsilon^n$ holds for some positive real ϵ, or alternatively, if $\mathcal{M}$ meets the nonquasianalyticity condition

$$\sum_{n=1}^{+\infty} \frac{M_n}{M_{n+1}} < +\infty.$$

1.6 The three phases

We mentioned already the phenomenon that the Carleman classes exhibit the phase transition associated with quasianalyticity. Here, the concept of quasianalyticity is usually defined in terms of the unique continuation property that the (formal) Taylor series at any given point determines the function uniquely. Under the regularity condition that the sequence $\mathcal{M} = \{M_n\}_n$ is logarithmically convex, it is known classically that the *Carleman class* $\mathscr{C}_{\mathcal{M}} = \mathscr{C}_{\mathcal{M}}^{\infty,0}$ *is quasianalytic if and only if*

$$\sum_{n=0}^{+\infty} \frac{M_n}{M_{n+1}} = +\infty.$$

In the small exponent range $0 < p < 1$ considered here, it turns out that we have actually *two phase transitions*:

(i) the Douady–Peetre disconnexion barrier, and
(ii) the quasianalyticity barrier.

Here, we shall attempt to explore both phenomena.

In the degenerate case when $M_n = 1$ for $n = 0, \ldots, k$ and $M_n = +\infty$ for $n > k$, the Carleman space $W_{\mathcal{M}}^{p,\theta}$ does not depend on the parameter $\theta \geq 0$, and is the same as the Sobolev space $W^{p,k}$, except that it is equipped with another (but equivalent) quasinorm. So in this instance, we get the Douady–Peetre disconnexion phenomenon (1.2) for $W_{\mathcal{M}}^{p,\theta}$, while under the bounded p-characteristic condition (1.10), the following result shows that it does not happen.

To properly formulate the result, we consider the canonical mapping $\boldsymbol{\pi}: W_{\mathcal{M}}^{p,\theta} \to L^p \times L^p \times \cdots$, defined initially on the test function space $\mathscr{S}_{p,\theta,\mathcal{M}}^{\circledast}$ by

$$\boldsymbol{\pi} f = (f, f', f'', \ldots).$$

It is natural to replace here the product space $L^p \times L^p \times \cdots$ by its linear subspace $\ell^\infty(L^p, \mathcal{M})$ supplied with the standard quasinorm:

$$\|(f_0, f_1, f_2, \ldots)\|_{\ell^\infty(L^p,\mathcal{M})} := \sup_{n\geq 0} \frac{\|f_n\|_p}{M_n}. \tag{1.15}$$

Indeed, the linear mapping $\boldsymbol{\pi}: W_{\mathcal{M}}^{p,\theta} \to \ell^\infty(L^p, \mathcal{M})$ then becomes an isometry. This is obvious for test functions in $\mathscr{S}_{p,\theta,\mathcal{M}}^{\circledast}$, and, then automatically holds for elements of the abstract completion as well. We denote the n-th component projection of $\boldsymbol{\pi}$ by $\boldsymbol{\pi}_n$: $\boldsymbol{\pi}_n(f_0, f_1, f_2, \dots) := f_n$.

Theorem 1.2 $(0 < p < 1)$ *Suppose that the weight sequence $\mathcal{M}$ is logarithmically convex and meets the finite p-characteristic condition (1.10). Then, for each $n = 0, 1, 2 \dots$, $\boldsymbol{\pi}_n$ maps $W^{p,\theta}_{\mathcal{M}}$ into $C^\infty(\mathbb{R})$, and the space $W^{p,\theta}_{\mathcal{M}}$ is coupled, in the sense that*

$$\partial \boldsymbol{\pi}_n f = \boldsymbol{\pi}_{n+1} f, \quad f \in W^{p,\theta}_{\mathcal{M}},$$

where ∂ stands for the differentiation operation. Moreover, the projection $\boldsymbol{\pi}_0$ is injective, and, in the natural sense, the space equals the collection of test functions: $W^{p,\theta}_{\mathcal{M}} = \mathscr{S}^{\circledast}_{p,\theta,\mathcal{M}}$.

The proof of this theorem is supplied in Sect. 4.2.

The conclusion of Theorem 1.2 is that under the strong finite p-characteristic condition, $W^{p,\theta}_{\mathcal{M}}$ is a space of smooth functions, and, indeed, it is identical with the test function class $\mathscr{S}^{\circledast}_{p,\theta,\mathcal{M}}$. The situation is drastically different when $\kappa(p, \mathcal{M}) = +\infty$. For reasons of convenience, in this regime we work with the smaller space of compactly supported test functions $\mathscr{S}^{\circledast}_{p,\theta,\mathcal{M}} \cap C^\infty_0$ and its closure $W^{p,\theta}_{\mathcal{M},0}$ in the space $W^{p,\theta}_{\mathcal{M}}$. It remains a possibility that the spaces $W^{p,\theta}_{\mathcal{M},0}$ and $W^{p,\theta}_{\mathcal{M}}$ actually coincide.

Theorem 1.3 $(0 < p < 1)$ *Suppose $\mathcal{M}$ is a p-regular sequence with infinite p-characteristic $\kappa(p, \mathcal{M}) = +\infty$. Then $\boldsymbol{\pi}_0 \colon W^{p,\theta}_{\mathcal{M}} \to L^p$ is such that already its restriction $\boldsymbol{\pi}_0 \colon W^{p,\theta}_{\mathcal{M},0} \to L^p$ is surjective onto L^p, and, moreover, $\boldsymbol{\pi}$ supplies an isomorphism*

$$W^{p,\theta}_{\mathcal{M},0} \cong L^p(\mathbb{R}) \oplus W^{p,\theta_1}_{\mathcal{M}_1,0},$$

where $\mathcal{M}_1$ denotes the shifted sequence $\mathcal{M}_1 := \{M_{n+1}\}_n$ and $\theta_1 = \theta/(1-p)$. In addition, $\mathcal{M}_1$ inherits the assumed properties of $\mathcal{M}$.

A sketch of the proof of Theorem 1.3 is supplied in Sect. 5.3. As for the formulation, the summand $W^{p,\theta_1}_{\mathcal{M}_1,0}$ on the right-hand side arises as the space of "derivatives" of functions in $W^{p,\theta}_{\mathcal{M},0}$. Here, the reason why $\mathcal{M}$ gets replaced by $\mathcal{M}_1$ is due to a one-unit shift in the sequence space $\ell^\infty(L^p, \mathcal{M})$. Moreover, the reason why θ gets replaced by $\theta_1 = \theta/(1-p)$ is the corresponding shift in the space of test functions $\mathscr{S}_{p,\theta,\mathcal{M}}$ when we take the derivative.

Remark 1.4 Since $\mathcal{M}_1$ inherits all relevant properties from $\mathcal{M}$, the theorem may be applied iteratively to obtain an isomorphism

$$W^{p,\theta}_{\mathcal{M},0} \cong L^p \oplus L^p \oplus \cdots \oplus L^p \oplus W^{p,\theta_k}_{\mathcal{M}_k,0}, \quad k \in \mathbb{N},$$

where $\theta_k = \theta(1-p)^{-k}$ and $\mathcal{M}_k$ denotes the k-shifted sequence $\{M_{k+n}\}_n$. Note that in principle, the space $W^{p,\theta}_{\mathcal{M}}$ may be even bigger than $W^{p,\theta}_{\mathcal{M},0}$.

We briefly comment on the remaining transition, between non-quasianalyticity and quasianalyticity. Here, for a general linear space of smooth functions, if for each point in the domain of definition, the Borel map is injective, we say that the space is *quasianalytic*. We recall that the Borel map for the given point a is $f \mapsto \{f(a), f'(a), f''(a), \ldots\}$. If the given linear space is not quasianalytic we call it *non-quasianalytic*. For the classical Carleman classes the structure is well understood, but we need a clearcut definition in the setting of the new L^p-Carleman classes $\mathscr{C}_{\mathcal{M}}^{p,\theta}$. Now, for $\theta = 0$, we characterize quasianalyticity for $\mathscr{C}_{\mathcal{M}}^{p,\theta} = \mathscr{C}_{\mathcal{M}}^{p,0}$ in terms of the quasianalyticity of the standard Carleman class $\mathscr{C}_{\mathcal{N}}$ for a certain related sequence $\mathcal{N}$. Moreover, the classical Denjoy–Carleman theorem supplies criteria for when the class $\mathscr{C}_{\mathcal{N}}$ is quasianalytic or non-quasianalytic. The associated sequence $\mathcal{N} = \{N_n\}_n$ is given by

$$N_n := \prod_{j=1}^{\infty} M_{n+j}^{(1-p)^{j-1}p}, \quad k = 0, 1, 2, \ldots,$$

which we recognize as coming from the L^∞-bound of the higher order derivatives in (1.13).

The result runs as follows.

Theorem 1.4 $(0 < p < 1)$ *Assume that* $\mathcal{M}$ *is logarithmically convex and bounded away from zero. If* $\kappa(p, \mathcal{M}) < +\infty$, *the following holds*:

- (i) *If* $\theta > 0$, *then* $\mathscr{C}_{\mathcal{M}}^{p,\theta}$ *is never quasianalytic.*
- (ii) *If* $\theta = 0$ *and* $\mathcal{M}$ *is decay-regular in the sense of Definition* 1.5, *then* $\mathscr{C}_{\mathcal{M}}^{p,0}$ *is quasianalytic if and only if* $\mathscr{C}_{\mathcal{N}}$ *is quasianalytic.*

Finally, we comment on the dependence of the classes on the parameter θ in the smooth context of Theorem 1.2.

Theorem 1.5 $(0 < p < 1)$ *Let* $\mathcal{M}$ *be an increasing, log-convex sequence such that* $\kappa(p, \mathcal{M}) < +\infty$. *Let* $0 \le \theta < \theta'$. *Then the inclusion* $W_{\mathcal{M}}^{p,\theta} \subset W_{\mathcal{M}}^{p,\theta'}$ *is strict*: $W_{\mathcal{M}}^{p,\theta} \neq W_{\mathcal{M}}^{p,\theta'}$.

We do not know whether such a strict inclusion holds in the uncoupled regime when $\kappa(p, \mathcal{M}) = +\infty$. It remains a possibility that the spaces are then so large that the parameter θ is not felt. In any case, we are able to show that (Proposition 5.3)

$$c_0(L^p, \mathcal{M}) \subset \boldsymbol{\pi} W_{\mathcal{M},0}^{p,\theta} \subset \boldsymbol{\pi} W_{\mathcal{M}}^{p,\theta} \subset \ell^\infty(L^p, \mathcal{M}). \tag{1.16}$$

Here, $c_0(L^p, \mathcal{M})$ denotes the subspace of $\ell^\infty(L^p, \mathcal{M})$ consisting of sequences $(f_0, f_1, f_2, \ldots)$ with

$$\lim_{n \to +\infty} \frac{\|f_n\|_p}{M_n} = 0.$$

Remark 1.5 (a) In view of the Douady–Peetre disconnexion phenomenon, the fact that for $0 < p < 1$, L^p functions fail to define distributions is a serious obstruction. An alternative approach is to consider the real Hardy spaces H^p in place of L^p, since H^p functions automatically define distributions. The drawback of that approach is that for $p = 1$, H^1 is substantially smaller than L^1. Our theme here is to keep L^p and to let a bootstrap argument (involving infinitely many higher order derivatives) take care of the smoothness, and to explore what happens when the bootstrap argument fails to supply appropriate bounds.

(b) A word on the title. The term *a critical topology* used here is borrowed from Beurling's work [5], where another phase transition is the object of study.

2 Sobolev spaces: Peetre's proof and failure of embedding

2.1 Sobolev spaces for $0 < p < 1$

We fix a number p with $0 < p < 1$ and an integer $k \geq 0$. Following Peetre [18] we consider the Sobolev space $W^{k,p} = W^{k,p}(\mathbb{R})$, defined as the abstract completion of $C_0^k(\mathbb{R})$ with respect to the quasinorm

$$\|f\|_{k,p} = \left(\|f\|_p^p + \|f'\|_p^p + \cdots + \|f^{(k)}\|_p^p\right)^{1/p}. \tag{2.1}$$

The resulting space $W^{k,p}$ is then a quasi-Banach space. Here, $C_0^k(\mathbb{R})$ denotes the space of compactly supported functions in $C^k(\mathbb{R})$.

Remark 2.1 In this paper we shall mostly work on the entire line. If at some place we consider spaces on bounded intervals, this will be explicitly mentioned. The definition of $W^{k,p}(I)$ for a general interval I is entirely analogous.

The space $W^{k,p}$ comes with two canonical mappings, $\boldsymbol{\alpha} = \boldsymbol{\alpha}_k : W^{k,p} \to L^p$, and $\boldsymbol{\delta} = \boldsymbol{\delta}_k : W^{k,p} \to W^{k-1,p}$. These are both initially defined for test functions $f \in C_0^k(\mathbb{R})$ by

$$\boldsymbol{\alpha} f = f, \quad \text{and} \quad \boldsymbol{\delta} f = f'.$$

The mappings $\boldsymbol{\alpha}$ and $\boldsymbol{\delta}$ are bounded and densely defined, and hence extend to bounded operators on the entire space $W^{k,p}$.

2.2 Douady–Peetre disconnexion

We begin this section by considering a simple example which explains how a crucial feature differs in the setting of $0 < p < 1$ as compared to the classical Sobolev space case.

We are used to thinking of $W^{k,p}$ as being a certain subspace of L^p, consisting of functions that are sufficiently smooth. As mentioned in the introduction, the first observation, made by Douady, is that this is not the right way to think when $0 < p < 1$. Indeed, the canonical map $\boldsymbol{\alpha}$ is not injective.

Proposition 2.1 (Douady) *There exists* $f \in W^{1,p}([0,1])$ *such that*

$$\boldsymbol{\alpha} f = 0, \quad \boldsymbol{\delta} f = 1.$$

This would suggest that there ought to exist functions that vanish identically but nevertheless the derivative equals the nonzero constant 1. This is of course absurd, and the right way to think about it is to realize that in the completion, the function and its derivative lose contact, they disconnect.

Proof A small argument (see [18, Lemma 2.1]) shows that we are allowed to work with functions whose derivatives have jumps. We let $\{\epsilon_j\}_j$ be a sequence of positive reals, such that $j\epsilon_j \to 0$ as $j \to +\infty$, and define f_j on the interval $[0, \frac{1}{j} + \epsilon_j]$ by

$$f_j(x) = \begin{cases} x, & 0 \le x \le \frac{1}{j}, \\ (\frac{1}{j} + \epsilon_j - x)/(j\epsilon_j), & \frac{1}{j} < x \le \frac{1}{j} + \epsilon_j, \end{cases}$$

and extend it *periodically* to $\mathbb{R}$ with period $\frac{1}{j} + \epsilon$. The resulting function f_j will be a skewed saw-tooth function that rises slowly with slope 1 and then drops steeply. By differentiating f_j, we have that

$$1 - f_j'(x) = \begin{cases} 0, & 0 \le x < \frac{1}{j}, \\ 1 + \frac{1}{j\epsilon_j}, & \frac{1}{j} < x < \frac{1}{j} + \epsilon_j. \end{cases}$$

Since f_j assumes values between 0 and $\frac{1}{j}$, it is clear that $f_j \to 0$ as $j \to +\infty$ in L^∞ and hence in L^p. Within the interval $[0, 1]$, there are at most j full periods of the function f_j, which allows us to estimate

$$\begin{aligned} \int_{[0,1]} |1 - f_j'(x)|^p \mathrm{d}x &\le j\epsilon_j \left(1 + \frac{1}{j\epsilon_j}\right)^p \le j\epsilon_j (j\epsilon_j)^{-p} \\ &= (j\epsilon_j)^{1-p} \to 0 \quad \text{as } j \to +\infty. \end{aligned}$$

In view of the above observations, $f_j \to 0$ while $f_j' \to 1$, both in the quasinorm of L^p, as $j \to +\infty$. In particular, $\{f_j\}_j$ is a Cauchy sequence, and if we let f denote the abstract limit in the completion, we find that $\boldsymbol{\alpha} f = 0$ while $\boldsymbol{\delta} f = 1$. □

2.3 The isomorphism and construction of the canonical lifts

We fix an integer $k = 1, 2, 3, \ldots$ and an exponent $0 < p < 1$.

The space $L^p \oplus W^{k-1,p}$ consists of pairs (g, h), where $g \in L^p$ and $h \in W^{k-1,p}$, and we equip it with the quasinorm

$$\|(g, h)\|^p = \|g\|_p^p + \|h\|_{k-1,p}^p, \quad g \in L^p,\ h \in W^{k-1,p}.$$

From the definition of the norm (2.1), we see that the operator $\mathbf{A}\colon W^{k,p} \to L^p \oplus W^{k-1,p}$ given by $\mathbf{A}f := (\boldsymbol{\alpha} f, \boldsymbol{\delta} f)$ is an *isometry*. Indeed, for $f \in C_0^k(I)$, we have that

$$\|\mathbf{A}f\|^p = \|(\boldsymbol{\alpha} f, \boldsymbol{\delta} f)\|^p = \|\boldsymbol{\alpha} f\|_p^p + \|\boldsymbol{\delta} f\|_{k-1,p}^p = \|f\|_p^p + \|f'\|_{k-1,p}^p = \|f\|_{k,p}^p, \tag{2.2}$$

and this property survives the completion process. If $\mathbf{A}$ can be shown to be surjective, then it is an isometric isomorphism $\mathbf{A}\colon W^{k,p} \to L^p \oplus W^{k-1,p}$. Proceeding iteratively with $W^{k-1,p}$, we obtain the desired decomposition, since clearly $W^{0,p} = L^p$.

To obtain the surjectivity of $\mathbf{A}$, we shall construct two canonical lifts, $\boldsymbol{\beta}\colon L^p \to W^{k,p}$ and $\boldsymbol{\gamma}\colon W^{k-1,p} \to W^{k,p}$ of $\boldsymbol{\alpha}$ and $\boldsymbol{\delta}$, respectively. These are injective mappings, from L^p and $W^{k-1,p}$ to $W^{k,p}$, respectively, satisfying certain relations with $\boldsymbol{\alpha}$ and $\boldsymbol{\delta}$. The properties of these are summarized in the following lemma (the notation id_X stands for the identity mapping on the space X). The details of the construction are postponed until Sect. 3.2.

Lemma 2.1 *For each $k = 1, 2, 3, \ldots$, there exist bounded linear mappings $\boldsymbol{\beta}\colon L^p \to W^{k,p}$ and $\boldsymbol{\gamma}\colon W^{k-1,p} \to W^{k,p}$, such that*

$$\boldsymbol{\alpha}\boldsymbol{\beta} = \mathrm{id}_{L^p}, \quad \boldsymbol{\delta}\boldsymbol{\gamma} = \mathrm{id}_{W^{k-1,p}}, \quad \boldsymbol{\delta}\boldsymbol{\beta} = 0, \quad \boldsymbol{\alpha}\boldsymbol{\gamma} = 0.$$

With this result at hand, the proof of the main theorem about the $W^{k,p}$-spaces becomes a simple exercise.

Proof of Theorem 1.1 As noted above, it will be enough to show that the isometry $\mathbf{A}\colon W^{k,p} \to L^p \oplus W^{k-1,p}$ given by $\mathbf{A}f := (\boldsymbol{\alpha} f, \boldsymbol{\delta} f)$ is surjective. To this end, we pick $(g, h) \in L^p \oplus W^{k-1,p}$. Then $\boldsymbol{\beta} g$ and $\boldsymbol{\gamma} h$ are both elements of $W^{k,p}$, and so is their sum

$$f = \boldsymbol{\beta} g + \boldsymbol{\gamma} h \in W^{k,p}.$$

It now follows from Lemma 2.1 that

$$\mathbf{A}f = (\boldsymbol{\alpha}(\boldsymbol{\beta} g + \boldsymbol{\gamma} h), \boldsymbol{\delta}(\boldsymbol{\beta} g + \boldsymbol{\gamma} h)) = (\boldsymbol{\alpha}\boldsymbol{\beta} g + \boldsymbol{\alpha}\boldsymbol{\gamma} h, \boldsymbol{\delta}\boldsymbol{\beta} g + \boldsymbol{\delta}\boldsymbol{\gamma} h) = (g, h).$$

As a consequence, $\mathbf{A}$ is surjective, and hence $\mathbf{A}$ induces an isometric isomorphism

$$W^{k,p} \cong L^p \oplus W^{k-1,p}.$$

By iteration of the same argument, the claimed decomposition of $W^{k,p}$ follows. □

Remark 2.2 The lift $\boldsymbol{\gamma}$ does not appear in Peetre's work [18]. Reading between the lines one can discern its role, but here we fill in the blanks and treat it explicitly.

3 Construction of lifts, and invisible mollifiers

3.1 A collection of smooth functions by iterated convolution

For an integer $k \geq 0$ and a real $0 \leq \alpha \leq 1$ let $C^{k,\alpha}$ denote the class of k times continuously differentiable functions, whose derivative of order k is Hölder continuous with exponent α. Given two functions $f, g \in L^1(\mathbb{R})$, their convolution $f * g \in L^1(\mathbb{R})$ is as usual given by

$$(f * g)(x) = \int_{\mathbb{R}} f(x-t)g(t)\mathrm{d}t, \quad x \in \mathbb{R}.$$

For $a > 0$, we let the function H_a denote the normalized characteristic function $H_a = a^{-1}1_{[0,a]}$. For a decreasing sequence of positive reals $a_1, a_2, a_3, \ldots,$ consider the associated repeated convolutions (for $j \leq k$)

$$\Phi_{j,k} := H_{a_j} * \cdots * H_{a_k}. \tag{3.1}$$

The function $\Phi_{j,k}$ then has compact support $[0, a_j + \cdots a_k]$ and belongs to the smoothness class $C^{k-j-1,1}$ which means that the derivative of order $k-j-1$ is Lipschitz continuous. We will assume that the sequence $a_1, a_2, a_3, \ldots$ decreases to 0 at least fast enough for $(a_j)_{j\geq 1} \in \ell^1$ to hold. Then we may form the limits

$$\Phi_{j,\infty} := \lim_{k\to+\infty} \Phi_{j,k}, \quad j = 1, 2, 3, \ldots,$$

and see that each such limit $\Phi_{j,\infty}$ is C^∞-smooth with support $[0, a_j + a_{j+1} + \ldots]$. Moreover, we have the sup-norm controls

$$\|\Phi_{j,k}\|_\infty \leq \frac{1}{a_j}, \quad \|\Phi_{j,\infty}\|_\infty \leq \frac{1}{a_j}. \tag{3.2}$$

Next, since for $l < k$

$$\Phi_{j,k} = \Phi_{j,l} * \Phi_{l+1,k} \quad \text{and} \quad \Phi_{j,\infty} = \Phi_{j,l} * \Phi_{l+1,\infty},$$

we may calculate the higher order derivatives by the formula

$$\Phi^{(n)}_{j,k} = \Phi^{(n)}_{j,j+n-1} * \Phi_{j+n,k} \quad \text{and} \quad \Phi^{(n)}_{j,\infty} = \Phi^{(n)}_{j,j+n-1} * \Phi_{j+n,\infty},$$

interpreted when needed in the sense of distribution theory. Here, we should ask that $j + n \leq k + 1$ in the first formula. By calculation,

$$\Phi^{(n)}_{j,j+n-1} = \frac{1}{a_j \cdots a_{j+n-1}}(\delta_{a_j} - \delta_0) * \cdots * (\delta_{a_{j+n-1}} - \delta_0),$$

which when expanded out is the sum of delta masses at 2^n (generically distinct) points, each with mass $(a_j \cdots a_{j+n-1})^{-1}$. By the convolution norm inequality $\|f * g\|_\infty \le \|f\|_1 \|g\|_\infty$, where the L^1 norm may be extended to the finite Borel measures, we have that

$$\left\|\Phi_{j,k}^{(n)}\right\|_\infty = \left\|\Phi_{j,j+n-1}^{(n)}\right\|_1 \left\|\Phi_{j+n,k}\right\|_\infty \le \frac{2^n}{a_j \cdots a_{j+n}},$$

where we used the estimate (3.2). The analogous estimate holds for $k = \infty$ as well:

$$\left\|\Phi_{j,\infty}^{(n)}\right\|_\infty \le \frac{2^n}{a_j \cdots a_{j+n}}. \tag{3.3}$$

We need to estimate the L^p-norm of the function $\Phi_{j,\infty}^{(n)}$ as well. The standard norm estimate for convolutions is $\|f * g\|_q \le \|f\|_1 \|g\|_q$ which holds provided that $1 \le q \le +\infty$. For our small exponents $0 < p < 1$ this is no longer true. However, there is a substitute, provided f is a finite sum of point masses:

$$\|f * g\|_p^p \le \|f\|_{\ell^p}^p \|g\|_p^p, \quad \text{where } \|f\|_{\ell^p}^p = \sum_j |b_j|^p \text{ if } f = \sum_j b_j \delta_{x_j}, \tag{3.4}$$

for some finite collection of reals x_j. This follows immediately from the p-triangle inequality and the translation invariance of the L^p-norm. In our present context we see that

$$\begin{aligned} \left\|\Phi_{j,k}^{(n)}\right\|_p^p &= \left\|\Phi_{j,j+n-1}^{(n)} * \Phi_{j+n,k}\right\|_p^p \le \left\|\Phi_{j,j+n-1}^{(n)}\right\|_{\ell^p}^p \left\|\Phi_{j+n,k}\right\|_p^p \\ &\le \frac{2^n}{(a_j \cdots a_{j+n})^p} \sum_{l=j+n}^{k} a_l, \end{aligned} \tag{3.5}$$

where $n + j \le k + 1$. Correspondingly for $k = +\infty$ we have that

$$\begin{aligned} \left\|\Phi_{j,\infty}^{(n)}\right\|_p^p &= \left\|\Phi_{j,j+n-1}^{(n)} * \Phi_{j+n,\infty}\right\|_p^p \le \left\|\Phi_{j,j+n-1}^{(n)}\right\|_{\ell^p}^p \left\|\Phi_{j+n,\infty}\right\|_p^p \\ &\le \frac{2^n}{(a_j \cdots a_{j+n})^p} \sum_{l=j+n}^{+\infty} a_l. \end{aligned} \tag{3.6}$$

3.2 Existence of invisible mollifiers in $W^{k,p}$

We now employ the repeated convolution procedure of Sect. 3.1, to exhibit mollifiers with L^p-vanishing properties.

Lemma 3.1 (Invisibility lemma) *Let $k \geq 0$ be an integer. Then for any given $\epsilon > 0$, there exists a non-negative function $\Phi \in C^{k-1,1}(\mathbb{R})$ such that*

$$\int_{\mathbb{R}} \Phi \, \mathrm{d}x = 1, \quad \mathrm{supp}(\Phi) \subset [0, \epsilon], \quad \|\Phi\|_{k,p} < \epsilon.$$

Proof For positive decreasing numbers $\{a_j\}_{j=1}^{k+1}$ (to be determined), let $\Phi = \Phi_{1,k+1}$ be given by (3.1). Then clearly $\Phi \in C^{k-1,1}(\mathbb{R})$ with $\Phi \geq 0$ and $\int_{\mathbb{R}} \Phi \, \mathrm{d}x = 1$. Moreover, the support of Φ equals $[0, a_1 + \cdots + a_{k+1}]$. By the estimate (3.5), it follows that

$$\|\Phi^{(n)}\|_p^p \leq \frac{2^n}{(a_1 \cdots a_{n+1})^p} \sum_{l=n+1}^{k+1} a_l. \tag{3.7}$$

We need to show that the finite sequence $\{a_l\}_{l=1}^{k+1}$ may be chosen such that the sum of the right-hand side in (3.7) over $0 \leq n \leq k$ is bounded by ϵ^p, while at the same time $\sum_{l=1}^{k+1} a_l \leq \epsilon$. As a first step, *we assume that $a_{j+1} \leq \frac{1}{2} a_j$ for integers $n \geq 0$*, and observe that it then follows that $\sum_{l=j}^{k+1} a_l \leq 2a_j$. Consequently, we get that $\mathrm{supp}\, \Phi \subset [0, 2a_0]$ and

$$\|\Phi^{(n)}\|_p^p \leq 2^{n+1} \frac{a_{n+1}}{(a_1 \cdots a_{n+1})^p}, \quad n = 0, \ldots, k.$$

We put

$$a_1 := \min \left\{ \left(\frac{\epsilon^p}{2(k+1)} \right)^{1/(1-p)}, \frac{\epsilon}{2} \right\}$$

and successively declare that

$$a_l := \min \left\{ \left(\frac{\epsilon^p (a_1 \cdots a_{l-1})^p}{2^l (k+1)} \right)^{1/(1-p)}, \frac{a_{l-1}}{2} \right\}, \quad l = 2, \ldots k+1.$$

It then follows that

$$\|\Phi^{(n)}\|_p^p \leq 2^{n+1} \frac{a_{n+1}}{a_1^p \cdots a_{n+1}^p} = a_{n+1}^{1-p} \frac{2^{n+1}}{(a_0 \cdots a_n)^p} \leq \frac{\epsilon^p}{k+1}, \quad n = 0, \ldots, k.$$

whence $\|\Phi\|_{p,k} \leq \epsilon$ and since also $\mathrm{supp}\, \Phi \subset [0, 2a_0] \subset [0, \epsilon]$, the constructed function Φ meets all the specifications. □

3.3 The definition of the lift $\boldsymbol{\beta}$ for $0 < p < 1$

The lift $\boldsymbol{\beta}$ maps boundedly $L^p \to W^{k,p}$, and we need to explain how it gets to be defined. Let $\mathscr{F}$ denote the collection of *step functions*, which we take to be the finite linear combination of characteristic functions of bounded intervals, and when also equip it with the quasinorm of L^p, we denote it by $\mathscr{F}_p := (\mathscr{F}, \|\cdot\|_p)$. We note that $\mathscr{F}_p$ is quasinorm dense in L^p. We first define $\boldsymbol{\beta} g$ for $g \in \mathscr{F}_p$. For $g \in \mathscr{F}_p$, we will write down a $W^{k,p}$-Cauchy sequence $\{g_j\}_j$ of test functions $g_j \in C_0^k(\mathbb{R})$, and declare $\boldsymbol{\beta} g \in W^{k,p}$ to be the abstract limit of the Cauchy sequence g_j as $j \to +\infty$.

We will require the following properties of the test functions g_j:

$$\lim_{j\to+\infty} \|g - g_j\|_p = 0 \quad \text{and} \quad \lim_{j\to+\infty} \|g_j'\|_{k-1,p} = 0. \tag{3.8}$$

If (3.8) can be achieved, then $\boldsymbol{\beta}\colon \mathscr{F}_p \to W^{k,p}$ becomes an isometry by the following calculation:

$$\|\boldsymbol{\beta} g\|_{k,p}^p = \lim_{j\to+\infty} \|g_j\|_{k,p}^p = \lim_{j\to+\infty} \left(\|g_j\|_p^p + \|g_j'\|_{k-1,p}^p\right) = \|g\|_p^p, \quad g \in \mathscr{F}.$$

These properties uniquely determine $\boldsymbol{\beta} g$ *for* $g \in \mathscr{F}$. Indeed, if $\tilde{g}_j$ were another Cauchy sequence satisfying (3.3), then $\{\tilde{g}_j\}_j$ and $\{g_j\}_j$ are equivalent as Cauchy sequences, in light of

$$\begin{aligned}\|\tilde{g}_j - g_j\|_{k,p}^p &= \|\tilde{g}_j - g_j\|_p^p + \|\tilde{g}_j' - g_j'\|_{k-1,p}^p \\ &\le \|\tilde{g}_j - g_j\|_p^p + \|\tilde{g}_j'\|_{k-1,p}^p + \|g_j'\|_{k-1,p}^p.\end{aligned}$$

In particular, $\boldsymbol{\beta}\colon \mathscr{F}_p \to W^{k,p}$ is then a well-defined bounded operator, and since $\mathscr{F}_p$ is dense in L^p it extends uniquely to a bounded operator $\boldsymbol{\beta}\colon L^p \to W^{k,p}$ which is actually an isometry.

In view of the above, it will be enough to define $\boldsymbol{\beta} g$ when g is the characteristic function of an interval $g = 1_{[a,b]}$ and to check (3.8) for it, since general step functions in $\mathscr{F}$ are obtained using finite linear combinations.

Let $\{\epsilon_j\}_{j=0}^{+\infty}$ be a sequence of numbers tending to zero with $0 < \epsilon_j < \frac{1}{2}(b-a)$. By Lemma 3.1 applied to $W^{k-1,p}$, there exists non-negative functions $\Phi_{\epsilon_j} \in C^{k-1}$, such that

$$\int_{\mathbb{R}} \Phi_{\epsilon_j} \mathrm{d}x = 1, \quad \operatorname{supp} \Phi_{\epsilon_j} \subset [0, \epsilon_j], \quad \left\|\Phi_{\epsilon_j}\right\|_{k-1,p} < \epsilon_j.$$

We define g_j by convolution: $g_j := \Phi_{\epsilon_j} * 1_{[a,b]}$. It is then clear that $g_j - g$ has support on $[a, a+\epsilon_j] \cup [b, b+\epsilon_j]$, and there, it is bounded by 1 in modulus. As a consequence,

$$\|g_j - g\|_p^p \le 2\epsilon_j,$$

so $g_j \to g$ in L^p. Next, we consider the derivative g_j', which we may express as $g_j' = (\tau_a - \tau_b)\Phi_{\epsilon_j}$, where we recall that τ with subscript is a translation operator. It is clear that

$$\|g_j'\|_{k-1,p}^p \le 2\|\Phi_{\epsilon_j}\|_{k-1,p}^p \le 2\epsilon_j.$$

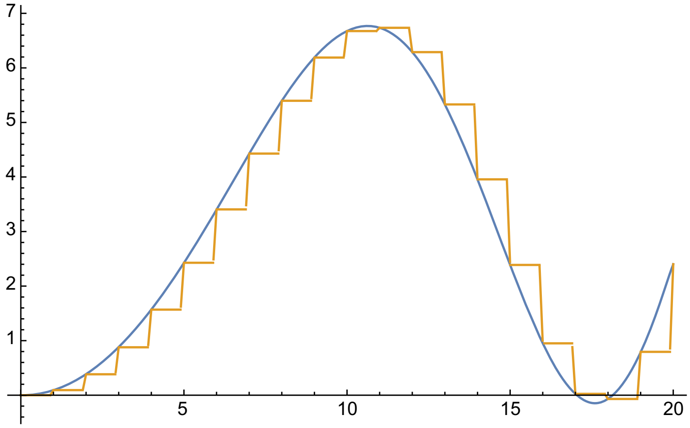

Fig. 1 The construction of $\boldsymbol{\beta}$ for $W^{1,p}$

Consequently, $\|g_j'\|_{k-1,p}^p$ also tends to zero, as needed. This establishes (3.8) (Fig. 1).

Proof of Lemma 2.1, part 1 We show that $\boldsymbol{\alpha}\boldsymbol{\beta} = \mathrm{id}_{L^p}$ and $\boldsymbol{\delta}\boldsymbol{\beta} = 0$. Since $\boldsymbol{\beta} g \in W^{k,p}$ is the abstract limit of the Cauchy sequence g_j with (3.8), and by definition $\boldsymbol{\alpha} g_j = g_j$ and $\boldsymbol{\delta} g_j = g_j'$, it follows from (3.8) that $\boldsymbol{\alpha}\boldsymbol{\beta} g = g$ and $\boldsymbol{\delta}\boldsymbol{\beta} g = 0$ for every $g \in L^p$. The assertion follows. □

3.4 The lift $\boldsymbol{\gamma}$

To construct $\boldsymbol{\gamma}$, we let $g \in W^{k-1,p}$ be an arbitrary element, which is by definition the abstract limit of some Cauchy sequence $\{g_j\}_j$, where $g_j \in C_0^{k-1}(\mathbb{R})$. For any given $\epsilon > 0$, Lemma 3.1 provides a function $\Phi_\epsilon \in C_0^{k-1}(\mathbb{R})$ with $\Phi_\epsilon \geq 0$, $\langle \Phi_\epsilon \rangle_{\mathbb{R}} := \int_{\mathbb{R}} \Phi_\epsilon(t)\mathrm{d}t = 1$, supported in $[0, \epsilon]$, while at the same time, $\|\Phi_\epsilon\|_{k-1,p} < \epsilon$. We use the functions Φ_{ϵ_j} to modify each $g_j(x)$ to have vanishing zeroth moment, by defining

$$\tilde{g}_j(x) := g_j(x) - \langle g_j \rangle_{\mathbb{R}} \Phi_{\epsilon_j}(x), \quad \langle g_j \rangle_{\mathbb{R}} := \int_{\mathbb{R}} g_j(t)\mathrm{d}t,$$

where the ϵ_j are chosen to tend to zero so quickly that

$$\lim_{j\to+\infty} \|g_j - \tilde{g}_j\|_{k-1,p} = \lim_{j\to+\infty} \|\Phi_{\epsilon_j}\|_{k-1,p} |\langle g_j \rangle_{\mathbb{R}}| = 0.$$

Next, we define the functions u_j as primitives:

$$u_j(x) = \int_{-\infty}^{x} \tilde{g}_j(t)\mathrm{d}t, \quad x \in \mathbb{R}.$$

Then as $\tilde{g}_j$ has integral 0, we see that $u_j \in C_0^k(\mathbb{R})$. We put $f_j := u_j - \boldsymbol{\beta} u_j \in W^{k,p}$, and observe that from the known properties of $\boldsymbol{\beta}$, it follows that

$$\boldsymbol{\alpha} f_j = \boldsymbol{\alpha} u_j - \boldsymbol{\alpha}\boldsymbol{\beta} u_j = u_j - u_j = 0 \quad \text{and} \quad \boldsymbol{\delta} f_j = \boldsymbol{\delta} u_j - \boldsymbol{\delta}\boldsymbol{\beta} u_j = \boldsymbol{\delta} u_j = u_j' = \tilde{g}_j. \tag{3.9}$$

Then, from the isometry of $\mathbf{A}\colon W^{k,p} \to L^p \oplus W^{k-1,p}$ (see (2.2)), we have that

$$\begin{aligned} \|f_j\|_{k,p}^p &= \|\boldsymbol{\alpha} f_j\|_p^p + \|\boldsymbol{\delta} f_j\|_{k-1,p}^p \\ &= \|\tilde{g}_j\|_{k-1,p}^p \le \|g_j\|_{k-1,p}^p + \|\Phi_{\epsilon_j}\|_{k-1,p}^p |\langle g_j\rangle_{\mathbb{R}}|^p, \end{aligned} \tag{3.10}$$

where in the last step, we applied the p-triangle inequality. A similar verification shows that $\{f_j\}_j$ is a Cauchy sequence, so that it has a limit $\boldsymbol{\gamma} g := \lim_{j\to+\infty} f_j$ in $W^{k,p}$. Moreover, in view of (3.9), it follows that

$$\boldsymbol{\alpha}\boldsymbol{\gamma} g = \lim_{j\to+\infty} \boldsymbol{\alpha} f_j = 0 \quad \text{and} \quad \boldsymbol{\delta}\boldsymbol{\gamma} g = \lim_{j\to+\infty} \boldsymbol{\delta} f_j = \lim_{j\to+\infty} \tilde{g}_j = g, \tag{3.11}$$

in L^p and $W^{k-1,p}$, respectively. In the construction of the sequence of functions f_j, there is some arbitrariness e.g. in the choice of the sequence of the ϵ_j (they were just asked to tend to 0 sufficiently quickly). To investigate whether this matters, we suppose another Cauchy sequence $\{F_j\}_j$ in $W^{k,p}$ is given, with properties that mimic (3.9): that $\boldsymbol{\alpha} F_j = 0$ in L^p, and that for some Cauchy sequence $\{G_j\}_j$ in $W^{k-1,p}$ converging to $g \in W^{k-1,p}$, we know that $\boldsymbol{\delta} F_j = G_j$, then

$$\begin{aligned} \|F_j - f_j\|_{k,p}^p &= \|\mathbf{A}(F_j - f_j)\|_{L^p\oplus W^{k-1,p}}^p = \|\boldsymbol{\alpha}(F_j - f_j)\|_p^p + \|\boldsymbol{\delta}(F_j - f_j)\|_{k-1,p}^p \\ &= \|G_j - \tilde{g}_j\|_{k-1,p}^p \le \|G_j - g_j\|_{k-1,p}^p + \|\Phi_{\epsilon_j}\|_{k-1,p}^p |\langle g_j\rangle_{\mathbb{R}}|^p \to 0, \\ &\quad \text{as } j \to +\infty, \end{aligned}$$

by the isometric properties of $\mathbf{A}$. If we let F denote the abstract limit of the Cauchy sequence F_j in $W^{k,p}$, we conclude that $F = f$ in $W^{k,p}$. In conclusion, it did not matter whether we used the prescribed Cauchy sequence or a competitor, and hence $\boldsymbol{\gamma} g \in W^{k,p}$ is well-defined for $g \in W^{k-1,p}$. Finally, we observe from (3.10) that

$$\|\boldsymbol{\gamma} g\|_{k,p} \le \|g\|_{k-1,p}, \quad g \in W^{k-1,p},$$

which makes $\boldsymbol{\gamma}\colon W^{k-1,p} \to W^{k,p}$ a linear contraction (Fig. 2).

Proof of Lemma 2.1, part 2 We show that the remaining properties, those involving $\boldsymbol{\gamma}$: $\boldsymbol{\alpha}\boldsymbol{\gamma} = 0$ and $\boldsymbol{\delta}\boldsymbol{\gamma} = \mathrm{id}_{W^{k-1,p}}$. We read off from (3.9) that $\boldsymbol{\alpha}\boldsymbol{\gamma} g = 0$ and $\boldsymbol{\delta}\boldsymbol{\gamma} g = g$ for $g \in W^{k-1,p}$, which does it. □

Remark 3.1 A classical theorem by Day [11] states that the L^p-spaces for $0 < p < 1$ have trivial dual. It follows from Douady–Peetre's isomorphism $W^{k,p} \cong L^p \oplus \cdots \oplus L^p$ that the same is true for the Sobolev spaces $W^{k,p}$. We note here that any space that

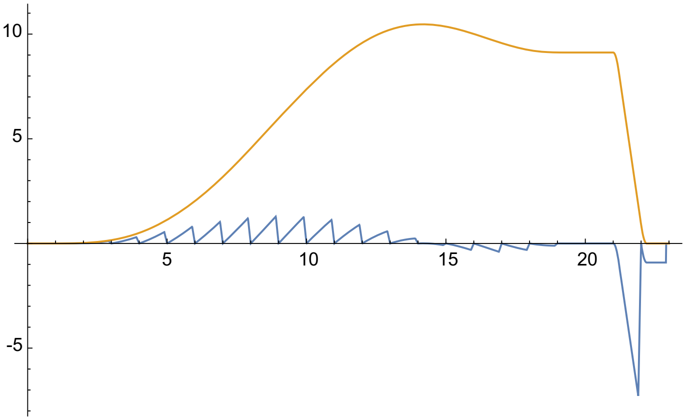

Fig. 2 The functions f_j and u_j, for $W^{1,p}$

could be realised as a space of distributions would necessarily admit nontrivial bounded functionals (the test functions, for instance). Hence it follows that the elements of $W^{k,p}$ cannot even be interpreted as distributions.

4 The smooth regime

4.1 Classes of test functions

Although we mainly focus on the classes $\mathscr{S}^{\circledast}_{p,\theta,\mathcal{M}}$, we also mention the Hermite class $\mathscr{S}^{\mathrm{Her}}$ of weighted polynomials

$$\mathscr{S}^{\mathrm{Her}} = \left\{ f : f(x) = q(x)\mathrm{e}^{-x^2}, \text{ where } q \in \mathrm{Pol}(\mathbb{R}) \right\},$$

where $\mathrm{Pol}(\mathbb{R})$ denotes the linear space of all polynomials of a real variable. We show that $\mathscr{S}^{\mathrm{Her}}$ consists of $(p,0)$-tame functions. To get a class which fits into the associated L^p-Carleman class, we also intersect the Hermite class with the space $\{f \in C^\infty(\mathbb{R}) : \|f\|_{p,\mathcal{M}} < +\infty\}$ to obtain the $\mathscr{S}^{\mathrm{Her}}_{p,\mathcal{M}}$.

Lemma 4.1 *The Hermite class* $\mathscr{S}^{\mathrm{Her}}$ *consists of* $(p,0)$*-tame functions.*

Proof Let $f \in \mathscr{S}^{\mathrm{Her}}$. Then $f(x) = q(x)\mathrm{e}^{-x^2}$ for some polynomial $q \in \mathrm{Pol}(\mathbb{R})$. Let d be the degree of q, and assume that all coefficients of the polynomial $q(x)$ are bounded by some number $m = m_q$. Let $\mathbf{L}$ be the operator on $\mathrm{Pol}(\mathbb{R})$ defined by the relation

$$\mathbf{L}q(x) = \mathrm{e}^{x^2}\frac{\mathrm{d}}{\mathrm{d}x}\left(q(x)\mathrm{e}^{-x^2}\right) = q'(x) - 2xq(x), \quad q \in \mathrm{Pol}(\mathbb{R}).$$

It follows that the Taylor coefficients $\widehat{\mathbf{L}q}(j)$ of $\mathbf{L}q$ can be estimated rather crudely in terms of d and m_q:

$$|\widehat{\mathbf{L}q}(j)| \le (d+2)m_q, \quad j = 0, 1, \dots, d+1, \tag{4.1}$$

and the coefficients vanish for $j > d+1$, so that the degree of $\mathbf{L}q$ is at most $d+1$. By repeating the same argument, with $\mathbf{L}q$ in place of q, we obtain

$$|\widehat{\mathbf{L}^2 q}(j)| \le (d+3)(d+2)m_q,$$

and, by iteration of (4.1), it follows more generally that for $j = 0, 1, 2, \dots,$

$$\begin{aligned} |\widehat{\mathbf{L}^n q}(j)| &\le (d+2+n-1)(d+n-2)\cdots(d+2)m_q \\ &= (d+2)_n m_q, \quad n = 1, 2, \dots, \end{aligned} \tag{4.2}$$

where we use the standard Pochhammer notation $(x)_k = x(x+1)\cdots(x+k-1)$ for $x \in \mathbb{R}$.

We proceed to estimate the L^∞-norm of $f^{(n)}$. By the way $\mathbf{L}$ was defined, we may estimate

$$\begin{aligned} \|f^{(n)}\|_\infty = \sup_{x\in\mathbb{R}} |\mathrm{e}^{-x^2}\mathbf{L}^n q(x)| &= \sup_{x\in\mathbb{R}} \left| \mathrm{e}^{-x^2} \sum_{j=0}^{d+n} \widehat{\mathbf{L}^n q}(j) x^j \right| \\ &\le \sum_{j=0}^{d+n} |\widehat{\mathbf{L}^n q}(j)| \sup_{x\in\mathbb{R}} |x|^j \mathrm{e}^{-x^2}. \end{aligned}$$

By trivial calculus, the supremum on the right-hand side is attained at $|x| = \sqrt{j/2}$, and this we may implement in the above estimate while we recall the coefficient estimate (4.2):

$$\begin{aligned} \|f^{(n)}\|_\infty &\le m_q (d+2)_n \sum_{j=0}^{d+n} \left(\frac{j}{2}\right)^{j/2} \mathrm{e}^{-j/2} \\ &\le m_q (d+2)_{n+1} \left(\frac{d+n}{2}\right)^{(d+n)/2} \mathrm{e}^{-(d+n)/2}, \end{aligned} \tag{4.3}$$

where in the last step we just estimated by the number of terms multiplied by the largest term. Next, we take logarithms and apply elementary estimates to arrive at

$$\begin{aligned} (1-p)^n \log \|f^{(n)}\|_\infty &\le (1-p)^n \\ &\times \left\{ \log m_q + (n+1)\log(d+n+2) + \frac{d+n}{2}\left(\log\frac{d+n}{2} - 1\right) \right\}. \end{aligned}$$

As the expression in brackets is of growth order $\mathrm{O}((n+d)\log(n+d))$, it follows that the right-hand side expression tends to 0 as $n \to +\infty$. It is then immediate that f is $(p,0)$-tame, and since f was an arbitrary element $f \in \mathscr{S}^{\mathrm{Her}}$, the claim follows. □

Next, we turn to the question of what is required of the weight sequence $\mathcal{M}$ in order for $\mathscr{S}^{\mathrm{Her}}_{p,\mathcal{M}}$ to contain nontrivial functions. We thus need to estimate the L^p-norms of $f^{(n)}$ for $f \in \mathscr{S}^{\mathrm{Her}}$. By performing the same estimates as in the above proof and by appealing to the p-triangle inequality (which says that $(a+b)^p \le a^p + b^p$ for $0 < p \le 1$ and positive a and b), we see that

$$\|f^{(n)}\|_p^p = \int_{\mathbb{R}} |f^{(n)}(x)|^p \mathrm{d}x \le m_q^p (d+2)_n^p \sum_{j=0}^{d+n} \int_{\mathbb{R}} |x|^{pj} \mathrm{e}^{-px^2} \mathrm{d}x.$$

The integrals on the right-hand side are easily evaluated:

$$\int_{\mathbb{R}} |x|^{pj} \mathrm{e}^{-px^2} \mathrm{d}x = p^{-(pj+1)/2} \Gamma((pj+1)/2),$$

so that the above gives the estimate

$$\begin{aligned}\|f^{(n)}\|_p^p &\le m_q^p (d+2)_n^p \sum_{j=0}^{d+n} p^{-(pj+1)/2} \Gamma((pj+1)/2) \\ &\le m_q^p (d+n+1)(d+2)_n^p p^{-(pd+pn+1)/2} \Gamma((pd+pn+1)/2).\end{aligned}$$

Next, using Stirling's formula yields the estimate that

$$\Gamma((pd+pn+1)/2) = \mathrm{O}\left((n!)^{p/2} n^{p(d-1)/2} \left(\frac{p}{2}\right)^{pn/2} \right),$$

whence

$$\begin{aligned}& m_q^p (d+n+1)(d+2)_n^p p^{-(pd+pn+1)/2} \Gamma((pd+pn+1)/2) \\ &= \mathrm{O}\left(\frac{n^{1+\frac{3p(d-1)}{2}+2p}}{2^{pn}} (n!)^{p+\frac{p}{2}} \right)\end{aligned}$$

Ignoring the specific constants, we find that

$$\|f^{(n)}\|_p^p = \mathrm{O}\left((n!)^{3p/2} \frac{n^\alpha}{\beta^n} \right),$$

where $\alpha > 0$ and $\beta > 1$ are some constants, and hence it follows that $\|f^{(n)}\|_p = \mathrm{O}\left((n!)^{3/2}\right)$. We do not proceed to analyse in full detail what $\mathcal{M} = \{M_n\}$ should have to fulfill in order for $\mathscr{S}^{\mathrm{Her}}_{p,M}$ to be a meaningful test class, but we note that the above implies that $\mathscr{S}^{\mathrm{Her}}_{p,M} = \mathscr{S}^{\mathrm{Her}}$ if e.g. $\mathcal{M} = \{M_n\}_n$ meets $M_n \geq (n!)^\sigma$ for any $\sigma \geq 3/2$, which we understand as a *Gevrey class* condition.

4.2 The smooth regime: Theorem 1.2

We have already presented the bootstrap argument, which is the main ingredient in the proof of Theorem 1.2, in the introduction. We proceed to fill in the remaining details.

Proof of Theorem 1.2 We consider the canonical mapping $\boldsymbol{\pi}: f \mapsto (f, f', f'', \ldots)$, defined initially from $\mathscr{S}^{\circledast}_{p,\theta,\mathcal{M}}$ into $\ell^\infty(L^p, \mathcal{M})$. The quasinorm on the sequence space $\ell^\infty(L^p, \mathcal{M})$ is supplied in Eq. (1.15). Then, as a matter of definition,

$$\|\boldsymbol{\pi} f\|_{\ell^\infty(L^p,\mathcal{M})} = \|f\|_{p,\mathcal{M}}, \quad f \in \mathscr{S}^{\circledast}_{p,\theta,\mathcal{M}}$$

and passing to the completion the mapping extends to an isometry $\boldsymbol{\pi}: W^{p,\theta}_{\mathcal{M}} \to \ell^\infty(L^p, \mathcal{M})$. In particular, the mapping $\boldsymbol{\pi}$ is injective. We recall that we think of an element $f \in W^{p,\theta}_{\mathcal{M}}$ as an abstract limit of a Cauchy sequence $\{f_j\}_j$ in the norm $\|\cdot\|_{p,\mathcal{M}}$, with $f_j \in \mathscr{S}^{\circledast}_{p,\theta,\mathcal{M}}$. For such an element, the mapping is obtained by taking the L^p-limit in all coordinates (that is, the sequence of higher order derivatives). This is well-defined, since if we were to take two Cauchy sequences $\{f_j\}$ and $\{\tilde{f}_j\}$ with the same abstract limit $f \in W^{p,\theta}_{\mathcal{M}}$, the images under the mapping would agree in each coordinate since

$$\left\| f_j^{(n)} - \tilde{f}_j^{(n)} \right\|_p \leq M_n \|f_j - \tilde{f}_j\|_{p,\mathcal{M}}.$$

Next, we show that the image of $W^{p,\theta}_{\mathcal{M}}$ under the mapping is actually inside $C^\infty \times C^\infty \times \cdots$. Indeed, since all the differences $f_j - f_k$ are in $\mathscr{S}^{\circledast}_{p,\theta,\mathcal{M}}$, it follows from (1.13) that

$$\left\| f_j^{(n)} - f_k^{(n)} \right\|_\infty \leq \left\{ \mathrm{e}^{\theta(1-p)^{-n} + p(1-p)^{-n-1}\kappa(p,\mathcal{M})} \prod_{l=1}^{n} M_l^{-(1-p)^{l-n-1}p} \right\} \|f_j - f_k\|_{p,\mathcal{M}},$$

where the right hand side tends to zero as $j, k \to +\infty$. It is now pretty obvious that under the finite p-characteristic condition $\kappa(p, \mathcal{M}) < +\infty$, each function $f_j^{(n)}$ has a limit $g_n \in C(\mathbb{R})$ as $j \to +\infty$. Moreover, as all the derivatives converge uniformly, we conclude that $g_n = g'_{n-1} = \cdots = g_0^{(n)}$ for each $n = 0, 1, 2, \ldots$, and hence that

$$\partial \boldsymbol{\pi}_n f = \boldsymbol{\pi}_{n+1} f, \quad n = 0, 1, 2, \ldots.$$

From this relation it follows that the first coordinate map $\boldsymbol{\pi}_0$ is injective. Indeed, if $\boldsymbol{\pi}_0 f = 0$, then by iteration $\boldsymbol{\pi}_n f = 0$ for $n = 0, 1, 2, \ldots$. Hence $\boldsymbol{\pi} f = 0$ and thus $f = 0$ by the injectivity of $\boldsymbol{\pi}$.

Finally, we show that the given test class is stable under the completion. We already saw that the limiting functions are of class C^∞, and naturally $\|f\|_{p,\mathcal{M}} < +\infty$ for each $f \in W^{p,\theta}_{\mathcal{M}}$. What remains is to show that

$$\limsup_{n\to+\infty}(1-p)^n \log \|f^{(n)}\|_\infty \le \theta.$$

However, the bound (1.13) tells us that

$$\|f^{(n)}\|_\infty \le \|f\|_{p,\mathcal{M}}\, \mathrm{e}^{\theta(1-p)^{-n}+p(1-p)^{-n-1}\kappa(p,\mathcal{M})} \prod_{j=1}^{n} M_j^{-(1-p)^{j-n-1}p},$$

so that

$$(1-p)^n \log \|f^{(n)}\|_\infty \le \theta + \frac{p}{(1-p)}\left\{\kappa(p,\mathcal{M}) - \sum_{j=1}^{n}(1-p)^j \log M_j\right\} + (1-p)^n \log \|f\|_{p,\mathcal{M}}.$$

By Remark 1.3(b), it follows that

$$\kappa(p,\mathcal{M}) - \sum_{j=1}^{n}(1-p)^j \log M_j \to 0, \quad n \to +\infty,$$

and hence

$$\limsup_{n\to+\infty}(1-p)^n \log \|f^{(n)}\|_\infty \le \theta,$$

as claimed. □

5 Independence of derivatives

5.1 Some notation and preparatory material

We previously considered the mapping $\boldsymbol{\pi}: f \mapsto (f, f', f'', \ldots)$. To conform with notation introduced earlier in this paper as well as in the work of Peetre, we will use the letter $\boldsymbol{\alpha}$ instead of $\boldsymbol{\pi}_0$ to refer to the first coordinate map $\boldsymbol{\alpha}: W^{p,\theta}_{\mathcal{M}} \to L^p$. Likewise, we write $\boldsymbol{\delta}$ for the mapping $W^{p,\theta}_{\mathcal{M}} \to W^{p,\theta(1-p)^{-1}}_{\mathcal{M}_1}$ of taking the derivative (on the test functions). Here, we recall that $\mathcal{M}_1$ is the shifted sequence $\mathcal{M}_1 = \{M_{n+1}\}_n$. More precisely, in terms of a Cauchy sequence $\{f_j\}_j$ of test functions in $\mathscr{S}^{\circledast}_{p,\theta,\mathcal{M}}$ which converges in quasinorm to an abstract limit $f \in W^{p,\theta}_{\mathcal{M}}$, we write

$$\boldsymbol{\alpha} f = \lim_j \boldsymbol{\alpha} f_j = \lim_j f_j \in L^p \quad \text{and} \quad \boldsymbol{\delta} f = \lim_j \boldsymbol{\delta} f_j = \lim_j f_j' \in W^{p,\theta(1-p)^{-1}}_{\mathcal{M}_1}.$$

As we saw in connection with the Sobolev $W^{k,p}$-spaces for $0 < p < 1$, the key step was the construction of the two lifts $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$. If we may find the analogous lifts in the present setting, the rest of the proof will carry over almost word for word from the Sobolev space case. It turns out that the lifts $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ are built in the same manner as before, using the existence of invisible mollifiers Φ_ϵ analogous to the case of $W^{k,p}$ (compare with Lemma 3.1), with slight technical obstacles in the construction of $\boldsymbol{\gamma}$, related to the unbounded support of test functions.

Before turning to the mollifiers, we make an observation regarding the weight sequence. We will make a dichotomy between the cases

$$\lim_{n\to+\infty}(1-p)^n \log M_n = 0 \quad \text{and} \quad \liminf_{n\to+\infty} \log(1-p)^n \log M_n > 0$$

in the definition of p-regularity (Definition 1.4).

Lemma 5.1 *Let $\mathcal{M}$ be a log-convex sequence such that $\kappa(p,\mathcal{M}) = +\infty$ and assume that*

$$\liminf_{n\to\infty}(1-p)^n \log M_n > 0.$$

Then there exists a p-regular minorant sequence $\mathcal{N}$ with $\kappa(p,\mathcal{N}) = +\infty$, such that

$$\lim_{n\to\infty}(1-p)^n \log N_n = 0.$$

For more complete details of this proof, see [4], as well as Behm's thesis [3], where the former paper is discussed.

Proof sketch Let $c_0 = \liminf_{n\to+\infty}(1-p)^n \log M_n > 0$. Then

$$c_0(1-p)^{-n} \le (1+\mathrm{o}(1)) \log M_n,$$

so that

$$\frac{c_0}{n+1}(1-p)^{-n} \le \frac{1}{n+1}(1+\mathrm{o}(1)) \log M_n \le \log M_n + \mathrm{O}(1).$$

We now put

$$N_n = C_1 \mathrm{e}^{c_0(n+1)^{-1}(1-p)^{-n}},$$

where the positive constant C_1 is chosen such that $N_n \le M_n$ for all integers $n \ge 0$. It follows that $\mathcal{N} = \{N_n\}_n$ is p-regular with $\kappa(p,\mathcal{N}) = +\infty$, and that $(1-p)^n \log N_n = \mathrm{o}(1)$, as needed. □

5.2 The construction of mollifiers

The invisibility lemma runs as follows.

Lemma 5.2 (Invisibility lemma for L^p-Carleman spaces) *Assume that $\mathcal{M}$ is p-regular and has infinite p-characteristic $\kappa(p,\mathcal{M}) = +\infty$. For any $\epsilon > 0$ there exists a non-negative function $\Phi \in \mathscr{S}^{\circledast}_{p,0,\mathcal{M}}$ (with $\theta = 0$) such that*

$$\int_{\mathbb{R}} \Phi\, dx = 1, \quad \operatorname{supp} \Phi \subset [0,\epsilon], \quad \|\Phi\|_{p,\mathcal{M}} < \epsilon.$$

The tools needed to prove the lemma were developed back in Sect. 3.1, in connection with the Invisibility Lemma for $W^{k,p}$. Given a sequence $\{a_l\}_{l=1}^{\infty}$, we form the associated convolution product $\Phi_{1,\infty}$ (see (3.1)). We recall that $\Phi_{1,\infty}$ has support $[0, \sum_{l\geq 1} a_l]$, and enjoys the estimates (3.3) and (3.6), provided that $\{a_j\}_j$ is a decreasing ℓ^1-sequence.

In order to bound the support of $\Phi_{1,\infty}$, will need to estimate the sums $\sum_{j\geq n+1} a_j$. If we write $a_j = c_j\alpha_j$ and ask for $\{\alpha_j\}_j$ to be a decreasing sequence of positive numbers, and that the c_j are positive with $\{c_j\}_j \in \ell^1$, we obtain the bound

$$\sum_{j=n+1}^{+\infty} a_j \leq c\alpha_{n+1}, \quad n \geq 0, \tag{5.1}$$

where $c = \|\{c_j\}_j\|_{\ell^1}$. Then the right-hand side of (3.6) may be estimated further:

$$\|\Phi_{1,\infty}^{(n)}\|_p^p \leq 2^n \frac{c\,\alpha_{n+1}}{(a_1\cdots a_{n+1})^p} = \frac{2^n c\,\alpha_{n+1}}{(c_1\cdots c_{n+1})^p(\alpha_1\cdots\alpha_{n+1})^p}. \tag{5.2}$$

We now briefly describe what we ask the sequence $a = \{a_l\}_l = \{c_l\alpha_l\}_l$ to satisfy to guarantee that $\Phi_{1,\infty}$ meets the conclusion of the lemma. The $(p,0)$-tameness is, in view of (3.3), ensured by

$$\limsup_{n\to\infty}(1-p)^n \log\frac{1}{a_1\cdots a_n} \leq 0. \tag{5.3}$$

Next, to get the norm control $\|\Phi_{1,\infty}\|_{p,\mathcal{M}} < \epsilon$, we require in view of (5.2) that

$$\frac{2^n c\,\alpha_{n+1}}{(c_1\cdots c_{n+1})^p(\alpha_1\cdots\alpha_{n+1})^p} \leq \epsilon^p M_n^p, \quad n = 0,1,2,\ldots. \tag{5.4}$$

Moreover, the assertion that $\operatorname{supp}\Phi_{1,\infty} \subset [0,\epsilon]$ is equivalent to having $\|a\|_{\ell^1} \leq \epsilon$, which follows if we assume that $\alpha_1 \leq \epsilon/c$.

Remark 5.1 We now digress on the structural consequences for positive sequences $\{\alpha_l\}_l$ satisfying a requirement of the form

$$\frac{\alpha_{l+1}}{(\alpha_1 \cdots \alpha_{l+1})^p} \leq \delta M_l^r,$$

where r and δ are positive constants. While r is fixed, we want to keep some flexibility in the choice of δ. Given the product structure, we make a quotient ansatz $\alpha_l = b_l/b_{l+1}$ with $b_1 = B^{1/p}$, where $B > 1$ is large. In terms of $\{b_l\}_{l\geq 1}$, the above relations read

$$\frac{b_{l+1}}{b_{l+2}^{1-p}} \leq \delta B M_l^r, \quad l \geq 0.$$

We assume now that $\delta B = 1$, and study the instance when all the inequalities are equalities. Then it follows that

$$\frac{b_{l+1}}{b_{l+2}^{1-p}} = M_l^r$$

which gives iteratively the sequence $\{b_l\}_l$ as

$$b_{l+2} = (b_{l+1} M_l^{-r})^{1/(1-p)},$$

for $l = 0, 1, 2, \ldots$. Proceeding iteratively, we would obtain

$$b_{l+2} = \big(B\, M_0^{-r} M_1^{-r(1-p)} \cdots M_l^{-r(1-p)^l}\big)^{(1-p)^{-l-1}}, \quad l = 0, 1, 2, \ldots.$$

Now, if $\kappa(p, \mathcal{M})$ were finite, we would put $B = \prod_{j\geq 0} M_j^{r(1-p)^j}$ and obtain the formula

$$b_l = \prod_{j\geq 0} M_{l+j-1}^{r(1-p)^j}, \quad l = 1, 2, 3, \ldots. \tag{5.5}$$

In the case at hand, we have $\kappa(p, \mathcal{M}) = +\infty$, so such a choice will not work directly. However, it does give a hint as to how to select these sequences.

With these preliminaries, we begin our construction.

Proof Without loss of generality we may assume that

$$\lim_{n\to+\infty} (1-p)^n \log M_n = 0.$$

Indeed, if $\mathcal{M}$ does not meet this requirement, then we would apply Lemma 5.1 and replace $\mathcal{M}$ by a minorant $\mathcal{N}$, and proceed as below. The result would then follow, since $\mathscr{S}^{\circledast}_{p,0,\mathcal{N}} \subset \mathscr{S}^{\circledast}_{p,0,\mathcal{M}}$, where the inclusion contracts the quasinorm.

By the assumption of p-regularity, the number

$$\beta := \liminf_{n\to\infty} \frac{\log M_n}{n \log n} \tag{5.6}$$

satisfies $1 < \beta \leq +\infty$. Let r and ρ be numbers with $0 < r \leq p$ and $\rho > 0$, such that

$$(p - r)\beta > (1 + \rho)p,$$

which is possible since $\beta > 1$. Guided by the considerations in Remark 5.1, we consider instead of (5.5) the truncated products

$$b_{l,k} = \prod_{j=0}^{k} M_{l-1+j}^{r(1-p)^j}, \quad l = 1, 2, 3, \ldots,$$

where the parameter k remains to be chosen. Next, we write

$$\alpha_{l,k} = \frac{b_{l,k}}{b_{l+1,k}}, \quad c_l = \frac{1}{l^{1+\rho}}, \qquad l = 1, 2, 3, \ldots,$$

and we put $a_{l,k} = c_l \alpha_{l,k}$ as before. Since $\mathcal{M}$ is logarithmically convex by assumption, we have that

$$\alpha_{l,k} = \prod_{j=0}^{k} \left(\frac{M_{l-1+j}}{M_{l+j}}\right)^{r(1-p)^j} \leq \prod_{j=0}^{k} \left(\frac{M_{l-2+j}}{M_{l+j-1}}\right)^{r(1-p)^j} = \alpha_{l-1,k}, \quad l = 2, 3, 4, \ldots,$$

so that the sequence $\{\alpha_{l,k}\}_l$ is decreasing. Moreover, the fact that $\{c_l\}_l \in \ell^1$ implies that the above discussion is applicable. In particular, the support of $\Phi_{1,\infty}$ is contained in the interval $[0, c\alpha_{1,k}]$, where $c = \|\{c_j\}_j\|_{\ell^1}$ and

$$c\alpha_{1,k} = c \prod_{j=0}^{k} \left(\frac{M_j}{M_{j+1}}\right)^{r(1-p)^j} = \frac{c\, M_0^r}{M_{k+1}^{r(1-p)^k} \prod_{j=1}^{k} M_j^{rp(1-p)^{j-1}}}.$$

Since $\kappa(p, \mathcal{M}) = +\infty$ is assumed and as $M_{k+1} \geq 1$ holds for large enough k, this expression tends to 0 as $k \to +\infty$.

To check that $\Phi_{1,\infty}$ is $(p,0)$-tame, we observe that

$$\begin{aligned}(1-p)^n \log\|\Phi_{1,\infty}^{(n)}\|_\infty &\le (1-p)^n \log \frac{2^n}{a_{1,k}\cdots a_{n+1,k}}\\ &= n(1-p)^n\log 2 + (1-p)^n \log\frac{1}{\alpha_{1,k}\cdots\alpha_{n+1,k}} + (1-p)^n\log\frac{1}{c_1\cdots c_{n+1}}\\ &= n(1-p)^n\log 2 + (1-p)^n\log\frac{b_{n+2,k}}{b_{1,k}} + (1+\rho)(1-p)^n\sum_{j=1}^{n+1}\log j\\ &= n(1-p)^n\log 2 - (1-p)^n\log b_{1,k} + (1+\rho)(1-p)^n\log((n+1)!)\\ &\quad + r\sum_{j=0}^{k}(1-p)^{j+n}\log M_{n+j+1}.\end{aligned}$$

Consequently, if k is fixed but large enough for $b_{1,k}\ge 1$ to hold, we have that

$$(1-p)^n\log\|\Phi^{(n)}\|_\infty \le r(1-p)^{-1}\sum_{j=0}^{k}(1-p)^{j+n+1}\log M_{n+j+1} + \mathrm{o}(1) = \mathrm{o}(1),$$

as $n\to+\infty$, since it is given that $\lim_n(1-p)^n\log M_n = 0$.

We turn to the property (5.4). We first observe that

$$\frac{2^n c}{(c_1\cdots c_{n+1})^p} = \mathrm{e}^{f(n)},$$

where $f(n) = \log c + n\log 2 + p(1+\rho)\log((n+1)!)$. In addition, we see that

$$\frac{\alpha_{n+1,k}}{(\alpha_{1,k}\cdots\alpha_{n+1,k})^p} = \frac{\frac{b_{n+1,k}}{b_{n+2,k}}}{\left(\frac{b_{1,k}}{b_{n+2,k}}\right)^p} = \frac{b_{n+1,k}}{b_{1,k}^p b_{n+2,k}^{1-p}} = \frac{M_n^r}{b_{1,k}^p M_{n+k+1}^{r(1-p)^{k+1}}}.$$

By the p-regularity, for large enough k, we have that $M_{n+k+1}\ge 1$, and hence the above calculation combined with (3.6) gives the estimate

$$\|\Phi_{1,\infty}^{(n)}\|_p^p \le \frac{1}{b_{1,k}^p}\mathrm{e}^{f(n)-(p-r)\log M_n} M_n^p.$$

In particular, observe that

$$\begin{aligned} f(n)-(p-r)\log M_n &\le \log c + n\log 2\\ &\quad + p(1+\rho)\left\{\log(n+1)! - \frac{p-r}{p(1+\rho)}\log M_n\right\} \le \Lambda\end{aligned}$$

for some positive constant Λ independent of k, if we use Stirling's formula and recall that $\frac{p-r}{(1+\rho)p} > \frac{1}{\beta}$, while taking into account the definition of the parameter β in (5.6).

In terms of quasinorm control, this means that

$$\|\Phi_{1,\infty}^{(n)}\|_p^p \le \frac{\mathrm{e}^{\Lambda}}{b_{1,k}^p} M_n^p.$$

Since we know that $b_{1,k} \to +\infty$ as $k \to +\infty$, while Λ is independent of k, the desired estimate on $\|\Phi_{1,\infty}\|_{p,\mathcal{M}}$ follows by choosing k large enough. The proof is complete. □

5.3 Construction of $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$

The existence of invisible mollifiers finally allows us to define the lifts $\boldsymbol{\beta}\colon L^p \to W_{\mathcal{M},0}^{p,\theta}$ and $\boldsymbol{\gamma}\colon W_{\mathcal{M}_1,0}^{p,\theta_1} \to W_{\mathcal{M},0}^{p,\theta}$.

Proposition 5.1 *Suppose $\mathcal{M}$ is p-regular with $\kappa(p,\mathcal{M}) = +\infty$. Then there exists a continuous linear mapping $\boldsymbol{\beta}\colon L^p \to W_{\mathcal{M},0}^{p,\theta}$ such that*

$$\boldsymbol{\alpha} \circ \boldsymbol{\beta} = \mathrm{id}_{L^p} \quad \textit{and} \quad \boldsymbol{\delta} \circ \boldsymbol{\beta} = 0.$$

Proof It is enough to demonstrate the result for $\theta = 0$, since the general case then follows by inclusion. The definition procedure remains the same as that in Sect. 3.3. We denote by $\mathscr{F}_p$ the space of step functions, and to each $g \in \mathscr{F}_p$ we aim to associate a Cauchy sequence $\{g_j\}_j \subset \mathscr{S}_{p,0,\mathcal{M}}^{\circledast} \cap C_0^\infty$ such that

$$\lim_{j\to+\infty} \|g - g_j\|_p = 0 \qquad \lim_{j\to+\infty} \|g_j'\|_{p,\mathcal{M}_1} = 0, \tag{5.7}$$

where $\mathcal{M}_1$ is the shifted sequence $\mathcal{M}_1 = \{M_{j+1}\}_{j\ge 0}$. We then declare $\boldsymbol{\beta} g$ to be the abstract limit $\lim_j g_j$ in $W_{\mathcal{M},0}^{p,0}$. If this can be achieved, these properties show that $\boldsymbol{\beta}$ is a rescaled isometry on $\mathscr{F}_p$: Indeed, the norm splits as

$$\|f\|_{p,\mathcal{M}} = \max\left\{\frac{\|f\|_p}{M_0},\ \|f\|_{p,\mathcal{M}_1}\right\}, \tag{5.8}$$

so we may write

$$\|\boldsymbol{\beta} g\|_{p,\mathcal{M}}^p = \lim_{j\to+\infty} \|g_j\|_{p,\mathcal{M}}^p = \lim_{j\to+\infty} \max\left\{\frac{\|g_j\|_p^p}{M_0^p},\ \|g_j'\|_{p,\mathcal{M}_1}^p\right\} = \frac{\|g_j\|_p^p}{M_0^p}$$

and they uniquely determine $\boldsymbol{\beta} g$. Indeed, any other Cauchy sequence $\{\tilde{g}_j\}_j$ for which (5.7) holds will be equivalent to $\{g_j\}$ in $W_{\mathcal{M},0}^{p,0}$:

$$\|g_j - \tilde{g}_j\|_{p,\mathcal{M}}^p \le \frac{\|g - g_j\|_p^p}{M_0} + \frac{\|g - \tilde{g}_j\|_p^p}{M_0} + \|g_j'\|_{p,\mathcal{M}_1}^p + \|\tilde{g}_j'\|_{p,\mathcal{M}_1}^p = \mathrm{o}(1),$$

as $j \to +\infty$. Moreover, (5.7) clearly shows that

$$(\boldsymbol{\alpha} \circ \boldsymbol{\beta})g = g \quad \text{and} \quad (\boldsymbol{\delta} \circ \boldsymbol{\beta})g = 0, \qquad g \in \mathscr{F}_p,$$

and since these properties are stable under extension by continuity to L^p, the proof is complete once the existence of a Cauchy sequence $\{g_j\}_j$ satisfying (5.7) is demonstrated.

The properties of $\mathcal{M}$ needed to apply Lemma 5.2 are inherited by $\mathcal{M}_1$, so for each j we may apply the lemma to $W^{p,0}_{\mathcal{M}_1,0}$ in order to produce functions $\Phi_j \in \mathscr{S}^{\circledast}_{p,0,\mathcal{M}_1}$ with support in $[0, \epsilon_j]$ such that $\|\Phi_j\|_{\mathcal{M}_1} \le \epsilon_j$, where ϵ_j is a decreasing sequence tending to 0. Let g be an arbitrary step function in $\mathscr{F}_p$, and put $g_j = g * \Phi_j$, so that g_j is smooth with uniformly bounded support. We observe that for $n \ge 1$,

$$(g * \Phi_j)^{(n)} = g' * \Phi_j^{(n-1)},$$

where g' is thought of as a finite sum of point masses. By (3.4), it follows that

$$\|g_j^{(n)}\|_p = \left\|(g * \Phi_j)^{(n)}\right\|_p \le \|g'\|_{\ell^p} \|\Phi_j^{(n-1)}\|_p \le \epsilon_j \|g'\|_{\ell^p} M_n, \quad n = 1, 2, 3, \ldots. \tag{5.9}$$

Clearly we have the convergence $g_j \to g$ in L^p, that is, $\lim_j \|g_j - g\|_p = 0$. It also follows from (5.9) that

$$\lim_{j\to+\infty} \|g_j'\|^p_{p,\mathcal{M}_1} = \lim_{j\to+\infty} \sup_{n\ge1} \frac{\|g_j^{(n)}\|_p}{M_n} \le \lim_{j\to+\infty} \epsilon_j \|g'\|_{\ell^p} = 0.$$

We conclude that for each j, $g_j \in \mathscr{S}^{\circledast}_{p,0,\mathcal{M}} \cap C_0^\infty$, and moreover, since $g_j \to g$ in L^p and $g_j' \to 0$ in $W^{p,0}_{\mathcal{M}_1,0}$, the sequence $\{g_j\}_j$ is Cauchy in the quasinorm of $W^{p,0}_{\mathcal{M}}$. We declare the abstract limit of this Cauchy sequence to be $\boldsymbol{\beta} g \in W^{p,0}_{\mathcal{M},0}$ for the given step function g. In view of the above calculations,

$$\|\boldsymbol{\beta} g\|_{p,\mathcal{M}} = \lim_j \|g_j\|_{p,\mathcal{M}} = \lim_j \sup_{n\ge0} \frac{\|g_j^{(n)}\|_p}{M_n} = \frac{\|g\|_p}{M_0}.$$

As of right now, $\boldsymbol{\beta}$ is a densely defined bounded operator $\mathscr{F}_p \to W^{p,0}_{\mathcal{M},0}$. By extending $\boldsymbol{\beta}$ to the entire space, we obtain a well-defined linear operator $\boldsymbol{\beta}\colon L^p \to W^{p,0}_{\mathcal{M},0}$ such that $\boldsymbol{\alpha}\boldsymbol{\beta} = \mathrm{id}$ and $\boldsymbol{\delta}\boldsymbol{\beta} = 0$. □

Recall the notation $\mathcal{M}_1 = \{M_{k+1}\}_k$ and $\theta_1 = \theta/(1-p)$.

Proposition 5.2 *Assume that $\mathcal{M}$ is p-regular with $\kappa(p, \mathcal{M}) = +\infty$. Then there exists a continuous linear mapping $\boldsymbol{\gamma}\colon W^{p,\theta_1}_{\mathcal{M}_1,0} \to W^{p,\theta}_{\mathcal{M},0}$ such that*

$$\boldsymbol{\delta} \circ \boldsymbol{\gamma} = \mathrm{id}_{W^{p,\theta_1}_{\mathcal{M}_1,0}} \quad \textit{and} \quad \boldsymbol{\alpha} \circ \boldsymbol{\gamma} = 0.$$

Proof To construct $\boldsymbol{\gamma}$, we let $g \in W^{p,\theta_1}_{\mathcal{M}_1,0}$ be an arbitrary element, which is by definition the abstract limit of some Cauchy sequence $\{g_j\}_j$ in $\mathscr{S}^{\circledast}_{p,\theta_1,\mathcal{M}_1} \cap C_0^\infty$. For any given $\epsilon > 0$, Lemma 5.2 provides a function $\Phi_\epsilon \in \mathscr{S}^{\circledast}_{p,\theta_1,\mathcal{M}_1} \cap C_0^\infty$ with the following properties: $\Phi_\epsilon \geq 0$, $\langle \Phi_\epsilon \rangle_{\mathbb{R}} := \int_{\mathbb{R}} \Phi_\epsilon(t)\mathrm{d}t = 1$, Φ_ϵ supported in $[0, \epsilon]$, while at the same time, $\|\Phi_\epsilon\|_{p,\mathcal{M}_1} \leq \epsilon$. For a sequence ϵ_j, we use the functions Φ_{ϵ_j} to modify each $g_j(x)$ to have vanishing zeroth moment, by defining

$$\tilde{g}_j(x) := g_j(x) - \langle g_j \rangle_{\mathbb{R}} \Phi_{\epsilon_j}(x), \quad \langle g_j \rangle_{\mathbb{R}} := \int_{\mathbb{R}} g_j(t)\mathrm{d}t,$$

where the ϵ_j are chosen to tend to zero so quickly that

$$\lim_{j \to +\infty} \|g_j - \tilde{g}_j\|_{p,\mathcal{M}_1} = \lim_{j \to +\infty} \|\Phi_{\epsilon_j}\|_{p,\mathcal{M}_1} |\langle g_j \rangle_{\mathbb{R}}| = 0.$$

Next, we define the functions u_j as primitives:

$$u_j(x) = \int_{-\infty}^{x} \tilde{g}_j(t)\mathrm{d}t, \quad x \in \mathbb{R}.$$

Then as $\tilde{g}_j$ has integral 0, we see that $u_j \in C_0^\infty(\mathbb{R})$, and it is easy to see that $u_j \in \mathscr{S}^{\circledast}_{p,\theta,\mathcal{M}}$. We put $f_j := u_j - \boldsymbol{\beta} u_j \in W^{p,\theta}_{\mathcal{M},0}$, and define $\boldsymbol{\gamma} g$ as the limit of $\{f_j\}_j$ in $W^{p,\theta}_{\mathcal{M},0}$. That this makes the mapping $\boldsymbol{\gamma}$ well-defined with the asserted properties follows in a fashion analogous to that of the Sobolev case (Sect. 3.4). □

Sketch of proof of Theorem 1.3 In principle, we just follow the algebraic approach from the proof of Theorem 1.1 supplied in Sect. 2.3. So, we define a linear operator $\mathbf{A}\colon W^{p,\theta}_{\mathcal{M},0} \to L^p \oplus W^{p,\theta_1}_{\mathcal{M}_1,0}$ given by $\mathbf{A} f := (\boldsymbol{\alpha} f, \boldsymbol{\delta} f)$ and analyze it using the properties of the four maps $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\delta}, \boldsymbol{\gamma}$. □

5.4 A remark on the size of $W^{p,\theta}_{\mathcal{M}}$

We conclude this section with a proposition which illustrates the size of $W^{p,\theta}_{\mathcal{M}}$ when $\kappa(p, \mathcal{M}) = +\infty$.

Proposition 5.3 *In the setting of Theorem 1.3, we have the inclusions*

$$c_0(L^p, \mathcal{M}) \subset \boldsymbol{\pi} W^{p,\theta}_{\mathcal{M},0} \subset \boldsymbol{\pi} W^{p,\theta}_{\mathcal{M}} \subset \ell^\infty(L^p, \mathcal{M})$$

Proof The second and third inclusions are clear. To obtain the first, let $(f_0, f_1, f_2, \ldots)$ be an arbitrary element of $c_0(L^p, \mathcal{M})$. By iterating Theorem 1.3 as in Remark 1.4, for each fixed k there exists an element $F_k \in W^{p,\theta}_{\mathcal{M},0}$ such that $\boldsymbol{\pi}(F_k) = (f_0, \cdots, f_k, 0, \ldots)$. We claim that $\lim_j F_j$ exists as an element of $W^{p,\theta}_{\mathcal{M},0}$, and that $\boldsymbol{\pi}(\lim_j F_j) = (f_0, f_1, f_2, \ldots)$. Since $(f_0, f_1, f_2, \ldots) \in c_0(L^p, \mathcal{M})$, the cut-off sequences $(f_0, \ldots, f_k, 0, \ldots)$ converge in the quasinorm of $l^\infty(L^p, \mathcal{M})$ to

$(f_0, f_1, f_2, \ldots)$ as $k \to +\infty$. Moreover, since $\boldsymbol{\pi}$ is an isometry $W^{p,\theta}_{\mathcal{M}} \to l^\infty(L^p, \mathcal{M})$, this means that the elements $F_k \in W^{p,\theta}_{\mathcal{M},0}$ converge in quasinorm as $k \to +\infty$. Finally, $\boldsymbol{\pi}$ applied to the limit coincides with the sequence $(f_0, f_1, f_2, \ldots)$. □

6 The quasianalyticity transition and the parameter θ

6.1 A remark on the case $1 \le p < +\infty$

It is of course a basic question is if the Denjoy–Carleman theorem remains valid in the setting of the L^p-Carleman Classes in the parameter range $1 \le p < +\infty$ (without any tameness requirement of course). This is of course true and should be well-known, but we have unfortunately not been able to find a suitable references for this fact. For this reason, we supply a short self-contained presentation. We begin with the following lemma.

Lemma 6.1 *If $\mathcal{M}$ is logarithmically convex, the Carleman class $\mathscr{C}_{\mathcal{M}}$ is an algebra.*

Proof By the logarithmic convexity of $\mathcal{M}$, it follows that

$$M_j M_{n-j} \le M_0 M_n, \quad 0 \le j \le n.$$

Indeed, if $j \le n - j$, since $M_0 = 1$,

$$M_j M_{n-j} = M_0 \left(\frac{M_1}{M_0} \cdots \frac{M_j}{M_{j-1}} \right) \left(\frac{M_{n-j}}{M_{n-j+1}} \cdots \frac{M_{n-1}}{M_n} \right) M_n.$$

We next observe that

$$\frac{M_k}{M_{k-1}} \le \frac{M_{k+1}}{M_k}, \quad k \ge 1,$$

so we may estimate the product

$$\begin{aligned} &\left(\frac{M_1}{M_0} \cdots \frac{M_j}{M_{j-1}} \right) \left(\frac{M_{n-j}}{M_{n-j+1}} \cdots \frac{M_{n-1}}{M_n} \right) \\ &\le \left(\frac{M_{n-j+1}}{M_{n-j}} \cdots \frac{M_n}{M_{n-1}} \right) \left(\frac{M_{n-j}}{M_{n-j+1}} \cdots \frac{M_{n-1}}{M_n} \right) = 1. \end{aligned}$$

The claim now follows.

Next, let $f, g \in \mathscr{C}_{\mathcal{M}}$, and suppose that the estimates

$$\|f^{(n)}\|_\infty \le A_f^n M_n \quad \text{and} \quad \|g^{(n)}\|_\infty \le A_g^n M_n,$$

hold for $n = 0, 1, 2, \ldots,$ where A_f and A_g are appropriate positive constants. Then, by the Leibniz rule, we have that

$$\|(fg)^{(n)}\|_\infty \le \sum_{j=0}^{n} \binom{n}{j} \|f^{(j)}\|_\infty \|g^{(n-j)}\|_\infty \le \sum_{j=0}^{n} \binom{n}{j} A_f^j A_g^{n-j} M_j M_{n-j}$$
$$\le (A_f + A_g)^n M_0 M_n, \tag{6.1}$$

which proves that $fg \in \mathscr{C}_{\mathcal{M}}$. By linearity, the conclusion extends to all $f, g \in \mathscr{C}_{\mathcal{M}}$. □

Theorem 6.1 *When $1 \le p < +\infty$ the p-Carleman class $\mathscr{C}^p_{\mathcal{M}}$ is quasianalytic if and only if $\mathscr{C}_{\mathcal{M}}$ is.*

Proof We obtain the contrapositive statements. Assume that $\mathscr{C}_{\mathcal{M}}$ is not quasianalytic. Then there exist functions in $\mathscr{C}_{\mathcal{M}}$ with compact support. Indeed, by the Denjoy–Carleman theorem it follows that

$$\sum_{n \ge 1} \frac{M_{n-1}}{M_n} < +\infty,$$

so $\Phi = \Phi_{1,\infty}$ constructed according to (3.1) with $a_n = \frac{M_{n-2}}{M_{n-1}}$ and $a_1 = \frac{1}{M_0}$, lies in the class $\mathscr{C}_{\mathcal{M}}$. As a consequence of the estimate

$$\|\Phi^{(n)}\|_p \le |\mathrm{supp}\,\Phi|^{1/p} \|\Phi^{(n)}\|_\infty \le |\mathrm{supp}\,\Phi|^{1/p} A_\Phi^n M_n,$$

where $|E|$ denotes the length of the set $E \subset \mathbb{R}$, we find that $\Phi \in \mathscr{C}^p_{\mathcal{M}}$ as well. In particular, the class $\mathscr{C}^p_{\mathcal{M}}$ cannot be quasianalytic.

As for the other direction, we assume that $\mathscr{C}^p_{\mathcal{M}}$ is not quasianalytic. Then there exists a function $f \in \mathscr{C}^p_{\mathcal{M}}$ which vanishes to infinite degree at a point a but is nontrivial at some other point b (so that $f(b) \ne 0$). In view of translation as well dilatation invariance, we may assume that $a = 0$ and $b = \frac{1}{2}$. We now estimate the supremum norm on the unit interval $I = [0, 1]$:

$$|f^{(n)}(x)| = |f^{(n)}(x) - f^{(n)}(a)| \le \|f^{(n+1)}|_I\|_1 \le \|f^{(n+1)}|_I\|_p \le C_f A_f^{n+1} M_{n+1}, \tag{6.2}$$

using Hölder's inequality, since we are considering $1 \le p < +\infty$. Next, we form $g(x) = f(x)f(1-x)$ for $0 \le x \le 1$ and put $g(x) = 0$ off this interval. It turns out that it follows from (6.1) that

$$\|g^{(n)}\|_\infty \le 2^n C_f^2 A_f^{n+2} M_1 M_{n+1}.$$

In particular, we have that $g \in \mathscr{C}_{\mathcal{M}_1}$, where $\mathcal{M}_1 = \{M_{n+1}\}_n$ is the shifted weight sequence, so that $C_{\mathcal{M}_1}$ is not quasianalytic. Since a simple weight shift does not affect whether or not a Carleman class is quasianalytic, it follows that $\mathscr{C}_{\mathcal{M}}$ is nonquasianalytic as well. □

6.2 Quasianalytic classes for $0 < p < 1$

For decay-regular sequences (see Definition 1.5), part (ii) of Theorem 1.4 asserts that the class $\mathscr{C}_{\mathcal{M}}^{p,0}$ is quasianalytic if and only if $\mathscr{C}_{\mathcal{N}}$ is. Here, the numbers N_n are the bounds that appeared in controlling $\|f^{(n)}\|_\infty$ by $\|f\|_{\mathcal{M}}$ where $f \in \mathscr{S}_{p,0,\mathcal{M}}^{\circledast}$, that is,

$$N_n = \prod_{j=0}^{+\infty} M_{n+j}^{p(1-p)^j}, \tag{6.3}$$

where the product converges because $\kappa(p, \mathcal{M}) < +\infty$. In addition, part (i) of Theorem 1.4 maintains that the class $\mathscr{C}_{\mathcal{M}}^{p,\theta}$ is never quasianalytic if $\theta > 0$. In the context of $\theta > 0$, we introduce the related sequence $\mathcal{N}_\theta = \{N_{n,\theta}\}_n$ given by

$$N_{n,\theta} = \mathrm{e}^{\theta(1-p)^{-n+1}} \prod_{j=0}^{+\infty} M_{n+j}^{(1-p)^j p}. \tag{6.4}$$

Proof of Theorem 1.4 We define the a sequence of positive numbers α_n by

$$\alpha_n = N_{n-1}/N_n, \quad n = 1, 2, 3, \ldots, \tag{6.5}$$

and observe that this sequence is decreasing. This follows from the logarithmic convexity of $\mathcal{M}$, since

$$\alpha_{n+1} = \prod_{j=0}^{+\infty} \left(\frac{M_{n+j}}{M_{n+1+j}}\right)^{(1-p)^j p} \leq \prod_{j=0}^{+\infty} \left(\frac{M_{n-1+j}}{M_{n+j}}\right)^{(1-p)^j p} = \alpha_n, \quad n = 1, 2, 3, \ldots.$$

A calculation shows that the numbers α_j enjoy the property

$$\frac{\alpha_{n+1}^{1-p}}{(\alpha_1 \cdots \alpha_n)^p} = \frac{N_n}{N_0^p N_{n+1}^{1-p}} = \frac{M_n^p}{N_0^p}. \tag{6.6}$$

We begin with the assertion (i). We put

$$a_n = \mathrm{e}^{-p\theta(1-p)^{-n+1}} \alpha_n, \quad n = 1, 2, 3, \ldots, \tag{6.7}$$

and observe that the numbers a_n may be expressed in terms of the sequence $\mathcal{N}_\theta$ by

$$a_n = \frac{N_{n-1,\theta}}{N_{n,\theta}}, \quad n = 1, 2, 3, \ldots.$$

Since

$$a_{n+1} = \mathrm{e}^{-p\theta(1-p)^{-n}}\alpha_{n+1} \leq \mathrm{e}^{-p\theta(1-p)^{-n}}\alpha_n = \mathrm{e}^{-p\theta(1-p)^{-n}+p\theta(1-p)^{-n+1}}a_n$$
$$= \mathrm{e}^{-p^2\theta(1-p)^{-n}}a_n, \tag{6.8}$$

and rather trivially,

$$0 < \mathrm{e}^{-p^2\theta(1-p)^{-n}} \leq \rho < 1$$

holds for some appropriate constant ρ so that $a_{n+1} \leq \rho a_n$, the sequence decays at least geometrically. We now consider the functions $\Phi_{1,\infty}$ defined as in Sect. 3.1 by the formula

$$\Phi_{1,\infty} = \lim_{j\to+\infty} a_1^{-1}1_{[0,a_1]} * \cdots a_j^{-1}1_{[0,a_j]}.$$

Then $\Phi_{1,\infty}$ is a nonnegative compactly supported C^∞-smooth function, since $\{a_j\}_j \in \ell^1$ and each convolution mollifies. In view of the estimate (3.6) and the geometric decay of the sequence $\{a_n\}_n$ as stated above, we find that

$$\|\Phi_{1,\infty}^{(n)}\|_p^p \leq \frac{2^n a_{n+1}}{(1-\rho)(a_1\cdots a_{n+1})^p} = \frac{2^n a_{n+1}^{1-p}}{(1-\rho)(a_1\cdots a_n)^p}$$
$$= \mathrm{e}^{p^2\theta+p^2\theta(1-p)^{-1}+\cdots+p^2\theta(1-p)^{-n+1}-p\theta(1-p)^{-n+1}}\frac{2^n\alpha_{n+1}^{1-p}}{(1-\rho)(\alpha_1\cdots\alpha_n)^p}$$
$$= \mathrm{e}^{-p\theta(1-p)}\frac{2^n M_n^p}{(1-\rho)N_0^p}, \tag{6.9}$$

where we have summed the finite geometric series and applied the identity (6.6). This means that the norm estimate associated with the L^p-Carleman classes $\mathscr{C}_{\mathcal{M}}^{p,\theta}$ is fulfilled for the function $\Phi_{1,\infty}$. It remains to verify that $\Phi_{1,\infty}$ is (p,θ)-tame. However, by (3.3), we see that

$$(1-p)^n \log\|\Phi_{1,\infty}^{(n)}\|_\infty \leq (1-p)^n \log\left(\frac{2^n}{a_1a_2\cdots a_{n+1}}\right)$$
$$= (1-p)^n n\log 2 + (1-p)^n \log N_{n+1,\theta} - (1-p)^n \log N_{0,\theta}$$
$$= \theta + (1-p)^n \sum_{j=0}^{+\infty} p(1-p)^j \log M_{n+j+1} + \mathrm{o}(1)$$
$$= \theta + p\sum_{j=n+1}^{+\infty}(1-p)^{j-1}\log M_j = \theta + \mathrm{o}(1), \tag{6.10}$$

as $n\to+\infty$, where we have used the factorization $N_{n+1,\theta} = \mathrm{e}^{\theta(1-p)^{-n}}N_{n+1,0}$ as well as the fact that the series on the right-hand side is the tail sum of the convergent series representing $\kappa(p, \mathcal{M}) < +\infty$. It now follows from (6.9) and (6.10) that $\Phi_{1,\infty} \in \mathscr{C}_{\mathcal{M}}^{p,\theta}$, and since $\Phi_{1,\infty}$ is a nontrivial compactly supported function, the class $\mathscr{C}_{\mathcal{M}}^{p,\theta}$ cannot be quasianalytic.

We continue with the assertion (ii). We derive the contrapositive statement, and begin by assuming that $\mathscr{C}_{\mathcal{M}}^{p,0}$ is not quasianalytic. Let $f \in W_{\mathcal{M}}^{p,0}$. After some inspection, the inequality (1.13) asserts that

$$\|f^{(n)}\|_\infty \leq N_{n+1}\|f\|_{p,\mathcal{M}}.$$

By applying appropriate dilations, it now follows that we have the inclusion

$$\mathscr{C}_{\mathcal{M}}^{p,0} \subset \mathscr{C}_{\mathcal{N}^\star},$$

where $\mathcal{N}^\star = \{N_{j+1}\}_j$ is the shifted sequence, and $\mathscr{C}_{\mathcal{N}^\star}$ denotes the usual Carleman class associated to $\mathcal{N}^\star$. Since by assumption, $\mathscr{C}_{\mathcal{M}}^{p,0}$ is not quasianalytic, $\mathscr{C}_{\mathcal{N}^\star}$ cannot be a quasianalytic class either. Finally, by the Denjoy–Carleman theorem, the same holds true for $\mathscr{C}_{\mathcal{N}}$.

Finally, as for the remaining implication, we assume that $\mathscr{C}_{\mathcal{N}}$ is non-quasianalytic. We single out the two cases when $\sum_n M_{n-1}/M_n = +\infty$ and when $\sum_n M_{n-1}/M_n < +\infty$. In the latter case, there are nontrivial compactly supported functions in $\mathscr{C}_{\mathcal{M}}$. Take one such function f, and set $B := |\mathrm{supp}\, f|$. We now claim that $f \in \mathscr{C}_{\mathcal{M}}^{p,0}$. First note that since $\kappa(p, \mathcal{M})$ is finite, the terms $(1-p)^n \log M_n$ must tend to zero as $n \to +\infty$. The norm estimate required for a function f to be in the Carleman class $\mathscr{C}_{\mathcal{M}}$ then gives that

$$(1-p)^n \log\|f^{(n)}\|_\infty \leq (1-p)^n \log(C_f A_f^n M_n) = \mathrm{o}(1),$$

from which it follows that f is automatically $(p, 0)$-tame. Next, it is clear that

$$\|f^{(n)}\|_p \leq B^{1/p} C_f A_f^n M_n,$$

since $\|f^{(n)}\|_\infty \leq C_f A_f^n M_n$ by assumption and, in addition, the support of f is bounded. This shows that $f \in \mathscr{C}_{\mathcal{M}}^{p,0}$, as needed.

It remains to investigate the case when $\sum_n M_{n-1}/M_n = +\infty$. Then our regularity assertion tells us that

$$M_{n-1}/M_n \geq \epsilon^n, \tag{6.11}$$

holds for some $\epsilon > 0$. We recall the decreasing sequence $\{\alpha_n\}$ defined by (6.5), and consider the corresponding convolution product $\Phi_{1,\infty}$, this time defined in terms of the numbers $\{\alpha_n\}_n$. In view of the Denjoy–Carleman theorem and the assumption that $\mathscr{C}_{\mathcal{N}}$ is non-quasianalytic, we know that the sequence $\{\alpha_n\}_n$ is in ℓ^1, from which it follows that $\Phi_{1,\infty}$ is a non-trivial compactly supported function. It follows from

(6.10) applied with $\theta = 0$ that $\Phi_{1,\infty}$ is $(p, 0)$-tame. Finally, we study the p-norms of $\Phi^{(n)}_{1,\infty}$. By the estimate (6.11), we find that

$$\alpha_n = \frac{N_{n-1}}{N_n} = \prod_{j=0}^{+\infty} \left(\frac{M_{n+j-1}}{M_{n+j}}\right)^{p(1-p)^j} \geq \prod_{j=0}^{+\infty} \epsilon^{(j+n)p(1-p)^j} \geq \epsilon^n.$$

Then, if R is large enough, it follows that $R^n\alpha_n \geq \|\{\alpha_j\}\|_{\ell^1}$, and consequently that $\sum_{j\geq n} \alpha_j \leq R^n\alpha_n$. Using this observation together with the estimate (3.6), we obtain

$$\|\Phi^{(n)}_{1,\infty}\|_p^p \leq 2^n \frac{\sum_{j=n+1}^{+\infty} \alpha_j}{(\alpha_1 \cdots \alpha_{n+1})^p} \leq (2R)^n M_n^p,$$

from which it follows that $\Phi_{1,\infty} \in \mathscr{C}^{p,0}_{\mathcal{M}}$, as we already know that $\Phi_{1,\infty}$ is $(p, 0)$-tame. Since $\Phi_{1,\infty}$ is nontrivial and compactly supported, this means that the class $\mathscr{C}^{p,0}_{\mathcal{M}}$ cannot be quasianalytic. □

We now conclude with the proof of our last remaining theorem.

Proof of Theorem 1.5 We are given a logarithmically convex increasing sequence $\mathcal{M}$ such that $\kappa(p, \mathcal{M}) < +\infty$, and numbers $0 \leq \theta < \theta'$, and aim to prove that the inclusion

$$W^{p,\theta}_{\mathcal{M}} \subset W^{p,\theta'}_{\mathcal{M}}$$

is strict for $\theta < \theta'$. First, we note that by Theorem 1.2 the space $W^{p,\theta}_{\mathcal{M}}$ equals the class $\mathscr{S}^{\circledast}_{p,\theta,\mathcal{M}}$ of test functions. Hence, in order to show that the containment $W^{p,\theta'}_{\mathcal{M}} \supset W^{p,\theta}_{\mathcal{M}}$ is strict, it is enough to exhibit a function $f \in W^{p,\theta'}_{\mathcal{M}}$ which is not (p, θ)-tame. To this end, we put $a_n = N_{n-1,\theta'}/N_{n,\theta'}$, where $\{N_{n,\theta'}\}_n$ is given by (6.4), and consider the associated convolution product $\Phi_{1,\infty} = \Phi_{1,\infty,\theta'} \in \mathscr{S}^{\circledast}_{p,\theta',\mathcal{M}}$ as in the proof of Theorem 1.4, part (i). It is a simple consequence of (6.8) that

$$\sum_{j>n} a_j = \mathrm{o}(a_n), \quad n \to +\infty.$$

It follows that there exists some n_0 such that

$$\|\Phi_{n,\infty}\|_\infty = \frac{1}{a_n}, \quad n \geq n_0.$$

Next, we write $\Phi^{(n)}_{1,\infty} = \Phi^{(n)}_{1,n} * \Phi_{n+1,\infty}$, and note that $\Phi^{(n)}_{1,n}$ is a sum of finitely many point masses, of mass $(a_1 \cdots a_n)^{-1}$, located at 2^n points $\{x_j\}_j$:

$$\Phi^{(n)}_{1,n} = \sum_{j=1}^{2^n} \epsilon(j) \frac{1}{a_1 \cdots a_n} \delta_{x_j},$$

where $\epsilon(j) \in \{1, -1\}$, and we have ordered the x_j in increasing order. Moreover, as each x_j is the sum $\sum_{k\in J_j} a_k$ over a subset $J_j \subset \{1, \ldots, n\}$, it follows that there exists at least one x_{j_0} such that $|x_j - x_{j_0}| \geq a_n$ for any $j \neq j_0$. For instance, we may take $x_1 = 0$ and note that $x_2 = a_n$ so that $|x_1 - x_2| = a_n$. As the support of $\Phi_{n+1,\infty}$ is contained in an interval of length $\mathrm{o}(a_n)$, no interference can occur on the interval $[x_{j_0}, x_{j_0+1})$, and we find that

$$\|\Phi_{1,\infty}^{(n)}\|_\infty \geq \frac{1}{a_1 \cdots a_n}\|\Phi_{n+1,\infty}\|_\infty = \frac{1}{a_1 \cdots a_{n+1}},$$

for n large enough. By a similar computation as carried out in (6.10), it follows from this estimate that $(1-p)^n \log\|\Phi_{1,\infty}^{(n)}\|_\infty \to \theta'$ as $n \to +\infty$, which completes the proof. □

7 Concluding remarks

A number of questions remain to be investigated. For instance, we would like to better understand the space $W_{\mathcal{M}}^{p,\theta}$ in the uncoupled regime. Note that when the p-characteristic $\kappa(p, \mathcal{M})$ is finite, we have a strict inclusion $W_{\mathcal{M}}^{p,\theta} \subset W_{\mathcal{M}}^{p,\theta'}$, $W_{\mathcal{M}}^{p,\theta} \neq W_{\mathcal{M}}^{p,\theta'}$ when $\theta < \theta'$ (see Proposition 1.5). The corresponding inclusions also hold for the spaces $W_{\mathcal{M},0}^{p,\theta}$. When $\kappa(p, \mathcal{M}) = +\infty$ we do not know whether this is the case. It is conceivable that these spaces are so large that the θ-dependence is lost. We note that Proposition 5.3, gives the inclusions

$$c_0(L^p, \mathcal{M}) \subset \boldsymbol{\pi} W_{\mathcal{M},0}^{p,\theta} \subset \boldsymbol{\pi} W_{\mathcal{M}}^{p,\theta} \subset \ell^\infty(L^p, \mathcal{M}).$$

Another issue is whether we can drop some of the regularity assumptions in Theorems 1.3 and 1.4.

In addition, it appears to be of interest to study the case when the maximal class $\mathscr{S}_{p,\theta,\mathcal{M}}^{\circledast}$ is replaced with some other smaller class of test functions, for which (1.4) remains valid. For instance, we might consider the Hermite class

$$\mathscr{S}_{p,\mathcal{M}}^{\mathrm{Her}} = \left\{ f(x) = \mathrm{e}^{-x^2} p(x) : \ p \text{ a polynomial such that } \|f\|_{p,\mathcal{M}} < +\infty \right\}$$

which was mentioned earlier, or, in the case of the unit circle, the class $\mathscr{P}_{\mathrm{trig}}$ of trigonometric polynomials, and ask whether the corresponding L^p-Carleman spaces undergo the same phase transitions.

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