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The Klein–Gordon equation, the Hilbert transform, and dynamics of Gauss-type maps

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Abstract. We study the uncertainty principle associated with the Klein–Gordon equation. As in the previous work [Ann. of Math. 173 (2011)], we consider vanishing along a lattice-cross. The following variants appear naturally: (1) vanishing only along "half" of the lattice-cross, where the "half" is defined as being on the boundary of a quarter-plane, and (2) that the function vanishes on the whole lattice-cross, but we require the function to have Fourier transform supported by one of the two branches of the hyperbola. In case (1) the critical phenomenon is whether the given condition forces the function to vanish on the quarter-plane in question. Here it turns out to be crucial whether the quarter-plane is space-like or time-like, and in short the answer is yes for space-like and no for time-like. The analysis brings us quite far, involving the orbit of the Hilbert kernel under the iterates of the transfer operator, and uses methods from the theory of totally positive matrices as well as Hurwitz zeta functions, and is partially postponed to a separate publication. In case (2), the critical phenomenon occurs at another density, and the dynamics then comes from the standard Gauss transformation $t \mapsto 1/t \bmod \mathbb{Z}$ on the interval [0, 1]. In the intermediate range of the density of the lattice-cross, we obtain unique extendability of the Fourier transform from one branch of the hyperbola to the other.

Keywords. Transfer operator, Hilbert transform, completeness, Klein–Gordon equation

1. Introduction

1.1. Heisenberg uniqueness pairs

Let μ be a finite complex-valued Borel measure in the plane $\mathbb{R}^2$, and associate with it the Fourier transform

$$\hat{\mu}(\xi) := \int_{\mathbb{R}^2} \mathrm{e}^{\mathrm{i}\pi\langle x,\xi\rangle}\, \mathrm{d}\mu(x),$$

where $x = (x_1, x_2)$ and $\xi = (\xi_1, \xi_2)$, with inner product

$$\langle x, \xi\rangle = x_1\xi_1 + x_2\xi_2.$$

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The Fourier transform $\hat{\mu}$ is a continuous and bounded function on $\mathbb{R}^2$. In [15], the concept of a Heisenberg uniqueness pair (HUP) was introduced. It is similar to the notion of weakly mutually annihilating pairs of Borel measurable sets having positive area measure, which appears, e.g., in the book by Havin and Jöricke [14]. For $\Gamma \subset \mathbb{R}^2$ which is a finite disjoint union of smooth curves in $\mathbb{R}^2$, let $\mathrm{M}(\Gamma)$ denote the Banach space of complex-valued finite Borel measures in $\mathbb{R}^2$, supported on Γ. Moreover, let $\mathrm{AC}(\Gamma)$ denote the closed subspace of $\mathrm{M}(\Gamma)$ consisting of the measures that are absolutely continuous with respect to arc length measure on Γ.

Definition 1.1.1. Let Γ be a finite disjoint union of smooth curves in $\mathbb{R}^2$. For a set $\Lambda \subset \mathbb{R}^2$, we say that (Γ, Λ) is a *Heisenberg uniqueness pair* provided that

$$\forall \mu \in \mathrm{AC}(\Gamma): \quad \hat{\mu}|_\Lambda = 0 \implies \mu = 0.$$

Heisenberg uniqueness pairs in which Γ is a straight line or the union of two parallel lines were described in [15]. Later, Blasi [3] solved particular cases of the union of three parallel lines. The ellipse case was considered independently by Lev and Sjölin in [20] and [26]; Sjölin [27] also considered the parabola. More recently, Jaming and Kellay [18] developed new tools to study Heisenberg uniqueness pairs for a variety of curves Γ, while Giri and Srivastava [11] studied four parallel lines among other things. As for higher dimensional analogues, Gröchenig and Jaming [13] connected the topic with the Cramér–Wold theorem on quadratic surfaces, while Srivastava [28] studied pairs composed of spheres and cones.

1.2. The Zariski closure

We turn to the notion of Zariski closure. Note that the Zariski topology (or hull-kernel topology) is a standard concept in algebraic geometry, in the setting of spaces of polynomials. We let $\mathrm{AC}(\Gamma, \Lambda)$ be the subspace of $\mathrm{AC}(\Gamma)$ consisting of those measures μ whose Fourier transform vanishes on Λ.

Definition 1.2.1. Let Γ be a finite disjoint union of smooth curves in $\mathbb{R}^2$, and let $\Lambda \subset \mathbb{R}^2$. With respect to $\mathrm{AC}(\Gamma)$, the *Zariski closure* of Λ is the set

$$\mathrm{zclos}_\Gamma(\Lambda) := \{\xi \in \mathbb{R}^2 : [\forall \mu \in \mathrm{AC}(\Gamma, \Lambda) : \hat{\mu}(\xi) = 0]\}.$$

Less formally, the Zariski closure (or hull) is the set where the Fourier transform of a measure $\mu \in \mathrm{AC}(\Gamma)$ must vanish given that it already vanishes on Λ. Now, as the Fourier image of $\mathrm{AC}(\Gamma)$ does not form an algebra with respect to pointwise multiplication of functions, we cannot expect the Zariski closure to correspond to a topology. This means in particular that the intersection of two Zariski closures need not be a closure itself. It is easy to see that the closure operation is idempotent, however: $\mathrm{zclos}_\Gamma^2 = \mathrm{zclos}_\Gamma$. In terms of Zariski closure, we may express the uniqueness pair property conveniently: (Γ, Λ) *is a Heisenberg uniqueness pair if and only if*

$$\mathrm{zclos}_\Gamma(\Lambda) = \mathbb{R}^2.$$

1.3. The Klein–Gordon equation

In natural units, the Klein–Gordon equation in one spatial dimension reads

$$\partial_t^2 u - \partial_x^2 u + M^2 u = 0.$$

In terms of the (preferred) coordinates

$$\xi_1 := t + x, \qquad \xi_2 := t - x,$$

the Klein–Gordon equation becomes

$$\partial_{\xi_1} \partial_{\xi_2} u + \frac{M^2}{4} u = 0. \tag{1.3.1}$$

Remark 1.3.1. Since $t^2 - x^2 = \xi_1 \xi_2$, the *time-like vectors* (those vectors $(t, x) \in \mathbb{R}^2$ with $t^2 - x^2 > 0$) correspond to the union of the first quadrant $\xi_1, \xi_2 > 0$ and the third quadrant $\xi_1, \xi_2 < 0$ in the (ξ, ξ_2)-plane. Likewise, the *space-like vectors* correspond to the union of the second quadrant $\xi_1 > 0, \xi_2 < 0$ and the fourth quadrant $\xi_1 < 0, \xi_2 > 0$.

1.4. Fourier-analytic treatment of the Klein–Gordon equation

We will not need to talk about the time and space coordinates (t, x) as such. So, e.g., we are free to use the notation $x = (x_1, x_2)$ for the Fourier dual coordinate to $\xi = (\xi_1, \xi_2)$.

Let $\mathcal{M}(\mathbb{R}^2)$ denote the Banach space of all finite complex-valued Borel measures in $\mathbb{R}^2$. We suppose that u is the Fourier transform of a $\mu \in \mathcal{M}(\mathbb{R}^2)$:

$$u(\xi) = \hat{\mu}(\xi) := \int_{\mathbb{R}^2} \mathrm{e}^{\mathrm{i}\pi \langle x, \xi \rangle} \, \mathrm{d}\mu(x), \qquad \xi \in \mathbb{R}^2. \tag{1.4.1}$$

The assumption that u solves the Klein–Gordon equation (1.3.1) would require that

$$\left(x_1 x_2 - \frac{M^2}{4\pi^2} \right) \mathrm{d}\mu(x) = 0$$

as a measure on $\mathbb{R}^2$, which we see is the same as a requirement on the support set of μ:

$$\operatorname{supp} \mu \subset \Gamma_M := \left\{ x \in \mathbb{R}^2 : \, x_1 x_2 = \frac{M^2}{4\pi^2} \right\}. \tag{1.4.2}$$

The set Γ_M is a hyperbola. We may use the x_1-axis to supply a global coordinate for Γ_M, and define a complex-valued finite Borel measure $\boldsymbol{\pi}_1 \mu$ on $\mathbb{R}$ by setting

$$\boldsymbol{\pi}_1 \mu(E) = \int_E \mathrm{d}\boldsymbol{\pi} \mu(x_1) := \mu(E \times \mathbb{R}) = \int_{E \times \mathbb{R}} \mathrm{d}\mu(x). \tag{1.4.3}$$

We shall at times refer to $\boldsymbol{\pi}_1 \mu$ as the *compression* of μ to the x_1-axis. It is easy to see that μ may be recovered from $\boldsymbol{\pi}_1 \mu$; indeed,

$$u(\xi) = \hat{\mu}(\xi) = \int_{\mathbb{R}^\times} \mathrm{e}^{\mathrm{i}\pi [\xi_1 t + M^2 \xi_2 / (4\pi^2 t)]} \, \mathrm{d}\boldsymbol{\pi}_1 \mu(t), \qquad \xi \in \mathbb{R}^2. \tag{1.4.4}$$

Here, we use the standard notational convention $\mathbb{R}^\times := \mathbb{R} \setminus \{0\}$. We note that μ is absolutely continuous with respect to arc length measure on Γ_M if and only if $\boldsymbol{\pi}_1 \mu$ is absolutely continuous with respect to Lebesgue length measure on $\mathbb{R}^\times$.

1.5. The lattice-cross as a uniqueness set for solutions to the Klein–Gordon equation

For positive reals α, β, let $\Lambda_{\alpha,\beta}$ denote the *lattice-cross*

$$\Lambda_{\alpha,\beta} := (\alpha\mathbb{Z} \times \{0\}) \cup (\{0\} \times \beta\mathbb{Z}), \tag{1.5.1}$$

so that the spacing along the ξ_1-axis is α, and along the ξ_2-axis it is β. In [15], we found the following.

Theorem 1.5.1 (Hedenmalm, Montes). *Fix positive reals M, α, β. Then $(\Gamma_M, \Lambda_{\alpha,\beta})$ is a Heisenberg uniqueness pair if and only if $\alpha\beta M^2 \le 4\pi^2$.*

In terms of Zariski closure, the theorem says that

$$\operatorname{zclos}_{\Gamma_M}(\Lambda_{\alpha,\beta}) = \mathbb{R}^2 \quad \text{if and only if} \quad \alpha\beta M^2 \le 4\pi^2.$$

By taking (1.4.4) into account, and by reducing the redundancy of the constants (i.e., we may without loss of generality consider $M = 2\pi$ and $\alpha = 1$ only), Theorem 1.5.1 is equivalent to the following statement: *the linear span of the functions*

$$\mathrm{e}^{\mathrm{i}\pi m t}, \quad \mathrm{e}^{-\mathrm{i}\pi\beta n/t}, \qquad m, n \in \mathbb{Z},$$

is weak-star dense in $L^\infty(\mathbb{R})$ if and only if $\beta \le 1$. Here, we supply new and unexpected insight into the theory of Heisenberg uniqueness pairs, such as a new connection with the standard Gauss map (motivated by Theorem 1.6.1), and more importantly we uncover, in the framework of Fourier analysis, profound connections between the Hilbert transform and the dynamics of transfer operators intimately related to Gauss-type maps leading up to Theorem 1.8.2.

1.6. Dynamic unique continuation from a branch of the hyperbola

Just looking at Theorem 1.5.1, one is immediately led to ask what happens if we replace the hyperbola Γ_M by one of its two branches, say

$$\Gamma_M^+ := \Gamma_M \cap (\mathbb{R}_+ \times \mathbb{R}_+) = \left\{ x \in \mathbb{R}^2 : x_1 x_2 = \frac{M^2}{4\pi^2} \text{ and } x_1 > 0 \right\}. \tag{1.6.1}$$

First, we will provide a uniqueness theorem for the branch Γ_M^+ of the hyperbola Γ_M, which turns out to be closely related to the famous Gauss–Kuz'min–Wirsing operator and the Gauss map $x \mapsto 1/x \bmod \mathbb{Z}$.

Theorem 1.6.1. *Fix positive reals α, β, M. Then $(\Gamma_M^+, \Lambda_{\alpha,\beta})$ is a Heisenberg uniqueness pair if and only if $\alpha\beta M^2 < 16\pi^2$. Moreover, in the critical case $\alpha\beta M^2 = 16\pi^2$, the space $\mathrm{AC}(\Gamma_M^+, \Lambda_{\alpha,\beta})$ is one-dimensional, spanned by the measure $\mu_0 \in \mathrm{AC}(\Gamma_M^+, \Lambda_{\alpha,\beta})$ whose x_1-compression is given by*

$$\mathrm{d}\boldsymbol{\pi}_1\mu_0(t) := \left\{ \frac{1_{[0,2/\alpha]}(t)}{2(2+\alpha t)} - \frac{1_{[2/\alpha,+\infty[}(t)}{\alpha t(2+\alpha t)} \right\} \mathrm{d}t.$$

The proof of Theorem 1.6.1 is presented in Section 6. In the same section, it is also shown that in the critical parameter regime $\alpha\beta = 16\pi^2$, the couple $(\Gamma_M^+, \Lambda_{\alpha,\beta}^\star)$ is indeed a Heisenberg uniqueness pair, where $\Lambda_{\alpha,\beta}^\star := \Lambda_{\alpha,\beta} \cup \{\xi^\star\}$, and $\xi^\star \in (\{0\} \times \mathbb{R}) \cup (\mathbb{R} \times \{0\})$ is any point off the lattice-cross $\Lambda_{\alpha,\beta}$ (see Theorem 6.1.1). The analysis of the proof of Theorem 6.1.1 involves a geometric object known as the *Nielsen spiral*.

Again, by taking (1.4.4) into account, and by reducing the redundancy of the constants (i.e., we consider $M = 2\pi$ and $\alpha = 1$ only), it is easy to see that Theorem 1.6.1 entails the following assertion: *the restriction to* $\mathbb{R}_+$ *of the linear span of the functions* $e^{i\pi m t}$, $e^{-i\pi\beta n/t}$, $m, n \in \mathbb{Z}$, *is weak-star dense in* $L^\infty(\mathbb{R}_+)$ *if and only if* $\beta < 4$. *Moreover, if* $\beta = 4$ *the weak-star closure of this linear span has codimension one in* $L^\infty(\mathbb{R}_+)$.

Theorem 1.6.1 has the following consequence in terms of unique continuation from the branch Γ_M^+, or the complementary branch $\Gamma_M^- := \Gamma_M \setminus \Gamma_M^+$, to the entire hyperbola Γ_M.

Corollary 1.6.2. *Fix positive reals* α, β, M. *Then* $\mu \in \mathrm{AC}(\Gamma_M, \Lambda_{\alpha,\beta})$ *is uniquely determined by its restriction to the hyperbola branch* Γ_M^- *if and only if* $\alpha\beta M^2 < 16\pi^2$. *The same holds with* Γ_M^- *replaced by* Γ_M^+.

1.7. The Zariski closures of the axes and semi-axes

We first consider the Zariski closure of the two axes $\mathbb{R} \times \{0\}$ and $\{0\} \times \mathbb{R}$ with respect to the space $\mathrm{AC}(\Gamma_M)$ of absolutely continuous measures, with respect to arc length, on the hyperbola Γ_M.

Proposition 1.7.1. *Fix a positive real* M. *If* $\mu \in \mathrm{AC}(\Gamma_M)$ *is such that* $\hat{\mu}$ *vanishes on one of the axes,* $\mathbb{R} \times \{0\}$ *or* $\{0\} \times \mathbb{R}$, *then* $\mu = 0$ *identically. In terms of Zariski closures, this means that*

$$\mathrm{zclos}_{\Gamma_M}(\mathbb{R} \times \{0\}) = \mathrm{zclos}_{\Gamma_M}(\{0\} \times \mathbb{R}) = \mathbb{R}^2.$$

The proof of Proposition 1.7.1 is supplied in Section 2.

The next proposition will show the difference between time-like and space-like quarter-planes. First, we need some notation. Let $\mathbb{R}_+ := \{t \in \mathbb{R} : t > 0\}$ and $\mathbb{R}_- := \{t \in \mathbb{R} : t < 0\}$ be the positive and negative half-lines, respectively. We write $\bar{\mathbb{R}}_+ := \{t \in \mathbb{R} : t \geq 0\}$ and $\bar{\mathbb{R}}_- := \{t \in \mathbb{R} : t \leq 0\}$ for the corresponding closed half-lines.

Proposition 1.7.2. *Fix a positive real* M. *Then the Zariski closures of each of the four semi-axes* $\mathbb{R}_+ \times \{0\}$, $\mathbb{R}_- \times \{0\}$, $\{0\} \times \mathbb{R}_+$, *and* $\{0\} \times \mathbb{R}_-$, *are as follows:*

$$\mathrm{zclos}_{\Gamma_M}(\mathbb{R}_+ \times \{0\}) = \mathrm{zclos}_{\Gamma_M}(\{0\} \times \mathbb{R}_-) = \bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_-,$$
$$\mathrm{zclos}_{\Gamma_M}(\mathbb{R}_- \times \{0\}) = \mathrm{zclos}_{\Gamma_M}(\{0\} \times \mathbb{R}_+) = \bar{\mathbb{R}}_- \times \bar{\mathbb{R}}_+.$$

The proof of Proposition 1.7.2 is also supplied in Section 2.

Remark 1.7.3. In each of the instances in Proposition 1.7.2, the Zariski closure of a semi-axis equals the topological closure of the adjacent quadrant of *space-like vectors*.

1.8. The Zariski closure of the lattice-cross restricted to a time-like or space-like quadrant

Let us write

$$\mathbb{Z}_+ := \{1, 2, 3, \ldots\}, \qquad \mathbb{Z}_- := \{-1, -2, -3, \ldots\},$$
$$\mathbb{Z}_{+,0} := \{0, 1, 2, \ldots\}, \qquad \mathbb{Z}_{-,0} := \{0, -1, -2, \ldots\}.$$

We consider the following four portions of the lattice-cross $\Lambda_{\alpha,\beta}$ given by (1.5.1):

$$\Lambda^{++}_{\alpha,\beta} := (\alpha\mathbb{Z}_{+,0} \times \{0\}) \cup (\{0\} \times \beta\mathbb{Z}_+), \qquad \Lambda^{+-}_{\alpha,\beta} := (\alpha\mathbb{Z}_{+,0} \times \{0\}) \cup (\{0\} \times \beta\mathbb{Z}_-),$$
$$\Lambda^{-+}_{\alpha,\beta} := (\alpha\mathbb{Z}_{-,0} \times \{0\}) \cup (\{0\} \times \beta\mathbb{Z}_+), \qquad \Lambda^{--}_{\alpha,\beta} := (\alpha\mathbb{Z}_{-,0} \times \{0\}) \cup (\{0\} \times \beta\mathbb{Z}_-).$$

We first calculate the Zariski closure of two of these (the first and the last), corresponding to the first and third quadrants, which are time-like.

Theorem 1.8.1 (time-like). *Fix positive reals α, β, M. Then for each $\xi^\star \in \mathbb{R}^2 \setminus \Lambda^{++}_{\alpha,\beta}$, there exists a measure $\mu \in \mathrm{AC}(\Gamma_M)$ such that $\hat{\mu} = 0$ on $\Lambda^{++}_{\alpha,\beta}$, while $\hat{\mu}(\xi^\star) \neq 0$. Moreover, the same holds with $\Lambda^{++}_{\alpha,\beta}$ replaced by $\Lambda^{--}_{\alpha,\beta}$. In terms of Zariski closures, this means*

$$\mathrm{zclos}_{\Gamma_M}(\Lambda^{++}_{\alpha,\beta}) = \Lambda^{++}_{\alpha,\beta}, \qquad \mathrm{zclos}_{\Gamma_M}(\Lambda^{--}_{\alpha,\beta}) = \Lambda^{--}_{\alpha,\beta}.$$

The proof of Theorem 1.8.1, which is presented in Section 5, requires a careful handling of the H^1-BMO duality and the explicit calculation of the Fourier transform of the unimodular function $t \mapsto \mathrm{e}^{\mathrm{i}/t}$ as a tempered distribution.

We turn to the Zariski closures of the remaining two portions of the lattice-cross. We first write down the statement in terms of weak-star closure of the linear span of a sequence of unimodular functions, and then explain what it means for the Zariski closure in the form of a corollary. This is our second main result.

Let $H^\infty_+(\mathbb{R})$ denote the weak-star closed subspace of $L^\infty(\mathbb{R})$ consisting of those functions whose Poisson extension to the upper half-plane is holomorphic.

Theorem 1.8.2. *Fix positive reals α, β. Then the functions*

$$\mathrm{e}^{\mathrm{i}\pi\alpha m t}, \quad \mathrm{e}^{-\mathrm{i}\pi\beta n/t}, \qquad m, n = 0, 1, 2, \ldots,$$

which are elements of $H^\infty_+(\mathbb{R})$, span together a weak-star dense subspace of $H^\infty_+(\mathbb{R})$ if and only if $\alpha\beta \le 1$.

A standard Möbius mapping brings the upper half-plane to the unit disk $\mathbb{D}$, and identifies $H^\infty_+(\mathbb{R})$ with $H^\infty(\mathbb{D})$, the space of all bounded holomorphic functions on $\mathbb{D}$. For this reason, Theorem 1.8.2 is equivalent to the following assertion.

Corollary 1.8.3. *Fix positive reals λ_1, λ_2. Then the linear span of the inner functions*

$$\phi_1(z)^m = \exp\left(m\lambda_1 \frac{z+1}{z-1}\right) \quad \text{and} \quad \phi_2(z)^n = \exp\left(n\lambda_2 \frac{z-1}{z+1}\right), \qquad m, n = 0, 1, 2, \ldots,$$

is weak-star dense set in $H^\infty(\mathbb{D})$ if and only if $\lambda_1\lambda_2 \le \pi^2$.

We omit the trivial proof of the corollary.

Remark 1.8.4. Clearly, Corollary 1.8.3 supplies a complete and affirmative answer to Problems 1 and 2 in [22]. We recall a question from [22]: is the algebra generated by the two inner functions

$$\phi_1(z) = \exp\left(\lambda_1 \frac{z+1}{z-1}\right) \quad \text{and} \quad \phi_2(z) = \exp\left(\lambda_2 \frac{z-1}{z+1}\right)$$

for some $0 < \lambda_1, \lambda_2 < +\infty$ weak-star dense in $H^\infty(\mathbb{D})$ if and only if $\lambda_1\lambda_2 \le \pi^2$? The "only if" was understood already in [22]. As pointed out in [22], it is a consequence of Corollary 1.8.3 that for $\lambda_1\lambda_2 \le \pi^2$, the lattice of closed subspaces invariant with respect to multiplication by the two inner functions ϕ_1, ϕ_2 coincides with the usual shift invariant subspaces in the Hardy space $H^p(\mathbb{D})$, where $1 < p < +\infty$.

Remark 1.8.5. It is impossible to derive the assertion of Theorem 1.8.2 from Theorem 1.5.1: the former is a *much finer statement*. In Section 11, we explain how the result relies on a hitherto unknown result, presented in [16], which extends the standard ergodic theory for certain Gauss-type transformations on the interval $I_1 :=]-1, 1[$, where the novelty is that we may handle distributions where the standard theory has only measures. The relevant space of distributions is obtained as the restriction to I_1 of $L^1(\mathbb{R})$ plus $\mathbf{H}L^1(\mathbb{R})$, where $\mathbf{H}$ is the Hilbert transform (i.e., convolution with the principal value distribution $\mathrm{pv}\,\frac{1}{\pi t}$ on the line). The issue has to do with the uniqueness of the absolutely continuous invariant measure in the larger space. Thinking physically, in the larger space, we have two types of particles, localized and delocalized. The localized particles are represented by δ_ξ, whereas delocalized particles are represented by $\mathbf{H}\delta_\xi$, for some real ξ. The state space allows for scalar multiples of localized and delocalized particles, and linear combinations of them. Finally, we are looking for such localized and delocalized particles smeared out in an absolutely continuous way, and call it an *invariant state* if it is preserved under the corresponding Gauss-type map. This generalizes the notion of absolutely continuous invariant measure which is standard in ergodic theory, and since uniqueness issues for the invariant measure translate to ergodic properties, we are left with a far-reaching generalization of ergodic theory. We have not been able to find any appropriate references for similar considerations in the literature.

All the effort in [16] is devoted to the "if" part of Theorem 1.8.2. On the other hand, the "only if" part is much simpler, as for instance the work in [4] shows that if $\alpha\beta > 1$, the weak-star closure of the linear span in question has infinite codimension in $H^\infty_+(\mathbb{R})$.

Theorem 1.8.2 can be restated in terms of uniqueness properties of solutions to the Klein–Gordon equation. Note that in the statement below, the pair $(\Lambda^{+-}_{\alpha,\beta}, \bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_-)$ can be replaced by $(\Lambda^{-+}_{\alpha,\beta}, \bar{\mathbb{R}}_- \times \bar{\mathbb{R}}_+)$ without affecting the validity of the result.

Corollary 1.8.6. *Fix positive reals α, β, M with $\alpha\beta M^2 \le 4\pi^2$. Suppose that $u = \hat{\mu}$ solves the Klein–Gordon equation (1.3.1), where μ is a finite complex Borel measure on $\mathbb{R}^2$, which is assumed to be absolutely continuous with respect to one-dimensional Hausdorff measure. Then the values of u on the space-like quarter-plane $\bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_-$ are determined by the values of u on the set $\Lambda^{+-}_{\alpha,\beta}$, which is the portion of the lattice-cross in the given quarter-plane. This does not hold for $\alpha\beta M^2 > 4\pi^2$.*

This formulation is actually a consequence of the Zariski closure result of Corollary 1.8.7 below, so we refer to the explanatory remarks that follow it.

Corollary 1.8.7 (space-like). *Fix positive reals* α, β, M. *The following assertions are equivalent:*

(i) $\mathrm{zclos}_{\Gamma_M}(\Lambda^{+-}_{\alpha,\beta}) = \bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_-$,
(ii) $\mathrm{zclos}_{\Gamma_M}(\Lambda^{-+}_{\alpha,\beta}) = \bar{\mathbb{R}}_- \times \bar{\mathbb{R}}_+$,
(iii) $\alpha\beta M^2 \le 4\pi^2$.

Here, the main part of the equivalence (i)⇔(iii) is the implication (iii)⇒(i′), where

(i′) $\mathrm{zclos}_{\Gamma_M}(\Lambda^{+-}_{\alpha,\beta}) \supset \bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_-$.

The latter implication can be understood in the following terms. Under the density condition (iii), any measure $\mu \in \mathrm{AC}(\Gamma_M)$ whose Fourier transform $\hat{\mu}$ vanishes on $\Lambda^{+-}_{\alpha,\beta}$ has the property that $\hat{\mu}$ actually vanishes on the entire space-like adjacent quarter-plane $\bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_-$. This assertion is seen to be equipotent with Theorem 1.8.2, after a scaling argument which permits us to assume that $M := 2\pi$. Finally, to obtain the equality (i) from the inclusion (i′) which results from Theorem 1.8.2, we may use e.g. Proposition 1.7.2. The remaining equivalence (ii)⇔(iii) is, by a symmetry argument, the same as (i)⇔(iii).

Remark 1.8.8. Let us now explain how Theorem 1.5.1 is an immediate consequence of the much deeper result of Corollary 1.8.7. First, an elementary argument (see [15], [4]) shows that $\mathrm{zclos}_{\Gamma_M}(\Lambda_{\alpha,\beta}) \ne \mathbb{R}^2$ for $\alpha\beta M^2 > 4\pi^2$, so that we just need to obtain the implication

$$\alpha\beta M^2 \le 4\pi^2 \implies \mathrm{zclos}_{\Gamma_M}(\Lambda_{\alpha,\beta}) = \mathbb{R}^2.$$

In view of Theorem 1.8.2,

$$\alpha\beta M^2 \le 4\pi^2$$
$$\implies \mathrm{zclos}_{\Gamma_M}(\Lambda_{\alpha,\beta}) = \mathrm{zclos}_{\Gamma_M}(\Lambda^{+-}_{\alpha,\beta} \cup \Lambda^{-+}_{\alpha,\beta}) \supset (\bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_-) \cup (\bar{\mathbb{R}}_- \times \bar{\mathbb{R}}_+) \supset \mathbb{R} \times \{0\},$$

and Theorem 1.5.1 becomes a consequence of Proposition 1.7.1 together with the idempotent property $\mathrm{zclos}^2_\Gamma = \mathrm{zclos}_\Gamma$.

2. The Zariski closures of the axes or semi-axes

2.1. *The standard Hardy spaces* $H^p_+(\mathbb{R})$

The Hardy space $H^\infty_+(\mathbb{R})$ consists of all functions $f \in L^\infty(\mathbb{R})$ whose Poisson extension to the upper half-plane

$$\mathbb{C}_+ := \{z \in \mathbb{C} : \mathrm{Im}\, z > 0\}$$

is holomorphic. Here, the *Poisson extension* of f is given by the expression

$$f(z) := \frac{\mathrm{Im}\, z}{\pi} \int_{\mathbb{R}} \frac{f(t)}{|z-t|^2}\, \mathrm{d}t, \qquad z \in \mathbb{C}_+.$$

In a similar fashion, for $1 \le p < +\infty$, we say that $f \in H^p_+(\mathbb{R})$ if $f \in L^p(\mathbb{R})$ and its Poisson extension is holomorphic in $\mathbb{C}_+$.

2.2. *The Zariski closures of the axes and semi-axes*

We now supply the proofs of Propositions 1.7.1 and 1.7.2. We should mention here that a more general version of Proposition 1.7.1 can be found in [18].

Proof of Proposition 1.7.1. By symmetry, it is enough to show that $\mathrm{zclos}_{\Gamma_M}(\mathbb{R} \times \{0\}) = \mathbb{R}^2$. More concretely, we need to show that if $\mu \in \mathrm{AC}(\Gamma_M)$ and

$$\hat{\mu}(\xi_1, 0) = 0, \qquad \xi_1 \in \mathbb{R},$$

then $\mu = 0$ as a measure. In view of (1.4.4),

$$\hat{\mu}(\xi_1, 0) = \int_{\mathbb{R}^\times} \mathrm{e}^{\mathrm{i}\pi\xi_1 t}\, \mathrm{d}\boldsymbol{\pi}_1\mu(t),$$

where $\boldsymbol{\pi}_1\mu$ is the compression of μ to the real line. The uniqueness theorem for the Fourier transform gives $\boldsymbol{\pi}_1\mu = 0$, and hence $\mu = 0$, since μ and its compression $\boldsymbol{\pi}_1\mu$ are in a one-to-one correspondence. □

Proof of Proposition 1.7.2. By symmetry, it is enough to show that

$$\mathrm{zclos}_{\Gamma_M}(\mathbb{R}_+ \times \{0\}) = \bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_-.$$

To this end, we consider a measure $\mu \in \mathrm{AC}(\Gamma_M)$ with (use (1.4.4))

$$\hat{\mu}(\xi_1, 0) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\pi\xi_1 t}\, \mathrm{d}\boldsymbol{\pi}_1\mu(t) = 0, \qquad \xi_1 \in \mathbb{R}_+.$$

This condition is equivalent to asking that $\mathrm{d}\boldsymbol{\pi}_1\mu(t) = f(t)\,\mathrm{d}t$, where $f \in H^1_+(\mathbb{R})$. It follows from standard arguments that

$$\int_{\mathbb{R}} g(t)\, \mathrm{d}\boldsymbol{\pi}_1\mu(t) = \int_{\mathbb{R}} f(t) g(t)\, \mathrm{d}t = 0$$

for all $g \in H^\infty_+(\mathbb{R})$. We observe that for $\xi_1 \ge 0$ and $\xi_2 \le 0$, the function

$$g(t) := \mathrm{e}^{\mathrm{i}\pi[\xi_1 t + M^2\xi_2/(4\pi^2 t)]}$$

is in $H^\infty_+(\mathbb{R})$, and so

$$\hat{\mu}(\xi_1, \xi_2) = \int_{\mathbb{R}^\times} \mathrm{e}^{\mathrm{i}\pi[\xi_1 t + M^2\xi_2/(4\pi^2 t)]}\, \mathrm{d}\boldsymbol{\pi}_1\mu(t) = 0, \qquad (\xi_1, \xi_2) \in \bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_-.$$

In conclusion, this argument proves the inclusion

$$\mathrm{zclos}_{\Gamma_M}(\mathbb{R}_+ \times \{0\}) \supset \bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_-.$$

To obtain equality, we need to show that if $(\xi_1, \xi_2) \in \mathbb{R}^2 \setminus (\bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_-)$, then there exists a $\mu \in \mathrm{AC}(\Gamma_M)$ with $\mathrm{d}\boldsymbol{\pi}_1\mu(t) = f(t)\,\mathrm{d}t$, where $f \in H^1_+(\mathbb{R})$, such that $\hat{\mu}(\xi_1, \xi_2) \neq 0$. But as the bounded function

$$g(t) = \mathrm{e}^{\mathrm{i}\pi[\xi_1 t + M^2\xi_2/(4\pi^2 t)]}, \qquad t \in \mathbb{R},$$

is not an element of $H^\infty_+(\mathbb{R})$, the standard Hardy space duality theory gives that

$$\sup\left\{\left|\int_{\mathbb{R}} f(t)g(t)\,\mathrm{d}t\right| : f \in \mathrm{ball}(H^1_+(\mathbb{R}))\right\} = \inf\{\|g - h\|_{L^\infty(\mathbb{R})} : h \in H^\infty_+(\mathbb{R})\} > 0.$$

In particular, there must exist an $f \in H^1_+(\mathbb{R})$ with

$$\hat{\mu}(\xi_1, \xi_2) = \int_{\mathbb{R}} f(t)g(t)\,\mathrm{d}t \neq 0. \qquad \square$$

3. Basic properties of the dynamics of Gauss-type maps on intervals

3.1. Notation for intervals

For a positive real γ, let $I_\gamma :=]-\gamma, \gamma[$ denote the corresponding symmetric open interval, and let $I^+_\gamma :=]0, \gamma[$ be the positive side of the interval I_γ. At times, we will need the half-open intervals $\tilde{I}_\gamma :=]-\gamma, \gamma]$ and $\tilde{I}^+_\gamma := [0, \gamma[$, as well as the closed intervals $\bar{I}_\gamma := [-\gamma, \gamma]$ and $\bar{I}^+_\gamma := [0, \gamma]$.

3.2. Dual action notation

For a Lebesgue measurable subset E of the real line $\mathbb{R}$, we write

$$\langle f, g\rangle_E := \int_E f(t)g(t)\,\mathrm{d}t$$

whenever $fg \in L^1(E)$. This will be of interest mainly when E is an open interval, and in this case, we use the same notation to describe the dual action of a distribution on a test function.

3.3. Gauss-type maps on intervals

For background material in ergodic theory, we refer to the book [5].

For $x \in \mathbb{R}$, let $\{x\}_1$ denote the *fractional part* of x, that is, the unique number in $\tilde{I}^+_1 = [0, 1[$ with $x - \{x\}_1 \in \mathbb{Z}$. Likewise, we let $\{x\}_2$ denote the *even-fractional part* of x, by which we mean the unique number in $\tilde{I}_1 =]-1, 1]$ with $x - \{x\}_2 \in 2\mathbb{Z}$. We will be interested in the Gauss-type maps $\sigma_\gamma : \tilde{I}^+_1 \to \tilde{I}^+_1$ and $\tau_\beta : \tilde{I}_1 \to \tilde{I}_1$ given by

$$\sigma_\gamma(x) := \{\gamma/x\}_1 \quad \text{and} \quad \tau_\beta(x) := \{-\beta/x\}_2.$$

Here, β, γ are reals with $0 < \beta, \gamma \leq 1$. Then σ_1 is the classical Gauss map of the unit interval I^+_1.

3.4. Transfer, subtransfer, and compressed Koopman operators

Fix two reals β, γ with $0 < \beta, \gamma \le 1$. Let $\mathbf{K}_\gamma : L^\infty(I_1^+) \to L^\infty(I_1^+)$ and $\mathbf{L}_\beta : L^\infty(I_1) \to L^\infty(I_1)$ and consider the *compressed Koopman operators* (or *sub-Koopman operators*)

$$\mathbf{K}_\gamma f(x) := 1_{I_\gamma^+}(x) f \circ \sigma_\gamma(x), \qquad \mathbf{L}_\beta f(x) := 1_{I_\beta}(x) f \circ \tau_\beta(x). \tag{3.4.1}$$

Here, as always, 1_E stands for the characteristic function of the set E, which equals 1 on E and vanishes elsewhere. The *subtransfer operators* $\mathbf{S}_\gamma : L^1(I_\gamma^+) \to L^1(I_\gamma^+)$ and $\mathbf{T}_\beta : L^1(I_1) \to L^1(I_1)$ are defined by

$$\mathbf{S}_\gamma f(x) := \sum_{j=1}^{+\infty} \frac{\gamma}{(j+x)^2} f\left(\frac{\gamma}{j+x}\right), \qquad \mathbf{T}_\beta f(x) := \sum_{j\in\mathbb{Z}^\times} \frac{\beta}{(2j+x)^2} f\left(-\frac{\beta}{2j+x}\right). \tag{3.4.2}$$

Here, we use the notation $\mathbb{Z}^\times := \mathbb{Z} \setminus \{0\}$. A standard argument shows that

$$\begin{cases} \langle \mathbf{S}_\gamma f, g\rangle_{I_1^+} = \langle f, \mathbf{K}_\gamma g\rangle_{I_1^+}, & f \in L^1(I_1^+),\ g \in L^\infty(I_1^+), \\ \langle \mathbf{T}_\beta f, g\rangle_{I_1} = \langle f, \mathbf{L}_\beta g\rangle_{I_1}, & f \in L^1(I_1),\ g \in L^\infty(I_1); \end{cases} \tag{3.4.3}$$

in other words, $\mathbf{S}_\gamma$ *is the preadjoint of* $\mathbf{K}_\gamma$*, and* $\mathbf{T}_\beta$ *is the preadjoint of* $\mathbf{L}_\beta$.

The *cone of positive functions* consists of all integrable functions f with $f \ge 0$ a.e. on the respective interval. Similarly, we say that f is *positive* if $f \ge 0$ a.e. on the given interval.

Proposition 3.4.1. *Fix* $0 < \beta, \gamma \le 1$*. The operators* $\mathbf{T}_\beta : L^1(I_1) \to L^1(I_1)$ *and* $\mathbf{S}_\gamma : L^1(I_\gamma^+) \to L^1(I_\gamma^+)$ *are both norm contractions which preserve the respective cones of positive functions. For* $\beta = \gamma = 1$*,* $\mathbf{T}_1$ *and* $\mathbf{S}_1$ *act isometrically on the positive functions. The associated adjoints* $\mathbf{L}_\beta : L^\infty(I_1) \to L^\infty(I_1)$ *and* $\mathbf{K}_\gamma : L^\infty(I_1^+) \to L^\infty(I_1^+)$ *are norm contractions as well.*

This is well-known for $\gamma = \beta = 1$ and very easy to obtain for $0 < \beta, \gamma < 1$.

3.5. An elementary observation and an estimate of the $\mathbf{T}_\beta$*-orbit of certain functions*

We begin with the following elementary observation.

Observation. The subtransfer operators $\mathbf{T}_\beta, \mathbf{S}_\gamma$, initially defined on L^1 functions, make sense for wider classes of functions. Indeed, if $f \ge 0$, then the formulae (3.4.2) make sense pointwise, with values in the extended nonnegative reals $[0, +\infty]$. More generally, if f is complex-valued, we may use the triangle inequality to dominate the convergence of $\mathbf{T}_\beta f$ by that of $\mathbf{T}_\beta|f|$. This entails that $\mathbf{T}_\beta f$ is well-defined a.e. if $\mathbf{T}_\beta|f| < +\infty$ a.e. The same holds for $\mathbf{S}_\gamma$ of course.

In view of the above observation, it is meaningful to try to control $\mathbf{T}_\beta f$ for $f \ge 0$. The following basic size estimate is useful.

Proposition 3.5.1. *Fix $0 < \beta \le 1$. If $f : I_1 \to \mathbb{R}$ is even and its restriction to I_1^+ is increasing, and if $f \ge 0$, then*

$$\beta C_0 f(0) \le \mathbf{T}_\beta f(x) - \frac{\beta}{(2-|x|)^2} f\left(\frac{\beta}{2-|x|}\right) \le \beta C_1 f\big(\tfrac{1}{2}\beta\big), \qquad x \in I_1,$$

where $C_0 := \frac{\pi^2}{6} - \frac{5}{4} = 0.3949\ldots$ and $C_1 := \frac{\pi^2}{6} - 1 = 0.6449\ldots$.

Proof. For convenience of notation, we write

$$s_j(x) := -\frac{\beta}{2j+x}, \tag{3.5.1}$$

which is an increasing function on I_1 for $j \in \mathbb{Z}^\times = \mathbb{Z} \setminus \{0\}$. We first consider the right half of the interval, i.e., $x \in I_1^+ =]0,1[$. As f is even, we see that

$$f(s_j(x)) = f\left(-\frac{\beta}{2j+x}\right) = f\left(\frac{\beta}{2j+x}\right),$$

and since f is increasing on I_1^+, we find that for integers $j \ge 1$,

$$f(0) \le f\left(\frac{\beta}{2j+1}\right) \le f(s_j(x)) = f\left(\frac{\beta}{2j+x}\right) \le f\left(\frac{\beta}{2j}\right) \le f\big(\tfrac{1}{2}\beta\big), \qquad x \in I_1^+,$$

while for $j \le -2$ we have a similar estimate:

$$\begin{aligned} f(0) \le f\left(\frac{\beta}{2|j|}\right) &\le f(s_j(x)) = f\left(\frac{\beta}{2|j|-x}\right) \le f\left(\frac{\beta}{2|j|-1}\right) \\ &\le f\big(\tfrac{1}{3}\beta\big) \le f\big(\tfrac{1}{2}\beta\big), \qquad x \in I_1^+. \end{aligned}$$

Since

$$\mathbf{T}_\beta f(x) - \frac{\beta}{(2-x)^2} f\left(\frac{\beta}{2-x}\right) = \frac{1}{\beta} \sum_{j \in \mathbb{Z}\setminus\{0,-1\}} [s_j(x)]^2 f(s_j(x)),$$

the claimed estimate follows from

$$\frac{\pi^2}{6} - \frac{5}{4} \le \frac{1}{\beta^2} \sum_{j \in \mathbb{Z}\setminus\{0,-1\}} [s_j(x)]^2 \le \frac{\pi^2}{6} - \frac{5}{4}, \qquad x \in I_1^+.$$

The remaining case when $x \in I_1^- :=]-1,0[$ is analogous. □

3.6. Symmetry preservation of the subtransfer operator $\mathbf{T}_\beta$

The fact that the action of $\mathbf{T}_\beta$ commutes with reflection in the origin will be needed. The precise formulation reads as follows. Let $\check{\mathbf{I}}$ be the antipodal operator $\check{\mathbf{I}} f(x) := f(-x)$, which is its own inverse: $\check{\mathbf{I}}^2 = \mathbf{I}$.

Proposition 3.6.1. *Fix* $0 < \beta \le 1$. *Suppose* $f : I_1 \to \mathbb{R}$ *is a function satisfying* $\mathbf{T}_\beta|f|(x) < +\infty$ *for some* $x \in I_1$. *Then*

$$\mathbf{T}_\beta f(x) = \check{\mathbf{I}}\mathbf{T}_\beta\check{\mathbf{I}} f(x).$$

Proof. We keep the notation $s_j(x) = -\beta/(2j + x)$ from the proof of Proposition 3.5.1, and note that

$$s_{-j}(-x) = -s_j(x),$$

which gives

$$\check{\mathbf{I}}\mathbf{T}_\beta\check{\mathbf{I}} f(x) = \frac{1}{\beta}\sum_{j\in\mathbb{Z}^\times}[s_{-j}(-x)]^2 f(-s_{-j}(-x)) = \frac{1}{\beta}\sum_{j\in\mathbb{Z}^\times}[s_j(x)]^2 f(s_j(x)) = \mathbf{T}_\beta f(x).$$

The assumption $\mathbf{T}_\beta|f|(x) < +\infty$ guarantees the absolute convergence of the above series. □

3.7. *Symmetry, monotonicity, convexity, and the operator* $\mathbf{T}_\beta$

We may now derive the property that $\mathbf{T}_\beta$ preserves the class of functions that are odd and increasing.

Proposition 3.7.1. *Fix* $0 < \beta \le 1$. *If* $f : I_1 \to \mathbb{R}$ *is odd and* [*strictly*] *increasing, then so is* $\mathbf{T}_\beta f : I_1 \to \mathbb{R}$.

Proof. If f is odd and increasing, then $|f|$ is even and its restriction to I_1^+ is increasing. From Proposition 3.5.1, we get $\mathbf{T}_\beta|f|(x) < +\infty$ for every $x \in I_1$, so that by Proposition 3.6.1, $\mathbf{T}_\beta f(x) = -\mathbf{T}_\beta f(-x)$, which means that $\mathbf{T}_\beta f$ is *odd*. Since

$$\mathbf{T}_\beta f(x) = \frac{1}{\beta}\sum_{j\in\mathbb{Z}^\times}[s_j(x)]^2 f(s_j(x)) = \frac{1}{\beta}\sum_{j\in\mathbb{Z}^\times} t^2 f(t)\big|_{t:=s_j(x)},$$

where $s_j(x) = -\beta/(2j+x)$ is known to be increasing on I_1 for each $j \in \mathbb{Z}^\times$, it is enough to check that $t^2 f(t)$ is increasing in $t \in I_1$, which in turn is an immediate consequence of the assumption that f is odd and increasing. The "strict" case is analogous. □

We can now derive the property that $\mathbf{T}_\beta$ preserves the class of functions that are positive, even, and convex.

Proposition 3.7.2. *Fix* $0 < \beta \le 1$. *If* $f : I_1 \to \mathbb{R}$ *is even and convex, and if* $f \ge 0$, *then so is* $\mathbf{T}_\beta f$.

Proof. From Proposition 3.5.1 we see that $0 \le \mathbf{T}_\beta f(x) < +\infty$ for each $x \in I_1$. We keep the notation $s_j(x) = -\beta/(2j + x)$. Since f is even, we know from Proposition 3.6.1 that $\mathbf{T}_\beta f$ is even as well. A direct calculation, based on $s_j'(x) = \beta^{-1}[s_j(x)]^2$, shows that

$$\frac{\mathrm{d}}{\mathrm{d}x}\{[s_j(x)]^2\, f(s_j(x))\} = \frac{1}{\beta}\big(2t^3 f(t) + t^4 f'(t)\big)\big|_{t:=s_j(x)}$$

where both sides are understood not in the pointwise sense but in the sense of distribution theory. Convexity means that the derivative is increasing, so we need to check that the left-hand side is increasing as a function of x. Now, since the function $x \mapsto s_j(x)$ is increasing on I_1 for each $j \in \mathbb{Z}^\times$, the above calculation shows that it is enough to check that the function $t \mapsto 2t^3 f(t) + t^4 f'(t)$ is increasing on I_1. By assumption, $f'(t)$ is odd and increasing, and hence $t^4 f'(t)$ is odd and increasing too. Moreover, as $f(t)$ is even and convex, f is increasing on I_1. Thus $t \mapsto t^3 f(t)$ is odd and increasing on I_1. The statement now follows from the fact that a sum of convex functions is convex as well. □

3.8. Preservation of continuous functions under $\mathbf{T}_\beta$

For γ with $0 < \gamma < +\infty$, let $C(\bar{I}_\gamma)$ denote the space of continuous functions on the compact symmetric interval $\bar{I}_\gamma = [-\gamma, \gamma]$. The following is a rather immediate observation (the proof is omitted).

Proposition 3.8.1. *Fix* $0 < \beta \le 1$. *If* $f \in C(\bar{I}_\beta)$, *then* $\mathbf{T}_\beta f \in C(\bar{I}_1)$.

Proposition 3.8.2. *Fix* $0 < \beta \le 1$. *If* $f \in C(\bar{I}_\beta)$ *is odd, then* $\mathbf{T}_\beta f(1) = \beta f(\beta)$.

Proof. By (3.4.2) and the assumption that f is odd, cancellation of all terms except for the one corresponding to index $j = -1$ gives

$$\mathbf{T}_\beta f(1) = \sum_{j\in\mathbb{Z}^\times} \frac{\beta}{(2j+1)^2} f\left(-\frac{\beta}{2j+1}\right) = \beta f(\beta). \qquad \square$$

3.9. Subinvariance of certain key functions

It is well-known that the Gauss map $\sigma_1(x) = \{1/x\}_1$ has the absolutely continuous invariant measure

$$\frac{\mathrm{d}t}{(1+t)\log 2}, \qquad t \in I_1^+,$$

normalized to be a probability measure. This suggests that we should analyze the behavior of the subtransfer operator $\mathbf{S}_\gamma$ on the function

$$\lambda_1(x) := \frac{1}{1+x}, \qquad x \in I_1^+.$$

Proposition 3.9.1. *Fix* $0 < \gamma \le 1$. *With* $\lambda_1(x) = 1/(1+x)$ *on* I_1, *for* $n = 1, 2, 3, \ldots$ *we have*

$$\mathbf{S}_\gamma^n \lambda_1(x) \le \left(\frac{2\gamma}{1+\gamma}\right)^n \lambda_1(x), \qquad x \in I_1^+.$$

Proof. We first establish the assertion for $n = 1$. It is elementary to show that for $j = 1, 2, \ldots,$

$$\frac{\gamma}{(j+x)(j+x+\gamma)} \le \frac{2\gamma}{1+\gamma}\,\frac{1}{(j+x)(j+x+1)}, \qquad x \in I_1^+,$$

and since

$$\mathbf{S}_\gamma \lambda_1(x) = \sum_{j=1}^{+\infty} \frac{\gamma}{(j+x)^2} \frac{1}{1+\frac{\gamma}{j+x}} = \sum_{j=1}^{+\infty} \frac{\gamma}{(j+x)(j+x+\gamma)},$$

the assertion for $n = 1$ now follows from the telescoping sum identity

$$\sum_{j=1}^{+\infty} \frac{1}{(j+x)(j+x+1)} = \sum_{j=1}^{+\infty} \left\{ \frac{1}{j+x} - \frac{1}{j+x+1} \right\} = \frac{1}{1+x}, \qquad x \in I_1^+.$$

Finally, the assertion for $n > 1$ follows by repeated application of the $n = 1$ case, using the fact that $\mathbf{S}_\gamma$ is positive, i.e., preserves the positive cone. □

Next, we consider the $\mathbf{T}_\beta$-iterates of the function (for $0 < \alpha \le 1$)

$$\kappa_\alpha(x) := \frac{\alpha}{\alpha^2 - x^2}, \qquad x \in I_1. \tag{3.9.1}$$

This function is not in $L^1(I_1)$, although it is in $L^{1,\infty}(I_1)$, the weak L^1-space; however, by the observation made in Subsection 3.5, we may still calculate the expression $\mathbf{T}_\beta \kappa_\alpha$ pointwise wherever $\mathbf{T}_\beta|\kappa_\alpha|(x) < +\infty$. Note that $\kappa_1(x)\,\mathrm{d}x$ is the invariant measure for the transformation $\tau_1(x) = \{-1/x\}_2$, which in terms of the transfer operator $\mathbf{T}_1$ means that $\mathbf{T}_1\kappa_1 = \kappa_1$. It is of fundamental importance in most of our considerations that this invariant measure has *infinite mass*, i.e., $\kappa_1 \notin L^1(I_1)$. The reason is that τ_1 has indifferent fixed points. The Gauss map σ_1, on the other hand, has only repelling fixed points, and an invariant measure $\lambda_1(x)\,\mathrm{d}x$ with finite mass. This is the main reason why the transfer operators $\mathbf{S}_1$ and $\mathbf{T}_1$ behave differently. We should add that control of the orbits is much more difficult and not so well understood in the case of indifferent fixed points, in contrast with the case of repelling fixed points where the theory is well developed.

Lemma 3.9.2. *Fix $0 < \beta \le 1$. For the function $\kappa_\beta(x) = \beta/(\beta^2 - x^2)$, we have*

$$\mathbf{T}_\beta\kappa_\beta(x) = \mathbf{T}_\beta|\kappa_\beta|(x) = \kappa_1(x) = \frac{1}{1-x^2}, \qquad \text{a.e. } x \in I_1,$$

For $\kappa_1(x) = (1-x^2)^{-1}$, we have

$$0 \le \mathbf{T}_\beta\kappa_1(x) \le \beta\,\kappa_1(x) = \frac{\beta}{1-x^2}, \qquad x \in I_1.$$

Proof. In view of (3.4.2), we have

$$\mathbf{T}_\beta\kappa_\alpha(x) = \sum_{j\in\mathbb{Z}^\times} \frac{\beta}{(x+2j)^2} \frac{\alpha}{\alpha^2 - [s_j(x)]^2}$$

$$= \sum_{j\in\mathbb{Z}^\times} \frac{\beta}{(x+2j)^2} \frac{\alpha}{\alpha^2 - \frac{\beta^2}{(x+2j)^2}} = \sum_{j\in\mathbb{Z}^\times} \frac{\alpha\beta}{\alpha^2(x+2j)^2 - \beta^2}, \tag{3.9.2}$$

where the series converges absolutely unless a term is undefined (as the result of division by 0). Since $s_j(x) \in I_\beta$ for $x \in I_1$, we see that each term is positive for $\alpha = \beta$, and hence

$$\mathbf{T}_\beta \kappa_\beta(x) = \mathbf{T}_\beta |\kappa_\beta|(x) = \sum_{j\in\mathbb{Z}^\times} \frac{1}{(x+2j)^2-1} = \frac{1}{2}\sum_{j\in\mathbb{Z}^\times}\left\{\frac{1}{x+2j-1} - \frac{1}{x+2j+1}\right\} = \frac{1}{1-x^2},$$

by telescoping sums, as claimed. Next, since for $0 < \beta \le 1$ and $j \in \mathbb{Z}^\times$,

$$0 \le \frac{\beta}{(x+2j)^2-\beta^2} \le \frac{\beta}{(x+2j)^2-1}, \qquad x \in I_1,$$

it follows that, by the same calculation,

$$0 \le \mathbf{T}_\beta \kappa_1(x) \le \sum_{j\in\mathbb{Z}^\times} \frac{\beta}{(x+2j)^2-1} = \frac{\beta}{1-x^2}, \qquad x \in I_1. \qquad \square$$

Remark 3.9.3. In particular, for $\beta = 1$, we have equality: $\mathbf{T}_1\kappa_1 = \kappa_1$.

We also obtain a uniform estimate of $\mathbf{T}_\beta^n \kappa_1$ for $0 < \beta < 1$ and $n = 1, 2, \ldots$.

Proposition 3.9.4. *Fix $0 < \beta < 1$. For $n = 1, 2, \ldots,$ we have*

$$\mathbf{T}_\beta^n \kappa_1(x) \le \frac{2\beta^n}{1-\beta}, \qquad x \in I_1.$$

Proof. We first establish the estimate for $n = 1$. As the function $\kappa_1(x) = (1-x^2)^{-1}$ is positive, even, and convex, Proposition 3.5.1 tells us that

$$\mathbf{T}_\beta \kappa_1(x) \le \beta C_1\kappa_1\big(\tfrac{1}{2}\beta\big) + \frac{\beta}{(2-|x|)^2}\kappa_1\left(\frac{\beta}{2-|x|}\right) \le \beta C_1 \kappa_1\big(\tfrac{1}{2}\big) + \beta\kappa_1(\beta) \le \frac{2\beta}{1-\beta}. \tag{3.9.3}$$

Here, we have used the fact that κ_1 is increasing on $I_1^+ =]0,1[$, and that $C_1\kappa_1\big(\frac{1}{2}\big) = \frac{4}{3}\big(\frac{\pi^2}{6}-1\big) \le 1$.

Next, by iteration of Lemma 3.9.2, since $\mathbf{T}_\beta$ is positive, we obtain $\mathbf{T}_\beta^{n-1}\kappa_1 \le \beta^{n-1}\kappa_1$, so that a single application of the estimate (3.9.3) gives

$$\mathbf{T}_\beta^n \kappa_1(x) = \mathbf{T}_\beta \mathbf{T}_\beta^{n-1}\kappa_1(x) \le \beta^{n-1}\mathbf{T}_\beta\kappa_1(x) \le \frac{2\beta^n}{1-\beta}, \qquad x \in I_1. \qquad \square$$

3.10. The associated transfer operators

For $0 < \beta \le 1$ and a function $f \in L^1(I_1)$, extended to vanish on $\mathbb{R} \setminus I_1$, we let $\boldsymbol{\mathcal{T}}_\beta f$ denote the function defined by

$$\boldsymbol{\mathcal{T}}_\beta f(x) := \begin{cases} \displaystyle\sum_{j\in\mathbb{Z}} \frac{\beta}{(x+2j)^2}\, f\Big(-\frac{\beta}{x+2j}\Big), & x \in I_1, \\ 0, & x \in \mathbb{R}\setminus I_1, \end{cases} \tag{3.10.1}$$

whenever the sum converges absolutely. Analogously, for $0 < \gamma \le 1$ and a function $f \in L^1(I_1^+)$, extended to vanish on $\mathbb{R} \setminus I_1^+$, we let $\boldsymbol{\mathcal{S}}_\gamma f$ denote the function defined by

$$\boldsymbol{\mathcal{S}}_\gamma f(x) := \begin{cases} \displaystyle\sum_{j=0}^{+\infty} \frac{\gamma}{(x+j)^2}\, f\Big(\frac{\gamma}{x+j}\Big), & x \in I_1^+, \\ 0, & x \in \mathbb{R}\setminus I_1^+, \end{cases} \tag{3.10.2}$$

whenever the sum converges absolutely. If we compare the definition of $\boldsymbol{\mathcal{T}}_\beta f$ with that of $\mathbf{T}_\beta f$, and the definition of $\boldsymbol{\mathcal{S}}_\gamma f$ with that of $\mathbf{S}_\gamma f$, we note that the index $j = 0$ is included in the summation this time. The operators $\boldsymbol{\mathcal{T}}_\beta$ and $\boldsymbol{\mathcal{S}}_\gamma$ are *transfer operators*.

Proposition 3.10.1. *Fix $0 < \beta \le 1$. The operator $\boldsymbol{\mathcal{T}}_\beta$ is norm contractive $L^1(I_1) \to L^1(I_1)$. Indeed,*

$$\int_{-1}^{1} |\boldsymbol{\mathcal{T}}_\beta f(x)|\,\mathrm{d}x \le \int_{-1}^{1} |f(x)|\,\mathrm{d}x, \qquad f \in L^1(I_1),$$

with equality if $f \ge 0$.

Proof. By definition, the function $\boldsymbol{\mathcal{T}}_\beta f$ vanishes off I_1. Next, by the triangle inequality and the change-of-variables formula, we have

$$\begin{aligned} \int_{-1}^{1} |\boldsymbol{\mathcal{T}}_\beta f(x)|\,\mathrm{d}x &\le \sum_{j\in\mathbb{Z}} \int_{-1}^{1} \Big| f\Big(-\frac{\beta}{x+2j}\Big)\Big| \frac{\beta\,\mathrm{d}x}{(x+2j)^2} \\ &= \int_{I_1\setminus I_\beta} |f(t)|\,\mathrm{d}t + \sum_{j\in\mathbb{Z}^\times} \int_{-\beta/(2j-1)}^{-\beta/(2j+1)} |f(t)|\,\mathrm{d}t = \int_{-1}^{1} |f(t)|\,\mathrm{d}t \end{aligned}$$

for $f \in L^1(I_1)$, understood to vanish off I_1. For $f \ge 0$, there is no loss in the triangle inequality, and we obtain equality. □

Proposition 3.10.2. *Fix $0 < \gamma \le 1$. The operator $\boldsymbol{\mathcal{S}}_\gamma$ is norm contractive $L^1(I_1^+) \to L^1(I_1^+)$. Indeed,*

$$\int_{0}^{1} |\boldsymbol{\mathcal{S}}_\gamma f(x)|\,\mathrm{d}x \le \int_{0}^{1} |f(x)|\,\mathrm{d}x, \qquad f \in L^1(I_1^+),$$

with equality if $f \ge 0$.

The proof is analogous to that of Proposition 3.10.1 and therefore omitted.

3.11. Aspects of dynamics of Gauss-type maps

We recall the interval notation from §3.1. For $0 < \beta, \gamma < 1$, the transformations $\tau_\beta(x) = \{-\beta/x\}_2$ and $\sigma_\gamma(x) = \{\gamma/x\}_1$ are rather degenerate on the sets $I_1 \setminus \bar{I}_\beta$ and $I_1^+ \setminus \bar{I}_\gamma^+$. Indeed, the set $I_1 \setminus \bar{I}_\beta$ is *invariant* for τ_β, as $\tau_\beta(I_1 \setminus \bar{I}_\beta) = I_1 \setminus \bar{I}_\beta$, and the points in $I_1 \setminus \bar{I}_\beta$ are 2-periodic:

$$\tau_\beta(\tau_\beta(x)) = x, \qquad x \in I_1 \setminus \bar{I}_\beta.$$

In the same vein, the set $I_1^+ \setminus \bar{I}_\gamma^+$ is invariant for σ_γ, and all points are 2-periodic:

$$\sigma_\gamma(\sigma_\gamma(x)) = x, \qquad x \in I_1^+ \setminus \bar{I}_\gamma^+.$$

Clearly, $I_1 \setminus \bar{I}_\beta$ acts as an attractor for τ_β, and similarly $I_1^+ \setminus \bar{I}_\gamma^+$ acts as an attractor for σ_γ. We would like to analyze the sets of points which remain outside the attractor in a given number of steps. To this end, we put, for $N = 2, 3, \ldots$,

$$\begin{aligned} \mathcal{E}_{\beta,N} &:= \{x \in \bar{I}_\beta : \tau_\beta^n(x) \in \bar{I}_\beta \text{ for } n = 1, \ldots, N-1\}, \\ \mathcal{F}_{\gamma,N} &:= \{x \in \bar{I}_\gamma^+ : \sigma_\gamma^n(x) \in \bar{I}_\gamma^+ \text{ for } n = 1, \ldots, N-1\}. \end{aligned} \tag{3.11.1}$$

where $\tau_\beta^n := \tau_\beta \circ \cdots \circ \tau_\beta$ and $\sigma_\gamma^n := \sigma_\gamma \circ \cdots \circ \sigma_\gamma$ (n-fold compositions). We also agree that $\mathcal{E}_{\beta,1} := \bar{I}_\beta$ and that $\mathcal{F}_{\gamma,1} := \bar{I}_\gamma^+$. The sets $\mathcal{E}_{\beta,N}$ and $\mathcal{F}_{\gamma,N}$ get smaller as N increases, and we form their intersections

$$\mathcal{E}_{\beta,\infty} := \bigcap_{N=1}^{+\infty} \mathcal{E}_{\beta,N}, \qquad \mathcal{F}_{\gamma,\infty} := \bigcap_{N=1}^{+\infty} \mathcal{F}_{\gamma,N}, \tag{3.11.2}$$

which are known as *wandering sets*, and consist of points whose orbits stay away from the attractor.

Proposition 3.11.1. $(0 < \beta, \gamma < 1)$ *For* $N = 1, 2, \ldots,$ *we have the estimates*

$$\int_{\mathcal{F}_{\gamma,N}} \frac{\mathrm{d}t}{1+t} \le \left(\frac{2\gamma}{1+\gamma}\right)^N \log 2 \quad \textit{and} \quad \int_{\mathcal{E}_{\beta,N}} \frac{\mathrm{d}t}{1-t^2} \le \frac{4\beta^N}{1-\beta}.$$

As a consequence, the one-dimensional Lebesgue measures of the sets $\mathcal{E}_{\beta,\infty}$ *and* $\mathcal{F}_{\gamma,\infty}$ *both vanish.*

Proof. By inspection of the definition of the Koopman operators (3.4.1), we see that a.e. on the respective interval,

$$\mathbf{L}_\beta^N 1 = 1_{\mathcal{E}_{\beta,N}}, \qquad \mathbf{K}_\gamma^N 1 = 1_{\mathcal{F}_{\beta,N}},$$

where 1 stands for the corresponding constant function. In view of the duality (3.4.3),

$$\int_{\mathcal{F}_{\beta,N}} \frac{\mathrm{d}t}{1+t} = \langle \lambda_1, \mathbf{K}_\gamma^N 1\rangle_{I_1^+} = \langle \mathbf{S}_\gamma^N \lambda_1, 1\rangle_{I_1^+} \le \left(\frac{2\gamma}{1+\gamma}\right)^N \langle \lambda_1, 1\rangle_{I_1^+} = \left(\frac{2\gamma}{1+\gamma}\right)^N \log 2$$

where $\lambda_1(x) = (1+x)^{-1}$ and the estimate comes from Proposition 3.9.1. It remains to obtain the corresponding estimate for the set $\mathcal{E}_{\beta,N}$. Let $\psi := 1_{I_\eta}\kappa_1$ for some η, $0<\eta<1$, where $\kappa_1(x) = (1-x^2)^{-1}$. Then $\psi \in L^1(I_1)$, and from the duality (3.4.3) together with Proposition 3.9.4 we obtain

$$\int_{I_\eta\cap\mathcal{E}_{\beta,N}} \frac{\mathrm{d}t}{1-t^2} = \langle \psi, \mathbf{L}_\beta^N 1\rangle_{I_1} = \langle \mathbf{T}_\beta^N \psi, 1\rangle_{I_1} \le \langle \mathbf{T}_\beta^N \kappa_1, 1\rangle_{I_1} \le \frac{2\beta^N}{1-\beta}\langle 1,1\rangle_{I_1} = \frac{4\beta^N}{1-\beta}.$$

If we let $\eta \to 1$, the remaining assertion follows by e.g. monotone convergence.

As for the sets $\mathcal{E}_{\beta,\infty}$ and $\mathcal{F}_{\gamma,\infty}$, we just need to observe that the right-hand sides converge to 0 geometrically, since $2\gamma/(1+\gamma) < 1$. □

The 2-periodicity of the points in the attractor of τ_β is reflected by the fact that the functions supported on the attractor are 2-periodic for the transfer operator $\boldsymbol{\mathcal{T}}_\beta$. Naturally, the same is true of σ_γ and $\boldsymbol{\mathcal{S}}_\gamma$. We omit the easy proof.

Proposition 3.11.2. *Fix $0<\beta,\gamma\le 1$. The operator $\boldsymbol{\mathcal{T}}_\beta$ maps $L^1(I_1\setminus I_\beta)$ contractively into itself. Likewise, $\boldsymbol{\mathcal{S}}_\gamma$ maps $L^1(I_1^+\setminus I_\gamma^+)$ contractively into itself. Moreover, $\boldsymbol{\mathcal{T}}_\beta^2 f = f$ for $f\in L^1(I_1\setminus I_\beta)$, and analogously $\boldsymbol{\mathcal{S}}_\gamma^2 f = f$ for $f\in L^1(I_1\setminus I_\gamma)$.*

We shall need the following result, which describes the interlacing of the iterates of $\boldsymbol{\mathcal{T}}_\beta$ with multiplication by characteristic functions.

Proposition 3.11.3. *Fix $0<\beta\le 1$. For $N=1,2,\dots$ and $f\in L^1(I_1)$ we have, a.e. on I_1*

$$1_{I_\beta}\boldsymbol{\mathcal{T}}_\beta^{N-1} f = \boldsymbol{\mathcal{T}}_\beta^{N-1}(1_{\mathcal{E}_{\beta,N}} f), \qquad \boldsymbol{\mathcal{T}}_\beta^{N}(1_{\mathcal{E}_{\beta,N}} f) = \mathbf{T}_\beta^N f.$$

Proof. To simplify the presentation, we replace the $L^1(I_1)$ function by a Dirac point mass $f=\delta_\xi$ at an arbitrary point $\xi\in I_1$. If we can show that the claimed equalities hold for $f=\delta_\xi$, i.e.,

$$1_{I_\beta}\boldsymbol{\mathcal{T}}_\beta^{N-1} \delta_\xi = \boldsymbol{\mathcal{T}}_\beta^{N-1}(1_{\mathcal{E}_{\beta,N}} \delta_\xi), \qquad \boldsymbol{\mathcal{T}}_\beta^{N}(1_{\mathcal{E}_{\beta,N}} \delta_\xi) = \mathbf{T}_\beta^N \delta_\xi,$$

for Lebesgue almost every $\xi\in I_1$, then they hold for every $f\in L^1(I_1)$ by "averaging". Indeed, a general $f\in L^1(I_1)$ may be written as

$$f(x) = \int_{I_1}\delta_x(t) f(t)\,\mathrm{d}t = \int_{I_1}\delta_t(x) f(t)\,\mathrm{d}t, \qquad x\in I_1, \tag{3.11.3}$$

where the integral is to be understood in the sense of distribution theory, so that, e.g.,

$$\boldsymbol{\mathcal{T}}_\beta f(x) = \int_{I_1}\boldsymbol{\mathcal{T}}_\beta\delta_t(x) f(t)\,\mathrm{d}t, \qquad x\in I_1.$$

We first focus on the claimed identity

$$1_{I_\beta}\boldsymbol{\mathcal{T}}_\beta^{N-1} \delta_\xi = \boldsymbol{\mathcal{T}}_\beta^{N-1}(1_{\mathcal{E}_{\beta,N}} \delta_\xi). \tag{3.11.4}$$

Here, we should remark that multiplication of a point mass and a characteristic function need only make sense for almost every $\xi \in I_1$. For $N = 1$, (3.11.4) holds trivially. In the following, we consider integers $N > 1$. The canonical extension of the transfer operator $\boldsymbol{\mathcal{T}}_\beta$ to such point masses reads

$$\boldsymbol{\mathcal{T}}_\beta \delta_\xi = \delta_{\tau_\beta(\xi)} = \delta_{\{-\beta/\xi\}_2}. \tag{3.11.5}$$

Note that by iteration of (3.11.5), we have

$$\boldsymbol{\mathcal{T}}_\beta^{N-1} \delta_\xi = \delta_{\tau_\beta^{N-1}(\xi)} \quad \text{for } \xi \in I_1. \tag{3.11.6}$$

By definition, we know that $\tau_\beta^{N-1}(\xi) \in \bar{I}_\beta$ for $\xi \in \mathcal{E}_{\beta,N}$, while for a.e. $\xi \in I_1 \setminus \mathcal{E}_{\beta,N}$, there exists an $n = 1, \ldots, N-1$ such that $\tau_\beta^n(\xi) \in I_1 \setminus \bar{I}_\beta$. As $J_\beta = I_1 \setminus \bar{I}_\beta$ is an attractor for τ_β, we conclude that for a.e. $\xi \in I_1 \setminus \mathcal{E}_{\beta,N}$, we have $\tau_\beta^{N-1}(\xi) \in I_1 \setminus \bar{I}_\beta$. The asserted identity (3.11.4) now follows from these observations.

We turn to the remaining assertion that

$$\boldsymbol{\mathcal{T}}_\beta^N (1_{E_{\beta,N}} f) = \mathbf{T}_\beta^N f, \qquad N = 1, 2, \ldots. \tag{3.11.7}$$

By inspection of the definition (3.4.2) of the subtransfer operator, the action of $\mathbf{T}_\beta$ lifts to a point mass at $\xi \in I_1$ for a.e. ξ in the following fashion:

$$\mathbf{T}_\beta \delta_\xi = \begin{cases} \delta_{\tau_\beta(\xi)} & \text{if } \xi \in \bar{I}_\beta, \\ 0 & \text{if } \xi \in I_1 \setminus \bar{I}_\beta, \end{cases}$$

so that by iteration, again for a.e. $\xi \in I_1$,

$$\mathbf{T}_\beta^N \delta_\xi = \begin{cases} \delta_{\tau_\beta^N(\xi)} & \text{if } \xi \in \mathcal{E}_{\beta,N}, \\ 0 & \text{if } \xi \in I_1 \setminus \mathcal{E}_{\beta,N}. \end{cases}$$

A comparison with the corresponding formula (3.11.6) shows that the identity (3.11.7) holds. □

The corresponding relations for $\mathbf{S}_\gamma$ and $\boldsymbol{\mathcal{S}}_\gamma$ read as follows.

Proposition 3.11.4. *Fix $0 < \gamma < 1$. For $N = 1, 2, \ldots$ and $f \in L^1(I_1^+)$ we have, a.e. on I_1^+,*

$$1_{I_\gamma^+} \boldsymbol{\mathcal{S}}_\gamma^{N-1} f = \boldsymbol{\mathcal{S}}_\beta^{N-1}(1_{\mathcal{F}_{\gamma,N}} f), \qquad \boldsymbol{\mathcal{S}}_\beta^N (1_{\mathcal{F}_{\gamma,N}} f) = \mathbf{S}_\gamma^N f.$$

The proof is similar to that of Proposition 3.11.3 and therefore omitted.

3.12. *Exact endomorphisms*

We need the concept of exactness. Here, we follow the abstract approach to this notion (see e.g. [21]) and say that τ_1 (and the transfer operator $\mathbf{T}_1$ as well) is *exact* if, in the a.e. sense,

$$\bigcap_{n=1}^{+\infty} \mathbf{L}_1^n L^\infty(I_1) = \{\text{constants}\}.$$

For $0 < \beta < 1$, however, τ_β has a nontrivial attractor, and the notion needs to be modified. So, for $0 < \beta < 1$, we say that τ_β (and $\mathbf{T}_\beta$) is *subexact* if, in the a.e. sense,

$$\bigcap_{n=1}^{+\infty} \mathbf{L}_\beta^n L^\infty(I_1) = \{0\}.$$

Mutatis mutandis, if we replace the triple $\mathbf{T}_\beta, \mathbf{L}_\beta, I_1$ by $\mathbf{S}_\gamma, \mathbf{K}_\gamma, I_1^+$, we also obtain the definition of exactness and subexactness for $\mathbf{S}_\gamma$ (and σ_γ).

Proposition 3.12.1. *Fix $0 < \beta, \gamma < 1$. The operators $\mathbf{T}_\beta : L^1(I_1) \to L^1(I_1)$ and $\mathbf{S}_\gamma : L^1(I_1^+) \to L^1(I_1^+)$ are subexact in the sense that*

$$\bigcap_{n=1}^{+\infty} \mathbf{L}_\beta^n L^\infty(I_1) = \{0\}, \qquad \bigcap_{n=1}^{+\infty} \mathbf{K}_\gamma^n L^\infty(I_1^+) = \{0\}.$$

Proof. By inspection of the compressed Koopman operator $\mathbf{L}_\beta^n$, an element of the intersection

$$\bigcap_{n=1}^{+\infty} \mathbf{L}_\beta^n L^\infty(I_1)$$

is a function in $L^\infty(I_1)$ which vanishes off the wandering set $\mathcal{E}_{\beta,\infty}$, but by Proposition 3.11.1, this is a null set, so the function vanishes a.e. The analogous argument applies in the case of $\mathbf{K}_\gamma$. □

Exactness in the case $\beta = \gamma = 1$ is known and can be derived from the work of Thaler [32] (see also Aaronson's book [2]):

Proposition 3.12.2. *Fix $\beta = \gamma = 1$. The operators $\mathbf{T}_1 : L^1(I_1) \to L^1(I_1)$ and $\mathbf{S}_1 : L^1(I_1^+) \to L^1(I_1^+)$ are exact in the sense that*

$$\bigcap_{n=1}^{+\infty} \mathbf{L}_1^n L^\infty(I_1) = \{\text{constants}\}, \qquad \bigcap_{n=1}^{+\infty} \mathbf{K}_1^n L^\infty(I_1^+) = \{\text{constants}\}.$$

Proof. The map τ_1 meets the conditions (1)–(4) of Thaler's paper [32, p. 69], so by [32, Theorem 1, p. 73], $\mathbf{T}_1$ is exact (or, in more standard terminology, τ_1 is exact). Let us check the conditions one by one, mutatis mutandis, as Thaler uses the interval $[0, 1]$ and not $\bar{I}_1 = [-1, 1]$ as we do.

Condition (1). The fundamental intervals are given by $B(j) := \left]\frac{1}{2j+1}, \frac{1}{2j-1}\right[$ for $j \in \mathbb{Z}^\times$ except when $j = \pm 1$, when we adjoin an end point: $B(-1) = [-1, -1/3[$ and $B(1) =]1/3, 1]$. The transformation τ_1 is of class C^2 on each fundamental interval $B(j)$, with $j \in$

$\mathbb{Z}^\times$, and has complete branches (it is "filling" in the terminology of [4]). Moreover, each fundamental interval $B(j)$ contains exactly one fixed point x_j, and $\tau_1'(x_j) > 1$, except on two fundamental intervals, $B(-1)$ and $B(1)$, where the fixed points are the boundary points 1 and -1. On each fundamental interval $B(j)$ we replace $\tau_1(x) = \{-1/x\}_2$ by the appropriate branch $\tau_{1,j}(x) = 2j - 1/x$ (this makes a difference only at the end points). The derivative at the remaining fixed points is then $\tau_{1,-1}'(-1) = \tau_{1,1}'(1) = 1$.

Condition (2). This condition is satisfied since $\tau_1'(x) = x^{-2} \geq (1-\epsilon)^{-2} > 1$ on the interval $I_{1-\epsilon}$ within each fundamental interval $B(j)$.

Condition (3). The derivative $\tau_1'(x) = x^{-2}$ is decreasing on $]1/3, 1[$ and increasing on $]-1, -1/3[$. The remaining requirements are void.

Condition (4). In each fundamental interval $B(j)$, the expression $|\tau_1''(x)|/\tau_1'(x)^2 = 2|x|$ is uniformly bounded.

We conclude from the definition of exactness in [32] that up to null sets, $\{\emptyset, I_1\}$ are the only measurable subsets of I_1 which for each $n = 1, 2, 3, \ldots$ may be written in the form $\tau_1^{-n}(E_n)$ for some measurable set $E_n \subset I_1$. This is equivalent to

$$\bigcap_{n=1}^{+\infty} \mathbf{L}_1^n L^\infty(I_1) = \{\text{constants}\}.$$

We turn to the Gauss map $\sigma_1(x) = \{1/x\}_2$, whose exactness is well-known. But we may derive it from [32, Theorem 1] as well. However, the condition (2) is not fulfilled, as $\sigma_1'(x) = -x^{-2} \leq -1$ in the interior of the fundamental intervals. But the iterate $\sigma_1 \circ \sigma_1$ is uniformly expanding with $\inf(\sigma_1 \circ \sigma_1)' > 1$, and the conditions (1)–(4) may be verified for it. So the exactness of $\sigma_1 \circ \sigma_1$ follows in the same fashion; this leads to

$$\bigcap_{n=1}^{+\infty} \mathbf{K}_1^{2n} L^\infty(I_1^+) = \{\text{constants}\}. \qquad \square$$

Remark 3.12.3. Some aspects of the work of Thaler [32] have been further developed by Melbourne and Terhesiu [23].

3.13. Asymptotical behavior of the orbits of $\mathbf{T}_\beta$ *and* $\mathbf{S}_\gamma$

We now apply the exactness obtained to show how the iterates of $\mathbf{T}_\beta$ and $\mathbf{S}_\gamma$ behave.

Proposition 3.13.1. *Fix* $0 < \beta, \gamma < 1$.

(a) *For* $f \in L^1(I_1^+)$, *we have* $\|\mathbf{S}_\gamma^n f\|_{L^1(I_1^+)} \to 0$ *as* $n \to +\infty$.
(b) *For* $f \in L^1(I_1)$, *we have* $\|\mathbf{T}_\beta^n f\|_{L^1(I_1)} \to 0$ *as* $n \to +\infty$.

Proof. This follows from Proposition 3.12.1 combined with [21, Theorem 4.3]. $\square$

Proposition 3.13.2. *Fix* $\beta = \gamma = 1$.

(a) *For* $f \in L^1(I_1^+)$ *with* $\langle f, 1\rangle_{I_1^+} = 0$, *we have* $\|\mathbf{S}_1^n f\|_{L^1(I_1^+)} \to 0$ *as* $n \to +\infty$.
(b) *For* $f \in L^1(I_1)$ *with* $\langle f, 1\rangle_{I_1} = 0$, *we have* $\|\mathbf{T}_1^n f\|_{L^1(I_1)} \to 0$ *as* $n \to +\infty$.

Proof. This follows from Proposition 3.12.2 combined with [21, Theorem 4.3]. □

There is a weak analogue of Proposition 3.13.1(b) which applies for $\beta = 1$. The proof is based on the fact that the absolutely continuous invariant measure has infinite mass.

Proposition 3.13.3. *Fix* $\beta = 1$. *For* $f \in L^1(I_1)$ *and fixed* η, $0 < \eta < 1$, *we have*

$$\lim_{n\to+\infty} \int_{-\eta}^{\eta} |\mathbf{T}_1^n f(x)|\,\mathrm{d}x = 0.$$

Proof. Since $|\mathbf{T}_1^n f| \le \mathbf{T}_1^n |f|$ pointwise, we may assume without loss of generality that $f \ge 0$. We recall the notation $\kappa_1(x) = (1 - x^2)^{-1}$, and pick a number ξ with $0 < \xi < 1$. Let g be the function

$$g(x) := \frac{\langle f, 1\rangle_{I_1}}{\langle 1_{I_\xi}\kappa_1, 1\rangle_{I_1}} 1_{I_\xi}(x)\kappa_1(x), \qquad x \in I_1.$$

Then $g \in L^1(I_1)$ and

$$\langle f - g, 1\rangle_{I_1} = 0.$$

By Proposition 3.13.2(b), we conclude that $\|\mathbf{T}_1^n(f - g)\|_{L^1(I_1)} \to 0$ as $n \to +\infty$. Moreover, by the triangle inequality,

$$\|\mathbf{T}_1^n f\|_{L^1(I_\eta)} \le \|\mathbf{T}_1^n(f - g)\|_{L^1(I_1)} + \|\mathbf{T}_1^n g\|_{L^1(I_\eta)}.$$

Since the function g is positive and

$$g(x) \le \frac{\langle f, 1\rangle_{I_1}}{\langle 1_{I_\xi}\kappa_1, 1\rangle_{I_1}} \kappa_1(x),$$

we see that

$$\|\mathbf{T}_1^n g\|_{L^1(I_\eta)} = \langle \mathbf{T}_1^n g, 1_{I_\eta}\rangle_{I_1} \le \frac{\langle f, 1\rangle_{I_1}}{\langle 1_{I_\xi}\kappa_1, 1\rangle_{I_1}} \langle \mathbf{T}_1^n \kappa_1, 1_{I_\eta}\rangle_{I_1} = \frac{\langle f, 1\rangle_{I_1}}{\langle 1_{I_\xi}\kappa_1, 1\rangle_{I_1}} \langle \kappa_1, 1_{I_\eta}\rangle_{I_1}, \tag{3.13.1}$$

because $\mathbf{T}_1\kappa_1 = \kappa_1$ (see Lemma 3.9.2). Moreover, since

$$\langle 1_{I_\xi}\kappa_1, 1\rangle_{I_1} \to +\infty \quad \text{as } \xi \to 1,$$

we may get the norm $\|\mathbf{T}_1^n g\|_{L^1(I_\eta)}$ as small as we like for fixed η by letting ξ be appropriately close to 1. This means that the right-hand side of (3.13.1) may be as close to 0 as we want, the first term by letting n be large, and the second by letting ξ be close to 1. □

4. Background material: the Hardy and BMO spaces on the line

4.1. The Hardy H^1-space: analytic and real

For a reference on the basics of Hardy spaces and BMO (bounded mean oscillation), we refer to, e.g., the monographs of Duren and Garnett [6], [10], as well as those of Stein [29], [30] and Stein and Weiss [31].

Let $H^1_+(\mathbb{R})$ and $H^1_-(\mathbb{R})$ be the subspaces of $L^1(\mathbb{R})$ consisting of those functions whose Poisson extensions to the upper half-plane $\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ are holomorphic and conjugate-holomorphic, respectively. Here, we use the term conjugate-holomorphic (or anti-holomorphic) to mean that the complex conjugate of the function in question is holomorphic.

It is well-known that any function $f \in H^1_+(\mathbb{R})$ has vanishing integral:

$$\langle f, 1\rangle_{\mathbb{R}} = \int_{\mathbb{R}} f(t)\,\mathrm{d}t = 0, \qquad f \in H^1_+(\mathbb{R}). \tag{4.1.1}$$

In other words, $H^1_+(\mathbb{C}) \subset L^1_0(\mathbb{R})$, where

$$L^1_0(\mathbb{R}) := \big\{f \in L^1(\mathbb{R}) : \langle f, 1\rangle_{\mathbb{R}} = 0\big\}. \tag{4.1.2}$$

In fact, there is a related Fourier-analytic characterization of $H^1_+(\mathbb{R})$ and $H^1_-(\mathbb{R})$: for $f \in L^1(\mathbb{R})$,

$$f \in H^1_+(\mathbb{R}) \iff \forall y \geq 0 : \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}yt} f(t)\,\mathrm{d}t = 0, \tag{4.1.3}$$

$$f \in H^1_-(\mathbb{R}) \iff \forall y \leq 0 : \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}yt} f(t)\,\mathrm{d}t = 0. \tag{4.1.4}$$

We will refer to the space

$$H^1_\circledast(\mathbb{R}) := H^1_+(\mathbb{R}) \oplus H^1_-(\mathbb{R})$$

as the *real H^1-space of the line* $\mathbb{R}$. Here, $\oplus$ means direct sum, i.e. the elements $f \in H^1_\circledast(\mathbb{R})$ are functions $f \in L^1_0(\mathbb{R})$ which may be written in the form

$$f = f_1 + f_2, \qquad \text{where} \qquad f_1 \in H^1_+(\mathbb{R}),\ f_2 \in H^1_-(\mathbb{R}), \tag{4.1.5}$$

plus the fact that $H^1_+(\mathbb{R}) \cap H^1_-(\mathbb{R}) = \{0\}$, which is a Fourier-analytic consequence of (4.1.3) and (4.1.4). Obviously, $H^1_\circledast(\mathbb{R}) \subset L^1_0(\mathbb{R})$; it is perhaps slightly less obvious that $H^1_\circledast(\mathbb{R})$ is dense in $L^1_0(\mathbb{R})$ in the norm of $L^1(\mathbb{R})$. It is clear that the decomposition (4.1.5) is unique. *We let $\mathbf{P}_+$ and $\mathbf{P}_-$ denote the projections $\mathbf{P}_+ f := f_1$ and $\mathbf{P}_- f := f_2$ in the decomposition* (4.1.5). These *Szegő projections* $\mathbf{P}_+, \mathbf{P}_-$ can of course be extended beyond the $H^1_\circledast(\mathbb{R})$ setting; more about this in the following subsection.

4.2. *The BMO space and the modified Hilbert transform*

With respect to the dual action

$$\langle f, g\rangle_{\mathbb{R}} = \int_{\mathbb{R}} f(t)g(t)\,\mathrm{d}t,$$

we may identify the dual space of $H^1_{\circledast}(\mathbb{R})$ with $\mathrm{BMO}(\mathbb{R})/\mathbb{C}$. Here, $\mathrm{BMO}(\mathbb{R})$ is the space of functions of *bounded mean oscillation*; this is the celebrated *Fefferman duality theorem* [7], [8]. We write "$\cdot/\mathbb{C}$" to express that we mod out by the constant functions. One of the main results in the theory is the theorem of Fefferman and Stein [8] which tells us that

$$\mathrm{BMO}(\mathbb{R}) = L^\infty(\mathbb{R}) + \tilde{\mathbf{H}}L^\infty(\mathbb{R}), \tag{4.2.1}$$

or, in words, a function g is in $\mathrm{BMO}(\mathbb{R})$ if and only if it may be written in the form $g = g_1 + \tilde{\mathbf{H}}g_2$, where $g_1, g_2 \in L^\infty(\mathbb{R})$. Here, $\tilde{\mathbf{H}}$ denotes the *modified Hilbert transform*, defined for $f \in L^\infty(\mathbb{R})$ by the formula

$$\begin{aligned}\tilde{\mathbf{H}}f(x) &:= \frac{1}{\pi}\mathrm{pv}\int_{\mathbb{R}} f(t)\left\{\frac{1}{x-t} + \frac{t}{1+t^2}\right\}\mathrm{d}t \\ &= \lim_{\epsilon\to 0^+}\int_{\mathbb{R}\setminus[x-\epsilon,x+\epsilon]} f(t)\left\{\frac{1}{x-t} + \frac{t}{1+t^2}\right\}\mathrm{d}t.\end{aligned} \tag{4.2.2}$$

The decomposition (4.2.1) is clearly not unique. The nonuniqueness of the decomposition is measured by

$$H^\infty_{\circledast}(\mathbb{R}) := L^\infty(\mathbb{R}) \cap \tilde{\mathbf{H}}L^\infty(\mathbb{R}), \tag{4.2.3}$$

the *real* H^∞*-space*.

We should compare the modified Hilbert transform $\tilde{\mathbf{H}}$ with the standard *Hilbert transform* $\mathbf{H}$, which acts boundedly on $L^p(\mathbb{R})$ for $1 < p < +\infty$, and maps $L^1(\mathbb{R})$ into $L^{1,\infty}(\mathbb{R})$ for $p = 1$. Here, $L^{1,\infty}(\mathbb{R})$ denotes the *weak* L^1*-space* (see Subsection 7.1 below). The Hilbert transform of a function f that is assumed to be integrable on $\mathbb{R}$ with respect to the measure $(1+t^2)^{-1/2}\,\mathrm{d}t$ is defined as the principal value integral

$$\mathbf{H}f(x) := \frac{1}{\pi}\,\mathrm{pv}\int_{\mathbb{R}} f(t)\frac{\mathrm{d}t}{x-t} = \lim_{\epsilon\to 0^+}\frac{1}{\pi}\int_{\mathbb{R}\setminus[x-\epsilon,x+\epsilon]} f(t)\,\frac{\mathrm{d}t}{x-t}. \tag{4.2.4}$$

If $f \in L^p(\mathbb{R})$, where $1 \le p < +\infty$, then both $\mathbf{H}f$ and $\tilde{\mathbf{H}}f$ are well-defined a.e., and it is easy to see that the difference $\tilde{\mathbf{H}}f - \mathbf{H}f$ *is a constant*. It is often useful to think of the natural harmonic extensions of the Hilbert transforms $\mathbf{H}f$ and $\tilde{\mathbf{H}}f$ to $\mathbb{C}_+$ given by

$$\mathbf{H}f(z) := \frac{1}{\pi}\int_{\mathbb{R}}\frac{\mathrm{Re}\,z - t}{|z-t|^2}f(t)\,\mathrm{d}t, \qquad \tilde{\mathbf{H}}f(z) := \frac{1}{\pi}\int_{\mathbb{R}}\left\{\frac{\mathrm{Re}\,z-t}{|z-t|^2} + \frac{t}{t^2+1}\right\}f(t)\,\mathrm{d}t. \tag{4.2.5}$$

So, as a matter of normalization, we have $\tilde{\mathbf{H}}f(\mathrm{i}) = 0$. This yields the value of the constant mentioned above: $\tilde{\mathbf{H}}f - \mathbf{H}f = -\mathbf{H}f(\mathrm{i})$.

Returning to the real H^1-space, we note the following characterization of the space in terms of the Hilbert transform: for $f \in L^1(\mathbb{R})$,

$$f \in H^1_{\circledast}(\mathbb{R}) \iff f \in L^1_0(\mathbb{R}) \text{ and } \mathbf{H}f \in L^1_0(\mathbb{R})$$

(see Proposition 7.1.1 later on).

The Szegő projections $\mathbf{P}_+$ and $\mathbf{P}_-$ mentioned in Subsection 4.1 are more generally defined in terms of the Hilbert transform:

$$\mathbf{P}_+ f := \tfrac{1}{2}(f + \mathrm{i}\mathbf{H}f), \qquad \mathbf{P}_- f := \tfrac{1}{2}(f - \mathrm{i}\mathbf{H}f). \tag{4.2.6}$$

In a similar manner, for $f \in L^\infty(\mathbb{R})$, based on the modified Hilbert transform $\tilde{\mathbf{H}}$ we may define the corresponding modified Szegő projections (which are actually projections modulo the constant functions)

$$\tilde{\mathbf{P}}_+ f := \tfrac{1}{2}(f + \mathrm{i}\tilde{\mathbf{H}}f), \qquad \tilde{\mathbf{P}}_- f := \tfrac{1}{2}(f - \mathrm{i}\tilde{\mathbf{H}}f), \tag{4.2.7}$$

so that, by definition, $f = \tilde{\mathbf{P}}_+ f + \tilde{\mathbf{P}}_- f$. If we are given two functions $f \in H^1_{\circledast}(\mathbb{R})$ and $g \in L^\infty(\mathbb{R})$, the dual action $\langle f, g\rangle_{\mathbb{R}}$ naturally splits into holomorphic and conjugate-holomorphic parts:

$$\langle f, g\rangle_{\mathbb{R}} = \langle \mathbf{P}_+ f, \tilde{\mathbf{P}}_- g\rangle_{\mathbb{R}} + \langle \mathbf{P}_- f, \tilde{\mathbf{P}}_+ g\rangle_{\mathbb{R}}. \tag{4.2.8}$$

Modulo the constants, the space $\mathrm{BMO}(\mathbb{R})$ naturally splits into holomorphic and conjugate-holomorphic components:

$$\mathrm{BMO}(\mathbb{R})/\mathbb{C} = [\mathrm{BMOA}^+(\mathbb{R})/\mathbb{C}] \oplus [\mathrm{BMOA}^-(\mathbb{R})/\mathbb{C}]. \tag{4.2.9}$$

Here $\mathrm{BMOA}^+(\mathbb{R})$ and $\mathrm{BMOA}^-(\mathbb{R})$ are the subspaces of $\mathrm{BMO}(\mathbb{R})$ consisting of functions with Poisson extensions to $\mathbb{C}_+$ that are holomorphic and conjugate-holomorphic, respectively.

The operator $\tilde{\mathbf{H}}$ also makes sense on functions from $\mathrm{BMO}(\mathbb{R})$. It is then natural to ask what is $\tilde{\mathbf{H}}^2$:

Lemma 4.2.1. *For $f \in L^p(\mathbb{R})$, $1 < p < +\infty$, we have $\mathbf{H}^2 f = -f$. Moreover, for $f \in L^\infty(\mathbb{R})$, we have $\tilde{\mathbf{H}}^2 f = -f + c(f)$, where*

$$c(f) := \frac{1}{\pi}\int_{\mathbb{R}} \frac{f(t)}{t^2+1}\,\mathrm{d}t.$$

Proof. The assertion for $1 < p < +\infty$ is completely standard (see any textbook in harmonic analysis). We turn to $p = +\infty$. First, we observe that without loss of generality, we may assume f is real-valued. Then $2\tilde{\mathbf{P}}_+ f$ is the holomorphic function in the upper half-plane whose real part is the Poisson extension of f, and the choice of the imaginary part is fixed by the requirement $2\,\mathrm{Im}\,\tilde{\mathbf{P}}_+ f(\mathrm{i}) = \tilde{\mathbf{H}}f(\mathrm{i}) = 0$. The function

$$-2\mathrm{i}\tilde{\mathbf{P}}_+ f = \tilde{\mathbf{H}}f - \mathrm{i}f$$

extends to a holomorphic function in $\mathbb{C}_+$ with real part $\tilde{\mathbf{H}}f$. So we may identify $-f$ with $\tilde{\mathbf{H}}^2 f$ up to an additive constant. The constant is determined by the requirement that $\tilde{\mathbf{H}}^2 f(\mathrm{i}) = 0$, and so $\tilde{\mathbf{H}}^2 f(\mathrm{i}) = -f + f(\mathrm{i}) = -f + c(f)$. Here, $f(\mathrm{i})$ is understood in terms of Poisson extension. □

4.3. BMO and the Fourier transform

The Fourier transform of a function $f \in L^1(\mathbb{R})$ is given by

$$\hat{f}(x) := \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\pi x t} f(t)\,\mathrm{d}t, \tag{4.3.1}$$

and it is well understood how to extend the Fourier transform to tempered distributions (see, e.g., [17]). It is well-known how to characterize in terms of the Fourier transform the spaces $\mathrm{BMOA}^+(\mathbb{R})$ and $\mathrm{BMOA}^-(\mathbb{R})$ as subspaces of $\mathrm{BMO}(\mathbb{R})$. We state these known facts as a lemma (without proof). We recall the notation $\bar{\mathbb{R}}_+ = [0, +\infty[$ and $\bar{\mathbb{R}}_- =]-\infty, 0]$.

Lemma 4.3.1. *Suppose* $f \in \mathrm{BMO}(\mathbb{R})$. *Then* $f \in \mathrm{BMOA}^+(\mathbb{R})$ *if and only if* $\hat{f}$ *is supported on* $\bar{\mathbb{R}}_-$. *Likewise,* $f \in \mathrm{BMOA}^-(\mathbb{R})$ *if and only if* $\hat{f}$ *is supported on* $\bar{\mathbb{R}}_+$.

4.4. The BMO space of 2-periodic functions

We shall need the space

$$\mathrm{BMO}(\mathbb{R}/2\mathbb{Z}) := \{f \in \mathrm{BMO}(\mathbb{R}) : f(t+2) \equiv f(t)\},$$

that is, the BMO space of 2-periodic functions. Via the complex exponential mapping $t \mapsto \mathrm{e}^{\mathrm{i}\pi t}$ ($\mathbb{R} \to \mathbb{T}$), we identify the unit circle $\mathbb{T}$ with $\mathbb{R}/2\mathbb{Z}$, and $\mathrm{BMO}(\mathbb{R}/2\mathbb{Z})$ is then just the standard BMO space on $\mathbb{T}$. Let us write

$$\mathrm{BMOA}^+(\mathbb{R}/2\mathbb{Z}) := \mathrm{BMOA}^+(\mathbb{R}) \cap \mathrm{BMO}(\mathbb{R}/2\mathbb{Z})$$

and

$$\mathrm{BMOA}^-(\mathbb{R}/2\mathbb{Z}) := \mathrm{BMOA}^-(\mathbb{R}) \cap \mathrm{BMO}(\mathbb{R}/2\mathbb{Z})$$

for the subspaces of $\mathrm{BMO}(\mathbb{R}/2\mathbb{Z})$ that consist of functions whose Poisson extensions to the upper half-plane $\mathbb{C}_+$ are holomorphic and conjugate-holomorphic, respectively.

As L^2-integrable functions on the "circle" $\mathbb{R}/2\mathbb{Z}$, the elements of $\mathrm{BMO}(\mathbb{R}/2\mathbb{Z})$ have (a.e. convergent) Fourier series expansions. This means that the Fourier transform $\hat{f}$ of a function $f \in \mathrm{BMO}(\mathbb{R}/2\mathbb{Z})$, defined by (4.3.1) and interpreted in the sense of distribution theory, is a sum of Dirac point masses along the integers $\mathbb{Z}$. We formalize this observation as a lemma.

Lemma 4.4.1. *Suppose* $f \in \mathrm{BMO}(\mathbb{R})$. *Then* $f \in \mathrm{BMO}(\mathbb{R}/2\mathbb{Z})$ *if and only if the distribution* $\hat{f}$ *is supported on* $\mathbb{Z}$, *and at each point of* $\mathbb{Z}$, *it is a Dirac point mass.*

This result is well-known.

5. The Zariski closures of two portions of the lattice-cross

5.1. An involution and the modified Hilbert transform on BMO

For a positive real parameter β, let $\mathbf{J}_\beta^*$ be the involutive operator defined by

$$\mathbf{J}_\beta^* f(x) := f(-\beta/x), \qquad x \in \mathbb{R}^\times. \tag{5.1.1}$$

Recall the definition (4.2.2) of the modified Hilbert transform $\tilde{\mathbf{H}}$.

Lemma 5.1.1. *For $f \in \mathrm{BMO}(\mathbb{R})$ and a positive real β, we have*

$$(\mathbf{J}_\beta^* \tilde{\mathbf{H}} f)(x) = (\tilde{\mathbf{H}} \mathbf{J}_\beta^* f)(x) + c_\beta(f),$$

where

$$c_\beta(f) := \tilde{\mathbf{H}} f(\mathrm{i}\beta) = (\beta^2 - 1) \int_{\mathbb{R}} \frac{t f(t)\, \mathrm{d}t}{(1+t^2)(\beta^2+t^2)}.$$

Proof. Without loss of generality, we may assume that f is real-valued. The mapping $x \mapsto -\beta/x$ extends to a conformal automorphism of $\mathbb{C}_+$ given by $z \mapsto -\beta/z$, and the function $2\tilde{\mathbf{P}}_+ f$ is a holomorphic function in $\mathbb{C}_+$ with real part equal to the Poisson extension of f. We realize that the functions $\mathbf{J}_\beta^* \tilde{\mathbf{P}}_+ f$ and $\mathbf{J}_\beta^* \tilde{\mathbf{P}}_+ f$ differ by an imaginary constant. The result follows by taking imaginary parts and plugging in $z = \mathrm{i}$. □

5.2. The Zariski closure of the portions of the lattice-cross on the space-like cone boundary

Recall that 1_E stands for the characteristic function of the set E, which equals 1 on E and 0 off E. The Fourier transform of the function $\mathrm{e}^{\mathrm{i}/t}$ in the sense of Schwartz distributions may be known, but we have no specific reference.

Proposition 5.2.1. *In the sense of distribution theory on $\mathbb{R}$ we have*

$$\lim_{\epsilon \to 0^+} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}/t + \mathrm{i}tx - \epsilon|t|} \frac{\mathrm{d}t}{2\pi} = \delta_0(x) - 1_{\mathbb{R}_+}(x) x^{-1/2} J_1(2x^{1/2}),$$

where δ_0 is the unit Dirac point mass at 0, and J_1 denotes the standard Bessel function, so that

$$x^{-1/2} J_1(2x^{1/2}) = \sum_{j=0}^{+\infty} \frac{(-1)^j}{j!(j+1)!} x^j.$$

Proof. A direct calculation can be made on the basis of [12, formula 3.324]. A less cumbersome approach is to compute the Fourier transform of the function $H_1(x) := 1_{\mathbb{R}_+}(x) x^{-1/2} J_1(2x^{1/2})$:

$$\hat{H}_1(y) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\pi xy} H_1(x)\, \mathrm{d}x = \int_0^{+\infty} \mathrm{e}^{\mathrm{i}\pi xy} x^{-1/2} J_1(2x^{1/2})\, \mathrm{d}x = 2 \int_0^{+\infty} \mathrm{e}^{\mathrm{i}\pi yt^2} J_1(2t)\, \mathrm{d}t,$$

where the integral is absolutely convergent for $\operatorname{Im} y > 0$ and has a well-defined interpretation on $\mathbb{R}$, e.g., in terms of nontangential limits. From the standard Bessel function asymptotics, we know that

$$|H_1(x)| = \mathrm{O}(x^{-3/4}) \quad \text{as } x \to +\infty,$$

so that, in particular, $H_1 \in L^2(\mathbb{R})$. By basic Hardy space theory, the nontangential limit interpretation from the upper half-plane agrees with the standard L^2 Fourier transform on $\mathbb{R}$. By an application of [12, formula 6.631] we see that, for $\operatorname{Im} y > 0$,

$$\hat{H}_1(y) = 2\int_0^{+\infty} \mathrm{e}^{\mathrm{i}\pi y t^2} J_1(2t)\,\mathrm{d}t = \mathrm{e}^{-\mathrm{i}/(2\pi y)} M_{0,1/2}\Big(\frac{\mathrm{i}}{\pi y}\Big),$$

where the function on the right-hand side is of *Whittaker type*. In view of the integral representation of such Whittaker functions [12, formula 9.221] we find that

$$\hat{H}_1(y) = 1 - \mathrm{e}^{-\mathrm{i}/(\pi y)}, \qquad \operatorname{Im} y > 0,$$

and, in a second step, that the above identification of the Fourier transform holds in the L^2-sense a.e. on $\mathbb{R}$. Since the Fourier transform of δ_0 is the constant function 1, the assertion now follows from the Fourier inversion formula. □

Proof of Theorem 1.8.1. We obviously have the inclusions

$$\Lambda_{\alpha,\beta}^{++} \subset \operatorname{zclos}_{\Gamma_M}(\Lambda_{\alpha,\beta}^{++}), \qquad \Lambda_{\alpha,\beta}^{--} \subset \operatorname{zclos}_{\Gamma_M}(\Lambda_{\alpha,\beta}^{--}),$$

and it remains to show that the Zariski closure contains no extraneous points. We will focus on the set $\Lambda_{\alpha,\beta}^{++}$; the treatment of $\Lambda_{\alpha,\beta}^{--}$ is analogous. In view of (1.4.4) (which relates $\hat{\mu}(\xi)$ to the compressed measure $\boldsymbol{\pi}_1\mu$), given a point $\xi^\star = (\xi_1^\star, \xi_2^\star) \in \mathbb{R}^2 \setminus \Lambda_{\alpha,\beta}^{++}$, we need to find a finite complex-valued absolutely continuous Borel measure ν on $\mathbb{R}^\times$ such that

$$\int_{\mathbb{R}^\times} \mathrm{e}^{\mathrm{i}\pi[\xi_1^\star t + M^2\xi_2^\star/(4\pi^2 t)]}\,\mathrm{d}\nu(t) \neq 0,$$

while at the same time

$$\int_{\mathbb{R}^\times} \mathrm{e}^{\mathrm{i}\pi\alpha m t}\,\mathrm{d}\nu(t) = \int_{\mathbb{R}^\times} \mathrm{e}^{\mathrm{i}M^2\beta n/(4\pi t)}\,\mathrm{d}\nu(t) = 0, \qquad m, n \in \mathbb{Z}_{+,0}.$$

By a scaling argument, we may restrict our attention to the normalized case $\alpha := 1$ and $M := 2\pi$. As ν is absolutely continuous, we may write $\mathrm{d}\nu(t) := g(t)\,\mathrm{d}t$, where $g \in L^1(\mathbb{R})$. Given the above normalization, we need g to satisfy

$$\int_{\mathbb{R}^\times} \mathrm{e}^{\mathrm{i}\pi[\xi_1^\star t + \xi_2^\star/t]} g(t)\,\mathrm{d}t \neq 0, \tag{5.2.1}$$

where

$$(\xi_1^\star, \xi_2^\star) \in \mathbb{R}^2 \setminus [(\mathbb{Z}_{+,0} \times \{0\}) \cup (\{0\} \times \beta\mathbb{Z}_+)],$$

while at the same time

$$\int_{\mathbb{R}^\times} \mathrm{e}^{\mathrm{i}\pi m t} g(t)\,\mathrm{d}t = \int_{\mathbb{R}^\times} \mathrm{e}^{\mathrm{i}\pi\beta n/t} g(t)\,\mathrm{d}t = 0, \qquad m,n \in \mathbb{Z}_{+,0}. \tag{5.2.2}$$

We will try to find such a function g in the slightly smaller space $H^1_{\circledast}(\mathbb{R})$. To analyze the condition (5.2.2), we might as well study the weak-star closures in the dual space $\mathrm{BMO}(\mathbb{R})/\mathbb{C}$ of the linear spans of (i) the functions $t \mapsto \mathrm{e}^{\mathrm{i}\pi m t}$ with $m \in \mathbb{Z}_{+,0}$, and of (ii) the functions $t \mapsto \mathrm{e}^{\mathrm{i}\pi\beta n/t}$ with $n \in \mathbb{Z}_{+,0}$. In the first case, we obtain the subspace $\mathrm{BMOA}^+(\mathbb{R}/2\mathbb{Z})/\mathbb{C}$ (see Subsection 4.4 for the notation). In the second case, we obtain instead the subspace $\mathrm{BMOA}^-_{\langle\beta\rangle}(\mathbb{R})/\mathbb{C}$, where $\mathrm{BMOA}^-_{\langle\beta\rangle}(\mathbb{R}) = \mathbf{J}^*_\beta \mathrm{BMOA}^-(\mathbb{R}/2\mathbb{Z})$ and the operator $\mathbf{J}^*_\beta$ is as in (5.1.1). Now, for $g \in H^1_{\circledast}(\mathbb{R})$, (5.2.2) expresses the fact that g annihilates the sum space $\mathrm{BMOA}^+(\mathbb{R}/2\mathbb{Z}) + \mathrm{BMOA}^-_{\langle\beta\rangle}(\mathbb{R})$.

To simplify the notation, we let $F_0 \in L^\infty(\mathbb{R})$ be the function $F_0(t) := \mathrm{e}^{\mathrm{i}\pi[\xi_1^\star t + \xi_2^\star/t]}$. Then, in view of (4.2.8),

$$\langle g, F_0\rangle_{\mathbb{R}} = \langle \mathbf{P}_+ g, \tilde{\mathbf{P}}_- F_0\rangle_{\mathbb{R}} + \langle \mathbf{P}_- g, \tilde{\mathbf{P}}_+ F_0\rangle_{\mathbb{R}}.$$

It follows that if we can show that

$$\tilde{\mathbf{P}}_+ F_0 \notin \mathrm{BMOA}^+(\mathbb{R}/2\mathbb{Z}) \quad \text{or} \quad \tilde{\mathbf{P}}_- F_0 \notin \mathrm{BMOA}^-_{\langle\beta\rangle}(\mathbb{R}), \tag{5.2.3}$$

then we are done, because we are free to choose $g \in H^1_{\circledast}(\mathbb{R})$ as we like. Indeed, if $\tilde{\mathbf{P}}_+ F_0 \notin \mathrm{BMOA}^+(\mathbb{R}/2\mathbb{Z})$, then we just pick a $g \in H^1_-(\mathbb{R})$ which does not annihilate $\mathrm{BMOA}^+(\mathbb{R}/2\mathbb{Z})$, and if $\tilde{\mathbf{P}}_- F_0 \notin \mathrm{BMOA}^-_{\langle\beta\rangle}(\mathbb{R})$, then we pick a $g \in H^1_+(\mathbb{R})$ which does not annihilate $\mathrm{BMOA}^-_{\langle\beta\rangle}(\mathbb{R})$. In each case, we achieve (5.2.1). Using $\mathbf{J}^*_\beta$, we see by Lemma 5.1.1 that (5.2.3) is equivalent to

$$\tilde{\mathbf{P}}_+ F_0 \notin \mathrm{BMOA}^+(\mathbb{R}/2\mathbb{Z}) \quad \text{or} \quad \tilde{\mathbf{P}}_- \mathbf{J}^*_\beta F_0 \notin \mathrm{BMOA}^-(\mathbb{R}/2\mathbb{Z}). \tag{5.2.4}$$

Moreover, the function $F_1 := \mathbf{J}^*_\beta F_0$ is of the same general type as F_0: $F_1(t) = \mathrm{e}^{-\mathrm{i}\pi[\eta_1^\star t + \eta_2^\star/t]}$, where $\eta_1^\star := \xi_2^\star/\beta$ and $\eta_2^\star := \beta\xi_1^\star$. We can bring this one step further, and consider $F_2(t) := \mathrm{e}^{\mathrm{i}\pi[\eta_1^\star t + \eta_2^\star/t]}$ (this is just the complex conjugate of $F_1(t)$), and express the requirement (5.2.4) in the form

$$\tilde{\mathbf{P}}_+ F_0 \notin \mathrm{BMOA}^+(\mathbb{R}/2\mathbb{Z}) \quad \text{or} \quad \tilde{\mathbf{P}}_+ F_2 \notin \mathrm{BMOA}^+(\mathbb{R}/2\mathbb{Z}). \tag{5.2.5}$$

By combining Lemmas 4.3.1 and 4.4.1 with Proposition 5.2.1 in the appropriate manner, using the fact that the Bessel function J_1 is real-analytic (so that its zero set is a discrete set of points), we find that

$$\tilde{\mathbf{P}}_+ F_0 \in \mathrm{BMOA}^+(\mathbb{R}/2\mathbb{Z}) \iff \xi^\star = (\xi_1^\star, \xi_2^\star) \in (\bar{\mathbb{R}}_- \times \bar{\mathbb{R}}_+) \cup (\mathbb{Z}_+ \times \{0\}).$$

The analogous case with F_2 in place of F_0 reads

$$\tilde{\mathbf{P}}_+ F_2 \in \mathrm{BMOA}^+(\mathbb{R}/2\mathbb{Z}) \iff \xi^\star = (\xi_1^\star, \xi_2^\star) \in (\bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_-) \cup (\{0\} \times \beta\mathbb{Z}_+).$$

If we put these assertions together, it becomes clear that

$$\tilde{\mathbf{P}}_+ F_0, \tilde{\mathbf{P}}_+ F_2 \in \mathrm{BMOA}^+(\mathbb{R}/2\mathbb{Z}) \iff (\xi_1^\star, \xi_2^\star) \in (\mathbb{Z}_{+,0} \times \{0\}) \cup (\{0\} \times \beta\mathbb{Z}_+).$$

The set of $\xi^\star$ in the right-hand side expression is precisely the excluded set of points on the lattice-cross, and we conclude that (5.2.5) must hold. □

6. Dynamic unique continuation from one branch of the hyperbola to the other

6.1. Dynamic unique continuation and the critical density case

We recall the definition of the hyperbola Γ_M and its branch Γ_M^+ from the introduction (see (1.4.2) and (1.6.1)). Here, we will supply the proof of Theorem 1.6.1. As Theorem 1.6.1 is somewhat defective in the critical regime $\alpha\beta M^2 = 16\pi^2$, we may ask whether adding an additional point to the lattice-cross $\Lambda_{\alpha,\beta}$ might improve the situation. Indeed, this turns out to be the case, provided that the point we add is on the cross (but not on the lattice-cross itself, of course):

Theorem 6.1.1. *Fix $0 < \alpha, \beta, M < +\infty$. Suppose $\alpha\beta M^2 = 16\pi^2$, and pick a point $\xi^\star \in (\mathbb{R} \times \{0\}) \times (\{0\} \times \mathbb{R})$ on the cross which is not in $\Lambda_{\alpha,\beta}$. Set $\Lambda_{\alpha,\beta}^\star := \Lambda_{\alpha,\beta} \cup \{\xi^\star\}$. Then $(\Gamma_M^+, \Lambda_{\alpha,\beta}^\star)$ is a Heisenberg uniqueness pair.*

Theorem 6.1.1 has a reformulation in terms of unique continuation from Γ_M^+ to Γ_M, which we think of as an example of *dynamic unique continuation*.

Corollary 6.1.2. *Fix $0 < \alpha, \beta, M < +\infty$. Suppose $\alpha\beta M^2 = 16\pi^2$, and pick a point $\xi^\star \in (\mathbb{R} \times \{0\}) \times (\{0\} \times \mathbb{R})$ on the cross which is not in $\Lambda_{\alpha,\beta}$. Set $\Lambda_{\alpha,\beta}^\star := \Lambda_{\alpha,\beta} \cup \{\xi^\star\}$. Then any measure $\mu \in \mathrm{AC}(\Gamma_M, \Lambda_{\alpha,\beta}^\star)$ is uniquely determined by its restriction to the hyperbola branch Γ_M^+.*

We first supply the proof of Theorem 1.6.1, and then prove Theorem 6.1.1.

Proof of Theorem 1.6.1. We pick an arbitrary measure $\mu \in \mathrm{AC}(\Gamma_M, \Lambda_{\alpha,\beta})$ and form its x_1-compression $\nu := \boldsymbol{\pi}_1\mu$, which is a finite absolutely continuous complex measure on $\mathbb{R}_+$. By a scaling argument, we may assume that

$$\alpha = 2, \qquad M = 2\pi.$$

Since ν is absolutely continuous, we may write $\mathrm{d}\nu(t) = f(t)\,\mathrm{d}t$, where $f \in L^1(\mathbb{R}_+)$. We observe that the vanishing condition $\hat{\mu} = 0$ on $\Lambda_{\alpha,\beta}$ with $\alpha = 2$ and $M = 2\pi$ amounts to

$$\int_{\mathbb{R}_+} \mathrm{e}^{\mathrm{i}2\pi mt} f(t)\,\mathrm{d}t = \int_{\mathbb{R}_+} \mathrm{e}^{\mathrm{i}2\pi\gamma n/t} f(t)\,\mathrm{d}t = 0, \qquad m, n \in \mathbb{Z}, \tag{6.1.1}$$

where $\gamma := \beta/2$. It was shown in [4] that for $2 < \beta < +\infty$, there is an infinite-dimensional space of solutions f. So, in what follows, *we will restrict the parameter β to $0 < \beta \le 2$, and hence γ to $0 < \gamma \le 1$*. To complete the proof of the theorem, we need to show that

(i) for $0 < \gamma < 1$, the condition (6.1.1) entails that $f = 0$ a.e. on $\mathbb{R}_+$, whereas
(ii) for $\gamma = 1$, (6.1.1) implies that $f = C_0 f_0$ a.e. on $\mathbb{R}_+$ for some constant C_0, where f_0 is the function

$$f_0(t) := \frac{1_{[0,1]}(t)}{1+t} - \frac{1_{[1,+\infty[}(t)}{t(1+t)}. \tag{6.1.2}$$

As a first step, we rewrite (6.1.1) in the form

$$\int_{\mathbb{R}_+} \mathrm{e}^{\mathrm{i}2\pi mt} f(t)\,\mathrm{d}t = \int_{\mathbb{R}_+} \mathrm{e}^{\mathrm{i}2\pi nt} f\Big(\frac{\gamma}{t}\Big)\frac{\mathrm{d}t}{t^2} = 0, \qquad m,n \in \mathbb{Z}. \tag{6.1.3}$$

Next, for $g \in L^1(\mathbb{R}_+)$ and $m \in \mathbb{Z}$ we have

$$\begin{aligned}\int_{\mathbb{R}_+} \mathrm{e}^{\mathrm{i}2\pi mt} g(t)\,\mathrm{d}t &= \sum_{j=0}^{+\infty} \int_{[j,j+1]} \mathrm{e}^{\mathrm{i}2\pi mt} g(t)\,\mathrm{d}t \\ &= \sum_{j=0}^{+\infty} \int_{[0,1]} \mathrm{e}^{\mathrm{i}2\pi mt} g(t+j)\,\mathrm{d}t = \int_{[0,1]} \mathrm{e}^{\mathrm{i}2\pi mt} \sum_{j=0}^{+\infty} g(t+j)\,\mathrm{d}t. \end{aligned} \tag{6.1.4}$$

Together with the uniqueness theorem for Fourier series, (6.1.4) now shows that

$$\int_{\mathbb{R}_+} \mathrm{e}^{\mathrm{i}2\pi mt} g(t)\,\mathrm{d}t = 0 \;\forall m \in \mathbb{Z} \iff \sum_{j=0}^{+\infty} g(t+j) = 0 \text{ a.e. on } \mathbb{R}_+. \tag{6.1.5}$$

If we apply (6.1.5) to the two cases $g(t) = f(t)$ and $g(t) = t^{-2} f(\gamma/t)$, the conditions of (6.1.3) find an equivalent formulation:

$$\sum_{j=0}^{+\infty} f(t+j) = \sum_{j=0}^{+\infty} \frac{1}{(t+j)^2} f\Big(\frac{\gamma}{t+j}\Big) = 0 \qquad \text{a.e. on } \mathbb{R}_+. \tag{6.1.6}$$

We single out the first term in each sum, and rewrite (6.1.6) further:

$$f(t) = -\sum_{j=1}^{+\infty} f(t+j), \qquad \frac{1}{t^2} f\Big(\frac{\gamma}{t}\Big) = -\sum_{j=1}^{+\infty} \frac{1}{(t+j)^2} f\Big(\frac{\gamma}{t+j}\Big), \tag{6.1.7}$$

in both cases a.e. on $\mathbb{R}_+$. After the change of variables $t \mapsto \gamma/t$ in the second condition, (6.1.7) becomes

$$f(t) = -\sum_{j=1}^{+\infty} f(t+j), \qquad f(t) = -\sum_{j=1}^{+\infty} \frac{\gamma^2}{(\gamma+jt)^2} f\Big(\frac{\gamma t}{\gamma+jt}\Big), \tag{6.1.8}$$

again a.e. on $\mathbb{R}_+$. By combining the conditions of equality in (6.1.8), we find that

$$f(t) = \sum_{j,l=1}^{+\infty} \frac{\gamma^2}{[\gamma + l(j+t)]^2} f\Big(\frac{\gamma(t+j)}{\gamma+l(t+j)}\Big) \qquad \text{a.e. on } \mathbb{R}_+. \tag{6.1.9}$$

Now, it is easy to check that after restriction to the interval $I_1^+ =]0,1[$, condition (6.1.9) amounts to

$$f = \mathbf{S}_\gamma^2 f \quad \text{a.e. on } I_1^+, \tag{6.1.10}$$

where $\mathbf{S}_\gamma$ is the subtransfer operator given by (3.4.2). If $0 < \gamma < 1$, Proposition 3.13.1(a) tells us that $\mathbf{S}_\gamma^{2n} f \to 0$ in $L^1(I_1^+)$ as $n \to +\infty$, so the only way the equality (6.1.10) is possible is that $f = 0$ a.e. on I_1^+. But then the second equality in (6.1.8) gives $f = 0$ a.e. on $\mathbb{R} \setminus I_1^+$, and hence $f = 0$ a.e. on $\mathbb{R}_+$, as desired. This settles (i).

We turn to the remaining case $\gamma = 1$. It is well-known that the function $\lambda_1(t) = (1+t)^{-1}$ is an invariant density on I_1^+ for the Gauss map $\theta_1(t) = \{1/t\}_1$ (see §3.9). In terms of the transfer operator $\mathbf{S}_1$, this means that $\mathbf{S}_1\lambda_1 = \lambda_1$, so that $\mathbf{S}_1^2\lambda_1 = \lambda_1$ as well. Next, we consider the function

$$h := f - \frac{\langle 1, f\rangle_{I_1^+}}{\log 2}\lambda_1 \in L^1(I_1^+),$$

which by construction has $\langle h, 1\rangle_{I_1^+} = 0$ and $h = \mathbf{S}_1^2 h$. By iteration, the latter property entails that $h = \mathbf{S}_1^{2n} h$ for $n = 1, 2, 3, \ldots$, so that in view of Proposition 3.13.2(a),

$$h = \mathbf{S}_1^{2n} h \to 0 \quad \text{as } n \to +\infty,$$

where the convergence is in the norm of $L^1(I_1^+)$, which implies that $h = 0$ a.e. on I_1^+. It is now immediate that

$$f = C_0\lambda_1 \quad \text{a.e. on } I_1^+, \quad \text{where} \quad C_0 := \frac{\langle 1, f\rangle_{I_1^+}}{\log 2} \in \mathbb{C}.$$

Next, the second identity in (6.1.8) with $\gamma = 1$ tells us what f equals on the remaining set $\mathbb{R}_+ \setminus I_1^+$:

$$\begin{aligned} f(t) &= -C_0 \sum_{j=1}^{+\infty} \frac{1}{(1+jt)^2}\lambda_1\left(\frac{t}{1+jt}\right) = -C_0 \sum_{j=1}^{+\infty} \frac{1}{(1+jt)^2}\,\frac{1}{1+\frac{t}{1+jt}} \\ &= -C_0 \sum_{j=1}^{+\infty} \frac{1}{(1+jt)(1+(j+1)t)} = -\frac{C_0}{t}\sum_{j=1}^{+\infty}\left\{\frac{1}{1+jt} - \frac{1}{1+(j+1)t}\right\} \\ &= -\frac{C_0}{t(1+t)}. \end{aligned}$$

The conclusion that $f = C_0 f_0$ a.e. on $\mathbb{R}_+$ is now immediate, where f_0 is given by (6.1.2) and $C_0 \in \mathbb{C}$ is a constant. Finally, it is an exercise to verify that the function f_0 indeed satisfies (6.1.6), so that f_0 (and its constant complex multiples) meets the vanishing condition for the Fourier transform, as expressed in (6.1.1). This settles (ii). □

Proof of Theorem 6.1.1. As before, rescaling allows us to fix the parameter values:

$$\alpha = \beta = 2, \qquad M = 2\pi,$$

which corresponds to $\gamma = \beta/2 = 1$ in the preceding proof. We need to show that if $\mu \in \mathrm{AC}(\Gamma_M, \Lambda^\star_{\alpha,\beta})$, then $\mu = 0$ as a measure. Since $\Lambda^\star_{\alpha,\beta} \supset \Lambda_{\alpha,\beta}$, and we are in the critical parameter regime in terms of Theorem 1.6.1, necessarily $\mathrm{d}\boldsymbol{\pi}_1\mu(t) = C_0 f_0(t)$, where f_0 is given by (6.1.2) and C_0 is a complex constant. We recall that $\Lambda^\star_{\alpha,\beta} = \Lambda^\star_{\alpha,\beta} \cup \{\xi^\star\}$ for some $\xi^\star = (\xi^\star_1, \xi^\star_2)$ with either $\xi^\star_1 = 0$ or $\xi^\star_2 = 0$, which is not on the lattice-cross $\Lambda_{\alpha,\beta}$. By symmetry, both cases are equivalent, and we consider $\xi^\star_2 = 0$, so that $\xi^\star = (\xi^\star_1, 0)$, where $\xi^\star_1 \in \mathbb{R} \setminus \alpha\mathbb{Z} = \mathbb{R} \setminus 2\mathbb{Z}$. The Fourier transform of μ restricted to $\mathbb{R} \times \{0\}$ equals (cf. (1.4.4))

$$
\begin{aligned}
\hat{\mu}(\xi_1, 0) &= \int_{\mathbb{R}^\times} \mathrm{e}^{\mathrm{i}\pi\xi_1 t}\, \mathrm{d}\boldsymbol{\pi}_1\mu(t) = C_0 \int_{\mathbb{R}^\times} \mathrm{e}^{\mathrm{i}\pi\xi_1 t} f_0(t)\, \mathrm{d}t \\
&= C_0 \left\{ \int_{[0,1]} \mathrm{e}^{\mathrm{i}\pi\xi_1 t} \frac{\mathrm{d}t}{1+t} - \int_{[1,+\infty[} \mathrm{e}^{\mathrm{i}\pi\xi_1 t} \frac{\mathrm{d}t}{t(1+t)} \right\} \\
&= C_0 \left\{ \int_{[0,1]} \mathrm{e}^{\mathrm{i}\pi\xi_1 t} \frac{\mathrm{d}t}{1+t} - \int_{[1,+\infty[} \mathrm{e}^{\mathrm{i}\pi\xi_1 t} \left(\frac{1}{t} - \frac{1}{1+t} \right) \mathrm{d}t \right\} \\
&= C_0 \left\{ \int_{[0,+\infty[} \mathrm{e}^{\mathrm{i}\pi\xi_1 t} \frac{\mathrm{d}t}{1+t} - \int_{[1,+\infty[} \mathrm{e}^{\mathrm{i}\pi\xi_1 t} \frac{\mathrm{d}t}{t} \right\} \\
&= C_0(\mathrm{e}^{-\mathrm{i}\pi\xi_1} - 1) \int_{[1,+\infty[} \mathrm{e}^{\mathrm{i}\pi\xi_1 t} \frac{\mathrm{d}t}{t}.
\end{aligned} \tag{6.1.11}
$$

Here, in the rightmost expression, the integral should be understood as a generalized Riemann integral. Since our additional vanishing condition is $\hat{\mu}(\xi^\star_1, 0) = 0$, the above calculation (6.1.11) tells us that this is the same as

$$
C_0(\mathrm{e}^{-\mathrm{i}\pi\xi^\star_1} - 1) \int_{[1,+\infty[} \mathrm{e}^{\mathrm{i}\pi\xi^\star_1 t} \frac{\mathrm{d}t}{t} = 0.
$$

Moreover, since $\xi^\star_1$ is real but not an even integer, we know that $\mathrm{e}^{\mathrm{i}\pi\xi^\star_1} \neq 1$, and the above equation simplifies to

$$
C_0 \int_{[1,+\infty[} \mathrm{e}^{\mathrm{i}\pi\xi^\star_1 t} \frac{\mathrm{d}t}{t} = 0. \tag{6.1.12}
$$

Splitting the above generalized Riemann integral into its real and imaginary parts, we see that

$$
\int_1^{+\infty} \mathrm{e}^{\mathrm{i}\pi\xi^\star_1 t} \frac{\mathrm{d}t}{t} = \int_1^{+\infty} \cos(\pi\xi^\star_1 t) \frac{\mathrm{d}t}{t} + \mathrm{i} \int_1^{+\infty} \sin(\pi\xi^\star_1 t) \frac{\mathrm{d}t}{t}.
$$

The real and imaginary parts may be expressed in terms of the rather standard functions "si" and "ci":

$$
\begin{aligned}
\int_1^{+\infty} \cos(\pi\xi^\star_1 t) \frac{\mathrm{d}t}{t} &= \int_{\pi|\xi^\star_1|}^{+\infty} \frac{\cos y}{y}\, \mathrm{d}y = -\operatorname{ci}(\pi|\xi^\star_1|), \\
\int_1^{+\infty} \sin(\pi\xi^\star_1 t) \frac{\mathrm{d}t}{t} &= \operatorname{sgn}(\xi^\star_1) \int_{\pi|\xi^\star_1|}^{+\infty} \frac{\sin y}{y}\, \mathrm{d}y = -\operatorname{sgn}(\xi^\star_1) \operatorname{si}(\pi|\xi^\star_1|),
\end{aligned}
$$

so that

$$\int_1^{+\infty} e^{i\pi\xi_1^\star t}\,\frac{dt}{t} = -\,\mathrm{ci}(\pi|\xi_1^\star|) - i\,\mathrm{sgn}(\xi_1^\star)\,\mathrm{si}(\pi|\xi_1^\star|).$$

Here, we write $\mathrm{sgn}(x) = x/|x|$ for the standard sign function. We now observe that it is rather well-known that the parametrization

$$\mathrm{ci}(\pi x) + i\,\mathrm{si}(\pi x), \qquad 0 < x < +\infty,$$

forms the *Nielsen* (or sici) *spiral* which converges to the origin as $x \to +\infty$, and whose curvature is proportional to x (see e.g. [1]). We will only need the fact that the spiral never intersects the origin:

$$\int_{[1,+\infty[} e^{i\pi\xi_1^\star t}\,\frac{dt}{t} \neq 0. \tag{6.1.13}$$

Given that (6.1.13) holds, (6.1.12) gives $C_0 = 0$ and consequently $\mu = 0$ as a measure, and the assertion of the theorem follows.

It remains to derive (6.1.13). For positive x, we put

$$\rho(x) := \left|\int_1^{+\infty} e^{ixt}\,\frac{dt}{t}\right|^2 = \left|\int_x^{+\infty} e^{it}\,\frac{dt}{t}\right|^2 = (\mathrm{ci}(x))^2 + (\mathrm{si}(x))^2,$$

and we need only show that $\rho(x) > 0$, as the condition (6.1.13) is invariant under complex conjugation. By the fundamental theorem of calculus and standard properties of the cosine,

$$\begin{aligned}
\rho'(x) &= -\frac{2}{x}\,\mathrm{Re}\int_x^{+\infty} e^{i(t-x)}\,\frac{dt}{t} = -\frac{2}{x}\,\mathrm{Re}\int_0^{+\infty} e^{it}\,\frac{dt}{t+x}\\
&= -\frac{2}{x}\int_0^{+\infty}\cos t\,\frac{dt}{t+x} = -\frac{2}{x}\sum_{k=0}^{+\infty}\left\{\int_{2k\pi}^{(2k+1)\pi}\cos t\,\frac{dt}{t+x} + \int_{(2k+1)\pi}^{(2k+2)\pi}\cos t\,\frac{dt}{t+x}\right\}\\
&= -\frac{2}{x}\sum_{k=0}^{+\infty}\int_0^{\pi}\left(\frac{1}{t+2k\pi+x} - \frac{1}{t+(2k+1)\pi+x}\right)\cos t\,dt\\
&= -\frac{2}{x}\sum_{k=0}^{+\infty}\int_0^{\pi}\psi_k(t,x)\cos t\,dt,
\end{aligned}$$

where the function

$$\psi_k(t,x) := \frac{\pi}{(t+2k\pi+x)(t+(2k+1)\pi+x)}$$

is strictly decreasing in t. Again by standard properties of the cosine and the strict monotonicity in t of $\psi_k(t,x)$, we find that

$$\int_0^{\pi}\psi_k(t,x)\cos t\,dt = \int_0^{\pi/2}\big(\psi_k(t,x) - \psi_k(\pi-t,x)\big)\cos t\,dt > 0$$

and hence $\rho'(x) < 0$, as the cosine factor is positive. It now follows from the mean value theorem of calculus that the function ρ is strictly decreasing, and since clearly $\rho(x) \geq 0$, we must have $\rho(x) > 0$ for all $x \geq 0$. In geometric terms, the modulus of the running point of the spiral is strictly decreasing and reaches the origin only as $x \to +\infty$. □

7. The Hilbert transform on L^1 and the predual of real H^∞ on the line

7.1. The Hilbert transform on L^1

For background material on the Hilbert transform and related topics, see, e.g. the monographs [6], [10] and [29]–[31].

Let $L^{1,\infty}(\mathbb{R})$ denote the *weak L^1-space*, i.e., the space of Lebesgue measurable functions $f : \mathbb{R} \to \mathbb{C}$ such that the set

$$E_f(\lambda) := \{x \in \mathbb{R} : |f(x)| > \lambda\}, \qquad \lambda \in \bar{\mathbb{R}}_+,$$

enjoys the estimate (the absolute value of a measurable subset of $\mathbb{R}$ stands for its Lebesgue measure)

$$|E_f(\lambda)| \le C_f/\lambda, \qquad \lambda \in \mathbb{R}_+;$$

the optimal constant C_f is written $\|f\|_{L^{1,\infty}(\mathbb{R})}$; it is the $L^{1,\infty}(\mathbb{R})$-*quasinorm* of f. If we identify functions that coincide almost everywhere, then $L^{1,\infty}(\mathbb{R})$ becomes a *quasi-Banach space*. It is well-known that the Hilbert transform as given by (4.2.4) maps $\mathbf{H} : L^1(\mathbb{R}) \to L^{1,\infty}(\mathbb{R})$. Note, however, that functions in $L^{1,\infty}(\mathbb{R})$ are rather wild and, e.g., it is not immediately clear how to associate a distribution to such a function. However, there is another interpretation of the Hilbert transform as a mapping from $L^1(\mathbb{R})$ into a space of distributions on $\mathbb{R}$, and it is good to know that these interpretations of $\mathbf{H}f$ for a given $f \in L^1(\mathbb{R})$ are in a one-to-one correspondence. The weak L^1-space associated with an interval I (or a set of positive Lebesgue measure), written $L^{1,\infty}(I)$, is defined analogously.

If for the moment we use the symbol $\mathbf{F}$ to denote the Fourier transform, then the Hilbert transform is $\mathbf{H} = -\mathrm{i}\mathbf{F}^{-1}\mathbf{M}_{\mathrm{sgn}}\mathbf{F}$, where $\mathbf{M}_{\mathrm{sgn}}$ stands for multiplication by the sign function sgn. Thus, after taking the Fourier transform, the distributional interpretation of the Hilbert transform is that of multiplication by the unimodular function which takes the value $-\mathrm{i}$ on the positive half-line, and the value i on the negative half-line. The distributional interpretation can also be implemented more directly:

$$\langle \varphi, \mathbf{H}f\rangle_{\mathbb{R}} := -\langle \mathbf{H}\varphi, f\rangle_{\mathbb{R}}, \tag{7.1.1}$$

where φ is a test function with compact support, and $f \in L^1(\mathbb{R})$. Note that $\mathbf{H}\varphi$, the Hilbert transform of the test function, may be defined without recourse to principal value integrals:

$$\mathbf{H}\varphi(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{\varphi(x-t)-\varphi(x+t)}{t}\,\mathrm{d}t;$$

it is a C^∞ function on $\mathbb{R}$ with decay $\mathbf{H}\varphi(x) = \mathrm{O}(|x|^{-1})$ as $|x| \to +\infty$. As a consequence, it is clear from (7.1.1) how to extend $\mathbf{H}f$ to functions f with $(|x|+1)^{-1}f(x)$ in $L^1(\mathbb{R})$.

Our next proposition characterizes the space $H^1_{\circledast}(\mathbb{R})$. For the proof, we need the notation for the *open unit disk*:

$$\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}.$$

Proposition 7.1.1. *Suppose $f \in L^1(\mathbb{R})$. Then the following are equivalent:*

(i) $f \in H^1_{\circledast}(\mathbb{R})$.
(ii) $\mathbf{H}f \in L^1(\mathbb{R})$, *where* $\mathbf{H}f$ *is understood as a distribution on* $\mathbb{R}$.
(iii) $\mathbf{H}f \in L^1(\mathbb{R})$, *where* $\mathbf{H}f$ *is understood as an almost everywhere defined function in* $L^{1,\infty}(\mathbb{R})$.

Proof. The implications (i)⇔(ii)⇒(iii) are trivial, so we turn to (iii)⇒(i). This result, however, is the real line analogue of the result for the circle in [19, p. 87]. The transfer to the unit disk is handled by an appropriate Möbius map from $\mathbb{D}$ to $\mathbb{C}_+$. □

A first application of Proposition 7.1.1 is the following result.

Corollary 7.1.2. *Suppose $f \in L^1(\mathbb{R})$, and $\mathbf{H}f = 0$ pointwise almost everywhere on $\mathbb{R}$. Then $f = 0$ almost everywhere.*

Proof. Without loss of generality, assume f is real-valued. In view of Proposition 7.1.1, $f \in H^1_{\circledast}(\mathbb{R})$, and as a consequence, the function $F := f + \mathrm{i}\mathbf{H}f$ is in $H^1_+(\mathbb{R})$. However, on the real line, F is real-valued, so that the Poisson extension of F to $\mathbb{C}_+$ is real-valued as well. But this Poisson extension is holomorphic in $\mathbb{C}_+$, so F must be constant, and the constant is seen to be 0. □

Remark 7.1.3. We note that there are closely related theories of reflectionless measures (see, e.g., [24]) and of real outer functions [9].

7.2. *The real H^∞-space*

The *real H^∞-space* is denoted by $H^\infty_{\circledast}(\mathbb{R})$, and it consists of all functions $f \in L^\infty(\mathbb{R})$ of the form

$$f = f_1 + f_2, \qquad f_1 \in H^\infty_+(\mathbb{R}),\ f_2 \in H^\infty_-(\mathbb{R}). \tag{7.2.1}$$

Here, $H^\infty_+(\mathbb{R})$ consists of all functions in $L^\infty(\mathbb{R})$ whose Poisson extension to the upper half-plane is holomorphic, while $H^\infty_-(\mathbb{R})$ consists of all functions in $L^\infty(\mathbb{R})$ whose Poisson extension to the upper half-plane is conjugate-holomorphic (alternatively, the Poisson extension to the lower half-plane is holomorphic). The decomposition (7.2.1) is unique up to additive constants. Equipped with the natural norm, $H^\infty_{\circledast}(\mathbb{R})$ is a Banach space.

The content of the next proposition is well-known. For the convenience of the reader, we supply the simple proof.

Proposition 7.2.1. *We have the equivalence*

$$f \in H^\infty_{\circledast}(\mathbb{R}) \iff f, \tilde{\mathbf{H}}f \in L^\infty(\mathbb{R}).$$

Proof. If $f \in H^\infty_{\circledast}(\mathbb{R})$, then $f = f_1 + f_2$, where $f_1 \in H^\infty_+(\mathbb{R})$ and $f_2 \in H^\infty_-(\mathbb{R})$. Since $\tilde{\mathbf{H}}f = \mathrm{i}(f_2 - f_1) + c$, where c is the constant that makes $\tilde{\mathbf{H}}f(\mathrm{i}) = 0$, we see that $\tilde{\mathbf{H}}f \in L^\infty(\mathbb{R})$.

On the other hand, if $f, \tilde{\mathbf{H}}f \in L^\infty(\mathbb{R})$, then $f + \mathrm{i}\tilde{\mathbf{H}}f \in H^\infty_+(\mathbb{R})$ and $f - \mathrm{i}\tilde{\mathbf{H}}f \in H^\infty_-(\mathbb{R})$, so that

$$2f = (f + \mathrm{i}\tilde{\mathbf{H}}f) + (f - \mathrm{i}\tilde{\mathbf{H}}f) \in H^\infty_{\circledast}(\mathbb{R}).$$ □

7.3. The predual of real H^∞

We shall be concerned with the following space of distributions on $\mathbb{R}$:

$$\mathfrak{L}(\mathbb{R}) := L^1(\mathbb{R}) + \mathbf{H}L^1_0(\mathbb{R}),$$

which we supply with the appropriate norm

$$\|u\|_{\mathfrak{L}(\mathbb{R})} := \inf\{\|f\|_{L^1(\mathbb{R})}+\|g\|_{L^1(\mathbb{R})} : u = f+\mathbf{H}g,\ f \in L^1(\mathbb{R}),\ g \in L^1_0(\mathbb{R})\}, \quad (7.3.1)$$

which makes $\mathfrak{L}(\mathbb{R})$ a Banach space.

We recall that $L^1_0(\mathbb{R})$ is a codimension-one subspace of $L^1(\mathbb{R})$ which consists of the functions whose integral over $\mathbb{R}$ vanishes. Given $f \in L^1(\mathbb{R})$ and $g \in L^1_0(\mathbb{R})$, the action of $u := f + \mathbf{H}g$ on a test function φ is (compare with (7.1.1))

$$\langle\varphi, f + \mathbf{H}g\rangle_{\mathbb{R}} = \langle\varphi, f\rangle_{\mathbb{R}} - \langle\mathbf{H}\varphi, g\rangle_{\mathbb{R}} = \langle\varphi, f\rangle_{\mathbb{R}} - \langle\tilde{\mathbf{H}}\varphi, g\rangle_{\mathbb{R}}; \quad (7.3.2)$$

we observe that the last identity uses $\langle 1, g\rangle_{\mathbb{R}} = 0$ and the fact that the functions $\tilde{\mathbf{H}}\varphi$ and $\mathbf{H}\varphi$ differ by a constant.

Observation. In view of Proposition 7.2.1, the right-hand side of (7.3.2) makes sense for $\varphi \in H^\infty_\circledast(\mathbb{R})$. To be more precise, in accordance with (7.3.2), every $\varphi \in H^\infty_\circledast(\mathbb{R})$ defines a continuous linear functional on $\mathfrak{L}(\mathbb{R})$.

It remains to identify the dual space of $\mathfrak{L}(\mathbb{R})$ with $H^\infty_\circledast(\mathbb{R})$.

Proposition 7.3.1. *Each continuous linear functional $\mathfrak{L}(\mathbb{R}) \to \mathbb{C}$ corresponds to a function $\varphi \in H^\infty_\circledast(\mathbb{R})$ in accordance with (7.3.2). In short, the dual space of $\mathfrak{L}(\mathbb{R})$ equals $H^\infty_\circledast(\mathbb{R})$.*

Proof. A standard approximation argument involving test functions can be used to establish that $L^1(\mathbb{R})$ is a dense subspace of $\mathfrak{L}(\mathbb{R})$. As the inclusion map $L^1(\mathbb{R}) \to \mathfrak{L}(\mathbb{R})$ is continuous, it follows that every continuous linear functional $\mathfrak{L}(\mathbb{R}) \to \mathbb{C}$ restricts to a continuous linear functional $L^1(\mathbb{R})$, which by standard functional analysis corresponds to an element $\varphi \in L^\infty(\mathbb{R})$. By density and continuity, φ determines the linear functional completely. As $\varphi \in L^\infty(\mathbb{R})$, we see that $\tilde{\mathbf{H}}\varphi \in \mathrm{BMO}(\mathbb{R})$. By (7.3.2), $\tilde{\mathbf{H}}\varphi$ must give a continuous linear functional $L^1_0(\mathbb{R}) \to \mathbb{C}$. It is easy to see that this is only possible if $\tilde{\mathbf{H}}\varphi \in L^\infty(\mathbb{R})$, which completes the proof, by Proposition 7.2.1. □

The space $\mathfrak{L}(\mathbb{R})$ is a Banach space, and Proposition 7.3.1 *asserts that its dual space is* $H^\infty_\circledast(\mathbb{R})$ (the real H^∞ space). For this reason, we will refer to $\mathfrak{L}(\mathbb{R})$ as the (canonical) *predual of real H^∞*.

Remark 7.3.2. Since an L^1-function f gives rise to an absolutely continuous measure $f(t)\,\mathrm{d}t$, it is natural to think of $\mathfrak{L}(\mathbb{R})$ as embedded into the space $\mathfrak{M}(\mathbb{R}) := \mathrm{M}(\mathbb{R}) + \mathbf{H}\mathrm{M}_0(\mathbb{R})$, where $\mathrm{M}(\mathbb{R})$ denotes the space of complex-valued finite Borel measures on $\mathbb{R}$, and $\mathrm{M}_0(\mathbb{R})$ is the subspace of measures $\mu \in \mathrm{M}(\mathbb{R})$ with $\mu(\mathbb{R}) = 0$. The Hilbert transforms of singular measures noticeably differ from those of absolutely continuous measures (see [25]).

7.4. The "valeur au point" function associated with an element of the predual of real H^∞

We recall that $\mathfrak{L}(\mathbb{R})$ consists of distributions on the real line. However, the definition

$$\mathfrak{L}(\mathbb{R}) = L^1(\mathbb{R}) + \mathbf{H}L^1_0(\mathbb{R})$$

would allow us to also think of this space as a subspace of $L^{1,\infty}(\mathbb{R})$, the weak L^1-space. It is natural to wonder about the relationship between the distribution and the $L^{1,\infty}$ function. We will stick to the distribution theory definition of $\mathfrak{L}(\mathbb{R})$, and associate with a given $u \in \mathfrak{L}(\mathbb{R})$ the "valeur au point" function $\mathrm{vap}[u]$ at almost all points of the line. The precise definition of $\mathrm{vap}[u]$ is as follows.

Definition 7.4.1. For a fixed $x \in \mathbb{R}$, let $\chi = \chi_x$ be a compactly supported C^∞ function on $\mathbb{R}$ with $\chi(t) = 1$ for all t in an open neighborhood of x. Also, let

$$P_{x+\mathrm{i}\epsilon}(t) := \pi^{-1}\frac{\epsilon}{\epsilon^2 + (x-t)^2}$$

be the Poisson kernel. The *valeur au point function* associated with the distribution u on $\mathbb{R}$ is the function $\mathrm{vap}[u] = \mathrm{vap}[u\chi]$ given by

$$\mathrm{vap}[u](x) := \lim_{\epsilon\to 0^+} \langle \chi P_{x+\mathrm{i}\epsilon}, u\rangle_{\mathbb{R}}, \qquad x \in \mathbb{R}, \tag{7.4.1}$$

wherever the limit exists.

In principle, $\mathrm{vap}[u](x)$ might depend on the choice of the cut-off function χ. The following lemma guarantees that this is not the case in the relevant situation.

Lemma 7.4.2. *For $u = f + \mathbf{H}g \in \mathfrak{L}(\mathbb{R})$, where $f \in L^1(\mathbb{R})$ and $g \in L^1_0(\mathbb{R})$, the valeur au point function* $\mathrm{vap}[u](x)$ *does not depend on the choice of the cut-off χ. Moreover,*

$$\mathrm{vap}[u](x) = f(x) + \mathbf{H}g(x), \qquad \text{a.e. } x \in \mathbb{R},$$

where on the right-hand side, the function $\mathbf{H}g(x)$ is defined pointwise as a principal value.

Proof. For $f \in L^1(\mathbb{R})$, it is a standard exercise involving Poisson integrals to show that $\mathrm{vap}[f](x) = f(x)$ for almost all $x \in \mathbb{R}$ (for details, see, e.g., [10, Chapter 1]), and the choice of χ does not affect the value of $\mathrm{vap}[f](x)$ for a given $x \in \mathbb{R}$.

We turn to the evaluation of $\mathrm{vap}[\mathbf{H}g](x)$. By translation invariance, we may as well consider only $x = 0$. By definition, we have

$$\begin{aligned}\mathrm{vap}[\mathbf{H}g](0) &= \lim_{\epsilon\to 0^+} \langle \chi P_{\mathrm{i}\epsilon}, \mathbf{H}g\rangle_{\mathbb{R}} = -\lim_{\epsilon\to 0^+} \langle \mathbf{H}[\chi P_{\mathrm{i}\epsilon}], g\rangle_{\mathbb{R}} \\ &= \lim_{\epsilon\to 0^+} \{\langle \mathbf{H}[\tilde{\chi} P_{\mathrm{i}\epsilon}], g\rangle_{\mathbb{R}} - \langle \mathbf{H}[P_{\mathrm{i}\epsilon}], g\rangle_{\mathbb{R}}\},\end{aligned} \tag{7.4.2}$$

where $\tilde{\chi} := 1 - \chi$ and χ is a smooth cut-off function with $\chi(t) = 1$ near $t = 0$. Here, as above, $P_{\mathrm{i}\epsilon}$ is the function

$$P_{\mathrm{i}\epsilon}(t) = \pi^{-1}\frac{\epsilon}{\epsilon^2 + t^2},$$

and its Hilbert transform is given by

$$\mathbf{H}[P_{\mathrm{i}\epsilon}](t) = \pi^{-1}\frac{t}{\epsilon^2+t^2}.$$

A calculation reveals that

$$\pi^{-1}\frac{t}{\epsilon^2+t^2} = \int_0^{+\infty}\frac{1_{\mathbb{R}\setminus[-\tau,\tau]}}{\pi t}\,\frac{2\epsilon^2\tau}{(\epsilon^2+\tau^2)^2}\,\mathrm{d}\tau,$$

which can be used to show that

$$-\lim_{\epsilon\to0^+}\langle\mathbf{H}[P_{\mathrm{i}\epsilon}](t), g\rangle_{\mathbb{R}} = -\lim_{\tau\to0^+}\int_{\mathbb{R}\setminus[-\tau,\tau]}\frac{g(t)}{\pi t}\,\mathrm{d}t = \mathbf{H}g(0),$$

where the rightmost equality sign is a matter of the pointwise definition of the Hilbert transform. The desired conclusion now follows from (7.4.2), once we have established that for fixed $\tilde\chi$,

$$\|\mathbf{H}[\tilde\chi P_{\mathrm{i}\epsilon}]\|_{L^\infty(\mathbb{R})} = \mathrm{O}(\epsilon) \quad \text{as } \epsilon\to0^+.$$

This is rather elementary and left to the interested reader; here, we only observe that the function $\tilde\chi$ is smooth and bounded, which equals 1 near infinity and vanishes near the origin, so that $\tilde\chi P_{\mathrm{i}\epsilon}$ becomes a very small and quite smooth function. □

Additional properties of the mapping vap are outlined below.

Proposition 7.4.3 (Kolmogorov). *The mapping* vap : $\mathfrak{L}(\mathbb{R})\to L^{1,\infty}(\mathbb{R})$, $u\mapsto \mathrm{vap}[u]$, *is continuous.*

Proof. This follows from the standard weak-type estimate for the Hilbert transform (see, e.g., [10]). □

The next result allows us to identify u with vap$[u]$.

Proposition 7.4.4 (Kolmogorov). *If* $u\in\mathfrak{L}(\mathbb{R})$ *and* $\mathrm{vap}[u]=0$ *almost everywhere on* $\mathbb{R}$, *then* $u=0$ *as a distribution.*

Proof. We write $u = f+\mathbf{H}g$, where $f\in L^1(\mathbb{R})$ and $g\in L^1_0(\mathbb{R})$. Since $g\in L^1_0(\mathbb{R})$ and, by assumption, $\mathrm{vap}[g] = -f\in L^1(\mathbb{R})$, it follows from Proposition 7.1.1 that $g\in H^1_{\circledast}(\mathbb{R})$ and consequently $\mathbf{H}g\in L^1(\mathbb{R})$ as a distribution. Since the Hilbert transform $\mathbf{H}$ leaves $H^1_{\circledast}(\mathbb{R})$ invariant, we also obtain $f\in H^1_{\circledast}(\mathbb{R})$, and then it is immediate from the assumption that $u=0$ as a distribution. □

The local version of Proposition 7.4.4 is as follows.

Proposition 7.4.5. *If* $u\in\mathfrak{L}(\mathbb{R})$ *and* $\mathrm{vap}[u]=0$ *almost everywhere on an open interval* $I\subset\mathbb{R}$, *then the distribution* u *is supported on* $\mathbb{R}\setminus I$.

Proof. We split $u = f + \mathbf{H}g$, where $f \in L^1(\mathbb{R})$ and $g \in L^1_0(\mathbb{R})$. Without loss of generality, we may assume that f and g are real-valued. Again, without loss of generality, the open interval I is assumed to be *bounded*. By the classical theorem of Kolmogorov [6], the function $G := g + \mathrm{i}\mathbf{H}g$ is in the H^p-space in the upper half-plane $\mathbb{C}_+$ (with respect to the Poissonian measure $\pi^{-1}(1+t^2)^{-1}\,\mathrm{d}t$ on the real line), for each p with $0 < p < 1$. In Kolomogorov's theorem, $\mathbf{H}g$ initially has the pointwise interpretation, but in a second step, it is valid with the distributional interpretation as well. By assumption, $\mathrm{vap}[\mathbf{H}g] = -f$ on the bounded open interval I, so that the boundary function for G is in L^1 on I. Essentially, this means that G is in H^1 near I in $\mathbb{C}_+$. This can be made precise in the following manner. We choose a slightly smaller interval $J \subset I$, whose both endpoints differ from those of I. Next, we choose a bounded simply connected Jordan domain Ω in $\mathbb{C}_+$ whose boundary curve $\partial\Omega$ is C^∞-smooth, with the property that $\partial\Omega \cap \mathbb{R} = J$. Then it is not difficult to see that G, restricted to Ω, belongs to the H^1-space on Ω, which is most conveniently defined in terms of a fixed conformal mapping from the unit disk $\mathbb{D}$ onto Ω. The remaining part of the proof is an exercise in Schwarzian reflection across the interval J. □

7.5. *Dual action on intervals*

If $I \subset \mathbb{R}$ is an open interval, and $f, g : I \to \mathbb{C}$ are Borel measurable functions with $fg \in L^1(I)$, then we may define the *dual action on I*:

$$\langle f, g\rangle_I := \int_I f(t)g(t)\,\mathrm{d}t;$$

this is a special case of dual action on a more general measurable set (see §3.2). For instance, if f is a test function with compact support in I, and g is locally integrable on I, then the dual action is well-defined. More generally, we will write $\langle\cdot,\cdot\rangle_I$ to denote the dual action of distributions on test functions on the given interval I. Naturally, this agrees with the notation we have introduced so far for $I = \mathbb{R}$.

7.6. *The restriction of $\mathfrak{L}(\mathbb{R})$ to an interval*

If u is a distribution on an open interval J, then the *restriction of u to an open subinterval I, denoted $u|_I$*, is the distribution defined by

$$\langle\varphi, u|_I\rangle_I := \langle\varphi, u\rangle_J,$$

where φ is a C^∞ test function whose support is compact and contained in I.

Definition 7.6.1. Let I be an open interval of the real line. Then $u \in \mathfrak{L}(I)$ means by definition that u is a distribution on I such that there exists a distribution $v \in \mathfrak{L}(\mathbb{R})$ such that $u = v|_I$.

Remark 7.6.2. The following observation is pretty trivial, but quite useful. If $I, J \subset \mathbb{R}$ are open intervals with $I \subset J$, then the restriction operation $v \mapsto v|_I$ acts $\mathfrak{L}(J) \to \mathfrak{L}(I)$.

Proposition 7.4.5 has a localized version on a given interval J.

Corollary 7.6.3. *Suppose $I, J \subset \mathbb{R}$ are open intervals with $I \subset J$. If $u \in \mathfrak{L}(J)$ and* $\mathrm{vap}[u] = 0$ *almost everywhere on I, then the support of the distribution u has empty intersection with I.*

Proof. The assertion is immediate from Proposition 7.4.5. □

The following result will prove quite useful.

Proposition 7.6.4. *Let I be a nonempty bounded open interval of $\mathbb{R}$. Then $L^1(I)$ is a norm dense subspace of $\mathfrak{L}(I)$.*

Proof. By definition, we have

$$\mathfrak{L}(I) = \mathfrak{L}(\mathbb{R})/\mathfrak{Z}(\mathbb{R}; I), \quad \text{where} \quad \mathfrak{Z}(\mathbb{R}; I) := \{u \in \mathfrak{L}(\mathbb{R}) : I \cap \operatorname{supp} u = \emptyset\}.$$

By elementary functional analysis, the dual space $\mathfrak{L}(I)^*$ is given by the annihilator

$$\mathfrak{L}(I)^* = \mathfrak{Z}(\mathbb{R}; I)^\perp = \{f \in H^\infty_\circledast(\mathbb{R}) : \forall u \in \mathfrak{Z}(\mathbb{R}; I) : \langle f, u\rangle_{\mathbb{R}} = 0\}.$$

Observation. We have $\mathfrak{Z}(\mathbb{R}; I)^\perp \subset \{f \in H^\infty_\circledast(\mathbb{R}) : f = 0 \text{ a.e. on } \mathbb{R} \setminus I\}$.

Proof of the observation. Indeed, if $f \in H^\infty_\circledast(\mathbb{R})$ and the restriction to $\mathbb{R} \setminus I$ is nonzero on a set of positive Lebesgue measure, we readily construct a function $u \in L^1(\mathbb{R})$ which vanishes on I such that $\langle f, u\rangle_{\mathbb{R}} \neq 0$. Since $u \in \mathfrak{Z}(\mathbb{R}; I)$, this proves the asserted inclusion.

We proceed with the proof of the proposition. If $f \in H^\infty_\circledast(\mathbb{R})$ vanishes a.e. on $\mathbb{R} \setminus I$, and as a functional on $\mathfrak{L}(I)$, f annihilates $L^1(I)$, then we may conclude that $f = 0$ a.e. on I as well. But now $f = 0$ a.e. on $\mathbb{R}$, so $f = 0$ as an element of $H^\infty_\circledast(\mathbb{R})$. By the Hahn–Banach theorem, we conclude that $L^1(I)$ is norm dense in $\mathfrak{L}(I)$. □

Remark 7.6.5. A more refined argument shows that in the context of the above observation, we actually have equality: $\mathfrak{Z}(\mathbb{R}; I)^\perp = \{f \in H^\infty_\circledast(\mathbb{R}) : f = 0 \text{ a.e. on } \mathbb{R} \setminus I\}$.

We may also translate Proposition 7.4.3 to this local context.

Corollary 7.6.6. *Let I be a nonempty open interval of $\mathbb{R}$. Then* $\mathrm{vap} : \mathfrak{L}(I) \to L^{1,\infty}(I)$ *is continuous.*

8. Background material: the Hardy and BMO spaces on the circle

8.1. The Hardy H^1-space on the circle: analytic and real

Let $L^1(\mathbb{R}/2\mathbb{Z})$ denote the space of 2-periodic Borel measurable functions $f : \mathbb{R} \to \mathbb{C}$ subject to the integrability condition

$$\|f\|_{L^1(\mathbb{R}/2\mathbb{Z})} := \int_{I_1} |f(t)|\, \mathrm{d}t < +\infty,$$

where $I_1 =]-1, 1[$ as before. As usual, we identify functions that agree except possibly on a null set. Via the exponential mapping $t \mapsto \mathrm{e}^{\mathrm{i}\pi t}$, which is 2-periodic and maps the real line $\mathbb{R}$ onto the unit circle $\mathbb{T}$, we may identify $L^1(\mathbb{R}/2\mathbb{Z})$ with the standard Lebesgue space $L^1(\mathbb{T})$ of the unit circle. This will allow us to develop the elements of Hardy space theory in the setting of 2-periodic functions. We shall need the subspace $L^1_0(\mathbb{R}/2\mathbb{Z})$ consisting of all $f \in L^1(\mathbb{R}/2\mathbb{Z})$ with

$$\langle f, 1\rangle_{I_1} = \int_{I_1} f(t)\,\mathrm{d}t = 0;$$

it has codimension 1 in $L^1(\mathbb{R}/2\mathbb{Z})$. The Hardy space $H^1_+(\mathbb{R}/2\mathbb{Z})$ is defined as the subspace of $L^1(\mathbb{R}/2\mathbb{Z})$ consisting of functions $g \in L^1(\mathbb{R}/2\mathbb{Z})$ with

$$\int_{-1}^{1} \mathrm{e}^{\mathrm{i}\pi nt} g(t)\,\mathrm{d}t = 0, \qquad n = 0, 1, 2, \ldots. \tag{8.1.1}$$

The space $H^1_+(\mathbb{R}/2\mathbb{Z})$ is the periodic analogue of the Hardy space $H^1_+(\mathbb{R})$, and it can be understood in terms of the Hardy H^1-space of the disk. If $H^1_+(\mathbb{T})$ denotes the standard Hardy space on the unit disk (restricted to the boundary unit circle), then $g \in H^1_+(\mathbb{R}/2\mathbb{Z})$ *means that* $g(x) = f(\mathrm{e}^{\mathrm{i}\pi x})$ *for some* $f \in H^1_+(\mathbb{T})$ *with* $f(0) = 0$. In particular, the functions in $H^1_+(\mathbb{R}/2\mathbb{Z})$ have holomorphic extensions to the upper half-plane which are 2-periodic. By definition, $H^1_-(\mathbb{R}/2\mathbb{Z})$ *consists of the functions* g *in* $L^1(\mathbb{R}/2Z)$ *whose complex conjugate* $\bar{g}$ *is in* $H^1_+(\mathbb{R}/2\mathbb{Z})$. Finally, we put

$$H^1_{\circledast}(\mathbb{R}/2\mathbb{Z}) := H^1_+(\mathbb{R}/2\mathbb{Z}) \oplus H^1_-(\mathbb{R}/2\mathbb{Z}),$$

where we think of the elements of the sum space as 2-periodic functions (as before the symbol $\oplus$ means direct sum, since $H^1_+(\mathbb{R}/2\mathbb{Z}) \cap H^1_-(\mathbb{R}/2\mathbb{Z}) = \{0\}$). We note that, for instance, $H^1_{\circledast}(\mathbb{R}/2\mathbb{Z}) \subset L^1_0(\mathbb{R}/2\mathbb{Z})$. We will think of $H^1_{\circledast}(\mathbb{R}/2\mathbb{Z})$ as the *real* H^1*-space of* 2*-periodic functions*.

8.2. The Hilbert transform on 2-periodic functions and distributions

For $f \in L^1(\mathbb{R}/2\mathbb{Z})$, we let $\mathbf{H}_2$ be the convolution operator

$$\mathbf{H}_2 f(x) := \frac{1}{2}\,\mathrm{pv}\int_{I_1} f(t)\cot\frac{\pi(x-t)}{2}\,\mathrm{d}t, \tag{8.2.1}$$

where again pv stands for principal value, which means we take the limit as $\epsilon \to 0^+$ of the integral over $I_1 =]-1, 1[$ with the set $\{x\} + 2\mathbb{Z} + [-\epsilon, \epsilon]$ removed. It is obvious from the periodicity of the cotangent function that $\mathbf{H}_2 f$, if it exists as a limit, is 2-periodic. By a standard trigonometric identity,

$$\frac{1}{2}\cot\frac{\pi y}{2} = \lim_{N\to+\infty}\frac{1}{\pi}\sum_{n=-N}^{N}\frac{1}{y+2n},$$

where the convergence is uniform on compact subsets of $\mathbb{R}$. By a change of variables,

$$\mathbf{H}_2 f(x) = \frac{1}{2}\lim_{\epsilon\to 0^+}\int_{I_1\setminus I_\epsilon} f(x-t)\cot\frac{\pi t}{2}\,\mathrm{d}t, \tag{8.2.2}$$

(here, as usual, $I_\epsilon =]-\epsilon, \epsilon[$) from which we conclude, by uniform convergence and periodicity, that

$$\begin{aligned}\mathbf{H}_2 f(x) &= \frac{1}{\pi} \lim_{N\to+\infty} \lim_{\epsilon\to 0^+} \sum_{n=-N}^{N} \int_{I_1\setminus I_\epsilon} f(x-t)\,\frac{\mathrm{d}t}{t+2n} \\ &= \frac{1}{\pi} \lim_{\epsilon\to 0^+} \int_{I_1\setminus I_\epsilon} f(x-t)\,\frac{\mathrm{d}t}{t} + \frac{1}{\pi} \lim_{N\to+\infty} \sum_{n:|n|\le N, n\neq 0} \int_{I_1} f(x-t)\,\frac{\mathrm{d}t}{t+2n} \\ &= \frac{1}{\pi} \lim_{\epsilon\to 0^+} \int_{I_1\setminus I_\epsilon} f(x-t)\,\frac{\mathrm{d}t}{t} + \frac{1}{\pi} \lim_{N\to+\infty} \sum_{n:|n|\le N, n\neq 0} \int_{[2n-1,2n+1]} f(x-t)\,\frac{\mathrm{d}t}{t} \\ &= \lim_{N\to+\infty} \lim_{\epsilon\to 0^+} \frac{1}{\pi} \int_{I_{2N+1}\setminus I_\epsilon} f(x-t)\,\frac{\mathrm{d}t}{t}. \end{aligned} \tag{8.2.3}$$

In other words, the operator $\mathbf{H}_2$ is just the natural extension of the Hilbert transform to the 2-periodic functions. We observe that $\mathbf{H}_2 1 = 0$, which contrasts with the nonperiodic case (where no nontrivial function is mapped to the zero function). It is well-known that the periodic Hilbert transform $\mathbf{H}_2$ maps $L^1(\mathbb{R}/2\mathbb{Z})$ into the weak L^1-space $L^{1,\infty}(\mathbb{R}/2\mathbb{Z})$. However, we prefer to work within the framework of distribution theory, so we proceed as follows.

Let $C^\infty(\mathbb{R}/2\mathbb{Z})$ denote the space of C^∞ 2-periodic functions on $\mathbb{R}$. It is easy to see that

$$\varphi \in C^\infty(\mathbb{R}/2\mathbb{Z}) \implies \mathbf{H}_2\varphi \in C^\infty(\mathbb{R}/2\mathbb{Z}).$$

To emphasize the importance of the circle $\mathbb{T} \cong \mathbb{R}/2\mathbb{Z}$, we write

$$\langle f, g\rangle_{\mathbb{R}/2\mathbb{Z}} := \int_{-1}^{1} f(t)g(t)\,\mathrm{d}t \tag{8.2.4}$$

for the dual action when f and g are 2-periodic.

Definition 8.2.1. For a test function $\varphi \in C^\infty(\mathbb{R}/2\mathbb{Z})$ and a distribution u on the circle $\mathbb{R}/2\mathbb{Z}$, we put

$$\langle \varphi, \mathbf{H}_2 u\rangle_{\mathbb{R}/2\mathbb{Z}} := -\langle \mathbf{H}_2\varphi, u\rangle_{\mathbb{R}/2\mathbb{Z}}.$$

This defines the Hilbert transform $\mathbf{H}_2 u$ for any distribution u on $\mathbb{R}/2\mathbb{Z}$.

The analogue of Proposition 7.1.1 for the circle reads as follows. Note that the formula defining the "valeur au point" function makes sense also for u in the space of distributions $L^1(\mathbb{R}/2\mathbb{Z}) + \mathbf{H}_2 L^1(\mathbb{R}/2\mathbb{Z})$. Moreover, the independence of the cut-off function is quite analogous to the real line case (Lemma 7.4.2) and left to the interested reader.

Proposition 8.2.2. *Suppose $f \in L^1_0(\mathbb{R}/2\mathbb{Z})$. Then the following are equivalent:*

(i) $f \in H^1_{\circledast}(\mathbb{R}/2\mathbb{Z})$.
(ii) $\mathbf{H}_2 f \in L^1(\mathbb{R}/2\mathbb{Z})$, *where $\mathbf{H}_2 f$ is understood as a distribution on $\mathbb{R}$.*
(iii) $\mathrm{vap}[\mathbf{H}_2 f] \in L^1(\mathbb{R}/2\mathbb{Z})$.

Proof. This is immediate from [19, p. 87]. □

8.3. The real H^∞-space of the circle

The *real H^∞-space on the circle* $\mathbb{R}/2\mathbb{Z}$ is denoted by $H^\infty_\circledast(\mathbb{R}/2\mathbb{Z})$, and consists of all the functions in $H^\infty_\circledast(\mathbb{R})$ that are 2-periodic. The analogue of Proposition 7.2.1 reads:

Proposition 8.3.1. *We have the equivalence*

$$f \in H^\infty_\circledast(\mathbb{R}/2\mathbb{Z}) \iff f, \mathbf{H}_2 f \in L^\infty(\mathbb{R}/2\mathbb{Z}).$$

This result is well-known.

8.4. A predual of 2-periodic real H^∞

We put

$$\mathfrak{L}(\mathbb{R}/2\mathbb{Z}) := L^1(\mathbb{R}/2\mathbb{Z}) + \mathbf{H}_2 L^1_0(\mathbb{R}/2\mathbb{Z}),$$

understood as a space of 2-periodic distributions on $\mathbb{R}$. More precisely, if $u = f + \mathbf{H}_2 g$, where $f \in L^1(\mathbb{R}/2\mathbb{Z})$ and $g \in L^1_0(\mathbb{R}/2\mathbb{Z})$, then the action on a test function $\varphi \in C^\infty(\mathbb{R}/2\mathbb{Z})$ is given by

$$\langle \varphi, u\rangle_{\mathbb{R}/2\mathbb{Z}} := \langle \varphi, f\rangle_{\mathbb{R}/2\mathbb{Z}} - \langle \mathbf{H}_2\varphi, g\rangle_{\mathbb{R}/2\mathbb{Z}}. \tag{8.4.1}$$

But it should be possible to think of a 2-periodic distribution as a distribution on the line, which means that we need to understand the action on standard test functions in $C^\infty_c(\mathbb{R})$. If $\psi \in C^\infty_c(\mathbb{R})$, we simply put

$$\langle \psi, u\rangle_{\mathbb{R}/2\mathbb{Z}} := \langle \mathbf{\Pi}_2\psi, u\rangle_{\mathbb{R}/2\mathbb{Z}}, \tag{8.4.2}$$

where $\mathbf{\Pi}_2\psi \in C^\infty(\mathbb{R}/2\mathbb{Z})$ is given by

$$\mathbf{\Pi}_2\psi(x) := \sum_{j\in\mathbb{Z}} \psi(x+2j). \tag{8.4.3}$$

We will refer to $\mathbf{\Pi}_2$ as the *periodization operator*.

As in the case of the line, we may identify $\mathfrak{L}(\mathbb{R}/2\mathbb{Z})$ with the predual of the real H^∞-space.

Proposition 8.4.1. *Each continuous linear functional $\mathfrak{L}(\mathbb{R}/2\mathbb{Z}) \to \mathbb{C}$ corresponds to a function $\varphi \in H^\infty_\circledast(\mathbb{R}/2\mathbb{Z})$ in accordance with (8.4.1). In short, the dual space of $\mathfrak{L}(\mathbb{R}/2\mathbb{Z})$ is isomorphic to $H^\infty_\circledast(\mathbb{R}/2\mathbb{Z})$.*

We omit the proof, which is analogous to that of Proposition 7.3.1.

The definition of vap$[u]$ makes sense for $u \in \mathfrak{L}(\mathbb{R}/2\mathbb{Z})$ and as in the case of the line, it does not depend on the choice of the cut-off function. We have the analogue of Proposition 7.4.3; as the result is standard, we omit the proof.

Proposition 8.4.2 (Kolmogorov). *The mapping* vap : $\mathfrak{L}(\mathbb{R}/2\mathbb{Z}) \to L^{1,\infty}(\mathbb{R}/2\mathbb{Z})$, $u \mapsto$ vap$[u]$, *is continuous.*

9. A sum of two preduals and its localization to intervals

9.1. The sum space $\mathfrak{L}(\mathbb{R}) \oplus \mathfrak{L}(\mathbb{R}/2\mathbb{Z})$

Suppose u is a distribution on $\mathbb{R}$ of the form

$$u = v + w, \quad \text{where} \quad v \in \mathfrak{L}(\mathbb{R}),\ w \in \mathfrak{L}(\mathbb{R}/2\mathbb{Z}). \tag{9.1.1}$$

A natural question is whether the distributions v, w on the right-hand side are unique. This is indeed so.

Proposition 9.1.1. $\mathfrak{L}(\mathbb{R}) \cap \mathfrak{L}(\mathbb{R}/2\mathbb{Z}) = \{0\}$.

This statement is pretty obvious in terms of the Fourier transform, which sends 2-periodic distributions to sums of point masses along the integers, while $\mathfrak{L}(\mathbb{R})$ is mapped to a space of bounded continuous functions.

In view of Proposition 9.1.1, it makes sense to write $\mathfrak{L}(\mathbb{R}) \oplus \mathfrak{L}(\mathbb{R}/2\mathbb{Z})$ for the space of tempered distributions u of the form (9.1.1). We endow $\mathfrak{L}(\mathbb{R}) \oplus \mathfrak{L}(\mathbb{R}/2\mathbb{Z})$ with the induced Banach space norm

$$\|u\|_{\mathfrak{L}(\mathbb{R})\oplus\mathfrak{L}(\mathbb{R}/2\mathbb{Z})} := \|v\|_{\mathfrak{L}(\mathbb{R})} + \|w\|_{\mathfrak{L}(\mathbb{R}/2\mathbb{Z})},$$

provided u, v, w are related via (9.1.1).

9.2. The localization of $\mathfrak{L}(\mathbb{R}) \oplus \mathfrak{L}(\mathbb{R}/2\mathbb{Z})$ *to a bounded open interval*

In the sense of Subsection 7.6, we may restrict a given distribution $u \in \mathfrak{L}(\mathbb{R}) \oplus \mathfrak{L}(\mathbb{R}/2\mathbb{Z})$ to a given open interval I. It is natural to wonder what the space of such restrictions looks like.

Proposition 9.2.1. *The restriction of* $\mathfrak{L}(\mathbb{R}) \oplus \mathfrak{L}(\mathbb{R}/2\mathbb{Z})$ *to a bounded open interval* I *equals* $\mathfrak{L}(I)$.

Proof. By definition, the restriction of $\mathfrak{L}(\mathbb{R})$ to I equals $\mathfrak{L}(I)$. It remains to show that the restriction to I of a distribution in $\mathfrak{L}(\mathbb{R}/2\mathbb{Z})$ is in $\mathfrak{L}(I)$ as well. Since

$$\mathfrak{L}(\mathbb{R}/2\mathbb{Z}) = L^1(\mathbb{R}/2\mathbb{Z}) + \mathbf{H}_2 L^1_0(\mathbb{R}/2\mathbb{Z}),$$

and the restriction of $L^1(\mathbb{R}/2\mathbb{Z})$ to the bounded interval I is contained in $L^1(I)$, the only thing we need to check is that the restriction of $\mathbf{H}_2 L^1_0(\mathbb{R}/2\mathbb{Z})$ to I is contained in $\mathfrak{L}(I)$. It will be enough to show that for each $f \in L^1_0(\mathbb{R}/2\mathbb{Z})$, there exist $g \in L^1(\mathbb{R})$, $h \in L^1_0(\mathbb{R})$, and a distribution $W \in \mathcal{D}'(\mathbb{R})$ with support in $\mathbb{R} \setminus I$, such that

$$\mathbf{H}_2 f = g + \mathbf{H}h + W.$$

We need two bounded open intervals J_1, J_2 such that $I \Subset J_1 \Subset J_2$. We first let h equal f on J_1, and put it equal to 0 on $\mathbb{R} \setminus J_2$. In the difference set $J_2 \setminus J_1$, we let h be constant, where the value of the constant is then determined by the requirement that $h \in L^1_0(\mathbb{R})$. As the cotangent kernel $\frac{1}{2}\cot\frac{\pi t}{2}$ used to define $\mathbf{H}_2$ and the Hilbert transform kernel $\frac{1}{\pi t}$ have the same singularity, it is easy to see that $\mathbf{H}_2 f - \mathbf{H}h$ is smooth on J_1, and we may declare g to equal $\mathbf{H}_2 f - \mathbf{H}h$ on I, and put it equal to 0 on the rest $\mathbb{R} \setminus I$. The distribution W is uniquely determined by these choices, and has the required properties. □

10. An involution, its adjoint, and the periodization operator

10.1. An involutive operator

For each positive real number β, let $\mathbf{J}_\beta$ denote the involution given by

$$\mathbf{J}_\beta f(x) := \frac{\beta}{x^2} f(-\beta/x), \qquad x \in \mathbb{R}^\times.$$

With respect to the dual action $\langle\cdot,\cdot\rangle_\mathbb{R}$, this operator $\mathbf{J}_\beta$ can be understood as the preadjoint of the involution $\mathbf{J}^*_\beta$ defined in (5.1.1).

We use the standard notation $\mathbb{R}^\times := \mathbb{R} \setminus \{0\}$. We now record some basic properties of this involution. For instance, by the change-of-variables formula, $\mathbf{J}_\beta : L^1(\mathbb{R}) \to L^1(\mathbb{R})$ is an *isometry*.

Proposition 10.1.1. *Fix $0 < \beta < +\infty$. The operator $\mathbf{J}_\beta$ is an isometric isomorphism $L^1(\mathbb{R}) \to L^1(\mathbb{R})$. In addition, $\mathbf{J}_\beta$ maps $H^1_+(\mathbb{R}) \to H^1_+(\mathbb{R})$ and $H^1_-(\mathbb{R}) \to H^1_-(\mathbb{R})$ and consequently $\mathbf{J}_\beta : H^1_\circledast(\mathbb{R}) \to H^1_\circledast(\mathbb{R})$ as well.*

Proof. The mapping $z \mapsto -\beta/z$ preserves the upper half-plane $\mathbb{C}_+$, and so functions holomorphic in $\mathbb{C}_+$ are sent to functions holomorphic in $\mathbb{C}_+$ under composition with $z \mapsto -\beta/z$. The isometric part is already settled, so it remains to check that $H^1_+(\mathbb{R})$ is preserved under $\mathbf{J}_\beta$, since the case of $H^1_-(\mathbb{R})$ is identical. This follows easily by checking the property on a dense subspace (e.g. consisting of rational functions). □

If $f \in L^1(\mathbb{R})$ and $\varphi \in L^\infty(\mathbb{R})$, the change-of-variables formula yields

$$\langle \varphi, \mathbf{J}_\beta f\rangle_\mathbb{R} = \int_\mathbb{R} \varphi(t) f(-\beta/t)\,\frac{\beta\,\mathrm{d}t}{t^2} = \int_\mathbb{R} \varphi(-\beta/t) f(t)\,\mathrm{d}t = \langle \mathbf{J}^*_\beta \varphi, f\rangle_\mathbb{R}, \qquad (10.1.1)$$

where $\mathbf{J}^*_\beta$ is the involution

$$\mathbf{J}^*_\beta \varphi(t) := \varphi(-\beta/t), \qquad t \in \mathbb{R}^\times.$$

We need to extend $\mathbf{J}_\beta$ to an operator $\mathfrak{L}(\mathbb{R}) \to \mathfrak{L}(\mathbb{R})$. To this end, we need to understand how to define $\mathbf{J}_\beta \mathbf{H} f$ as a distribution in $\mathfrak{L}(\mathbb{R})$ when $f \in L^1_0(\mathbb{R})$. First, following (10.1.1), we put

$$\langle \varphi, \mathbf{J}_\beta \mathbf{H} f\rangle_\mathbb{R} := -\langle \mathbf{H}\mathbf{J}^*_\beta \varphi, f\rangle_\mathbb{R} \qquad (10.1.2)$$

for $f \in L^1_0(\mathbb{R})$ and $\varphi \in C^\infty_c(\mathbb{R}^\times)$, since such test functions vanish near the origin. Note here that if $\varphi \in C^\infty_c(\mathbb{R}^\times)$, then necessarily $\mathbf{J}^*_\beta \varphi \in C^\infty_c(\mathbb{R}^\times)$ as well, so the right-hand side of (10.1.2) is well-defined.

Proposition 10.1.2. *For $\varphi \in C^\infty_c(\mathbb{R}^\times)$, we have the identity*

$$\mathbf{H}\mathbf{J}^*_\beta \varphi(x) = \mathbf{J}^*_\beta \mathbf{H}\varphi(x) - \big\langle \varphi, t \mapsto \tfrac{1}{\pi t}\big\rangle_\mathbb{R}, \qquad x \in \mathbb{R}^\times.$$

Proof. By a change of variables in the corresponding integral, we have

$$\mathbf{J}_\beta^*\mathbf{H}\varphi(x) = -\frac{1}{\pi}\,\mathrm{pv}\int_{\mathbb{R}} \frac{x}{\beta+tx}\,\varphi(t)\,\mathrm{d}t, \qquad \mathbf{H}\mathbf{J}_\beta^*\varphi(x) = \frac{1}{\pi}\,\mathrm{pv}\int_{\mathbb{R}} \frac{\beta}{t(\beta+tx)}\varphi(t)\,\mathrm{d}t,$$

so the asserted equality is a simple consequence of the algebraic identity

$$-\frac{x}{\beta+tx} = \frac{\beta}{t(\beta+tx)} - \frac{1}{t}.$$

□

As $f \in L_0^1(\mathbb{R})$, its action on constants vanishes, so by a combination of (10.1.1), (10.1.2), and Proposition 10.1.2, we obtain

$$\langle\varphi, \mathbf{J}_\beta\mathbf{H}f\rangle_{\mathbb{R}} = -\langle\mathbf{H}\mathbf{J}_\beta^*\varphi, f\rangle_{\mathbb{R}} = -\langle\mathbf{J}_\beta^*\mathbf{H}\varphi, f\rangle_{\mathbb{R}} = -\langle\mathbf{H}\varphi, \mathbf{J}_\beta f\rangle_{\mathbb{R}} = \langle\varphi, \mathbf{H}\mathbf{J}_\beta f\rangle_{\mathbb{R}} \tag{10.1.3}$$

for $\varphi \in C_c^\infty(\mathbb{R}^\times)$. As $f \in L_0^1(\mathbb{R})$, we also have $\mathbf{J}_\beta f \in L_0^1(\mathbb{R})$, so $\mathbf{H}\mathbf{J}_\beta f \in \mathbf{H}L_0^1(\mathbb{R}) \subset \mathfrak{L}(\mathbb{R})$. This means that as distributions on $\mathbb{R}^\times = \mathbb{R}\setminus\{0\}$, $\mathbf{J}_\beta\mathbf{H}f$ and $\mathbf{H}\mathbf{J}_\beta f$ coincide. In particular, their "valeur au point" functions, which are well-defined almost everywhere, coincide on $\mathbb{R}^\times$. However, the distribution $\mathbf{H}\mathbf{J}_\beta f$ makes sense on test functions $\varphi \in C_c^\infty(\mathbb{R})$, and actually, more generally for $\varphi \in H_\circledast^\infty(\mathbb{R})$. This allows us to extend the action of $\mathbf{J}_\beta\mathbf{H}f$ from $C_c^\infty(\mathbb{R}^\times)$ to $H_\circledast^\infty(\mathbb{R})$ (compare with (7.3.2)).

Definition 10.1.3. For $u \in \mathfrak{L}(\mathbb{R})$ of the form $u = f + \mathbf{H}g \in \mathfrak{L}(\mathbb{R})$, where $f \in L^1(\mathbb{R})$ and $g \in L_0^1(\mathbb{R})$, we define $\mathbf{J}_\beta u$ to be the distribution on $\mathbb{R}$ given by the formula

$$\langle\varphi, \mathbf{J}_\beta u\rangle_{\mathbb{R}} = \langle\varphi, \mathbf{J}_\beta(f+\mathbf{H}g)\rangle_{\mathbb{R}} := \langle\varphi, \mathbf{J}_\beta f\rangle_{\mathbb{R}} + \langle\varphi, \mathbf{H}\mathbf{J}_\beta g\rangle_{\mathbb{R}} = \langle\varphi, \mathbf{J}_\beta f\rangle_{\mathbb{R}} - \langle\tilde{\mathbf{H}}\varphi, \mathbf{J}_\beta g\rangle_{\mathbb{R}}$$

for test functions $\varphi \in H_\circledast^\infty(\mathbb{R})$.

As already noted, this is in complete agreement with the way we would previously understand $\mathbf{J}_\beta u$ as a distribution on $\mathbb{R}^\times$, using smooth test functions having compact support on the punctured line $\mathbb{R}^\times$; see (10.1.1) and (10.1.2).

Proposition 10.1.4. *Fix $0 < \beta < +\infty$. The map $\mathbf{J}_\beta$ acts continuously $\mathfrak{L}(\mathbb{R}) \to \mathfrak{L}(\mathbb{R})$, and $\mathbf{J}_\beta^*$ acts continuously $H_\circledast^\infty(\mathbb{R}) \to H_\circledast^\infty(\mathbb{R})$. Moreover, on the respective spaces, $\mathbf{J}_\beta^2$ and $\mathbf{J}_\beta^{*2}$ both equal the identity operator.*

Proof. Let $u \in \mathfrak{L}(\mathbb{R})$ be of the form $u = f + \mathbf{H}g$, where $f \in L^1(\mathbb{R})$ and $g \in L_0^1(\mathbb{R})$. Then, by definition, $\mathbf{J}_\beta u = \mathbf{J}_\beta f + \mathbf{H}\mathbf{J}_\beta g \in \mathfrak{L}(\mathbb{R})$, and it is clear that $\mathbf{J}_\beta$ acts continuously. Moreover, by iteration

$$\mathbf{J}_\beta^2 u = \mathbf{J}_\beta^2 f + \mathbf{H}\mathbf{J}_\beta^2 g = f + \mathbf{H}g = u$$

since $\mathbf{J}_\beta^2 F = F$ for all $F \in L^1(\mathbb{R})$. The assertions concerning $\mathbf{J}_\beta^*$ follow by duality. □

10.2. The periodization operator

We recall the definition of the *periodization operator* $\mathbf{\Pi}_2$:

$$\mathbf{\Pi}_2 f(x) := \sum_{j\in\mathbb{Z}} f(x+2j).$$

In (8.4.3), we defined $\mathbf{\Pi}_2$ on test functions. It is however clear that it remains well-defined with much less smoothness required of f. The terminology comes from the property that whenever it is well-defined, the function $\mathbf{\Pi}_2 f$ is 2-periodic automatically. A first result is the following.

Proposition 10.2.1. *The operator* $\mathbf{\Pi}_2$ *acts contractively* $L^1(\mathbb{R}) \to L^1(\mathbb{R}/2\mathbb{Z})$. *Moreover,* $\mathbf{\Pi}_2$ *maps* $H^1_+(\mathbb{R})$ *onto* $H^1_+(\mathbb{R}/2\mathbb{Z})$ *and* $H^1_-(\mathbb{R})$ *onto* $H^1_-(\mathbb{R}/2\mathbb{Z})$.

Proof. By the triangle inequality and Fubini's theorem, $\mathbf{\Pi}_2$ is a contraction $L^1(\mathbb{R}) \to L^1(\mathbb{R}/2\mathbb{Z})$:

$$\int_{-1}^{1} |\mathbf{\Pi}_2 f(x)|\,\mathrm{d}x \le \sum_{j\in\mathbb{Z}} \int_{-1}^{1} |f(x+2j)|\,\mathrm{d}x = \sum_{j\in\mathbb{Z}} \int_{2j-1}^{2j+1} |f(x)|\,\mathrm{d}x = \int_{\mathbb{R}} |f(x)|\,\mathrm{d}x,$$

It remains to check the mapping properties, which are immediate from the characterizations (4.1.3), (4.1.4) for the line and (8.1.1) for the circle, combined with the calculation

$$\int_{-1}^{1} \mathrm{e}^{\mathrm{i}\pi n t}\,\mathbf{\Pi}_2 f(t)\,\mathrm{d}t = \sum_{j\in\mathbb{Z}} \int_{-1}^{1} \mathrm{e}^{\mathrm{i}\pi n t} f(t+2j)\,\mathrm{d}t = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\pi n t} f(t)\,\mathrm{d}t, \qquad n \in \mathbb{Z}. \tag{10.2.1}$$

□

The identity (10.2.1) is a special case of a more general identity, for $f \in L^1(\mathbb{R})$ and $F \in L^\infty(\mathbb{R}/2\mathbb{Z})$ (compare with (8.4.2)):

$$\begin{aligned}\langle F, \mathbf{\Pi}_2 f\rangle_{\mathbb{R}/2\mathbb{Z}} &= \int_{-1}^{1} F(t)\mathbf{\Pi}_2 f(t)\,\mathrm{d}t = \sum_{j\in\mathbb{Z}} \int_{-1}^{1} F(t) f(t+2j)\,\mathrm{d}t \\ &= \int_{\mathbb{R}} F(t) f(t)\,\mathrm{d}t = \langle F, f\rangle_{\mathbb{R}}, \qquad n \in \mathbb{Z}. \end{aligned} \tag{10.2.2}$$

We need to extend $\mathbf{\Pi}_2$ in a natural fashion to the space $\mathfrak{L}(\mathbb{R})$. If $\varphi \in C^\infty(\mathbb{R}/2\mathbb{Z})$ is a test function on the circle, we glance at (10.2.2), and for $u \in \mathfrak{L}(\mathbb{R})$ with $u = f + \mathbf{H}g$, where $f \in L^1(\mathbb{R})$ and $g \in L^1_0(\mathbb{R})$, we set

$$\langle \varphi, \mathbf{\Pi}_2 u\rangle_{\mathbb{R}/2\mathbb{Z}} := \langle \varphi, u\rangle_{\mathbb{R}} = \langle \varphi, f\rangle_{\mathbb{R}} - \langle \tilde{\mathbf{H}}\varphi, g\rangle_{\mathbb{R}}. \tag{10.2.3}$$

This defines $\mathbf{\Pi}_2 u$ as a distribution on the circle (compare with (7.3.2)).

Proposition 10.2.2. *For* $u \in \mathfrak{L}(\mathbb{R})$ *of the form* $u = f + \mathbf{H}g$, *where* $f \in L^1(\mathbb{R})$ *and* $g \in L^1_0(\mathbb{R})$, *we have* $\mathbf{\Pi}_2 u = \mathbf{\Pi}_2 f + \mathbf{H}_2 \mathbf{\Pi}_2 g$. *In particular,* $\mathbf{\Pi}_2$ *maps* $\mathfrak{L}(\mathbb{R}) \to \mathfrak{L}(\mathbb{R}/2\mathbb{Z})$ *continuously.*

Proof. For a 2-periodic test function $\varphi \in C^\infty(\mathbb{R}/2\mathbb{Z})$, we check that

$$\langle \varphi, \mathbf{\Pi}_2 f + \mathbf{H}_2 \mathbf{\Pi}_2 g \rangle_{\mathbb{R}/2\mathbb{Z}} = \langle \varphi, \mathbf{\Pi}_2 f \rangle_{\mathbb{R}/2\mathbb{Z}} - \langle \mathbf{H}_2 \varphi, \mathbf{\Pi}_2 g \rangle_{\mathbb{R}/2\mathbb{Z}} = \langle \varphi, f \rangle_{\mathbb{R}} - \langle \mathbf{H}_2 \varphi, g \rangle_{\mathbb{R}},$$

where we applied the identity (10.2.2) twice. If we compare this with (10.2.3), we realize we have the same expression, because $\tilde{\mathbf{H}}\varphi$ and $\mathbf{H}_2\varphi$ differ by a constant. After all, they are two harmonic conjugates of one and the same function, and g annihilates constants. □

11. The spanning problem formulation of Theorem 1.8.2

11.1. A reformulation of Theorem 1.8.2

Let us consider the following problem.

Problem 11.1.1. For which values of the positive real parameter β is the linear span of the functions

$$e_n(t) := \mathrm{e}^{\mathrm{i}\pi n t}, \quad e_m^{\langle\beta\rangle}(t) := \mathrm{e}^{-\mathrm{i}\pi\beta m/t}, \quad m, n \in \mathbb{Z}_{+,0},$$

weak-star dense in $H_+^\infty(\mathbb{R})$?

We first remark that the functions $\mathrm{e}^{\mathrm{i}\pi n t}$ and $\mathrm{e}^{-\mathrm{i}\pi\beta m/t}$ for $m, n \in \mathbb{Z}_{+,0}$ belong to $H_+^\infty(\mathbb{R})$ (they have bounded holomorphic extensions to $\mathbb{C}_+$), so that the problem makes sense. A simple scaling argument allows us to take $\alpha := 1$, so that *Theorem 1.8.2 is equivalent to Problem 11.1.1 having an affirmative answer if and only if* $\beta \le 1$.

With respect to the dual action $\langle\cdot,\cdot\rangle_{\mathbb{R}}$ on the line, the understood predual of $H_+^\infty(\mathbb{R})$ is the quotient space $L^1(\mathbb{R})/H_+^1(\mathbb{R})$. So, in terms of duality, the question raised in Problem 11.1.1 is: *When, provided that $f \in L^1(\mathbb{R})$, do we have the implication*

$$\langle e_n, f \rangle_{\mathbb{R}} = \langle e_m^{\langle\beta\rangle}, f \rangle_{\mathbb{R}} = 0 \; \forall m, n \in \mathbb{Z}_{+,0} \implies f \in H_+^1(\mathbb{R})? \tag{11.1.1}$$

The argument involving point separation in $\mathbb{C}_+$ from [15] applies here as well, which makes $\beta \le 1$ a necessary condition for the implication (11.1.1) to hold. Actually, as mentioned in the introduction, the methods of [4] supply infinitely many linearly independent counterexamples for $\beta > 1$.

Also, by testing with $n = 0$, we note that we might as well assume that $f \in L_0^1(\mathbb{R})$ in (11.1.1). In view of (10.2.1),

$$\langle e_n, f \rangle_{\mathbb{R}} = \int_{-1}^{1} \mathrm{e}^{\mathrm{i}\pi n t} \mathbf{\Pi}_2 f(t)\, \mathrm{d}t = \langle e_n, \mathbf{\Pi}_2 f \rangle_{\mathbb{R}/2\mathbb{Z}}, \tag{11.1.2}$$

so that for $f \in L^1(\mathbb{R})$ we have the equivalence

$$\{\forall n \in \mathbb{Z}_{+,0} : \langle e_n, f \rangle_{\mathbb{R}} = 0\} \iff \mathbf{\Pi}_2 f \in H_+^1(\mathbb{R}/2\mathbb{Z}).$$

Since $\mathbf{J}_\beta^* e_m = e_m^{\langle\beta\rangle}$, where $\mathbf{J}_\beta^*$ is the involutive operator studied in §§5.1 and 10.1, we have

$$\langle f, e_m^{\langle\beta\rangle} \rangle_{\mathbb{R}} = \langle f, \mathbf{J}_\beta^* e_m \rangle_{\mathbb{R}} = \langle \mathbf{J}_\beta f, e_m \rangle_{\mathbb{R}},$$

which leads for $f \in L^1(\mathbb{R})$ to the equivalence

$$\{\forall m \in \mathbb{Z}_{+,0} : \langle e_m^{\langle\beta\rangle}, f\rangle_{\mathbb{R}} = 0\} \iff \mathbf{\Pi}_2\mathbf{J}_\beta f \in H^1_+(\mathbb{R}/2\mathbb{Z}).$$

We can now rephrase the question (11.1.1) and hence Problem 11.1.1.

Problem 11.1.2. Fix $0 < \beta \leq 1$. Is it true that for $f \in L^1_0(\mathbb{R})$,

$$\mathbf{\Pi}_2 f, \mathbf{\Pi}_2\mathbf{J}_\beta f \in H^1_+(\mathbb{R}/2\mathbb{Z}) \implies f \in H^1_+(\mathbb{R})?$$

It is rather obvious that the reverse implication holds (use, e.g., Propositions 10.1.1 and 10.2.1). If we think of $\mathbf{\Pi}_2 f$ and $\mathbf{\Pi}_2\mathbf{J}_\beta f$ as 2-periodic "shadows" of f and $\mathbf{J}_\beta f$, the issue at hand is whether knowing that the two shadows are in the right space is enough to conclude that the function comes from the space $H^1_+(\mathbb{R})$. We note here that the main result of [15] may be understood as the assertion that *f is uniquely determined by the two "shadows" $\mathbf{\Pi}_2 f$ and $\mathbf{\Pi}_2\mathbf{J}_\beta f$ if and only if $\beta \leq 1$*. This offers some rather weak support for the plausibility of the implication of Problem 11.1.2.

11.2. An alternative reformulation in terms of the space $\mathfrak{L}(\mathbb{R})$

We begin with a function $f \in L^1_0(\mathbb{R})$, and form the conjugate-analytic Szegő projection (cf. (4.2.6))

$$u := \mathbf{P}_- f = \tfrac{1}{2}(f - \mathrm{i}\mathbf{H}f) \in L^1_0(\mathbb{R}) + \mathbf{H}L^1_0(\mathbb{R}) \subset \mathfrak{L}(\mathbb{R}).$$

Then, by Definition 10.1.3,

$$\mathbf{J}_\beta u = \mathbf{J}_\beta\mathbf{P}_- f = \tfrac{1}{2}(\mathbf{J}_\beta f - \mathrm{i}\mathbf{J}_\beta\mathbf{H}f) = \tfrac{1}{2}(\mathbf{J}_\beta f - \mathrm{i}\mathbf{H}\mathbf{J}_\beta f) \in L^1_0(\mathbb{R}) + \mathbf{H}L^1_0(\mathbb{R}) \subset \mathfrak{L}(\mathbb{R}),$$

and we calculate that (use Lemma 10.2.2)

$$\mathbf{\Pi}_2 u = \tfrac{1}{2}(\mathbf{\Pi}_2 f - \mathrm{i}\mathbf{\Pi}_2\mathbf{H}f) = \tfrac{1}{2}(\mathbf{\Pi}_2 f - \mathrm{i}\mathbf{H}_2\mathbf{\Pi}_2 f) = \tfrac{1}{2}(\mathbf{I} - \mathrm{i}\mathbf{H}_2)\mathbf{\Pi}_2 f \in \mathfrak{L}(\mathbb{R}/2\mathbb{Z}), \tag{11.2.1}$$

and that (use Proposition 10.2.2 again)

$$\begin{aligned}\mathbf{\Pi}_2\mathbf{J}_\beta u &= \tfrac{1}{2}(\mathbf{\Pi}_2\mathbf{J}_\beta f - \mathrm{i}\mathbf{\Pi}_2\mathbf{H}\mathbf{J}_\beta f) = \tfrac{1}{2}(\mathbf{\Pi}_2\mathbf{J}_\beta f - \mathrm{i}\mathbf{H}_2\mathbf{\Pi}_2\mathbf{J}_\beta f)\\ &= \tfrac{1}{2}(\mathbf{I} - \mathrm{i}\mathbf{H}_2)\mathbf{\Pi}_2\mathbf{J}_\beta f \in \mathfrak{L}(\mathbb{R}/2\mathbb{Z}).\end{aligned} \tag{11.2.2}$$

Here, we write $\mathbf{I}$ for the identity operator. Modulo the constants, the operator $\mathbf{P}_{2,-} := \frac{1}{2}(\mathbf{I} - \mathrm{i}\mathbf{H}_2)$ projects to the 2-periodic conjugate-holomorphic functions in $\mathbb{C}_+$, and $H^1_+(\mathbb{R}/2\mathbb{Z})$ is indeed mapped to $\{0\}$:

$$\mathbf{P}_{2,-}H^1_+(\mathbb{R}/2\mathbb{Z}) = \{0\}. \tag{11.2.3}$$

Hence we conclude from (11.2.1) and (11.2.2) that

$$\mathbf{\Pi}_2 f,\ \mathbf{\Pi}_2\mathbf{J}_\beta f \in H^1_+(\mathbb{R}/2\mathbb{Z}) \implies \mathbf{\Pi}_2 u = \mathbf{\Pi}_2\mathbf{J}_\beta u = 0.$$

We are led to consider the following problem. Let $\mathfrak{L}_0(\mathbb{R})$ be the one-codimensional subspace of $\mathfrak{L}(\mathbb{R})$ given by

$$\mathfrak{L}_0(\mathbb{R}) := L^1_0(\mathbb{R}) + \mathbf{H}L^1_0(\mathbb{R}) \subset \mathfrak{L}(\mathbb{R}).$$

Problem 11.2.1. Fix $0 < \beta \le 1$. Is it true that for $u \in \mathfrak{L}_0(\mathbb{R})$,

$$\mathbf{\Pi}_2 u = \mathbf{\Pi}_2 \mathbf{J}_\beta u = 0 \implies u = 0?$$

Proposition 11.2.2. *If the answer to Problem 11.2.1 is affirmative, then the answers to Problems 11.1.1 and 11.1.2 are affirmative as well, and the assertion of Theorem 1.8.2 is valid.*

Proof. We already know that Problems 11.1.1 and 11.1.2 are equivalent. Let $f \in L^1(\mathbb{R})$ be such that $\mathbf{\Pi}_2 f \in H^1_+(\mathbb{R}/2\mathbb{Z})$ and $\mathbf{\Pi}_2 \mathbf{J}_\beta f \in H^1_+(\mathbb{R}/2\mathbb{Z})$. Then, as a first step, $f \in L^1_0(\mathbb{R})$ by the identity (10.2.1) with $n = 0$. We recall the notation $\mathbf{P}_- := \frac{1}{2}(\mathbf{I} - \mathrm{i}\mathbf{H})$ for the Szegő projection to the conjugate-holomorphic functions in $\mathbb{C}_+$. Next, we consider the distribution $u := \mathbf{P}_- f = \frac{1}{2}(f - \mathrm{i}\mathbf{H}f) \in \mathfrak{L}_0(\mathbb{R})$, and use the identities (11.2.1) and (11.2.2) together with (11.2.3) to see that $\mathbf{\Pi}_2 u = \mathbf{\Pi}_2 \mathbf{J}_\beta u = 0$. Now, given that Problem 11.2.1 has an affirmative answer, we find that $\mathbf{P}_- f = u = 0$, which is only possible for $f \in L^1(\mathbb{R})$ if $f \in H^1_+(\mathbb{R})$. We conclude that Problems 11.1.1 and 11.1.2 have affirmative answers as well. Finally, given the discussion in §1.8, the correctness of the assertion of Theorem 1.8.2 follows as well. □

11.3. The connection with an extension of ergodic theory

In [16], the following result is obtained as an application of an extension of ergodic theory in the setting of Gauss-type maps.

Theorem 11.3.1 (see [16]). *For* $0 < \beta \le 1$ *and* $u \in \mathfrak{L}_0(\mathbb{R})$,

$$\mathbf{\Pi}_2 u = \mathbf{\Pi}_2 \mathbf{J}_\beta u = 0 \implies u = 0.$$

Modulo this result, we may now conclude the proof of Theorem 1.8.2.

Proof of Theorem 1.8.2. As observed right after the formulation of Theorem 1.8.2, a scaling argument allows us to reduce the redundancy and *fix* $\alpha = 1$, *in which case the condition* $0 < \alpha\beta \le 1$ *reads* $0 < \beta \le 1$. Now, in view of §11.1 and Proposition 11.2.2, the assertion is an immediate consequence of Theorem 11.3.1. □

It remains to explain how Theorem 11.3.1 connects with an extension of ergodic theory. The connection is strongest for $\beta = 1$, which is why we restrict our attention to this value of β. For $u \in \mathfrak{L}_0(\mathbb{R})$, we need to show that if $\mathbf{\Pi}_2 u = 0$ and $\mathbf{\Pi}_2 \mathbf{J}_1 u = 0$, then $u = 0$ is the only possibility. We split $\mathbf{\Pi}_2 = \mathbf{I} + \mathbf{\Sigma}_2$, so that

$$\mathbf{\Sigma}_2 u(t) = \sum_{j \in \mathbb{Z}^\times} u(t + 2j),$$

where the two sides are to be understood liberally (compare with (10.2.3)). Then $\mathbf{\Pi}_2 u = 0$ is the same as $u = -\mathbf{\Sigma}_2 u$, while $\mathbf{\Pi}_2 \mathbf{J}_1 u = 0$ means that $\mathbf{J}_1 u = -\mathbf{\Sigma}_2 \mathbf{J}_1 u$. Since $\mathbf{J}_1$ is an involution, we could write the latter as $u = -\mathbf{J}_1 \mathbf{\Sigma}_2 \mathbf{J}_1 u$. We may need to be careful with the interpretation of the right-hand side, but let us not worry about that now. So, the two pieces of information we have about $u \in \mathfrak{L}_0(\mathbb{R})$ are $u = -\mathbf{\Sigma}_2 u$ and $u = -\mathbf{J}_1\mathbf{\Sigma}_2\mathbf{J}_1 u$. We are free to combine them:

$$u = \mathbf{\Sigma}_2\mathbf{J}_1\mathbf{\Sigma}_2\mathbf{J}_1 u \quad \text{and} \quad u = \mathbf{J}_1\mathbf{\Sigma}_2\mathbf{J}_1\mathbf{\Sigma}_2 u. \tag{11.3.1}$$

If we write $\mathbf{T}_1 := \mathbf{\Sigma}_2\mathbf{J}_1$ and $\mathbf{V}_1 := \mathbf{J}_1\mathbf{\Sigma}_2$, (11.3.1) maintains that $u = \mathbf{T}_1^2 u$ and $u = \mathbf{V}_1^2 u$. The operator $\mathbf{T}_1$ behaves like the transfer operator associated with the Gauss-type transformation $\tau_1(x) = \{-1/x\}_2$ (see, e.g., (3.4.2)), but to get a precise fit we need to restrict our space of distributions to the symmetric standard interval I_1, and consider $\mathfrak{L}(I_1)$. Of course $\mathbf{T}_1$ acts contractively on $L^1(I_1)$ (see Proposition 3.4.1), but on the larger space $\mathfrak{L}(I_1)$ it is no longer a norm contraction (but it does define a bounded operator, see [16]). This is a serious complication, which is overcome only by a careful analysis of the action of the iterates of the transfer operator on the Hilbert kernel. We remark that on the interval I_1, the equality $u = \mathbf{T}_1^2 u$ asks for u to be an "invariant observable" in the space $\mathfrak{L}(I_1)$ of "extended observables" for the composition square of the Gauss-type transformation. In the considerably simpler $L^1(I_1)$ setting, this is the same as being a scalar multiple of the invariant measure (this observation uses ergodicity). From a functional analysis perspective, in the case of a finite mass invariant measure, ergodicity can be understood as the property that the given invariant measure is an extreme point in the convex body of all the invariant probability measures. In the case at hand, the absolutely continuous invariant measure is $(1 - t^2)^{-1}\,\mathrm{d}t$, which is ergodic but has infinite mass, so it does not fit in the standard functional analysis interpretation. Then we would still know from ergodicity that the only possible solution to $u = \mathbf{T}_1^2 u$ with $u \in L^1(I_1)$ is the function $u = 0$ (see e.g. [15]). In this sense, the assertion that $u = 0$ is the only possibility in the larger space $\mathfrak{L}(I_1)$ of extended observables is stronger than standard ergodicity. The analogue for a transformation without an indifferent fixed point would be the statement that the given invariant observable is unique up to scalar multiples within the extended observables space $\mathfrak{L}(I_1)$. We may think of $\mathfrak{L}(I_1)$ as arising from a mix of absolutely continuous signed densities of two types of particles, (i) point particles (represented by δ_ξ) and (ii) fuzzy particles (represented by $\mathbf{H}\delta_\xi$). In the fuzzy case, we need to include source points ξ located outside the basic interval I_1; if we prefer to consider only $\xi \in I_1$, the Hilbert transform needs a slight modification to give the whole space $\mathfrak{L}(I_1)$ in this manner.

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