Proc. London Math. Soc. Page 1 of 49 © 2012 London Mathematical Society
doi:10.1112/plms/pds032

Coulomb gas ensembles and Laplacian growth

Håkan Hedenmalm and Nikolai Makarov

Abstract

We consider weight functions $Q:\mathbb{C}\to\mathbb{R}$ that are locally in a suitable Sobolev space and impose a logarithmic growth condition from below. We use Q as a confining potential in the model of one-component plasma (2-dimensional Coulomb gas) and study the configuration of the electron cloud as the number n of electrons tends to infinity, while the confining potential is rescaled: we use mQ in place of Q and let m tend to infinity as well. We show that if m and n tend to infinity in a proportional fashion, with $n/m\to t$, where $0<t<+\infty$ is fixed, then the electrons accumulate on a compact set S_t, which we call the *droplet*. The set S_t can be obtained as the coincidence set of an obstacle problem, if we remove a small set (the shallow points). Moreover, on the droplet S_t, the density of electrons is asymptotically ΔQ. The growth of the droplets S_t as t increases is known as the Laplacian growth. It is well known that Laplacian growth is unstable. To analyse this feature, we introduce the notion of a *local droplet*, which involves removing part of the obstacle away from the set S_t. The local droplets are no longer uniquely determined by the time parameter t, but at least they may be partially ordered. We show that the growth of the local droplets may be terminated in a maximal local droplet or by the droplets' growing to infinity in some direction ('fingering').

1. *Overview*

1.1. *Outline of the paper*

In Section 2 and Appendix A, we study the one-component plasma (OCP)– also called the Coulomb gas ensemble – in two dimensions and find the quasi-classical limit as the number n of electrons tends to infinity while the confining potential is rescaled: mQ replaces Q, where m tends to infinity, so that $n/m\to t$. The limit distribution is given by the *equilibrium measure*, as was shown by Johansson [18] in the 1-dimensional context. It turns out that Johansson's proof carries through with only minor modifications also in the 2-dimensional case, as was explained earlier in our arXiv preprint [14]. Here, we make an effort to obtain the result under minimal smoothness and growth assumptions on the potential Q.

In Section 3, we connect the equilibrium measure with an obstacle problem and show how to apply the Kinderlehrer–Stampacchia–Caffarelli theory to obtain *a priori* smoothness of the solutions to the obstacle problem. We also show that the density of the equilibrium measure is given by ΔQ on the droplet, which permits us to reduce the complexity of the equilibrium measure to the study of its support (the droplet). Here, $\Delta:=\partial\bar\partial$ is a quarter of the usual Laplacian. The droplet is shown to equal the coincidence set for the associated obstacle problem, if we remove the so-called shallow points. For smooth strictly convex Q, the topology of the droplets is shown to be simple.

In Section 4, we introduce the notion of local droplets, which are obtained when we pass from the potential Q to its localization Q_Σ for subsets $\Sigma\subset\mathbb{C}$ (cf., for example [8]). The local

Received 11 May 2011; revised 13 February 2012.

2010 *Mathematics Subject Classification* 15A52, 31B20, 35R35, 62E20, 82B44 (primary), 76M40, 76D27, 31A05 (secondary).

The first author was supported by the Göran Gustafsson Foundation (KVA) and Vetenskapsrådet (VR). The second author was supported by NSF Grant no. 0201893.

droplets are partially ordered and in Section 5, we study maximal domination chains of local droplets. The maximal domination chains either end in a maximal local droplet, or grow to infinity. The local droplets appear to be natural from the point of view of physics (see, for example, [**23**]). They are also natural from the mathematical point of view: the description of all possible local droplets is exactly the inverse problem of potential theory.

One purpose with the material on domination chains of droplets in Section 5 is to provide a natural setting to analyse Laplacian growth (that is, the Hele-Shaw equation), which is known to be unstable in the forward time direction. This is explained in Section 6. The domination chains of droplets are interesting in part because of their integrability nature, especially in the case of potentials Q with $\Delta Q = \text{constant} > 0$ near the local droplet (such Q will be called constant strength potentials). This will be the topic of a forthcoming paper, where we will discuss the algebraic–geometric nature of maximal local droplets for constant strength potentials.

We should mention the recent paper [**22**] which studies Coulomb gases inthe low temperature limit (i.e., as the inverse temperature parameter β approaches infinity). We should also mention the recent paper [**13**] which considers a polyanalytically excited Vandermondian for the standard quadratic weight $Q(z) = |z|^2$ and for $\beta = 2$.

1.2. *Comments on the exposition*

While a few of the results covered in this paper are essentially understood, we believe the reader will appreciate a rather self-contained and easily accessible exposition. As for the treatment of Johansson's theorems in Appendix A, the extension to the 2-dimensional setting requires some care about details and, as far as we know, no general proof has been available so far beyond the arXiv preprint [**14**], where an excessive regularity condition was made to simplify the presentation (here, we remove that condition by modifying the smoothing argument of Johansson's paper [**18**]; see Subsection A.2).

The connection between equilibrium measures and obstacle problems is known (see, for example [**21**]). However, it is perhaps less well known that the Kinderlehrer–Stampacchia–Caffarelli theory (see [**19**]; cf. also [**17**], where the same technique was used) allows us to develop an understanding of the equilibrium measures in terms of their supports, the droplets. This contrasts with the 1-dimensional theory, where a lot of difficulty is involved in determining the density of the equilibrium measure. As for the treatment of the Hele-Shaw equation, our approach based on equilibrium measures and obstacle problems allows us to develop the theory with low regularity. The standard approach to Hele-Shaw flow theory is to use (partial) balayage and variational inequalities, see, for example, [**12**]. We prefer the obstacle problem approach because it is more intuitive and geometrically appealing.

2. *Quasi-classical limit of Coulomb gas ensembles*

2.1. *One-component plasma*

In the 2-dimensional Coulomb gas model (or rather the OCP model), we have n electrons located at points $\{z_j\}_{j=1}^n$ in the complex plane, influenced by an external field. The potential of interaction is

$$\log \frac{1}{|z_j - z_k|^2}, \quad j \neq k, \;\; j,k \in \{1,\ldots,n\},$$

while the external field potential is denoted by $V(z)$. The function

$$V : \mathbb{C} \longrightarrow \mathbb{R} \cup \{+\infty\}$$

is lower semi-continuous and sufficiently large to keep the electrons at finite distances. We shall supply the precise condition shortly. The combined potential energy resulting from particle interaction and the external potential is the function $\mathcal{E}_V : \mathbb{C}^n \to \mathbb{R} \cup \{+\infty\}$ given by

$$\mathcal{E}_V(z) = \frac{1}{2} \sum_{j,k:j\neq k} \log \frac{1}{|z_j - z_k|^2} + \sum_j V(z_j), \quad z = (z_1, \ldots, z_n) \in \mathbb{C}^n,$$

where the summation indices j and k are assumed confined to the set $\{1, \ldots, n\}$. In any reasonable gas dynamics model, the low-energy states are supposed to be more likely than the high-energy states. For a positive constant β, let $Z_n = Z_{n,\beta,V}$ denote the constant

$$Z_n = \int_{\mathbb{C}^n} e^{-(\beta/2)\mathcal{E}_V} \, \mathrm{dvol}_{2n},$$

where vol_{2n} denotes the standard volume measure in $\mathbb{C}^n \cong \mathbb{R}^{2n}$. We suppose that $0 < Z_n < +\infty$, which means that the potential V imposes a weak localization restraint on the plasma cloud. The corresponding Gibbs model then gives the joint density of states

$$\frac{1}{Z_n} \, e^{-(\beta/2)\mathcal{E}_V(z)},$$

where β has the interpretation as the inverse temperature. In terms of the usual van der Monde expression

$$\triangle(z) = \prod_{j,k:j<k} (z_k - z_j),$$

we may write

$$Z_n = \int_{\mathbb{C}^n} |\triangle(z)|^\beta e^{-(\beta/2)\sum_j V(z_j)} \, \mathrm{dvol}_{2n}(z).$$

We thus introduce a probability point process

$$\Pi_n \equiv \Pi_{n,\beta,V} \in \mathrm{prob}(\mathbb{C}^n)$$

($\mathrm{prob}(\mathbb{C}^n)$ is the convex set of all Borel probability measures on $\mathbb{C}^n$) by setting

$$d\Pi_n(z) = \frac{e^{-(\beta/2)\mathcal{E}_V(z)}}{Z_n} \, \mathrm{dvol}_{2n}(z) = \frac{|\triangle(z)|^\beta}{Z_n} e^{-(\beta/2)\sum_j V(z_j)} \, \mathrm{dvol}_{2n}(z), \quad z \in \mathbb{C}^n.$$

2.2. *Marginal measures*

For integers $k = 1, \ldots, n$, we define the marginal probability measure $\Pi_n^{(k)} \in \mathrm{prob}(\mathbb{C}^k)$ by setting

$$\Pi_n^{(k)}(e) = \Pi_n(e \times \mathbb{C}^{n-k}),$$

for Borel measurable subsets $e \subset \mathbb{C}^k$; in particular, $\Pi_n^{(n)} = \Pi_n$. The associated measures

$$\Gamma_n^{(k)} = \frac{n!}{(n-k)!} \Pi_n^{(k)}$$

are known as *intensity* (*or correlation*) *measures*. For $k = n$, we have $\Gamma_n^{(n)} = n!\Pi_n$, which is why we simplify the notation and write $\Gamma_n = \Gamma_n^{(n)}$. On the other hand, for $k = 1$, we have ($\mathbb{E}$ is the expectation operation)

$$\Gamma_n^{(1)}(e) = \mathbb{E}[\#\{j : z_j \in e\}],$$

where it is tacitly assumed that j is confined to the set $\{1, \ldots, n\}$, and $\#$ denotes counting measure. In a more explicit form, we have, for $n = 2$ and $k = 1$,

$$d\Gamma_2^{(1)}(\zeta) = \frac{2 \int_{\mathbb{C}} |\zeta - \xi|^\beta \, d\mu(\xi)}{\int_{\mathbb{C}^2} |\xi - \eta|^\beta \, d\mu(\xi) \, d\mu(\eta)} \, d\mu(\zeta), \quad \zeta \in \mathbb{C}, \tag{2.1}$$

where $d\mu(\xi) = e^{-(\beta/2)V(\xi)}\,d\mathrm{vol}_2(\xi)$. More generally, for $1 \leqslant k \leqslant n$ and a Borel subset $e \subset \mathbb{C}^k$, we have

$$\Gamma_n^{(k)}(e) = \mathbb{E}[\#\{(j_1, \ldots, j_k) \in \mathrm{perm}(k,n) : \ (z_{j_1}, \ldots, z_{j_k}) \in e\}],$$

where $\mathrm{perm}(k,n)$ stands for the collection of all permutations of length k of the set $\{1, \ldots, n\}$.

REMARK 2.1. In the above definition of the probability measure Π_n, we realize that

$$e^{-(\beta/2)\sum_j V(z_j)}\,d\mathrm{vol}_{2n}(z) = d\mu(z_1)\cdots d\mu(z_n), \quad z = (z_1, \ldots, z_n),$$

where

$$d\mu(\xi) = e^{-(\beta/2)V(\xi)}\,d\mathrm{vol}_2(\xi), \quad \xi \in \mathbb{C}.$$

Most of the above discussion does not depend on this particular structure of the measure μ, and we are free to consider more general measures. For instance, this allows us to include the 1-dimensional theory in the model.

2.3. *The random normal matrix model*

If $\beta = 2$, then the probability measure

$$d\Pi_n(z) = \frac{|\triangle(z)|^2}{Z_n} e^{-\sum_j V(z_j)}\,d\mathrm{vol}_{2n}(z)$$

with normalization constant

$$Z_n = \int_{\mathbb{C}^n} |\triangle(z)|^2 e^{-\sum_j V(z_j)}\,d\mathrm{vol}_{2n}(z)$$

describes the distribution of the eigenvalues of $n \times n$ random normal matrices with a joint probability measure proportional to

$$e^{-\mathrm{tr}\,V(M)}\,dM,$$

where 'tr' is the trace, and dM stands for the natural 'Haar-type' measure on the submanifold of all complex-valued $n \times n$ matrices M with $M^*M = MM^*$ (the normal matrices). In this case, the point process is determinantal:

$$d\Gamma_n^{(k)}(z) = \det[K_n(z_i, z_j)]_{i,j=1}^k e^{-\sum_j V(z_j)}\,d\mathrm{vol}_{2k}(z), \tag{2.2}$$

where K_n is the reproducing kernel in the polynomial Bargmann–Fock space

$$\mathrm{Pol}_n = \mathrm{span}\{1, z, \ldots, z^{n-1}\} \subset L^2(\mathbb{C}, e^{-V}).$$

We thus consider Pol_n as a finite-dimensional linear subspace of $L^2(\mathbb{C}, e^{-V})$ (linearity is always with respect to the field $\mathbb{C}$), and the Gram–Schmidt procedure supplies, for $j = 0, \ldots, n-1$, polynomials p_j of degree j and norm 1 such that $p_j \perp p_k$ for $j \neq k$. In terms of these orthogonal polynomials, we have

$$K_n(z, w) = \sum_{j=0}^{n-1} p_j(z)\bar{p}_j(w). \tag{2.3}$$

The algebraic mechanism behind the formula for the correlation measure $\Gamma_n^{(m)}$ is well understood. See, for instance, the book by Mehta [20].

2.4. *Aggregation of quantum droplets*

For reasons that will become clearer later on, we shall regard the point process $\Gamma_n = n!\Pi_n$ (or, equivalently, Π_n) as a quantum droplet. We are interested in the transition $\Gamma_n \to \Gamma_{n+1}$, which corresponds to adding one more electron to the droplet. A direct comparison of the processes Γ_n and Γ_{n+1} is not possible, and we are led to consider marginal intensities. The following lemma for $\beta = 2$ has the interpretation that if we add an electron, the expected number of k-tuples of electrons increases everywhere in $\mathbb{C}^k$.

LEMMA 2.2. *If* $\beta = 2$, *then*

$$\forall k \quad \Gamma_n^{(k)} \leqslant \Gamma_{n+1}^{(k)}.$$

Proof. In view of (2.3), we have

$$[K_{n+1}(z_i, z_j)]_{i,j=1}^k = [K_n(z_i, z_j)]_{i,j=1}^k + [p_n(z_i)\bar{p}_n(z_j)]_{i,j=1}^k,$$

where all matrices involved are positive (semi-)definite (the rightmost matrix has rank 1). As we compare with (2.2), we realize that the desired assertion

$$\det[K_n(z_i, z_j)]_{i,j=1}^k \leqslant \det[K_{n+1}(z_i, z_j)]_{i,j=1}^k$$

is an immediate consequence of the minimax principle (see, for example, the books of Dunford and Schwartz [**7**] and Gohberg and Krein [**11**]). □

REMARK 2.3. This 'aggregation' property might well be true for all $\beta \leqslant 2$ but it certainly fails for $\beta > 2$. We consider the illuminating special case $\Gamma_1^{(1)} \leqslant \Gamma_2^{(1)}$, which in the notation of (2.1) asserts that

$$\int_{\mathbb{C}^2} |\xi - \eta|^\beta \, d\mu(\xi)\, d\mu(\eta) \leqslant 2\mu(\mathbb{C}) \int_{\mathbb{C}} |\zeta - \xi|^\beta \, d\mu(\xi), \quad \zeta \in \mathbb{C}. \tag{2.4}$$

The measure $d\mu(\xi) = e^{-(\beta/2)V(\xi)}\, d\mathrm{vol}_2(\xi)$ can essentially be replaced by an fairly arbitrary positive Borel measure (with finite moments). As we plug in the choice $d\mu = d\delta_0 + d\delta_1$, we see that (2.4) is equivalent to

$$|\zeta|^\beta + |\zeta - 1|^\beta \geqslant \tfrac{1}{2}, \quad \zeta \in \mathbb{C}.$$

With $\zeta = \frac{1}{2}$ this gives $\beta \leqslant 2$. In fact, it is possible to show that the inequality $\Gamma_1^{(1)} \leqslant \Gamma_2^{(1)}$ holds generally for $0 < \beta \leqslant 2$. We outline the argument. It suffices to consider $\zeta = 0$ in (2.4), and to show that

$$\int_{\mathbb{C}^2} \{|\xi|^\beta + |\eta|^\beta - |\xi - \eta|^\beta\} \, d\mu(\xi)\, d\mu(\eta) \geqslant 0 \tag{2.5}$$

for all positive measures μ with finite moments. For $0 < \beta \leqslant 1$, the L^β triangle inequality shows that the integrand on the left-hand side is positive point-wise, and the assertion is immediate. We turn to the remaining case $1 < \beta < 2$. One first establishes with the methods of Calculus that

$$(1 + t + 2x)^{\beta/2} \leqslant 1 + t^{\beta/2} + \beta x, \quad -\sqrt{t} \leqslant x \leqslant \sqrt{t}, \ \ 0 < t < +\infty,$$

which in a complex form becomes

$$|1 + \tau|^\beta \leqslant 1 + |\tau|^\beta + \beta \operatorname{Re} \tau, \quad \tau \in \mathbb{D},$$

where $\mathbb{D}$ denotes the open unit disk in $\mathbb{C}$. By homogenization, this inequality leads to

$$|\xi - \eta|^\beta \leqslant |\xi|^\beta + |\eta|^\beta - \beta \min\{|\xi|^{\beta-2}, |\eta|^{\beta-2}\} \operatorname{Re}(\bar{\eta}\xi), \quad \xi, \eta \in \mathbb{C},$$

so that

$$|\xi|^\beta + |\eta|^\beta - |\xi - \eta|^\beta \geqslant \beta \min\{|\xi|^{\beta-2}, |\eta|^{\beta-2}\} \operatorname{Re}(\bar{\eta}\xi), \quad \xi, \eta \in \mathbb{C}.$$

So, to get (2.5) it suffices to obtain

$$\int_{\mathbb{C}^2} \min\{|\xi|^{\beta-2}, |\eta|^{\beta-2}\} \operatorname{Re}(\bar{\eta}\xi)\, d\mu(\xi)\, d\mu(\eta) \geqslant 0.$$

But this is an immediate consequence of Schur's product theorem for positive definite matrices (in this case, we have 'continuous' matrices), as both $\min\{|\xi|^{\beta-2}, |\eta|^{\beta-2}\}$ and $\operatorname{Re}(\bar{\eta}\xi)$ express positive definite kernels.

2.5. *Scaling and the class of weights*

If we keep the confining potential V fixed, and let n (the number of electrons) grow, the process Π_n will generically grow beyond any confinement. For this reason, it is necessary to jack up the confinement as n grows. This is achieved by putting $V = mQ$, where m is a scaling parameter and

$$Q: \ \mathbb{C} \longrightarrow \mathbb{R} \cup \{+\infty\}$$

is a fixed potential, assumed to be lower semi-continuous. To avoid degeneracy, we must suppose that $Q < +\infty$ at least on a set of positive area. From well-known physical considerations, it is natural to let m be essentially proportional to n. As we are free to pick Q as we like, we may assume that the proportionality constant is 1, that is, that $m = n + \mathrm{o}(n)$ as $n \to +\infty$. The growth requirement on Q which fits with this normalization is

$$\liminf_{|z|\to+\infty} \frac{Q(z)}{\log|z|^2} > 1. \tag{2.6}$$

In some potential-theoretic considerations the assumption (2.6) may be relaxed to

$$Q(z) - \log|z|^2 \to +\infty \quad \text{as } \ |z| \to +\infty. \tag{2.7}$$

For this reason, in the subsection below we assume generally that (2.7) holds, and when we need (2.6), we mention this explicitly. Please note that later on, the property (2.6) is called *extra growth*.

2.6. *The equilibrium measure*

We consider the limit of the point processes

$$\Pi_{mQ,n} \quad \text{as } n \longrightarrow +\infty \text{ while } \frac{n}{m} \longrightarrow 1,$$

while assuming that Q grows in accordance with (2.6). In this case, we have convergence of the saddle point configurations. More precisely, the probability measures

$$\sigma_n = \frac{1}{n}\sum_j \delta_{z_j}, \tag{2.8}$$

which minimize the functionals (we write $z = (z_1, \ldots, z_n)$)

$$I^{\#}_{mQ,n}[\sigma_n] := \frac{2}{n(n-1)}\mathcal{E}_{mQ}(z) = \frac{1}{n(n-1)} \sum_{j,k:j\neq k} \log\frac{1}{|z_j - z_k|^2} + \frac{2m}{n(n-1)}\sum_j Q(z_j),$$

converge as $n \to +\infty$ while $m = n + \mathrm{o}(n)$ in the weak-star sense of measures to the unique probability measure $\sigma = \hat{\sigma}_Q$ which minimizes the weighted logarithmic energy

$$I_Q[\sigma] := \int_{\mathbb{C}^2} \log\frac{1}{|\xi-\eta|^2}\, d\sigma(\xi)\, d\sigma(\eta) + 2\int_{\mathbb{C}} Q\, d\sigma. \tag{2.9}$$

This comes as no big surprise given the striking similarity of the expressions $I^{\#}_{mQ,n}[\sigma_n]$ and $I_Q[\sigma]$. The configuration of points corresponding to a minimizer σ_n is known as a collection of *weighted Fekete points*, and the measure $\hat{\sigma}_Q$ is called the *equilibrium measure*. The existence and uniqueness of the minimizing measure $\hat{\sigma}_Q$ is due to Frostman. Let $\mathrm{prob}_c(\mathbb{C})$ denote the convex body of all compactly supported Borel probability measures on $\mathbb{C}$.

THEOREM 2.4 (Frostman). *There exists a unique equilibrium measure $\hat{\sigma} = \hat{\sigma}_Q$ in $\mathrm{prob}_c(\mathbb{C})$ such that*

$$I_Q[\hat{\sigma}] = \inf_{\sigma} I_Q[\sigma],$$

the infimum being taken over all compactly supported probability measures σ.

For the proof, we refer to [**21**]. We will write

$$\gamma(Q) := I_Q[\hat{\sigma}_Q], \quad \gamma^*(Q) := \gamma(Q) - \int_{\mathbb{C}} Q\, d\hat{\sigma}_Q, \tag{2.10}$$

for the (modified) Robin constants involved. Let

$$L_Q(\xi,\eta) := \log \frac{1}{|\xi-\eta|^2} + Q(\xi) + Q(\eta), \tag{2.11}$$

and observe that for probability measures σ, we have

$$I_Q[\sigma] = \int_{\mathbb{C}^2} L_Q(\xi,\eta)\; d\sigma(\xi)\, d\sigma(\eta). \tag{2.12}$$

Next, we introduce the weighted potential

$$U_Q^\sigma(\xi) = \int_{\mathbb{C}} L_Q(\xi,\eta)\, d\sigma(\eta)$$

and observe that since

$$I_Q[\sigma] = \int_{\mathbb{C}} U_Q^\sigma(\xi)\, d\sigma(\xi) \tag{2.13}$$

we expect that the energy minimizer $\sigma = \hat{\sigma}_Q$ should have U_Q^σ constant on the support

$$S = S_Q := \operatorname{supp} \hat{\sigma}_Q,$$

and that constant should also equal the minimum value of U_Q^σ. We will at times use the notation $\hat{\sigma}_Q = \hat{\sigma}[Q]$ and $S_Q = S[Q]$. We use q.e. as short-hand for *quasi-everywhere.*

THEOREM 2.5 (Frostman). *The support S_Q of the equilibrium measure $\hat{\sigma}_Q$ is compact. Moreover, if $\gamma(Q)$ is as in (2.10), then $U_Q^{\hat{\sigma}_Q} \geqslant \gamma(Q)$ q.e. on $\mathbb{C}$, while $U_Q^{\hat{\sigma}_Q} \leqslant \gamma(Q)$ at each point of S_Q. The value $\gamma(Q)$ equals the minimal energy $I_Q[\hat{\sigma}_Q]$.*

For the proof, we refer to [**21**]. The number $e^{-\gamma(Q)}$ is said to be the weighted capacity.

In terms of the usual logarithmic potential

$$U^\sigma(\xi) = \int_{\mathbb{C}} \log \frac{1}{|\xi-\eta|^2}\, d\sigma(\eta),$$

we see that for a compactly supported probability measure σ,

$$U_Q^\sigma(\xi) = \int_{\mathbb{C}} L_Q(\xi,\eta)\, d\sigma(\eta) = U^\sigma(\xi) + Q(\xi) + \int_{\mathbb{C}} Q\, d\sigma,$$

which allows us to write Frostman's Theorem 2.5 in the following form.

THEOREM 2.6 (Frostman). *The support S_Q of the equilibrium measure $\hat\sigma_Q$ is compact. Moreover, if $\gamma^*(Q)$ is as in (2.10), then $U^{\hat\sigma_Q}+Q\geqslant\gamma^*(Q)$ q.e. on $\mathbb{C}$, while $U^{\hat\sigma_Q}+Q\leqslant\gamma^*(Q)$ at each point of S_Q.*

Let $\hat\sigma_{mQ,n}$ denote the probability measure σ_n given by (2.8) corresponding to a weighted Fekete point configuration (that is, a minimizing configuration). The convergence to the global energy minimizing measure is as follows.

THEOREM 2.7 (Fekete and Totik). *We assume that (2.6) holds. We then have the convergence*

$$\hat\sigma_{mQ,n}\longrightarrow\hat\sigma_Q\quad\text{as } n\longrightarrow+\infty \text{ while } m=n+\mathrm{o}(n)$$

in the weak-star sense of measures. Moreover, we have convergence in energy:

$$I^{\#}_{mQ,n}[\hat\sigma_{mQ,n}]\longrightarrow I_Q[\hat\sigma_Q]=\gamma(Q)\quad\text{as } n\longrightarrow+\infty \text{ while } m=n+\mathrm{o}(n).$$

For the proof, we refer to [**21**, p. 145].

2.7. *Johansson's marginal measure theorem for the plane*

For a probability measure $\sigma\in\mathrm{prob}(\mathbb{C})$ and an integer $k=1,2,3,\ldots$, we denote by $\sigma^{\otimes k}\in\mathrm{prob}(\mathbb{C}^k)$ the product measure given by

$$d\sigma^{\otimes k}(z_1,\ldots,z_k)=d\sigma(z_1)\cdots d\sigma(z_k).$$

DEFINITION 2.8. We say that Q has *extra growth* provided that

$$Q(z)\geqslant(1+\delta_0)\log(1+|z|^2)-C_0,\quad z\in\mathbb{C},\tag{2.14}$$

holds for some small but positive value of δ_0 and some (positive) real constant C_0. Moreover, we say that Q is *regular* provided that it is bounded and continuous in an open neighbourhood of $\mathcal{S}_Q=\mathrm{supp}\,\hat\sigma_Q$.

THEOREM 2.9. *Suppose Q is regular with extra growth. Then, for every $k=1,2,3,\ldots$, we have the convergence*

$$\Pi^{(k)}_{mQ,n}\longrightarrow\hat\sigma_Q^{\otimes k}\quad\text{as } n\longrightarrow+\infty \text{ while } m=n+\mathrm{o}(n),$$

in the weak-star sense of measures.

REMARK 2.10. (i) Johansson [**18**] proves his theorem in the degenerate real line case when $Q(\xi)=+\infty$ for $\xi\in\mathbb{C}\setminus\mathbb{R}$ (the Hermitian matrix case). This can be viewed as a limit case of our considerations. However, the approach of Johansson's proof can be modified so as to include the complex plane case stated here. We indicate the necessary modifications in an appendix below.

(ii) An alternative formulation of Theorem 2.9 runs as follows. As $n\to+\infty$ while $m=n+\mathrm{o}(n)$, the random variables $z_1,\ldots,z_k$ on $(\mathbb{C}^n,\Pi_{mQ,n})$ are asymptotically independent and identically distributed with law $\hat\sigma_Q$.

(iii) We now find an application of Theorem 2.9 to linear statistics. Let $\mathrm{C_b}(\mathbb{C}^k)$ denote the Banach space of bounded continuous functions in $\mathbb{C}^k$. Moreover, let the trace $\mathrm{tr}_n f$ of the function $f \in \mathrm{C_b}(\mathbb{C})$ be given by

$$\mathrm{tr}_n f = \sum_j f(z_j),$$

where the sum as usual runs over $j = 1, \ldots, n$ and $z_1, \ldots, z_n$ are random variables with joint probability $(\mathbb{C}^n, \Pi_{mQ,n})$. For each $j = 1, \ldots, n$, we have, in view of Johansson's marginal measure theorem, for $f \in \mathrm{C_b}(\mathbb{C})$,

$$\mathbb{E}[f(z_j)] = \int_{\mathbb{C}} f(\xi)\, d\Pi^{(1)}_{mQ,n}(\xi) \longrightarrow \int_{\mathbb{C}} f(\xi)\, d\hat{\sigma}_Q(\xi) =: \langle f, \hat{\sigma}_Q \rangle,$$

as $n \to +\infty$ while $m = n + \mathrm{o}(n)$. By forming the average over j, we obtain, for $f \in \mathrm{C_b}(\mathbb{C})$,

$$\mathbb{E}\left[\frac{1}{n}\mathrm{tr}_n f\right] = \int_{\mathbb{C}} f(\xi)\, d\Pi^{(1)}_{mQ,n}(\xi) \longrightarrow \langle f, \hat{\sigma}_Q \rangle,$$

as $n \to +\infty$ while $m = n + \mathrm{o}(n)$. There is an analogous statement which holds for functions $f \in \mathrm{C_b}(\mathbb{C}^k)$ and involves the measure $\hat{\sigma}_Q^{\otimes k}$ in place of $\hat{\sigma}_Q$. This more general statement allows us to obtain that for real-valued $f \in \mathrm{C_b}(\mathbb{C})$, ($k$ is a fixed nonnegative integer)

$$\mathbb{E}\left[\left(\frac{1}{n}\mathrm{tr}_n f\right)^k\right] \longrightarrow (\langle f, \hat{\sigma}_Q \rangle)^k,$$

as $n \to +\infty$ while $m = n + \mathrm{o}(n)$. This expresses that $(1/n)\mathrm{tr}_n f$ tends to the constant value $\langle f, \hat{\sigma}_Q \rangle$ in all moments as $n \to +\infty$ while $m = n + \mathrm{o}(n)$, and hence, in particular, we have convergence in distribution (as in the weak law of large numbers).

(iv) We remark that Theorem 2.9 holds independently of the value of the inverse temperature β. However, for $\beta = 2$, much more precise statements have been obtained recently in [**1**–**3**]. The reason why this is possible is the determinantal property (2.2). To give some hints about the results, we introduce the fluctuation

$$\mathrm{fl}_n f = \mathrm{tr}_n f - n\langle f, \hat{\sigma}_Q \rangle.$$

In view of (iii), we know that $(1/n)\mathrm{fl}_n f \to 0$ in moments and hence in distribution as $n \to +\infty$ while $m = n + \mathrm{o}(n)$. Next, suppose $n \to +\infty$ while $m = n + \mathrm{o}(1)$, which means that m is kept much closer to n than before, and suppose also that the function Q is real-analytically smooth with $\Delta Q > 0$ in the interior of S_Q (we recall that S_Q is the support of the equilibrium measure $\hat{\sigma}_Q$). In analogy with the CLT (central limit theorem), it is shown in [**1**, **2**] that under some additional assumptions, the stochastic variable $\mathrm{fl}_n f$ converges in distribution to a real-valued Gaussian with expectation e_f and variance v_f,

$$e_f = \frac{1}{2\pi}\int_{S_Q} f \Delta \log \Delta Q \, \mathrm{dvol}_2, \quad v_f = \frac{1}{4\pi}\int_{S_Q} |\nabla f|^2 \, \mathrm{dvol}_2,$$

provided the function f is real-valued, C^∞-smooth, and is supported in the interior of S_Q. The extension to general test functions f is obtained in [**3**]; the general formulae for e_f and v_f include boundary effects.

2.8. *Johansson's free energy theorem for the plane*

We recall the expression for the normalization constant

$$Z_{m,n} = \int_{\mathbb{C}^n} |\triangle(z)|^\beta e^{-(\beta/2)m\sum_j Q(z_j)} \, \mathrm{dvol}_{2n}(z),$$

which we write in the form

$$Z_{m,n} = \int_{\mathbb{C}^n} \exp\left\{-\frac{\beta}{4}\sum_{j,k:j\neq k} L_Q(z_j, z_k) + \frac{\beta}{2}(n - m - 1)\sum_j Q(z_j)\right\} \mathrm{dvol}_{2n}(z), \qquad (2.15)$$

where L_Q is as in (2.11). The quantity

$$\frac{1}{n(n-1)}\log Z_{m,n}$$

has in the physics literature acquired the name *free energy* (frequently n^2 is used in place of $n(n-1)$; asymptotically, there is no difference). See Definition 2.8 for the terms *regular* and *extra growth*.

THEOREM 2.11. *Suppose Q is regular with extra growth. Then*

$$\frac{1}{n(n-1)}\log Z_{m,n}\longrightarrow -\frac{\beta}{4}\gamma(Q)=-\frac{\beta}{4}I_Q[\hat\sigma_Q]\quad \text{as } n\longrightarrow+\infty \text{ while } m=n+\mathrm{o}(n).$$

REMARK 2.12. As with Theorem 2.9, Johansson [**18**] proves his theorem in the degenerate real line case when $Q(\xi)=+\infty$ for $\xi\in\mathbb{C}\setminus\mathbb{R}$ (the Hermitian matrix case). This can be viewed as a limit case of our considerations. However, the approach of Johansson's proof can be modified so as to include the complex plane case stated here. We indicate the necessary modifications in an appendix below.

2.9. *Aggregation of equilibrium measures*

We now look at the quasi-classical limit of the evolution of quantum droplets (the addition of more electrons to the droplet). This will allow us to understand how the quantum process is related to a growth process of Hele-Shaw type for compact sets in the plane.

We restrict our attention to the potentials Q that satisfy a scale invariant version of the growth condition (2.6), namely

$$\liminf_{|z|\to+\infty}\frac{Q(z)}{\log|z|^2}\geqslant A,$$

no matter how big the positive parameter A gets. We will be interested in the evolution of positive measures

$$\hat\sigma_t\equiv\hat\sigma_t[Q]:=t\hat\sigma_{Q/t},$$

where t ranges over $0<t<+\infty$. We write $S_t=S_t[Q]$ for the support of the measure $\hat\sigma_t[Q]$ (that is, $S_t=S_{Q/t}$). Note that $\hat\sigma_t[Q]$ has total mass t. The process of increasing the parameter t has the following interpretation. We consider the limit process of letting $n\to+\infty$ while $m=n/t+\mathrm{o}(n)$. In other words, $m\to+\infty$ while $n=mt+\mathrm{o}(m)$. An increase of t therefore has the interpretation of increasing the total number of electrons n for fixed m. To rescale, we introduce $m'=mt$, so that the relationship reads $n=m'+\mathrm{o}(m')$. Since $mQ=m'Q/t$, rescaling also means that we must replace Q by Q/t. In view of Johansson's theorem, we find that

$$\lim\mathbb{E}\,\frac{\#\{\text{electrons in } e\}}{m}=t\lim\mathbb{E}\,\frac{\#\{\text{electrons in } e\}}{m'}=t\hat\sigma_{Q/t}(e)=\hat\sigma_t[Q](e).$$

In other words, the growth process $\hat\sigma_t[Q]$ is the quasi-classical limit of the growth process of adding electrons to the quantum droplet. We shall see that if Q is C^2-smooth, the measure $\hat\sigma_t[Q]$ is determined uniquely by its support $S_t[Q]$. We understand the set $S_t[Q]$ as a (classical) droplet; cf. Subsection 4.1 for a precise definition.

COROLLARY 2.13. *The family of measures $\hat\sigma_t[Q]$ is monotonically increasing in t.*

Proof. This is true for quantum droplets if $\beta=2$; the quasi-classical limit does not depend on β. □

REMARK 2.14. It is not hard to write down a potential theoretic proof of this fact; see Proposition 3.15.

Appendix A. *Proof of Johansson's marginal measure and free energy theorems for the plane*

A.1. *Fekete configurations*

The approach to prove Johansson's theorem in this setting is to show that point configurations $(z_1, \dots, z_n)$ whose associated energy functional

$$I^\sharp_{n,(n-1)Q}[\sigma_n] = \frac{1}{n(n-1)} \sum_{j,k:j\neq k} L_Q(z_j, z_k)$$

deviate substantially from the minimum are highly unlikely. We note that since $m = n + \mathrm{o}(n)$ is assumed, the choice to replace m by $n-1$ in the energy is reasonable. Let us write

$$\hat\sigma_n := \hat\sigma_{n,(n-1)Q}$$

for the minimizing (Fekete) measure in the context of Theorem 2.7 and also write

$$I^\sharp_n[\hat\sigma_n] := I^\sharp_{n,(n-1)Q}[\hat\sigma_{n,(n-1)Q}]$$

for the associated energy. By Saff and Totik [**21**, pp. 143–145], the sequence of energies $I^\sharp_n[\hat\sigma_n]$ is decreasing in n, and converges to $I_Q[\hat\sigma_Q] = \gamma(Q)$ as $n \to +\infty$ (cf. Theorem 2.7).

A.2. *An entropy estimate*

We introduce an auxiliary Borel measurable function $\phi : \mathbb{C} \to [0, +\infty)$ with

$$\int_{\mathbb{C}} \phi \, d\mathrm{vol}_2 = 1, \qquad \int_{\mathbb{C}} (Q + |\log \phi|)\phi \, d\mathrm{vol}_2 < +\infty, \tag{A.1}$$

with the understanding that $\phi \log \phi = 0$ at points where $\phi = 0$. We artificially smuggle it into the expression (2.15) for $Z_{m,n}$:

$$\begin{aligned} Z_{m,n} = \int_{\mathbb{C}^n} \exp &\left\{ -\frac{\beta}{4} \sum_{j,k:j\neq k} L_Q(z_j, z_k) + \frac{\beta}{2}(n-m-1) \sum_j Q(z_j) - \sum_j \log \phi(z_j) \right\} \\ &\times \prod_j \phi(z_j) \, d\mathrm{vol}_{2n}(z). \end{aligned}$$

Now, by Jensen's inequality, we have, with $d\sigma_\phi = \phi \, d\mathrm{vol}_2$,

$$\begin{aligned} \log Z_{m,n} \geqslant \int_{\mathbb{C}^n} &\left\{ -\frac{\beta}{4} \sum_{j,k:j\neq k} L_Q(z_j, z_k) + \frac{\beta}{2}(n-m-1) \sum_j Q(z_j) - \sum_j \log \phi(z_j) \right\} \\ &\times \prod_j \phi(z_j) \, d\mathrm{vol}_{2n}(z) \\ &= -\frac{\beta n(n-1)}{4} \int_{\mathbb{C}^2} L_Q(\xi, \eta) \, d\sigma_\phi(\xi) \, d\sigma_\phi(\eta) + \frac{\beta}{2} n(n-m-1) \int_{\mathbb{C}} Q \, d\sigma_\phi \\ &\quad - n \int_{\mathbb{C}} \log \phi \, d\sigma_\phi, \end{aligned} \tag{A.2}$$

where we used repeatedly that σ_ϕ is a probability measure. We rewrite this as

$$\frac{1}{n(n-1)}\log Z_{m,n} \geqslant -\frac{\beta}{4}I_Q[\sigma_\phi] + \beta\frac{n-m-1}{2(n-1)}\int_{\mathbb{C}} Q\,d\sigma_\phi - \frac{1}{n-1}\int_{\mathbb{C}}\log\phi\,d\sigma_\phi,$$

This gives (as $n\to+\infty$ while $m = n + \mathrm{o}(n)$)

$$\liminf \frac{1}{n(n-1)}\log Z_{m,n} \geqslant -\frac{\beta}{4}I_Q[\sigma_\phi]. \tag{A.3}$$

The condition on ϕ that $\phi\log\phi \in L^1(\mathbb{C})$ is of entropy type and this is the reason why we call (A.3) an *entropy estimate*. We would like to plug in the choice $\sigma_\phi = \hat\sigma_Q$ into the entropy estimate (A.3) to obtain an effective bound. At this point, we do not know enough about $\hat\sigma_Q$ to be sure whether it is of the form σ_ϕ with ϕ meeting (A.1). To remedy this, we consider the function $\phi_r : \mathbb{C}\to[0,+\infty)$ given by $(0<r\leqslant 1)$

$$\phi_r(\xi) = \frac{1}{\pi r^2}\int_{\mathbb{D}(\xi,r)} d\hat\sigma_Q(\eta) = \frac{1}{\pi r^2}\int_{\tau\in\mathbb{D}(0,r)} d\hat\sigma_Q(\xi-\tau);$$

this amounts to convolution with the normalized characteristic function of the disk $\mathbb{D}(0,r)$. The corresponding measure

$$d\sigma_r := d\sigma_{\phi_r} = \phi_r\,d\mathrm{vol}_2$$

is a compactly supported (Borel) probability measure, with density $\phi_r\in L^\infty(\mathbb{C})$, so that (A.1) holds with ϕ_r in place of ϕ. By the standard properties of convolutions, $\sigma_r\to\hat\sigma_Q$ in the weak-star sense of measures as $r\to 0$. We claim that we also have convergence in energy,

$$I_Q[\sigma_r] \longrightarrow I_Q[\hat\sigma_Q] = \gamma(Q) \quad \text{as } r\longrightarrow 0. \tag{A.4}$$

Suppose for the moment that we have obtained (A.4). Then we find from (A.3) and (A.4) that

$$\liminf \frac{1}{n^2}\log Z_{m,n} \geqslant -\frac{\beta}{4}I_Q[\hat\sigma_Q] = -\frac{\beta}{4}\gamma(Q). \tag{A.5}$$

To obtain (A.4), we note that interchanging the order of integration gives

$$I_Q[\sigma_r] - I_Q[\hat\sigma_Q] = 2\int_{\mathbb{C}} Q(d\sigma_r - d\hat\sigma_Q) + \int_{\mathbb{C}^2}\Lambda_r(\xi,\eta)\,d\hat\sigma_Q(\xi)\,d\hat\sigma_Q(\eta),$$

where

$$\Lambda_r(\xi,\eta) = \frac{2}{\pi^2 r^4}\int_{(\tau,\tau')\in\mathbb{D}(0,r)^2}\left[\log|(\xi+\tau)-(\eta+\tau')| - \log|\xi-\eta|\right] d\mathrm{vol}_2(\tau)\,d\mathrm{vol}_2(\tau').$$

The support of σ_r is at most within distance r from the support $\mathcal{S}_Q$ of $\hat\sigma_Q$, so in view of the assumption that Q be bounded and continuous in a fixed neighbourhood of $\mathcal{S}_Q$, we obtain

$$\int_{\mathbb{C}} Q(d\sigma_r - d\hat\sigma_Q) = \int_{\mathbb{C}} Q\,d\sigma_r - \int_{\mathbb{C}} Q\,d\hat\sigma_Q \longrightarrow 0 \quad \text{as } r\longrightarrow 0. \tag{A.6}$$

Next, we rewrite the expression for Λ_r:

$$\begin{aligned}\Lambda_r(\xi,\eta) &= \frac{2}{\pi^2 r^4}\int_{(\tau,\tau')\in\mathbb{D}(0,r)^2}\log\left|1+\frac{\tau-\tau'}{\xi-\eta}\right| d\mathrm{vol}_2(\tau)\,d\mathrm{vol}_2(\tau')\\ &= \frac{2}{\pi^2 r^4}\int_{\tau''\in\mathbb{D}(0,2r)}\mathrm{vol}_2(\mathbb{D}(0,r)\cap\mathbb{D}(\tau'',r))\log\left|1+\frac{\tau''}{\xi-\eta}\right| d\mathrm{vol}_2(\tau'').\end{aligned}$$

We use that the common area of the two intersecting circular disks is

$$\mathrm{vol}_2(\mathbb{D}(0,r)\cap\mathbb{D}(\tau'',r)) = 2r^2\arccos\frac{|\tau''|}{2r} - r|\tau''|\sqrt{1-\frac{|\tau''|^2}{4r^2}}$$

to obtain

$$\Lambda_r(\xi,\eta)=\frac{4}{\pi^2 r^2}\int_{\tau''\in\mathbb{D}(0,2r)}\left\{\arccos\frac{|\tau''|}{2r}-\frac{|\tau''|}{2r}\sqrt{1-\frac{|\tau''|^2}{4r^2}}\right\}\log\left|1+\frac{\tau''}{\xi-\eta}\right|\mathrm{dvol}_2(\tau'').$$

The identity

$$\int_{-\pi}^{\pi}\log|1+\lambda\,e^{\mathrm{i}\theta}|\,d\theta=2\pi\log^+|\lambda|,\quad \lambda\in\mathbb{C},$$

where for real $x\geqslant 0$, $\log^+ x=\max\{0,\log x\}$, shows that

$$\Lambda_r(\xi,\eta)=0\quad\text{if } 2r\leqslant|\xi-\eta|,$$

while

$$\Lambda_r(\xi,\eta)=\frac{8}{\pi r^2}\int_{|\xi-\eta|}^{2r}\left\{\arccos\frac{s}{2r}-\frac{s}{2r}\sqrt{1-\frac{s^2}{4r^2}}\right\}\log\frac{s}{|\xi-\eta|}\,s\,ds\quad\text{if } |\xi-\eta|<2r.$$

In the latter case, we may use that for $|\xi-\eta|<s<2r$,

$$0\leqslant\arccos\frac{s}{2r}-\frac{s}{2r}\sqrt{1-\frac{s^2}{4r^2}}\leqslant\frac{\pi}{2},\quad 0\leqslant\log\frac{s}{|\xi-\eta|}\leqslant\log\frac{2r}{|\xi-\eta|},$$

to conclude that

$$0\leqslant\Lambda_r(\xi,\eta)\leqslant 8\log\frac{2r}{|\xi-\eta|}\quad\text{if } |\xi-\eta|<2r.$$

It follows that generally, we have

$$0\leqslant\Lambda_r(\xi,\eta)\leqslant 8\log^+\frac{2r}{|\xi-\eta|}.\tag{A.7}$$

The measure $\hat\sigma_Q$ has compact support and finite logarithmic energy,

$$\int_{\mathbb{C}^2}\log\frac{1}{|\xi-\eta|}\,d\hat\sigma_Q(\xi)\,d\hat\sigma_Q(\eta)<+\infty,$$

so that if we use (A.7) and the Lebesgue's domintated convergence theorem, we see that

$$\int_{\mathbb{C}^2}\Lambda_r(\xi,\eta)\,d\hat\sigma_Q(\xi)\,d\hat\sigma_Q(\eta)\longrightarrow 0\quad\text{as } r\longrightarrow 0.$$

As we combine this with (A.6), the claimed energy convergence (A.4) is immediate, and hence (A.5) follows.

A.3. *Low probability of high-energy configurations*

In view of (A.5), we have

$$\frac{1}{n(n-1)}\log Z_{m,n}\geqslant-\frac{\beta}{4}(\gamma(Q)+\varepsilon),\tag{A.8}$$

for fixed positive ε and large enough n. In this context, we think of $m=m_n$ as (fixed) sequence which depends on n, with $m=m_n=n+\mathrm{o}(n)$.

We put

$$G(\xi,\eta):=\log\frac{(1+|\xi|^2)(1+|\eta|^2)}{|\xi-\eta|^2}\geqslant 0,\quad \xi,\eta\in\mathbb{C}.\tag{A.9}$$

In view of the assumed extra growth (2.14), we have

$$
\begin{aligned}
L_Q(\xi,\eta) &= \log\frac{1}{|\xi-\eta|^2} + Q(\xi) + Q(\eta) \\
&\geqslant \log\frac{1}{|\xi-\eta|^2} + \frac{\delta_0}{1+\delta_0}[Q(\xi)+Q(\eta)] + \log[(1+|\xi|^2)(1+|\eta|^2)] - \frac{2C_0}{1+\delta_0} \\
&= G(\xi,\eta) + \frac{\delta_0}{1+\delta_0}[Q(\xi)+Q(\eta)] - \frac{2C_0}{1+\delta_0}, \qquad \text{(A.10)}
\end{aligned}
$$

where δ_0 and C_0 are as in (2.14). To simplify the notation, we write, with $z=(z_1,\ldots,z_n)$,

$$
L_Q^{\langle\!\langle n\rangle\!\rangle}(z) = \sum_{j,k:j\neq k} L_Q(z_j,z_k) \quad \text{and} \quad G^{\langle\!\langle n\rangle\!\rangle}(z) = \sum_{j,k:j\neq k} G(z_j,z_k) \geqslant 0,
$$

where it is assumed that j and k range over $\{1,\ldots,n\}$. These expressions are of 'double trace type' associated with the functions L_Q and G (see (2.11) and (A.9)). We also have the 'trace type' expressions (with $z=(z_1,\ldots,z_n)$)

$$
Q^{\langle n\rangle}(z) = \sum_j Q(z_j) \quad \text{and} \quad \Lambda^{\langle n\rangle}(z) := \sum_j \log(1+|z_j|^2).
$$

It now follows from (A.10) that

$$
L_Q^{\langle\!\langle n\rangle\!\rangle}(z) \geqslant G^{\langle\!\langle n\rangle\!\rangle}(z) + \frac{2\delta_0(n-1)}{1+\delta_0} Q^{\langle n\rangle}(z) - \frac{2C_0}{1+\delta_0} n(n-1), \qquad \text{(A.11)}
$$

while a direct application of the extra growth condition (2.14) leads to

$$
Q^{\langle n\rangle}(z) \geqslant (1+\delta_0)\Lambda^{\langle n\rangle}(z) - C_0 n. \qquad \text{(A.12)}
$$

The point with introducing this notation is that (2.15) simplifies to

$$
Z_{m,n} = \int_{\mathbb{C}^n} \exp\left\{-\frac{\beta}{4} L_Q^{\langle\!\langle n\rangle\!\rangle}(z) + \frac{\beta}{2}(n-m-1) Q^{\langle n\rangle}(z)\right\} \mathrm{dvol}_{2n}(z), \qquad \text{(A.13)}
$$

while the probability density becomes

$$
d\Pi_{mQ,n}(z) = \frac{1}{Z_{m,n}} \exp\left\{-\frac{\beta}{4} L_Q^{\langle\!\langle n\rangle\!\rangle}(z) + \frac{\beta}{2}(n-m-1) Q^{\langle n\rangle}(z)\right\} \mathrm{dvol}_{2n}(z), \qquad \text{(A.14)}
$$

As mentioned in Subsection A.1, we have the estimate

$$
\frac{1}{n(n-1)} L_Q^{\langle\!\langle n\rangle\!\rangle}(z) \geqslant \gamma(Q), \quad z=(z_1,\ldots,z_n)\in\mathbb{C}^n. \qquad \text{(A.15)}
$$

We introduce the set

$$
\mathcal{A}(n,\epsilon) = \left\{ z\in\mathbb{C}^n : \ \frac{1}{n(n-1)} L_Q^{\langle\!\langle n\rangle\!\rangle}(z) \leqslant \gamma(Q)+\epsilon \right\}, \qquad \text{(A.16)}
$$

where ϵ is a positive real number.

PROPOSITION A.1. *There exists a positive integer N_0, which depends on $\epsilon>0$ but not on $a\geqslant 0$, such that*

$$
\Pi_{mQ,n}(\mathbb{C}^n\setminus\mathcal{A}(n,\epsilon+a)) \leqslant e^{-\beta a n(n-1)/8}, \quad n\geqslant N_0,
$$

provided the sequence $m=m_n=n+\mathrm{o}(n)$ is kept fixed.

Proof. By definition, we have

$$
\frac{1}{n(n-1)} L_Q^{\langle\!\langle n\rangle\!\rangle}(z) > \gamma(Q)+\epsilon+a, \quad z\in\mathbb{C}^n\setminus\mathcal{A}(n,\epsilon+a). \qquad \text{(A.17)}
$$

We rewrite (A.11) as

$$\frac{1}{n(n-1)}L_Q^{\langle\langle n\rangle\rangle}(z) \geqslant \frac{2\delta_0}{(1+\delta_0)n}Q^{\langle n\rangle}(z) - \frac{2C_0}{1+\delta_0}, \quad z\in\mathbb{C}^n, \tag{A.18}$$

and form a convex combination of (A.17) and (A.18) (we keep θ fixed with $0<\theta<1$)

$$\frac{1}{n(n-1)}L_Q^{\langle\langle n\rangle\rangle}(z) \geqslant (1-\theta)(\gamma(Q)+\epsilon+a) + \frac{\theta}{1+\delta_0}\left\{\frac{2\delta_0}{n}Q^{\langle n\rangle}(z) - 2C_0\right\}, \\ z\in\mathbb{C}^n\setminus\mathcal{A}(n,\epsilon+a). \tag{A.19}$$

The exponent in the density defining $\Pi_{mQ,n}$ is (cf. (A.13))

$$-\frac{\beta}{4}\sum_{j,k:j\neq k}L_Q(z_j,z_k) + \frac{\beta}{2}(n-m-1)\sum_j Q(z_j) = -\frac{\beta}{4}L_Q^{\langle\langle n\rangle\rangle}(z) + \frac{\beta}{2}(n-m-1)Q^{\langle n\rangle}(z),$$

and in view of the estimate (A.19), we obtain

$$\begin{aligned}-\frac{\beta}{4}L_Q^{\langle\langle n\rangle\rangle}(z) + \frac{\beta}{2}(n-m-1)Q^{\langle n\rangle}(z) \leqslant & -\frac{\beta}{4}n(n-1)(1-\theta)(\gamma(Q)+\epsilon+a) \\ & -\frac{\beta}{2}\left\{\theta(n-1)\frac{\delta_0}{1+\delta_0} - (n-m-1)\right\}Q^{\langle n\rangle}(z) \\ & + \frac{C_1\theta\beta}{4}n(n-1), \quad z\in\mathbb{C}^n\setminus\mathcal{A}(n,\epsilon+a).\end{aligned} \tag{A.20}$$

If

$$\frac{m}{n-1} > 1 - \frac{\theta\delta_0}{1+\delta_0}, \tag{A.21}$$

holds, which is bound to be the case for big enough n (provided θ is kept away from 0), since $m = n + \mathrm{o}(n)$, the expression in front of $Q^{\langle n\rangle}(z)$ on the right-hand side of (A.20) is negative, and we may apply (A.12) to (A.20), and arrive at

$$\begin{aligned}-\frac{\beta}{4}L_Q^{\langle\langle n\rangle\rangle}(z) + \frac{\beta}{2}(n-m-1)Q^{\langle n\rangle}(z) \leqslant & -\frac{\beta}{4}n(n-1)(1-\theta)(\gamma(Q)+\epsilon+a) \\ & -\frac{\beta}{2}\{\theta(n-1)\delta_0 - (1+\delta_0)(n-m-1)\}\Lambda^{\langle n\rangle}(z) \\ & + \frac{C_0\beta\theta}{2}n(n-1) - \frac{C_0\beta}{2}n(n-m-1), \\ & z\in\mathbb{C}^n\setminus\mathcal{A}(n,\epsilon+a).\end{aligned} \tag{A.22}$$

As a consequence, we find that

$$\begin{aligned}&\Pi_{mQ,n}(\mathbb{C}^n\setminus\mathcal{A}(n,\epsilon+a)) \\ &\quad = \frac{1}{Z_{m,n}}\int_{\mathbb{C}^n\setminus\mathcal{A}(n,\epsilon+a)}\exp\left\{-\frac{\beta}{4}L_Q^{\langle\langle n\rangle\rangle}(z) + \frac{\beta}{2}(n-m-1)Q^{\langle n\rangle}(z)\right\}\mathrm{dvol}_{2n}(z) \\ &\quad \leqslant \frac{1}{Z_{m,n}}\exp\left\{-\frac{\beta}{4}n(n-1)(1-\theta)(\gamma(Q)+\epsilon+a)\right. \\ &\qquad \left. + \frac{C_0\beta\theta}{2}n(n-1) - \frac{C_0\beta}{2}n(n-m-1)\right\} \\ &\qquad \times\left\{\int_{\mathbb{C}}(1+|\xi|^2)^{-(\beta/2)\{\theta(n-1)\delta_0-(1+\delta_0)(n-m-1)\}}\,\mathrm{dvol}_2(\xi)\right\}^n.\end{aligned} \tag{A.23}$$

An exercise involving polar coordinates convinces us that, for $\alpha>1$,

$$\int_{\mathbb{C}}(1+|\xi|^2)^{-\alpha}\,\mathrm{dvol}_2(\xi) = \frac{\pi}{\alpha-1}, \tag{A.24}$$

and we see that (A.23) entails that

$$\Pi_{mQ,n}(\mathbb{C}^n\setminus\mathcal{A}(n,\epsilon+a))\leqslant\frac{1}{Z_{m,n}}\exp\bigg\{-\frac{\beta}{4}n(n-1)(1-\theta)(\gamma(Q)+\epsilon+a)+\frac{C_0\beta\theta}{2}n(n-1)-\frac{C_0\beta}{2}n(n-m-1)\bigg\}\times\bigg\{\frac{2\pi}{\beta\{\theta(n-1)\delta_0-(1+\delta_0)(n-m-1)\}-2}\bigg\}^n, \tag{A.25}$$

provided that

$$\frac{m}{n-1}>1-\frac{\theta\delta_0}{1+\delta_0}+\frac{2}{\beta(1+\delta_0)(n-1)}.$$

Let us assume slightly more, namely that

$$\frac{m}{n-1}>1-\frac{\theta\delta_0}{1+\delta_0}+\frac{2(1+\pi)}{\beta(1+\delta_0)(n-1)}, \tag{A.26}$$

which is a little stronger than (A.21), and holds for big enough n (as long as θ is kept away from 0), since $m=n+\mathrm{o}(n)$. This allows us to get rid of the last factor on the right-hand side of (A.25):

$$\Pi_{mQ,n}(\mathbb{C}^n\setminus\mathcal{A}(n,\epsilon+a))\leqslant\frac{1}{Z_{m,n}}\exp\bigg\{-\frac{\beta}{4}n(n-1)(1-\theta)(\gamma(Q)+\epsilon+a)+\frac{C_0\beta\theta}{2}n(n-1)-\frac{C_0\beta}{2}n(n-m-1)\bigg\}. \tag{A.27}$$

We finally implement the estimate (A.8), and obtain

$$\Pi_{mQ,n}(\mathbb{C}^n\setminus\mathcal{A}(n,\epsilon+a))\leqslant\exp\bigg\{\frac{\beta}{4}n(n-1)\bigg[\theta\gamma(Q)-(1-\theta)(\epsilon+a)+\varepsilon+2\theta C_0-2C_0\bigg(1-\frac{m}{n-1}\bigg)\bigg]\bigg\}. \tag{A.28}$$

The constant C_0 is assumed positive and we may therefore pick a small θ, $0<\theta<\frac{1}{2}$, such that

$$\theta[\gamma(Q)+2C_0]\leqslant\frac{\epsilon}{2}.$$

Since $m=n+\mathrm{o}(n)$, it follows from (A.28) that

$$\Pi_{mQ,n}(\mathbb{C}^n\setminus\mathcal{A}(n,\epsilon+a))\leqslant\exp\bigg\{\frac{\beta}{4}n(n-1)\bigg[-(1-\theta)a-\bigg(\frac{1}{2}-\theta\bigg)\epsilon+\varepsilon+\mathrm{o}(1)\bigg)\bigg]\bigg\}. \tag{A.29}$$

Also, by choosing ε sufficiently small, we can make sure that

$$-(\tfrac{1}{2}-\theta)\epsilon+\varepsilon+\mathrm{o}(1)\leqslant 0$$

for big n, so that (A.29) gives

$$\Pi_{mQ,n}(\mathbb{C}^n\setminus\mathcal{A}(n,\epsilon+a))\leqslant\exp\bigg\{-\frac{\beta}{4}(1-\theta)an(n-1)\bigg\}\leqslant\exp\bigg\{-\frac{\beta}{8}an(n-1)\bigg\}, \tag{A.30}$$

as claimed. □

A.4. *The proof of Johansson's free energy theorem*

The claim is that

$$\frac{1}{n(n-1)}\log Z_{m,n}\longrightarrow-\frac{\beta}{4}\gamma(Q)\quad\text{as } n\longrightarrow+\infty \text{ while } m=n+\mathrm{o}(n). \tag{A.31}$$

Note that by (A.5), we only need to show that lim sup converges to a number at most $-\beta\gamma(Q)/4$. To this end, we begin by establishing that, for $0<\theta<1$, we have

$$-\frac{1}{2}L_Q^{\langle\langle n\rangle\rangle}(z)+(n-m-1)Q^{\langle n\rangle}(z)\leqslant-\frac{1-\theta}{2}n(n-1)\gamma(Q)\\ -\left[(n-1)\frac{\delta_0\theta}{1+\delta_0}-(n-m-1)\right]Q^{\langle n\rangle}(z)\\ +\frac{C_1\theta}{2}n(n-1), \tag{A.32}$$

by forming a convex combination of (A.15) and (A.18). By applying (A.12) to (A.32), we obtain that (since the expression in front of $Q^{\langle n\rangle}(z)$ is negative for big m and n with $m=n+\mathrm{o}(n)$)

$$-\frac{1}{2}L_Q^{\langle\langle n\rangle\rangle}(z)+(n-m-1)Q^{\langle n\rangle}(z)\leqslant-\frac{1-\theta}{2}n(n-1)\gamma(Q)\\ -[(n-1)\delta_0\theta-(1+\delta_0)(n-m-1)]\\ \times\left(\Lambda^{\langle n\rangle}(z)-\frac{C_0 n}{1+\delta_0}\right)+\frac{C_1\theta}{2}n(n-1)\\ =-\frac{1-\theta}{2}n(n-1)\gamma(Q)\\ -[(n-1)\delta_0\theta-(1+\delta_0)(n-m-1)]\Lambda^{\langle n\rangle}(z)\\ +C_0\theta n(n-1)-C_0 n(n-m-1). \tag{A.33}$$

We multiply by $\beta/2$ on the left- and right-hand sides, to obtain

$$Z_{m,n}=\int_{\mathbb{C}^n}\exp\left\{-\frac{\beta}{4}L_Q^{\langle\langle n\rangle\rangle}(z)+\frac{\beta}{2}(n-m-1)Q^{\langle n\rangle}(z)\right\}\mathrm{dvol}_{2n}(z)\\ \leqslant e^{-(\beta/4)(1-\theta)n(n-1)\gamma(Q)+\frac{\beta}{2}C_0\theta n(n-1)-C_0 n(n-m-1)}\\ \times\left\{\int_{\mathbb{C}}(1+|\xi|^2)^{-(\beta/2)[\theta\delta_0(n-1)-(1+\delta_0)(n-m-1)]}\,\mathrm{dvol}_2(\xi)\right\}^n, \tag{A.34}$$

so that in view of (A.24), we have

$$Z_{m,n}\leqslant\exp\left\{-\frac{\beta}{4}(1-\theta)\gamma(Q)n(n-1)+\frac{C_0\beta}{2}\theta n(n-1)-C_0 n(n-m-1)\right\},$$

provided (A.26) is assumed. Taking logarithms, we find that

$$\frac{1}{n(n-1)}\log Z_{m,n}\leqslant-\frac{\beta}{4}(1-\theta)\gamma(Q)+\frac{C_0\beta}{2}\theta-C_0\left(1-\frac{m}{n-1}\right),$$

for big enough m and n with $m=n+\mathrm{o}(n)$, since (A.26) is fulfilled then. As θ, $0<\theta<1$, can be taken as close to 0 as we like, it follows that

$$\limsup\frac{1}{n(n-1)}\log Z_{m,n}\leqslant-\frac{\beta}{4}\gamma(Q).$$

The claim is an immediate consequence.

A.5. *The proof of Johansson's marginal probability theorem*

For a positive real R (a radius), we put

$$n_R(z)=\sharp\{j\in\{1,\ldots,n\}:|z_j|\geqslant R\},$$

where $\sharp$ counts the number of elements, and $z=(z_1,\ldots,z_n)$, as before. We let R_0 be a positive real with

$$\delta_0\log(1+R_0^2)\geqslant\gamma(Q)+2C_0+1. \tag{A.35}$$

Proposition A.2. *We have the estimate*

$$\frac{n_{R_0}(z)}{n} \leqslant \epsilon, \quad z \in \mathcal{A}(n,\epsilon).$$

Proof. We split the integer interval:

$$\{1,\ldots,n\} = \mathfrak{n}(z,R_0) \cup \mathfrak{m}(z,R_0), \quad \mathfrak{n}(z,R_0) \cap \mathfrak{m}(z,R_0) = \emptyset,$$

where

$$\mathfrak{n}(z,R_0) = \{j \in \{1,\ldots,n\} : |z_j| \geqslant R\},$$

so that $n_{R_0}(z) = \sharp[\mathfrak{n}(z,R_0)]$. We split the sum defining $L^{\langle\langle n\rangle\rangle}(z)$ accordingly (we use the symmetry $L_Q(\xi,\eta) = L_Q(\eta,\xi)$):

$$\begin{aligned} L^{\langle\langle n\rangle\rangle}(z) &= L_Q^{\mathrm{I}}(z) + 2L_Q^{\mathrm{II}}(z) + L_Q^{\mathrm{III}}(z) \\ &= \sum_{j,k\in\mathfrak{m}(z,R_0):\, j\neq k} L_Q(z_j,z_k) + 2\sum_{j\in\mathfrak{m}(z,R_0),\ k\in\mathfrak{n}(z,R_0)} L_Q(z_j,z_k) \\ &\quad + \sum_{j,k\in\mathfrak{n}(z,R_0):\, j\neq k} L_Q(z_j,z_k), \end{aligned}$$

with the obvious interpretation of $L_Q^{\mathrm{I}}(z)$, $L_Q^{\mathrm{II}}(z)$, and $L_Q^{\mathrm{III}}(z)$. From the extra growth condition (2.14), we see that

$$L_Q(\xi,\eta) \geqslant G(\xi,\eta) + \delta_0 \log[(1+|\xi|^2)(1+|\eta|^2)] - 2C_0,$$

so that by (A.35),

$$L_Q(\xi,\eta) \geqslant \delta_0 \log[(1+R_0^2)] - 2C_0 \geqslant \gamma(Q) + 1 \quad \text{if } |\eta| \geqslant R_0.$$

This allows us to conclude that

$$2L_Q^{\mathrm{II}}(z) + L_Q^{\mathrm{III}}(z) \geqslant 2n_{R_0}(z)[n - n_{R_0}(z)](\gamma(Q)+1) + n_{R_0}(z)[n_{R_0}(z) - 1](\gamma(Q)+1).$$

As regards the term $L_Q^{\mathrm{I}}(z)$, we may apply (A.15) to the remaining $(n - n_{R_0}(z))$-tuple:

$$L_Q^{\mathrm{I}}(z) \geqslant (n - n_{R_0}(z))(n - n_{R_0}(z) - 1)\gamma(Q).$$

By adding up the terms, we find that

$$L_Q^{\langle\langle n\rangle\rangle}(z) = L_Q^{\mathrm{I}}(z) + 2L_Q^{\mathrm{II}}(z) + L_Q^{\mathrm{III}}(z) \geqslant n(n-1)\gamma(Q) + (n-1)n_{R_0}(z).$$

For $z \in \mathcal{A}(n,\epsilon)$, we then obtain

$$\gamma(Q) + \frac{n_{R_0}(z)}{n} \leqslant \frac{1}{n(n-1)} L^{\langle\langle n\rangle\rangle}(z) \leqslant \gamma(Q) + \epsilon,$$

from which the assertion is immediate. □

For a point $z \in \mathbb{C}^n$, we define the associated weighted sum of point masses $\sigma_z \in \mathcal{P}_{\mathrm{c}}(\mathbb{C})$ by the formula

$$d\sigma_z(\xi) = \frac{1}{n}\sum_{j=1}^{n} d\delta_{z_j}(\xi), \quad \xi \in \mathbb{C}, \tag{A.36}$$

where δ_w means the Dirac point mass at $w \in \mathbb{C}$. Also, let $C_b(\mathbb{C}) = C(\mathbb{C}) \cap L^\infty(\mathbb{C})$ denote the space of bounded complex-valued continuous functions on $\mathbb{C}$.

Proposition A.3. *Suppose $\sigma_n = \sigma_z$ is as above, with $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$. Suppose, moreover, that*

$$I_Q^\sharp[\sigma_n] = \frac{1}{n(n-1)} L^{\langle\langle n \rangle\rangle}(z) \longrightarrow \gamma(Q)$$

as $n \to +\infty$. Then, as $n \to +\infty$, we have $\sigma_n \to \hat{\sigma}_Q$ weak-star. In other words, for each $f \in C_b(\mathbb{C})$, we have

$$\int_{\mathbb{C}} f \, d\sigma_n \longrightarrow \int_{\mathbb{C}} f \, d\hat{\sigma}_Q \quad \text{as } n \longrightarrow +\infty.$$

Proof. The proof is standard. We choose a weak-star convergent subsequence and call the limit σ^*. From the assumptions on the probability measure σ_n, we find that almost all its mass is concentrated to a fixed compact subset of $\mathbb{C}$ (cf. Proposition A.3), and that $I_Q[\sigma^*] \leqslant I_Q[\hat{\sigma}_Q] = \gamma(Q)$, by considering a cut-off of the logarithmic kernel. We leave the details to the interested reader. □

Let $\omega : \{1, \ldots, n\} \to \{1, \ldots, n\}$ be a permutation. For $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, we let $z^\omega = (z_{\omega(1)}, \ldots, z_{\omega(n)}) \in \mathbb{C}^n$ be point induced by the permutation. Suppose for the moment that $f \in C_b(\mathbb{C}^n)$, and write $f^\omega(z) = f(z^\omega)$. By symmetry, we then have

$$\int_{\mathbb{C}^n} f \, d\Pi_{mQ,n} = \int_{\mathbb{C}^n} f^\omega \, d\Pi_{mQ,n},$$

which gives

$$\int_{\mathbb{C}^n} f \, d\Pi_{mQ,n} = \frac{1}{n!} \int_{\mathbb{C}^n} \sum_\omega f^\omega \, d\Pi_{mQ,n},$$

where the sum runs over all permutations ω. We next split the integral:

$$\int_{\mathbb{C}^n} f \, d\Pi_{mQ,n} = \frac{1}{n!} \int_{\mathcal{A}(n,\epsilon)} \sum_\omega f^\omega \, d\Pi_{mQ,n} + \frac{1}{n!} \int_{\mathbb{C}^n \setminus \mathcal{A}(n,\epsilon)} \sum_\varsigma f^\omega \, d\Pi_{mQ,n}. \tag{A.37}$$

By Propositions A.1 and A.2, the last term is $\mathrm{o}(1)$ as $m, n \to +\infty$ while $m = n + \mathrm{o}(n)$. In order to understand the remaining term, we should study

$$\frac{1}{n!} \sum_\omega f^\omega \quad \text{on } \mathcal{A}(n, \epsilon). \tag{A.38}$$

We now focus on the $k = 1$ case of Johansson's theorem, and restrict our attention to f which only depend on the first coordinate, $f(z) = f(z_1)$ with some slight abuse of notation. Then (A.38) amounts to the linear statistic

$$\frac{1}{n} \sum_{j=1}^n f(z_j), \quad (z_1, \ldots, z_n) \in \mathcal{A}(n, \epsilon). \tag{A.39}$$

By Proposition A.2, only an ϵ proportion of the points z_j may fall outside the disk $\mathbb{D}(0, R_0)$, and by Proposition A.3, the expression (A.39) is close to (the constant!)

$$\int_{\mathbb{C}} f \, d\hat{\sigma}_Q$$

for small ϵ and large n. The weak-star convergence $\Pi^{(1)}_{mQ,n} \to \hat{\sigma}_Q$ follows, if we let ϵ approach 0 slowly as $n \to +\infty$. The remaining case $k > 1$ is analogous.

3. *An obstacle problem. Smooth potentials*

3.1. *Equilibrium measure in terms of an obstacle problem*

We consider the cone $\mathrm{Sub}(\mathbb{C})$ of all subharmonic functions in the plane $\mathbb{C}$ and its convex subset ($0 < t < +\infty$ is assumed fixed)

$$\mathrm{Sub}_t(\mathbb{C}) := \left\{ v \in \mathrm{Sub}(\mathbb{C}) : \limsup_{|z|\to+\infty} \, [v(z) - t \log |z|^2] < +\infty \right\}.$$

Given $Q : \mathbb{C} \to \mathbb{R} \cup \{+\infty\}$, the obstacle problem is to find

$$\mathrm{Obst}_t[Q](z) := \sup\{v(z) : \; v \in \mathrm{Sub}_t(\mathbb{C}) \text{ and } v \leqslant Q \text{ on } \mathbb{C}\}. \tag{3.1}$$

Here, we assume of Q, as before, that it is lower semi-continuous, bounded on a set of positive area, and that

$$\lim_{|z|\to+\infty} \, [Q(z) - t \log |z|^2] = +\infty. \tag{3.2}$$

We think of both Q and t as fixed; we observe, however, that if (3.2) is fulfilled for one value of t, then any smaller positive value works as well. It is easy to check that the supremum in (3.1) is taken over a non-empty collection of functions v (for example, a large negative constant will satisfy the requirements). See, for example, Doob [**6**] for the potential theory pertaining to obstacle problems of this type. For instance, after possibly redefining the function $\mathrm{Obst}_t[Q]$ on a negligible set (here, this is a set of logarithmic capacity 0), we obtain a subharmonic function. We need to connect the obstacle problem (3.1) with the equilibrium measure theory of Subsections 2.6 and 2.9. To this end, let

$$\hat\sigma_t = \hat\sigma_t[Q] := t\hat\sigma_{Q/t}, \quad S_t = S_t[Q] := S_{Q/t} = \mathrm{supp}\, \hat\sigma_t,$$

be the scaled equilibrium measure of Subsection 2.9 and its associated support set. We write

$$\gamma_t(Q) = t\gamma(Q/t) \quad \text{and} \quad \gamma_t^*(Q) = \gamma_t(Q) - \frac{1}{t}\int_{\mathbb{C}} Q \, d\hat\sigma_t.$$

For a compactly supported finite positive Borel measure σ, let U^σ denote the logarithmic potential

$$U^\sigma(\xi) = \int_{\mathbb{C}} \log \frac{1}{|\xi - \eta|^2} \, d\sigma(\eta),$$

and put

$$\hat Q_t(\xi) = \gamma_t^*(Q) - U^{\hat\sigma_t}(\xi). \tag{3.3}$$

The function $\hat Q_t$ is then subharmonic in $\mathbb{C}$ and harmonic in $\mathbb{C} \setminus S_t$, where $S_t = \mathrm{supp}\, \hat\sigma_t$. Moreover, as it is the total mass of the measure which determines the decay of the logarithmic potential at infinity, we have

$$\hat Q_t(z) = t \log |z|^2 + \mathrm{O}(1) \quad \text{as } |z| \longrightarrow +\infty. \tag{3.4}$$

The following lemma supplies a criterion which allows us to solve the obstacle problem. We recall that the logarithmic energy $I_0[\sigma]$ is given by (2.9) with Q replaced by 0.

Lemma 3.1. *Let σ be a compactly supported finite positive Borel measure in $\mathbb{C}$ of finite logarithmic energy $I_0[\sigma] < +\infty$ with total mass $\|\sigma\| = t$. Suppose $W = c - U^\sigma$, where $c \in \mathbb{R}$ is a constant. If W has both $W \leqslant Q$ q.e. on $\mathbb{C}$ and $W = Q$ q.e. on $\mathrm{supp}\, \sigma$, then $W = \mathrm{Obst}_t[Q]$ q.e.*

Proof. Without loss of generality, we may assume that $Q \geqslant 1$ on $\mathbb{C}$. The function W is in $\mathrm{Sub}_t(\mathbb{C})$ while $W \leqslant Q$ q.e. on $\mathbb{C}$. So $W \leqslant Q$ on $\mathbb{C} \setminus E$, where $E \subset \mathbb{C}$ is polar (that is, has logarithmic capacity 0). Let ρ be a compactly supported Borel probability measure on $\mathbb{C}$ such that the corresponding potential has $U^\rho = +\infty$ on E (see, for example [6]). Put $W' := (1-\epsilon)W - \epsilon' U^\rho$ where ϵ and ϵ' are two small positive numbers. If ϵ' is very small (also relative to ϵ), we can make sure that $W' \leqslant Q$ on $\mathbb{C}\setminus E$, by using that $W \leqslant Q$ on $\mathbb{C}\setminus E$, the standard properties of potentials and the given properties of Q. Then $W' \leqslant Q$ throughout $\mathbb{C}$ automatically as $W' = -\infty$ on E. We conclude that $W' \leqslant \mathrm{Obst}_t[Q]$ on $\mathbb{C}$. By letting $\epsilon' \to 0$ first and then $\epsilon \to 0$ second, we get that $W \leqslant \mathrm{Obst}_t[Q]$ on $\mathbb{C}\setminus E$, and consequently, $W \leqslant \mathrm{Obst}_t[Q]$ q.e. on $\mathbb{C}$. As a side remark, we observe that W is locally of Sobolev class $W^{1,2}$, which means that in the sense of distributions, its gradient is locally in L^2 with respect to area measure. It remains to show the reverse inequality $\mathrm{Obst}_t[Q] \leqslant W$ q.e. on $\mathbb{C}$. To this end, we pick a function $v \in \mathrm{Sub}_t(\mathbb{C})$ with $v \leqslant Q$ q.e. on $\mathbb{C}$. It will suffice to show that $v \leqslant W$ on $\mathbb{C}$. The potential U^σ is harmonic in $\mathbb{C}\setminus S$, where $S := \operatorname{supp}\sigma$, and therefore W is harmonic there as well. The assumption on the total mass of σ gives that

$$W(z) = t\log|z|^2 + \mathrm{O}(1) \quad \text{as } |z| \longrightarrow +\infty. \tag{3.5}$$

Next, we consider the difference $u = v - W$, which is subharmonic in $\mathbb{C}\setminus S$ and has $u \leqslant 0$ q.e. on S. Moreover, the assumption that $v \in \mathrm{Sub}_t(\mathbb{C})$ together with (3.5) shows that u is bounded from above near infinity. We should like to apply the maximum principle in the open set $\mathbb{C}\setminus S$ and obtain that $u \leqslant 0$ on $\mathbb{C}\setminus S$ since $u \leqslant 0$ q.e. on the boundary. However, this is a little delicate as the functions are not necessarily continuous up to the boundary. The so-called Principle of Domination [21, p. 104], is a good substitute. To apply it, we need to make the technical assumption that v is harmonic in a punctured neighbourhood of infinity, because it allows us to represent v q.e. in the form $b - U^\nu$, where b is a constant and ν is finite compactly supported positive Borel measure. The assumption $v \in \mathrm{Sub}_t(\mathbb{C})$ then gives that ν has total mass $\|\nu\| \leqslant t$, so that $\|\nu\| \leqslant \|\sigma\|$. From the assumptions we read off that $U^\sigma \leqslant U^\nu + c - b$ holds q.e. on $S = \operatorname{supp}\sigma$, and hence σ-almost everywhere. (because σ has finite logarithmic energy), so by the Principle of Domination (which again uses that σ has finite logarithmic energy, and that $\|\nu\| \leqslant \|\sigma\|$), we find that $U^\sigma \leqslant U^\nu + c - b$ holds throughout $\mathbb{C}$. The desired conclusion that $v \leqslant W$ follows. Next, to justify the conclusion that $v \leqslant W$ on $\mathbb{C}$ holds when we only assume that $v \in \mathrm{Sub}_t(\mathbb{C})$ with $v \leqslant Q$ q.e. on $\mathbb{C}$, we proceed as follows. If we let ε be a small positive real number, and put

$$\tilde v(z) := \max\left\{\frac{v(z)}{1+\varepsilon}, t\log|z|^2 - C\right\},$$

then $\tilde v \in \mathrm{Sub}_t(\mathbb{C})$ is harmonic in a punctured neighbourhood of the point at infinity, and $\tilde v \leqslant Q$ q.e. on $\mathbb{C}$ holds if the constant C is big enough positive. So we have the conclusion $\tilde v \leqslant W$ on $\mathbb{C}$ from the previous argument. Finally, we first let $C \to +\infty$ and afterwards let $\varepsilon \to 0$, and obtain $v \leqslant W$, as claimed. □

We have the following characterization.

Proposition 3.2. *We have*

$$\mathrm{Obst}_t[Q](z) = \hat Q_t(z), \quad \text{q.e. } z \in \mathbb{C}.$$

Proof. In view of Lemma 3.1, we just need to check that $\hat Q_t = Q$ q.e. on S_t while $\hat Q_t \leqslant Q$ q.e. on $\mathbb{C}$. By Frostman's Theorem 2.6, we have $U^{\hat\sigma_{Q/t}} + Q/t \geqslant \gamma^*(Q/t)$ q.e. on $\mathbb{C}$, while $U^{\hat\sigma_{Q/t}} + Q/t = \gamma^*(Q/t)$ q.e. on $S_{Q/t}$. Since $\hat\sigma_t = t\hat\sigma_{Q/t}$, $S_t = S_{Q/t}$ and $\gamma_t^*(Q) = t\gamma^*(Q/t)$, this means that $U^{\hat\sigma_t} + Q \geqslant \gamma_t^*(Q)$ q.e. on $\mathbb{C}$, while $U^{\hat\sigma_t} + Q = \gamma_t^*(Q)$ q.e. on S_t. With $\hat Q_t = \gamma_t^*(Q) - U^{\hat\sigma_t}$, this is the same as having $\hat Q_t \leqslant Q$ q.e. on $\mathbb{C}$, while $\hat Q_t = Q$ q.e. on S_t, as needed. □

Remark 3.3. The assertion of Proposition 3.2 is essentially equivalent to that of Theorem I.4.1 (see [**21**]).

We easily recover the density from the potential; we write $dA := \pi^{-1} d\mathrm{vol}_2$ for the normalized area measure. We remark that for a positive distribution f, we use the (formal) convention to let $f dA$ be the associated positive Borel measure (which need not be absolutely continuous with respect to area).

Corollary 3.4. *We have, in the sense of distribution theory,* $d\hat{\sigma}_t = \Delta \hat{Q}_t \, dA$.

3.2. *The super-coincidence and coincidence sets*

We keep the setting of the previous subsection, and assume $Q : \mathbb{C} \to \mathbb{R} \cup \{+\infty\}$ is lower semi-continuous, and bounded on a set of positive area, subject to the growth condition (3.2). The potential U^σ is superharmonic for a given finite positive compactly supported measure σ, and therefore the function $\hat{Q}_t$ defined by (3.4) is automatically subharmonic. In particular, $\hat{Q}_t$ is upper semi-continuous, and we find that the difference $Q - \hat{Q}_t$ is lower semi-continuous. It follows that the *super-coincidence set*

$$S_t^* = S_t^*[Q] = \{z \in \mathbb{C} : \hat{Q}_t(z) \geqslant Q(z)\} \tag{3.6}$$

is compact: it is closed by semi-continuity, while (3.4) and (3.5) show that it is bounded. We note that by Proposition 3.2, $\hat{Q}_t \leqslant Q$ q.e., so that S_t^* equals, up to a set of logarithmic capacity zero, the *coincidence set*

$$\{z \in \mathbb{C} : \hat{Q}_t(z) = Q(z)\}.$$

In all cases when we have a little regularity, $\hat{Q}_t$ is continuous, and then the super-coincidence set S_t^* is the same as the coincidence set. We therefore refrain from introducing separate notation for the coincidence set.

Proposition 3.5. *The function* $\hat{Q}_t$ *is harmonic in* $\mathbb{C} \setminus S_t^*$*, and as a consequence,* $S_t \subset S_t^*$. *In particular,* S_t^* *is non-empty.*

Proof. We pick a point $z_0 \in \mathbb{C} \setminus S_t^*$, so that $\hat{Q}_t(z_0) < Q(z_0)$. By semi-continuity, we get that $\hat{Q}_t < Q$ in a neighbourhood of z_0. We claim that $\hat{Q}_t$ is harmonic near z_0. If not, we could use Perron's lemma and replace $\mathrm{Obst}_t[Q]$ (which equals $\hat{Q}_t$ q.e., by Proposition 3.2) on a small disk around z_0 by the harmonic function which has the same boundary values, and obtain a function which is in $\mathrm{Sub}_t(\mathbb{C})$, and bigger than $\mathrm{Obst}_t[Q]$ while being at most Q. This violates the extremality of $\mathrm{Obst}_t[Q]$ and the claim follows. Next, by Corollary 3.4, we see that $z_0 \in \mathbb{C} \setminus S_t$. Since z_0 was an arbitrary point in $\mathbb{C} \setminus S_t^*$, the proof is complete. □

3.3. *A priori smoothness for the obstacle problem for smooth potentials*

As before, $Q : \mathbb{C} \to \mathbb{R} \cup \{+\infty\}$ is lower semi-continuous with (3.2) where t is a (fixed) positive real. If Q has some degree of smoothness, say, for example, $Q : \mathbb{C} \to \mathbb{R}$ is C^2-smooth, it is natural to wonder to what extent that carries over to $\mathrm{Obst}_t[Q] = \hat{Q}_t$. Since we sometimes need to work with slightly less smooth weights, the (local) Sobolev classes $W^{2,p}$ are sometimes more appropriate. By Sobolev imbedding, functions in $W^{2,p}$ are continuous provided $1 < p < +\infty$. The case $p = +\infty$ gives the space $W^{2,\infty} = C^{1,1}$ of functions whose first-order partial derivatives are locally Lipschitz continuous. We use the notation $W^{2,p}$ and $C^{1,1}$ for *local* classes unless otherwise stated.

The following *a priori* smoothness result is standard in connection with the constrained obstacle problem discussed below [9] and associated with the names such as Lewy, Stampacchia, Brezis, Lions, Kinderlehrer, and Caffarelli. We present the elementary approach recently found by Berman [4], which gives the $C^{1,1}$-smoothness part. Berman's approach also applies in the several complex variables context.

PROPOSITION 3.6. *If $Q \in C^2$, then $\mathrm{Obst}_t[Q] \in C^{1,1}$. More generally, if $Q \in C^{1,1}$, then we still have $\mathrm{Obst}_t[Q] \in C^{1,1}$. Finally, if $Q \in W^{2,p}$ for some p, $1 < p < +\infty$, then $\mathrm{Obst}_t[Q] \in W^{2,p}$.*

Proof. We first show how $Q \in C^{1,1}$ implies that $\mathrm{Obst}_t[Q] \in C^{1,1}$. We begin by noting that by (3.4) and Proposition 3.2,

$$\mathrm{Obst}_t[Q](z) = \hat{Q}_t(z) \leqslant t\log(1 + |z|^2) + C, \quad z \in \mathbb{C},$$

for a suitable real constant C. By (3.2), the growth of $Q(z)$ is faster than that of $\mathrm{Obst}_t[Q]$, which we can use to show that for some possibly big value of the radius r_0,

$$\mathrm{Obst}_t[Q](z + w) \leqslant Q(z), \quad |z| \geqslant r_0, \ |w| \leqslant 1. \tag{3.7}$$

Let

$$M_r := \sup\{|\partial_t^2\{Q(z + t\zeta)\}| : |z| \leqslant r, \ |\zeta| = 1, \ 0 \leqslant t \leqslant 1\},$$

which is finite for each radius r due to the assumption that $Q \in C^{1,1}$, and note that by Taylor's formula,

$$\mathrm{Obst}_t[Q](z + w) \leqslant Q(z + w) \leqslant Q(z) + 2\,\mathrm{Re}[w\partial Q(z)] + \tfrac{1}{2}M_r|w|^2, \quad |z| \leqslant r, \ |w| \leqslant 1. \tag{3.8}$$

We fix $w \in \mathbb{C}$ with $|w| \leqslant 1$ and put

$$\tilde{Q}_w(z) := \tfrac{1}{2}\mathrm{Obst}_t[Q](z + w) + \tfrac{1}{2}\mathrm{Obst}_t[Q](z - w) - \tfrac{1}{2}M_{r_0}|w|^2.$$

By a combination of (3.7) and (3.8), $\tilde{Q}_w \leqslant Q$ on $\mathbb{C}$, while it is obvious that $\tilde{Q}_w \in \mathrm{Sub}_t(\mathbb{C})$. So, from the definition of the obstacle problem, we see that $\tilde{Q}_w \leqslant \mathrm{Obst}_t[Q]$ on $\mathbb{C}$. In other words,

$$\mathrm{Obst}_t[Q](z + w) + \mathrm{Obst}_t[Q](z - w) - 2\mathrm{Obst}_t[Q](z) \leqslant M_{r_0}|w|^2, \quad z \in \mathbb{C}, \ |w| \leqslant 1. \tag{3.9}$$

Next, if we divide both sides of (3.9) by $|w|^2$ and then let $w \to 0$, we obtain

$$\partial_t^2\,\mathrm{Obst}_t[Q](z + t\zeta)|_{t=0} \leqslant M_{r_0}, \quad z \in \mathbb{C}, \ |\zeta| = 1.$$

In particular, if $z = x + \mathrm{i}y$, we have

$$\partial_x^2\mathrm{Obst}_t[Q](z) \leqslant M_{r_0}, \quad \partial_y^2\mathrm{Obst}_t[Q](z) \leqslant M_{r_0}.$$

Since $\mathrm{Obst}_t[Q]$ is subharmonic, that is,

$$\partial_x^2\mathrm{Obst}_t[Q](z) + \partial_y^2\mathrm{Obst}_t[Q](z) \geqslant 0$$

holds in the sense of distribution theory, we must then also have

$$-M_{r_0} \leqslant \partial_x^2\mathrm{Obst}_t[Q](z) \leqslant M_{r_0}, \quad -M_{r_0} \leqslant \partial_y^2\mathrm{Obst}_t[Q](z) \leqslant M_{r_0}.$$

In particular, then, $\mathrm{Obst}_t[Q] \in C^{1,1}$.

As for the remaining case when we have less smoothness, that is, when $Q \in W^{2,p}$, the assertion follows from the smoothness theory of constrained obstacle problems (see Lemma 3.7 and Theorem 3.9). □

3.4. *A constrained obstacle problem*

Let $Q : \mathbb{C} \to \mathbb{R} \cup \{+\infty\}$ be lower semi-continuous with (3.2) where t is a (fixed) positive real, as before. Let Ω be a (bounded) Jordan domain, and $\varrho : \partial\Omega \to \mathbb{R}$ a continuous function with $\varrho \leqslant Q|_{\partial\Omega}$. Consider the constrained obstacle problem

$$\mathrm{Obst}_{\Omega,\varrho}[Q](z) := \sup\{v(z) : \ v \in \mathrm{Sub}(\Omega),\ v \leqslant Q \text{ on } \Omega,\ v = \varrho \text{ on } \partial\Omega\}, \quad z \in \bar{\Omega}.$$

We would like to model the obstacle problem associated with $\mathrm{Obst}_t[Q]$ in the form of such a constrained obstacle problem. The natural way to do this is to put $\varrho := \mathrm{Obst}_t[Q]|_{\partial\Omega}$.

LEMMA 3.7. *If Ω is a C^∞-smooth bounded Jordan domain and $\varrho = \mathrm{Obst}_t[Q]|_{\partial\Omega}$, then*

$$\mathrm{Obst}_{\Omega,\varrho}[Q] = \mathrm{Obst}_t[Q] \quad \text{on } \Omega.$$

Proof. We put $R_0 = \mathrm{Obst}_t[Q]$, $R_1 = \mathrm{Obst}_t[Q]|_{\bar{\Omega}}$, and $R_2 = \mathrm{Obst}_{\Omega,\varrho}[Q]$. The function R_1 is subharmonic with $R_1 \leqslant Q$ in Ω, and has boundary values $R_1|_{\partial\Omega} = \varrho$. It is now immediate that $R_1 \leqslant R_2$. We proceed to show that $R_2 \leqslant R_1$. To this end, we let $v \in \mathrm{Sub}(\Omega)$ have $v \leqslant Q$ on Ω and boundary data $v = \varrho$ on $\partial\Omega$; we are to check that $v \leqslant R_1$. Next, we put $\tilde{v} = \max\{v, R_1\}$; the function $\tilde{v}$ is in $\mathrm{Sub}(\Omega)$, has $R_1 \leqslant \tilde{v} \leqslant Q$ on Ω, and boundary data $\tilde{v}|_{\partial\Omega} = \varrho$. We consider its extension

$$V = \begin{cases} \tilde{v} & \text{in } \Omega, \\ R_0 & \text{in } \mathbb{C} \setminus \Omega. \end{cases}$$

The way things are set up, $R_0 \leqslant V \leqslant Q$ in $\mathbb{C}$, with $V = R_0$ on $\partial\Omega$. We claim that $V \in \mathrm{Sub}(\mathbb{C})$. It is enough to check the mean value inequality along $\partial\Omega$. For points $a \in \partial\Omega$, we have ($\epsilon > 0$ is a small real parameter)

$$V(a) = R_0(a) \leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} R_0(a + \epsilon e^{\mathrm{i}\theta})\, d\theta \leqslant \int_{-\pi}^{\pi} V(a + \epsilon e^{\mathrm{i}\theta})\, d\theta.$$

It follows that $V \in \mathrm{Sub}(\mathbb{C})$ and *a fortiori* $V \in \mathrm{Sub}_t(\mathbb{C})$ (because of the growth at infinity). We conclude that V is a function which we may plug into the optimization problem defining $R_0 = \mathrm{Obst}_t[Q]$ and so $V \leqslant R_0$ on $\mathbb{C}$. In fact, due to the reverse inequality, we must have $V = R_0$. In particular, $\tilde{v} = V|_{\bar{\Omega}} = R_1$ and so $v \leqslant R_1$. □

3.5. *Kinderlehrer–Stampacchia–Caffarelli theory*

For our purposes, it would be enough to consider the case of C^2 or even C^∞ potentials Q but since we sometimes have to modify them (see, for example, [**2**]), the Sobolev classes $W^{2,p}$ seem to be more appropriate. We generally assume that $Q : \mathbb{C} \to \mathbb{R}$ is continuous subject to the growth condition (3.2) for some (fixed) positive real t.

We start with a simple observation.

LEMMA 3.8. *Let $S_t = S_t[Q]$ and suppose $Q \in W^{2,1}(\mathrm{int}\, S_t)$. Then $\hat{\sigma}_t$ is absolutely continuous in $\mathrm{int}\, S_t$ and in fact*

$$d\hat{\sigma}_t = \Delta Q\, dA \quad \text{on } \mathrm{int}\, S_t.$$

Proof. As we know that $d\sigma_t = \Delta\hat{Q}_t\, dA$ in the sense of distributions, and so the same is true if we restrict the distributions to the open set $\mathrm{int}\, S_t$, where $\hat{Q}_t = Q$, and therefore $\Delta\hat{Q}_t = \Delta Q$ as distributions. □

The following two theorems are adapted from the theory of constrained obstacle problems (variational inequalities); this theory is, as mentioned previously, associated with the names of Lewy, Stampacchia, Brezis, Lions, Kinderlehrer, and Caffarelli *et al.* A standard reference is [9], Chapter 1 (see also [5]).

THEOREM 3.9. *Fix p, $1 < p < +\infty$, and let Ω be a C^∞-smooth bounded Jordan domain. We suppose Q is $W^{2,p}$-smooth in $\mathbb{C}$, and that $\varrho : \partial\Omega \to \mathbb{R}$ is a function which is the restriction to $\partial\Omega$ of a function in $W^{2,p}(\mathbb{C})$, with $\varrho \leqslant Q$ on $\partial\Omega$. Then* $\mathrm{Obst}_{\Omega,\varrho}[Q] \in W^{2,p}(\Omega)$.

Proof. This is explained in Chapter 1 of Friedman's book [9], see Theorem 1.3.2 and Problem 1 on p. 29. □

Together with Lemma 3.7, this justifies the $W^{2,p}$ part of the assertion of Proposition 3.6. Now, in view of Proposition 3.6, if we suppose $Q \in W^{2,p}$ for some $1 < p < +\infty$, the function $\hat{Q}_t = \mathrm{Obst}_t[Q]$ is in $W^{2,p}$, and by Corollary 3.4, the measure $\hat{\sigma}_t$ is absolutely continuous (with respect to area), and the density is locally in L^p. By Lemma 3.8, we obtain

$$d\hat{\sigma}_t = \Delta Q\, dA \quad \text{on int}\, S_t.$$

If the boundary ∂S_t has a zero area, we can conclude that $d\hat{\sigma}_t = 1_{S_t}\Delta Q\, dA$. It is remarkable that this conclusion holds even when ∂S_t has a positive area.

THEOREM 3.10. *If, for some $1 < p < +\infty$, we have $Q \in W^{2,p}$, then $\hat{Q}_t \in W^{2,p}$ and*

$$d\hat{\sigma}_t = 1_{S_t}\Delta Q\, dA.$$

Proof. As S_t is the support of $\hat{\sigma}_t$, the measure $\hat{\sigma}_t$ vanishes off S_t. On S_t, however, the two $W^{2,p}$-smooth functions Q and $\hat{Q}_t$ coincide, and by Kinderlehrer and Stampacchia [19, p. 53], this entails that their partial derivatives of order at most 2 coincide almost everywhere on S_t. In particular, $\Delta\hat{Q}_t = \Delta Q$ on S_t as L^p functions. In view of Corollary 3.4, the assertion is immediate. □

REMARK 3.11. (a) In the context of Proposition 3.6, it does not help to add more smoothness to Q. For example, if $Q \in C^\infty$ is assumed, we still cannot do better than $\mathrm{Obst}_t[Q] \in C^{1,1}$, at least near ∂S_t.

(b) The smoothness assumptions of this subsection are excessive, in the sense that it suffices to have the required smoothness of Q in a neighbourhood of the droplet S_t. All the statements are valid under this weaker assumption.

(c) By the properties of the 2-dimensional Hilbert transform, the assertion that $\hat{Q}_t \in W^{2,p}$ is equivalent to the property that the density of the absolutely continuous measure $d\hat{\sigma}_t$ is in L^p (locally).

3.6. *The coincidence set and shallow points*

As in the previous subsection, $Q : \mathbb{C} \to \mathbb{R}$ is assumed to be of (local) Sobolev class $W^{2,p}$, with $1 < p < +\infty$. We assume that Q meets the growth assumption (3.2) for all t with $0 < t < T$, where $T = T(Q)$ has $0 < T \leqslant +\infty$.

We recall that we introduced the parameter t to consider the evolution of the renormalized equilibrium measures $\hat{\sigma}_t = t\hat{\sigma}_{Q/t}$ as t moves. The conclusion of Theorem 3.10 allows us to reduce the complexity and just study the evolution of the droplets $S_t = S_t[Q] = S_{Q/t}$. The super-coincidence set $S_t^* = S_t^*[Q]$ defined by (3.6) will be referred to as the coincidence set, because the smoothness of Q makes $\hat{Q}_t$ continuous.

We should explain the relationship between the sets S_t and S_t^* (we already know that $S_t \subset S_t^*$). To this end, we say that a point $z_0 \in S_t^*$ is *Q-shallow* (with respect to S_t^*) if there exists an open disk D centred at z_0 such that

$$\int_{S_t^* \cap D} |\Delta Q| \, dA = 0.$$

The Q-shallow point in S_t^* form a relatively open subset. We mention in passing that it follows from Theorem 3.10 that $\Delta Q \geqslant 0$ almost everywhere on S_t.

PROPOSITION 3.12. *The set S_t is obtained from S_t^* by removal of all the Q-shallow points.*

Proof. Since $\hat{Q}_t$ and Q are both in $C^{1,1}$ and coincide on S_t^*, we obtain from [**19**, p. 53], that $\Delta \hat{Q}_t = \Delta Q$ holds almost everywhere on S_t^*, so that (in the same way as Lemma 3.8 was obtained)

$$d\hat{\sigma}_t = 1_{S_t^*} \Delta Q \, dA.$$

By comparing with Lemma 3.8, we see that $\Delta Q = 0$ almost everywhere on $S_t^* \setminus S_t$. To calculate the support of $\hat{\sigma}_t$, we must remove all the points of S_t^* where there is no $|\Delta Q| dA$-mass nearby, that is, the Q-shallow points. □

3.7. *Coincidence sets and the dynamics of droplets*

As in the previous subsection, $Q : \mathbb{C} \to \mathbb{R}$ is assumed to be of class $W^{2,p}$, with $1 < p < +\infty$. We assume that Q meets the growth assumption (3.2) for all t with $0 < t < T$, where $T = T(Q)$ has $0 < T \leqslant +\infty$.

The coincidence set $S_t^* = S_t[Q]$ defined by (3.6) is just a little bigger than S_t (we remove the Q-shallow points), but it contains essential information which helps us understand the evolution of S_t as t grows.

We begin with some elementary properties.

LEMMA 3.13. *If $0 < t_1 \leqslant t_2 < T$, then $\hat{Q}_{t_1} \leqslant \hat{Q}_{t_2}$ and, in particular, $S_{t_1}^* \subset S_{t_2}^*$.*

Proof. Since $\mathrm{Sub}_{t_1}(\mathbb{C}) \subset \mathrm{Sub}_{t_2}(\mathbb{C})$, we clearly have $\mathrm{Obst}_{t_1}[Q] \leqslant \mathrm{Obst}_{t_2}[Q]$, from the definition of the obstacle problem. The first assertion, $\hat{Q}_{t_1} \leqslant \hat{Q}_{t_2}$, now follows from Proposition 3.2. The second assertion, the inclusion $S_{t_1}^* \subset S_{t_2}^*$, is an easy consequence of the first assertion. □

Let S_0^* denote the (non-empty compact) set where the global minimum of Q is attained.

PROPOSITION 3.14. *We have $S_0^* \subset S_t^*$ for all $0 < t < T$.*

Proof. Pick a point $a \in S_0^*$ and observe that the function $\max\{Q(a), \hat{Q}_t\}$ is subharmonic (in fact, in $\mathrm{Sub}_t(\mathbb{C})$) and therefore competes with $\hat{Q}_t$ for the obstacle problem. We conclude that $Q(a) \leqslant \hat{Q}_t$. As $\hat{Q}_t \leqslant Q$, it follows that $\hat{Q}_t(a) = Q(a)$, so that $a \in S_t^*$. The proof is complete. □

PROPOSITION 3.15. *If $0 < t_1 \leqslant t_2 < T$, we have $S_{t_1} \subset S_{t_2}$ and $\hat{\sigma}_{t_1} \leqslant \hat{\sigma}_{t_2}$.*

Proof. If a point $a \in S^*_{t_1} \cap S^*_{t_2}$ is Q-shallow with respect to $S^*_{t_2}$, then it is also Q-shallow with respect to $S^*_{t_1}$, since $S^*_{t_1} \subset S^*_{t_2}$, by Lemma 3.13. The first assertion, $S_{t_1} \subset S_{t_2}$, now follows from a second application of $S^*_{t_1} \subset S^*_{t_2}$. The second assertion, $\hat{\sigma}_{t_1} \leqslant \hat{\sigma}_{t_2}$, is a consequence of the first assertion combined with Theorem 3.10. □

REMARK 3.16. This supplies the potential theoretical proof of Corollary 2.13 alluded to in Remark 2.14.

We need the following two lemmas from [21, pp. 227–228].

LEMMA 3.17 ($0 < t_0 < T$). *The map $t \mapsto S_t$ is monotonically increasing and left-continuous in the Hausdorff metric*:

$$S_t \nearrow S_{t_0} \quad \text{as } t \nearrow t_0.$$

This means that S_{t_0} is in a small neighbourhood of S_t for $t < t_0$ close to t_0 or, equivalently, that

$$S_{t_0} = \operatorname{clos} \bigcup_{t<t_0} S_t.$$

LEMMA 3.18 ($0 < t_0 < T$). *We have*

$$\bigcap_{t_0<t<T} S_t \subset S^*_{t_0}.$$

In particular, if $S_{t_0} = S^*_{t_0}$, then

$$S_{t_0} = \bigcap_{t>t_0} S_t.$$

It is easy to construct examples which show how $S^*_{t_0}$ may contain 'seed points' outside the main body of S_{t_0} which grow into (small) components of S_t for $t > t_0$. The next lemma gives a criterion which guarantees that this phenomenon takes place. For a compact set $E \subset \mathbb{C}$, let phull(E) denote its polynomially convex hull, that is,

$$\operatorname{phull}(E) := \left\{ z \in \mathbb{C} : \ |p(z)| \leqslant \max_E |p| \text{ for all polynomials } p \right\}.$$

The (compact) set phull(E) adds to E all the points of $\mathbb{C} \setminus E$ which belong to bounded connectivity components of $\mathbb{C} \setminus E$ (that is, points invisible to Brownian motion in $\mathbb{C} \setminus E$ starting at ∞).

LEMMA 3.19. *For all t_0, t with $0 < t_0 < t < T$, we have the inclusion*

$$\partial[\operatorname{phull}(S^*_{t_0})] \subset S_t.$$

Proof. The standard geometric interpretation of the polynomially convex hull gives that

$$\partial[\operatorname{phull}(S^*_{t_0})] \subset \partial S^*_{t_0} \subset S^*_{t_0}.$$

We need to show that $\partial[\operatorname{phull}(S^*_{t_0})] \subset S_t$ for all t, $t_0 < t < T$. We argue by contradiction, and suppose that there exists a point $a \in \partial[\operatorname{phull}(S^*_{t_0})]$ such that $a \in \mathbb{C} \setminus S_{t_1}$ for some t_1, with $t_0 < t_1 < T$. Then $a \in \mathbb{C} \setminus S_t$ for all t with $t_0 < t \leqslant t_1$, and if $t > t_0$ is sufficiently close to

t_0, the point a belongs to the unbounded component of $\mathbb{C}\setminus S_t$. Indeed, choose a small open neighbourhood U of a avoiding S_{t_1} and since, by assumption, $a\in\partial[\mathrm{phull}(S^*_{t_0})]$, we may assure that there exists a point $b\in U\setminus\mathrm{phull}(S^*_{t_0})$. The point b belongs to the unbounded component of $\mathbb{C}\setminus S^*_{t_0}$, so we may connect b with ∞ by a curve γ in $\mathbb{C}\setminus S^*_{t_0}$; γ is at a positive distance from $S^*_{t_0}$. By Lemma 3.18, $\gamma\subset\mathbb{C}\setminus S_t$ for all $t>t_0$ close to t_0, and so the point b, and *a fortiori* a, is in the unbounded component of $\mathbb{C}\setminus S_t$. Next, we consider (for t with $t>t_0$ close to t_0) the function $u=\hat{Q}_{t_0}-\hat{Q}_t$. Then, by Lemma 3.13, we have $u\leqslant 0$. Moreover, since $a\in S^*_{t_0}\subset S^*_t$, we have $\hat{Q}_{t_0}(a)=\hat{Q}_t(a)=Q(a)$, and therefore, $u(a)=0$. The function $\hat{Q}_{t_0}$ is harmonic in $\mathbb{C}\setminus S_{t_0}$, and, likewise, $\hat{Q}_t$ is harmonic in $\mathbb{C}\setminus S_t$, so we conclude that u is harmonic in $\mathbb{C}\setminus S_t$. The function u then has a local maximum at the interior point a, so by the strong maximum principle, we obtain that $u=0$ throughout $\mathbb{C}\setminus\mathrm{phull}(S_t)$. This does not agree with the known asymptotics (3.4). We conclude that the initial assumption must be false, so that $a\in S_{t_1}$ for all t_1 with $t_0<t_1<T$. □

REMARK 3.20. The above assertions extend to the case $t_0=0$ if as before S^*_0 is the set where the global minimum of Q is attained, and we put $S_0=\emptyset$.

3.8. *Subharmonic potentials*

As before, $Q:\mathbb{C}\to\mathbb{R}$ is assumed to be of class $W^{2,p}$, so that, for example, $\Delta Q\in L^p_{\mathrm{loc}}(\mathbb{C})$. We suppose that there exists $T=T(Q)$ with $0<T\leqslant+\infty$ such that (3.2) holds for $0<t<T$ while it fails for $t>T$.

LEMMA 3.21 ($0<t<T$). *Let D be a bounded domain in $\mathbb{C}$ and suppose $\Delta Q\geqslant 0$ in D. Then $\partial D\subset S^*_t$ implies $D\subset S^*_t$.*

Proof. The assumption $\partial D\subset S^*_t$ means that $\hat{Q}_t=Q$ on ∂D. We write $R_0=\hat{Q}_t$, and let R_1 be the function which equals Q in D and equals $\hat{Q}_t$ elsewhere. We observe that $R_0\leqslant R_1\leqslant Q$ on $\mathbb{C}$, while $R_0=R_1=Q$ on ∂D. Also, the function R_1 is subharmonic. Indeed, $\Delta R_1=\Delta Q\geqslant 0$ on D (by assumption), and $\Delta R_1=\Delta R_0\geqslant 0$ on $\mathbb{C}\setminus\bar{D}$. It remains to observe that, for $a\in\partial D$,

$$R_1(a)=R_0(a)\leqslant\frac{1}{2\pi}\int_{-\pi}^{\pi}R_0(a+\varepsilon e^{\mathrm{i}\theta})\,d\theta\leqslant\frac{1}{2\pi}\int_{-\pi}^{\pi}R_1(a+\varepsilon e^{\mathrm{i}\theta})\,d\theta,\quad 0<\varepsilon<+\infty.$$

We see that R_1 is subharmonic in $\mathbb{C}$ and the conclusion $R_1=R_0$ follows. □

COROLLARY 3.22. *If Q is subharmonic in $\mathbb{C}$, then $\mathbb{C}\setminus S^*_t$ is connected.*

A continuous function $h:\mathbb{C}\to\mathbb{R}$ is said to be *nowhere harmonic* if for every open set $D\subset\mathbb{C}$ the restriction $h|_D$ fails to be harmonic.

COROLLARY 3.23. *Suppose Q is subharmonic in $\mathbb{C}$, and that Q is nowhere harmonic. Then $S^*_{t_0}\subset S_t$ for all t_0,t with $0<t_0<t<T$.*

Proof. By Corollary 3.22, the set $\mathbb{C}\setminus S^*_{t_0}$ is connected, and so $\mathrm{phull}(S^*_{t_0})=S^*_{t_0}$. By Lemma 3.19, then, we arrive at $\partial S^*_{t_0}\subset S_t$ for all t with $t_0<t<T$. It remains to check that $\mathrm{int}S^*_{t_0}\subset S_t$ for all t with $t_0<t<T$. By Proposition 3.12, we just need to show that no point

in $\mathrm{int}S^*_{t_0}$ is Q-shallow with respect to S^*_t. This is guaranteed by the requirement that Q be nowhere harmonic. □

3.9. *Convex potentials*

We say that a convex function $q : \mathbb{C} \to \mathbb{R}$ is *locally uniformly convex* if

$$|\xi|^2 \Delta q(z) + \mathrm{Re}[\xi^2 \partial^2 q(z)] \geqslant \epsilon(z)|\xi|^2, \quad \xi \in \mathbb{C},$$

for some continuous $\epsilon : \mathbb{C} \to]0, +\infty[$. For C^2-smooth q, this just says that the Hessian of q is (strictly) positive definite everywhere.

In [**19**, Chapter V], coincidence sets for constrained obstacle problems are considered, and under suitable convexity assumptions, the coincidence set is simply connected with $C^{1,\alpha}$-smooth boundary (here, $0 < \alpha < 1$). The setting is the following. Suppose Ω is a strictly convex bounded C^∞-smooth domain, and let $q : \bar{\Omega} \to \mathbb{R}$ be C^2-smooth and locally uniformly convex, with $q > 0$ on $\partial\Omega$ and $\min_\Omega q < 0$. Then, if we put $\varrho = 0$ in the constrained obstacle problem (see Subsection 3.4), the coincidence set

$$S^*_{\Omega,q} := \{z \in \Omega : \mathrm{Obst}_{\Omega,0}[q](z) = q(z)\}$$

is non-empty, compact, simply connected, and equal to the closure of its interior. Moreover, if q is $C^{2,\alpha}$-smooth for some α, $0 < \alpha < 1$, then the boundary $\partial S^*_{\Omega,q}$ is a $C^{1,\alpha'}$-smooth Jordan curve, for some α', $0 < \alpha' < 1$.

Applied to our setting (cf. Subsection 3.3), the results of [**19**, Chapter V] give us Theorem 3.24. Before we formulate the theorem, we note that if $Q : \mathbb{C} \to \mathbb{R}$ is convex, and (3.2) holds for some positive t, then Q must grow faster (radially, the growth is at least linear), so that (3.2) holds for all positive reals t (which makes $T = T(Q) = +\infty$).

Theorem 3.24 ($0 < t < T = +\infty$). *Suppose $Q : \mathbb{C} \to \mathbb{R}$ is C^2-smooth and locally uniformly convex with (3.2). Then the droplet S_t is simply connected, and equal to the closure of its interior. Moreover, if Q is $C^{2,\alpha}$-smooth for some α, $0 < \alpha < 1$, then ∂S_t is a $C^{1,\alpha'}$-smooth Jordan curve, for some α', $0 < \alpha' < 1$.*

Proof. We claim that for big enough c, the compact set

$$\bar{\Omega}_c := \{z \in \mathbb{C} : \hat{Q}_t(z) \leqslant c\}$$

is strictly convex with C^∞-smooth boundary. In fact, we know from (3.4) and the fact that $\hat{Q}_t$ has the form

$$\hat{Q}_t(z) = t \log |z|^2 + h(z),$$

where h is real-valued, bounded, and harmonic in a neighbourhood of infinity. As c increases the sets $\bar{\Omega}_c$ cover bigger and bigger portions of the plane $\mathbb{C}$, and the boundary $\partial\bar{\Omega}_c$ is contained in a fixed neighbourhood of infinity for big enough c. The equation defining the boundary is

$$|z|\, e^{h(z)} = e^c,$$

and an argument using the harmonic conjugate of h shows that this equation may be written in the form

$$|z + a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots| = e^c,$$

where the series converges for big $|z|$. In other words, using the inverse mapping, $\partial\bar{\Omega}_c$ is (for big c) the image of the circle $|z| = e^c$ under a mapping

$$z \mapsto z + b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots,$$

which also converges for big $|z|$. After rescaling by a factor of e^{-c}, we are talking about the image of the unit circle $|z| = 1$ under the mapping

$$z \mapsto z + b_0 e^{-c} + b_1 e^{-2c} z^{-1} + b_2 e^{-3c} z^{-2} + \cdots,$$

which, for large values of c, constitutes a very slight perturbation of the circle $|z| = 1$, and it is then easy to check that the domain inside the curve is strictly convex with C^∞-smooth boundary. As a consequence, $\bar\Omega_c$ is strictly convex with C^∞-smooth boundary for big c. To finish the proof, we observe that (cf. Lemma 3.7)

$$\mathrm{Obst}_{\Omega_c,0}[q] + c = \mathrm{Obst}_t[Q] \quad \text{on } \bar\Omega_c,$$

if $q = Q - c$. It is immediate that $S_t = S_{\Omega_c,q}$. The rest follows from [**19**, Chapter V]. □

4. *Local droplets*

4.1. *Localization*

We often localize the field $Q : \mathbb{C} \to \mathbb{R} \cup \{+\infty\}$ (which we assume to be lower semi-continuous) to a closed set $\Sigma \subset \mathbb{C}$ and write

$$Q_\Sigma = \begin{cases} Q & \text{on } \Sigma, \\ +\infty & \text{on } \mathbb{C} \setminus \Sigma. \end{cases}$$

The function Q_Σ is then also lower semi-continuous. We will assume that Q_Σ meets the growth condition (3.2) (which is the t-scaled version of (2.6); in case Σ is compact, this is automatic). To avoid triviality, we also need to require that $Q_\Sigma < +\infty$ on a set of positive area. We will refer to the closed set Σ as a *localization*.

We will use the notation (which corresponds to the special parameter choice $t = 1$)

$$\hat\sigma_{Q,\Sigma} = \hat\sigma[Q,\Sigma] := \hat\sigma_{Q_\Sigma}, \quad S_{Q,\Sigma} = S[Q,\Sigma] := \operatorname{supp} \hat\sigma[Q,\Sigma];$$

this conforms with the convention to write $\hat\sigma_Q = \hat\sigma[Q]$ and $S_Q = S[Q]$. We will focus on the t-scaled variants $(0 < t < +\infty)$

$$\hat\sigma_t[Q,\Sigma] := t\,\hat\sigma_{Q/t,\Sigma}, \quad S_t[Q,\Sigma] := \operatorname{supp} \hat\sigma_t[Q,\Sigma] = \operatorname{supp} \hat\sigma[Q/t,\Sigma] = S[Q/t,\Sigma].$$

We shall also need the modified Robin constant

$$\gamma_t^*(Q,\Sigma) = \gamma_t^*(Q_\Sigma)$$

from Subsection 3.1.

Lemma 4.1 $(0 < t < +\infty)$. *Suppose $\Sigma \subset \mathbb{C}$ is closed, and that Q_Σ meets the growth condition* (3.2)*, while $Q_\Sigma < +\infty$ holds on a set of positive area. Then $S_t[Q,\Sigma] \subset \Sigma$.*

Proof. If the probability measure $\sigma_0 := \hat\sigma[Q/t,\Sigma]$ were to have support outside Σ, the corresponding energy

$$I_{Q_\Sigma/t}[\sigma_0] = \int_{\mathbb{C}^2} \log \frac{1}{|\xi - \eta|^2}\, d\sigma_0(\xi)\, d\sigma_0(\eta) + \frac{2}{t} \int_{\mathbb{C}} Q_\Sigma\, d\sigma_0$$

would necessarily equal $+\infty$, which does not agree with the energy minimizing property of the equilibrium measure. □

We now compare two different localizations, one contained in the other.

Lemma 4.2 ($0 < t < +\infty$). *Suppose $\Sigma_1, \Sigma_2 \subset \mathbb{C}$ are closed with $\Sigma_1 \subset \Sigma_2$, and that Q_{Σ_2} meets the growth condition (3.2), while $Q_{\Sigma_1} < +\infty$ holds on a set of positive area. If $S_t[Q, \Sigma_2] \subset \Sigma_1$, then*

$$\hat{\sigma}_t[Q, \Sigma_1] = \hat{\sigma}_t[Q, \Sigma_2], \quad S_t[Q, \Sigma_1] = S_t[Q, \Sigma_2].$$

Proof. For $j = 1, 2$, we write $\sigma_j = \hat{\sigma}[Q/t, \Sigma_j]$ and $Q_j = Q_{\Sigma_j}/t$. If L_Q is as in (2.11), we then obtain that (since $\Sigma_1 \subset \Sigma_2$)

$$L_{Q_1}(\xi, \eta) = L_{Q_2}(\xi, \eta), \quad (\xi, \eta) \in \Sigma_1 \times \Sigma_1,$$

and so (since $S_t[Q, \Sigma_2] = \operatorname{supp} \hat{\sigma}[Q/t, \Sigma_2] = \operatorname{supp} \sigma_2 \subset \Sigma_1$)

$$I_{Q_1}[\sigma_2] = \int_{\mathbb{C}^2} L_{Q_1}(\xi, \eta)\, d\sigma_2(\xi)\, d\sigma_2(\eta) = \int_{\mathbb{C}^2} L_{Q_2}(\xi, \eta)\, d\sigma_2(\xi)\, d\sigma_2(\eta) = I_{Q_2}[\sigma_2], \tag{4.1}$$

where we have used the identity (2.12). As $Q_1 \geqslant Q_2$, we have

$$I_{Q_1}[\sigma_1] = \inf_{\sigma} I_{Q_1}[\sigma] \geqslant \inf_{\sigma} I_{Q_2}[\sigma] = I_{Q_2}[\sigma_2],$$

where both infima run over $\sigma \in \operatorname{prob}_c(\mathbb{C})$. Combined with (4.1), this gives $I_{Q_1}[\sigma_1] \leqslant I_{Q_1}[\sigma_1]$, which is only possible if $\sigma_1 = \sigma_2$, by Frostman's theorem (Theorem 2.4). Finally, if the measures coincide, their supports coincide as well. □

The typical application of Lemma 4.1 will be when both Σ_1 and Σ_2 are compact. However, already the case when $\Sigma_1 = S_Q$ and $\Sigma_2 = \mathbb{C}$ is interesting.

Corollary 4.3 ($0 < t < +\infty$). *If Q meets the growth condition (3.2), then with $\Sigma := S_t[Q] = S_{Q/t}$, we have*

$$\hat{\sigma}_t[Q] = \hat{\sigma}_t[Q, \Sigma], \quad S_t[Q, \Sigma] = S_t[Q].$$

4.2. *Local droplets and the obstacle problem*

We let Σ be a localization and suppose that $Q < +\infty$ on a subset of Σ with positive area. We require that Q_Σ meets the growth condition (3.2) for a positive t, which is kept fixed for the moment (this requirement is void if $\Sigma \subset \mathbb{C}$ is compact). Then the Borel measure $\hat{\sigma}_t[Q, \Sigma]$ is a well-defined positive measure of total mass t and its support $S_t[Q, \Sigma]$ is compact with $S \subset \Sigma$. The following lemma is immediate from Lemma 3.8.

Lemma 4.4. *Suppose $Q \in W^{2,1}(\operatorname{int} S)$ where $S = S_t[Q, \Sigma]$. Then $\hat{\sigma}_t[Q, \Sigma]$ is absolutely continuous in $\operatorname{int} S$ and in fact*

$$d\hat{\sigma} = \Delta Q\, dA \quad \text{on } \operatorname{int} S.$$

Proof. This follows from Lemma 3.8. □

We are led to the following three definitions.

Definition 4.5. Suppose Q is in $W^{2,1}$ on a neighbourhood of $S = S_t[Q, \Sigma]$. We say that S is a *local (Q, t)-droplet with localization* Σ if the following equality holds ($\hat{\sigma} = \hat{\sigma}_t[Q, \Sigma]$):

$$d\hat{\sigma} = 1_S \Delta Q\, dA.$$

Definition 4.6. A compact set $S \subset \mathbb{C}$ is a *local* (Q,t)*-droplet* if it is a local (Q,t)-droplet with respect to some localization Σ.

Definition 4.7. A compact set $S \subset \mathbb{C}$ is a *global* (Q,t)*-droplet* if it is a local (Q,t)-droplet with respect to the localization $\Sigma = \mathbb{C}$.

Remark 4.8. (a) There is at most one global (Q,t)-droplet S, as it is given by $S = S_t[Q]$. Theorem 3.10 guarantees that it exists under the growth requirement (3.2) and the additional regularity $Q \in W^{2,p}$ (this is true also if Q is in $W^{2,p}$ only in a neighbourhood of S). In contrast, there may exist several local (Q,t)-droplets.

(b) If the boundary ∂S of the set $S = S[Q,\Sigma]$ has a zero area, then S is a local (Q,t)-droplet if and only if and only if $\hat\sigma = \hat\sigma_t[Q,\Sigma]$ is absolutely continuous.

(c) Two local (Q,t)-droplets S_1 and S_2 cannot have the containment $S_1 \subset S_2$ unless $S_1 = S_2$.

(d) The point with the definition of local droplets is that we may focus on the support $S = \operatorname{supp}\hat\sigma$ rather than the (generally more complicated) equilibrium measure $\hat\sigma$ (with respect to the weight Q_Σ).

(e) In the above definition, it is possible to weaken the smoothness assumption on Q to $W^{\Delta,1}$ smoothness, which just asks that the function and its Laplacian are both locally integrable.

Proposition 4.9. *If S is a local (Q,t)-droplet with respect to some localization $\Sigma = \Sigma_0$, then it is a local (Q,t)-droplet with respect to the minimal localization $\Sigma = S$.*

Proof. This follows from Lemma 4.2 with $\Sigma_1 = S$ and $\Sigma_2 = \Sigma_0$. □

Remark 4.10. If $S = S_t[Q,\Sigma]$ is a local (Q,t)-droplet, then the associated measure $\hat\sigma = \hat\sigma_t[Q,\Sigma]$ is absolutely continuous. It is possible that the converse might be true (cf. Lemma 4.4). For the moment, we have a weaker statement. Suppose first that Q is in $W^{2,p}$ in a neighbourhood of S, for some p, $1 < p < +\infty$. The statement now runs as follows: if $\hat\sigma$ is absolutely continuous with density in L^p for some p, $1 < p < +\infty$, then S is a local (Q,t)-droplet. Indeed, from the properties of the 2-dimensional Hilbert transform, we get that the function

$$\widehat{(Q_\Sigma)}_t(\xi) = \gamma_t^*(Q,\Sigma) - U^{\hat\sigma}(\xi),$$

is in $W^{2,p}$ and from Proposition 3.5, we have that

$$\widehat{(Q_\Sigma)}_t(\xi) = Q(\xi), \quad \xi \in S,$$

so that by [19, p. 53], we obtain

$$\Delta\widehat{(Q_\Sigma)}_t(\xi) = \Delta Q(\xi), \quad \xi \in S,$$

as distributions, which leads to the desired result.

For a compact $S \subset \mathbb{C}$, we define the corresponding (weighted) logarithmic potential

$$U^{Q,S}(\xi) = \int_S \log\frac{1}{|\xi-\eta|^2}\,\Delta Q(\eta)\,dA(\eta). \tag{4.2}$$

We have the following characterization of local $(Q,1)$-droplets. We recall the notion of Q-shallow points from Subsection 3.6.

Theorem 4.11. *Suppose that $S \subset \Sigma \subset \mathbb{C}$, where S is compact and Σ closed, and that Q is in $W^{2,1}$ in a neighbourhood of S. Then S is a local (Q,t)-droplet with localization Σ if and only if:*

(i) $\Delta Q \geqslant 0$ *almost everywhere on S,*
(ii) *S contains no Q-shallow points,*
(iii) $\int_S \Delta Q \, dA = t$,
(iv) $U^{S,Q} + Q = \gamma_t^*(Q,S)$ *q.e. on S, for some real constant $\gamma_t^*(Q,S)$ (the modified Robin constant), and*
(v) $U^{S,Q} + Q \geqslant \gamma_t^*(Q,S)$ *q.e. on Σ.*

Proof. We first establish the necessity of conditions (i)–(iv). So, we suppose that S is a local (Q,t)-droplet. Since $d\hat{\sigma} = 1_S \Delta Q \, dA$ is positive with mass t and S is its support set, the conditions (i)–(iii) are necessary. The necessity of the conditions (iv) and (v) follows from Frostman's Theorem 2.6 (with Q_S/t in place of Q, where S is used as a localization).

We turn to the sufficiency of the conditions (i)–(v). We write $W := c - U^{Q,S}$, where the constant $c = \gamma_t^*(Q,S)$ is as in (iv). By (iv), we then have $W = Q_\Sigma$ q.e. on S while (v) gives $W \leqslant Q_\Sigma$ q.e. on $\mathbb{C}$. By Lemma 3.1, we obtain that $W = \mathrm{Obst}_t[Q_\Sigma]$. Next, Proposition 3.2 and Corollary 3.4 show that

$$d\hat{\sigma}_t[Q,S] = \Delta W \, dA = -\Delta U^{Q,S} \, dA = 1_S \Delta Q \, dA.$$

So we have a local (Q,t)-droplet with localization Σ. □

Remark 4.12. To characterize the local (Q,t)-droplets, we use the minimal localization S. We see that condition (v) becomes vacuous and may be removed.

Corollary 4.13. *Suppose that $S \subset \Sigma \subset \mathbb{C}$, where S is compact and Σ closed, and that Q is in $W^{2,1}$ in a neighbourhood of S. Consider the function*

$$\hat{Q}_S := \gamma_t^*(Q,S) - U^{Q,S},$$

where $\gamma_t^*(Q,S)$ is the constant in Theorem 4.11. Then S is a local (Q,t)-droplet with localization Σ if and only if:

(i) $\Delta Q \geqslant 0$ *almost everywhere on S,*
(ii) *S contains no Q-shallow points,*
(iii) $\int_S \Delta Q \, dA = t$,
(iv) $\hat{Q}_S = Q$ *q.e. on S, and*
(v) $\hat{Q}_S \leqslant Q$ *q.e. on Σ.*

Moreover, if (i)–(v) *are assumed, then $\hat{Q}_S \in \mathrm{Sub}_t(\mathbb{C})$ is harmonic on $\mathbb{C} \setminus S$, with asymptotics*

$$\hat{Q}_S(z) = t \log |z|^2 + \mathrm{O}(1) \quad \text{as } |z| \longrightarrow +\infty.$$

As a consequence, we have q.e.

$$\hat{Q}_S = \widehat{(Q_\Sigma)}_t = \mathrm{Obst}_t[Q_\Sigma].$$

Moreover, if, for some p with $1 < p < +\infty$, we have $Q \in W^{2,p}$ in a neighbourhood of S, then $\hat{Q}_S \in W^{2,p}$ as well.

Proof. It is clear from the properties of logarithmic potentials that $U^{Q,S}$ is subharmonic in $\mathbb{C}$ and harmonic in $\mathbb{C} \setminus S$, with the corresponding asymptotics at infinity as a consequence of condition (ii) of Theorem 4.11. Moreover, the properties of the 2-dimensional Hilbert transform show that if $Q \in W^{2,p}$ in a neighbourhood of S, then $U^{Q,S} \in W^{2,p}$, for $1 < p < +\infty$. These properties are then inherited by $\hat{Q}_S$. □

If there is some room to wiggle between the set $S_t[Q,\Sigma]$ and the localization Σ, then the set $S_t[Q,\Sigma]$ is automatically a local (Q,t)-droplet:

THEOREM 4.14. *Suppose $Q \in W^{2,p}$ for some p, $1 < p < +\infty$. If for a localization Σ, we have $S = S_t[Q,\Sigma] \subset \operatorname{int}\Sigma$, then S is a local (Q,t)-droplet with localization Σ.*

Proof. This is Theorem 3.10 for Q_Σ in place of Q. □

REMARK 4.15. The modified Robin constant $\gamma^*(Q,S)$ may be written out explicitly:

$$\gamma_t^*(Q,S) = \frac{1}{t}\int_{S\times S} \log\frac{1}{|\xi-\eta|^2}\Delta Q(\xi)\Delta Q(\eta)\,dA(\xi)\,dA(\eta) + \int_S Q\Delta Q\,dA. \tag{4.3}$$

4.3. *Characterization of local droplets*

We need the concept of local Q-droplets. We consider compact localizations Σ only, which means that no requirement on Q near infinity is needed, just that $Q : \mathbb{C} \to \mathbb{R} \cup \{+\infty\}$ is lower semi-continuous and has $Q < +\infty$ on a subset of Σ with positive area. We recall the concept of a (Q,t)-droplet, which presupposed that Q was $W^{2,1}$-smooth near S.

DEFINITION 4.16. A compact set $S \subset \mathbb{C}$ is a (local) Q-droplet if it is a local (Q,t)-droplet for some t with $0 < t < +\infty$.

We see that Theorem 4.11 has the following consequence.

COROLLARY 4.17. *Suppose that $S \subset \mathbb{C}$ is compact, and that Q is in $W^{2,1}$ in a neighbourhood of S. Then S is a local Q-droplet if and only if:*

(i) *$\Delta Q \geqslant 0$ almost everywhere on S,*
(ii) *S contains no Q-shallow points, and*
(iii) *$U^{S,Q} + Q$ is constant q.e. on S.*

By Sobolev imbedding, we have $W^{2,p} \subset C^1$ for $2 < p \leqslant +\infty$. The following characterization will prove useful later.

PROPOSITION 4.18 ($0 < t < +\infty$). *Suppose $S \subset \mathbb{C}$ is compact with $S = \operatorname{clos}\operatorname{int} S$, and that $Q \in W^{2,p}$ in a neighbourhood of S, for some p, $2 < p < +\infty$. We then have:*

(i) *f S is a local Q-droplet, then $\bar\partial(U^{Q,S} + Q) = 0$ on S, and*
(ii) *if S is connected and $\bar\partial(U^{Q,S} + Q) = 0$ on ∂S, and if S has no Q-shallow points, then S is a local Q-droplet.*

Proof. We first treat part (i). So, we assume that S is a local Q-droplet. By Corollary 4.17, $U^{Q,S} + Q$ is constant q.e. on S. As both Q and $U^{Q,S}$ are in $W^{2,p}$ in a neighbourhood of

S, we conclude from [19, p. 53], that $\bar\partial(U^{Q,S}+Q)=0$ almost everywhere on S. By Sobolev imbedding, $\bar\partial(U^{Q,S}+Q)$ is continuous in a neighbourhood of S, and so $\bar\partial(U^{Q,S}+Q)=0$ on $\operatorname{int} S$ and *a fortiori* (by the topological assumption) on S.

We turn to part (ii). Consider the function $F:=\bar\partial(U^{Q,S}+Q)$, which is in $W^{1,p}$ in a neighbourhood of S, and therefore continuous. We have

$$\partial F=\Delta(U^{Q,S}+Q)=-1_S\Delta Q+\Delta Q=0 \quad \text{almost everywhere on } S.$$

Hence F is conjugate holomorphic in the interior of S and since $F=0$ on the boundary, we have $F\equiv 0$ on S. If S is connected, then this implies that U_S+Q is constant on S, so by Corollary 4.17, S is a local Q-droplet. □

5. *Chains of local droplets*

5.1. *A partial ordering of local droplets*

We recall that S is a local Q-droplet if it is a local (Q,t)-droplet for some t with $0<t<+\infty$. For the concept to make sense, we need to ask that $Q:\mathbb{C}\to\mathbb{R}\cup\{+\infty\}$ is lower semi-continuous and $W^{2,1}$-smooth near S. Given a local Q-droplet S, the corresponding value of (the evolution parameter) t is easily calculated:

$$t=t(Q,S):=\int_S \Delta Q\, dA.$$

We note that by Corollary 4.17, $\Delta Q\geqslant 0$ on S. To simplify the presentation, we shall *assume that Q is $W^{2,p}$-smooth in $\mathbb{C}$ for some p, $1<p<+\infty$.*

LEMMA 5.1. *Let S_2 be a local Q-droplet, with $t_2=t(Q,S_2)$. If t_1 has $0<t_1<t_2$ we put $S_1:=S_{t_1}[Q,S_2]$. Then S_1 is a local Q-droplet, with $t_1=t(Q,S_1)$.*

Proof. We should study the measure $\sigma_1:=\sigma_{t_1}[Q,S_2]$, which by Proposition 3.2 and Corollary 3.4 is obtained from $\operatorname{Obst}_{t_1}[Q_{S_2}]$ by applying the Laplacian. From $t_1<t_2$ and the definition of the obstacle problem, we see that

$$\operatorname{Obst}_{t_1}[Q_{S_2}]\leqslant \operatorname{Obst}_{t_2}[Q_{S_2}]=\hat Q_{S_2},$$

where we use Corollary 4.13 to obtain the rightmost identity. A moments reflection, using that $\hat Q_{S_2}\leqslant Q$, reveals that in fact

$$\operatorname{Obst}_{t_1}[Q_{S_2}]=\operatorname{Obst}_{t_2}[\hat Q_{S_2}].$$

Since ΔQ is in L^p locally, $\hat Q_{S_2}$ is $W^{2,p}$-smooth and by Theorem 3.10 with $\hat Q_{S_2}$ in place of Q, we obtain that S_1 is a local Q-droplet. □

Lemma 5.1 allows us to introduce a partial ordering in the set of all local Q-droplets.

DEFINITION 5.2. Let S_1 and S_2 be two local Q-droplets, and write $t_j=t(Q,S_j)$, $j=1,2$. We write $S_1\prec S_2$ if $S_1\subset S_2$ and $S_1=S_{t_1}[Q,S_2]$.

REMARK 5.3. (a) In other words, $S_1\prec S_2$ if S_1 and S_2 are local Q-droplets and S_1 is a local (Q,t_1)-droplet with localization S_2, where $t_1=t(Q,S_1)$.

(b) It follows from the definition that if S_1 and S_2 are Q-droplets with $S_1 \prec S_2$, then $t(Q, S_1) \leqslant t(Q, S_2)$.

(c) If $S_1 \prec S_2$ and $S_2 \prec S_1$ for two local Q-droplets, then $S_1 \subset S_2$ and $S_2 \subset S_2$, and so $S_1 = S_2$.

Proposition 5.4. *Let S_1 and S_2 be two local Q-droplets with $S_1 \subset S_2$. Then $S_1 \prec S_2$ if and only if $\hat{Q}_{S_1} \leqslant Q$ holds on S_2.*

Proof. This follows from Corollary 4.13. After all, for local Q-droplets we do not need to check conditions (i)–(iv); only (v) remains. Moreover, by continuity and the fact that Q-droplets lack Q-shallow points, the q.e. statements hold everywhere. □

Lemma 5.5. *For a local Q-droplet S, we have $S \prec S$.*

Proof. This follows from the definition of the '$\prec$' relation together with Proposition 4.9. □

There is one more property we need to check to show that '$\prec$' defines a partial ordering.

Lemma 5.6. *If S_1, S_2 and S_3 are three local Q-droplets with $S_1 \prec S_2$ and $S_2 \prec S_3$, then $S_1 \prec S_3$.*

Proof. If we use that $S_2 \prec S_3$, we see from Lemma 5.1 that $S_{t_1}[Q, S_3]$ is a local Q-droplet with

$$S_{t_1}(Q, S_3) \subset S_2 = S_{t_2}(Q, S_3).$$

Using that $S_1 \prec S_2$, we appeal to Lemma 4.2 and obtain

$$S_{t_1}[Q, S_3] = S_{t_1}[Q, S_2] = S_1,$$

so that $S_1 \prec S_3$, as claimed. □

Remark 5.7. The containment $S_1 \subset S_2$ does not imply $S_1 \prec S_2$. For example, suppose Q has two global minima at the points 0 and 2 and suppose the minima are non-degenerate. Consider

$$S_1 := S_{t_1}(Q, \Sigma_1), \quad S_2 := S_{t_2}(Q, \Sigma_2) \quad \text{where } 0 < t_1 \ll t_2 \ll 1,$$

with $\Sigma_1 = \bar{\mathbb{D}}(0, 1)$ and $\Sigma_2 = \bar{\mathbb{D}}(0, 3)$. Then $S_1 \subset S_2$ but $S_1 \not\prec S_2$. This is easy to see using the characterization of Proposition 5.4.

5.2. *A comparison principle*

We keep the setting of the previous subsection. We recall the definition of the polynomially convex hull $\operatorname{phull}(E)$ of a compact set E from Subsection 3.7. The set $\operatorname{phull}(E) \setminus E$ is the union of all the bounded components of $\mathbb{C} \setminus E$.

Proposition 5.8. *Suppose S_1 and S_2 are two local Q-droplets with $S_1 \subset S_2$. We then have $\hat{Q}_{S_2} \leqslant \hat{Q}_{S_1}$ on* $\operatorname{phull}(S_1)$, *with equality on S_1. Moreover, if for some $z_0 \in \operatorname{int}[\operatorname{phull}(S_1)]$,* *we have* $\hat{Q}_{S_2}(z_0) = \hat{Q}_{S_1}(z_0)$*, then* $\hat{Q}_{S_2} = \hat{Q}_{S_1}$ *holds on the component of* $\mathrm{int}[\mathrm{phull}(S_1)]$ *that contains* z_0*.*

Proof. The difference $\hat{Q}_{S_2} - \hat{Q}_{S_1}$ is in $W^{2,p}$ and therefore continuous, and it is subharmonic, as

$$\Delta[\hat{Q}_{S_2} - \hat{Q}_{S_1}] = 1_{S_2 \setminus S_1} \Delta Q \geqslant 0 \quad \text{almost everywhere on } \mathbb{C}.$$

Moreover, by Corollary 4.13, $\hat{Q}_{S_2} = Q$ on S_2 and $\hat{Q}_{S_1} = Q$ on S_1, and so $\hat{Q}_{S_2} - \hat{Q}_{S_1} = 0$ on S_1 as $S_1 \subset S_2$. The inequality $\hat{Q}_{S_2} - \hat{Q}_{S_1} \leqslant 0$ now follows from the maximum principle. The last assertion follows from the strong maximum principle. □

We see that a local Q-droplet S_2 with $S_1 \prec S_2$ does not grow in the direction of the interior holes of S_1:

COROLLARY 5.9. *Suppose* S_1 *and* S_2 *are two local* Q*-droplets with* $S_1 \prec S_2$*. We then have* $[S_2 \setminus S_1] \cap \mathrm{phull}(S_1) = \emptyset$ *and* $\hat{Q}_{S_2} = \hat{Q}_{S_1}$ *on* $\mathrm{phull}(S_1)$*.*

Proof. If $S_1 \prec S_2$, we have $\hat{Q}_{S_1} \leqslant Q$ on S_2 (cf. Proposition 5.4) and $\hat{Q}_{S_2} = Q$ on S_2 (cf. Corollary 4.13). In view of Proposition 5.8, it follows that if $z_0 \in S_2 \cap \mathrm{phull}(S_1)$, then $\hat{Q}_{S_2}(z_0) = \hat{Q}_{S_1}(z_0)$. So, if $z_0 \in S_2 \cap \mathrm{int}[\mathrm{phull}(S_1)]$, another application of Proposition 5.8 shows that $\hat{Q}_{S_2} = \hat{Q}_{S_1}$ holds on the component of $\mathrm{int}[\mathrm{phull}(S_1)]$ which contains z_0. Taking the Laplacian, we find that $1_{S_1} \Delta Q = 1_{S_2} \Delta Q$ almost everywhere on the component $\mathrm{Comp}(z_0)$ of $\mathrm{int}[\mathrm{phull}(S_1)]$ which contains z_0, which leads to

$$S_1 \cap \mathrm{Comp}(z_0) = S_2 \cap \mathrm{Comp}(z_0).$$

Since $z_0 \in S_2$ we also must have $z_0 \in S_1$. We conclude that $[S_2 \setminus S_1] \cap \mathrm{int}[\mathrm{phull}(S_1)] = \emptyset$ and *a fortiori* $[S_2 \setminus S_1] \cap \mathrm{phull}(S_1) = \emptyset$. But then $\hat{Q}_{S_2} - \hat{Q}_{S_1}$ is harmonic in $\mathrm{int}[\mathrm{phull}(S_1)]$ and vanishes on $\partial[\mathrm{phull}(S_1)] \subset \partial S_1 \subset S_1$, and the conclusion $\hat{Q}_{S_2} - \hat{Q}_{S_1} = 0$ on $\mathrm{phull}(S_1)$ is immediate. □

5.3. *Domination chains of local droplets*

We are interested in chains of local Q-droplets.

DEFINITION 5.10. A *domination chain* of local Q-droplets is a (continuously indexed) family of Q-droplets $\{S_t\}_t$, where the index t ranges over a non-empty interval $I \subset \mathbb{R}_+$, with left endpoint 0, such that $t = t(Q, S_t)$ and

$$t_1 \leqslant t_2 \iff S_{t_1} \prec S_{t_2}.$$

The domination chain is *terminating* if the interval I is given by $0 < t \leqslant t_*$, for some T_* with $0 < t_* < +\infty$, and *non-terminating* if it is given by $0 < t < t_*$ for some t_* with $0 < t_* \leqslant +\infty$. In case the domination chain is terminating, we say that it *terminates at* S_{t_*}.

LEMMA 5.11. *Given a local* Q*-droplet* S_**, there is exactly one domination chain of local* Q*-droplets that terminates at* S_**.*

Proof. By Lemma 5.1, $S_t := S_t[Q, S_*]$ for $0 < t \leqslant t_* := t(Q, S_*)$ defines a continuously indexed collection of local Q-droplets and by Lemma 4.2 it is a (terminating) domination chain. Finally, if $S_\sharp$ is a local Q-droplet with $S_\sharp \prec S_*$, then by definition, it is of the form $S_\sharp = S_{t_\sharp}[Q, S_*]$ with $t_\sharp := t(Q, S_\sharp) \leqslant t_*$, so the domination chain is unique. □

5.4. *Maximal domination chains of local Q-droplets*

We keep the setting of the previous subsection. We shall need the concept of a maximal domination chain of local Q-droplets.

DEFINITION 5.12. A domination chain of Q-droplets is *maximal* if it is contained in no larger domination chain of local Q-droplets.

Maximal domination chains of Q-droplets can be either terminating or non-terminating. If the chain is indexed by the unbounded interval $I = \mathbb{R}_+$, then it is automatically non-terminating. If the chain is indexed by a bounded interval, then it can be non-terminating only if the droplets develop 'arms' or 'islands' that tend to infinity:

THEOREM 5.13. *Let $\{S_t\}_{t\in I}$ be a maximal non-terminating domination chain of local Q-droplets. Then the union*

$$S_\cup := \bigcup_{t\in I} S_t$$

is an unbounded subset of $\mathbb{C}$.

Proof. We suppose $S_\cup$ is bounded, and form $S_* = \operatorname{clos} S_\cup$, which is then compact. We are to show that the non-terminating domination chain $\{S_t\}_{t\in I}$ cannot be maximal. The interval I is given by $0 < t < t_*$ for some t_* with $0 < t_* < +\infty$. For $t \in I$, we let σ_t be the positive measure $d\sigma_t = 1_{S_t}\Delta Q\, dA$, which has total mass $\|\sigma_t\| = t$. Let σ_* be given by $d\sigma_* = 1_{S_\cup}\Delta Q\, dA$, which has total mass $\|\sigma_*\| = t_*$. Then $\sigma_t \to \sigma_*$ in norm as $t \to t_*$, and in fact the corresponding densities converge in L^p:

$$1_{S_t}\Delta Q\, dA \longrightarrow 1_{S_\cup}\Delta Q\, dA \quad \text{in } L^p(\mathbb{C}) \text{ as } t \longrightarrow t_*.$$

By the well-known properties of the 2-dimensional Hilbert transform, we find that the associated potentials converge in $W^{2,p}$: $U^{Q,S_t} \to U^{Q,S_\cup}$ as $t \to t_*$. Also, we easily check that if the constants $\gamma^*(Q, S_t)$ and $\gamma^*(Q, S_\cup)$ are as in (4.3), we have $\gamma^*(Q, S_t) \to \gamma^*(Q, S_\cup)$ as $t \to t_*$. As a consequence,

$$\hat{Q}_{S_t} = \gamma_t^*(Q, S_t) - U^{Q,S_t} \longrightarrow \hat{Q}_{S_\cup} = \gamma_{t_*}^*(Q, S_t) - U^{Q,S_\cup} \quad \text{in } W^{2,p} \text{ as } t \longrightarrow t_*.$$

By Sobolev imbedding the convergence is locally uniform. Since $\hat{Q}_{S_t} = Q$ on S_t we obtain that $\hat{Q}_{S_\cup} = Q$ on $S_\cup$. By continuity, then, we find that $\hat{Q}_{S_\cup} = Q$ on $S_* = \operatorname{clos} S_\cup$. Next, by [**19**, p. 53], we see that $\Delta\hat{Q}_{S_\cup} = \Delta Q$ almost everywhere on S_*, that is, $1_{S_\cup}\Delta Q = \Delta Q$ almost everywhere on S_*. Expressed differently, we have $1_{S_\cup}\Delta Q = 1_{S_*}\Delta Q$ as elements of $L^p(\mathbb{C})$. In particular, $\Delta Q \geqslant 0$ holds almost everywhere on S_*. By construction, S_* has no Q-shallow points, a property this set inherits from the individual droplets S_t, $t \in I$. In view of Corollary 4.17, S_* is a local Q-droplet. It remains to show that we may add S_* as a terminal local Q-droplet for the domination chain, thereby defeating the maximality of the non-terminating domination chain. To this end, it suffices to obtain that $S_t \prec S_*$ for $t \in I$. We pick a t' with $t < t' < t_*$ and use $S_t \prec S_{t'}$ to deduce that $\hat{Q}_{S_t} \leqslant Q$ on $S_{t'}$ (Proposition 5.4). By letting $t' \to t_*$, we get that $\hat{Q}_{S_t} \leqslant Q$ on $S_\cup$, and by continuity that $\hat{Q}_{S_t} \leqslant Q$ on S_*. By Proposition 5.4, this means that $S_t \prec S_*$. The proof is finished. □

The following definition is useful.

Definition 5.14. A local Q-droplet S is *maximal* if for any other local Q-droplet S' the relation $S \prec S'$ implies that $S = S'$.

Corollary 5.15. *A maximal domination chain $\{S_t\}_{t\in I}$ of local Q-droplets either terminates at a maximal local Q-droplet, or is non-terminating, in which case the set $S_\cup$ of Theorem 5.13 is unbounded.*

5.5. *Richardson's formula*

We keep the setting of the previous two subsections. We would like to understand the flow evolution $t \mapsto S_t$ of domination chains (or containment chains, see the next subsection) of local Q-droplets. A natural way to do this is to analyse the effect of the flow when we use harmonic functions as test function (that is, we calculate 'harmonic moments').

Proposition 5.16. *Suppose S and S' are two local Q-droplets with $S \subset S'$. Then for all $h \in W^{2,1}(\mathbb{C})$ (local Sobolev class) that are harmonic in $\mathbb{C} \setminus S$ and bounded near infinity, we have (with $t = t(Q,S)$ and $t' = t(Q,S')$)*

$$\int_{S'\setminus S} h\Delta Q\, dA = (t'-t)\, h(\infty).$$

Proof. The formula holds for constant h, by the choice of t and t'. So, by subtracting a constant, we may take $h(\infty) = 0$. We have

$$\int_S h\Delta Q\, dA = \int_{\mathbb{C}} h\, 1_S \Delta Q\, dA = \int_{\mathbb{C}} h\Delta\hat{Q}_S\, dA = \int_{\mathbb{C}} \hat{Q}_S\, \Delta h\, dA, \tag{5.1}$$

and the analogous identity holds for S' as well. By forming the difference between (5.1) for S and S' we see that

$$\int_{S'\setminus S} h\Delta Q\, dA = \int_{\mathbb{C}} [\hat{Q}_{S'} - \hat{Q}_S]\Delta h\, dA = 0,$$

because $\Delta h = 0$ on $\mathbb{C}\setminus S$ while $\hat{Q}_{S'} - \hat{Q}_S = Q - Q = 0$ on S (see, for example, Corollary 4.13). To finish the proof, we only need to justify (5.1). By Green's formula, we have

$$\int_{\mathbb{D}(0,R)} [h\Delta\hat{Q}_S - \hat{Q}_S\Delta h]\, dA = 2\int_{\mathbb{T}(0,R)} [h\partial_n\hat{Q}_S - \hat{Q}_S\partial_n h]\, ds, \tag{5.2}$$

where ds is the normalized arc length (that is, arc length divided by 2π) and ∂_n is the exterior normal derivative. Next, we observe that as $|z| \to +\infty$, we have the asymptotics

$$h = \mathrm{O}(|z|^{-1}), \quad |\nabla h| = \mathrm{O}(|z|^{-2}), \quad \hat{Q}_S = \mathrm{O}(\log|z|), \quad |\nabla\hat{Q}_S| = \mathrm{O}(|z|^{-1}),$$

because both h and $\hat{Q}_S$ are harmonic in $\mathbb{C}\setminus S$ with given asymptotical behaviour. By letting $R \to +\infty$ in (5.2), we obtain (5.1). The proof is complete. □

Corollary 5.17. *Suppose S and S' are two local Q-droplets with $S \subset S'$. Also suppose that the interior $\operatorname{int} S$ has finitely many components. Then for all h continuous and bounded in $\mathbb{C}\setminus \operatorname{int} S$, which are harmonic in $\mathbb{C}\setminus \operatorname{clos}\operatorname{int} S$, we have (with $t = t(Q,S)$ and $t' = t(Q,S')$)*

$$\int_{S'\setminus S} h\Delta Q\, dA = (t'-t)\, h(\infty).$$

Proof. By Mergelyan-type approximation, we can find a sequence of bounded C^∞-smooth functions h_n that are harmonic in $\mathbb{C}\setminus\operatorname{clos}\operatorname{int}S$, such that $h_n\to h$ uniformly on $\mathbb{C}\setminus\operatorname{int}S$ as $n\to+\infty$. The assertion now follows from Proposition 5.16. □

5.6. *Differential form of Richardson's formula*

We keep the setting of the previous subsections and introduce the concept of a containment chain.

DEFINITION 5.18. A *containment chain* of local Q-droplets is a (continuously indexed) family of Q-droplets $\{S_t\}_t$, where the index t ranges over a non-empty interval $I\subset\mathbb{R}_+$, with left endpoint 0, such that $t=t(Q,S_t)$ and

$$t_1\leqslant t_2 \iff S_{t_1}\subset S_{t_2}.$$

Let $\{S_t\}_{t\in I}$ be a containment chain of local Q-droplets and let t_* denote the right endpoint of I. Let I^- be the interval obtained from I by removal of t_* (if $t_*\notin I$, we put $I^-:=I$).

LEMMA 5.19. *The map $t\mapsto S_t$, $t\in I$, is continuous in the Hausdorff metric except for a countable subset of the interval I.*

Proof. For $t_0\in I^-$, we form the compact sets

$$S_{t_0}^-=\operatorname{clos}\bigcup_{t:t<t_0}S_t,\quad S_{t_0}^+=\bigcap_{t:t>t_0}S_t.$$

Then $S_t^-\subset S_t\subset S_t^+$ holds for each for $t\in I^-$. Note that $S_{t_0}^-$ is well defined also when $t_0=t_*$. For $t\in I^-$, we put

$$\delta^-(t):=\max_{z\in S_t}\operatorname{dist}_{\mathbb{C}}(z,S_t^-),\quad \delta^+(t):=\max_{z\in S_t^+}\operatorname{dist}_{\mathbb{C}}(z,S_t).$$

Next, for positive ϵ, we consider the sets

$$D_\epsilon^-:=\{t\in I^-:\,\delta^-(t)>\epsilon\}\quad\text{and}\quad D_\epsilon^+:=\{t\in I^-:\,\delta^+(t)>\epsilon\}.$$

We argue that for each positive ϵ, the sets D_ϵ^- and D_ϵ^+ are countable and that the only possible accumulation point is t_* (and if t_* is an accumulation point, then the set $S_{t_*}^-$ must be unbounded). Indeed, if, for some t' with $0<t'<t_*$, the set $D_\epsilon^-\cap]0,t']$ has $N=N(t',\epsilon)$ elements, then the local Q-droplet $S_{t'}$ contains N points which are ϵ-separated (the distance between any two different points is at least ϵ). We obtain an effective bound on N in terms of the diameter of the compact set $S_{t'}$. The analogous argument applies to D_ϵ^+ in place of D_ϵ^-. □

We let $\omega_\infty^{(t)}$ denote harmonic measure for the open set $\mathbb{C}\setminus S_t$ with respect to the point at infinity. This is a probability measure whose support is contained in $\partial\,\mathrm{phull}(S_t)\subset\partial S_t$ (the effect on a test function is that we get the value at infinity of the harmonic extension).

PROPOSITION 5.20. *Suppose the map $t\mapsto S_t$, $t\in I$, is right continuous at $t_0\in I^-$. Suppose moreover that $\operatorname{int}S_{t_0}$ has finitely many components and that $S_{t_0}=\operatorname{clos}\operatorname{int}S_{t_0}$. Then, for all $g\in C(\mathbb{C})$, we have*

$$\lim_{t\to t_0^+}\frac{1}{t-t_0}\int_{S_t\setminus S_{t_0}}g\,\Delta Q\,dA=\int_{\partial S_{t_0}}g\,d\omega_\infty^{(t_0)},$$

that is, we have the weak-star convergence of measures

$$\lim_{t\to t_0^+} \frac{1}{t-t_0} 1_{S_t\setminus S_{t_0}} \Delta Q\, dA = d\omega_\infty^{(t_0)}.$$

Proof. Let h denote the function which coincides with g on S_{t_0} and extends harmonically (and boundedly) to $\mathbb{C}\setminus S_t$, so that, in particular,

$$h(\infty) = \int_{\partial S_{t_0}} g d\omega_\infty^{(t_0)}.$$

Then h is continuous and bounded in $\mathbb{C}$ (see, for example [10] for a discussion of the Dirichlet problem). Since $S_{t_0} = \operatorname{clos}\operatorname{int} S_{t_0}$, Corollary 5.17 applied to h gives

$$\frac{1}{t-t_0}\int_{S_t\setminus S_{t_0}} g\Delta Q\, dA = h(\infty) + \frac{1}{t-t_0}\int_{S_t\setminus S_{t_0}} (g-h)\Delta Q dA.$$

It remains to show that the last term on the right-hand side tends to zero as $t\to t_0$. This follows from the fact that $h(z)-g(z)\to 0$ as $z\to\partial S_{t_0}$ and that $S_t \searrow S_{t_0}$ by the right continuity assumption. □

REMARK 5.21. Proposition 5.20 states that (under regularity assumptions) the infinitesimal growth of the local Q-droplets is in the exterior direction only. If the containment chain of local Q-droplets were to grow in the direction of the internal holes, the containment chain could not possibly be a domination chain (cf. Corollary 5.9).

5.7. *Richardson's inequality*

We now show that under modest regularity conditions, containment chains of local Q-droplets are in fact domination chains.

THEOREM 5.22. *Suppose S and S' are two local Q-droplets, with $S\subset S'$. Then the following are equivalent*:

(i) $S \prec S'$.
(ii) *For all real-valued functions $h\in W^{2,1}(\mathbb{C})$ (local Sobolev class) that are subharmonic in $\mathbb{C}\setminus S$, harmonic in $\mathbb{C}\setminus S'$, and bounded near infinity, we have (with $t=t(Q,S)$ and $t'=t(Q,S')$)*

$$(t'-t)h(\infty) \leqslant \int_{S'\setminus S} h\Delta Q\, dA.$$

Proof. We first show that (i) $\Longrightarrow$ (ii). We note that the inequality is an equality when h is constant (see, for example, Proposition 5.16). This allows us to restrict our attention to h with $h(\infty)=0$. As in the proof of Richardson's formula (Proposition 5.16), we find that

$$\int_{S'\setminus S} h\Delta Q\, dA = \int_{\mathbb{C}} [\hat{Q}_{S'} - \hat{Q}_S]\Delta h\, dA = \int_{S'\setminus S} [\hat{Q}_{S'} - \hat{Q}_S]\Delta h\, dA \geqslant 0,$$

since $\hat{Q}_S \leqslant Q = \hat{Q}_{S'}$, by Proposition 5.4 and Corollary 4.13.

We turn to the implication (ii) $\Longrightarrow$ (i). We take $h(\infty)=0$ and get (as above) from (ii) that

$$0 \leqslant \int_{S'\setminus S} h\Delta Q\, dA = \int_{S'\setminus S} [\hat{Q}_{S'} - \hat{Q}_S]\Delta h\, dA. \tag{5.3}$$

The question now is what kind of functions Δh are possible here. We have automatically $\Delta h \in L^1(S')$ while $\Delta h \geqslant 0$ almost everywhere on $S' \setminus S$. We also need to impose that

$$\int_{S'} \Delta h \, dA = 0,$$

as a consequence of the behaviour of h near infinity. In fact, any real-valued function $g \in L^q(S')$ for some q with $1 < q < +\infty$ with $g \geqslant 0$ almost everywhere on $S' \setminus S$ and

$$\int_{S'} g \, dA = 0, \tag{5.4}$$

is of the form $g = \Delta h$ for an h as in (ii) with $h(\infty) = 0$. It follows from (5.3) that

$$0 \leqslant \int_{S' \setminus S} [\hat{Q}_{S'} - \hat{Q}_S] g \, dA. \tag{5.5}$$

As it is easy to fulfil (5.4) by placing an L^q-integrable negative mass on S to compensate for the positive mass on $S' \setminus S$, on $S' \setminus S$ the function g is basically any positive L^q function on $S' \setminus S$. This is only possible if $\hat{Q}_S \leqslant \hat{Q}_{S'}$ on $S' \setminus S$, and as $\hat{Q}_{S'} = Q$ on S', we obtain $\hat{Q}_S \leqslant Q$ on $S' \setminus S$. Since $\hat{Q}_S \leqslant Q$ holds automatically on S, we see that $\hat{Q}_S \leqslant Q$ on S'. The conclusion $S \prec S'$ now follows from Proposition 5.4. The proof is complete. □

The proof of Theorem 5.22 has the following consequence.

Corollary 5.23. *Suppose S' is a local Q-droplet, and that $S \subset S'$, where S is compact and lacks Q-shallow points. If condition* (ii) *of Theorem* 5.22 *is fulfilled, then S is a local Q-droplet, and $S \prec S'$.*

Proof. As in the proof of Theorem 5.22, we get from condition (ii) of that theorem

$$\begin{aligned} 0 \leqslant \int_{S' \setminus S} h \Delta Q \, dA &= \int_{\mathbb{C}} [U^{Q,S} - U^{Q,S'}] \Delta h \, dA \\ &= \int_S [U^{Q,S} - U^{Q,S'}] \Delta h \, dA + \int_{S' \setminus S} [U^{Q,S} - U^{Q,S'}] \Delta h \, dA, \end{aligned}$$

provided that $h \in W^{2,1}(\mathbb{C})$ (local Sobolev class) is subharmonic in $\mathbb{C} \setminus S$ and harmonic in $\mathbb{C} \setminus S'$, and bounded near infinity, with $h(\infty) = 0$. As in the proof of Theorem 5.22, we choose h as (minus) the logarithmic potential of g, where $g \in L^q(S')$ has $g \geqslant 0$ on $S' \setminus S$ and

$$\int_{S'} g \, dA = 0$$

so that

$$0 \leqslant \int_S [U^{Q,S} - U^{Q,S'}] g \, dA + \int_{S' \setminus S} [U^{Q,S} - U^{Q,S'}] g \, dA. \tag{5.6}$$

If we choose g such that $g = 0$ on $S' \setminus S$, we have equality (since then the inequality applies to $-g$ as well):

$$\int_S [U^{Q,S} - U^{Q,S'}] g \, dA = 0.$$

As g is now arbitrary except that its integral over S vanishes, we conclude that $U^{Q,S} - U^{Q,S'}$ is constant on S. Call the constant c: $U^{Q,S} = c + U^{Q,S'}$ on S. Since S' is a local Q-droplet, the function $U^{Q,S'} + Q$ is constant on S' (cf. Corollary 4.17) and consequently, $U^{Q,S}$ is constant on S (after all, $S \subset S'$). We conclude that S is a local Q-droplet. That $S \prec S'$ now follows from Theorem 5.22. □

THEOREM 5.24. *Let $\{S_t\}_{t\in I}$ be a containment chain of local Q-droplets. If, for almost every $t \in I$, the set $\operatorname{int} S_t$ has finitely many components and $S_t = \operatorname{clos}\operatorname{int} S_t$, then $\{S_t\}_{t\in I}$ is a domination chain.*

Proof. We consider the $W^{2,p}$-smooth function

$$V(\xi,\eta;t) := U^{Q,S_t}(\xi) - U^{Q,S_t}(\eta) = \int_{S_t} \log\left|\frac{z-\xi}{z-\eta}\right|^2 \Delta Q(z)\, dA(z), \quad t \in I,$$

and note that

$$V(\xi,\eta;t) = \hat{Q}_{S_t}(\eta) - \hat{Q}_{S_t}(\xi), \quad t \in I. \tag{5.7}$$

Let μ and ν be two compactly supported Borel probability measures which are absolutely continuous with densities in L^q for some q, $1 < q < +\infty$. We need the expression

$$V^{\mu,\nu}(t) := \int_{C^2} V(\xi,\eta;t)\, d\mu(\xi)\, d\nu(\eta) = \int_{S_t} U^{\nu-\mu}\Delta Q\, dA, \quad t \in I,$$

where $U^{\nu-\mu} := U^\nu - U^\mu$, and U^μ and U^ν are the usual logarithmic potentials. The functions U^μ and U^ν are in $W^{2,q}$ and therefore continuous (and bounded). The function $U^{\nu-\mu}$ is harmonic off $\operatorname{supp}(\nu-\mu)$, and its value at infinity is $U^{\nu-\mu}(\infty) = 0$. We have

$$V^{\mu,\nu}(t') - V^{\mu,\nu}(t) = \int_{S_{t'}\setminus S_t} U^{\nu-\mu}\Delta Q\, dA \quad \text{for } t,t' \in I \text{ with } t < t', \tag{5.8}$$

which gives

$$\begin{aligned}|V^{\mu,\nu}(t') - V^{\mu,\nu}(t)| &\leqslant \|U^{\nu-\mu}\|_{L^\infty(\mathbb{C})} \int_{S_{t'}\setminus S_t} \Delta Q\, dA \\ &= (t'-t)\|U^{\nu-\mu}\|_{L^\infty(\mathbb{C})} \quad \text{for } t,t' \in I \text{ with } t < t',\end{aligned}$$

since $\Delta Q \geqslant 0$ almost everywhere on a local Q-droplet. It follows that the function $V^{\mu,\nu}$ is Lipschitz continuous, and therefore differentiable almost everywhere. In view of (5.8), its right derivative is

$$[V^{\mu,\nu}]'(t^+) = \lim_{t'\to t^+} \frac{1}{t'-t}\int_{S_{t'}\setminus S_t} U^{\nu-\mu}\Delta Q\, dA = \int_{\partial S_t^\infty} U^{\nu-\mu}\, d\omega_\infty^{(t)},$$

by Proposition 5.20, with the possible exception of a countable set of values of t. If now $\operatorname{supp}\nu \subset S_t$, the function $U^{\nu-\mu}$ becomes subharmonic (and bounded) in $\mathbb{C}\setminus S_t$, so by the maximum principle

$$0 = U^{\nu-\mu}(\infty) \leqslant \int_{\partial S_t^\infty} U^{\nu-\mu}\, d\omega_\infty^{(t)}.$$

We conclude that $[V^{\mu,\nu}]'(t) \geqslant 0$ for almost everywhere t with $\operatorname{supp}\nu \subset S_t$. Put

$$t_\nu := \inf\{t \in I : \operatorname{supp}\nu \subset S_t\},$$

and note that for $t \in I$ with $t > t_\nu$ we have $[V^{\mu,\nu}]'(t) \geqslant 0$ almost everywhere, and hence $V^{\mu,\nu}$ is increasing on that sub-interval:

$$V^{\mu,\nu}(t) \leqslant V^{\mu,\nu}(t') \quad \text{for } t,t' \in I \text{ with } t_\nu < t < t'. \tag{5.9}$$

Next, we let the probability measures μ and ν get more and more concentrated, so that $\operatorname{supp}\mu \to \{\xi\}$ and $\operatorname{supp}\nu \to \{\eta\}$. The inequality (5.9) survives the limit process and we obtain that

$$V(\xi,\eta;t) \leqslant V(\xi,\eta;t') \quad \text{for } t,t' \in I \text{ with } t_\xi < t < t', \tag{5.10}$$

where

$$t_\xi := \inf\{t \in I : \xi \in S_t\}.$$

The short argument which justifies this involves choosing the support of ν cleverly, and this is made possible by the fact that a local Q-droplet lacks Q-shallow points. If we use (5.7), we see that (5.10) expresses that

$$\hat{Q}_{S_t}(\eta) - \hat{Q}_{S_t}(\xi) \leqslant \hat{Q}_{S_{t'}}(\eta) - \hat{Q}_{S_{t'}}(\xi) \quad \text{for } t, t' \in I \text{ with } t_\xi < t < t'. \tag{5.11}$$

Since for $t_\xi < t < t'$ we have $\xi \in S_t \subset S_{t'}$, we obtain that (cf. Proposition 3.5)

$$\hat{Q}_{S_t}(\xi) = \hat{Q}_{S_{t'}}(\xi) = Q(\xi),$$

so that (5.11) simplifies:

$$\hat{Q}_{S_t}(\eta) \leqslant \hat{Q}_{S_{t'}}(\eta) \quad \text{for } t, t' \in I \text{ with } t_\xi < t < t'.$$

By making clever choices of the point ξ we can obtain t_ξ to be as close to 0 as we need, and so

$$\hat{Q}_{S_t}(\eta) \leqslant \hat{Q}_{S_{t'}}(\eta) \quad \text{for } t, t' \in I \text{ with } t < t'.$$

For $\eta \in S_{t'}$ we have $\hat{Q}_{S_{t'}}(\eta) = Q(\eta)$, and we derive that for $t, t' \in I$ with $t < t'$, we have

$$\hat{Q}_{S_t}(\eta) \leqslant Q(\eta), \quad \eta \in S_{t'}.$$

By Proposition 5.4, we obtain $S_t \prec S_{t'}$ for all $t, t' \in I$ with $t < t'$, and $\{S_t\}_{t \in I}$ is a domination chain. □

6. *The Hele-Shaw equation*

6.1. *Smooth curve families (laminations)*

We need the following definition.

Definition 6.1. A family of simple curves Γ_t (where t runs over some interval) in $\mathbb{C}$ is a C^∞-smooth *lamination* if

(i) $\Gamma_t \cap \Gamma_{t'} = \emptyset$ holds for $t \neq t'$, and
(ii) each curve has a local parametrization $z = \gamma_t(\theta)$ (θ runs over some interval), such that the function $\gamma(\theta, t) := \gamma_t(\theta)$ is a local C^∞-diffeomorphism.

We will alternatively use the term *C^∞-smooth curve family* as synonymous to C^∞-smooth lamination. We mention that it is of course also possible to define laminations with a lower degree of smoothness than C^∞. The normal velocity $v_n = v_n(z)$, $z \in \Gamma_t$, may be defined as follows:

$$v_n := \langle \partial_t \gamma, n \rangle = \frac{1}{|\partial_\theta \gamma|} \operatorname{Im}[\partial_t \gamma \partial_\theta \bar{\gamma}],$$

where the inner product is that of $\mathbb{C} \cong \mathbb{R}^2$ and n is a unit normal to Γ_t. It is easy to see that the definition does not depend on the choice of parametrization γ. Indeed, if we write

$$\tilde{\gamma}_t(\vartheta) = \tilde{\gamma}(\vartheta, t) := \gamma(\theta(\vartheta, t), t),$$

where $\vartheta \mapsto \theta(\vartheta, t)$ is a local diffeomorphism, then

$$\partial_t \tilde{\gamma} = \partial_\theta \gamma \, \partial_t \theta + \partial_t \gamma, \quad \partial_\vartheta \tilde{\gamma} = \partial_\theta \gamma \, \partial_\vartheta \theta,$$

so that

$$\frac{1}{|\partial_\vartheta\tilde\gamma|}\operatorname{Im}[\partial_t\tilde\gamma\partial_\vartheta\bar{\tilde\gamma}] = \frac{1}{|\partial_\theta\gamma\partial_\vartheta\theta|}\operatorname{Im}[|\partial_\theta\gamma|^2\partial_t\theta\partial_\vartheta\theta + \partial_t\gamma\partial_\theta\bar\gamma\partial_\vartheta\theta]$$
$$= \frac{\partial_\vartheta\theta}{|\partial_\vartheta\theta|}\frac{1}{|\partial_\theta\gamma|}\operatorname{Im}[\partial_t\gamma\partial_\theta\bar\gamma] = \pm\frac{1}{|\partial_\theta\gamma|}\operatorname{Im}[\partial_t\gamma\partial_\theta\bar\gamma],$$

where there is a sign change if the coordinate change reverses the direction of the unit normal vector.

Lemma 6.2. *Let Γ_t be a C^∞-smooth lamination of Jordan curves, such that the domain D_t interior to Γ_t increases with t. Then, for continuous $f : \mathbb{C} \to \mathbb{C}$, we have*

$$\frac{d}{dt}\int_{D_t} f\,dA = 2\int_{\Gamma_t} f v_n\,ds.$$

Proof. We identify the area form with the area measure according to, for example, $dz\wedge d\bar z = 2\pi\mathrm{i}\,dA(z)$. We may assume that for t and t_0 close to one another with $t_0 < t$, $D_t \setminus D_{t_0}$ is parametrized by $\gamma_\tau(\theta) = \gamma(\theta,\tau)$ where $0 \leqslant \theta \leqslant 1$ and $t_0 \leqslant \tau < t$, with periodicity boundary conditions in θ: $\gamma(0,\tau) = \gamma(1,\tau)$. We let $R(t_0,t)$ denote the rectangle $[0,1]\times[t_0,t]$, so that

$$\int_{D_t\setminus D_{t_0}} f\,dA = \frac{1}{2\pi\mathrm{i}}\int_{R(t_0,t)} f(\gamma(\theta,\tau))\,d\gamma\wedge d\bar\gamma.$$

We calculate:

$$d\gamma\wedge d\bar\gamma = [\partial_\theta\gamma\partial_t\bar\gamma - \partial_\theta\bar\gamma\partial_t\gamma]\,d\theta\wedge dt = 2\mathrm{i}\operatorname{Im}[\partial_\theta\gamma\partial_t\bar\gamma]\,d\theta\wedge dt = 2\mathrm{i}|\partial_\theta\gamma|v_n d\theta\,dt,$$

where we have identified a form with the corresponding measure. We identify $|\partial_\theta\gamma|d\theta$ as arc length along Γ_t, so that $|\partial_\theta\gamma|d\theta = 2\pi ds(\theta)$, and therefore,

$$\frac{1}{2\pi\mathrm{i}}\,d\gamma\wedge d\bar\gamma = 2v_n ds(\theta)\,dt.$$

The assertion is now immediate. □

6.2. *The Hele-Shaw flow equation*

We assume that we have a C^∞-smooth lamination of Jordan curves Γ_t, and let D_t denote the interior domain while Ω_t is the exterior (unbounded) domain. We also write $K_t := \operatorname{clos} D_t = \mathbb{C}\setminus\Omega_t$, so that K_t is compact. The classical *Hele-Shaw equation* relates the normal velocity v_n to the normal derivative of the Green function (for the Laplacian) of the exterior domain Ω_t when one of the two coordinates is the point at infinity (the factor $\frac14$ comes from our choice of normalizations):

$$v_n = \tfrac14\partial_n G_t \quad \text{on } \Gamma_t \text{ where } G_t = G(\cdot,\infty;\Omega_t). \tag{6.1}$$

The Green function G_t is always positive in Ω_t and vanishes along the boundary Γ_t, and n is taken in the exterior direction, so that $\partial_n G_t > 0$ on Γ_t. Actually, $\partial_n G_t$ is the Poisson kernel of Ω for the point at infinity, so that $\frac12\partial_n G_t$ times normalized arc length measure has the interpretation of $d\omega_\infty^{(t)}$, harmonic measure at infinity for the domain Ω_t. There is also a weighted analogue of (6.1): the *weighted Hele-Shaw equation* is

$$\rho\,v_n = \tfrac14\partial_n G_t \quad \text{on } \Gamma_t. \tag{6.2}$$

The function ρ is the weight and is assumed to be C^∞-smooth with $\rho > 0$ point-wise. It is possible to interpret the introduction of the weight as a change of the geometry (cf. [**17**, **15**, **16**]). In what follows, we will use $\rho = \Delta Q$.

Definition 6.3. We say that an increasing family of compact sets $\{K_t\}_t$ (where t ranges over some interval) is a *generalized* solution of the weighted Hele-Shaw equation with weight $\rho = \Delta Q$ if

(i) $\Delta Q \geqslant 0$ on $\cup_t K_t$, if
(ii) for each $t \in I$, K_t lacks Q-shallow points, and if,
(iii) for all $f \in C(\mathbb{C})$, the function
$$t \mapsto \int_{K_t} f \Delta Q \, dA$$
is absolutely continuous and for almost every t we have ($\Gamma_t = \partial\Omega_t^\infty$ where $\Omega_t^\infty := \mathbb{C} \setminus \mathrm{phull}(K_t)$ is the unbounded component of the complement $\mathbb{C} \setminus K_t$ and $\omega_\infty^{(t)}$ is harmonic measure at infinity for Ω_t^∞)
$$\frac{d}{dt} \int_{K_t} f \Delta Q \, dA = \int_{\Gamma_t} f \, d\omega_\infty^{(t)}.$$

Note that no smoothness requirement is imposed on the compact sets K_t as in the standard formulation of the weighted Hele-Shaw equation (6.2). The way things are set up, strong solutions of the weighted Hele-Shaw equation (that is, solutions of (6.2)) are automatically generalized solutions. In short, the equation asks that the compact sets K_t grow according to the law
$$\frac{d}{dt} [1_{K_t} \Delta Q \, dA] = d\omega_\infty^{(t)}.$$

Proposition 6.4. *Let $\{K_t\}_t$ be an increasing family of compact sets, where t ranges over an open interval I, and suppose $\Delta Q \geqslant 0$ on $\bigcup_{t \in I} K_t$, and that K_t lacks Q-shallow points, for each $t \in I$. Then $\{K_t\}_{t \in I}$ is a generalized solution of the weighted Hele-Shaw equation with weight ΔQ if and only if, for all $t, t' \in I$ with $t < t'$, and for all real-valued $f \in C(\mathbb{C})$,*
$$\int_{K_{t'} \setminus K_t} f \Delta Q \, dA = \int_t^{t'} \int_{\Gamma_\tau} f \, d\omega_\infty^{(\tau)} \, d\tau.$$

Proof. This is just an application of Calculus. □

So, the weighted Hele-Shaw equation corresponds to the disintegration of measures
$$1_{K_{t'} \setminus K_t} \Delta Q \, dA = \int_t^{t'} d\omega_\infty^{(\tau)} \, d\tau.$$
It follows from the standard properties of the harmonic measure that if $f \in C(\mathbb{C})$ is real-valued, bounded, and subharmonic in $\mathbb{C} \setminus K_t$ while it is harmonic near infinity, then
$$\int_{K_{t'} \setminus K_t} f \Delta Q \, dA = \int_t^{t'} \int_{\Gamma_\tau} f \, d\omega_\infty^{(\tau)} \, d\tau \geqslant \int_t^{t'} f(\infty) \, d\tau = (t' - t) f(\infty). \tag{6.3}$$
This strongly resembles Richardson's inequality for local Q-droplets (Theorem 5.22). The comparison with Theorem 5.22 suggests the concept of a weak solution to the Hele-Shaw equation.

Definition 6.5. We say that an increasing family of compact sets $\{K_t\}_t$ (where t ranges over some interval) is a *weak* solution of the weighted Hele-Shaw equation with weight ΔQ if

(i) $\Delta Q \geqslant 0$ on $\bigcup_t K_t$, if

(ii) for each $t \in I$, K_t lacks Q-shallow points, and if,
(iii) for all real-valued $f \in W^{2,1}(\mathbb{C})$ (local Sobolev class),

$$(t'-t)f(\infty) \leqslant \int_{K_{t'}\setminus K_t} f\Delta Q\, dA \quad \text{for } t,t' \in I \text{ with } t<t',$$

provided f is subharmonic in $\mathbb{C}\setminus K_t$, harmonic in $\mathbb{C}\setminus K_{t'}$, and bounded near infinity.

Proposition 6.6. *A generalized solution of the Hele-Shaw equation is a weak solution.*

Proof. It is known that it suffices to have condition (iii) of Definition 6.5 fulfilled for $f \in W^{2,q}$ for some q slightly bigger than 1. Such functions are continuous, so the assertion is immediate from (6.3). □

We note that the sets K_t need not be local Q-droplets, although that is one particular instance. The analogy with that case suggest the following.

Definition 6.7. An increasing family of compact sets $\{K_t\}_t$ is *correctly indexed* if

$$t = t(Q, K_t) = \int_{K_t} \Delta Q\, dA.$$

This is in agreement with (6.3) (or with Definition 6.5) for $f \equiv 1$ (since the inequality applies to $f \equiv -1$ as well the inequality is of course an equality).

It is known that the Hele-Shaw equation behaves like the heat equation, in that one direction of time t is stable and the other is unstable. Here, the evolution $t \mapsto K_t$ is unstable when t increases, and stable when t decreases.

Theorem 6.8. *Let $K_* \subset \mathbb{C}$ be compact, with $\Delta Q \geqslant 0$ almost everywhere on K_*. We assume that $t_* := t(Q, K_*) > 0$, and that K_* lacks Q-shallow points. Then there exists a correctly indexed weak solution $t \mapsto K_t$ of the weighted Hele-Shaw equation with weight ΔQ on the interval $0 < t \leqslant t_*$, such that $K_{t_*} = K_*$. The solution is unique.*

Proof. We consider the function $\tilde{Q}_* := -U^{Q,K_*}$, where U^{Q,K_*} is as in (4.2). It has

$$\Delta\tilde{Q}_* = 1_{K_*}\Delta Q, \quad \tilde{Q}_*(z) = t_* \log|z|^2 + \mathrm{O}(1) \text{ as } |z| \longrightarrow +\infty,$$

and we can define

$$K_t := S_t[\tilde{Q}_*, K_*], \quad 0 < t < t_*, \tag{6.4}$$

and $K_{t_*} := K_*$. The way things are set up, K_* becomes a $\tilde{Q}_*$-droplet (cf. Corollary 4.17). Moreover, in view of Lemma 5.1 (with $\tilde{Q}_*$ in place of Q), the sets K_t are local $\tilde{Q}_*$-droplets. We see from Theorem 5.22 that the sets K_t form a domination chain of local $\tilde{Q}_*$-droplets if and only if they form a weak solution of the Hele-Shaw equation.

It remains to establish that the weak solution $t \mapsto K_t$ unique. So, suppose $t \mapsto K_t$ is a weak solution, which need not be of the form (6.4). We claim that K_t is a $\tilde{Q}_*$-droplet for $0 < t < t_*$. We know that K_* is a local $\hat{Q}_*$-droplet, that K_t has no Q-shallow points, and that $K_t \subset K_*$. In addition, the weak solution condition entails

$$(t_* - t)f(\infty) \leqslant \int_{K_*\setminus K_t} f\Delta Q\, dA, \quad 0 < t < t_*,$$

provided $f \in W^{2,1}(\mathbb{C})$ (local Sobolev class) is real-valued, subharmonic in $\mathbb{C} \setminus K_t$, harmonic in $\mathbb{C} \setminus K_*$, and bounded near infinity. An application of Corollary 5.23 shows that K_t must also be a local $\tilde{Q}_*$-droplet, with $K_t \prec K_*$ with respect to the weight $\tilde{Q}_*$. The uniqueness part is now a consequence of Lemma 5.11. □

REMARK 6.9. (a) A key element of the proof of Theorem 6.8 is the identification of the weak solutions of the Hele-Shaw equation $t \mapsto K_t$ with domination chains with respect to the weight $\tilde{Q}_*$.

(b) Theorem 6.8 supplies existence and uniqueness in the backward time direction. It is not difficult to see that there is even *local* uniqueness in the backward time direction. However, in the forward time direction, there is generally neither existence nor uniqueness. An example of non-uniqueness can be based on, for example, the setting of Remark 5.7. We now discuss non-existence. In the context of Theorem 6.8, the difference $U^{Q,K_*} - U^{Q,K_t}$ is constant on K_t for $0 < t < t_*$ (see, for example, the proof of Corollary 5.23); let $c(t)$ be that constant. We consider the functions $H_t := Q - U^{Q,K_t}$ and $H_* := Q - U^{Q,K_*}$, which have $\Delta H_t = 0$ almost everywhere on K_t and $\Delta H_* = 0$ almost everywhere on K_*, respectively. We have $H_t - H_* = U^{Q,K_*} - U^{Q,K_t} = c(t)$ on K_t, we write as $H_t = c(t) + H_*$ on K_t. So H_t restricted to K_t is supposed to have an extension to K_*, the function $\tilde{H}_t := c(t) + H_*$, with $\Delta \tilde{H}_t = 0$ almost everywhere on K_*. This adds an additional smoothness requirement on H_t for $0 < t < t_*$, which suggests that K_t cannot be an arbitrary compact subset of $\mathbb{C}$ with $\Delta Q \geqslant 0$ almost everywhere on K_t which lacks Q-shallow points. But K_t is uniquely given for $0 < t < t_*$ (the backward direction) for arbitrary compacts K_* lacking Q-shallow points. So with very irregular K_* we should be able to arrange that we have non-existence in the forward time direction. Another reason for non-existence in the forward direction is the existence of maximal local Q-droplets, at least for some Q with $\Delta Q \equiv 1$.

A proof of the following statement can be based on Proposition 5.20. The only part that needs checking is the absolute continuity requirement, which we leave to the interested reader.

PROPOSITION 6.10. *Suppose $t \mapsto K_t$ is a weak solution to the Hele-Shaw equation and that for almost all t the sets $\operatorname{int} K_t$ have finitely many components. Then $t \mapsto K_t$ is a generalized solution.*

Acknowledgements. We thank Kurt Johansson for helpful comments in connection with the previous arXiv preprint [**14**], and Serguei Shimorin for help with the proofreading.

References

1. Y. AMEUR, H. HEDENMALM and N. MAKAROV, 'Berezin transform in polynomial Bergman spaces', *Comm. Pure Appl. Math.* 63 (2010) 1533–1584.

2. Y. AMEUR, H. HEDENMALM and N. MAKAROV, 'Fluctuations of eigenvalues of random normal matrices', *Duke Math. J.* 159 (2011) 31–81.

3. Y. AMEUR, H. HEDENMALM and N. MAKAROV, 'Random normal matrices and Ward identities', "Submitted. arXiv: 1109.5941".

4. R. J. BERMAN, 'Bergman kernels for weighted polynomials and weighted equilibrium measures of $\mathbb{C}^n$', *Indiana Univ. Math. J.* 58 (2009) 1921–1946.

5. L. A. CAFFARELLI and D. KINDERLEHRER, 'Potential methods in variational inequalities', *J. Anal. Math.* 37 (1980) 285–295.

6. J. L. DOOB, *Classical potential theory and its probabilistic counterpart*, Classics in Mathematics (Springer, Berlin, 2001), Reprint of the 1984 edition.

7. N. Dunford and J. T. Schwartz, *Linear operators. Part II. Spectral theory. Selfadjoint operators in Hilbert space*, Wiley classics library, A Wiley-Interscience Publication (Wiley, New York, 1988), With the assistance of William G. Bade and Robert G. Bartle. Reprint of the 1963 original.

8. P. Elbau and G. Felder, 'Density of eigenvalues of random normal matrices', *Comm. Math. Phys.* 259 (2005) 433–450.

9. A. Friedman, *Variational principles and free-boundary problems*, 2nd edn (Robert E. Krieger Publishing Co., Inc., Malabar, FL, 1988).

10. J. B. Garnett and D. E. Marshall, *Harmonic measure*, New Mathematical Monographs 2 (Cambridge University Press, Cambridge, 2005).

11. I. C. Gohberg and M. G. Krein, *Introduction to the theory of linear nonselfadjoint operators*, Translations of Mathematical Monographs 18 (American Mathematical Society, Providence, RI, 1969), translated from the Russian by A. Feinstein.

12. B. Gustafsson, 'Applications of variational inequalities to a moving boundary problem for Hele-Shaw flows', *SIAM J. Math. Anal.* 16 (1985) 279–300.

13. A. Haimi and H. Hedenmalm, 'The polyanalytic Ginibre ensembles', Preprint, 2011, arXiv:1106. 2975.

14. H. Hedenmalm and N. Makarov, 'Quantum Hele-Shaw flow', Preprint, 2004, arXiv: math/0411437.

15. H. Hedenmalm and A. Olofsson, 'Hele-Shaw flow on weakly hyperbolic surfaces', *Indiana Univ. Math. J.* 54 (2005) 1161–1180.

16. H. Hedenmalm and Y. Perdomo G., 'Mean value surfaces with prescribed curvature form', *J. Math. Pures Appl.* (9) 83 (2004) 1075–1107.

17. H. Hedenmalm and S. Shimorin, 'Hele-Shaw flow on hyperbolic surfaces', *J. Math. Pures Appl.* (9) 81 (2002) 187–222.

18. K. Johansson, 'On fluctuations of eigenvalues of random Hermitian matrices', *Duke Math. J.* 91 (1998) 151–204.

19. D. Kinderlehrer and G. Stampacchia, *An introduction to variational inequalities and their applications*, Pure and Applied Mathematics 88 (Academic Press, 1980).

20. M. L. Mehta, *Random matrices*, 3rd edn. Pure and Applied Mathematics (Amsterdam) 142 (Elsevier/Academic Press, Amsterdam, 2004).

21. E. B. Saff and V. Totik, *Logarithmic potentials with external fields*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences] 316 (Springer, Berlin, 1997), Appendix B by Thomas Bloom.

22. E. Sandier and S. Serfaty, *2D Coulomb gases and the renormalized energy*. arXiv:1201.3503.

23. P. B. Wiegmann, 'Aharonov–Bohm effect in the quantum Hall regime and Laplacian growth problems', *Statistical field theories (Como, 2001)*, NATO Science Series II Mathematical and Physical Chemistry 73 (Kluwer Academic Publishers, Dordrecht, 2002) 337–349.

Håkan Hedenmalm
Department of Mathematics
KTH Royal Institute of Technology
S - 100 44 Stockholm
Sweden

haakanh@math.kth.se

Nikolai Makarov
Department of Mathematics
California Institute of Technology
Pasadena, CA 91125
USA

makarov@caltech.edu