PLANAR ORTHOGONAL POLYNOMIALS AND BOUNDARY UNIVERSALITY IN THE RANDOM NORMAL MATRIX MODEL

HAAKAN HEDENMALM AND ARON WENNMAN

Contents

1. Introduction

1.1. **Orthogonal polynomials.** We consider polynomials in one complex variable of the form

$$P(z) = c_n z^n + c_{n-1} z^{n-1} + \ldots + c_0, \tag{1.1.1}$$

where $c_0, c_1, \ldots, c_n$ are complex numbers. If $c_n \neq 0$, we say that P has degree n, and call c_n is the *leading coefficient*. We denote the $(n+1)$-dimensional space of all polynomials of the form (1.1.1) by Pol_{n+1}. Given a positive Borel measure μ with infinite support on the complex plane $\mathbb{C}$, with finite moments

$$\int_{\mathbb{C}} |z|^{2k} d\mu(z) < \infty, \qquad 0 \le k \le N, \tag{1.1.2}$$

for some positive integer N, we define the *system* $\{P_n(z)\}_{n=0}^N$ *of normalized orthogonal polynomials (ONPs)* with respect to μ recursively by applying the Gram-Schmidt algorithm to the sequence $\{z^n\}_{n=0}^N$ of monomials. Equivalently, the orthogonal polynomial P_n is the unique element in Pol_{n+1} of unit norm in $L^2(\mathbb{C}, \mu)$ with positive leading coefficient $c_n > 0$, such that for all lower degree polynomials $q \in \mathrm{Pol}_n$ we have

$$\int_{\mathbb{C}} P_n(z)\overline{q(z)} d\mu(z) = 0.$$

Date: April 14, 2021.

H. H. acknowledges support from Vetenskapsrådet (VR) Grant No. 2016-04912. A. W. acknowledges support from VR Grant No. 2016-04912, Knut and Alice Wallenberg (KAW) Foundation Grant No. 017.0389 and ERC Advanced Grant No. 692616.

When the measure $\mu = \mu_m$ depends on a parameter m, the orthogonal polynomials will be denoted by $P_{m,n}$, where the first index is the parameter for the measure, and the second is the degree of the polynomial.

For additional definitions and notation we refer the reader to §1.9.

1.2. **Carleman-Szegő asymptotics.** The 1920s witnessed a rapid development in the understanding of orthogonal polynomials and related kernel functions. Among the pioneers were Gabor Szegő, Stefan Bergman and Torsten Carleman. One of the early results is that of Szegő [55] (see also [56]), who considered the orthogonal polynomials in $L^2(\Gamma, ds)$, where Γ a real-analytically smooth Jordan curve in the complex plane $\mathbb{C}$ supplied with normalized arc length measure $ds = (2\pi)^{-1}|dz|$. Let $\mathbb{C} \setminus \Gamma = \Omega \cup \Omega_e$ be the decomposition of the complement into disjoint connected components, where Ω is bounded and Ω_e is unbounded, and denote by ϕ the conformal mapping of the exterior domain Ω_e onto the exterior disk $\mathbb{D}_{\mathrm{e}} := \{z \in \mathbb{C} : |z| > 1\}$, which fixes the point at infinity with positive derivative. Szegő's theorem asserts that

$$P_n(z) = \sqrt{\phi'(z)}[\phi(z)]^n \left(1 + \mathrm{O}(\rho^n)\right), \qquad z \in \Omega_e, \tag{1.2.1}$$

where ρ is some number with $0 < \rho < 1$. Due to the real-analytically smooth boundary, the conformal mapping ϕ extends conformally past the boundary $\partial\Omega$. With the extended mapping still denoted by ϕ, the asymptotic formula (1.2.1) remains valid in a neighborhood of $\Omega_e \cup \Gamma$.

Slightly later, Carleman [13, 14] – inspired by the work of Szegő – considered instead the orthogonal polynomials in $L^2(\Omega, dA)$, where $dA = (2\pi i)^{-1} dz \wedge d\bar{z}$ denotes the normalized area element and Ω is a simply connected domain with real-analytic boundary curve Γ. He found an analogous asymptotic formula for the planar orthogonal polynomials, which holds in a neighborhood $\tilde{\Omega}_e$ of the closure of the exterior domain Ω_e and is expressed in terms of the conformal mapping ϕ:

$$P_n(z) = (n+1)^{\frac{1}{2}}\, \phi'(z)[\phi(z)]^n \left(1 + \mathrm{O}(\rho^n)\right), \qquad z \in \tilde{\Omega}_e,$$

for some ρ with $0 < \rho < 1$. In the 1960s, Suetin extended Carleman's result to domains whose boundary has a lower degree of smoothness, as well as to weighted cases (see the monograph [54]). We should also mention the more recent work of Dragnev and Miña-Diaz ([18], [19], and [43]) which strengthens Carleman's theorem on orthogonal polynomials, and gives information on the asymptotic distribution of the zeros. In the work [35], which expands on ideas developed here, we derive a complete asymptotic expansion for the orthogonal polynomials in a weighted Carleman setting. Earlier, only the first term of the expansion was known.

In the above asymptotic formulæ a Jacobian factor appears, it is $(\phi')^{\frac{1}{2}}$ in the case of Szegő's theorem and ϕ' in Carleman's case. By inspection, the orthogonal polynomials are asymptotically push-forwards of the monomials under the conformal mapping in the relevant L^2-space.

We wish to contrast the above-mentioned results with the more classical study of orthogonal polynomials on the real line $\mathbb{R}$. Here, the earliest work is associated with Legendre, Jacobi, Chebyshev, Hermite, Laguerre, and Gegenbauer, with further contributions by

Markov, Stieltjes, Szegő, Bernstein, and Akhiezer. The structure of orthogonal polynomials on the line is rather rigid with the appearance of a three-term recursion relation, which comes from the fact that multiplication by the independent variable is self-adjoint on the weighted L^2-space. Analogous rigidity applies to the orthogonal polynomials on the unit circle $\mathbb{T}$ as well. These facts are basic in many of the standard approaches to the asymptotics of orthogonal polynomials, see e.g. [51, 52]. Going beyond measures supported on the line or the circle, the rigidity is lost (except in some special cases, including arc length measure on ellipses [20]). For planar orthogonal polynomials, recursion formulæ are rare, even if we allow any finite number of terms [44].

1.3. **Exponentially varying weights.** For a C^2-smooth function $Q : \mathbb{C} \to \mathbb{R} \cup \{+\infty\}$ called the *potential*, subject to the growth bound

$$\liminf_{z\to\infty} \frac{Q(z)}{\log|z|} > 1 \tag{1.3.1}$$

and a real parameter $m > 0$, we consider the weighted area measures of the form

$$d\mu_{2mQ}(z) = e^{-2mQ(z)}\, dA(z), \qquad z \in \mathbb{C}$$

where we recall that dA denotes the normalized planar area element. The condition (1.3.1) guarantees that the measure $\mu = \mu_{2mQ}$ has finite moments (1.1.2), with upper range given by $N = N_m := \lceil (1+\epsilon_1)m \rceil - 2$ for some $\epsilon_1 > 0$. Here, $\lceil \cdot \rceil$ denotes the standard ceiling function. This allows us to consider the sequence $\{P_{m,n}\}_{0\le n\le N_m}$ of normalized orthogonal polynomials (ONPs) with respect to the measure $d\mu_{2mQ}$ where n denotes the degree (cf. §1.1). Under certain additional assumptions on the regularity of the weight Q, we will obtain an asymptotic expansion of $P_{m,n}$ valid as m and n tend to infinity with the ratio $\tau = \frac{n}{m}$ confined to an open interval around $\tau = 1$.

The motivation for studying this particular class of orthogonal polynomials comes from the theory of Random Normal Matrix (RNM) ensembles, a particular instance of two-dimensional Coulomb gas. If m is a positive integer, the connection is that the eigenvalue process associated to an $m \times m$ matrix from the RNM-ensemble with potential Q is determinantal with correlation kernel K_m given by

$$\mathrm{K}_m(z,w) = K_m(z,w)\, e^{-m(Q(z)+Q(w))} \quad \text{where} \quad K_m(z,w) = \sum_{j=0}^{m-1} P_{m,j}(z)\overline{P_{m,j}(w)},$$

see §5.1 below for details. Analogous families of exponentially varying weights confined to the real line appear in connection with the study of random Hermitian matrices. In the 1980s, successive progress was made towards understanding the asymptotics of weighted ONPs on the real line, with important contributions by Freud, Nevai, Lubinsky, Mhaskar, Saff, and Totik, to mention a few (see e.g. the monographs [42], [53], and [59]). A deeper understanding came through the efforts of Fokas, Its, Kitaev, and Deift and Zhou, whose work brought novel methods into play. Their approach analyzes the ONPs with respect to rather general potentials Q on the real line in terms of solutions to matrix Riemann-Hilbert problems, see, e.g., [16, 17, 23, 24].

In the work [39] of Its and Takhtajan a natural *soft Riemann-Hilbert problem*, or matrix $\bar\partial$-problem, is considered, whose solution would give us the orthogonal polynomial $P_{m,n}$ for the planar measure μ_{2mQ}. However, unlike the one-dimensional situation, it is not clear

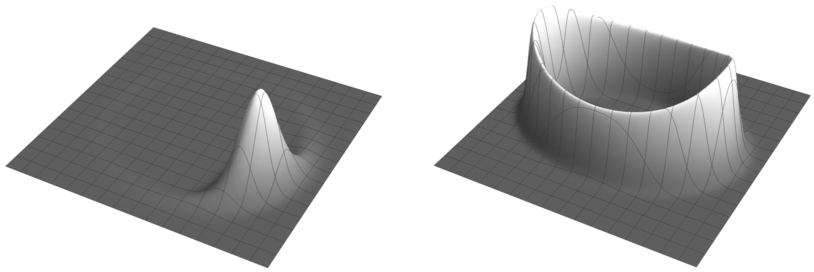

FIGURE 1.1. (left) The Berezin density $\mathrm{K}_m(z_0, z_0)^{-1}|\mathrm{K}_m(z, z_0)|^2$ with $Q(z) = \frac{1}{2}|z|^2$ for the boundary point $z_0 = 1$ and $m = 30$. (right) The orthogonal polynomial density $|P_{m,n}(z)|^2 e^{-2mQ(z)}$ for $n = 25, m = 20$ and $Q(z) = \frac{1}{2}|z|^2 - \mathrm{Re}(tz^2)$, where $t = 0.2$.

how to constructively solve these soft Riemann-Hilbert problems. The main obstruction appears to be the complex conjugation of the matrix, which results from the sesquilinearity of the inner product. While our analysis of the asymptotics of the ONPs is different, we try to connect with the Its-Takhtajan approach later on in §7.

1.4. **The boundary universality conjecture.** We return to the study of random normal matrix ensembles with the associated correlation kernel K_m. Macroscopically, the situation is well understood. For instance, in the limit as $m \to +\infty$ the eigenvalues condensate to a certain compact set $\mathcal{S}_1$, called the *droplet*, or alternatively *spectral droplet* (see §5.1 below). For simplicity, we assume below that Q is C^2-smooth with positive Laplacian $\varDelta Q > 0$ in a neighborhood of $\mathcal{S}_1$. An interesting question is how the process behaves at the microscopic level, which we express in rescaled coordinates as follows. For a point $z_0 \in \mathbb{C}$ with $\varDelta Q(z_0) > 0$ and a direction $\mathrm{n} \in \mathbb{T}$, we let

$$z_m(\xi) = z_0 + \mathrm{n}\frac{\xi}{\sqrt{2m\varDelta Q(z_0)}} \tag{1.4.1}$$

where $\varDelta_z = \partial_z \bar{\partial}_z$ denotes the (quarter) Laplacian, and consider

$$\rho_m(\xi) = \frac{1}{2m\varDelta Q(z_0)} \mathrm{K}_m(z_m(\xi), z_m(\xi)). \tag{1.4.2}$$

We introduce the notation $\mathcal{E}^\circ$ for the interior and $\overline{\mathcal{E}}$ for the closure of a subset $\mathcal{E} \subset \mathbb{C}$, while $\mathcal{E}^c = \mathbb{C} \setminus \mathcal{E}$ denotes the complement. Near any bulk point z_0, i.e., a point in the interior $\in \mathcal{S}_1^\circ$ of the droplet, there exists a full asymptotic expansion of the kernel K_m, see e.g. [3, 4]. In this case $\lim_m \rho_m(\xi) = 1$, uniformly on compact subsets. Away from the droplet, i.e. for $z_0 \in \mathcal{S}_1^c$ we instead have $\lim_m \rho_m(\xi) = 0$. It remains to analyze the boundary points $z_0 \in \partial\mathcal{S}_1$. An illustration of this blow-up procedure for a boundary point in the context of RNM-ensembles is supplied in Figure 1.2.

A natural simplifying assumption is that the boundary $\partial\mathcal{S}_1$ is smooth near z_0, in which case we let n be the outer normal to $\mathcal{S}_1$ at z_0. It is not known what is the limit of the density ρ_m, but the following universal behavior is expected.

Conjecture 1.4.1 (boundary universality)**.** Let $z_0 \in \partial\mathcal{S}_1$ and assume that $\partial\mathcal{S}_1$ is smooth in a neighborhood of z_0. Then the density ρ_m converges as $m \to \infty$ to the limit

$$\rho(\xi) = \operatorname{erf}(2 \operatorname{Re} \xi).$$

Here, we write erf for the complex *error function*

$$\operatorname{erf}(z) = \frac{1}{\sqrt{2\pi}} \int_z^\infty e^{-t^2/2} \, dt,$$

where the integral is taken along a suitable contour from z to the origin and then from the origin to ∞ along the positive real line. This conjecture, which has circulated in the community at least since 2008, may have appeared in print for the first time in Riser's thesis [46]. It has been verified in some specific cases, and partial results have appeared recently. In connection with this we want to mention the work by Ameur, Kang, and Makarov [5] who used a limiting form of the Ward identities to show that if $\rho(\xi)$ is a priori known to only depend on $\operatorname{Re}\xi$, then it must necessarily be of the form predicted by Conjecture 1.4.1. The full conjecture however remains open. In the setting of Kähler manifolds, a similar problem appears in the context of partial Bergman kernels defined by vanishing to high order along a divisor. Under the assumption of S^1-invariance around the divisor, Ross and Singer [47] obtain the error function asymptotics near the emergent interface around the divisor (see also the work of Zelditch and Zhou [62]). In recent work, Zelditch and Zhou [63] find that this is a universal edge phenomenon along interfaces in the context of partial Bergman kernels defined by a quantized Hamiltonian.

Let us briefly motivate why the interface asymptotics for the RNM-ensembles should be approached via the orthogonal polynomials. The standard methods to obtain the asymptotics of Bergman kernels are local in nature, both the peak section approach of Tian (see [58]) as well as the microlocal approach of Boutet de Monvel and Sjöstrand, as explained by Berman, Berndtsson, and Sjöstrand [9] (see also [28]). The same applies to older work of Hörmander [37] and Fefferman [22]. One reason to expect the boundary universality conjecture to be difficult is the apparent nonlocality of the correlation kernel. To illustrate this, we consider the Berezin density of [2], associated with secondary quantization and complementary to the Palm measure, cf. [12], given by

$$B_m^{\langle z_0 \rangle}(z) = K_m(z_0, z_0)^{-1} |K_m(z, z_0)|^2 e^{-2mQ(z)}.$$

We find numerically that for boundary points $z_0 \in \partial\mathcal{S}_1$, this probability density develops a noticeable ridge with slow decay along the whole boundary of the spectral droplet (see Figure 1.1 (left)). For this reason we focus our analysis on the orthogonal polynomials, which have an even more pronounced nonlocal behavior (see Figure 1.1 (right)). Indeed, for rather general potentials Q, the mean field approximation of the random normal matrix model [3, 4] supplies information regarding the individual orthogonal polynomials, and gives the weak-star convergence of measures

$$|P_{m,n}|^2 e^{-2mQ} \to \varpi(\cdot, \widehat{\mathbb{C}} \setminus \mathcal{S}_1, \infty),$$

as $n, m \to \infty$ with $n = m + \mathrm{O}(1)$. Here, the left-hand side is the density of a probability measure, and the right-hand side expression $\varpi(\cdot, \widehat{\mathbb{C}} \setminus \mathcal{S}_1, \infty)$ stands for harmonic measure of the domain $\widehat{\mathbb{C}} \setminus \mathcal{S}_1$ evaluated at the point at infinity, which has the interpretation of hitting probability of Brownian motion starting at ∞. We observe that harmonic measure

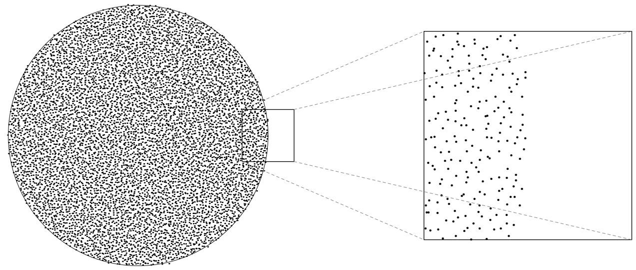

FIGURE 1.2. The RNM process associated to a quadratic potential (The Ginibre ensemble) with blow-up at a boundary point (courtesy of Nam-Gyu Kang).

is concentrated to the boundary, so that the above convergence may be interpreted as *boundary concentration*. Within the random normal matrix model, the addition of a new particle has the net effect of adding a term $|P_{m,n}|^2 e^{-2mQ}$ of highest degree. This means that the net effect of adding a particle is felt primarily along the droplet boundary. As a consequence, we obtain a growing chain of spectral droplets $\mathcal{S}_\tau$, so that the probability wave $|P_{m,n}|^2 e^{-2mQ}$ concentrates along $\partial\mathcal{S}_\tau$ as $m, n \to \infty$ with $n = m\tau$.

Finally, we mention that the orthogonal polynomial approach has proven to be successful in several special cases. For instance, when $Q(z) = \frac{1}{2}|z|^2 + a\,\mathrm{Re}(z^2)$ with $a > 0$, Lee and Riser [41] obtain the orthogonal polynomials in explicit form, and verify Conjecture 1.4.1 in this case. Along the same lines, in [10], Balogh, Bertola, Lee and McLaughlin consider potentials Q which are perturbations of the standard quadratic potential of the form

$$Q(z) = \frac{1}{2}|z|^2 - c\log|z - a|^2,$$

for some $a \in \mathbb{R}$, $c > 0$. For this Q, they obtain an asymptotic expansion of the orthogonal polynomials. For parameters a and c such that the droplet $\mathcal{S}_\tau$ does not divide the plane, the expansion is expressed in terms of the properly normalized conformal mapping of the complement $\mathcal{S}_\tau^c$ onto the exterior disk $\mathbb{D}_e$, denoted ϕ_τ. After some rewriting, their formula reads

$$P_{m,n}(z) = \left(\frac{m}{2\pi}\right)^{1/4} \sqrt{\phi_\tau'(z)}[\phi_\tau(z)]^n e^{m\mathcal{Q}_\tau(z)} \left(1 + \mathrm{O}(m^{-1})\right), \tag{1.4.3}$$

valid in a neighborhood of the closed exterior of the droplet for $\frac{n}{m} = \tau + \mathrm{O}(m^{-1})$, where $\mathcal{Q}_\tau$ is the bounded holomorphic function on $\mathcal{S}_\tau^c$, with real part equal to Q on the boundary $\partial\mathcal{S}_\tau$, extended analytically across the boundary. Using the asymptotics (1.4.3), they verify Conjecture 1.4.1 for the given collection of potentials. The analysis in [10] is based on Riemann-Hilbert problem methods, which are accessible due to a miraculous identity which transforms the Hermitian orthogonality over the plane into bilinear orthogonality relations along curves. The latter approach should be compared with the work of Bleher and Kuijlaars [11] in the context of a cubic potential.

At the physical level, it is understood that the asymptotic formula (1.4.3) should hold for the wider class of potentials of the form $Q(z) = \frac{1}{2}|z|^2 + H(z)$, where H is harmonic in a neighborhood of the droplet (the so-called Hele-Shaw potentials) [1, 57, 60]. To the best of our knowledge, no higher order correction terms have been identified earlier.

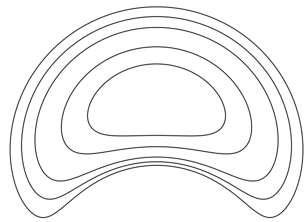

FIGURE 1.3. Laplacian growth of the compacts $\mathcal{S}_\tau$ for the potential $Q(z) = \frac{1}{2}|z|^2 - 2^{-\frac{1}{2}} \log|z+i|$ (boundary curves indicated).

1.5. **Summary of the results.** Here, we study the orthogonal polynomials with respect to a rather general exponentially varying weight e^{-2mQ} in the complex plane. To be more precise, we will work with potentials Q that are admissible in the sense of the definition below. Under C^2-smoothness and some growth assumption on Q, we consider for $\tau > 0$ the coincidence set

$$\mathcal{S}_\tau^\star := \{z \in \mathbb{C} : \hat{Q}_\tau(z) = Q(z)\},$$

where $\hat{Q}_\tau$ solves the obstacle problem

$$\hat{Q}_\tau(z) = \sup\{q(z) : q \in \mathrm{Subh}_\tau(\mathbb{C}),\ q \le Q \text{ on } \mathbb{C}\}.$$

Here, $\mathrm{Subh}_\tau(\mathbb{C})$ denotes the convex body of subharmonic functions in the plane which grow at most like $\tau \log|z|$ at infinity. The function $\hat{Q}_\tau$ is $C^{1,1}$-smooth and harmonic outside the set $\mathcal{S}_\tau^\star$. Moreover, if Q has sufficient growth, $\mathcal{S}_\tau^\star$ is compact. For a subset $\mathcal{E} \subset \mathbb{C}$ we write $1_\mathcal{E}$ for the corresponding indicator function. The support of the probability measure μ_τ given by

$$d\mu_\tau = 2\tau^{-1} 1_{\mathcal{S}_\tau^\star} \Delta Q \, dA$$

is denoted by $\mathcal{S}_\tau$ and called the *droplet*. Clearly, $\mathcal{S}_\tau \subset \mathcal{S}_\tau^\star$, and $\mathcal{S}_\tau^\star \setminus \mathcal{S}_\tau$ is a null-set for the measure $|\Delta Q| dA$. We note that μ_τ is the equilibrium measure for the weighted logarithmic energy problem in the external field $\tau^{-1}Q$. More details are supplied in §2.1 below.

Definition 1.5.1. The potential $Q : \mathbb{C} \to \mathbb{R}$ is said to be τ*-admissible* at $\tau = \tau_0$ (or, in short, τ_0-admissible) if $\mathcal{S}_{\tau_0} = \mathcal{S}_{\tau_0}^*$ and the following conditions are satisfied:

(i) Q is C^2-smooth in the entire complex plane,
(ii) Q is real-analytic and strictly subharmonic (i.e. $\Delta Q > 0$) in a neighborhood of $\mathcal{S}_{\tau_0}$,
(iii) Q is grows sufficiently fast at infinity:

$$\liminf_{|z|\to+\infty} \frac{Q(z)}{\log|z|} > \tau_0. \tag{1.5.1}$$

(iv) The boundary $\partial\mathcal{S}_{\tau_0}$ is a smooth Jordan curve.

Note that under these conditions, it follows that $\hat{Q}_{\tau_0}(z) < Q(z)$ on $\mathcal{S}_{\tau_0}^c$. As a consequence, we may exclude the immediate birth of additional components of $\mathcal{S}_\tau$ as τ increases from τ_0.

In the sequel, we consider $\tau_0 = 1$, and *assume that* Q *is* τ*-admissible* at $\tau = 1$. As observed in §1.3, the condition (1.5.1) with $\tau = 1$ guarantees that all polynomials of degree up to $\lceil(1+\epsilon_1)m\rceil - 2$ belong to the space $L^2(\mathbb{C}, e^{-2mQ}dA)$, for some fixed small $\epsilon_1 > 0$. As Q is assumed 1-admissible, the curve $\partial\mathcal{S}_1$ is smooth, simple and closed. By known properties of Laplacian growth, this assumption implies that the same holds for the boundaries $\partial\mathcal{S}_\tau$

for $\tau \in I_{\epsilon_0} := [1-\epsilon_0, 1+\epsilon_0]$ for some $\epsilon_0 > 0$ (cf. [32, 29]). By considering a smaller ϵ_0, we can make sure this property holds on the larger interval $I_{2\epsilon_0}$ as well, so that in particular Q grows at least like $(1+2\epsilon_0)\log|z| + \mathrm{O}(1)$ at infinity. Moreover, the assumption of 1-admissibility entails that the smooth curves $\partial\mathcal{S}_\tau$ are actually real-analytically smooth for $\tau \in I_{\epsilon_0}$. This follows from the work of Sakai [50] on boundaries with a one-sided Schwarz function, as observed in [32].

We now proceed to present our main theorem. To set things up, we denote for $\tau \in I_{\epsilon_0}$ by ϕ_τ the conformal mapping $\phi_\tau : \mathcal{S}_\tau^c \to \mathbb{D}_\mathrm{e}$, normalized by $\phi_\tau(\infty) = \infty$ and $\phi_\tau'(\infty) > 0$. As a consequence of 1-admissibility, ϕ_τ extends to a conformal mapping $\mathcal{K}_{\tau,0}^c \to \mathbb{D}_\mathrm{e}(0, \rho_{0,0})$, where $0 < \rho_{0,0} < 1$ and $\mathcal{K}_{\tau,0} \subset \mathcal{S}_\tau$ denotes an appropriate compact continuum. Here, we use the notation $\mathbb{D}_\mathrm{e}(0,r) := \{z \in \mathbb{C} : |z| > r\}$ for the exterior disk of radius r centered at the origin. We let $\mathcal{Q}_\tau$ denote the bounded holomorphic function on $\mathcal{S}_\tau^c$ whose real part equals the potential Q along the boundary $\partial\mathcal{S}_\tau$, and whose imaginary part vanishes at infinity. By possibly adjusting $\rho_{0,0}$, we may ensure that $\mathcal{Q}_\tau$ extends holomorphically to $\mathcal{K}_{\tau,0}^c$.

For a subset $\mathcal{E} \subset \mathbb{C}$, we use the notation $\mathrm{dist}_\mathbb{C}(z, \mathcal{E}) = \inf_{w\in\mathcal{E}} |z-w|$ for the Euclidean distance from z to the set $\mathcal{E}$.

Theorem 1.5.2. *Assume that Q is 1-admissible. Given a positive integer κ there exist bounded holomorphic functions $\mathcal{B}_{\tau,j}$ defined in a fixed neighborhood of $\mathcal{S}_\tau^c$ such that for any positive real A, the asymptotic formula*

$$P_{m,n}(z) = m^{\frac14}[\phi_\tau'(z)]^{\frac12}[\phi_\tau(z)]^n e^{m\mathcal{Q}_\tau(z)}\left(\sum_{j=0}^{\kappa} m^{-j}\mathcal{B}_{\tau,j}(z) + \mathrm{O}\left(m^{-\kappa-1}\right)\right),$$

holds, where the error term is uniform over all $z \in \mathbb{C}$ with

$$\mathrm{dist}_\mathbb{C}(z, \mathcal{S}_\tau^c) \le A(m^{-1}\log m)^{\frac12}$$

as $n = \tau m \to +\infty$ along the integers with $\tau \in I_{\epsilon_0}$.

In other words, the orthogonal polynomials $P_{m,n}$ enjoy an asymptotic expansion

$$P_{m,n}(z) \sim m^{\frac14}[\phi_\tau'(z)]^{\frac12}[\phi_\tau(z)]^n e^{m\mathcal{Q}_\tau(z)}\Big(\mathcal{B}_{\tau,0}(z) + \tfrac{1}{m}\mathcal{B}_{\tau,1}(z) + \dots\Big),$$

valid provided that $\mathrm{dist}_\mathbb{C}(z, \mathcal{S}_\tau^c) \le A(m^{-1}\log m)^{\frac12}$ as $n = \tau m \to +\infty$ and $\tau \in I_{\epsilon_0}$, for any given $A > 0$.

Remark 1.5.3. (a) We derive Theorem 1.5.2 from an L^2-version of the asymptotic expansion, given in Theorem 3.1.2 below. An advantage of the L^2-version is that it holds in a fixed ϵ-neighborhood of the exterior $\mathcal{S}_\tau^c$.
(b) It is curious to note that the expansion of $P_{m,n}$ contains the factor $(\phi_\tau')^{\frac12}$, rather than ϕ_τ' as one might expect from Carleman's theorem. The square root is more reminiscent of Szegő's theorem. We have no satisfactory explanation for this fact, other than appealing to heuristics based on the steepest descent method. Naturally, the expansion could be written with ϕ_τ' as a factor, by adjusting the terms $\mathcal{B}_{\tau,j}$ accordingly. However, the term $\mathcal{B}_{\tau,0}$ takes on the simplest possible form with the former choice, as shown in Theorem 1.5.4.

In the context of Theorem 1.5.2, we would like to know the coefficient functions $\mathcal{B}_{\tau,j}$. How to do find them is explained in the following theorem. For the formulation, we need the Szegő projection $\mathbf{P}_{H^2_{-,0}}$ of $L^2(\mathbb{T})$ onto the conjugate Hardy space $H^2_{-,0} = L^2(\mathbb{T}) \ominus H^2$

(cf. Subsection 2.5 below). In addition, we need the *effective weight* R_τ which takes into account the growth behavior of polynomials and a conformal change-of-variables. It is defined in a neighborhood of $\bar{\mathbb{D}}_e$ by

$$R_\tau = (Q - \check{Q}_\tau) \circ \phi_\tau^{-1}, \tag{1.5.2}$$

where we need to explain what is the function $\check{Q}_\tau$. The solution $\hat{Q}_\tau$ to the obstacle problem is a $C^{1,1}$-smooth function which equals Q on $\mathcal{S}_\tau$, while it is strictly smaller and harmonic in the exterior $\mathcal{S}_\tau^c$. As a consequence of the smoothness of Q and the boundary curve $\partial\mathcal{S}_\tau$, the restriction $\hat{Q}_\tau|_{\mathcal{S}_\tau^c}$ to the exterior extends harmonically across the boundary for each $\tau \in I_{\epsilon_0}$. We denote the extended function by $\check{Q}_\tau$.

Theorem 1.5.4. *In the asymptotic expansion of Theorem 1.5.2, we have that* $\mathcal{B}_{\tau,0} = \pi^{-\frac{1}{4}} e^{H_{Q,\tau}}$, *where* $H_{Q,\tau}$ *is bounded and holomorphic in* $\mathcal{S}_\tau^c$ *and satisfies* $\operatorname{Im} H_{Q,\tau}(\infty) = 0$, *as well as*

$$\operatorname{Re} H_{Q,\tau} = \frac{1}{4} \log \Delta Q, \quad \text{on} \quad \partial\mathcal{S}_\tau.$$

Moreover, if H_{R_τ} *denotes the bounded holomorphic function on* $\mathbb{D}_{\mathrm{e}}$ *with*

$$\operatorname{Re} H_{R_\tau} = \frac{1}{4} \log(4\Delta R_\tau) \quad \text{on } \mathbb{T},$$

and $\operatorname{Im} H_{R_\tau}(\infty) = 0$, *then for* $j = 1, 2, 3, \ldots$, *the coefficients* $\mathcal{B}_{\tau,j}$ *have the form*

$$\mathcal{B}_{\tau,j} = [\phi_\tau']^{\frac{1}{2}} \, B_{\tau,j} \circ \phi_\tau,$$

where the functions $B_{\tau,j}$ *are bounded and holomorphic in* $\mathbb{D}_{\mathrm{e}}$, *and given by*

$$B_{\tau,j} = c_{\tau,j} \, e^{H_{R_\tau}} - e^{H_{R_\tau}} \mathbf{P}_{H^2_{-,0}}[e^{\bar{H}_{R_\tau}} F_{\tau,j}]$$

for some real-analytic functions $F_{\tau,j}$ *on the circle* $\mathbb{T}$ *and constants* $c_{\tau,j} \in \mathbb{R}$. *The functions* $F_{\tau,j}$ *as well as the constants* $c_{\tau,j}$ *may be computed algorithmically in terms of the potential* R_τ *and the functions* $B_{\tau,0}, \ldots, B_{\tau,j-1}$, *where* $B_{\tau,0} = (4\pi)^{-\frac{1}{4}} e^{H_{R_\tau}}$.

Remark 1.5.5. (a) In the above theorem, all the functions $\mathcal{B}_{\tau,j}$, $B_{\tau,j}$ as well as $H_{Q,\tau}$ and H_{R_τ} extend holomorphically across their respective boundaries.
(b) The functions $H_{Q,\tau}$ and H_{R_τ} are related by

$$H_{R_\tau} \circ \phi_\tau = \frac{1}{2} \log(2\phi_\tau') + H_{Q,\tau}.$$

(c) We point out that Theorems 1.5.2 and 1.5.4 together imply that for large enough m, and for $\tau = \frac{n}{m} \in I_{\epsilon_0}$, all the zeros of the polynomial $P_{m,n}(z)$ lie inside $\mathcal{S}_\tau$, and stay away from the boundary curve $\partial\mathcal{S}_\tau$ by a distance of at least $A(m^{-1} \log m)^{\frac{1}{2}}$.

While Theorem 1.5.4 gives the asymptotic structure of the orthogonal polynomials, it remains to specify how to algorithmically obtain the real-analytic functions $F_{\tau,j}$ and the constants $c_{\tau,j}$, for $j = 1, 2, 3, \ldots$. For $k = 0, 1, 2, \ldots$, let $\mathbf{L}_k$ be the differential operator given by

$$\mathbf{L}_k[f] = \sum_{\nu=k}^{3k} \frac{(-1)^{\nu-k} 2^{-\nu}}{\nu!(\nu-k)![\partial_r^2 R_\tau(re^{i\theta})]^\nu} \partial_r^{2\nu} \Big(\Big[R_\tau - \frac{1}{2}(r-1)^2 \partial_r^2 R_\tau(e^{i\theta}) \Big]^{\nu-k} f(re^{i\theta}) \Big). \tag{1.5.3}$$

This is a differential operator of order $6k$, acting on a smooth function f defined in a neighborhood of the unit circle. We are specifically interested in the restriction $\mathbf{L}_k[f](re^{i\theta})\big|_{r=1}$, which expression only involves derivatives of order at most $2k$. The operator $\mathbf{L}_k$ results from the asymptotic analysis of definite integrals using Laplace's method, as in Proposition 2.6.1 below. Later on, in Lemma 4.1.1, we show the existence of differential operators $\mathbf{M}_k$ with the property that

$$\int_{\mathbb{T}} e^{il\theta}\big(\partial_r^2 R_\tau(re^{i\theta})\big)^{-\frac{1}{2}}\mathbf{L}_k[r^{1-l}f(re^{i\theta})]\Big|_{r=1} d\theta = \int_{\mathbb{T}} e^{il\theta}\mathbf{M}_k[f](e^{i\theta})]d\theta,$$

for $l = 1, 2, 3, \ldots$. We use these operators to rid the left-hand side of any unwanted dependence on the parameter l. In terms of the operators $\mathbf{L}_k$ and $\mathbf{M}_k$, we may now express $F_{\tau,j}$ and $c_{\tau,j}$ as follows:

$$F_{\tau,j}(\theta) = \sum_{k=1}^{j}\mathbf{M}_k[B_{\tau,j-k}](e^{i\theta}), \qquad j \ge 1, \tag{1.5.4}$$

and the real constants $c_{\tau,j}$ are given by $c_{\tau,0} = (4\pi)^{-1/4}$ while for $j = 1, 2, 3, \ldots$,

$$c_{\tau,j} = -\frac{1}{2}(4\pi)^{\frac{1}{4}}\sum_{(i,k,l)\in\mathfrak{n}_j}\int_{\mathbb{T}}\mathbf{M}_k\big[B_{\tau,i}\bar{B}_{\tau,l}\big](e^{i\theta})ds(e^{i\theta}) \tag{1.5.5}$$

where $\mathfrak{n}_j = \{(i,k,l) \in \mathbb{N}^3 : \ i,l < j, \ k \ge 0, \ i+k+l = j\}$ and $\mathbb{N} := \{0, 1, 2, \ldots\}$. The way this algorithm works is that we start with the known function $B_{\tau,0}$, which in its turn gives the function $F_{\tau,1}$ and the constant $c_{\tau,1}$ via (1.5.4) and (1.5.5), respectively. This then gives $B_{\tau,1}$ from the expression in Theorem 1.5.4. In the next round, we obtain $F_{\tau,2}$ and $c_{\tau,2}$ followed by $B_{\tau,2}$ in a similar fashion. An inductive procedure gives $F_{\tau,j}$, $c_{\tau,j}$, and $B_{\tau,j}$ for all $j \ge 2$ as well. Knowing $B_{\tau,j}$ then gives the coefficient function $\mathcal{B}_{\tau,j}$ as well, by Theorem 1.5.4.

As a direct consequence of Theorems 1.5.2 and 1.5.4, we resolve the boundary universality conjecture (Conjecture 1.4.1) for 1-admissible potentials. For the convenience of the reader, we recall some notation. For $z_0 \in \partial\mathcal{S}_1$ we denote by n the outward unit normal to $\partial\mathcal{S}_\tau$ at z_0, and write $z_m(\xi)$ for the rescaled variable around z_0 given by (1.4.1).

Corollary 1.5.6. *Assume that Q is 1-admissible, and denote by k_m the rescaled kernel*

$$\mathrm{k}_m(\xi,\eta) = \frac{1}{2m\Delta Q(z_0)}\mathrm{K}_m(z_m(\xi), z_m(\eta)).$$

Then, there exist unimodular continuous functions $c_m : \mathbb{C} \to \mathbb{T}$ such that we have the convergence

$$\lim_{m\to\infty} c_m(\xi)\bar{c}_m(\eta)\mathrm{k}_m(\xi,\eta) = \mathrm{k}(\xi,\eta),$$

locally uniformly on $\mathbb{C}^2$, where the limiting kernel is the Faddeeva plasma kernel

$$\mathrm{k}(\xi,\eta) = e^{\xi\bar{\eta}-\frac{1}{2}(|\xi|^2+|\eta|^2)}\ \mathrm{erf}(\xi+\bar{\eta}).$$

The terminology *Faddeeva plasma kernel* comes from the plasma dispersion function, which was tabulated by Faddeeva and Terent'ev in [21].

Remark 1.5.7. The above kernel convergence has an interpretation in terms of determinantal point processes in the plane. More precisely, the blow-up of the eigenvalue process for the RNM-ensemble around z_0 converges to the Faddeeva plasma point field, with correlation kernel $\mathrm{k}(\xi,\eta)$. The unimodular continuous functions c_m are irrelevant, as they do not affect determinantal point processes.

To complement the present exposition on planar orthogonal polynomials, we explain in [34] how the ideas developed here also apply to give a full asymptotic expansion of the Bergman kernel for exponentially varying weights when one of the variables is away from the corresponding droplet. In that setting, the droplet arises typically from the repulsive effect of patches where $\Delta Q < 0$. This result gives error function transition behavior along smooth loops of the droplet boundary.

In the follow-up work [36], we intend to explore further the implications of Theorem 3.1.2 and 1.5.4 for the theory of random normal matrices. In particular, we analyze the asymptotics of the free energy $\log \mathcal{Z}_{m,Q}$, where $\mathcal{Z}_{m,Q}$ denotes the partition function of the RNM-ensemble, and relate the analysis to the planar analogue of the classical Szegő limit theorem on Toeplitz determinants.

1.6. **Sketch of the main ideas.** The first step towards obtaining Theorem 1.5.2 is the construction of a family of approximately orthogonal quasipolynomials, defined outside a compact subset $\mathcal{K}_\tau$ of the interior of the droplet $\mathcal{S}_\tau$. This family of functions have the property that they are approximately orthogonal to the collection of lower degree polynomials, have the correct polynomial growth at infinity, but need not be well-defined globally (i.e. on $\mathcal{K}_\tau$). In a second step, these quasipolynomials may be corrected to true polynomials using Hörmander's $\bar\partial$-estimates. The actual construction depends on our key lemma (Lemma 3.4.1) which establishes the existence of what we call the *orthogonal foliation flow.*

We turn to the underlying ideas for the orthogonal foliation flow. Our approach will take a slightly different point of view than what is used later on. It has the advantage of being more intuitively direct. The approach begins with the following disintegration formula: let $\{\gamma_{m,n,t}\}_t$ denote a smoothly varying family of closed simple curves, which foliate a region $\Omega_{m,n}$ when t runs through an interval J_m. If $\nu(z)$ denotes the scalar normal velocity of the curve flow as it passes through the point z, then for a suitably integrable function F we have

$$\int_{\Omega_{m,n}} F(z)\, e^{-2mQ(z)}\, dA(z) = 2\int_{J_m}\int_{\gamma_{m,n,t}} F(z)\, e^{-2mQ(z)}\nu(z)\, ds(z)\, dt \tag{1.6.1}$$

We consider the weighted arc length measure $e^{-2mQ}\nu\, ds$ restricted to the curve $\gamma_{m,n,t}$, and the associated orthogonal polynomial $P_{m,n,t}$ of degree n. We would like to find a foliation $\{\gamma_{m,n,t}\}_t$ of the region $\Omega_{m,n}$ such that $P_{m,n,t} = c(t)P_{m,n,0}$, where $P_{m,n,0}$ is independent of the flow parameter t and $c(t)$ is an appropriate positive constant. As a consequence of (1.6.1), the polynomial $P_{m,n,0}$ is then orthogonal to Pol_n with respect to the measure $1_{\Omega_{m,n}} e^{-2mQ}\, dA$. Now, if the foliation covers a sufficiently large enough region $\Omega_{m,n}$, then the resulting normalized orthogonal polynomial ought to be close to $P_{m,n}$ itself. In other words, the two-dimensional orthogonality relations *foliate* into lower-dimensional orthogonality relations along a curve family $\{\gamma_{m,n,t}\}_t$.

The stationarity condition $P_{m,n,t} = c(t)P_{m,n,0}$ is quite demanding, and in fact we do not know that such a foliation exists, at least if we require it to foliate the entire plane. Instead, we obtain the foliation in an approximate sense, up to any given precision, so that $\Omega_{m,n}$ covers a band around $\partial\mathcal{S}_\tau$ of width $\asymp m^{-\frac{1}{2}}\log m$. We remark that the stationarity condition may be thought of as a Hele-Shaw flow condition (see [27], [32]) for the curves $\gamma_{m,n,t}$, with respect to the weight $|P_{m,n,0}|^2 e^{-2mQ}$. Hele-Shaw flows are notorious for singularity formation, after which the foliation flow cannot be continued. The requirement not to develop such singularities puts a strong requirement on the weight $|P_{m,n,0}|^2 e^{-2mQ}$. This is used in an approximate fashion in §6 to devise an algorithm to construct $P_{m,n,0}$ together with the foliation iteratively in a self-improving manner. For technical reasons, we work with the flow curves $\Gamma_{m,n,t} = \phi_\tau(\gamma_{m,n,t})$ after applying the conformal mapping ϕ_τ, and consider quasipolynomials rather than polynomials.

1.7. **An outline of the presentation.** In §2, we introduce some auxiliary material which will be needed later on. In particular, we discuss some aspects of weighted logarithmic potential theory and obstacle problems, and introduce the concept of weighted Laplacian growth. Moreover, we collect some results on Hörmander-type L^2-estimates for the $\bar\partial$-operator, and the asymptotic analysis of integrals based on Laplace's method.

In §3, we introduce the notion of quasipolynomials, and state our key lemma on the orthogonal foliation flow (Lemma 3.4.1). Using Hörmander-type $\bar\partial$-techniques we obtain the L^2-analogue of the main theorem (Theorem 3.1.2) from the key lemma. The main theorem (Theorem 1.5.2) then follows from Theorem 3.1.2 by a weighted Bernstein-Walsh lemma.

In §4, we obtain Theorem 1.5.4, which identifies the coefficient functions in the asymptotic expansion. The proof is based on steepest descent analysis. The starting point is the existence of the expansion of Theorem 1.5.2 which tells us that the probability distribution $|P_{m,n}|^2 e^{-2mQ}$ is approximately a Gaussian ridge centered around $\partial\mathcal{S}_\tau$, so by composing with the conformal mapping ϕ_τ we obtain a Gaussian ridge around the unit circle. By writing the relevant integrals in polar coordinates and applying Laplace's method in the radial direction, this structure allows us to *collapse* the orthogonality conditions into integral equations on the unit circle. The collapsed orthogonality conditions then reduce to inhomogeneous Toeplitz kernel equations. The algorithm arises when we solve those equations.

In §5, we supply more details on determinantal point processes, and give the proof of Corollary 1.5.6 on boundary universality in the random normal matrix model for 1-admissible potentials.

In §6, we supply the proof of key lemma on the existence of the orthogonal foliation flow. The proof is based on an algorithm, which determines both the flow and the asymptotic expansion of the approximately orthogonal quasipolynomials in an iterative and intertwined fashion. An outline of the algorithm is provided in Subsections 6.2 and 6.4.

Finally, in Section 7, we connect our orthogonal foliation flow with the Its and Takhtajan approach involving soft Riemann-Hilbert problems (2×2 matrix $\bar\partial$-problems).

1.8. **Acknowledgments.** We wish to thank the anonymous referees for several helpful and insightful comments, which have significantly improved the manuscript. In addition, we would like to thank Gernot Akemann, Yacin Ameur, Alexander Aptekarev, Robert Berman, Maurice Duits, Nam-Gyu Kang, Simon Larson, Nikolai Makarov, Julius Ross, Seong-Mi Seo, and Ofer Zeitouni for stimulating discussions.

1.9. **Notation and conventions.** We denote by ∂_z and $\bar\partial_z$ the standard Wirtinger derivatives, given by

$$\partial_z = \frac{1}{2}\big(\partial_x - i\partial_y\big) \quad \text{and} \quad \bar\partial_z = \frac{1}{2}\big(\partial_x + i\partial_y\big), \qquad z = x + iy.$$

When the dependence on z is clear we will omit the subscript and simply write ∂ and $\bar\partial$. The Laplacian factorizes as $\Delta = \partial\bar\partial$ (notice that this is a quarter of the usual Laplacian).

The Riemann sphere is denoted by $\widehat{\mathbb{C}}$, and we identify it with the extended complex plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ via stereographic projection. If Γ is a bounded Jordan curve, and Ω_e denotes the unbounded component of $\mathbb{C} \setminus \Gamma$, then the domain Ω_e is simply connected if regarded as a domain on the Riemann sphere $\widehat{\mathbb{C}}$. As a consequence, the Riemann mapping theorem guarantees that there exists a conformal mapping $\phi : \Omega_e \to \mathbb{D}_{\mathrm{e}}$ onto the exterior disk $\mathbb{D}_{\mathrm{e}}$. This mapping is uniquely determined if we require that

$$\phi(\infty) = \infty \quad \text{and} \quad \phi'(\infty) > 0. \tag{1.9.1}$$

A conformal mapping of unbounded domains which is subject to the normalization (1.9.1) at infinity is called *orthostatic*. Unless specified otherwise, a conformal mapping $\phi : \Omega_1 \to \Omega_2$ is tacitly assumed to be onto.

We use the standard Landau notation for control of asymptotic quantities. Namely, if $f(t)$ and $g(t)$ denote two positive functions defined for $t \in (0, 1]$, we say that $f(t) = \mathrm{O}(g(t))$ as $t \to 0$ if there exists a constant C with $0 < C < \infty$ such that $f(t) \le Cg(t)$ for all $t > 0$ sufficiently small. Moreover, we say that $f(t) = \mathrm{o}(g(t))$ as $t \to 0$ if $\lim_{t\to0} f(t)/g(t) = 0$. Moreover, we use the notation $f(t) \asymp g(t)$ to say that $f(t) = \mathrm{O}(g(t))$ and $g(t) = \mathrm{O}(f(t))$, as $t \to 0$. Similar comparisons when f and g are functions defined on more general sets are understood analogously.

For a positive Borel measure μ supported on the set $\Omega \subset \mathbb{C}$, we denote by $L^2(\Omega, \mu)$ the standard L^2-space of square integrable functions with respect to μ, with inner product

$$\langle f, g\rangle_\mu = \int_\Omega f(z)\overline{g(z)}\, d\mu(z).$$

For a domain $\Omega \subset \mathbb{C}$, we define the Bergman space $A^2(\Omega, \mu)$ as the subspace of $L^2(\Omega, \mu)$ consisting of all $f \in L^2(\Omega, \mu)$ which are holomorphic on Ω. For an integer n and unbounded Ω, we denote by $L^2_n(\Omega, \mu)$ and $A^2_n(\Omega, \mu)$ the subspaces of functions f with

$$|f(z)| = \mathrm{O}(|z|^{n-1}), \qquad z \in \Omega,\ \ |z| \to +\infty.$$

If $\Omega = \mathbb{C}$ is the entire complex plane, we drop it from the notation. Measures of the form $d\mu = e^{-\phi} dA$ play a major role in our analysis, and for such measures we use the shorthand notation $A^2_\phi(\Omega)$, $L^2_\phi(\Omega)$, $A^2_{\phi,n}(\Omega)$, and $L^2_{\phi,n}(\Omega)$ for the spaces discussed above. The L^2 norm and inner products are denoted by $\|\cdot\|_\mu$ and $\langle\cdot,\cdot\rangle_\mu$, or simply by $\|\cdot\|_\phi$ and $\langle\cdot,\cdot\rangle_\phi$ in the case of measure of the form $d\mu = e^{-\phi} dA$.

Frequently used notation. For the convenience of the reader, we supply a list of frequently used notation.

$\mathbb{C}$, $\mathbb{R}$, $\mathbb{T}$	Complex plane, real line, and unit circle, respectively.
$\mathbb{D}$, $\mathbb{D}_{\mathrm{e}}$	Open unit disk $\mathbb{D} = \{z : \lvert z\rvert < 1\}$ and exterior disk $\mathbb{D}_{\mathrm{e}} = \{z : \lvert z\rvert > 1\}$, also for arguments (z_0, r) denoting center and radius of the boundary circle.
$\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Z}_+$	Integers, natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$ and positive integers $\mathbb{Z}_+ = \{1, 2, 3, \ldots\}$, respectively.
$\mathcal{E}^c, \mathcal{E}^\circ, \overline{\mathcal{E}}$	Complement, interior, and closure of the set $\mathcal{E}$. The complement is understood as $\mathbb{C} \setminus \mathcal{E}$, unless specified otherwise.

$1_{\mathcal{E}}$	Indicator function for the set $\mathcal{E}$.
$\partial_z, \bar\partial_z$	Wirtinger derivatives, given by $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$, $\bar\partial_z = \frac{1}{2}(\partial_x + i\partial_y)$, where $z = x + iy$.
$\varDelta$	Laplacian, which factorizes as $\varDelta = \partial\bar\partial$. N.B.: this equals one-quarter of the usual Laplacian.
Pol_n	Space of polynomials of degree at most $n-1$.
Q, $\hat{Q}_\tau$	The potential and the solution to obstacle problem with growth $\tau \log\lvert z\rvert$ at infinity, respectively.
$\breve{Q}_\tau$	Harmonic extension of $\hat{Q}_\tau\big\rvert_{\mathcal{S}_\tau^c}$ across $\partial\mathcal{S}_\tau$.
$Q_\tau^{\circledast}$	Bounded harmonic extension of $Q\big\rvert_{\partial\mathcal{S}_\tau}$ to $\mathcal{S}_\tau^c$.
$\mathcal{Q}_\tau$	Holomorphic function on $\mathcal{S}_\tau^c$ with $\operatorname{Re}\mathcal{Q}_\tau = Q_\tau^{\circledast}$ and $\operatorname{Im}\mathcal{Q}_\tau(\infty) = 0$.
$\mathcal{S}_\tau, \mathcal{S}_\tau^\star$	The droplet and the coincidence set for the obstacle problem, respectively. These are equal under the τ_0-admissibility assumption, for $\lvert\tau - \tau_0\rvert$ small.
$\mathcal{K}_{0,\tau}$, $\mathcal{K}_\tau$	Compact subsets of $\mathcal{S}_\tau^\circ$ related with the radii $\rho_{0,0}$ and ρ_0, respectively.
I_{ϵ_0}	$I_{\epsilon_0} = [1-\epsilon_0, 1+\epsilon_0]$ for a small parameter $\epsilon_0 > 0$.
ϕ_τ	Conformal mapping $\phi_\tau : \mathcal{S}_\tau^c \to \mathbb{D}_{\mathrm{e}}$ with $\phi_\tau(\infty) = \infty$ and $\phi_\tau'(\infty) > 0$.
R_τ	The modified potential, given by $(Q - \breve{Q}_\tau) \circ \phi_\tau^{-1}$.
$\chi_{\tau,0}, \chi_{\tau,1}$	Smooth cut-off functions related via $\chi_{\tau,0} = \chi_{\tau,1} \circ \phi_\tau$.
$\varpi(E, \Omega, z_0)$	Harmonic measure of E relative to (Ω, z_0).
H^2, $H^2_-, H^2_{-,0}$	Hardy spaces, cf. §2.5.
$\mathbf{H}_\Omega$	The Herglotz operator for a domain Ω containing the point at infinity.
$\mathbf{P}_{H^2}, \mathbf{P}_{H^2_-}$	Orthogonal projection onto Hardy spaces.
ת, ס, ט, ץ	Index sets, appearing with various subscripts and superscripts. See pp. 36, 43 and 49.
$\mathbf{L}_k$, $\mathbf{M}_k$	Differential operators arising in steepest descent calculations.
$\mathcal{B}_{\tau,j}$, $B_{\tau,j}$	Coefficient functions in asymptotic expansions of ONPs, related through the conformal mapping ϕ_τ (see Theorem 1.5.4).
$\psi_{s,t}$, $\hat\psi_{j,l}$, b_j	Conformal mappings related to the orthogonal foliation flow, their Taylor coefficients in (s,t), and bounded holomorphic coefficient functions.
$\Gamma_{m,n,t}$, $\mathcal{D}_{m,n}$	The curves of the orthogonal foliation and the foliated region, respectively.
$\Pi_{s,t}$, $\hat\Pi_{j,l}$	The logarithmic density in the master equation and its Taylor coefficients in (s,t), see §6.4.
$\mathbf{\Lambda}_{m,n}$	Canonical positioning operator, cf. §3.3.
$F_{m,n}^{\langle\kappa\rangle}$, $f_{m,n}^{\langle\kappa\rangle}$	Quasipolynomial and analogous bounded function, related through $\mathbf{\Lambda}_{m,n}$.
δ_m	The number $\delta_m = m^{-\frac{1}{2}}\log m$.
$\hat{\mathbb{A}}(\varrho, \sigma)$	The 2σ-fattened diagonal annulus, cf. §6.1.
$\prec_{\mathrm{L}}$, $\prec_{\mathrm{OL}}$	Lexicographic and order-lexicographic orderings.
$\mathrm{POL}(\cdot)$	Polynomial complexity classes, cf. §6.7.
$\mathcal{G}_{\boldsymbol{\mu},\boldsymbol{\nu}}$, $\mathcal{H}_{\boldsymbol{\mu},\boldsymbol{\nu}}$	Non-linear differential expressions for Faà di Bruno's formula.

2. Preliminaries

2.1. **An obstacle problem and logarithmic potential theory.** In this section, we follow the presentation of [29]. The standard reference for the potential theoretic aspects of this material is the monograph [49] by Saff and Totik.

For a positive real parameter τ, let $\mathrm{Subh}_\tau(\mathbb{C})$ denote the convex set of all subharmonic functions $q : \mathbb{C} \to \mathbb{R} \cup \{-\infty\}$ on the complex plane $\mathbb{C}$ which meet the growth bound

$$q(z) \le \tau \log |z| + \mathrm{O}(1)$$

as $|z| \to \infty$. For lower semicontinuous potentials Q subject to the growth condition (1.5.1) and for $0 < \tau \le \tau_0$, we let $\hat{Q}_\tau$ be the solution to the obstacle problem

$$\hat{Q}_\tau(z) := \sup\big\{q(z) :\ q \in \mathrm{Subh}_\tau(\mathbb{C}) \ \text{ and } \ q \le Q \ \text{ on } \ \mathbb{C}\big\},$$

and observe that trivially $\hat{Q}_\tau \le Q$, and if we regularize $\hat{Q}_\tau$ on a set of logarithmic capacity 0 (and keep the same notation for the regularized function) then $\hat{Q}_\tau \in \mathrm{Subh}_\tau(\mathbb{C})$ holds. Suppose now that Q is C^2-smooth. Standard regularity results then give that $\hat{Q}_\tau$ is $C^{1,1}$-smooth, so that the partial derivatives of order 2 of $\hat{Q}_\tau$ are locally bounded (in the sense of distribution theory), see e.g. [8] for a simple argument to this effect. As a consequence of the growth condition (1.5.1) on Q, the *coincidence set* defined by

$$\mathcal{S}_\tau^\star := \{z \in \mathbb{C} :\ \hat{Q}_\tau(z) = Q(z)\}$$

is compact, and moreover, a Perron-type argument shows that $\hat{Q}_\tau$ is harmonic off $\mathcal{S}_\tau^\star$. It now follows from the $C^{1,1}$-smoothness that $\Delta \hat{Q}_\tau = 1_{\mathcal{S}_\tau^\star} \Delta Q$ holds in the sense of distribution theory (see [40, p. 53]).

The above obstacle problem has a direct relation with weighted potential theory. The weighted logarithmic energy, with respect to a continuous weight function $V : \mathbb{C} \to \mathbb{R}$, of a compactly supported finite real Borel measure μ is defined as

$$I_V[\mu] = \int_{\mathbb{C}\times\mathbb{C}} \log \frac{1}{|z-w|}\, d\mu(z)\, d\mu(w) + 2\int_{\mathbb{C}} V(z)\, d\mu(w).$$

With $V = \tau^{-1}Q$, we set out to minimize the energy $I_{\tau^{-1}Q}[\mu]$ over all compactly supported Borel *probability* measures μ. There is a unique such minimizer, called the *equilibrium measure*, which we denote by μ_τ. The connection with the obstacle problem is via the relation

$$d\mu_\tau(z) = 2\tau^{-1} \Delta \hat{Q}_\tau\, dA = 2\tau^{-1} 1_{\mathcal{S}_\tau^\star} \Delta Q(z)\, dA. \tag{2.1.1}$$

As a consequence, we may recover the logarithmic potential for the equilibrium measure from $\hat{Q}_\tau$ and a real constant $F_{Q,\tau}$:

$$U^{\mu_\tau}(z) := \int_{\mathbb{C}} \log \frac{1}{|z-w|}\, d\mu_\tau(z) = -\tau^{-1}\hat{Q}_\tau(z) + F_{Q,\tau}, \qquad z \in \mathbb{C}.$$

Since μ_τ is a probability measure by definition, we see from (2.1.1) that $\Delta Q \ge 0$ a.e. on $\mathcal{S}_\tau^\star$. So, the coincidence set $\mathcal{S}_\tau^\star$ will avoid the open subset of the plane where $\Delta Q < 0$, which may be nonempty. We call the support (as a distribution) of the equilibrium measure μ_τ the *droplet*, and denote it by $\mathcal{S}_\tau$. We alternatively call it the *spectral droplet*, due to the spectral interpretation as the accumulation set for the eigenvalues of random matrices. In general this is a subset of the coincidence (or contact) set $\mathcal{S}_\tau^\star$. However, the difference set $\mathcal{S}_\tau^\star \setminus \mathcal{S}_\tau$ is small, in the sense that it is a null set with respect to the weighted area measure $|\Delta Q|\, dA$. In this presentation, we will assume throughout that the potential Q is 1-admissible. Under this assumption, we have the equality $\mathcal{S}_\tau = \mathcal{S}_\tau^\star$ for $\tau \in I_{\epsilon_0} := [1-\epsilon_0, 1+\epsilon_0]$ with some small but positive ϵ_0.

The function $\hat{Q}_\tau$ is $C^{1,1}$-smooth, with $\hat{Q}_\tau = Q$ on the droplet $\mathcal{S}_\tau$, whereas in the complement $\mathcal{S}_\tau^c$ it is harmonic and determined by the boundary data that $\hat{Q}_\tau = Q$ on $\partial\mathcal{S}_\tau$ and the growth $\hat{Q}_\tau(z) = \tau\log|z| + \mathrm{O}(1)$ as $|z|\to+\infty$. We proceed to introduce some further functions related to the potential Q.

Definition 2.1.1. Assume that Q is 1-admissible, and let $\tau\in I_{\epsilon_0}$. Then

(i) $\breve{Q}_\tau$ is defined as the harmonic extension of the restriction of $\hat{Q}_\tau$ to $\mathcal{S}_\tau^c$ across $\partial\mathcal{S}_\tau$.

(ii) $Q_\tau^{\circledast}$ is the bounded harmonic harmonic function on $\mathcal{S}_\tau^c$ which equals Q on $\partial\mathcal{S}_\tau$, extended harmonically across $\partial\mathcal{S}_\tau$.

(iii) $\mathcal{Q}_\tau$ is the bounded holomorphic function in $\mathcal{S}_\tau^c$ such that $\operatorname{Re}\mathcal{Q}_\tau = Q_\tau^{\circledast}$ on $\mathcal{S}_\tau^c$ with $\operatorname{Im}\mathcal{Q}_\tau(\infty)=0$, extended analytically across $\partial\mathcal{S}_\tau$.

It is clear that the functions $\breve{Q}_\tau$ and $Q_\tau^{\circledast}$ are related via

$$\breve{Q}_\tau = \tau\log|\phi_\tau| + Q_\tau^{\circledast}. \tag{2.1.2}$$

2.2. **A weighted Bernstein-Walsh lemma.** The significance of the set $\mathcal{S}_\tau$ in relation to orthogonal polynomials is made clear by Proposition 2.2.2 below. We begin with a useful lemma taken from [2], see Lemma 3.2.

Lemma 2.2.1. *Let u be holomorphic in a disk $\mathbb{D}(z, m^{-1/2}\delta)$. Then*

$$|u(z)|^2 e^{-2mQ(z)} \le \frac{m e^{A\delta^2}}{\delta^2}\int_{\mathbb{D}(z,m^{-1/2})}|u|^2 e^{-2mQ}\,dA,$$

where A denotes the essential supremum of ΔQ on $\mathbb{D}(z,m^{-1/2}\delta)$.

This lemma is used in [2] to obtain growth bounds for polynomials of degree at most n. The approach works more generally, for functions of polynomial growth in the space $A^2_{2mQ}(\mathcal{K}^c)$ defined in §1.9, where $\mathcal{K}$ is a compact subset of the interior of the droplet $\mathcal{S}_\tau$. The following result generalizes the classical Bernstein-Walsh lemma, see e.g. Chapter III.2 in [49].

Proposition 2.2.2. *Let $\tau=\frac{n}{m}$, and suppose $\mathcal{K}$ is a compact subset of the interior of $\mathcal{S}_\tau$. Then there exists a positive constant C such that for any $u\in A^2_{2mQ}(\mathcal{K}^c)$ with the polynomial growth control $|u(z)| = \mathrm{O}\left(|z|^n\right)$ as $|z|\to\infty$, we have that*

$$|u(z)|\le Cm^{\frac{1}{2}}\|u\|_{L^2(\mathcal{K}^c,e^{-2mQ})}e^{m\hat{Q}_\tau(z)},\qquad \operatorname{dist}_{\mathbb{C}}(z,\mathcal{K})\ge\delta m^{-1/2}.$$

Proof. Assume that $z\in\mathcal{S}_\tau\setminus\mathcal{K}$ lies at a distance of at least $m^{-1/2}\delta$ from $\mathcal{K}$. By Lemma 2.2.1, we have the estimate

$$|u(z)|^2\le\frac{me^{2A\delta^2}}{\delta^2}e^{2mQ(z)}\|u\|^2_{L^2(\mathcal{K}^c,e^{-2mQ})},$$

which yields the claim for $z\in\mathcal{S}_\tau\setminus\mathcal{K}$ with the constant $C=C_\delta=\delta^{-1}e^{A\delta^2}$. Next, suppose that u has norm equal to 1, and let $q(z)$ be the subharmonic function

$$q(z)=\frac{1}{2m}\log\frac{|u(z)|^2}{mC_\delta^2},\qquad z\in\mathcal{K}^c.$$

It follows from the above estimate on $|u(z)|^2$ that $q(z)\le Q$ for $z\in\mathcal{S}_\tau\setminus\mathcal{K}$, and the growth bound on $|u(z)|$ as $|z|\to\infty$ entails that $q(z)\le\tau\log|z|+\mathrm{O}(1)$ as $|z|\to\infty$. Now, we consider the difference $q-\hat{Q}_\tau$ and observe that it is harmonic in $\mathcal{S}_\tau^c$ and that $q-\hat{Q}_\tau\le 0$ holds on the boundary $\partial\mathcal{S}_\tau$, since $\hat{Q}_\tau=Q$ there. Moreover, we see from the growth bound on q that the difference $q-\hat{Q}_\tau$ is bounded from above in $\mathcal{S}_\tau^c$. It now follows from the maximum principle for subharmonic functions that $q(z)-\hat{Q}_\tau(z)\le 0$ throughout $z\in\mathcal{S}_\tau^c$, which completes the proof. □

In particular, Proposition 2.2.2 tells us that $|P_{m,n}(z)|^2e^{-2mQ}$ decays exponentially off the droplet $\mathcal{S}_\tau$ if $\tau=\frac{n}{m}$. As alluded to in the introduction, it is possible to also locate the mass of the probability density $|P_{m,n}(z)|^2e^{-2mQ(z)}$. We recall the notation $\varpi(\cdot,\widehat{\mathbb{C}}\setminus\mathcal{S}_t,\infty)$ for the harmonic measure of $\widehat{\mathbb{C}}\setminus\mathcal{S}_t$ relative to the point at infinity. The following is from [3].

Theorem 2.2.3. *As $m,n\to\infty$ with $\tau=\frac{n}{m}=\tau_0+\mathrm{O}(m^{-1})$ for some τ_0 with $0<\tau_0\le 1$, we have the convergence*

$$|P_{m,n}|^2e^{-2mQ}\to\varpi(\cdot,\widehat{\mathbb{C}}\setminus\mathcal{S}_{\tau_0},\infty),$$

in the sense of weak-star convergence of measures.

See Figure 1.1 (right) above for an illustration of this convergence.

2.3. **Weighted Laplacian growth.** Weighted Laplacian growth (or weighted Hele-Shaw flow) describes the movement of the boundary of a viscous fluid droplet in a porous medium, as fluid is injected into the droplet. The weight appears as a result of the variable permeability of the medium, or, alternatively, as a result of curved geometry. For the mathematical formulation, consider a simply connected domain Ω_0 on the Riemann sphere $\widehat{\mathbb{C}}:=\mathbb{C}\cup\{\infty\}$ containing the point at infinity. A smoothly increasing family $\{\Omega_t\}_t$ of domains is said to be a Hele-Shaw flow with weight ω, relative to the injection point at infinity, if the infinitesimal change of the measure $1_{\Omega_\tau}\omega(z)dA$ equals harmonic measure (the derivative is as usual taken in the sense of distribution theory):

$$\partial_t(1_{\Omega_t}\omega dA)=d\varpi(\cdot,\Omega_t,\infty). \tag{2.3.1}$$

Alternatively, we can think in terms of the weak formulation, which amounts to the requirement that

$$\int_{\Omega_t\setminus\Omega_s}h\,\omega dA=(t-s)h(\infty),\quad s<t,$$

holds for all bounded harmonic functions h on Ω_t. At times, we prefer to think of the flow of the boundary loops $\{\partial\Omega_t\}_t$ rather than the flow of domains itself. A basic reference on Hele-Shaw flow is the book [27] by Gustafsson, Teodorescu and Vasili'ev. The weighted Hele-Shaw flow problem appears to have been treated first in the paper [32] by Hedenmalm and Shimorin, where the weight was interpreted as a Riemannian metric, motivated by considerations in the potential theory of clamped plates [33]. This line of work is continued by [31], [30]. In this connection, we mention the work [48] by Ross and Witt-Nyström, which deals with a less regular situation.

In the present work, weighted Laplacian growth appears for two distinct families of weights that arise naturally. For instance, the complement $\mathcal{S}_\tau^c$ evolves according to Laplacian growth with the weight $2\Delta Q$ and injection point at infinity, with τ as backward time. The second type of Laplacian growth occurs with the weight $\omega=|P|^2e^{-2mQ}$, where P is an approximation of the orthogonal polynomial $P_{m,n}$, see the discussion in §1.6. The latter flow of loops is what we call the orthogonal foliation flow.

We will need the following lemma, about the movement of the loops $\partial\mathcal{S}_\tau$ as τ varies.

Lemma 2.3.1. *Fix $\tau \in I_{\epsilon_0} = [1-\epsilon_0, 1+\epsilon_0]$. Denote by $\mathrm{n}_\tau(\zeta)$ the outer unit normal to $\partial\mathcal{S}_\tau$ at a point $\zeta \in \partial\mathcal{S}_\tau$, and let $\mathrm{n}_\tau(\zeta)\mathbb{R}$ denote the straight line which contains $\mathrm{n}_\tau(\zeta)$ and the origin. Then, if for real ε the point ζ_ε is closest to ζ in the intersection*

$$(\zeta - \mathrm{n}_\tau(\zeta)\mathbb{R}) \cap \partial\mathcal{S}_{\tau-\varepsilon},$$

we have as $\varepsilon \to 0$ that

$$\zeta_\varepsilon = \zeta - \varepsilon \mathrm{n}_\tau(\zeta)\frac{|\phi_\tau'(\zeta)|}{4\varDelta Q(\zeta)} + \mathrm{O}(\varepsilon^2)$$

and the outer normal $\mathrm{n}_{\tau-\varepsilon}(\zeta_\varepsilon)$ satisfies

$$\mathrm{n}_{\tau-\varepsilon}(\zeta_\varepsilon) = \mathrm{n}_\tau(\zeta) + \mathrm{O}(\varepsilon).$$

Proof. We recall that the compact sets $\mathcal{S}_\tau$ evolve according to weighted Laplacian growth with respect to the weight $2\varDelta Q$, so that we have (2.3.1) with $\Omega_\tau = \mathcal{S}_\tau^c$. For the details, we refer to Theorem 5.22 and Proposition 6.10 in [29]. This means that

$$\partial_\tau(1_{\mathcal{S}_\tau} 2\varDelta Q\, dA) = d\varpi(\cdot, \mathcal{S}_\tau^c, \infty) = |\phi_\tau'|\, ds, \tag{2.3.2}$$

where we recall that ϕ_τ is the (surjective) conformal mapping $\mathcal{S}_\tau^c \to \mathbb{D}_\mathrm{e}$. Informally, the boundary $\partial\mathcal{S}_\tau$ moves at local speed $(4\varDelta Q)^{-1}|\phi_\tau'|$ in the exterior normal direction, where the number 4 appears in place of 2 as a result of the different normalizations associated with ds and dA. It is known by Theorem 6.2 in [32], which is based on the Nishida-Nirenberg version of the Cauchy-Kovalevskaya theorem, that the loops $\partial\mathcal{S}_\tau$ deform real-analytically as τ varies. In view of this fact and the evolution equation (2.3.2), the claimed assertions follow from Taylor's formula. □

2.4. **Polynomial $\bar\partial$-methods.** Let ϕ be a strictly subharmonic function on $\mathbb{C}$. Hörmander's classical result states that the inhomogeneous $\bar\partial$-equation

$$\bar\partial u = f$$

can be solved for any datum $f \in L^2_{\mathrm{loc}}(\mathbb{C})$ with the estimate

$$\int_{\mathbb{C}} |u|^2 e^{-\phi}\, dA \le \int_{\mathbb{C}} |f|^2 \frac{e^{-\phi}}{\varDelta\phi}\, dA.$$

Taking this a starting point, in [2], Ameur, Hedenmalm, and Makarov investigate the case when the solution u is constrained by an additional polynomial growth condition at infinity. We now describe this result. Recall from §1.9 that $L^2_{\phi,n}(\mathbb{C})$ denotes the subspace of $L^2_\phi(\mathbb{C})$ subject to the growth restraint

$$f(z) = \mathrm{O}(|z|^{n-1})$$

near infinity. The polynomial growth Bergman space $A^2_{\phi,n}(\mathbb{C})$ is analogously defined there. We will consider these spaces with $\phi = 2mQ$.

The following is a direct consequence of Theorem 4.1 in [2].

Proposition 2.4.1. *Let $f \in L^\infty(\mathbb{C})$ be supported on $\mathcal{S}_\tau$. Then the $L^2_{2mQ,n}(\mathbb{C})$-minimal solution $u_{0,n}$ to the problem*

$$\bar\partial u_{0,n} = f$$

satisfies

$$\int_{\mathbb{C}} |u_{0,n}|^2 e^{-2mQ} dA \le \frac{1}{2m} \int_{\mathcal{S}_\tau} |f|^2 \frac{e^{-2mQ}}{\Delta Q} dA,$$

provided that the right-hand side is finite.

Proof. We apply Theorem 4.1 of [2] with $\mathcal{T} = \mathcal{S}_\tau$, $\phi = 2mQ$, $\varrho = 0$, and

$$\hat\phi = 2m\Big(1 - \frac{\varepsilon}{\tau}\Big)\hat Q_\tau + \varepsilon m \log(1 + |z|^2).$$

Then all conditions except (ii) are trivially satisfied with $a, b = \mathrm{o}(1)$ as $\varepsilon \to 0^+$. To see why (ii) holds, it is enough to observe that

$$\hat\phi(z) = 2m\tau\big(1 - \frac{\varepsilon}{\tau}\big) \log|z| + 2\varepsilon m \log|z| + \mathrm{O}(1) = \log(|z|^{2n}) + \mathrm{O}(1)$$

as $|z| \to \infty$. Hence the inclusion $A^2_{\hat\phi} \subset \mathrm{Pol}_n$ follows. Letting $\varepsilon \to 0^+$ for fixed m and n completes the proof. $\square$

Remark 2.4.2. In Theorem 4.1 of [2] there is an additional freedom to modify the weight with a function ϱ, which we set to equal $\varrho = 0$ in the above. The conditions on ϱ are such that there is flexibility in the interior direction inside the droplet, but none in the exterior or along the boundary. As ϱ is used to control the norm-minimal solution to the $\bar\partial$-equation, this flexibility tells us that decay of the datum f in the interior of the droplet translates to a corresponding decay of the solution $u_{0,n}$. On the other hand, decay of the datum near a boundary point in the tangential direction will not necessarily have the same effect.

2.5. **Holomorphic boundary value problems and Toeplitz operators.** For the reader's convenience, we include some elementary facts from the theory of Herglotz kernels and Hardy spaces. Let f be holomorphic in the unit disk $\mathbb{D}$ with continuous extension to the boundary. The classical Herglotz integral formula [25, pp. 52] asserts that

$$f(z) = \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \mathrm{Re}(f(\zeta))\, ds(\zeta) + \mathrm{Im}(f(0)), \qquad z \in \mathbb{D}.$$

If $F \in L^1(\mathbb{T})$ is real-valued, this allows us to solve the boundary value problem

$$\mathrm{Re}\, f\big|_{\mathbb{T}} = F$$

where f is holomorphic in the disk by the integral formula

$$f(z) = \mathbf{H}_{\mathbb{D}} F(z) := \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} F(\zeta)\, ds(\zeta), \quad z \in \mathbb{D}.$$

Moreover, the solution is unique up to an additive imaginary constant. For us, it is more natural to work in the exterior disk. By reflection in the unit circle, we obtain the formula

$$f(z) = \mathbf{H}_{\mathbb{D}_\mathrm{e}} F(z) := \int_{\mathbb{T}} \frac{z + \zeta}{z - \zeta} F(\zeta)\, ds(\zeta), \quad z \in \mathbb{D}_\mathrm{e},$$

which we refer to as the *Herglotz transform* of F. If F is $L^2(\mathbb{T})$-integrable, its Herglotz transform is in the Hardy space H^2. If we assume slightly more smoothness, e.g. that F is C^1-smooth, then its Herglotz transform is continuous and bounded in the closed exterior disk $\bar{\mathbb{D}}_\mathrm{e}$. Analogously, if we have a lot of smoothness, e.g. F is C^ω-smooth, then its Herglotz transform extends to a bounded analytic function on a slightly bigger exterior disk $\mathbb{D}_{\rm e}(0,\rho)$ with $\rho<1$. We recall the definition of the Hardy space $H^2=H^2(\mathbb{D})$ mentioned above. A function f is in H^2 if it is holomorphic in $\mathbb{D}$ with

$$\sup_{0<r<1}\int_{\mathbb{T}}|f(r\zeta)|^2\,ds(\zeta)<+\infty.$$

Alternatively, in terms of the boundary values, H^2 is the closed subspace of $L^2(\mathbb{T})$ defined by the property that the Fourier coefficients with negative index all vanish. The conjugate Hardy space H^2_- consists of all functions of the form $\bar f$, where $f\in H^2$, which may also be viewed as Hardy space on the exterior disk $\mathbb{D}_{\rm e}$. In a similar fashion, the standard H^p-spaces can be defined as well. For instance, for $p=\infty$ the space H^∞ consists of the bounded holomorphic functions in the unit disk $\mathbb{D}$ equipped with the supremum norm.

Associated with the Hardy and conjugate Hardy subspaces of $L^2(\mathbb{T})$ there are the orthogonal projections $\mathbf{P}_{H^2}:L^2(\mathbb{T})\to H^2$ and $\mathbf{P}_{H^2_-}:L^2(\mathbb{T})\to H^2_-$. These are associated with the Szegő integral kernel:

$$\mathbf{P}_{H^2}f(z)=\int_{\mathbb{T}}\frac{f(\zeta)}{1-z\bar\zeta}\,ds(\zeta),\qquad z\in\mathbb{D},$$

and

$$\mathbf{P}_{H^2_-}f(z)=\int_{\mathbb{T}}\frac{zf(\zeta)}{z-\zeta}\,ds(\zeta),\qquad z\in\mathbb{D}_{\rm e}.$$

We will also be interested in the subspace $H^2_{-,0}$ of H^2_- consisting of all functions that vanish at infinity (or equivalently, have average 0 on the unit circle). The associated projection is

$$\mathbf{P}_{H^2_{-,0}}f(z)=\int_{\mathbb{T}}\frac{\zeta f(\zeta)}{z-\zeta}\,ds(\zeta),\qquad z\in\mathbb{D}_{\rm e}.$$

It is clear from the above concrete formulæ that the Herglotz transform $\mathbf{H}_{\mathbb{D}_{\rm e}}$ can be expressed in terms of projections: $\mathbf{H}_{\mathbb{D}_{\rm e}}=\mathbf{P}_{H^2_-}+\mathbf{P}_{H^2_{-,0}}$. For an $L^\infty(\mathbb{T})$-function Θ, we define the (exterior) Toeplitz operator $\mathbf{T}_\Theta:H^2_-\to H^2_-$ by

$$\mathbf{T}_\Theta f=\mathbf{P}_{H^2_-}[\Theta f],\qquad f\in H^2_-.$$

The nullspace (kernel) of this operator consists of all solutions in H^2_- to $\mathbf{T}_\Theta f=0$. Assuming that Θ is nonzero almost everywhere on the circle $\mathbb{T}$, it follows that the condition that f belongs to the nullspace is equivalent to $f\in H^2_-\cap\Theta^{-1}H^2_0$, where H^2_0 consists of the functions in H^2 with mean 0. If we implicitly define the function ϑ by $\Theta(z)=z\vartheta(z)$, we may rephrase this condition as

$$f\in H^2_-\cap\vartheta^{-1}H^2,$$

which we refer to as a *homogeneous (exterior) Toeplitz kernel condition.* For a function F in the space $L^2(\mathbb{T})$, we also consider the related condition

$$f\in H^2_-\cap\vartheta^{-1}(-F+H^2),$$

which we refer to as an *inhomogeneous Toeplitz kernel condition.* In terms of Toeplitz operators, this condition may be written as $\mathbf{T}_{z\vartheta}f+\mathbf{P}_{H^2_-}[zF]=0$. The following proposition provides the structure of solutions to the homogeneous and inhomogeneous Toeplitz kernel conditions for sufficiently regular symbols ϑ.

Proposition 2.5.1. *Suppose that ϑ can be written in the form $\vartheta = e^{u+\bar{v}}$, where u and v are in H^∞, and let F be a function in $L^2(\mathbb{T})$. Then f solves*

$$f \in H^2_- \cap \vartheta^{-1}(-F + H^2)$$

if and only if

$$f = Ce^{-\bar{v}} - e^{-\bar{v}}\mathbf{P}_{H^2_{-,0}}[e^{-u}F],$$

for some constant C.

Proof. That $f \in H^2_- \cap \vartheta^{-1}(-F + H^2)$ is equivalent to having

$$e^{\bar{v}} f \in e^{\bar{v}} H^2_- \cap (-e^{-u}F + e^{-u}H^2) = H^2_- \cap (-e^{-u}F + H^2). \tag{2.5.1}$$

Since $e^{\bar{v}} f \in H^2_-$, an application of the projection $\mathbf{P}_{H^2_{-,0}}$ gives

$$\mathbf{P}_{H^2_{-,0}}[e^{\bar{v}} f] = e^{\bar{v}} f - C$$

for some constant C. On the other hand, since $e^{\bar{v}} f \in -e^{-u}F + H^2$ holds by (2.5.1), it is immediate that

$$\mathbf{P}_{H^2_{-,0}}[e^{\bar{v}} f] = -\mathbf{P}_{H^2_{-,0}}[e^{-u}F],$$

since H^2 projects to $\{0\}$. It follows that

$$e^{\bar{v}} f = C + \mathbf{P}_{H^2_{-,0}}[e^{\bar{v}} f] = C - \mathbf{P}_{H^2_{-,0}}[e^{-u}F],$$

as claimed. □

Remark 2.5.2. The Toeplitz kernel equation (2.5.1) may be viewed as a scalar Riemann-Hilbert problem with jump from the inside $\mathbb{D}$ to the outside $\mathbb{D}_e$ equal to $e^{-u}F$. Later, we will use the conformal mapping from the complement of the droplet $\mathcal{S}_\tau^c$ to the exterior disk $\mathbb{D}_e$, and the interpretation of the Toeplitz kernel equation in that context is as a scalar Riemann-Hilbert problem on the Schottky double of $\mathcal{S}_\tau^c$.

2.6. **Steepest descent analysis.** For our computational algorithm in §4, we will need the following result ([38], p. 220, Theorem 7.7.5). The formulation requires some notation. For an open subset Ω of $\mathbb{R}$, we let $C^k(\Omega)$ denote the space of k times differentiable functions on Ω, and for a compact subset K of $\mathbb{R}$, we let $C^k_0(K)$ denote the space k times differentiable, compactly supported functions on $\mathbb{R}$ whose support is contained in K. The norm in the space $C^k(\Omega)$ is defined as

$$\|u\|_{C^k(\Omega)} = \sum_{j=0}^{k} \|u^{(j)}\|_{L^\infty(\Omega)},$$

and the norm in $C^k_0(K)$ is analogously defined.

Proposition 2.6.1. *Let $\mathrm{K} \subset \mathbb{R}$ be a compact interval, Ω an open neighborhood of K, x_0 an interior point of K, and k a positive integer. If $u \in C^{2k}_0(\mathrm{K})$, $V \in C^{3k+1}(\Omega)$ and $V \geq 0$ in Ω, $V'(x_0) = 0$, $V''(x_0) > 0$, and $V' \neq 0$ in $\mathrm{K} \setminus \{x_0\}$, then, for $\omega > 0$, we have*

$$\left| e^{\omega V(x_0)} \int_{\mathrm{K}} u(x) e^{-\omega V(x)} dx - \left(\frac{2\pi}{\omega V''(x_0)} \right)^{\frac{1}{2}} \sum_{j=0}^{k-1} \omega^{-j} \mathbf{L}_j u(x_0) \right| \leq C\omega^{-k} \|u\|_{C^{2k}(\mathrm{K})}.$$

Here, C is bounded when V stays in a bounded set in $C^{3k+1}(\Omega)$, and $|x-x_0|/|V'(x)|$ has a uniform bound. With

$$W_{x_0}(x) := V(x) - V(x_0) - \frac{1}{2}(x-x_0)^2 V''(x_0),$$

we have

$$\mathbf{L}_j u(x) := \sum_{(k,l):l-k=j,\, 2l\geq 3k} \frac{(-1)^k 2^{-l}}{k!l![V''(x_0)]^l} \partial_x^{2l}(W_{x_0}^k u)(x).$$

In the definition of the above differential operator $\mathbf{L}_j$, it is implicit that the summation takes place over nonnegative integers k and l. The differential operator (1.5.3) mentioned in connection with Theorem 1.5.4 is obtained from this formula.

The following proposition is tailored to our needs, based on Proposition 2.6.1.

Proposition 2.6.2. *Let three reals ρ_0, ρ_1, ρ_2 be given, with $0 < \rho_0 < 1 < \rho_1 < \rho_2$. Assume that $V : [\rho_0, \infty) \to \mathbb{R}$ is C^{3k+1}-smooth, and that V has a unique minimum at 1, with $V(1) = V'(1) = 0$. Suppose furthermore that we have*
(a) *the convexity bound $V'' \geq \alpha$ on (ρ_0, ρ_2) for some real $\alpha > 0$,*
(b) *and that V has a bound from below of the form $V(x) \geq \vartheta \log x$ on the interval $[\rho_1, \infty)$, for some real constant $\vartheta > 0$.*
If the function $u : (\rho_0, \infty) \to \mathbb{C}$ is bounded and continuous throughout, and in addition u is C^{2k}-smooth on the interval $[0, \rho_2]$ and vanishes on $[0, \rho_0]$, then we have

$$\int_{\rho_0}^{\infty} u(x) e^{-\omega V(x)}\, dx = \left(\frac{2\pi}{\omega V''(1)}\right)^{\frac{1}{2}} \sum_{j=0}^{k-1} \omega^{-j} \mathbf{L}_j[u](1) + E,$$

where the error term $E = E(\omega, k, u, \vartheta, \rho_0, \rho_1, \rho_2)$ enjoys the bound

$$|E| \leq C_1 \omega^{-k} \|u\|_{C^{2k}([\rho_0,\rho_2])} + \|u\|_{L^\infty([\rho_1,\infty))} \rho_1^{-\omega\vartheta+1},$$

provided that $\omega > \frac{2}{\vartheta}$, where C_1 remains uniformly bounded when V stays in a bounded set of $C^{3k+1}([\rho_0, \rho_2])$.

Sketch of proof. Let χ be a smooth cut-off function with $0 \leq \chi \leq 1$ throughout, which equals 1 on the interval $[\rho_0, \rho_1]$, and vanishes on $[\rho_2, \infty)$. We use the cut-off function to split the integral

$$\int_{\rho_0}^{\infty} u(x) e^{-\omega V(x)}\, dx = \int_{\rho_0}^{\rho_2} \chi(x) u(x) e^{-\omega V(x)}\, dx + \int_{\rho_1}^{\infty} (1-\chi(x)) u(x) e^{-\omega V(x)}\, dx.$$

The first integral gives the main contribution, which is estimated using Proposition 2.6.1. The other two integrals are estimated using the given bounds from below on V. The details are omitted. □

3. Existence of an asymptotic expansion

3.1. **An L^2-version of the main theorem.** The proof of Theorem 1.5.2 goes via an expansion valid in weighted L^2-space, which is of independent interest. Modulo the key lemma (Lemma 3.4.1) concerning the orthogonal foliation flow, we first obtain the weighted L^2-expansion, and then obtain Theorem 1.5.2 as a consequence. The proof of the key lemma is deferred to §6.

For two sets $\mathcal{E}, \mathcal{F} \subset \mathbb{C}$ we define the distance between them as

$$\mathrm{dist}_{\mathbb{C}}(\mathcal{E}, \mathcal{F}) = \inf_{z\in\mathcal{E},\, w\in\mathcal{F}} |z-w|.$$

We shall need the following notion.

Definition 3.1.1. If $\mathcal{K}$ and $\mathcal{S}$ are compact sets in the plane with $\mathcal{K} \subset \mathcal{S}$ and

$$\mathrm{dist}_{\mathbb{C}}(\mathcal{K}, \mathcal{S}^c) = \varepsilon,$$

we say that a compact set $\mathcal{X}$ is *intermediate* between $\mathcal{K}$ and $\mathcal{S}$ if $\mathcal{K} \subset \mathcal{X} \subset \mathcal{S}$ with

$$\mathrm{dist}_{\mathbb{C}}(\mathcal{K}, \mathcal{X}^c) \geq \frac{\varepsilon}{1000} \quad \text{and} \quad \mathrm{dist}_{\mathbb{C}}(\mathcal{X}, \mathcal{S}^c) \geq \frac{\varepsilon}{1000}.$$

We recall from the discussion following Definition 1.5.1 the notation $I_{\epsilon_0} = [1-\epsilon_0, 1+\epsilon_0]$, where ϵ_0 is fixed and positive, with the property that the curves $\partial\mathcal{S}_\tau$ form a smooth flow of simple loops for $\tau \in I_{\epsilon_0}$.

Theorem 3.1.2. *Assume that Q is 1-admissible, and fix the precision parameter $\kappa \in \mathbb{N}$. Then, for each $\tau \in I_{\epsilon_0}$ there exists a compact subset $\mathcal{K}_\tau \subset \mathcal{S}_\tau$ with $\mathrm{dist}_{\mathbb{C}}(\mathcal{K}_\tau, \partial\mathcal{S}_\tau) \geq \varepsilon$ for some positive real number ε, such that the following holds. On the complement $\mathcal{K}_\tau^c$, there are bounded holomorphic functions $\mathcal{B}_{\tau,j}$ such that the associated function*

$$F_{m,n}^{\langle\kappa\rangle} = m^{\frac{1}{4}} \sqrt{\phi_\tau'}\, [\phi_\tau]^n e^{m\mathcal{Q}_\tau} \sum_{j=0}^{\kappa} m^{-j} \mathcal{B}_{\tau,j},$$

approximates well the normalized orthogonal polynomials $P_{m,n}$ in the sense that we have the norm control

$$\left\| P_{m,n} - \chi_{\tau,0} F_{m,n}^{\langle\kappa\rangle} \right\|_{2mQ} = \mathrm{O}(m^{-\kappa-1})$$

as $n, m \to \infty$ while $\tau = \frac{n}{m} \in I_{\epsilon_0}$. Here, $\chi_{\tau,0}$ denotes a smooth cut-off function with $0 \leq \chi_{\tau,0} \leq 1$ and uniformly bounded gradient. In addition the function $\chi_{\tau,0}$ vanishes on $\mathcal{K}_\tau$, and equals 1 on the set $\mathcal{X}_\tau^c$ where $\mathcal{X}_\tau$ is an intermediate set between $\mathcal{K}_\tau$ and $\mathcal{S}_\tau$. In the above estimate, the implicit constant is uniform for $\tau \in I_{\epsilon_0}$.

In the above theorem, the products $\chi_{\tau,0} F_{m,n}^{\langle\kappa\rangle}$ are understood to vanish on the set $\mathcal{K}_\tau$, where $F_{m,n}^{\langle\kappa\rangle}$ may be undefined.

Remark 3.1.3. (a) By inserting a further family $\mathcal{X}_\tau'$ of intermediate sets between $\mathcal{K}_\tau$ and $\mathcal{S}_\tau$ such that $\mathcal{X}_\tau$ is intermediate between $\mathcal{X}_\tau'$ and $\mathcal{S}_\tau$, we can make sure that the cut-off function $\chi_{\tau,0}$ vanishes on $\mathcal{X}_\tau'$ (and not just on $\mathcal{K}_\tau$). We mention that the compact sets $\mathcal{K}_\tau$, $\mathcal{X}_\tau'$ and $\mathcal{X}_\tau$ may be obtained, e.g., as the complements of the conformal images under ϕ_τ^{-1} of the exterior disks $\mathbb{D}_{\mathrm{e}}(0,\rho)$ with $\rho = \rho_0$, $\rho_{0,1}$ and $\rho_{0,2}$, where $0 < \rho_0 < \rho_{0,1} < \rho_{0,2} < 1$. As for the intermediate property of Definition 3.1.1 regarding the sets $\mathcal{K}_\tau$, $\mathcal{X}_\tau'$, $\mathcal{X}_\tau$, and $\mathcal{S}_\tau$, this is a little subtle, and depends on making a correct choice of the parameters ρ_0, $\rho_{0,1}$, and $\rho_{0,2}$. At our disposal, we have the Koebe distortion theorem and the fact that $\log(\phi_\tau^{-1})'$ is a Lipschitz function in the hyperbolic metric with known Lipschitz constant (see, e.g., Corollary 1.4 and Proposition 1.2 in [45], respectively). We omit the necessary details.
(b) Without loss of generality, we may assume that the cut-off function $\chi_{\tau,0}$ is uniformly smooth in the sense that for any fixed positive integer k the $C^k(\mathbb{C})$-norm of $\chi_{\tau,0}$ is uniformly bounded for $\tau \in I_{\epsilon_0}$.

(c) Our method of proof involves Toeplitz kernel problems and the construction of an approximate orthogonal foliation flow of loops. The underlying idea is inspired by an approach to the local expansion of Bergman kernels, which involves a flow of loops emanating from the point of expansion [26].

3.2. **Introduction of quasipolynomials.** We turn to the *approximate orthogonal quasipolynomials* $F_{m,n}$, by which we mean certain functions which behave like orthogonal polynomials with respect to the measure $e^{-2mQ}dA$, in a sense specified below. Let $\mathcal{K}_\tau$ be an appropriately chosen compact subset of the droplet $\mathcal{S}_\tau$, which lies at a fixed positive distance from $\partial\mathcal{S}_\tau$. Moreover, we require that the conformal mapping $\phi_\tau : \mathcal{S}_\tau \to \mathbb{D}_{\mathrm{e}}$ extends to a (surjective) conformal mapping

$$\phi_\tau \colon \mathcal{K}_\tau^c \to \mathbb{D}_{\mathrm{e}}(0,\rho_0), \qquad \tau \in I_{\epsilon_0},$$

for some ρ_0 with $0<\rho_{0,0}<\rho_0<1$, where we recall that $\rho_{0,0}$ was defined in the discussion preceding Theorem 1.5.2. In what follows, we will disregard the behavior on the compact set $\mathcal{K}_\tau$. We will justify this a posteriori, using $\bar\partial$-methods.

Definition 3.2.1. We say that a function F is a *quasipolynomial* on $\mathcal{K}_\tau^c$ of degree n if it is defined and holomorphic on $\mathcal{K}_\tau^c$, with polynomial growth near infinity: $|F(z)| \asymp |z|^n$ as $|z|\to\infty$.

In the context of this definition, a quasipolynomial F of degree n has $F(z) = az^n + \mathrm{O}(|z|^{n-1})$ near infinity, for some complex number $a \neq 0$. We refer to the number a as the *leading coefficient* of the quasipolynomial F.

We now fix a positive integer κ, which we think of as an precision parameter. Moreover, we denote by $\chi_{\tau,0}$ a smooth cut-off function that vanishes on $\mathcal{X}_\tau'$ and equals 1 on $\mathcal{X}_\tau^c$, where $\mathcal{X}_\tau'$ denotes an intermediate set between $\mathcal{K}_\tau$ and $\mathcal{S}_\tau$, while $\mathcal{X}_\tau$ is an intermediate set between $\mathcal{X}_\tau'$ and $\mathcal{S}_\tau$. In addition, we shall require that the $C^{2(\kappa+1)}$-norm of $\chi_{\tau,0}$ remains uniformly bounded for $\tau \in I_{\epsilon_0}$.

Definition 3.2.2. We say that a sequence $\{F_{m,n}\}_{m,n}$ of quasipolynomials of degree n on $\mathcal{K}_\tau^c$ is *normalized and approximately orthogonal (of accuracy κ)* if the following asymptotic conditions (i)-(iii) are met as $m\to\infty$ while $\tau = \frac{n}{m} \in I_{\epsilon_0}$:
(i) we have the approximate orthogonality

$$\forall p \in \mathrm{Pol}_n: \quad \int_{\mathbb{C}} \chi_{\tau,0} F_{m,n}(z)\overline{p(z)}\, e^{-2mQ(z)}\, dA(z) = \mathrm{O}\left(m^{-\kappa-\frac{1}{3}}\|p\|_{2mQ}\right),$$

(ii) the quasipolynomials $F_{m,n}$ have approximately unit norm,

$$\int_{\mathbb{C}} \chi_{\tau,0}^2(z)|F_{m,n}(z)|^2 e^{-2mQ(z)}\, dA(z) = 1 + \mathrm{O}(m^{-\kappa-\frac{1}{3}}),$$

(iii) and the quasipolynomial $F_{m,n}$ has leading coefficient $c_{m,n}$ at infinity which is approximately real and positive, in the sense that

$$\frac{\operatorname{Im} c_{m,n}}{\operatorname{Re} c_{m,n}} = \mathrm{O}(m^{-\kappa-\frac{1}{12}})$$

where all the implied constants are uniform.

In terms of the above definition, Theorem 3.1.2 implies in particular that $F_{m,n}^{\langle\kappa\rangle}$ is a sequence of approximately orthogonal quasipolynomials with accuracy κ. The fraction $\frac{1}{3}$ which appears in the definition is convenient in our calculations. The concept would be meaningful even if this number were replaced by e.g. $\frac{1}{5}$.

3.3. **The renormalizing ansatz.** Since Q is assumed 1-admissible, the curves $\Gamma := \partial\mathcal{S}_\tau$ remain real-analytically smooth and simple for $\tau \in I_{\epsilon_0} = [1-\epsilon_0, 1+\epsilon_0]$. In view of the requirement that $\mathcal{K}_{\tau,0} \subset \mathcal{K}_\tau$, the functions $Q_\tau^{\circledast}$ and $\breve{Q}_\tau$ are harmonic while $\mathcal{Q}_\tau$ is holomorphic in the domain $\mathcal{K}_\tau^c$ (see Definition 2.1.1). We define the operator $\mathbf{\Lambda}_{m,n}$ by

$$\mathbf{\Lambda}_{m,n} f(z) := \phi_\tau'(z)\,[\phi_\tau(z)]^n\, e^{m\mathcal{Q}_\tau(z)}\,(f\circ\phi_\tau)(z), \qquad \tau = \frac{n}{m}. \tag{3.3.1}$$

If f, g are well-defined in $\mathbb{D}_{\mathrm{e}}(0,\rho_0)$, then $\mathbf{\Lambda}_{m,n} f$ and $\mathbf{\Lambda}_{m,n} g$ are well-defined in $\mathcal{K}_\tau^c$. We observe that by a change-of-variables,

$$\begin{aligned}\int_{\mathcal{K}_\tau^c} \mathbf{\Lambda}_{m,n} f\, \overline{\mathbf{\Lambda}_{m,n} g}\, e^{-2mQ}\, dA &= \int_{\mathcal{K}_\tau^c} (f\circ\phi_\tau)\overline{(g\circ\phi_\tau)}\, e^{-2m(Q-\tau\log|\phi_\tau| - \operatorname{Re}\mathcal{Q}_\tau)} |\phi_\tau'|^2\, dA \\ &= \int_{\mathbb{D}_{\mathrm{e}}(0,\rho_0)} f\,\overline{g}\, e^{-2mR_\tau}\, dA,\end{aligned} \tag{3.3.2}$$

where we write

$$R_\tau := (Q - \breve{Q}_\tau)\circ\phi_\tau^{-1},$$

and the first equality holds by (2.1.2).

The function R_τ given by (1.5.2) is a central object in our analysis, and we turn to some of its basic properties.

Proposition 3.3.1. *The function R_τ is defined on $\mathbb{D}_{\mathrm{e}}(0,\rho_0)$, and is real-analytic in a neighborhood of $\mathbb{T}$. Moreover, near the unit circle R_τ satisfies*

$$R_\tau(re^{i\theta}) = 2\Delta R_\tau(e^{i\theta})(1-r)^2 + \mathrm{O}((1-r)^3), \qquad r\to 1,$$

where the implied constant is uniform for $e^{i\theta}\in\mathbb{T}$ and $\tau \in I_{\epsilon_0}$. Furthermore, R_τ has the growth bound from below

$$R_\tau(z) \ge \vartheta \log|z|, \qquad z\in\mathbb{D}_{\mathrm{e}}(0,\rho_1),$$

for some real parameters $\vartheta > 0$ and $\rho_1 > 1$, which do not depend on $\tau \in I_{\epsilon_0}$.

Remark 3.3.2. In particular, $R_\tau(z) \asymp (1-|z|)^2$ near the unit circle. Indeed, since $\breve{Q}_\tau$ is harmonic on $\mathcal{K}_\tau^c$, we find that

$$\Delta R_\tau = \Delta(Q-\breve{Q}_\tau)\circ\phi_\tau^{-1} = |(\phi_\tau^{-1})'|^2\,(\Delta Q)\circ\phi_\tau^{-1},$$

which shows that near the circle $\mathbb{T}$, we have uniform bound of ΔR_τ from below by a positive constant. As a consequence, the same holds for $\partial_r^2 R_\tau(re^{i\theta})$ for r close to 1, which will be useful in the context of Proposition 2.6.2.

Sketch of proof. The assertion on the local behavior near the circle $\mathbb{T}$ results from an application of Taylor's formula, using that along the boundary $\partial\mathcal{S}_\tau$ we have $Q = \breve{Q}_\tau$, $\nabla Q = \nabla\breve{Q}_\tau$ while

$$\partial_{\mathrm{n}}^2(Q-\breve{Q}_\tau) = (\partial_{\mathrm{n}}^2 + \partial_{\mathrm{t}}^2)(Q-\breve{Q}_\tau) = 4\Delta Q.$$

Here, ∂_{n} and ∂_{t} denote the normal and tangential derivatives, respectively. We turn to the global estimate from below on R_τ. By the assumption (1.5.1) with $\tau = 1$ on the growth of Q near infinity, and the growth control

$$\check{Q}_\tau(z) = \hat{Q}_\tau(z) = \tau \log|z| + \mathrm{O}(1), \qquad \text{as } |z| \to \infty,$$

it follows from the choice of the interval I_{ϵ_0} that

$$\liminf_{|z|\to\infty} \frac{(Q - \check{Q}_\tau)(z)}{\log|z|} \geq 1 + 2\epsilon_0 - \tau > 0$$

for $\tau \in I_{\epsilon_0}$. Since $|\phi_\tau^{-1}(z)| \asymp |z|$ near infinity, we see that

$$\lim_{|z|\to\infty} \frac{R_\tau(z)}{\log|z|} \geq 1 + 2\epsilon_0 - \tau > 0.$$

There is no point in $\mathbb{D}_e$ where R_τ vanishes, since the coincidence set (where $\hat{Q}_\tau$ and Q coincide) equals $\mathcal{S}_\tau$ (see Definition 1.5.1). We may conclude that the ratio $\frac{R_\tau(z)}{\log|z|}$ is bounded below by a positive constant ϑ on the exterior disk $\mathbb{D}_{\mathrm{e}}(0, \rho_1)$, independently of τ in I_{ϵ_0}. □

Informally, Proposition 3.3.1 tells us that near the unit circle, the function e^{-2mR_τ} may be thought of as a Gaussian wave around the unit circle $\mathbb{T}$.

We return to the operator $\mathbf{\Lambda}_{m,n}$, defined in (3.3.1). It renormalizes the weight, and transports holomorphic functions in the exterior disk $\mathbb{D}_{\mathrm{e}}(0, \rho_0)$ to holomorphic functions in the region $\mathcal{K}_\tau^c$. In the sequel, we will refer to $\mathbf{\Lambda}_{m,n}$ as *the canonical positioning operator.* Its basic properties are summarized in the following proposition, which involves the spaces $L^2_\phi(\mathcal{X}^c)$ and $A^2_\phi(\mathcal{X})$, as well as the restricted growth subspaces $L^2_{\phi,k}(\mathcal{X}^c)$ and $A^2_{\phi,k}(\mathcal{X}^c)$, all defined in §1.9. Below, these appear for various choices of the weight ϕ, the parameter k, and the compact set $\mathcal{X}$.

Proposition 3.3.3. *The canonical positioning operator $\mathbf{\Lambda}_{m,n}$ is an isometric isomorphism $L^2_{2mR_\tau}(\mathbb{D}_{\mathrm{e}}(0, \rho_0)) \to L^2_{2mQ}(\mathcal{K}_\tau^c)$, and the inverse operator is given by*

$$\mathbf{\Lambda}_{m,n}^{-1} g(z) = z^{-n} [\phi_\tau^{-1}]'(z)\, e^{-m(\mathcal{Q}_\tau \circ \phi_\tau^{-1})(z)} (g \circ \phi_\tau^{-1})(z), \qquad g \in L^2_{2mQ}(\mathcal{K}_\tau^c).$$

Moreover, the operator $\mathbf{\Lambda}_{m,n}$ preserves holomorphicity, and in addition, it maps the subspace $A^2_{2mR_\tau,0}(\mathbb{D}_{\mathrm{e}}(0, \rho_0))$ onto $A^2_{2mQ,n}(\mathcal{K}_\tau^c)$.

Proof. As direct consequence of the (3.3.2), we see that $L^2_{2mR_\tau}(\mathbb{D}_{\mathrm{e}}(0, \rho_0))$ is mapped isometrically into $L^2_{2mQ}(\mathcal{K}_\tau^c)$, and moreover if $\mathbf{\Lambda}_{m,n}^{-1}$ is given by the above formula, we see that it is actually the inverse to $\mathbf{\Lambda}_{m,n}$. By definition, $\mathbf{\Lambda}_{m,n} f$ is holomorphic in $\mathcal{K}_\tau^c$, if f is holomorphic in $\mathbb{D}_{\mathrm{e}}(0, \rho_0)$. It follows that $\mathbf{\Lambda}_{m,n}$ is actually an isometric isomorphism $A^2_{2mR_\tau}(\mathbb{D}_{\mathrm{e}}(0, \rho_0)) \to A^2_{2mQ}(\mathcal{K}_\tau^c)$. It remains to note that $\mathbf{\Lambda}_{m,n}$ maps bijectively

$$A^2_{2mR_\tau,0}(\mathbb{D}_{\mathrm{e}}(0, \rho_0)) \to A^2_{2mQ,n}(\mathcal{K}_\tau^c),$$

which is a direct consequence of the fact that $|\phi_\tau(z)| \asymp |z|$ as $|z| \to \infty$. □

3.4. **The orthogonal foliation flow.** We will obtain our main result, Theorem 3.1.2, as a consequence of the existence of what we call the *approximate orthogonal foliation flow of simple loops* $\Gamma_{m,n,t}$, parameterized by the parameter t. For a brief sketch of the intuition that lies behind the construction of this flow of curves, we refer to the discussion in §1.6 above.

We recall from §1.9 that a conformal mapping ψ of the exterior disk $\mathbb{D}_{\mathrm{e}}$ onto a domain containing the point at infinity is said to be *orthostatic* if it maps ∞ to ∞, and has $\psi'(\infty) > 0$. Given a smooth family ψ_t of orthostatic conformal mappings on the exterior disk, indexed by a real parameter t close to 0, such that the image domains $\Omega_t := \psi_t(\mathbb{D}_{\mathrm{e}})$ increase with t, we put $\Gamma_t = \psi_t(\mathbb{T})$ and denote by $\mathcal{D} = \bigcup_t \Gamma_t$ the region covered by the flow. We may form the *foliation mapping* Ψ by the formula

$$\Psi(z) = \psi_{1-|z|}\Big(\frac{z}{|z|}\Big),$$

for z in some annulus $\mathbb{A}$ containing the unit circle. The foliation mapping Ψ maps $\mathbb{A}$ onto the domain $\mathcal{D}$ covered by the boundaries. Moreover, the Jacobian J_Ψ of the foliation mapping is given by

$$J_\Psi(r\zeta) = -\frac{1}{r}\operatorname{Re}\big(\bar{\zeta}\partial_t\psi_t(\zeta)\overline{\psi_t'(\zeta)}\big)\big|_{t=1-r}, \qquad \zeta \in \mathbb{T},$$

for r near 1. We may integrate over a flow encoded by a foliation mapping Ψ as follows: If we denote by $\mathbb{A}_\epsilon$ the annulus $\mathbb{A}_\epsilon = \mathbb{D}(0, 1+\epsilon) \setminus \bar{\mathbb{D}}(0, 1-\epsilon)$, we have for integrable f,

$$\int_{\Psi(\mathbb{A}_\epsilon)} f\, dA = \int_{\mathbb{A}_\epsilon} f \circ \Psi\ J_\Psi\, dA = 2\int_{-\epsilon}^{\epsilon}\int_{\mathbb{T}} f \circ \psi_t(\zeta)(1-t)J_\Psi\big((1-t)\zeta\big)\, ds(\zeta)\, dt. \tag{3.4.1}$$

The existence of the foliation flow may be phrased as follows. We call the relation (3.4.2) below the *master equation for the orthogonal foliation flow.* For convenience of notation, let δ_m be the number

$$\delta_m := m^{-1/2}\log m.$$

Lemma 3.4.1. *Fix the precision parameter κ to be a positive integer. For $\tau = \frac{n}{m} \in I_{\epsilon_0}$, there exist $0 < \rho_0 < 1$ and bounded holomorphic functions $B_{\tau,j}$ on $\mathbb{D}_{\mathrm{e}}(0, \rho_0)$ for $j = 0, \ldots, \kappa$, such that the the following properties hold. The function $B_{\tau,0}$ is bounded away from zero with $B_{\tau,0}(\infty) > 0$, while for $j = 1, \ldots, \kappa$ we have $\operatorname{Im} B_{\tau,j}(\infty) = 0$. Moreover, there exists a smooth family of orthostatic conformal mappings $\{\psi_{m,n,t}\}_{m,n,t}$ on $\bar{\mathbb{D}}_e$, such that if we write $f_{m,n}^{\langle\kappa\rangle} = \sum_{j=0}^{\kappa} m^{-j}B_{\tau,j}$, we have that*

$$m^{\frac{1}{2}}\big|f_{m,n}^{\langle\kappa\rangle} \circ \psi_{m,n,t}(\zeta)\big|^2 e^{-2m(R_\tau\circ\psi_{m,n,t})(\zeta)}\,(1-t)J_{\Psi_{m,n}}((1-t)\zeta) = \frac{m^{\frac{1}{2}}}{(4\pi)^{\frac{1}{2}}}e^{-mt^2}\big(1 + \mathrm{O}\big(m^{-\kappa-\frac{1}{3}}\big)\big), \qquad \zeta \in \mathbb{T}, \tag{3.4.2}$$

provided that $|t| \le \delta_m$. Here, the implicit constant is uniform in $\tau \in I_{\epsilon_0}$. Moreover, if $\mathcal{D}_{m,n}$ denotes the union $\mathcal{D}_{m,n} = \bigcup_{|t|\le\delta_m}\psi_{m,n,t}(\mathbb{T})$, then $\operatorname{dist}_{\mathbb{C}}(\mathcal{D}_{m,n}^c, \mathbb{T}) \ge c_0\delta_m$ for some positive constant c_0.

Remark 3.4.2. The equation (3.4.2) may be understood as an approximate *weighted Polubarinova-Galin equation* with weight $|f_{m,n}^{\langle\kappa\rangle}|^2 e^{-2mR_\tau}$, and variable speed of expansion. Indeed, we should compare with equation (6.11) in [32], which states in a similar context that along concentric circles,

$$J_\Psi = \omega^{-1} \circ \Psi,$$

where Ψ is a foliation mapping, and ω denotes a weight. In comparison, our factor $(4\pi)^{-\frac{1}{2}}e^{-mt^2}$ appears as consequence of the variable speed.

In what follows, we take this key lemma for granted. The proof is supplied in §6.

3.5. **The L^2-expansion for quasipolynomials.** We first find a sequence of approximately orthogonal quasipolynomials with an asymptotic expansion.

Lemma 3.5.1. *Let $\kappa\in\mathbb{N}$ be given and let $f_{m,n}^{\langle\kappa\rangle}=\sum_{j=0}^{\kappa}m^{-j}B_{\tau,j}(z)$ be the functions defined in Lemma 3.4.1. Then the functions*

$$F_{m,n}^{\langle\kappa\rangle}(z)=m^{\frac14}\mathbf{\Lambda}_{m,n}[f_{m,n}^{\langle\kappa\rangle}]=m^{\frac14}\phi_\tau'(z)[\phi_\tau(z)]^n e^{m\mathcal{Q}_\tau(z)}(f_{m,n}^{\langle\kappa\rangle}\circ\phi_\tau)(z) \tag{3.5.1}$$

constitute a family of approximately orthogonal quasipolynomials to accuracy κ in the sense of Definition 3.2.2.

Proof. We denote by $\chi_{\tau,1}$ a radial smooth cut-off function which vanishes on $\mathbb{D}(0,\rho_{0,1})$ and equals 1 on $\mathbb{D}_{\mathrm{e}}(0,\rho_{0,2})$, where the parameters $0<\rho_0<\rho_{0,1}<\rho_{0,2}<1$ are chosen in accordance with Remark 3.1.3. The cut-off function $\chi_{\tau,0}$ is then given by $\chi_{\tau,0}=\chi_{\tau,1}\circ\phi_\tau$. The intermediate sets $\mathcal{X}_\tau'$ and $\mathcal{X}_\tau$ are given as the complements of the conformal images of $\mathbb{D}_{\mathrm{e}}(0,\rho_{0,1})$ and $\mathbb{D}_{\mathrm{e}}(0,\rho_{0,2})$ under ϕ_τ, respectively.

By Lemma 3.4.1, the functions $f_{m,n}^{\langle\kappa\rangle}$ are bounded and holomorphic on the exterior disk $\mathbb{D}_{\mathrm{e}}(0,\rho_0)$, with $f_{m,n}^{\langle\kappa\rangle}(\infty)>0$. As the leading term $B_{\tau,0}$ is bounded away from 0 on $\mathbb{D}_{\mathrm{e}}(0,\rho_0)$, it follows that for large enough m, the same can be said for $f_{m,n}^{\langle\kappa\rangle}$. In view of this, the functions $F_{m,n}^{\langle\kappa\rangle}$ given by (3.5.1) are quasipolynomials of order n on $\mathcal{K}_\tau^{c}:=\phi_\tau^{-1}(\mathbb{D}_{\mathrm{e}}(0,\rho_0))$ in the sense of Definition 3.2.1.

It remains to verify the properties (i), (ii), and (iii) of Definition 3.2.2. To this end, we recall the definition of the domain $\mathcal{D}_{m,n}$ from Lemma 3.4.1, which is a certain closed neighborhood of the unit circle which arises from our orthogonal foliation flow. We recall that

$$\mathrm{dist}_{\mathbb{C}}(\mathcal{D}_{m,n}^{c},\mathbb{T}))\geq c_0\delta_m$$

holds for some fixed constant $c_0>0$, where $\delta_m=m^{-\frac12}\log m$. We first check property (ii) of Definition 3.2.2. As a step in this direction, we claim that most of the weighted L^2-mass of the function $\chi_{\tau,1}f_{m,n}^{\langle\kappa\rangle}$ lies in the domain $\mathcal{D}_{m,n}$. Indeed, a computation based on the change-of-variables formula (3.4.1) reveals that

$$\begin{aligned}(3.5.2)\quad & m^{\frac12}\int_{\mathcal{D}_{m,n}}|f_{m,n}^{\langle\kappa\rangle}|^2e^{-2mR_\tau}\,dA\\ &=2m^{\frac12}\int_{-\delta_m}^{\delta_m}\int_{\mathbb{T}}\big|f_{m,n}^{\langle\kappa\rangle}\circ\psi_{m,n,t}(\zeta)\big|^2e^{-2m\,R_\tau\circ\psi_{m,n,t}(\zeta)}\,\mathrm{Re}\,\big(-\bar\zeta\partial_t\psi_{m,n,t}\overline{\psi_{m,n,t}'}\big)\,ds(\zeta)\,dt\\ &=2m^{\frac12}\int_{-\delta_m}^{\delta_m}\big((4\pi)^{-\frac12}+\mathrm{O}(\delta_m^{2\kappa+1})\big)e^{-mt^2}dt=1+\mathrm{O}(\delta_m^{2\kappa+1})=1+\mathrm{O}(m^{-\kappa-\frac13}),\end{aligned}$$

where we move the integration to the flow coordinates $(t,\zeta)\in[-\delta_m,\delta_m]\times\mathbb{T}$.

We know that the functions $f_{m,n}^{\langle\kappa\rangle}$ are bounded uniformly in $\mathbb{D}_{\mathrm{e}}(0,\rho_0)$ independently of m and n while $\tau\in I_{\epsilon_0}$, so that

$$\chi_{\tau,1}|f_{m,n}^{\langle\kappa\rangle}|\leq C_0 \tag{3.5.3}$$

holds in the whole plane $\mathbb{C}$, for some constant C_0. Let $\mathcal{D}_{\circledast}$ denote a fixed bounded domain which contains $\mathbb{D}\cup\mathcal{D}_{m,n}$, such that the bound from below $R_\tau(z)\geq\theta_0\log|z|$ holds outside $\mathcal{D}_{\circledast}$, for some $\theta_0>0$ and all $\tau\in I_{\epsilon_0}$. That such a domain exists for sufficiently large m

is shown in Proposition 3.3.1. On the other hand, in view of Remark 3.3.2 we have the estimate

$$e^{-2mR_\tau} \le e^{-\alpha_0(\log m)^2}, \qquad \text{on } \mathcal{D}_\circledast \cap \mathbb{D}_{\mathrm{e}}(0,\rho_0) \setminus \mathcal{D}_{m,n}$$

for some constant $\alpha_0 > 0$ (if necessary we adjust ρ_0 and $\mathcal{D}_\circledast$). As a consequence, we have

$$\begin{aligned}(3.5.4)\quad m^{\frac12}\int_{\mathbb{C}\setminus\mathcal{D}_{m,n}} \chi_{\tau,1}^2 |f_{m,n}^{\langle\kappa\rangle}|^2 e^{-2mR_\tau}\,dA \le C_0^2 m^{\frac12}\int_{\mathbb{C}\setminus\mathcal{D}_\circledast} e^{-2m\theta_0\log|z|}\,dA \\ + C_0^2 m^{\frac12}\int_{\mathcal{D}_\circledast\cap\mathbb{D}(0,\rho_0)\setminus\mathcal{D}_{m,n}} e^{-\alpha_0(\log m)^2}\,dA = \mathrm{O}(m^{\frac12}e^{-\alpha_0(\log m)^2}) = \mathrm{O}(m^{-\alpha_0\log m+\frac12}).\end{aligned}$$

It now follows from (3.5.2) and (3.5.4) that

$$\begin{aligned}m^{\frac12}\int_{\mathbb{C}} \chi_{\tau,1}^2|f_{m,n}^{\langle\kappa\rangle}|^2 e^{-2mR_\tau}\,dA = m^{\frac12}\int_{\mathcal{D}_{m,n}} |f_{m,n}^{\langle\kappa\rangle}|^2 e^{-2mR_\tau}\,dA \\ + m^{\frac12}\int_{\mathbb{C}\setminus\mathcal{D}_{m,n}} \chi_{\tau,1}^2|f_{m,n}^{\langle\kappa\rangle}|^2 e^{-2mR_\tau}\,dA = 1 + \mathrm{O}(m^{-\kappa-\frac13}),\end{aligned}$$

where we use that $\chi_{\tau,1} = 1$ holds on the set $\mathcal{D}_{m,n}$ together with our foliation flow Lemma 3.4.1 and the estimate (3.5.4). Hence, by the isometric property of $\mathbf{\Lambda}_{m,n}$ from Proposition 3.3.3, it follows that

$$\int_{\mathbb{C}} \chi_{\tau,0}^2 |F_{m,n}^{\langle\kappa\rangle}|^2 e^{-2mQ}\,dA = 1 + \mathrm{O}(m^{-\kappa-\frac13}),$$

as required by property (ii) of Definition 3.2.2.

We turn to property (i) of Definition 3.2.2, the approximate orthogonality property. For a polynomial $p \in \mathrm{Pol}_n$ of degree at most $n-1$, we put $g = \mathbf{\Lambda}_{m,n}^{-1}[p]$ and note that $g(\infty) = 0$. For all large enough n and m with $\tau = \frac{n}{m} \in I_{\epsilon_0}$, the function $f_{m,n}^{\langle\kappa\rangle}$ is zero-free in a neighborhood of the extended exterior disk $\bar{\mathbb{D}}_e \cup \{\infty\}$, which we may assume to be a fixed exterior disk $\mathbb{D}_{\mathrm{e}}(0,\rho_0)\cup\{\infty\}$ for some fixed $\rho_0 < 1$. By the isometric property of $\mathbf{\Lambda}_{m,n}$, we find that

$$\begin{aligned}(3.5.5)\quad \int_{\mathbb{C}} \chi_{\tau,0}\, p\, \overline{F_{m,n}^{\langle\kappa\rangle}} e^{-2mQ}\,dA = m^{\frac14}\int_{\mathbb{C}} \chi_{\tau,1}\, g\, \overline{f_{m,n}^{\langle\kappa\rangle}} e^{-2mR_\tau}\,dA(z) \\ = m^{\frac14}\int_{\mathcal{D}_{m,n}} \frac{g}{f_{m,n}^{\langle\kappa\rangle}} |f_{m,n}^{\langle\kappa\rangle}|^2 e^{-2mR_\tau}\,dA + \mathrm{O}(m^{-\frac{\alpha_0}{2}\log m+\frac34}\|p\|_{2mQ}),\end{aligned}$$

where we are required to justify the indicated error term estimate. To do this, we need Proposition 2.2.2, or more accurately, Lemma 3.5 in [2], which gives the estimate for $p \in \mathrm{Pol}_n$

$$(3.5.6)\qquad |p| \le C_1 m^{\frac12}\|p\|_{2mQ}\, e^{m\hat{Q}_\tau}$$

in the whole plane $\mathbb{C}$ for some constant C_1, independent of $\tau = \frac{n}{m} \in I_{\epsilon_0}$. The missing term on the right-hand side of (3.5.5) equals

$$m^{\frac14}\int_{\mathbb{C}\setminus\mathcal{D}_{m,n}} \chi_{\tau,1}\, g\, \overline{f_{m,n}^{\langle\kappa\rangle}} e^{-2mR_\tau}\,dA = \int_{\mathbb{C}\setminus\phi_\tau^{-1}(\mathcal{D}_{m,n})} \chi_{\tau,0}\, p\, \overline{F_{m,n}^{\langle\kappa\rangle}} e^{-2mQ}\,dA,$$

and if we apply the pointwise estimate (3.5.6), we obtain

$$\begin{aligned}\int_{\mathbb{C}\setminus\phi_\tau^{-1}(\mathcal{D}_{m,n})} &\chi_{\tau,0}|p\, F_{m,n}^{\langle\kappa\rangle}|\, e^{-2mQ}\, dA \\ &\le C_1 m^{\frac12}\|p\|_{2mQ}\int_{\mathbb{C}\setminus\phi_\tau^{-1}(\mathcal{D}_{m,n})}\chi_{\tau,0}|F_{m,n}^{\langle\kappa\rangle}|\, e^{-2mQ+m\hat{Q}_\tau}\, dA \\ &= C_1 m^{\frac34}\|p\|_{2mQ}\int_{\mathbb{C}\setminus\mathcal{D}_{m,n}}\chi_{\tau,1}|f_{m,n}^{\langle\kappa\rangle}|\, e^{m(\hat{Q}_\tau-Q)\circ\phi_\tau^{-1}-mR_\tau}\, dA \\ &\le C_0C_1 m^{\frac34}\|p\|_{2mQ}\int_{\mathbb{D}_{\mathrm{e}}(0,\rho_0)\setminus\mathcal{D}_{m,n}} e^{-mR_\tau}\, dA,\end{aligned}$$

where in the last step, we applied the estimate (3.5.3) and the fact that $\hat{Q}_\tau \le Q$. The rest of the argument that gives (3.5.5) involves splitting the domain of integration using the set $\mathcal{D}_\tau$, and proceeds as in (3.5.4). This establishes (3.5.5), although we still need to control the main term on the right-hand side. To this end, we denote by h the ratio $h = g/f_{m,n}^{\langle\kappa\rangle}$. In view of the stated properties of $f_{m,n}^{\langle\kappa\rangle}$ and g, the function h is holomorphic in the exterior disk $\mathbb{D}_{\mathrm{e}}(0,\rho_0)$ and vanishes at infinity. Using the foliation flow as coordinates on $\mathcal{D}_{m,n}$ in terms of $(t,\zeta) \in [-\delta_m, \delta_m]\times\mathbb{T}$, we find from Lemma 3.4.1 that

$$\begin{aligned}m^{\frac14}\int_{\mathcal{D}_{m,n}} & h(z)|f_{m,n}^{\langle\kappa\rangle}(z)|^2 e^{-2mR_\tau(z)}\, dA(z) \\ &= 2m^{\frac14}\int_{-\delta_m}^{\delta_m}\int_{\mathbb{T}} h\circ\psi_{m,n,t}(\zeta)\big|f_{m,n}^{\langle\kappa\rangle}\circ\psi_{m,n,t}(\zeta)\big|^2 e^{-2m\, R_\tau\circ\psi_{m,n,t}(\zeta)} \\ &\qquad\times \operatorname{Re}\big(-\bar\zeta\partial_t\psi_{m,n,t}(\zeta)\overline{\psi'_{m,n,t}(\zeta)}\big)\, ds(\zeta)\, dt \\ &= 2m^{\frac14}\int_{-\delta_m}^{\delta_m}\int_{\mathbb{T}} h\circ\psi_{m,n,t}(\zeta)\Big((4\pi)^{-\frac12}e^{-mt^2} + \mathrm{O}\big(m^{-\kappa-\frac13}e^{-mt^2}\big)\Big)\, ds(\zeta)\, dt \\ &= \mathrm{O}\bigg(m^{-\kappa-\frac{1}{12}}\int_{-\delta_m}^{\delta_m}\int_{\mathbb{T}}|h\circ\psi_{m,n,t}(\zeta)|\, ds(\zeta)\, e^{-mt^2}\, dt\bigg).\end{aligned}\tag{3.5.7}$$

Here, the crucial reduction in the last step of (3.5.7) is based on the fact that the function $h\circ\psi_{m,n,t}$ is holomorphic in $\bar{\mathbb{D}}_{\mathrm{e}}$ and vanishes at infinity, so that by the mean value property

$$\int_{\mathbb{T}} h\circ\psi_{m,n,t}\, ds = 0.$$

Now that (3.5.7) is established, we need to simplify the error term further. We will use the observation that all the steps before the last in (3.5.7) apply to a fairly general sufficiently integrable function in place of h, for instance $|h|$ will work. It then follows from (3.5.7) with $|h|$ instead that large enough m, we have

$$\begin{aligned}\int_{-\delta_m}^{\delta_m}\int_{\mathbb{T}}|h\circ\psi_{m,n,t}(\zeta)|\, e^{-mt^2}\, ds(\zeta)\, dt &\le 2\int_{\mathcal{D}_{m,n}}|h(z)|\,|f_{m,n}^{\langle\kappa\rangle}(z)|^2 e^{-2mR_\tau(z)}\, dA(z) \\ &= 2\int_{\mathcal{D}_{m,n}}|g(z)\, f_{m,n}^{\langle\kappa\rangle}(z)|e^{-2mR_\tau(z)}\, dA(z) \le 2C_0\int_{\mathcal{D}_{m,n}}|g(z)|\, e^{-2mR_\tau(z)}\, dA(z),\end{aligned}$$

where in the last step we applied the bound (3.5.3). Finally, we apply the Cauchy-Schwarz inequality, and recall that recall that $g = \mathbf{\Lambda}_{m,n}^{-1}[p]$ where $\mathbf{\Lambda}_{m,n}$ has the isometry property of Proposition 3.3.3:

$$\int_{-\delta_m}^{\delta_m}\int_{\mathbb{T}}|h\circ\psi_{m,n,t}(\zeta)|\,e^{-mt^2}\,ds(\zeta)dt\le 2C_0\int_{\mathcal{D}_{m,n}}|g(z)|\,e^{-2mR_\tau(z)}\,dA(z)$$
$$\le 2C_0\|g\|_{L^2(\mathcal{D}_{m,n},e^{-2mR_\tau})}\bigg(\int_{\mathcal{D}_{m,n}}e^{-2mR_\tau}\,dA\bigg)^{1/2}=\mathrm{O}\big(m^{-\frac14}\|p\|_{2mQ}\big). \tag{3.5.8}$$

Here, we used a simple decay estimate of the integral of the *Gaussian ridge* e^{-2mR_τ}. Next, we write $g/f^{\langle\kappa\rangle}_{m,n}$ in place of h, and combine the estimates (3.5.7) and (3.5.8), and arrive at

$$m^{\frac14}\int_{\mathcal{D}_{m,n}}g\,\overline{f^{\langle\kappa\rangle}_{m,n}}\,e^{-2mR_\tau(z)}\,dA(z)=m^{\frac14}\int_{\mathcal{D}_{m,n}}h(z)|f^{\langle\kappa\rangle}_{m,n}(z)|^2e^{-2mR_\tau(z)}\,dA(z)$$
$$=\mathrm{O}\big(m^{-\kappa-\frac13}\|p\|_{2mQ}\big). \tag{3.5.9}$$

In view of (3.5.5) and (3.5.9), we find that for all polynomials $p\in\mathrm{Pol}_n$,

$$\int_{\mathbb{C}}\chi_{\tau,0}\,p\,\overline{F^{\langle\kappa\rangle}_{m,n}}\,e^{-2mQ}\,dA=\mathrm{O}\big(m^{-\kappa-\frac13}\|p\|_{2mQ}\big), \tag{3.5.10}$$

as required. Since in addition, $f^{\langle\kappa\rangle}_{m,n}(\infty)>0$, while $\mathcal{Q}_\tau(\infty)\in\mathbb{R}$ and $\phi'_\tau(\infty)>0$ hold, the leading coefficient of the quasipolynomial $F^{\langle\kappa\rangle}_{m,n}$ is now positive, which settles property (iii) of Definition 3.2.2 as well. This completes the proof. □

3.6. **Polynomialization of quasipolynomials and proof of Theorem 3.1.2.** We have applied Lemma 3.4.1 to obtain the existence of quasipolynomials $F^{\langle\kappa\rangle}_{m,n}$, of degree n and accuracy κ with an asymptotic expansion, and shown that they are approximately orthogonal and normalized. To obtain the full L^2-expansion, it remains to show that they are indeed good approximations of the true normalized orthogonal polynomials $P_{m,n}$.

Proof of Theorem 3.1.2. We retain the above notation, and consider the $\bar\partial$-problem

$$\bar\partial_z u(z)=F^{\langle\kappa\rangle}_{m,n}(z)\bar\partial_z\chi_{\tau,0}(z).$$

In view of Proposition 2.4.1, the $L^2_{2mQ,n}$-norm minimal solution u_0, which then has the growth $u_0(z)=\mathrm{O}(|z|^{n-1})$ near infinity, enjoys the norm bound

$$\int_{\mathbb{C}}|u_0|^2e^{-2mQ}\,dA\le\frac{1}{\alpha_1 m}\int_{\mathcal{S}_\tau}|F^{\langle\kappa\rangle}_{m,n}|^2|\bar\partial\chi_{\tau,0}|^2e^{-2mQ}\,dA, \tag{3.6.1}$$

where $\alpha_1>0$ stands for the minimum of $\varDelta Q$ on the biggest droplet $\mathcal{S}_\tau$ with $\tau\in I_{\epsilon_0}$ (which is attained for the rightmost endpoint $\tau=1+\epsilon_0$). Next, given that the quasipolynomials of degree n are of the form $F^{\langle\kappa\rangle}_{m,n}=m^{\frac14}\mathbf{\Lambda}_{m,n}[f^{\langle\kappa\rangle}_{m,n}]$, where the functions $f^{\langle\kappa\rangle}_{m,n}$ are uniformly bounded in $\mathbb{D}_{\mathrm{e}}(0,\rho_0)$ for some radius $\rho_0<1$, we find that

$$\int_{\mathcal{S}_\tau}|F^{\langle\kappa\rangle}_{m,n}|^2|\bar\partial\chi_{\tau,0}|^2e^{-2mQ}\,dA=m^{\frac12}\int_{\mathbb{D}}|f^{\langle\kappa\rangle}_{m,n}|^2|\bar\partial\chi_{\tau,1}|^2|\phi'_\tau\circ\phi_\tau^{-1}|^2e^{-2mR_\tau}\,dA$$
$$=\mathrm{O}(m^{\frac12}e^{-\alpha_2 m}) \tag{3.6.2}$$

for some $\alpha_2>0$ such that $2R_\tau\ge\alpha_2$ on the support of $\bar\partial\chi_{\tau,1}$. This exponential decay estimate is possible since the support of $\bar\partial\chi_{\tau,1}$ is located inside $\mathbb{D}$ away from the boundary. Note that in the context of the estimate (3.6.2) it is important as well that the expression

$|\phi_\tau' \circ \phi_\tau^{-1}|^2$ is uniformly bounded on the support of $\bar\partial\chi_{\tau,1}$ as well. If we combine the above estimates (3.6.1) and (3.6.2), we find that

$$\int_{\mathbb{C}} |u_0|^2 e^{-2mQ} dA = \mathrm{O}(m^{-\frac{1}{2}} e^{-\alpha_2 m}), \tag{3.6.3}$$

as $m \to \infty$ while $\tau = \frac{n}{m} \in I_{\epsilon_0}$, with a uniform implicit constant. Next, we put

$$P_{m,n}^\star := F_{m,n}^{\langle\kappa\rangle}\chi_{\tau,0} - u_0$$

which is then automatically a polynomial of degree n, since the function is entire and has growth $|P_{m,n}^\star(z)| \asymp |z|^n$ near infinity. Moreover, in view of (3.6.3), this polynomial is very close to the function $F_{m,n}^{\langle\kappa\rangle}\chi_{\tau,0}$ in the norm of $L^2(\mathbb{C}, e^{-2mQ})$:

$$\int_{\mathbb{C}} |P_{m,n}^\star - F_{m,n}^{\langle\kappa\rangle}\chi_{\tau,0}|^2 e^{-2mQ} dA = \int_{\mathbb{C}} |u_0|^2 e^{-2mQ} dA = \mathrm{O}(m^{-\frac{1}{2}} e^{-\alpha_2 m}). \tag{3.6.4}$$

It now follows from (3.5.10) and (3.6.4) that for all polynomials $p \in \mathrm{Pol}_n$ of degree at most $n-1$, we have that

$$\int_{\mathbb{C}} p\, \bar P_{m,n}^\star\, e^{-2mQ} dA = \mathrm{O}(m^{-\kappa-\frac{1}{3}} \|p\|_{2mQ}), \tag{3.6.5}$$

while

$$\int_{\mathbb{C}} |P_{m,n}^\star|^2 e^{-2mQ} dA = 1 + \mathrm{O}(m^{-\kappa-\frac{1}{3}}). \tag{3.6.6}$$

We observe that by duality, (3.6.5) asserts that

$$\|\mathbf{P}_{m,n} P_{m,n}^\star\|_{2mQ} = \mathrm{O}(m^{-\kappa-\frac{1}{3}}), \tag{3.6.7}$$

where $\mathbf{P}_{m,n}$ denotes the orthogonal projection in $L^2(\mathbb{C}, e^{-2mQ})$ onto the subspace Pol_n of polynomials of degree at most $n-1$. If we use this to correct the polynomial $P_{m,n}^\star$, and put $\tilde P_{m,n} := \mathbf{P}_{m,n}^\perp P_{m,n}^\star = P_{m,n}^\star - \mathbf{P}_{m,n} P_{m,n}^\star$, then automatically $\tilde P_{m,n}$ has degree n and it is also orthogonal to all the lower degree polynomials. As a consequence, $\tilde P_{m,n}$ must be a scalar multiple of $P_{m,n}$, the orthogonal polynomial we are looking for, which we write as $\tilde P_{m,n} = cP_{m,n}$ for a constant c. Putting things together so far, we have obtained that

$$\big\|\tilde P_{m,n} - F_{m,n}^{\langle\kappa\rangle}\chi_{\tau,0}\big\|_{2mQ} = \mathrm{O}(m^{-\kappa-\frac{1}{3}}) \tag{3.6.8}$$

with a uniform implied constant. Moreover, by (3.6.6) and (3.6.7), the norm of $\tilde P_{m,n}$ equals

$$|c| = \|cP_{m,n}\|_{2mQ} = \big\|\tilde P_{m,n}\big\|_{2mQ} = 1 + \mathrm{O}(m^{-\kappa-\frac{1}{3}}), \tag{3.6.9}$$

Next, by our version of the Bernstein-Walsh lemma (Proposition 2.2.2), it follows from (3.6.8) that

$$\big|cP_{m,n} - F_{m,n}^{\langle\kappa\rangle}\big| = \big|\tilde P_{m,n} - F_{m,n}^{\langle\kappa\rangle}\big| = \mathrm{O}(m^{-\kappa+\frac{1}{6}} e^{m\hat Q_\tau})$$

holds in $\mathcal{S}_\tau^c$, which after division by $F_{m,n}^{\langle\kappa\rangle}$ gives that

$$\left| c\frac{P_{m,n}}{F_{m,n}^{\langle\kappa\rangle}} - 1\right| = \mathrm{O}(m^{-\kappa-\frac{1}{12}}), \tag{3.6.10}$$

since $f_{m,n}^{\langle\kappa\rangle}$ is uniformly bounded away from zero. Next, we let $|z| \to +\infty$ and observe that both the functions $F_{m,n}^{\langle\kappa\rangle}$ and $P_{m,n}$ have positive leading coefficients, whose quotient is denoted by $\gamma_{m,n}$. Since $\gamma_{m,n} > 0$ we obtain from (3.6.10) that

$$\frac{|\operatorname{Im} c|}{|c|} \le \big|c\gamma_{m,n} - 1\big| = \mathrm{O}(m^{-\kappa-\frac{1}{12}}),$$

where the left-hand side inequality is elementary. Moreover, we can also realize from the above that $\mathrm{Re}(c) > 0$. But then it follows from (3.6.9) that

$$c = 1 + \mathrm{O}(m^{-\kappa-\frac{1}{12}}).$$

It now follows from this observation combined with (3.6.8) that

$$\|P_{m,n} - \chi_{\tau,0} F_{m,n}^{\langle\kappa\rangle}\|_{2mQ} = \mathrm{O}(m^{-\kappa-\frac{1}{12}}).$$

This falls slightly short of allowing us to obtain Theorem 3.1.2 right away. The problem is that our error term is larger than what is claimed. However, since the precision κ is arbitrary, we might as well replace κ by $\kappa+1$ and see what we get. This would give that

$$\|P_{m,n} - \chi_{\tau,0} F_{m,n}^{\langle\kappa+1\rangle}\|_{2mQ} = \mathrm{O}(m^{-\kappa-1-\frac{1}{12}}). \tag{3.6.11}$$

By analyzing the last term in the asymptotic expansion, it is easy to verify that

$$\|\chi_{\tau,0} F_{m,n}^{\langle\kappa+1\rangle} - \chi_{\tau,0} F_{m,n}^{\langle\kappa\rangle}\|_{2mQ} = \mathrm{O}(m^{-\kappa-1}),$$

and hence the assertion of the theorem immediate from this estimate and (3.6.11). □

3.7. **Proof of the main theorem.** We are now ready to obtain the pointwise asymptotic expansion of the orthogonal polynomials. We still work under the assumption that Lemma 3.4.1 holds.

Proof of Theorem 1.5.2. The quasipolynomials $F_{m,n}^{\langle\kappa\rangle}$ obtained in Theorem 3.1.2 may be written in the form

$$F_{m,n}^{\langle\kappa\rangle} = m^{\frac{1}{4}} \sqrt{\phi_\tau'}[\phi_\tau]^n e^{m\mathcal{Q}_\tau} \sum_{j=0}^{\kappa} m^{-j} \mathcal{B}_{\tau,j},$$

where $\mathcal{B}_{\tau,j} = [\phi_\tau']^{\frac{1}{2}} B_{\tau,j} \circ \phi_\tau$ are uniformly bounded, and holomorphic in the exterior domain $\mathcal{K}_\tau^c$. To obtain the theorem, we need to show that $F_{m,n}^{\langle\kappa\rangle}$ is close to $P_{m,n}$ pointwise in the complement of the set

$$\mathcal{K}_{\tau,A,m} = \left\{z \in \mathbb{C} : \mathrm{dist}_{\mathbb{C}}(z, \mathcal{S}_\tau^c) \ge A(m^{-1}\log m)^{\frac{1}{2}}\right\}. \tag{3.7.1}$$

On the complement $\mathcal{K}_{\tau,A,m}^c$ we have the estimate

$$0 \le m(\hat{Q}_\tau - \check{Q}_\tau)(z) \le D \log m,$$

and hence

$$e^{m(\hat{Q}_\tau - \check{Q}_\tau)} \le e^{D\log m} = m^D$$

where D is some positive constant, which is uniformly bounded while $\tau \in I_{\epsilon_0}$. To see this, a simple Taylor expansion of the difference $\hat{Q}_\tau - \check{Q}_\tau$ in the interior direction suffices. In view of Theorem 3.1.2, and the pointwise estimate of Proposition 2.2.2 applied to the intermediate set $\mathcal{X}_\tau$ between $\mathcal{K}_\tau$ and $\mathcal{S}_\tau^c$ where the cut-off function $\chi_{\tau,0}$ assumes the value 1, we find that

$$|P_{m,n}(z) - F_{m,n}^{\langle\kappa\rangle}(z)| = \mathrm{O}\big(m^{-\kappa-\frac{1}{2}} e^{m\hat{Q}_\tau(z)}\big) = \mathrm{O}\big(m^{-\kappa-\frac{1}{2}+D} e^{m\check{Q}_\tau(z)}\big), \qquad z \in \mathcal{K}_{\tau,A,m}^c,$$

where the implicit constant again is uniform in the relevant parameter range. We may rephrase this as saying that

$$P_{m,n}(z) = F^{\langle\kappa\rangle}_{m,n}(z) + \mathrm{O}\big(m^{-\kappa-\frac{1}{2}+D}\, e^{m\breve{Q}_\tau(z)}\big)$$
$$= m^{\frac14}\sqrt{\phi_\tau'}[\phi_\tau]^n e^{m\mathcal{Q}_\tau}\Big(\sum_{j=0}^{\kappa}\mathcal{B}_{\tau,j} + \mathrm{O}\big(m^{-\kappa-\frac34+D}\big)\Big),$$

for $z\in\mathcal{K}^c_{\tau,A,m}$. This essentially proves the theorem, except that the error term is now worse than claimed. However, we may fix this by replacing κ by $\kappa' := \kappa + \lceil D\rceil + 1$ in the above argument, to obtain on $\mathcal{K}^c_{\tau,A,m}$ that

$$P_{m,n}(z) = m^{\frac14}\sqrt{\phi_\tau'}[\phi_\tau]^n e^{m\mathcal{Q}_\tau}\Big(\sum_{j=0}^{\kappa'} m^{-j}\mathcal{B}_{\tau,j} + \mathrm{O}(m^{-\kappa-\frac74})\Big)$$
$$= m^{\frac14}\sqrt{\phi_\tau'}[\phi_\tau]^n e^{m\mathcal{Q}_\tau}\Big(\sum_{j=0}^{\kappa} m^{-j}\mathcal{B}_{\tau,j} + \mathrm{O}(m^{-\kappa-1})\Big),$$

where the last step follows since the functions $m^{-j}\mathcal{B}_{\tau,j}$ are all $\mathrm{O}(m^{-\kappa-1})$ for j in the range $\kappa+1\le j\le\kappa'$. The proof is complete. □

4. Algorithmic determination of the coefficient functions

4.1. **Implementation of the radial Laplace method.** We turn to the algorithm of Theorem 1.5.4. To proceed, we need two families of differential operators. We recall the differential operators $\mathbf{L}_k$ defined in (1.5.3) appearing in the application of Laplace's method in Proposition 2.6.1. We need to apply these operators to functions defined in a neighborhood of the unit circle, and we apply them in the radial direction. So, for functions $f(re^{i\theta})$, we put

$$\mathbf{L}_k[f](re^{i\theta}) = \sum_{\nu=k}^{3k}\frac{(-1)^{\nu-k}2^{-\nu}}{\nu!(\nu-k)![\partial_r^2 R_\tau(re^{i\theta})]^\nu}\partial_r^{2\nu}\left(\big[W_\tau(re^{i\theta})\big]^{\nu-k} f(re^{i\theta})\right),$$

where

$$W_\tau(re^{i\theta}) = R_\tau(re^{i\theta}) - \frac12(r-1)^2\partial_x^2 R_\tau(xe^{i\theta})\Big|_{x=1}.$$

The second family of operators is defined implicitly in the following lemma, which turns explicit appearances of the parameter l into differential operators.

Lemma 4.1.1. *Let k be a nonnegative integer. Then there exist partial differential operators $\mathbf{M}_k$ of order $2k$ with real-analytic coefficients, such that for any integer $l\ge 0$ and any function smooth function f defined in a neighborhood of $\mathbb{T}$, we have that*

$$\int_{\mathbb{T}} e^{il\theta}\big(\partial_r^2 R_\tau(re^{i\theta})\big)^{-\frac12}\mathbf{L}_k[r^{1-l}f(re^{i\theta})]\Big|_{r=1} d\theta = \int_{\mathbb{T}} e^{il\theta}\mathbf{M}_k[f](e^{i\theta})\,d\theta.$$

Proof. We first observe that by integration by parts, multiplication by l corresponds to applying the differential operator $i\partial_\theta$ inside the integral:

$$l\int_{\mathbb{T}} f(\theta)e^{il\theta}d\theta = \int_{\mathbb{T}} i\partial_\theta f(\theta)e^{il\theta}d\theta.$$

From this it is immediate that the formula

$$p(l)\int_{\mathbb{T}} f(\theta)e^{il\theta}d\theta = \int_{\mathbb{T}} p(i\partial_\theta)f(\theta)e^{il\theta}d\theta \tag{4.1.1}$$

holds for polynomials p. Structurally, $\mathbf{L}_k[r^{1-l}f(re^{i\theta})]$ can be written as

$$\mathbf{L}_k[r^{1-l}f(re^{i\theta})] = \sum_{\nu=k}^{3k} b_\nu(re^{i\theta})\partial_r^{2\nu}\big[[W_\tau(re^{i\theta})]^{\nu-k}r^{1-l}f(re^{i\theta})\big], \tag{4.1.2}$$

where b_ν is the real-analytic function given by

$$b_\nu(re^{i\theta}) = \frac{(-1)^{\nu-k}2^{-\nu}}{\nu!(\nu-k)![\partial_r^2 R_\tau(re^{i\theta})]^\nu}.$$

We observe that by the Leibniz rule

$$\begin{aligned}\partial_r^j(r^{1-l}f(re^{i\theta}))\Big|_{r=1} &= \sum_{i=0}^{j}\binom{j}{i}(-1)^{j-i}(l-1)_{j-i}r^{1-l-j+i}\partial_r^i f(re^{i\theta})\Big|_{r=1} \\ &= \sum_{i=0}^{j}\binom{j}{i}(-1)^{j-i}(l-1)_{j-i}\partial_r^i f(re^{i\theta})\Big|_{r=1},\end{aligned} \tag{4.1.3}$$

where $(x)_i = x(x+1)\cdots(x+i-1)$ denotes the standard Pochhammer symbol. We return to the formula (4.1.2) for $\mathbf{L}_k$. Again by the Leibniz formula we have that

$$\begin{aligned}&\partial_r^{2\nu}[W_\tau^{\nu-k}(e^{i\theta})r^{1-l}f(re^{i\theta})]\Big|_{r=1} \\ &\quad= \sum_{j=0}^{2\nu}\binom{2\nu}{j}\partial_r^{2\nu-j}\Big([W_\tau(re^{i\theta})]^{\nu-k}\Big)\partial_r^j\Big(r^{1-l}f(re^{i\theta})\Big)\Big|_{r=1} \\ &\quad= \sum_{j=0}^{3k-\nu}\binom{2\nu}{j}\partial_r^{2\nu-j}\Big([W_\tau(re^{i\theta})]^{\nu-k}\Big)\partial_r^j\Big(r^{1-l}f(re^{i\theta})\Big)\Big|_{r=1} \\ &\quad= \sum_{j=0}^{3k-\nu}\sum_{i=0}^{j}(-1)^{j-i}\binom{2\nu}{j}\binom{j}{i}(l-1)_{j-i}\partial_r^{2\nu-j}\Big([W_\tau(re^{i\theta})]^{\nu-k}\Big)\partial_r^i f(re^{i\theta})\Big|_{r=1},\end{aligned}$$

where the truncation of the sum follows from an application of the flatness of W_τ near the unit circle $\mathbb{T}$, and the last equality is due to (4.1.3). We write the expression for $\mathbf{L}_k[r^{1-l}f(re^{i\theta})]$ as

$$\mathbf{L}_k[r^{1-l}f(re^{i\theta})]\Big|_{r=1} = \sum_{\nu=k}^{3k}\sum_{j=0}^{3k-\nu}\sum_{i=0}^{j}(l-1)_{j-1}c_{i,j,\nu}(e^{i\theta})\partial_r^i f(re^{i\theta})\Big|_{r=1},$$

where

$$c_{i,j,\nu}(e^{i\theta}) = (-1)^{j-i}\binom{2\nu}{j}\binom{j}{i}(l-1)_{j-i}b_\nu(e^{i\theta})\partial_r^{2\nu-j}\Big([W_\tau(re^{i\theta})]^{\nu-k}\Big)\Big|_{r=1}.$$

Changing the order of summation, we arrive at

$$\begin{aligned}&\Big(\partial_r^2 R_\tau(re^{i\theta})\Big)^{-\frac12}\mathbf{L}_k[r^{1-l}f(re^{i\theta})]\Big|_{r=1} \\ &\quad= \sum_{i=0}^{2k}\sum_{j=i}^{2k}(-1)^{j-i}\binom{j}{i}(l-1)_{i-j}\Big(\partial_r^2 R_\tau(re^{i\theta})\Big)^{-\frac12}d_j(e^{i\theta})\partial_r^i f(re^{i\theta})\Big|_{r=1},\end{aligned}$$

where

$$d_j(e^{i\theta}) = \sum_{\nu=k}^{3k-j}\binom{2\nu}{j}b_\nu(e^{i\theta})\partial_r^{2\nu-j}\Big([W_\tau(re^{i\theta})]^{\nu-k}\Big)\Big|_{r=1}.$$

It follows from (4.1.1) that the asserted identity holds with $\mathbf{M}_k$ given by

$$\mathbf{M}_k[f](e^{i\theta}) = \sum_{i=0}^{2k}\sum_{j=i}^{2k}(-1)^{j-i}\binom{j}{i}(i\partial_\theta - 1)_{i-j}\Big[(\partial_r^2 R_\tau(re^{i\theta}))^{-\frac{1}{2}} d_j(e^{i\theta})\partial_r^i f(re^{i\theta})\Big]\Big|_{r=1}.$$

The proof of the lemma is complete. □

4.2. **Algorithmic computation of the coefficients in the asymptotic expansion.** In this section we supply the proof of Theorem 1.5.4, and explain the underlying computational algorithm. The main point is that we show how to iteratively obtain the coefficients, given that an asymptotic expansion exists, as formulated in Theorem 3.1.2.

Proof of Theorem 1.5.4. Fix the precision κ to be a positive integer. Let $F_{m,n}^{\langle\kappa\rangle}$ be the approximate orthogonal quasipolynomials from Theorem 3.1.2 with the expansion

$$F_{m,n}^{\langle\kappa\rangle}(z) = m^{\frac{1}{4}}\sqrt{\phi_\tau'(z)}[\phi_\tau(z)]^n e^{m\mathcal{Q}_\tau(z)}\sum_{j=0}^{\kappa} m^{-j}\mathcal{B}_{\tau,j}(z),$$

where the functions $\mathcal{B}_{\tau,j}$ are bounded and holomorphic on $\mathcal{K}_\tau^c$ for some compact subset $\mathcal{K}_\tau$ of $\mathcal{S}_\tau^\circ$, which we may assume to be the conformal image of the exterior disk $\mathbb{D}_e(0,\rho_0)$ under the mapping ϕ_τ^{-1}. If we make the ansatz

$$\mathcal{B}_{\tau,j}(z) = \sqrt{\phi_\tau'(z)}(B_{\tau,j}\circ\phi_\tau)(z),$$

we may express $F_{m,n}$ using the canonical positioning operator $F_{m,n}^{\langle\kappa\rangle} = m^{\frac{1}{4}}\mathbf{\Lambda}_{m,n}[f_{m,n}^{\langle\kappa\rangle}]$, where

$$f_{m,n}^{\langle\kappa\rangle}(z) = \sum_{j=0}^{\kappa} m^{-j}B_{\tau,j}(z), \qquad z\in\mathbb{D}_e(0,\rho_0). \tag{4.2.1}$$

According to Theorem 3.1.2, the functions $F_{m,n}^{\langle\kappa\rangle}$ have the approximate orthogonality property

$$\int_{\mathbb{C}} \chi_{\tau,0}F_{m,n}^{\langle\kappa\rangle}\bar{p}\, e^{-2mQ}\, dA = \mathrm{O}(m^{-\kappa-1}\|p\|_{2mQ}), \qquad p\in\mathrm{Pol}_n. \tag{4.2.2}$$

The function $\chi_{\tau,0}$ is a cut-off function with $0\le\chi\le 1$ throughout $\mathbb{C}$, such that $\chi_{\tau,0}$ vanishes on $\mathcal{K}_\tau$ and equals 1 on $\mathcal{X}_\tau^c$, where $\mathcal{K}_\tau$ lies at a fixed positive distance from $\partial\mathcal{S}_\tau$, and $\mathcal{X}_\tau$ is an intermediate set between them (cf. Definition 3.1.1). We consider the associated cut-off function $\chi_{\tau,1} = \chi_{\tau,0}\circ\phi_\tau^{-1}$, tacitly extended to vanish where it is undefined. Without loss of generality, we may assume that $\chi_{\tau,1}$ is radial. By Remark 3.1.3, we may assume that $\chi_{\tau,1}$ vanishes on $\mathbb{D}(0,\rho_0')$ for some number ρ_0' with $\rho_0<\rho_0'<1$. In order to compute the functions $B_{\tau,j}$, we would like to apply equation (4.2.2) to

$$q(z) = \mathbf{\Lambda}_{m,n}[z^{-l}] = \phi_\tau'(z)[\phi_\tau(z)]^{n-l}e^{m\mathcal{Q}_\tau(z)}$$

for a positive integer l, but this function is unfortunately not a polynomial. To fix this, we consider the $L^2_{2mQ,n}$-minimal solution v to the $\bar\partial$-problem

$$\bar\partial v = \bar\partial(\chi_{\tau,0}q) = q\,\bar\partial\chi_{\tau,0}.$$

If v is the solution, then the difference $\chi_{\tau,0}q - v$ will be an entire function with the polynomial growth bound $\mathrm{O}(|z|^{n-1})$ at infinity, and hence a polynomial of degree less than or equal to $n-1$. By the estimate of Proposition 2.4.1, we have the norm control

$$\int_{\mathbb{C}}|v|^2 e^{-2mQ}\, dA \le \frac{1}{2m}\int_{\mathbb{C}}|q|^2|\bar\partial\chi_{\tau,0}|^2\frac{e^{-2mQ}}{\Delta Q}\, dA \le \frac{A^2}{2m\alpha_1}\int_{\mathcal{X}_\tau\setminus\mathcal{K}_\tau}|q|^2 e^{-2mQ}\, dA,$$

where we have used that there exists a positive real α_1 such that $\Delta Q \geq \alpha_1$ holds on $\mathcal{S}_\tau$, which contains the support of $\bar\partial\chi_{\tau,0}$, and that we have the bound $|\bar\partial\chi_{\tau,0}| \leq A$. Since the support of $\bar\partial\chi_{\tau,0}$ lies in $\mathcal{K}_\tau^c$, we may use the structure of q as $q = \mathbf{\Lambda}_{m,n}[z^{-l}]$ and Proposition 3.3.3

$$\int_{\mathcal{X}_\tau\setminus\mathcal{K}_\tau} |q|^2 e^{-2mQ}dA = \int_{\rho_0\leq|z|\leq\rho_0''}|z|^{-2l}e^{-2mR_\tau(z)}dA(z),$$

where ρ_0'' is associated with a natural choice of the intermediate set $\mathcal{X}_\tau$ as the image of an exterior disk under ϕ_τ^{-1}, and satisfies $\rho_0 < \rho_0' < \rho_0'' < 1$. Due to Proposition 3.3.1, this immediately gives that for any fixed positive integer l

$$\int_{\mathbb{C}}|v|^2e^{-2mQ}dA = \mathrm{O}(e^{-\epsilon_1 m})$$

as m, n tend to infinity while $\tau = \frac{n}{m} \in I_{\epsilon_0}$, for some positive real ϵ_1. This means that for a fixed positive integer l, we have for $q = \mathbf{\Lambda}[z^{-l}]$ the approximate orthogonality

$$\int_{\mathbb{C}} \chi_{\tau,0}^2 F_{m,n}^{\langle\kappa\rangle}\bar q\, e^{-2mQ}dA = \mathrm{O}(m^{-\kappa-1}), \tag{4.2.3}$$

where we have used that $\chi_{\tau,0}q - v$ is a polynomial of degree at most $n-1$, and the above smallness of v. If we use the canonical positioning operator as in Proposition 3.3.3 in polarized form, (4.2.3) reads in polar coordinates

$$m^{\frac14}\int_{\mathbb{T}} e^{il\theta}\int_{\rho_0}^{\infty} r^{1-l}\chi_{\tau,1}^2(r) f_{m,n}^{\langle\kappa\rangle}(re^{i\theta})e^{-2mR_\tau(re^{i\theta})}dr\,ds(e^{i\theta}) = \mathrm{O}\left(m^{-\kappa-1}\right), \tag{4.2.4}$$

for fixed l. We now apply Proposition 2.6.2 to the radial integral, with $V(r) = 2R_\tau(re^{i\theta})$. Note that $\partial_r^2 R_\tau(re^{i\theta})|_{r=1} = 4\Delta R_\tau(e^{i\theta})$. As a consequence, the inner integral in (4.2.4) has an expansion

$$\begin{aligned}\int_{\rho_0}^{\infty} & r^{1-l}\chi_{\tau,1}^2(r) f_{m,n}^{\langle\kappa\rangle}(re^{i\theta})e^{-2mR_\tau(re^{i\theta})}dr \\ &= \left(\frac{\pi}{4m\Delta R_\tau(e^{i\theta})}\right)^{\frac12}\sum_{j=0}^{\kappa} m^{-j}\mathbf{L}_j[r^{1-l}f_{m,n}^{\langle\kappa\rangle}(re^{i\theta})]\Big|_{r=1} \\ &+ \mathrm{O}\bigg(m^{-\kappa-1}\big\|r^{1-l}\chi_{\tau,1}^2 f_{m,n,\theta}^{\langle\kappa\rangle}\big\|_{C^{2(\kappa+1)}([\rho_0,\rho_2])} + \big\|r^{1-l}\chi_{\tau,1}^2 f_{m,n,\theta}^{\langle\kappa\rangle}\big\|_{L^\infty([\rho_1,\infty))}\rho_1^{-m\vartheta+1}\bigg),\end{aligned}$$

where we to simplify the notation we use the subscript θ to denote the radial restriction $f_\theta(r) = f(re^{i\theta})$. Here, ϑ, α and ρ_1 are some real numbers with $\vartheta > 0$, $\alpha > 0$ and $1 < \rho_1 < \rho_2$, which are independent of $\tau \in I_{\epsilon_0}$. By applying the standard Cauchy estimates to the functions $f_{m,n}^{\langle\kappa\rangle}$, and by Remark 3.1.3 (both part (a) and (b) are needed) we have uniform control on the norms

$$\big\|r^{1-l}\chi_{\tau,1}^2 f_{m,n,\theta}^{\langle\kappa\rangle}\big\|_{C^{2(\kappa+1)}([\rho_0,\rho_2])} \quad\text{and}\quad \big\|r^{1-l}\chi_{\tau,1}^2 f_{m,n,\theta}^{\langle\kappa\rangle}\big\|_{L^\infty([\rho_1,\infty))}$$

provided that l is fixed, and that $f_{m,n}^{\langle\kappa\rangle}$ are uniformly bounded. For fixed l, it follows that

$$\begin{aligned}\int_{\rho_0}^{\infty} & r^{1-l}\chi_{\tau,1}^2(r) f_{m,n}^{\langle\kappa\rangle}(re^{i\theta})e^{-2mR_\tau(re^{i\theta})}dr \\ &= \left(\frac{\pi}{4m\Delta R_\tau(e^{i\theta})}\right)^{\frac12}\sum_{j=0}^{\kappa} m^{-j}\mathbf{L}_j[r^{1-l}f_{m,n}^{\langle\kappa\rangle}(re^{i\theta})]\Big|_{r=1} + \mathrm{O}\big(m^{-\kappa-1}\big),\end{aligned} \tag{4.2.5}$$

where the implied constant is uniformly bounded as long as $f_{m,n}^{\langle\kappa\rangle}$ is uniformly bounded on $\mathbb{D}_{\mathrm{e}}(0,\rho_0)$. By expanding the expression (4.2.1) for $f_{m,n}^{\langle\kappa\rangle}$, it follows from (4.2.5) that

$$\begin{aligned}(4.2.6)\quad &\int_{\rho_0}^{\infty} r^{1-l}\chi_{\tau,1}^2(r) f_{m,n}^{\langle\kappa\rangle}(re^{i\theta})e^{-2mR_\tau(re^{i\theta})}dr \\ &\quad= \bigg(\frac{\pi}{4m\varDelta R_\tau(e^{i\theta})}\bigg)^{\frac12}\sum_{k=0}^{\kappa} m^{-k}\mathbf{L}_k[r^{1-l}f_{m,n}^{\langle\kappa\rangle}(re^{i\theta})]\Big|_{r=1} + \mathrm{O}(m^{-\kappa-1}) \\ &\quad= \bigg(\frac{\pi}{4m\varDelta R_\tau(e^{i\theta})}\bigg)^{\frac12}\sum_{j=0}^{\kappa} m^{-j}\sum_{k=0}^{j}\mathbf{L}_k[r^{1-l}B_{\tau,j-k}(re^{i\theta})]\Big|_{r=1} + \mathrm{O}(m^{-\kappa-1}),\end{aligned}$$

as $m\to\infty$. We multiply the expression (4.2.6) by $e^{il\theta}$ and integrate with respect to θ to get

$$\begin{aligned}&m^{\frac14}\int_{\mathbb{T}} e^{il\theta}\int_{\rho_0}^{\infty} r^{1-l}\chi_{\tau,1}^2(r) f_{m,n}^{\langle\kappa\rangle}(re^{i\theta})e^{-2mR_\tau(re^{i\theta})}dr\,ds(e^{i\theta}) \\ &= \sum_{j=0}^{\kappa} m^{-j-\frac14}\int_{\mathbb{T}} e^{il\theta}\bigg(\frac{\pi}{4\varDelta R_\tau(e^{i\theta})}\bigg)^{\frac12}\sum_{k=0}^{j}\mathbf{L}_k[r^{1-l}B_{\tau,j-k}(re^{i\theta})]\Big|_{r=1} ds(e^{i\theta}) + \mathrm{O}(m^{-\kappa-\frac34}),\end{aligned}$$

as $m\to\infty$. This is an asymptotic series, and so is (4.2.4), only that all the coefficients vanish in the latter, and only the error term remains. Since two asymptotic series coincide only if they coincide term by term, we find that for integers $j=0,\ldots,\kappa$,

$$\int_{\mathbb{T}} e^{il\theta}\big(4\varDelta R_\tau(e^{i\theta})\big)^{-\frac12}\sum_{k=0}^{j}\mathbf{L}_k[r^{1-l}B_{\tau,j-k}(re^{i\theta})]\Big|_{r=1} ds(e^{i\theta}) = 0, \qquad l=1,2,3,\ldots.$$

This condition looks like the standard condition membership in the Hardy space H^2. The problem with this is that the functions unfortunately depend on the parameter l, so the criterion does not apply. To remedy this, we apply Lemma 4.1.1, which gives

$$(4.2.7)\qquad \int_{\mathbb{T}} e^{il\theta}\sum_{k=0}^{j}\mathbf{M}_k[B_{\tau,j-k}](e^{i\theta})ds(e^{i\theta}) = 0, \qquad l=1,2,3,\ldots,$$

which is now of the desired form. So, by the standard Fourier analytic characterization of the Hardy space, the equation (4.2.7) is equivalent to having

$$(4.2.8)\qquad \sum_{k=0}^{j}\mathbf{M}_k[B_{\tau,j-k}]\Big|_{\mathbb{T}} \in H^2, \qquad j=0,\ldots,\kappa.$$

We look at the case $j=0$ first. Then (4.2.8) says that $\mathbf{M}_0[B_{\tau,0}]\big|_{\mathbb{T}}\in H^2$. The operator $\mathbf{M}_0$, with the defining property given by Lemma 4.1.1, has the form

$$(4.2.9)\qquad \mathbf{M}_0[f](e^{i\theta}) = (4\varDelta R_\tau(e^{i\theta}))^{-\frac12} f(e^{i\theta}).$$

We recall that it is given that $B_{\tau,0}$ is bounded and holomorphic in a neighborhood of the closed exterior disk $\bar{\mathbb{D}}_e$, so that in particular $B_{\tau,0}\big|_{\mathbb{T}}\in H^2_-$. If we combine this with the observation that $\mathbf{M}_0[B_{\tau,0}]\big|_{\mathbb{T}}\in H^2$ together with the explicit expression (4.2.9) for $\mathbf{M}_0$, we arrive at

$$(4.2.10)\qquad B_{\tau,0}\big|_{\mathbb{T}} \in (4\varDelta R_\tau)^{\frac12}H^2\cap H^2_-.$$

Let H_{R_τ} be the bounded holomorphic function in $\mathbb{D}_e$ such that

$$\operatorname{Re} H_{R_\tau} = \frac{1}{2}\log(4\varDelta R_\tau)^{\frac{1}{2}} = \frac{1}{4}\log(4\varDelta R_\tau), \qquad \text{on } \mathbb{T} \tag{4.2.11}$$

with $\operatorname{Im} H_{R_\tau}(\infty) = 0$. It follows from the given regularity of R_τ that H_{R_τ} is a bounded holomorphic function in the exterior disk, which extends holomorphically to a neighborhood of $\bar{\mathbb{D}}_e$. We may rewrite (4.2.10) in the form

$$B_{\tau,0}\Big|_{\mathbb{T}} \in e^{2\operatorname{Re} H_{R_\tau}} H^2 \cap H^2_-.$$

By Proposition 2.5.1 applied with $u = v = -\bar{H}_{R_\tau}$ and $F = 0$, it follows that $B_{\tau,0}$ is of the form

$$B_{\tau,0} = c_{\tau,0}\, e^{H_{R_\tau}} \tag{4.2.12}$$

for some constant $c_{\tau,0}$, which must be positive by our normalization.

We proceed to consider more generally $j = 1, 2, 3, \ldots$. If we separate out the term corresponding to $k = 0$ from equation (4.2.8), we find that

$$\frac{B_{\tau,j}}{(4\varDelta R_\tau)^{\frac{1}{2}}} + \sum_{k=1}^{j} \mathbf{M}_k[B_{\tau,j-k}]\Big|_{\mathbb{T}} \in H^2, \qquad j = 1, \ldots, \kappa. \tag{4.2.13}$$

This equation allows us to compute $B_{\tau,j}$, given that we have already obtained the functions $B_{\tau,0}, \ldots, B_{\tau,j-1}$. Indeed, if we put

$$F_{\tau,j} = \sum_{k=1}^{j} \mathbf{M}_k[B_{\tau,j-k}],$$

which involves only the functions $B_{\tau,0}, \ldots, B_{\tau,j-1}$, we may write (4.2.13) in the form

$$B_{\tau,j}\big|_{\mathbb{T}} \in H^2_- \cap (4\varDelta R_\tau)^{\frac{1}{2}}(-F_{\tau,j} + H^2) = H^2_- \cap e^{2\operatorname{Re} H_{R_\tau}}(-F_{\tau,j} + H^2),$$

which by Proposition 2.5.1 has the solution

$$B_{\tau,j} = c_{\tau,j}\, e^{H_{R_\tau}} - e^{H_{R_\tau}} \mathbf{P}_{H^2_{-,0}}[e^{\bar{H}_{R_\tau}} F_{\tau,j}], \tag{4.2.14}$$

for some constant $c_{\tau,j}$, which have to be real in view of our normalization $f^{\langle\kappa\rangle}_{m,n}(\infty) > 0$. Since $B_{\tau,0}$ is known up to a constant multiple, this allows us to iteratively derive $B_{\tau,j}$ for $j = 1, \ldots, \kappa$. The only remaining freedom is the choice of the constants $c_{\tau,j}$ for $j = 0, \ldots, \kappa$. We proceed to determine them. Since the orthogonal polynomials $P_{m,n}$ are normalized, it follows from Theorem 3.1.2 together with the triangle inequality that

$$\big\|\chi_{\tau,0} F^{\langle\kappa\rangle}_{m,n}\big\|_{2mQ} = 1 + \mathrm{O}(m^{-\kappa-1})$$

as $m \to \infty$. Since $\chi_{\tau,0} F^{\langle\kappa\rangle}_{m,n} = m^{\frac{1}{4}} \mathbf{\Lambda}_{m,n}[\chi_{\tau,1} f^{\langle\kappa\rangle}_{m,n}]$, it follows from the isometric property described in Proposition 3.3.3 that

$$m^{\frac{1}{2}} \int_{\mathbb{C}} \chi^2_{\tau,1} |f^{\langle\kappa\rangle}_{m,n}|^2 e^{-2mR_\tau}\, dA = \int_{\mathbb{C}} \chi^2_{\tau,0} |F^{\langle\kappa\rangle}_{m,n}|^2 e^{-2mQ}\, dA = 1 + \mathrm{O}(m^{-\kappa-1}). \tag{4.2.15}$$

Here, the integrals are over the whole plane, although the isometry is only over the the complements of certain compact subsets. However, since we interpret the products with the cut-off functions as vanishing where the cut-off function vanishes itself, this is of no concern to us. We now expand $f_{m,n}^{\langle\kappa\rangle}$ according to (4.2.1), so that by equation (4.2.15),

$$2m^{\frac{1}{2}}\sum_{j,k=0}^{\kappa} m^{-(j+k)}\int_{\mathbb{T}}\int_{\rho_0}^{\infty}\chi_{\tau,1}^2(r)B_{\tau,j}(re^{i\theta})\bar{B}_{\tau,k}(re^{i\theta})e^{-2mR_\tau(re^{i\theta})}r\,dr\,ds(e^{i\theta}) = 1+\mathrm{O}(m^{-\kappa-1}), \tag{4.2.16}$$

where the factor 2 appears as a result of our normalizations. This equation is what will give us the values of the constants $c_{\tau,j}$. We turn first to the case $j=0$. By a trivial version of Proposition 2.6.2, for any integers j,k with $0\le j,k\le\kappa$ we have the rough estimate

$$\int_{\rho_0}^{\infty}\chi_{\tau,1}^2(r)B_{\tau,j}(re^{i\theta})\bar{B}_{\tau,k}(re^{i\theta})e^{-2mR_\tau(re^{i\theta})}r\,dr\,ds(re^{i\theta}) = \mathrm{O}(m^{-\frac{1}{2}}),$$

where the implicit constant is uniform for $\tau\in I_{\epsilon_0}$. If we disregard all the contributions in (4.2.16) which are of order $\mathrm{O}(m^{-\frac{1}{2}})$, we see that only $j=k=0$ gives a nontrivial contribution. The term corresponding to $j=k=0$ in (4.2.16) can be expanded using the Laplace method of Proposition 2.6.2 (recall the formula (4.2.12) for $B_{\tau,0}$), to give

$$2m^{\frac{1}{2}}\int_{\mathbb{T}}\int_{\rho_0}^{\infty}\chi_{\tau,1}^2(r)|B_{\tau,0}(re^{i\theta})|^2e^{-2mR_\tau(re^{i\theta})}r\,dr\,ds(e^{i\theta})$$
$$= 2m^{\frac{1}{2}}|c_{\tau,0}|^2\int_{\mathbb{T}}\Big(\frac{\pi}{4m\varDelta R_\tau(e^{i\theta})}\Big)^{\frac{1}{2}}\mathbf{L}_0[re^{2\,\mathrm{Re}\,H_{R_\tau}(re^{i\theta})}]\Big|_{r=1}ds+\mathrm{O}(m^{-\frac{1}{2}}).$$

Since in general, for a smooth function f we have that $\mathbf{L}_0[f(r)]\big|_{r=1}=f(1)$, the leading contribution simplifies to (recall the definition (4.2.11) of H_{R_τ}),

$$2m^{\frac{1}{2}}|c_{\tau,0}|^2\int_{\mathbb{T}}\Big(\frac{\pi}{4m\varDelta R_\tau(e^{i\theta})}\Big)^{\frac{1}{2}}\mathbf{L}_0[re^{2\,\mathrm{Re}\,H_{R_\tau}(re^{i\theta})}]\Big|_{r=1}ds$$
$$=2\pi^{\frac{1}{2}}|c_{\tau,0}|^2\int_{\mathbb{T}}\big(4\varDelta R_\tau(e^{i\theta})\big)^{-\frac{1}{2}}e^{2\,\mathrm{Re}\,H_{R_\tau}(e^{i\theta})}ds(e^{i\theta})$$
$$=2\pi^{\frac{1}{2}}|c_{\tau,0}|^2\int_{\mathbb{T}}ds(e^{i\theta})=2\pi^{\frac{1}{2}}|c_{\tau,0}|^2.$$

As this is the leading contribution to (4.2.16), we must have $2\pi^{\frac{1}{2}}|c_{\tau,0}|^2=1$. This determines the constant $c_{\tau,0}$ up to a unimodular factor, and by positivity we find that $c_{\tau,0}=(4\pi)^{-\frac{1}{4}}$.

We turn to the remaining coefficients $c_{\tau,j}$, for $j=1,\ldots,\kappa$. By applying the Laplace method of Proposition 2.6.1 to the radial integral in the formula (4.2.16), we arrive at

$$2\pi^{\frac{1}{2}}\sum_{j=0}^{\kappa}m^{-j}\sum_{(i,k,l)\in\mathfrak{n}_j^\star}\int_{\mathbb{T}}(4\varDelta R_\tau(e^{i\theta}))^{-\frac{1}{2}}\mathbf{L}_k[rB_{\tau,i}(re^{i\theta})\bar{B}_{\tau,l}(re^{i\theta})]\Big|_{r=1}ds(e^{i\theta}) = 1+\mathrm{O}(m^{-\kappa-\frac{1}{2}}),$$

where the index set is $\mathfrak{n}_j^\star:=\big\{(i,k,l)\in\mathbb{N}^3\ :\ i+k+l=j\big\}$. Here, $\mathbb{N}=\{0,1,2,\ldots\}$ as usual. As this represents an equality of asymptotic series, we may identify term by term.

The term with $j = 0$ was already analyzed, and it follows that for $j = 1, \ldots, \kappa$ we have

$$
\begin{aligned}
(4.2.17)\qquad & \sum_{(i,k,l)\in\mathfrak{n}_j^\star} \int_{\mathbb{T}} (4\mathit{\Delta} R_\tau(e^{i\theta}))^{-\frac12}\mathbf{L}_k[rB_{\tau,i}(re^{i\theta})\bar{B}_{\tau,l}(re^{i\theta})]\Big|_{r=1} ds(e^{i\theta}) \\
&= 2\operatorname{Re}\int_{\mathbb{T}} (4\mathit{\Delta} R_\tau(e^{i\theta}))^{-\frac12}\mathbf{L}_0[rB_{\tau,j}(re^{i\theta})\bar{B}_{\tau,0}(re^{i\theta})]\Big|_{r=1} ds(e^{i\theta}) \\
&\quad + \sum_{(i,k,l)\in\mathfrak{n}_j} \int_{\mathbb{T}} (4\mathit{\Delta} R_\tau(e^{i\theta}))^{-\frac12}\mathbf{L}_k[rB_{\tau,i}(re^{i\theta})\bar{B}_{\tau,l}(re^{i\theta})]\Big|_{r=1} ds(e^{i\theta}) = 0,
\end{aligned}
$$

where $\mathfrak{n}_j$ denotes the restricted index set $\mathfrak{n}_j := \big\{(i,k,l) \in \mathfrak{n}_j^\star : i, l < j\big\}$, and where we separate out the terms involving the leading term $B_{\tau,j}$. We successfully resolve the first term on the right-hand side of (4.2.17), while the second term is much more complicated. However, we may observe that it only depends on the functions $B_{\tau,\nu}$ with $\nu = 0, \ldots, j-1$, and hence only on the constants $c_{\tau,\nu}$ with $\nu = 0, \ldots, j-1$. This allows us to algorithmically determine these constants, albeit with increasing degree of complexity. As for the first term on the right-hand side, we observe that the operator $\mathbf{L}_0\big|_{r=1}$ only evaluates at $r = 1$. Using the structure of $B_{\tau,j}$ as given by (4.2.14), we find that

$$
\begin{aligned}
&\int_{\mathbb{T}} (4\mathit{\Delta} R_\tau(e^{i\theta}))^{-\frac12}\mathbf{L}_0[rB_{\tau,j}(re^{i\theta})\bar{B}_{\tau,0}(re^{i\theta})]\Big|_{r=1} ds(e^{i\theta}) \\
&\qquad = \int_{\mathbb{T}} \big(4\mathit{\Delta} R_\tau(e^{i\theta})\big)^{-\frac12} B_{\tau,j}(e^{i\theta})\bar{B}_{\tau,0}(e^{i\theta}) ds(e^{i\theta}) \\
&= c_{\tau,0}\int_{\mathbb{T}} \big(4\mathit{\Delta} R_\tau(e^{i\theta})\big)^{-\frac12} e^{2\operatorname{Re} H_{R_\tau}(re^{i\theta})}\big(c_{\tau,j} - \mathbf{P}_{H^2_{-,0}}[e^{\bar{H}_{R_\tau}}F_{\tau,j}](e^{i\theta})\big) ds(e^{i\theta}) \\
&\qquad = c_{\tau,0}\int_{\mathbb{T}} \big(c_{\tau,j} - \mathbf{P}_{H^2_{-,0}}[e^{\bar{H}_{R_\tau}}F_{\tau,j}](e^{i\theta})\big) ds(e^{i\theta}) = c_{\tau,0}c_{\tau,j}.
\end{aligned}
$$

Here we use the definition (4.2.11) of H_{R_τ} and the fact that the projection $\mathbf{P}_{H^2_{-,0}}$ maps into a subspace of functions with mean 0. Assume now that j is given, and that we have determined $c_{\tau,k}$ for $k = 0, \ldots, j-1$. The above equality together with (4.2.17) then gives that

$$
2\operatorname{Re} c_{\tau,j}c_{\tau,0} = -\sum_{(i,k,l)\in\mathfrak{n}_j} \int_{\mathbb{T}} \big(4\mathit{\Delta} R_\tau(e^{i\theta})\big)^{-\frac12}\mathbf{L}_k[rB_{\tau,i}(re^{i\theta})\bar{B}_{\tau,l}(re^{i\theta})]\Big|_{r=1} ds(e^{i\theta}).
$$

Since $c_{\tau,0} = (4\pi)^{-\frac14}$ and moreover since the constants $c_{\tau,j}$ must be real by our normalization, we obtain that

$$
c_{\tau,j} = -\frac12(4\pi)^{\frac14}\sum_{(i,k,l)\in\mathfrak{n}_j} \int_{\mathbb{T}} \big(4\mathit{\Delta} R_\tau(e^{i\theta})\big)^{-\frac12}\mathbf{L}_k[rB_{\tau,i}(re^{i\theta})\bar{B}_{\tau,l}(re^{i\theta})]\Big|_{r=1} ds(e^{i\theta}),
$$

where the integral may be expressed in terms of the operator $\mathbf{M}_k$ by

$$
\begin{aligned}
&\int_{\mathbb{T}} \big(4\mathit{\Delta} R_\tau(e^{i\theta})\big)^{-\frac12}\mathbf{L}_k[rB_{\tau,i}(re^{i\theta})\bar{B}_{\tau,l}(re^{i\theta})]\Big|_{r=1} ds(e^{i\theta}) \\
&\qquad\qquad = \int_{\mathbb{T}} \mathbf{M}_k\big[B_{\tau,i}(re^{i\theta})\bar{B}_{\tau,l}(re^{i\theta})\big] ds(e^{i\theta}).
\end{aligned}
$$

This completes the proof. □

5. Applications to random matrix theory

5.1. **The random normal matrix model.** For extensive treatments of the random normal matrix ensembles, see e.g. [29, 3, 4, 5, 6, 61]. Here we only briefly discuss the topic, in order to fix the notation and recall some basic concepts.

Let M be a matrix, picked with respect to the probability measure ("tr" stands for trace)

$$d\mu_m(M) = \frac{1}{Z_{m,Q}}\, e^{-2m\,\mathrm{tr}(Q(M))}\, dM,$$

where dM denotes the measure induced by the flat Euclidean metric of $\mathbb{C}^{m^2}$ on the submanifold of normal $m\times m$ matrices, where $Z_{m,Q}$ is a normalizing constant. Such a matrix M has a set of m random eigenvalues, which we denote by $\Phi_m = \{z_{1,m},\ldots,z_{m,m}\}$. It is known that the eigenvalues follow the law

$$d\mathbb{P}_m(z_1,\ldots,z_m) = \frac{1}{\mathcal{Z}_{m,Q}}\left[\prod_{j<k}|z_j-z_k|^2\right] e^{-2m\sum_{j=1}^m Q(z_j)}\, dA^{\otimes n}(z_1,\ldots,z_m), \tag{5.1.1}$$

where $\mathcal{Z}_{m,Q}$ is a related normalizing constant, known as the *partition function* of the ensemble. Here, $dA^{\otimes n}$ stands for Euclidean volume measure in $\mathbb{C}^n$ normalized by the factor π^{-n}. We recognize this as the law for the Coulomb gas with m particles at the inverse temperature $\beta = 2$ in the external field Q. Courtesy of the fact that the product expression in (5.1.1) may be written as the square modulus of a Vandermondian determinant, these ensembles are determinantal. That is, if the k-point intensities $R_{k,m}(z_1,\ldots z_k)$ are defined as the intensities associated to finding points simultaneously at the locations $z_1,\ldots,z_k$, then we may compute $R_{k,m}$ by

$$R_{k,m}(z_1,\ldots,z_k) = \det\left(\mathrm{K}_m(z_j,z_l)\right)_{1\le j,l\le k}. \tag{5.1.2}$$

Here K_m is the *correlation kernel*

$$\mathrm{K}_m(z,w) = K_m(z,w)\, e^{-m(Q(z)+Q(w))}, \qquad z,w\in\mathbb{C}$$

where K_m is the reproducing kernel for the space Pol_m, supplied with the inner product of the space $L^2_{2mQ}(\mathbb{C})$. We remark that the correlation kernel K_m is not uniquely determined by the above-mentioned intensities, since any kernel modified by a cocycle

$$\mathrm{K}^c_m(z,w) = c(z)\bar{c}(w)\mathrm{K}_m(z,w),$$

will generate the same point process by the determinantal formula (5.1.2). Here, the cocycle is associated with a continuous unimodular function $c:\mathbb{C}\to\mathbb{T}$. This means that in terms of convergence of point processes, we need only correlation kernel convergence modulo cocycles. It is known (see [29, 61]) that the process Φ_m condensates to the droplet $\mathcal{S}_1$ as $m\to+\infty$. Indeed, if ν_m denotes the empirical measure

$$\nu_m = \frac{1}{m}\sum_{z\in\Phi_m}\delta_z,$$

then almost surely, ν_m converges weakly to the equilibrium measure μ_τ with $\tau = 1$, the support of which equals $\mathcal{S}_1$. We rescale the point process near a boundary point z_0, in the outer normal direction n, in order to understand the microscopic behavior of Φ_m. To rescale we use the linear transformation

$$z_m(\zeta) := z_0 + \mathrm{n}\frac{\zeta}{\sqrt{2m\mathit{\Delta} Q(z_0)}}.$$

Writing $\Phi_m = \{z_{j,m}\}_j$, we introduce the rescaled local process by $\Psi_m = \{\zeta_{j,m}\}_j$, where

$$z_{j,m} = z_m(\zeta_{j,m}), \qquad j = 1, \ldots, m.$$

Similarly, we denote by k_m the rescaled correlation kernel

$$\mathrm{k}_m(\xi, \eta) = \frac{1}{2m\varDelta Q(z_0)} \mathrm{K}_m(z_m(\xi), z_m(\eta)).$$

We recall the familiar notion that a function $F(\xi, \eta)$ is *Hermitian entire* if it is an entire function of the two variables $(\xi, \bar\eta)$ with the symmetry property $F(\xi, \eta) = \bar F(\eta, \xi)$. The following is from [5].

Theorem 5.1.1. *There exists a sequence of continuous unimodular functions $c_m : \mathbb{C} \to \mathbb{T}$, such that for any given infinite sequence of positive integers $\mathcal{N}$, there exist an infinite subsequence $\mathcal{N}^* \subset \mathcal{N}$ and an Hermitian entire function $F(\xi, \eta)$ such that*

$$\lim_{\mathcal{N}^* \ni m \to \infty} c_m(\xi)\bar c_m(\eta)\, \mathrm{k}_m(z_m(\xi), z_m(\eta)) = e^{\xi\bar\eta - \frac12(|\xi|^2 + |\eta|^2)} F(\xi, \eta).$$

5.2. **Uniform asymptotics near $\tau = 1$.** We take as our starting point the first term of the asymptotic expansion of Theorem 1.5.2. Recall the definition of the compact set $\mathcal{K}_{\tau,A,m}$ in (3.7.1).

Corollary 5.2.1. *Let $\mathcal{H}_{Q,\tau}$ be the bounded holomorphic function in the set $\mathcal{K}_\tau^c$ with real part $\operatorname{Re}\mathcal{H}_{Q,\tau} = \frac14 \log(2\varDelta Q)$ on the boundary $\partial\mathcal{S}_\tau$, which is real-valued at infinity. Then, in the limit as $m, n \to \infty$ while $\tau = \frac{n}{m} \in I_{\epsilon_0}$, we have the asymptotics*

$$|P_{m,n}(z)|^2 e^{-2mQ(z)} = m^{\frac12} |\phi_\tau'(z)|\, e^{-2m(Q - \breve Q_\tau)(z)} \Big(\pi^{-\frac12} e^{2\operatorname{Re}\mathcal{H}_{Q,\tau}(z)} + \mathrm{O}\big(m^{-1}\big) \Big),$$

where the implied constant is uniform for $z \in \mathcal{K}^c_{\tau,A,m}$.

Proof. We recall that

$$\breve Q_\tau = \operatorname{Re}\mathcal{Q}_\tau + \tau \log|\phi_\tau| = \operatorname{Re}\mathcal{Q}_\tau + \tfrac{n}{m}\log|\phi_\tau|,$$

and in view of Theorems 1.5.2 and 1.5.4, we may write

$$\begin{aligned} |P_{m,n}|^2 &= m^{\frac12} |\phi_\tau'(z)| |\phi_\tau|^{2n} e^{2m\operatorname{Re}\mathcal{Q}_\tau} \big|\mathcal{B}_{\tau,0} + \mathrm{O}(m^{-1})\big|^2 \\ &= m^{\frac12} |\phi_\tau'(z)|\, e^{2m\breve Q_\tau} \Big(\pi^{-\frac12} e^{2\operatorname{Re}\mathcal{H}_{Q,\tau}(z)} + \mathrm{O}(m^{-1}) \Big), \end{aligned}$$

and the assertion follows. □

5.3. **Error function asymptotics.** In view of Corollary 5.2.1, we observe that the probability density $|P_{m,n}|^2 e^{-2mQ}$ resembles a Gaussian wave which crests around the boundary $\partial\mathcal{S}_\tau$ of the droplet, where $\tau = \frac{n}{m}$. As a consequence, we expect the density to be obtained as the sum of such Gaussians. Near the droplet boundary, this effect is the strongest, and adding a large but finite number of such Gaussian waves crested along boundary curves $\partial\mathcal{S}_\tau$ which move with the degree parameter n results in error function asymptotics.

Proposition 5.3.1. *If Q is 1-admissible and $z_0 \in \partial\mathcal{S}_1$ is a boundary point, then if ρ_m is the blow-up density given by (1.4.1) and (1.4.2), we have the convergence*

$$\lim_{m\to\infty} \rho_m(\zeta) = \operatorname{erf}(2\zeta),$$

locally uniformly on $\mathbb{C}$.

Proof. We recall the rescaled variable from the introduction

$$z_m(\xi) = z_0 + \mathrm{n}\frac{\xi}{\sqrt{2m\Delta Q(z_0)}},$$

where $z_0 \in \partial\mathcal{S}_\tau$ and n is the outward unit normal to $\mathcal{S}_\tau$ at z_0, and the rescaled density $\rho_m(\xi)$ given by (1.4.2). In terms of orthogonal polynomials, the object of study is the function

$$\rho_m(\xi) = \frac{1}{2m\Delta Q(z_0)}\sum_{n=0}^{m-1}|P_{m,n}(z_m(\xi))|^2 e^{-2mQ(z_m(\xi))}.$$

We begin by noting that $z_m(\xi)$ is in the set $\mathcal{K}^c_{\tau,A,m}$ (see Theorem 1.5.2), provided that ξ is confined to the disk $\mathbb{D}(0,r_m)$, where $r_m = A\sqrt{\Delta Q(z_0)\,\log m}$, and that m is large enough. We shall assume throughout that $\xi \in \mathbb{D}(0,r_m)$.

Next, we write

$$\rho_{m_1,m}(\xi) = \frac{1}{2m\Delta Q(z_0)}\sum_{n=0}^{m_1-1}|P_{m,n}(z_m(\xi))|^2 e^{-2mQ(z_m(\xi))}$$

and split accordingly for $m_1 < m$

$$\rho_m(\xi) = \frac{1}{2m\Delta Q(z_0)}\sum_{n=m_1}^{m-1}|P_{m,n}(z_m(\xi))|^2 e^{-2mQ(z_m(\xi))} + \rho_{m_1,m}(\xi). \tag{5.3.1}$$

We choose m_1 to be the integer part of $m - m^{\frac12}\log m$.

By Proposition 2.2.2 it follows that for $n \le m_1$,

$$|P_{m,n}(z)|^2 e^{-2mQ(z)} \le Cm\, e^{-2m(Q-\hat{Q}_{\tau_1})(z)}, \tag{5.3.2}$$

where $\tau_1 = \frac{m_1}{m} \in I_{\epsilon_0}$ for m large enough. By Taylor's formula applied to the relative potential $Q - \check{Q}_{\tau_1} = R_{\tau_1}\circ\phi_{\tau_1}$ in $\mathcal{S}^c_{\tau_1}$ (Proposition 3.3.1), it follows that

$$(Q-\hat{Q}_{\tau_1})(z) \ge \beta_0 \mathrm{dist}_{\mathbb{C}}(z,\partial\mathcal{S}_{\tau_1})^2 \tag{5.3.3}$$

for some constant $\beta_0 > 0$, provided that $z \in \mathcal{S}^c_{\tau_1}$ is close enough to $\partial\mathcal{S}_{\tau_1}$. For instance, this estimate holds for $z \in \mathcal{S}_1 \setminus \mathcal{S}_{\tau_1}$. Moreover, as $\tau_1 = \frac{m_1}{m}$ eventually is in I_{ϵ_0}, the function $Q - \hat{Q}_{\tau_1}$ does not vanish on $\mathcal{S}^c_{\tau_1}$, and tends to infinity at infinity. The latter observation shows that further away from the boundary $\partial\mathcal{S}_{\tau_1}$, the right-hand side of (5.3.2) decays exponentially.

If $n \le m_1$ and $\tau = \frac{n}{m}$, then $1-\tau \ge m^{-\frac12}\log m = \delta_m$. As a consequence of Lemma 2.3.1 we obtain that the boundary $\partial\mathcal{S}_\tau$ moves at a positive speed in τ. In particular, for $\tau = \frac{n}{m}$ where $n \le m_1$ we have that the distance $\mathrm{dist}_{\mathbb{C}}(\partial\mathcal{S}_\tau, \partial\mathcal{S}_1)$ is at least $2\alpha_0\delta_m$, for some fixed positive α_0. Since $\mathrm{dist}_{\mathbb{C}}(\partial\mathcal{S}_{\tau_1}, \partial\mathcal{S}_1)$ is at least $2\alpha_0\delta_m$, we have that

$$\mathrm{dist}_{\mathbb{C}}(z,\partial\mathcal{S}_{\tau_1}) \ge \alpha_0\delta_m, \qquad z \in \mathbb{D}(z_0,\alpha_0\delta_m). \tag{5.3.4}$$

Next, we note that if $\zeta \in \mathbb{D}(0,r_m)$, then for large enough m we have $z_m(\zeta) \in \mathbb{D}(z_0,\alpha_0\delta_m)$. This follows from the obvious fact that $(\log m)^{\frac12} = \mathrm{o}(\log m)$. By a combination of (5.3.3) and (5.3.4) it follows that

$$(Q-\hat{Q}_{\tau_1})(z_m(\zeta)) \ge \beta_0\alpha_0^2\delta_m^2.$$

Now, it follows from the above estimates (5.3.2) and (5.3.3) that for $n \le m_1$

$$|P_{m,n}(z_m(\xi))|^2 e^{-2mQ(z_m(\xi))} = \mathrm{O}(m\, e^{-2\beta_0\alpha_0^2(\log m)^2}),$$

where the implicit constant is uniform in $\xi \in \mathbb{D}(0, r_m)$. It follows that

$$\rho_{m_1,m}(\xi) = \mathrm{O}\big(m^2 e^{-\beta_0\alpha_0^2(\log m)^2}\big), \qquad \xi \in \mathbb{D}(0, r_m)$$

which shows in particular $\rho_{m_1,m}(\xi) = \mathrm{O}(m^{-M})$ for arbitrarily large M.

As a result of the above considerations, it follows that we may focus on the remaining sum in (5.3.1) over the degrees n with $m_1 \le n \le m-1$, that is, $\tau = \frac{n}{m}$ with $\tau_1 \le \tau \le 1$. In particular, the asymptotics of Corollary 5.2.1 applies in the whole range. Set $\tau(j) = \tau_m(j) = 1 - \frac{j}{m}$, where j ranges from 1 to $m - m_1$, which is approximately $m^{\frac12}\log m$. We obtain

$$\rho_m(\xi) = \frac{(\pi m)^{-\frac12}}{2\Delta Q(z_0)} \sum_{j=1}^{m-m_1} \big|\phi'_{\tau(j)}(z_m(\xi))\big|\, e^{-2m(Q-\check{Q}_{\tau(j)})(z_m(\xi))+2\operatorname{Re}\mathcal{H}_{Q,\tau(j)}(z_m(\xi))} + \mathrm{O}(m^{-M}).$$

By Taylor's formula, it follows that

$$|\phi'_{\tau(j)}(z_m(\xi))| = |\phi'_1(z_0)| + \mathrm{O}((m^{-1}\log m)^{\frac12}),$$

and by the same token that

$$2\operatorname{Re}\mathcal{H}_{Q,\tau(j)}(z_m(\xi)) = \frac12 \log \Delta Q(z_0) + \mathrm{O}((m^{-1}\log m)^{\frac12})$$

as $m \to \infty$ for all $j \le m - m_1$. The next thing to consider is the movement of $\partial\mathcal{S}_\tau$, where $\tau = \tau(j)$ and j increases. As n denotes the outward pointing unit normal to $\partial\mathcal{S}_1$ at the point z_0, Lemma 2.3.1 tells us that the line $z_0 + \mathrm{n}\mathbb{R}$ intersects $\partial\mathcal{S}_{\tau(j)}$ at the nearest point

$$z_j = z_0 - \mathrm{n}\frac{j}{m}\frac{|\phi'_1(z_0)|}{4\Delta Q(z_0)} + \mathrm{O}\Big(\Big(\frac{j}{m}\Big)^2\Big),$$

and the outer unit normal n_j to $\partial\mathcal{S}_{\tau(j)}$ at the point z_j will satisfy

$$\mathrm{n}_j = \mathrm{n} + \mathrm{O}\Big(\frac{j}{m}\Big) = \mathrm{n} + \mathrm{O}(m^{-\frac12}\log m).$$

We may hence write

$$(Q - \check{Q}_{\tau(j)})(z_m(\xi)) = (Q - \check{Q}_{\tau(j)})\left(z_j + \mathrm{n}_j \frac{\xi + \frac{j}{2}\frac{|\phi'_1(z_0)|}{\sqrt{2m\Delta Q(z_0)}} + \mathrm{O}\big(m^{-\frac12}(\log m)^2\big)}{\sqrt{2m\Delta Q(z_0)}}\right).$$

A simple Taylor series expansion in normal and tangential coordinates at the point z_j gives that

$$(Q - \check{Q}_{\tau_j})(z_j + \mathrm{n}_j\lambda) = 2\Delta Q(z_j)\,(\operatorname{Re}\lambda)^2 + \mathrm{O}(|\lambda|^3) = 2\Delta Q(z_0)(\operatorname{Re}\,\lambda)^2 + \mathrm{O}\Big(|\lambda|^2\frac{j}{m} + |\lambda|^3\Big),$$

for λ close to 0. From this we deduce that for η with $|\eta| = \mathrm{O}(\log m)$ we have

$$2m(Q - \check{Q}_{\tau(j)})\Big(z_j + \mathrm{n}_j\frac{\eta}{\sqrt{2m\Delta Q(z_0)}}\Big) = \frac12(2\operatorname{Re}\eta)^2 + \mathrm{O}(m^{-1/2}(\log m)^3), \quad m \to \infty.$$

We apply this with η given by

$$\eta = \xi + \frac{j}{2}\frac{|\phi'_1(z_0)|}{\sqrt{2m\Delta Q(z_0)}} + \mathrm{O}\big(m^{-\frac12}(\log m)^2\big),$$

which then gives that

$$(2\operatorname{Re}\eta)^2 = \Big(2\operatorname{Re}\xi + j\frac{|\phi'_1(z_0)|}{\sqrt{2m\Delta Q(z_0)}}\Big)^2 + \mathrm{O}\big(m^{-\frac12}(\log m)^3\big).$$

Putting these asymptotic relations together, we find that

$$(5.3.5)\quad \rho_m(\xi) = \frac{1}{\sqrt{2\pi}}\Big(1 + \mathrm{O}\big(m^{-\frac{1}{2}}(\log m)^3\big)\Big) \times \sum_{j=1}^{m-m_1} \frac{|\phi_1'(z_0)|}{\sqrt{2m\Delta Q(z_0)}}\ \exp\bigg(-\frac{1}{2}\Big(2\operatorname{Re}\xi + j\frac{|\phi'(z_0)|}{\sqrt{2m\Delta Q(z_0)}}\Big)^2\bigg) + \mathrm{O}(m^{-M}).$$

We recognize immediately (5.3.5) as an approximate Riemann sum for

$$\operatorname{erf}(2\operatorname{Re}\xi) = \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-\frac{1}{2}(2\operatorname{Re}\xi + t)^2}\,dt$$

with respect to a partition of the interval $[0, \gamma_0 \log m]$, with step length $m^{-\frac{1}{2}}\gamma_0$, where

$$\gamma_0 = \frac{|\phi'(z_0)|}{\sqrt{2\Delta Q(z_0)}}.$$

Since such Riemann sums converge to the corresponding integral with small error, this implies that

$$\lim_{m\to\infty} \rho_m(\xi) = \operatorname{erf}\big(2\operatorname{Re}\xi\big),$$

which completes the proof. □

5.4. **Convergence of correlation kernels to the Faddeeva plasma kernel.** Finally, we turn to the convergence of the rescaled kernels $\mathrm{k}_m(z_m(\xi), z_m(\eta))$ as $m \to \infty$. In principle, this should follow from our expansion of the orthogonal polynomials, but to do this directly seems a bit tricky. However, given the work of Ameur, Kang, and Makarov [5], it turns out to be enough to obtain the more straightforward diagonal convergence of the correlation kernel.

Proof of Corollary 1.5.6. We denote by $G(\xi, \eta)$ the Ginibre-∞ kernel

$$G(\xi, \eta) = e^{\xi\bar\eta - \frac{1}{2}(|\xi|^2 + |\eta|^2)},$$

which is the correlation kernel of a translation invariant planar point process. We now present some material from [5]. An important concept is that of cocycles. By Theorem 5.1.1, there exists a sequence of continuous functions $c_m : \mathbb{C} \to \mathbb{T}$ such that, for any subsequence $\mathcal{N}$ of the natural numbers $\mathbb{N}$, there exists a Hermitian entire function $F(\xi, \eta)$ and a further subsequence $\mathcal{N}^\star \subset \mathcal{N}$ such that

$$(5.4.1)\qquad c_m(\xi)\bar c_m(\eta)\,\mathrm{k}_m(z_m(\xi), z_m(\eta)) \to G(\xi, \eta)F(\xi, \eta), \quad m \in \mathcal{N}^\star, m \to \infty,$$

where the convergence is uniform on compact subsets of $\mathbb{C}^2$. For Hermitian entire functions, the diagonal restriction $F(\xi, \xi)$ determines the function uniquely. Indeed, the polarization of the diagonal restriction gives back our function $F(\xi, \eta)$. We denote by $\rho(\xi)$ the limiting density

$$\rho(\xi) = \lim_{m\to\infty, m\in\mathcal{N}^\star} \mathrm{k}_m(z_m(\xi), z_m(\xi)) = G(\xi, \xi)F(\xi, \xi),$$

and since $G(\xi, \xi) \equiv 1$, it follows that $F(\xi, \xi) = \rho(\xi)$. By Proposition 5.3.1 we have that

$$\rho(\xi) = \operatorname{erf}(2\operatorname{Re}\xi).$$

Moreover, by the uniqueness property of diagonal restriction, the only possibility for the Hermitian entire function is

$$F(\xi, \eta) = \operatorname{erf}(\xi + \bar\eta).$$

This shows that the limit along some subsequence of any given sequence of positive integers is always the same. We claim that this means that the whole sequence converges. Indeed, in case the convergence (5.4.1) were to fail along the positive integers, by a normal families argument, we could distill a sequence $\mathcal{N}_0$ such that the left-hand side of (5.4.1) would converge to something else along the subsequence $\mathcal{N}_0$. This would contradict what we have already established, which is that the we have diagonal convergence to the error function. The assertion of the corollary follows. □

6. The existence of the orthogonal foliation flow

6.1. **Smoothness classes and polarization of functions.** In order to proceed with less obscuring notation, we consider a smooth family of bounded holomorphic functions $f_s(z)$, and a smooth family of orthostatic conformal mappings $\psi_{s,t}$. Here, f_s corresponds to $f_{m,n}^{\langle\kappa\rangle}$ where $s = m^{-1}$, and $\psi_{s,t}$ corresponds to the mappings $\psi_{m,n,t}$ appearing in Lemma 3.4.1. We suppress τ and κ in the notation, because κ is thought of as fixed, and we work with uniformity in the parameter τ. Moreover, we denote by R a weight whose properties are analogous to those of R_τ, as captured in Definition 6.1.1 below.

We denote by $\mathbb{A}(\varrho_1, \varrho_2)$ the annulus

$$\mathbb{A}(\varrho_1, \varrho_2) := \mathbb{D}(0, \varrho_2) \setminus \bar{\mathbb{D}}(0, \varrho_1),$$

for positive real numbers ϱ_1 and ϱ_2 with $\varrho_1 < \varrho_2$ (notice that we distinguish between the symbols ρ and ϱ). In addition, for parameters ϱ_0 and σ_0, we denote by $\hat{\mathbb{A}}(\varrho_0, \sigma_0)$ the $2\sigma_0$-fattened diagonal annulus in $\mathbb{C}^2$:

$$\hat{\mathbb{A}}(\varrho_0, \sigma_0) := \big\{(z, w) \in \mathbb{A}(\varrho_0, \varrho_0^{-1}) \times \mathbb{A}(\varrho_0, \varrho_0^{-1}) \ : \ |z - w| \le 2\sigma_0\big\},$$

For a real-analytic function R there exists a *polarization* $R(z, w)$, which is holomorphic in $(z, \bar{w})$ and has $R(z, z) = R(z)$. This is easy to see using convergent local Taylor series expansions of $R(z)$ in the coordinates which are the real and imaginary parts, $\operatorname{Re} z$ and $\operatorname{Im} z$. By replacing $\operatorname{Re} z$ by $\frac{1}{2}(z + \bar{w})$ and $\operatorname{Im} z$ by $\frac{1}{2i}(z - \bar{w})$ in this expansion, we obtain the polarization $R(z, w)$. We observe that if $R(z, w)$ is such a polarization of a function $R(z)$ which is real-analytically smooth near the circle $\mathbb{T}$ and in addition is quadratically flat there, then $R(z) = (1 - |z|^2)^2 R^\sharp(z)$, where $R^\sharp(z)$ is real-analytic near the circle $\mathbb{T}$. In polarized form, $R(z, w)$ factors as

$$R(z, w) = (1 - z\bar{w})^2 R^\sharp(z, w), \tag{6.1.1}$$

where $R^\sharp(z, w)$ is holomorphic in $(z, \bar{w})$ in a neighborhood of the part of the diagonal where both variables are near $\mathbb{T}$.

Definition 6.1.1. For positive real numbers ϱ_0, σ_0 where $\varrho_0 < 1$, we denote by $\mathcal{W}(\varrho_0, \sigma_0)$ the class of C^2-smooth non-negative functions R on $\mathbb{D}_{\mathrm{e}}(0, \varrho_0)$ such that the following holds:

(i) The functions R and ∇R both vanish on $\mathbb{T}$, while $\Delta R > 0$ holds on $\mathbb{T}$.
(ii) R is real-analytic on $\mathbb{A}(\varrho_0, (\varrho_0)^{-1})$ and both $R(z, w)$ and $R^\sharp(z, w)$ given by (6.1.1) polarizes to bounded holomorphic functions in $(z, \bar{w})$ on the diagonal annulus $\hat{\mathbb{A}}(\varrho_0, \sigma_0)$, such that $R^\sharp(z, w)$ remains bounded away from 0 there.
(iii) In addition,

$$R^\sharp(z, z) \ge \alpha(R) > 0, \qquad z \in \mathbb{A}(\varrho_0, \varrho_0^{-1}),$$

and further away,

$$\inf_{z\in\mathbb{D}_{\mathrm{e}}(0,\varrho_0^{-1})} \frac{R(z)}{\log|z|} = \theta(R) > 0.$$

We say that a subset $\mathfrak{S} \subset \mathcal{W}(\varrho_0, \sigma_0)$ is a *uniform family*, provided that for each $R \in \mathfrak{S}$, the corresponding $R^\sharp(z, w)$ is uniformly bounded and bounded away from 0 on $\hat{\mathbb{A}}(\varrho_0, \sigma_0)$ while the controlling constants such as $\alpha(R)$ and $\theta(R)$ are uniformly bounded away from 0.

If a function $f(z, w)$ is holomorphic in $(z, \bar{w})$, we may consider the associated function

$$f_{\mathbb{T}}(z) = f\Big(z, \frac{1}{\bar{z}}\Big) \tag{6.1.2}$$

which is then holomorphic in z, wherever it is well-defined. We note that $f_{\mathbb{T}}(z) = f(z, z)$ on the circle $\mathbb{T}$. We recall the notation of Definition 6.1.1.

Proposition 6.1.2. *Suppose that $f(z, w)$ is holomorphic in $(z, \bar{w})$ on the domain $\hat{\mathbb{A}}(\varrho, \sigma)$, where $0 < \varrho < 1$ and $\sigma > 0$. Then the function $f_{\mathbb{T}}(z)$, which extends the restriction of the diagonal function $f(z, z)$ to $\mathbb{T}$, has a holomorphic extension to the annulus*

$$\varrho' < |z| < \frac{1}{\varrho'}$$

where

$$\varrho' = \max\Big\{\varrho, (\sqrt{1+\sigma^2} + \sigma)^{-1}\Big\}.$$

Proof. The function $f_{\mathbb{T}}(z) = f(z, \bar{z}^{-1})$ is automatically holomorphic and bounded in the variable z in the domain

$$\Big|z - \frac{1}{\bar{z}}\Big| < 2\sigma,$$

provided that $z \in \mathbb{A}(\varrho, \varrho^{-1})$. The displayed condition is equivalent to the requirement that

$$-2\sigma|z| < |z|^2 - 1 < 2\sigma|z|,$$

from which the claim follows by solving two quadratic equations. $\square$

Remark 6.1.3. We note that if ϱ is close enough to 1 to guarantee that $\varrho \geq (\sqrt{1+\sigma^2}+\sigma)^{-1}$, then $\varrho' = \varrho$.

Remark 6.1.4. Suppose a real-analytic function $F(z)$ admits a polarization $F(z, w)$ which is holomorphic in $(z, \bar{w})$ for $(z, w) \in \hat{\mathbb{A}}(\varrho, \sigma)$, and let f be given in terms of the Herglotz kernel by $f = \mathbf{H}_{\mathbb{D}_{\mathrm{e}}}[F|_{\mathbb{T}}]$ (cf. §2.5). We note that by the properties of the Herglotz kernel, f may be obtained by the formula $f = 2\mathbf{P}_{H^2_-}[F|_{\mathbb{T}}] - \langle F\rangle_{\mathbb{T}}$, where $\langle F\rangle_{\mathbb{T}}$ denotes the average of F on the unit circle. Let $F_{\mathbb{T}}$ be as in (6.1.2), and express it in terms of its Laurent series, which by Proposition 6.1.2 converges in the annulus $\mathbb{A}(\varrho', (\varrho')^{-1})$:

$$F_{\mathbb{T}}(z) = \sum_{n\in\mathbb{Z}} a_n z^n.$$

In terms of the Laurent series, $\mathbf{P}_{H^2_-}[F|_{\mathbb{T}}]$ equals $\sum_{n\leq 0} a_n z^n$ and $\langle F\rangle_{\mathbb{T}} = a_0$. As a consequence, $\mathbf{P}_{H^2_-}[F|_{\mathbb{T}}]$ defines a holomorphic function on the exterior disk $\mathbb{D}_{\mathrm{e}}(0, \varrho')$ and hence, f is holomorphic on $\mathbb{D}_{\mathrm{e}}(0, \varrho')$ as well.

The setting which will prove useful to us is when we may control certain related quantities and their polarizations, which is possible on thinner $\mathbb{C}^2$-complexified annuli. The polarization of $\log \varDelta R$ appears later in the induction algorithm, while $\log(z\partial\hat{R})$ is important for the control associated with the implicit function theorem.

Proposition 6.1.5. *If R belongs to a uniform family $\mathfrak{S}\subset\mathcal{W}(\varrho_0,\sigma_0)$ for some positive reals ϱ_0,σ_0 with $\varrho_0<1$, and if $\hat R=\sqrt{R}$ is chosen so that $\hat R(z)$ is positive for $|z|>1$ and negative for $|z|<1$, then there exist positive ϱ_1,σ_1 with $\varrho_0\le\varrho_1<1$, $\sigma_1\le\sigma_0$ and $\varrho_1\ge(\sqrt{1+\sigma_1^2}+\sigma_1)^{-1}$, such that the polarizations of the functions $\log\Delta R$, $\hat R$, $\log(z\partial\hat R)$ are all holomorphic in $(z,\bar w)$ and uniformly bounded on the $2\sigma_1$-fattened diagonal annulus $\hat{\mathbb{A}}(\varrho_1,\sigma_1)$.*

Proof sketch. This follows from the assumptions on the uniform family, if we use the standard Cauchy estimates plus the fact that $\log\Delta R=\log(2(R^\sharp)^2)$ and $\log(z\partial\hat R)=\frac12\log R^\sharp$ hold on the unit circle $\mathbb{T}$. The condition $\varrho_1\ge(\sqrt{1+\sigma_1^2}+\sigma_1)^{-1}$ is achieved by choosing ϱ_1 large enough, but still in the range $\varrho_0\le\varrho_1<1$. □

6.2. **The master equation for the orthogonal foliation flow.** For an integer n, we denote by Υ_n the triangular index set

$$\Upsilon_n=\big\{(j,l)\in\mathbb{N}^2:\ 2j+l\le n\big\},\tag{6.2.1}$$

and supply it with the inherited lexicographic ordering $\prec_{\mathrm{L}}$:

$$(i,k)\prec_{\mathrm{L}}(j,l)\ \text{ if }\ i<j\ \text{ or }\ i=j\ \text{ and }\ k<l.$$

We recall the notation of the pair (ϱ_1,σ_1) from Proposition 6.1.5.

The following is an analogue of Lemma 3.4.1. We introduce a parameter s, which is supposed to be close to 0, and plays the role of the Planck constant $\hbar$. Later on, we will put $s=1/m$.

Proposition 6.2.1. *Let κ be a given positive integer and let $R\in\mathcal{W}(\varrho_0,\sigma_0)$, for some ϱ_0,σ_0 with $0<\varrho_0<1$ and $\sigma_0>0$. Then there exist a radius ϱ_2 with $\varrho_1<\varrho_2<1$, bounded holomorphic functions b_j on the exterior disk $\mathbb{D}_{\mathrm{e}}(0,\varrho_1)$ for $j=0,\dots,\kappa$, and orthostatic conformal mappings*

$$\psi_{s,t}=\psi_{0,t}+\sum_{\substack{(j,l)\in\Upsilon_{2\kappa+1}\\ j\ge1}}s^jt^l\hat\psi_{j,l}$$

defined on $\mathbb{D}_{\mathrm{e}}(0,\varrho_2)$ with $\psi_{s,t}(\mathbb{D}_{\mathrm{e}}(0,\varrho_2))\subset\mathbb{D}_{\mathrm{e}}(0,\varrho_1)$ for s and t close to 0, such that the following holds. For fixed s, the domains $\psi_{s,t}(\mathbb{D}_{\mathrm{e}})$ increase with t: $\psi_{s,t}(\mathbb{D}_{\mathrm{e}})\subset\psi_{s,t'}(\mathbb{D}_{\mathrm{e}})$ for $t<t'$, and if we put $h_s=\sum_{j=0}^{\kappa}s^jb_j$ and $f_s=\exp(h_s)$, the functions f_s and $\psi_{s,t}$ have the property that for $\zeta\in\mathbb{T}$

$$\begin{aligned}|f_s\circ\psi_{s,t}(\zeta)|^2e^{-2s^{-1}R\circ\psi_{s,t}(\zeta)}\,\mathrm{Re}\big(-\bar\zeta\partial_t\psi_{s,t}(\zeta)\overline{\psi'_{s,t}(\zeta)}\big)&\\ =e^{-t^2/s}\Big((4\pi)^{-\frac12}+\mathrm{O}\big(|s|^{\kappa+\frac12}+|t|^{2\kappa+1}\big)\Big)&.\end{aligned}\tag{6.2.2}$$

Here, the implicit constant remains uniformly bounded as long as R is confined to a uniform family in $\mathcal{W}(\varrho_0,\sigma_0)$, for fixed ϱ_0,σ_0.

Remark 6.2.2. (a) Strictly speaking, the functions $\psi_{s,t}$ and h_s we write down depend on the precision parameter κ, while the coefficient functions b_j and $\hat\psi_{j,l}$ do not. We observe that the orthostaticity of $\psi_{s,t}$ gives that $\hat\psi'_{0,0}(\infty)>0$, and moreover that $\mathrm{Im}\,\hat\psi'_{j,l}(\infty)=0$ for all $j,l\ge0$.

(b) The function f_s will also admit an asymptotic expansion of the form

$$f_s(\zeta) = \sum_{j=0}^{\kappa} s^j B_j(\zeta) + \mathrm{O}(s^{\kappa+1}), \qquad \zeta \in \mathbb{D}_{\mathrm{e}}(0, \varrho_1)),$$

where the coefficient functions B_j may be obtained algorithmically as multivariate polynomials in the functions $b_0, \ldots, b_j$.

The first step towards finding the conformal mappings $\psi_{s,t}$ is to note the following: we find by taking logarithms that

$$2\,\mathrm{Re}\, h_s \circ \psi_{s,t}(\zeta) - 2s^{-1}(R \circ \psi_{s,t})(\zeta) + \log \mathrm{Re}\Big(-\bar\zeta \partial_t \psi_{s,t} \overline{\psi'_{s,t}(\zeta)}\Big) = -s^{-1}t^2 + \mathrm{O}(1),$$

as $s, t \to 0$. Next, we multiply both sides by s, to obtain

$$2s\,\mathrm{Re}\, h_s \circ \psi_{s,t}(\zeta) - 2\, R \circ \psi_{s,t}(\zeta) + s \log \mathrm{Re}\Big(-\bar\zeta \partial_t \psi_{s,t} \overline{\psi'_{s,t}(\zeta)}\Big) = -t^2 + \mathrm{O}(s). \tag{6.2.3}$$

Finally, we take the limit as $s \to 0$ in (6.2.3), expecting that $\mathrm{Re}\, h_s \circ \psi_{s,t}$ and $\log \mathrm{Re}(-\bar\zeta \partial_t \psi_{s,t} \bar\psi'_{s,t})$ remain bounded, and arrive at the equation

$$R \circ \psi_{0,t}(\zeta) = \frac{t^2}{2}.$$

As a consequence, $\psi_{0,t}$ should be a conformal mapping of $\mathbb{D}_{\mathrm{e}}$ onto the exterior of the appropriate level curve of the weight R.

Proposition 6.2.3. *Let R be as in Proposition 6.2.1. There exists a positive number t_0, and a real-analytically smooth family $\{\psi_{0,t}\}_{t \in (-t_0, t_0)}$ of orthostatic conformal mappings $\mathbb{D}_{\mathrm{e}} \to \Omega_t$, where Ω_t is the unbounded component of $\mathbb{C} \setminus \Gamma_t$, and where Γ_t are real-analytically smooth, simple closed level curves of R:*

$$R|_{\Gamma_t} = \frac{t^2}{2}.$$

Moreover, $\Omega_0 = \mathbb{D}_{\mathrm{e}}$ and Ω_t increases with t.

Proof. The assumed strict subharmonicity of R gives that there exists a neighborhood U of $\mathbb{T}$ such that $\nabla R\big|_{U \setminus \mathbb{T}} \neq 0$. This shows that the level sets must be simple and closed curves, for $|t|$ sufficiently small. Indeed, if a curve would possess a loop, then R would have to have a local extremal point inside the loop, which is impossible. Since ∇R vanishes on $\mathbb{T}$, we cannot apply the implicit function theorem directly to R to obtain the result. However, the function

$$\tilde R(re^{i\theta}) := \frac{R(re^{i\theta})}{(r-1)^2}$$

is, in view of Proposition 3.3.1, strictly positive and real-analytic in a neighborhood of the unit circle $\mathbb{T}$. We form the square root $\hat R = \sqrt{R}$ by

$$\hat R(re^{i\theta}) = (r-1)\sqrt{\tilde R(re^{i\theta})},$$

where the square root on the right-hand side is the standard square root of a positive number. We may now apply the implicit function theorem to the function $\hat R$. The result follows immediately by applying the Riemann mapping theorem to the exterior of the resulting analytic level curves of $\hat R$. $\square$

Remark 6.2.4. Proposition 6.2.3 tells us that the conformal mappings $\psi_{0,t}$ extend to some domain containing $\bar{\mathbb{D}}_e$, but supplies little information on how much bigger such a domain is allowed to be. We will discuss this issue in §6.3 below. Along the way, we also obtain an alternative proof of Proposition 6.2.3, which may be viewed as a quantitative version of the implicit function theorem in the given context.

The Taylor coefficients $\hat{\psi}_{0,l}$ (in the flow variable t) of the conformal mappings $\psi_{0,t}$ may be explicitly computed in terms of the weight R, using a higher order version of Nehari's formula for conformal mappings to nearly circular domains. We will return to this in §6.8. Before we carry on, we formulate the following lemma, which allows us to draw the conclusion that the mappings $\psi_{s,t}$ of Proposition 6.2.1 are actually conformal.

Lemma 6.2.5. *Assume that ψ is a holomorphic function on $\mathbb{D}_{\mathrm{e}}(0,\varrho)$ of the form*

$$\psi(z) = z + F(z),$$

such that $|F'| \le \frac{1}{2}$ and

$$2|zF''(z)| \le \frac{\varrho^2}{|z|^2 - \varrho^2}, \qquad z \in \mathbb{D}_{\mathrm{e}}(0,\varrho).$$

Then ψ is univalent on $\mathbb{D}_{\mathrm{e}}(0,\varrho)$.

Proof. This is immediate from the Becker-Pommerenke univalence criterion [7]. □

It is clear that the mappings $\psi_{s,t}$ meet this criterion for some $\varrho < 1$, for small enough s and t for a fixed precision parameter κ.

6.3. **The smoothness of level curves and the implicit function theorem.** In this subsection, we analyze the extension properties of conformal mappings from $\mathbb{D}_{\mathrm{e}}$ onto the exterior of the level curves of R near the unit circle. In a sense, this may be viewed as a quantitative version of the implicit function theorem.

The function R is assumed to belong to the class $\mathcal{W}(\varrho_0,\sigma_0)$ of Definition 6.1.1, which is a quantitative way to say that R is real-analytic near the unit circle $\mathbb{T}$, and vanishes along with its normal derivative on $\mathbb{T}$, while ΔR is positive on $\mathbb{T}$. We recall the definition of the choice of square root $\hat{R}$ of R from the proof of Proposition 6.2.3. This function is also real-analytic near the circle, vanishes on $\mathbb{T}$ but its gradient is nonzero and points in the direction of the outward normal. To make this more quantitative, we let ϱ_1 and σ_1 be the parameters of Proposition 6.1.5. Then, in the $2\sigma_1$-fattened diagonal annulus $\hat{\mathbb{A}}(\varrho_1,\sigma_1)$, we have the control

$$\sup_{(z,w)\in\hat{\mathbb{A}}(\varrho_1,\sigma_1)} |\log(z\partial_z\hat{R}(z,w))| < +\infty. \tag{6.3.1}$$

We recall that the mappings $\psi_{0,t}$ are defined by the requirement of orthostaticity and

$$\hat{R}\circ\psi_{0,t}(\zeta) = -\frac{t}{\sqrt{2}}, \qquad \zeta\in\mathbb{T}. \tag{6.3.2}$$

By differentiating the relation (6.3.2) with respect to t, we obtain from the chain rule

$$[(\partial_r\hat{R})\circ\psi_{0,t}]\,\partial_t|\psi_{0,t}| + [(\partial_\theta\hat{R})\circ\psi_{0,t}]\,\partial_t\arg\psi_{0,t} = -\frac{1}{\sqrt{2}},$$

which we may rewrite as

$$[(r\partial_r\hat{R})\circ\psi_{0,t}]\,\partial\log|\psi_{0,t}|+[(\partial_\theta\hat{R})\circ\psi_{0,t}]\,\partial_t\arg\psi_{0,t}\\ =\operatorname{Re}\Big(\big[\big(r\partial_r\hat{R}-i\partial_\theta\hat{R}\big)\circ\psi_{0,t}\big]\,\partial_t\log\frac{\psi_{0,t}}{\zeta}\Big)=-\frac{1}{\sqrt{2}}.$$

Here, we divided by the coordinate function ζ in order to avoid issues with branch cuts of the logarithm. The differential operator acting on $\hat{R}$ may be written as $r\partial_r-i\partial_\theta=2z\partial_z$, so the above expression simplifies further to

$$\operatorname{Re}\Big(\big[\big(2z\partial_z\hat{R}\big)\circ\psi_{0,t}\big]\,\partial_t\log\frac{\psi_{0,t}}{\zeta}\Big)=-\frac{1}{\sqrt{2}}\quad\text{on }\mathbb{T}.\tag{6.3.3}$$

It is on the basis of the relation (6.3.3) that we will try to recover information on the mappings $\psi_{0,t}$. We introduce the notation

$$\mu(\zeta):=\log\big(2z\partial_z\hat{R}\big),\quad \mu_t=\mu\circ\psi_{0,t},\quad F_t(\zeta)=\partial_t\log\frac{\psi_{0,t}(\zeta)}{\zeta},\tag{6.3.4}$$

and observe that (6.3.3) may be written in the form

$$e^{\mu_t}F_t+e^{\bar{\mu}_t}\bar{F}_t=-\sqrt{2}\quad\text{on }\mathbb{T}.\tag{6.3.5}$$

We note that along the unit circle $\mathbb{T}$, the function $e^{\mu}=2z\partial_z\hat{R}$ equals the positive function $\sqrt{2\Delta R}$, so there are no problems with taking the logarithm in the definition of μ in a neighborhood of $\mathbb{T}$. In particular, if $\psi_{0,t}$ is a perturbation of the identity, the function μ_t is well-defined and smooth. Next, we decompose $\mu_t=\mu_t^++\mu_t^-$, where $\mu_t^+\in H^2$ and $\mu_t^-\in H^2_{-,0}$ are both smooth, and write $G_t=e^{\mu_t^-}F_t$. Given that $F_t\in H^2_-$, it is clear that $G_t\in H^2_-$. If we multiply the above equation (6.3.5) by $e^{-2\operatorname{Re}\mu_t^+}$, we arrive at

$$e^{-\bar{\mu}_t{}^+}G_t+e^{-\mu_t^+}\bar{G}_t=2\operatorname{Re}\big(e^{-\bar{\mu}_t^+}G_t\big)=-\sqrt{2}\,e^{-2\operatorname{Re}\mu_t^+},$$

where we point out that $e^{-\bar{\mu}_t^+}G_t\in H^2_-$ while $e^{-\mu_t^+}\bar{G}_t\in H^2$. This is equation is solved by applying the Herglotz kernel, and yields the solution

$$G_t=-\frac{1}{\sqrt{2}}\,e^{\bar{\mu}_t^+}\mathbf{H}_{\mathbb{D}_e}\big[e^{-2\operatorname{Re}\mu_t^+}\big],$$

where we use the fact that F_t and μ_t^+ are real-valued at infinity (cf. Remark 6.2.2 (a)). That is,

$$F_t=-\frac{1}{\sqrt{2}}\,e^{\bar{\mu}_t^+-\mu_t^-}\mathbf{H}_{\mathbb{D}_e}[e^{-2\operatorname{Re}\mu_t^+}].\tag{6.3.6}$$

Let us write

$$g_t(\zeta)=\log\frac{\psi_{0,t}(\zeta)}{\zeta},$$

so that $\partial_t g_t=G_t$ and $g_0=0$. Here, the logarithm is understood as the principal branch. In terms of these functions, the equation (6.3.6) becomes the following nonlinear differential equation in t:

$$\partial_t g_t=-\frac{1}{\sqrt{2}}\exp\Big(\overline{\mathbf{P}_{H^2}[\mu\circ\psi_{0,t}]}-\mathbf{P}_{H^2_{-,0}}[\mu\circ\psi_{0,t}]\Big)\mathbf{H}_{\mathbb{D}_e}\big[\exp\big(-2\operatorname{Re}\mathbf{P}_{H^2}[\mu\circ\psi_{0,t}]\big)\big].\tag{6.3.7}$$

It is not difficult to see that the equation (6.3.7) may be solved by an iterative procedure, if we rewrite it in integral form

(6.3.8)
$$g_t = -\frac{1}{\sqrt{2}}\int_0^t \exp\Big(\overline{\mathbf{P}_{H^2}[\mu\circ\psi_{0,\theta}]} - \mathbf{P}_{H^2_{-,0}}[\mu\circ\psi_{0,\theta}]\Big)\mathbf{H}_{\mathbb{D}_e}\big[\exp\big(-2\operatorname{Re}\mathbf{P}_{H^2}[\mu\circ\psi_{0,\theta}]\big)\big]\,d\theta.$$

As a first order approximation, we start with $\psi_{0,t}^{[0]}(\zeta)=\zeta$, and use the formula (6.3.8) to define $g_t^{[j+1]}$ in terms of $\psi_{0,t}^{[j]}$, for $j=0,1,2,\ldots$ by integration. The process is interlaced with computing $\psi_{0,t}^{[j+1]} := \zeta\exp(g_t^{[j+1]})$, and results in convergent sequences $g_t^{[j]}$ and $\psi_{0,t}^{[j]}$.

We are interested in analyzing where the function $\psi_{0,t}$ extends as a holomorphic mapping. To this end, we recall that the function μ given by (6.3.4) has a well-defined polarization to $\hat{\mathbb{A}}(\varrho_1,\sigma_1)$. It is clear that if for some $\hat\varrho_t<1$, $\psi_{0,t}$ maps the annulus $\mathbb{A}(\hat\varrho_t,(\hat\varrho_t)^{-1})$ into $\mathbb{A}(\varrho_1,\varrho_1^{-1})$, we obtain the estimate

$$\|\partial_t g_t\|_{H^\infty(\mathbb{A}(\hat\varrho_t,(\hat\varrho_t)^{-1}))} \le \frac{\sqrt{2}}{1-\hat\varrho_t^2}\exp\left(5\frac{\|\mu\|_{H^\infty(\mathbb{A}(\varrho_1,\varrho_1^{-1}))}}{1-\hat\varrho_t^2}\right),$$

where we use the estimate

$$\|\mathbf{P}_{H^2}[f]\|_{H^\infty(\mathbb{D}(0,(\hat\varrho_t)^{-1}))} \le \frac{\|f\|_{H^\infty(\mathbb{A}(\hat\varrho_t,(\hat\varrho_t)^{-1}))}}{1-\hat\varrho_t^2},$$

and the analogous estimate for $\mathbf{P}_{H^2_{-,0}}[f]$. Assume for the moment that $\hat\varrho_t<1$ is monotonically increasing in $|t|$, and recall that $\psi_{0,t}(\zeta)=\zeta\exp(g_t)$. In light of the above estimate of $\partial_t g_t$, we obtain

$$\|g_t\|_{H^\infty(\mathbb{A}(\hat\varrho_t,(\hat\varrho_t)^{-1}))} \le \frac{\sqrt{2}|t|}{1-\varrho_t^2}\exp\left(5\frac{\|\mu\|_{H^\infty(\mathbb{A}(\varrho_1,\varrho_1^{-1}))}}{1-\hat\varrho_t^2}\right) =: C_t|t|,$$

where C_t is defined implicitly by the last relation. This leads to the control

$$e^{-C_t|t|}\hat\varrho_t \le |\psi_{0,t}(\zeta)| \le e^{C_t|t|}(\hat\varrho_t)^{-1}, \qquad \zeta\in\mathbb{A}(\hat\varrho_t,(\hat\varrho_t)^{-1}),$$

which means that $\psi_{0,t}$ maps the annulus $\mathbb{A}(\hat\varrho_t,(\hat\varrho_t)^{-1})$ into $\mathbb{A}(\varrho_1,\varrho_1^{-1})$, provided that

$$e^{-C_t|t|}\hat\varrho_t \ge \varrho_1.$$

Let us make the ansatz $\hat\varrho_t=\varrho_1 e^{M|t|}$, for some positive constant M. The above requirement is then satisfied provided that $M\ge C_t$. If we restrict t to the interval

(6.3.9)
$$|t| \le \frac{\log\frac{1}{\varrho_1}}{2M},$$

it is immediate that

$$\frac{1}{1-\hat\varrho_t^2} \le \frac{1}{1-\varrho_1}.$$

This then gives the estimate for C_t

$$C_t \le \frac{\sqrt{2}}{1-\varrho_1}\exp\left(5\frac{\|\mu\|_{H^\infty(\mathbb{A}(\varrho_1,\varrho_1^{-1})}}{1-\varrho_1}\right),$$

where the right-hand side does not depend on t. We may finally choose M to be this constant,

(6.3.10)
$$M = \frac{\sqrt{2}}{1-\varrho_1}\exp\left(5\frac{\|\mu\|_{H^\infty(\mathbb{A}(\varrho_1,\varrho_1^{-1})}}{1-\varrho_1}\right)$$

and obtain that $\psi_{0,t}$ is holomorphic in the exterior disk $\mathbb{D}_{\mathrm{e}}(0,\hat{\varrho}_t)$, where

$$\hat{\varrho}_t = \varrho_1 e^{M|t|},$$

provided that t satisfies (6.3.9). For t close to 0, $\hat{\varrho}_t$ is then close to ϱ_1 in a quantifiable fashion. We gather these observations in a proposition.

Proposition 6.3.1. *Suppose R is in the class $\mathcal{W}(\rho_0,\sigma_0)$ and that* (6.3.1) *holds. Then the conformal maps $\psi_{0,t}$, initially defined on $\bar{\mathbb{D}}_{\mathrm{e}}$, extend to holomorphic functions on the exterior disk $\mathbb{D}_{\mathrm{e}}(0,\hat{\rho}_t)$, where $\hat{\rho}_t = \rho_1 e^{M|t|} \le \sqrt{\rho_1}$ and M is given by* (6.3.10)*, provided that t is in the interval* (6.3.9).

6.4. **An outline of the orthogonal foliation flow algorithm.** We now proceed to describe an outline of the algorithm. With the notation

$$(6.4.1)\quad \Pi_{s,t}(\zeta) = 2\,\mathrm{Re}\, h_s\circ\psi_{s,t}(\zeta) - \tfrac{2}{s}\big((R\circ\psi_{s,t})(\zeta) - \tfrac{1}{2}t^2\big) + \log\Big(\mathrm{Re}\,\big\{-\bar{\zeta}\partial_t\psi_{s,t}(\zeta)\overline{\psi'_{s,t}(\zeta)}\big\}\Big),$$

we may rewrite the master equation (6.2.2) for the orthogonal foliation flow as

$$(6.4.2)\qquad \hat{\Pi}_{j,l}(\zeta) = \frac{\partial_s^j\partial_k^l\Pi_{s,t}(\zeta)}{j!l!}\Big|_{s=t=0} = \begin{cases} -\frac{1}{2}\log(4\pi) & \text{for}\quad \zeta\in\mathbb{T}\ \text{ and }\ (j,l)=(0,0),\\ 0 & \text{for}\quad \zeta\in\mathbb{T}\ \text{ and }\ (j,l)\in \boldsymbol{\gamma}_{2\kappa}\setminus\{(0,0)\}. \end{cases}$$

provided that the functions h_s and $\psi_{s,t}$ obtained by solving these equations do not degenerate, as long as R remains in a bounded set of $\mathcal{W}(\varrho_0,\sigma_0)$ for some ϱ_0 with $0<\varrho_0<1$ and some $\sigma_0>0$. We recall that h_s is defined by the finite expansion

$$h_s = \sum_{j=0}^{\kappa} s^j b_j.$$

As it turns out later on in Proposition 6.11.1, we have for $j,l\ge 1$,

$$(6.4.3)\qquad \hat{\Pi}_{j-1,l}(\zeta) = -2(4\Delta R(\zeta))^{\frac{1}{2}}\,\mathrm{Re}(\bar{\zeta}\hat{\psi}_{j,l-1}(\zeta)) + \mathfrak{T}_{j-1,l}(\zeta),\qquad \zeta\in\mathbb{T},$$

where $\mathfrak{T}_{j-1,l}$ is real-valued and real-analytic, and depends only on $b_0,\ldots,b_{j-1}$ and $\hat{\psi}_{p,q}$ where $(p,q)\prec_{\mathrm{L}}(j,l-1)$, where we recall that $\prec_{\mathrm{L}}$ denotes the standard lexicographic ordering. Moreover, when $l=0$ we get

$$(6.4.4)\qquad \hat{\Pi}_{j,0}(\zeta) = 2\,\mathrm{Re}\, b_j(\zeta) + \mathfrak{T}_{j,0}(\zeta),\qquad \zeta\in\mathbb{T},$$

where $\mathfrak{T}_{j,0}$ depends only on $b_0,\ldots,b_{j-1}$ and $\hat{\psi}_{p,q}$ for $(p,q)\prec_{\mathrm{L}}(j+1,0)$. Such dependencies will be encoded in terms of complexity classes introduced in §6.7.

Step 1. We let $\psi_{0,t}$ be the orthostatic conformal mappings to the exterior of level curves of R, as given by Proposition 6.2.3. In particular, this determines uniquely the coefficient functions $\hat{\psi}_{0,l}$, for $l=0,\ldots,2\kappa+1$ (for the details, see Proposition 6.8.1 below). For instance, we find that $\hat{\psi}_{0,0}(\zeta)=\zeta$, while $\hat{\psi}_{0,1}(\zeta) = -\zeta\mathbf{H}_{\mathbb{D}_{\mathrm{e}}}[(4\Delta R)^{-\frac{1}{2}}]$.

Step 2. By evaluating $\hat{\Pi}_{0,0}(\zeta) = \Pi_{s,t}(\zeta)\big|_{s=t=0}$, we obtain from (6.4.2) that

$$2\,\mathrm{Re}\, b_0(\zeta) + \log\mathrm{Re}(-\bar{\zeta}\hat{\psi}_{0,1}(\zeta)) = -\frac{1}{2}\log(4\pi),\qquad \zeta\in\mathbb{T}.$$

As $\hat\psi_{0,1}$ is already known and the above real part is strictly positive on $\mathbb{T}$ (see Proposition 6.8.1 below), this gives the value of $2\operatorname{Re} b_0$ on the unit circle $\mathbb{T}$, which gives that

$$b_0 = -\frac{1}{4}\log(4\pi) + \frac{1}{4}\mathbf{H}_{\mathbb{D}_{\mathrm{e}}}\big[\log(4\Delta R)\big].$$

We proceed from Step 2 to Step 3 with $j = 1$.

Step 3. We have determined $b_0, \ldots, b_{j_0-1}$ and $\hat\psi_{j,l}$ for all $(j,l) \prec_{\mathrm{L}} (j_0, 0)$, and in this step we intend to determine all the coefficient functions $\hat\psi_{j,l}$ for $(j,l) \prec_{\mathrm{L}} (j_0+1, 0)$. In view of Proposition 6.11.1 below, we may obtain explicitly $\mathfrak{T}_{j_0-1,1}$ in terms of this known data set, which by the equations (6.4.2) and (6.4.3) gives an equation for $\hat\psi_{j_0,0}$. More generally, the equation which gives $\hat\psi_{j_0,l_0}$ takes the form

$$\operatorname{Re}(\bar\zeta\hat\psi_{j_0,l_0}) = \tfrac{1}{2}(4\Delta R)^{-\frac{1}{2}}\mathfrak{T}_{j_0-1,l_0+1} \quad \text{on } \ \mathbb{T},$$

and we solve it with the formula

$$\hat\psi_{j_0,l_0}(\zeta) = \tfrac{1}{2}\zeta\,\mathbf{H}_{\mathbb{D}_{\mathrm{e}}}\big[(4\Delta R)^{-\frac{1}{2}}\mathfrak{T}_{j_0-1,l_0+1}\big](\zeta).$$

If we apply this solution formula with $l = 0$, the background data gets extended to all $\hat\psi_{j,l}$ with $(j,l) \prec_{\mathrm{L}} (j_0, 1)$. Continuing in the same fashion, Proposition 6.11.1 shows that $\mathfrak{T}_{j_0-1,2}$ may be expressed in terms of this extended data set. Consequently, the above solution formula also determines $\hat\psi_{j_0,1}$. More generally, as we proceed iteratively in the same manner, we obtain all the coefficient functions $\hat\psi_{j,l}$ with $j = j_0$ and $(j,l) \prec_{\mathrm{L}} (j_0+1, 0)$.

Step 4. At this stage, using Step 3, we have at our disposal the functions $b_0, \ldots, b_{j_0-1}$, and $\hat\psi_{j,l}$ for all $(j,l) \prec_{\mathrm{L}} (j_0+1, 0)$. Proposition 6.11.1 now allows us to compute $\mathfrak{T}_{j_0,0}$ in terms of this data, and from (6.4.2) and (6.4.4), we derive an equation for b_{j_0}:

$$2\operatorname{Re} b_{j_0} = -\mathfrak{T}_{j_0,0}, \qquad \text{on } \mathbb{T}.$$

We solve this equation explicitly by

$$b_{j_0}(\zeta) = -\frac{1}{2}\mathbf{H}_{\mathbb{D}_{\mathrm{e}}}\big[\mathfrak{T}_{j_0,0}\big](\zeta), \qquad \zeta \in \mathbb{D}_{\mathrm{e}}.$$

After completing this step in the algorithm, we have extended the data set to contain $b_0, \ldots, b_{j_0}$ and all coefficient functions $\hat\psi_{j,l}$ with $(j,l) \prec_{\mathrm{L}} (j_0+1, 0)$.

Step 5. Finally, we iterate Steps 3 and 4 with j_0 replaced by $j_0 + 1$, until all coefficient functions b_k and $\hat\psi_{j,l}$ have been determined, for $0 \le k \le \kappa$ and $(j,l) \in \Upsilon_{2\kappa+1}$. This also means that the flow equation (6.4.2) is met with the given choices of coefficient functions.

Remark 6.4.1. If we apply the above algorithm to the function $R = R_\tau$, the coefficient functions B_j in the expansion of $f_s = \exp(h_s)$ obtained here are the same as those appearing in Theorem 1.5.4. There, the algorithm was based on Laplace's method and inhomogeneous Toeplitz kernel equations. The algorithm presented here is in principle an alternative route towards finding the coefficient functions. However, a drawback is that the algorithm requires us to compute the additional functions $\hat\psi_{j,l}$, which adds further complexity.

6.5. **The general multivariate Faà di Bruno formula.** We recall the Faà di Bruno formula in several variables, and study some of its properties. To prepare for the formulation, we introduce the well-ordering used in [15], which we call the order-lexicographical ordering (OL for short). Given two multi-indices

$$\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d) \quad \text{and} \quad \boldsymbol{\beta} = (\beta_1, \ldots, \beta_d),$$

we write that $\boldsymbol{\alpha} \prec_{\mathrm{OL}} \boldsymbol{\beta}$ if:
(i) $|\boldsymbol{\alpha}| < |\boldsymbol{\beta}|$, or if
(ii) $|\boldsymbol{\alpha}| = |\boldsymbol{\beta}|$ and $\boldsymbol{\alpha} \prec_{\mathrm{L}} \boldsymbol{\beta}$ (lexicographically).
Here, we recall that in the lexicographical ordering $\boldsymbol{\alpha} \prec_{\mathrm{L}} \boldsymbol{\beta}$ holds if either $\alpha_1 < \beta_1$ or $\alpha_1 = \beta_1, \ldots, \alpha_{k-1} = \beta_{k-1}$ while $\alpha_{k+1} < \beta_{k+1}$ holds for some $1 \le k \le d$. As a matter of notation, $\boldsymbol{\alpha} \preceq \boldsymbol{\beta}$ means that either $\boldsymbol{\alpha} \prec \boldsymbol{\beta}$ or $\boldsymbol{\alpha} = \boldsymbol{\beta}$; this applies to both the lexicographical and order-lexicographical orderings. We use some elements of standard multi-index notation. For instance, if $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)$ is a d-dimensional multi-index, that is, a d-vector of integers in $\mathbb{N} := \{0, 1, 2, \ldots\}$, we write

$$\begin{aligned} |\boldsymbol{\alpha}| &= \sum_j \alpha_j, \\ \boldsymbol{\alpha}! &= \prod_j (\alpha_j!), \\ \boldsymbol{\xi}^{\boldsymbol{\alpha}} &= \prod_j \xi_j^{\alpha_j}, \qquad \boldsymbol{\xi} = (\xi_1, \ldots, \xi_d) \in \mathbb{C}^d, \\ \partial^{\boldsymbol{\alpha}} f(\boldsymbol{x}) &= \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d} f(\boldsymbol{x}), \qquad \boldsymbol{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d. \end{aligned}$$

We will need the index set

$$(6.5.1) \quad \mathfrak{d}_{k;d',d}^{\mathrm{OL}} = \big\{ (\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_k; \boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_k) \in (\mathbb{N}^{d'})^k \times (\mathbb{N}^d)^k : \\ 0 \prec_{\mathrm{OL}} \boldsymbol{\alpha}_1 \prec_{\mathrm{OL}} \cdots \prec_{\mathrm{OL}} \boldsymbol{\alpha}_k \text{ and } \forall j = 1, \ldots, k : \; |\boldsymbol{\beta}_j| > 0 \big\}.$$

We now formulate the multivariate Faà di Bruno's formula as it appears in [15].

Proposition 6.5.1. *Let $\Omega \subset \mathbb{R}^d$ and $\Omega' \subset \mathbb{R}^{d'}$ be domains in the respective Euclidean space. Let $\mathbf{g} = (g_1, \ldots, g_d) : \Omega' \to \Omega$ and $f : \Omega \to \mathbb{R}$ be C^n-smooth, so that the composition $f \circ \mathbf{g} : \Omega' \to \mathbb{R}$ is C^n-smooth as well. Then, for any d'-dimensional multi-index $\boldsymbol{\nu}$ with $|\boldsymbol{\nu}| = n$, we have on Ω'*

$$\partial^{\boldsymbol{\nu}}(f \circ \mathbf{g}) = \sum_{1 \le |\boldsymbol{\mu}| \le n} [(\partial^{\boldsymbol{\mu}} f) \circ \mathbf{g}] \, \mathcal{G}_{\boldsymbol{\mu},\boldsymbol{\nu}}(\mathbf{g}),$$

where $\boldsymbol{\mu}$ runs over the d'-dimensional multi-indices, and the function $\mathcal{G}_{\boldsymbol{\mu},\boldsymbol{\nu}}(\mathbf{g})$ is given by

$$\mathcal{G}_{\boldsymbol{\mu},\boldsymbol{\nu}}(\mathbf{g}) = \boldsymbol{\nu}! \sum_{k=1}^{n} \sum_{(\boldsymbol{\alpha};\boldsymbol{\beta}) \in \mathfrak{d}_k^{\mathrm{OL}}(\boldsymbol{\mu},\boldsymbol{\nu})} \prod_{j=1}^{k} \frac{[\partial^{\boldsymbol{\alpha}_j} \mathbf{g}]^{\boldsymbol{\beta}_j}}{\boldsymbol{\beta}_j! [\boldsymbol{\alpha}_j!]^{|\boldsymbol{\beta}_j|}}.$$

Here, the indicated index set is given by

$$\mathfrak{d}_k^{\mathrm{OL}}(\boldsymbol{\mu},\boldsymbol{\nu}) := \Big\{ (\boldsymbol{\alpha};\boldsymbol{\beta}) = (\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_k; \boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_k) \in \mathfrak{d}_{k;d',d}^{\mathrm{OL}} : \sum_j \boldsymbol{\beta}_j = \boldsymbol{\mu}, \; \sum_j |\boldsymbol{\beta}_j| \boldsymbol{\alpha}_j = \boldsymbol{\nu} \Big\}.$$

Note that since $\mathbf{g}$ is assumed vector-valued, the multi-index partial derivative $\partial^{\boldsymbol{\alpha}_j}\mathbf{g}$ is vector-valued as well, and the multi-index power $[\partial^{\boldsymbol{\alpha}_j}\mathbf{g}]^{\boldsymbol{\beta}_j}$ produces a real-valued function.

Remark 6.5.2. Both the order-lexicographical and the lexicographical ordering are well-orderings of the multi-indices. If in (6.5.1) we replace $\prec_{\mathrm{OL}}$ by the lexicographic ordering $\prec_{\mathrm{L}}$ to obtain the analogous index set $\mathfrak{v}^{\mathrm{L}}_{k;d',d}$, this amounts to a reshuffling of the multi-indices $\boldsymbol{\alpha}_1,\ldots,\boldsymbol{\alpha}_k$ to get them ordered with respect to $\prec_{\mathrm{L}}$ instead. This allows us to define the index set $\mathfrak{o}^{\mathrm{L}}_k(\boldsymbol{\mu},\boldsymbol{\nu})$ as well, based on $\mathfrak{v}^{\mathrm{L}}_{k;d',d}$ instead. It is important to note that the assertion of Proposition 6.5.1 holds with the index set $\mathfrak{o}^{\mathrm{OL}}_k(\boldsymbol{\mu},\boldsymbol{\nu})$ replaced by $\mathfrak{o}^{\mathrm{L}}_k(\boldsymbol{\mu},\boldsymbol{\nu})$. The reason why this is so is that if we reshuffle both the $\boldsymbol{\alpha}_j$ and the $\boldsymbol{\beta}_j$, then nothing really happened and the involved sum remains the same.

6.6. **The multivariate Faà di Bruno formula adapted to our setting.** We specialize Proposition 6.5.1 to the situation that we need to analyze. We will consider only the case of $d=d'=2$. We work in terms of polar coordinates (r,θ), and put $\text{Я}(r,\theta)=R(re^{i\theta})$. Although still not specified completely, we assume the function $\psi_{s,t}$ is sufficiently smooth in both (s,t), and introduce the function $\boldsymbol{\Psi}_{s,t}$,

$$\boldsymbol{\Psi}_{s,t}=(|\psi_{s,t}|,\arg\psi_{s,t}), \tag{6.6.1}$$

which maps to polar coordinates, so that

$$R\circ\psi_{s,t}=\text{Я}\circ\boldsymbol{\Psi}_{s,t}.$$

Accordingly, we denote by $D^{\boldsymbol{\mu}}_{r,\theta}$ the differential operator

$$D^{\boldsymbol{\mu}}_{r,\theta}=\partial_r^{\mu_1}\partial_\theta^{\mu_2},\quad \boldsymbol{\mu}=(\mu_1,\mu_2),$$

and obtain by applying Proposition 6.5.1 to $\text{Я}\circ\boldsymbol{\Psi}_{s,t}$ with $\boldsymbol{\nu}=(j,l)$ that along the circle $\mathbb{T}$,

$$\begin{aligned}\partial_s^j\partial_t^l(R\circ\psi_{s,t})\big|_{s=t=0}&=\partial_s^j\partial_t^l(\text{Я}\circ\boldsymbol{\Psi}_{\boldsymbol{s},\boldsymbol{t}})\big|_{s=t=0}\\ &=\sum_{2\le|\boldsymbol{\mu}|\le j+l}[(D^{\boldsymbol{\mu}}_{r,\theta}\text{Я})\circ\boldsymbol{\Psi}_{\boldsymbol{s},\boldsymbol{t}}]\,\mathcal{G}_{\boldsymbol{\mu},(j,l)}(\boldsymbol{\Psi}_{s,t})\Big|_{s=t=0}\\ &=\sum_{2\le|\boldsymbol{\mu}|\le j+l}[(D^{\boldsymbol{\mu}}_{r,\theta}R)(\psi_{s,t})]\,\mathcal{G}_{\boldsymbol{\mu},(j,l)}(\boldsymbol{\Psi}_{s,t})\Big|_{s=t=0}\\ &=\sum_{2\le|\boldsymbol{\mu}|\le j+l}(D^{\boldsymbol{\mu}}_{r,\theta}R)\,\mathcal{G}_{\boldsymbol{\mu},(j,l)}(\boldsymbol{\Psi}_{s,t})\Big|_{s=t=0},\end{aligned} \tag{6.6.2}$$

where the terms corresponding to indices $\boldsymbol{\mu}$ with $|\boldsymbol{\mu}|\le 1$ vanish and hence get dropped. The reason for this is that $\psi_{0,0}(\zeta)=\zeta$ preserves $\mathbb{T}$ and that the function R together with its gradient vanish along the unit circle $\mathbb{T}$. More generally, we find that

$$D^{\boldsymbol{\mu}}_{r,\theta}R|_{\mathbb{T}}=\partial_r^{\mu_1}\partial_\theta^{\mu_2}R|_{\mathbb{T}}=0,\qquad \boldsymbol{\mu}=(\mu_1,\mu_2)\in\{0,1\}\times\mathbb{N}. \tag{6.6.3}$$

In the context of (6.6.2), it is important to point out that the multi-index derivatives that appear in the expression $\mathcal{G}_{\boldsymbol{\mu},(j,l)}(\boldsymbol{\Psi}_{s,t})$ (as defined in Proposition 6.5.1) are taken with respect to the variables (s,t). Moreover, in the equality (6.6.2) we have suppressed the variable $\zeta\in\mathbb{T}$, and consider it to be fixed.

We will be interested in identifying the *maximal* index (p,q) with respect to the lexicographical ordering, such that the partial derivative $\partial_s^p\partial_t^q\boldsymbol{\Psi}_{s,t}$ appears nontrivially in the right-hand side expression of (6.6.2).

Proposition 6.6.1. *Let $\boldsymbol{\nu}, \boldsymbol{\mu} \in \mathbb{N}^2$ be double-indices with $2 \le |\boldsymbol{\mu}| \le |\boldsymbol{\nu}|$ and $\boldsymbol{\mu} \notin \{(1,1),(0,2)\}$, and let $(\boldsymbol{\alpha};\boldsymbol{\beta}) \in \mathfrak{o}_k^{\mathrm{L}}(\boldsymbol{\mu},\boldsymbol{\nu})$. Then*

(i) *If $\boldsymbol{\nu} = (j,l)$, where $j,l \ge 1$, then for all $i = 1,\ldots,k$, we have that $\boldsymbol{\alpha}_i \preceq_{\mathrm{L}} (j,l-1)$. Moreover, the equality $\boldsymbol{\alpha}_i = (j,l-1)$ holds if and only if $i = k = 2$, $\boldsymbol{\mu} = (2,0)$, and*

$$(\boldsymbol{\alpha};\boldsymbol{\beta}) = ((0,1),(j,l-1);(1,0),(1,0)).$$

(ii) *If $\boldsymbol{\nu} = (j,0)$ with $j \ge 3$, then each $\boldsymbol{\alpha}_i$ is of the form $(a,0)$ with $a \le j-1$. Moreover, the equality $a = j-1$ holds if and only if $i = k = 2$, $\boldsymbol{\mu} = (2,0)$, and*

$$(\boldsymbol{\alpha};\boldsymbol{\beta}) = ((1,0),(j-1,0);(1,0),(1,0)).$$

(iii) *If $\boldsymbol{\nu} = (0,l)$ with $l \ge 3$, then $\boldsymbol{\alpha}_i$ is of the form $(0,b)$ with $b \le l-1$. Moreover, the equality $b = l-1$ holds if and only if $\boldsymbol{\mu} = (2,0)$ and*

$$(\boldsymbol{\alpha};\boldsymbol{\beta}) = ((0,1),(0,l-1);(1,0),(1,0)).$$

(iv) *If $\boldsymbol{\nu} = (2,0)$, then necessarily $\boldsymbol{\mu} = (2,0)$ and the only nontrivial index $(\boldsymbol{\alpha};\boldsymbol{\beta})$ is*

$$(\boldsymbol{\alpha};\boldsymbol{\beta}) = ((1,0);(2,0)).$$

(v) *If $\boldsymbol{\nu} = (0,2)$, then necessarily $\boldsymbol{\mu} = (2,0)$ and the only nontrivial index $(\boldsymbol{\alpha};\boldsymbol{\beta})$ is*

$$(\boldsymbol{\alpha};\boldsymbol{\beta}) = ((0,1);(2,0)).$$

Note that since $|\boldsymbol{\nu}| \ge 2$, the above list covers all the possibilities. We will denote by $(\boldsymbol{\alpha}^{\circledast};\boldsymbol{\beta}^{\circledast})$ the indicated extremal index $(\boldsymbol{\alpha};\boldsymbol{\beta})$ in each of the cases (i)–(v).

Proof of Proposition 6.6.1. We show how to obtain (i), (ii) and (iv). The remaining cases (iii) and (v) are analogous and omitted. We recall the compatibility conditions on the index set $\mathfrak{o}_k^{\mathrm{L}}(\boldsymbol{\mu},\boldsymbol{\nu})$. After all, the assertion $(\boldsymbol{\alpha};\boldsymbol{\beta}) \in \mathfrak{o}_k^{\mathrm{L}}(\boldsymbol{\mu},\boldsymbol{\nu})$ means that $(\boldsymbol{\alpha};\boldsymbol{\beta}) \in \mathfrak{w}_{k;2,2}^{\mathrm{L}}$ plus the

$$\sum_{i=1}^{k} |\boldsymbol{\beta}_i|\boldsymbol{\alpha}_i = \boldsymbol{\nu}, \qquad \sum_{i=1}^{k} \boldsymbol{\beta}_i = \boldsymbol{\mu}, \tag{6.6.4}$$

where each $\boldsymbol{\beta}_i$ has $|\boldsymbol{\beta}_i| \ge 1$, and the multi-indices α_{i} are strictly increasing with i in the lexicographical ordering. From these assumptions it is immediate that each $\boldsymbol{\alpha}_i$ satisfies $\boldsymbol{\alpha}_i \preceq_{\mathrm{L}} \boldsymbol{\nu}$.

As for assertion (i), we see that equality $\boldsymbol{\alpha}_i = (j,l)$ could hold only if $k = 1$, with $\boldsymbol{\alpha}_1 = (j,l)$ and $|\boldsymbol{\beta}_1| = 1$. But then $|\boldsymbol{\mu}| = 1$, which would contradict our assumption that $|\boldsymbol{\mu}| \ge 2$. Hence, given the structure of the lexicographic ordering, for any index i, we have $\boldsymbol{\alpha}_i \preceq_{\mathrm{L}} (j,l-1)$. However, if equality holds here, that is, if for some i_0 we have $\boldsymbol{\alpha}_{i_0} = (j,l-1)$, we find from (6.6.4) that $|\boldsymbol{\beta}_{i_0}| = 1$, whereas the sum on the left-hand side, taken over all other indices $i \ne i_0$, must equal $(0,1)$. As a consequence, only $k = 2$ is possible, and then $\boldsymbol{\alpha} = ((0,1),(j,l-1))$. In addition, we get that $|\boldsymbol{\beta}_1| = |\boldsymbol{\beta}_2| = 1$, so that by the second relation in (6.6.4), $|\boldsymbol{\mu}| = 2$ holds. Given the assumptions on $\boldsymbol{\mu}$, the only remaining possibility is $\boldsymbol{\mu} = (2,0)$, and then $\boldsymbol{\beta}_1 = \boldsymbol{\beta}_2 = (1,0)$.

We turn our attention to the assertion (ii). In a similar manner as above, since the weighted sum of the multi-indices $\boldsymbol{\alpha}_i$ equals $(j,0)$, we see that for each index $i = 1,\ldots,k$, $\boldsymbol{\alpha}_i = (a_i,0)$ for some $a_i \in \mathbb{N}$ with $0 < a_i \le j$. It is clear that $a_{i_0} = j$ could occur for some i_0 only if $i_0 = k = 1$, $|\boldsymbol{\beta}_1| = 1$ and $|\boldsymbol{\mu}| = 1$, which again would contradict our assumption $|\boldsymbol{\mu}| \ge 2$. It follows that $a_i \le j-1$ for each i. Next, the only way we could have

$\boldsymbol{\alpha}_{i_0} = (j-1, 0)$ for some i_0 is if $i_0 = k = 2$ and correspondingly $\boldsymbol{\alpha} = ((1,0),(j-1,0))$. The remaining properties are immediate.

Finally, to see why (iv) holds, we analogously find that each $\boldsymbol{\alpha}_i$ is of the form $\boldsymbol{\alpha}_i = (a_i, 0)$, where $0 < a_i \le 2$. In view of (6.6.4),

$$a_1|\boldsymbol{\beta}_1| + \cdots + a_k|\boldsymbol{\beta}_k| = 2,$$

with $|\boldsymbol{\beta}_i| \ge 1$ for each i. This is possible only if $1 \le k \le 2$. If $k = 2$, we get that $a_1 = a_2 = 1$ and $|\boldsymbol{\beta}_1| = |\boldsymbol{\beta}_2| = 1$, which leads to $\boldsymbol{\alpha}_1 = \boldsymbol{\alpha}_2 = (1,0)$. This gets excluded on the basis of the monotonicity requirement $\boldsymbol{\alpha}_1 \prec_{\mathrm{L}} \boldsymbol{\alpha}_2$, so $k = 1$ is the only possibility. So the requirement (6.6.4) now reads $a_1|\boldsymbol{\beta}_1| = 2$ and $\boldsymbol{\beta}_1 = \boldsymbol{\mu}$. If $a_1 = 2$, then $|\boldsymbol{\beta}_1| = 1$, and consequently $|\boldsymbol{\mu}| = 1$, which is contrary to our assumption that $|\boldsymbol{\mu}| \ge 2$. The only remaining alternative is that $\boldsymbol{\alpha}_1 = (1,0)$ and $|\boldsymbol{\beta}_1| = 2$. Since $\boldsymbol{\beta}_1 = \boldsymbol{\mu}$, and the only admissible $\boldsymbol{\mu}$ of length 2 is $\boldsymbol{\mu} = (2,0)$, it follows that $\boldsymbol{\beta}_1 = (2,0)$, and the claim follows. □

We observe that in each of the cases (i)-(v), the lexicographically maximal $\boldsymbol{\alpha}_i$ occurs as the index $i = k$, where $k \in \{1,2\}$ and $(\boldsymbol{\alpha};\boldsymbol{\beta}) \in \mathfrak{d}_k^{\mathrm{L}}(\boldsymbol{\mu},\boldsymbol{\nu})$ and $\boldsymbol{\mu} = \boldsymbol{\mu}_0 := (2,0)$ while $|\boldsymbol{\nu}| \ge 2$. If we put

$$A(\boldsymbol{\nu}) = \max_k \max_{(\boldsymbol{\alpha};\boldsymbol{\beta})\in\mathfrak{d}_k^{\mathrm{L}}(\boldsymbol{\mu}_0,\boldsymbol{\nu})} \boldsymbol{\alpha}_k$$

where the maximum is taken lexicographically over the entire range $k = 1, \ldots, |\boldsymbol{\nu}|$, then the maximum occurs for $k = 2$ unless if $\boldsymbol{\nu} = (2,0)$ or $\boldsymbol{\nu} = (0,2)$. Moreover, if $\boldsymbol{\nu} = (2,0)$ or $\boldsymbol{\nu} = (0,2)$, the maximum occurs for $k = 1$. Let $k_{\boldsymbol{\nu}} \in \{1,2\}$ be the parameter value for which the maximum is attained, depending on $\boldsymbol{\nu}$, as just explained. In any of the instances (i)-(v), there exists a unique extremal pair $(\boldsymbol{\alpha}^{\circledast};\boldsymbol{\beta}^{\circledast}) \in \mathfrak{d}_k^{\mathrm{L}}(\boldsymbol{\mu}_0,\boldsymbol{\nu})$ provided that $k = k_{\boldsymbol{\nu}}$. Next, let $\mathfrak{d}_{k,\circledast}^{\mathrm{L}}(\boldsymbol{\mu}_0,\boldsymbol{\nu})$ denote the depleted index set

$$\mathfrak{d}_{k,\circledast}^{\mathrm{L}}(\boldsymbol{\mu}_0,\boldsymbol{\nu}) = \begin{cases} \mathfrak{d}_k^{\mathrm{L}}(\boldsymbol{\mu}_0,\boldsymbol{\nu}), & \text{if } \; k \ne k_{\boldsymbol{\nu}}, \\ \mathfrak{d}_k^{\mathrm{L}}(\boldsymbol{\mu}_0,\boldsymbol{\nu}) \setminus \{(\boldsymbol{\alpha}^{\circledast};\boldsymbol{\beta}^{\circledast})\}, & \text{if } \; k = k_{\boldsymbol{\nu}}, \end{cases}$$

and consider the associated expression in the context of the multivariate Faà di Bruno formula:

$$\mathcal{G}^{\circledast}_{\boldsymbol{\mu}_0,\boldsymbol{\nu}}(\boldsymbol{\Psi}_{s,t}) := \boldsymbol{\nu}! \sum_{k=1}^{|\boldsymbol{\nu}|} \sum_{(\boldsymbol{\alpha};\boldsymbol{\beta})\in\mathfrak{d}_{k,\circledast}^{\mathrm{L}}(\boldsymbol{\mu}_0,\boldsymbol{\nu})} \prod_{j=1}^{k} \frac{[\partial^{\boldsymbol{\alpha}_j}\boldsymbol{\Psi}_{s,t}]^{\boldsymbol{\beta}_j}}{\boldsymbol{\beta}_j![\boldsymbol{\alpha}_j!]^{|\boldsymbol{\beta}_j|}}. \tag{6.6.5}$$

Then $\mathcal{G}_{\boldsymbol{\mu}_0,\boldsymbol{\nu}}(\boldsymbol{\Psi}_{s,t})$ splits as follows (where $(\boldsymbol{\alpha}^{\circledast};\boldsymbol{\beta}^{\circledast}) = (\boldsymbol{\alpha}_1^{\circledast},\ldots,\boldsymbol{\alpha}_{k_{\boldsymbol{\nu}}}^{\circledast};\boldsymbol{\beta}_1^{\circledast},\ldots,\boldsymbol{\beta}_{k_{\boldsymbol{\nu}}}^{\circledast})$):

$$\mathcal{G}_{\boldsymbol{\mu}_0,\boldsymbol{\nu}}(\boldsymbol{\Psi}_{s,t}) = \mathcal{G}^{\circledast}_{\boldsymbol{\mu}_0,\boldsymbol{\nu}}(\boldsymbol{\Psi}_{s,t}) + \mathcal{H}_{\boldsymbol{\mu}_0,\boldsymbol{\nu}}(\boldsymbol{\Psi}_{s,t}), \quad \mathcal{H}_{\boldsymbol{\mu}_0,\boldsymbol{\nu}}(\boldsymbol{\Psi}_{s,t}) := \boldsymbol{\nu}! \prod_{j=1}^{k_{\boldsymbol{\nu}}} \frac{[\partial^{\boldsymbol{\alpha}_j^{\circledast}}\boldsymbol{\Psi}_{s,t}]^{\boldsymbol{\beta}_j^{\circledast}}}{\boldsymbol{\beta}_j^{\circledast}![\boldsymbol{\alpha}_j^{\circledast}!]^{|\boldsymbol{\beta}_j^{\circledast}|}}. \tag{6.6.6}$$

If $\boldsymbol{\nu} = (2,0)$, the depleted index set $\mathfrak{d}_{k,\circledast}^{\mathrm{L}}(\boldsymbol{\mu}_0,\boldsymbol{\nu})$ is empty for $k \in \{1,2\}$, which gives that

$$\mathcal{G}^{\circledast}_{\boldsymbol{\mu}_0,\boldsymbol{\nu}}(\boldsymbol{\Psi}_{s,t}) = 0 \quad \text{if } \; \boldsymbol{\nu} = (2,0).$$

6.7. **Polynomial complexity classes.** In order to make sure that the algorithm outlined above in §6.4 does not break down, we need to carefully keep track of the dependency structure of the coefficient functions involved. In particular, when solving for the coefficient function $\hat{\psi}_{j,l}$ in terms of a Herglotz operator applied to a function $g_{j,l}$, we need to know that $g_{j,l}$ may be computed in terms of functions already determined in previous steps of the algorithm. To help with this, we introduce for a nonnegative integer j and a subset $\Sigma\subset\mathbb{N}^2$ the *polynomial complexity class* $\mathrm{POL}(j,\Sigma)$, defined as the following function class on the unit circle $\mathbb{T}$:

$$\begin{aligned}\mathrm{POL}(j,\Sigma)\\ =\mathbb{R}\Big[&\operatorname{Re}\zeta,\operatorname{Im}\zeta,D^{\boldsymbol{\alpha}}_{r,\theta}R(\zeta),\operatorname{Re}b^{(k)}_\nu,\operatorname{Im}b^{(k)}_\nu,\operatorname{Re}\hat\psi_{p,q},\operatorname{Im}\hat\psi_{p,q},\operatorname{Re}\hat\psi'_{p,q},\operatorname{Im}\hat\psi'_{p,q}\\ &\text{such that }\;k\in\mathbb{N},\;0\le\nu\le j,\;(p,q)\in\Sigma,\;\boldsymbol{\alpha}\in\mathbb{N}^2\Big].\end{aligned}$$

Here, $\mathbb{R}[X:Y]$ denotes the class of multivariate polynomials with real coefficients in the variables X, restricted by the condition Y. In other words, $\mathrm{POL}(j,\Sigma)$ is the collection of multivariate polynomials in the expressions

$$\operatorname{Re}\zeta,\operatorname{Im}\zeta,D^{\boldsymbol{\alpha}}_{r,\theta}R(\zeta),\operatorname{Re}b^{(k)}_\nu,\operatorname{Im}b^{(k)}_\nu,\operatorname{Re}\hat\psi_{p,q},\operatorname{Im}\hat\psi_{p,q},\operatorname{Re}\hat\psi'_{p,q},\;\text{and}\;\operatorname{Im}\hat\psi'_{p,q},$$

under the conditions $k\in\mathbb{N}$, $0\le\nu\le j$, $(p,q)\in\Sigma$, and $\boldsymbol{\alpha}\in\mathbb{N}^2$. If there is no dependence on any of the functions b_j, we simplify the notation and write $\mathrm{POL}(\Sigma)$ for the polynomial complexity class. In connection with these classes, we will find it useful to introduce for nonnegative integers p and q the rectangular index sets

$$\Sigma_{p,q}=\{(a,b)\in\mathbb{N}^2\,:\,a\le p\;\text{ and }\;b\le q\}.$$

6.8. **The semiclassical case of the orthogonal foliation flow.** We first explore Step 1 of the algorithmic procedure outlined in §6.4. We recall the notation $\boldsymbol{\Psi}_{0,t}=(|\psi_{0,t}|,\arg\psi_{0,t})$ from (6.6.1). Moreover, we recall that ϱ_1 is as in Proposition 6.1.5 (see also Proposition 6.1.2). We have already established the regularity of $\psi_{0,t}$ in the implicit function theorem of §6.3. We proceed to compute the Taylor coefficients in t, and highlight the algorithmic aspects. We use the notation introduced in §6.5 and §6.6 freely.

Proposition 6.8.1. *The Taylor coefficients $\hat\psi_{0,l}$ in the variable t near $t=0$ of the conformal mapping $\psi_{0,t}$ with*

$$\psi_{0,t}(\zeta)=\sum_{l=0}^{2\kappa+1}t^l\hat\psi_{0,l}(\zeta)+\mathrm{O}(t^{2\kappa+2}),$$

are uniquely determined by the level-curve requirement

$$R\circ\psi_{0,t}(\zeta)=\frac{t^2}{2},\qquad\zeta\in\mathbb{T},$$

together with the monotonicity condition that the images $\psi_{0,t}(\mathbb{D}_{\mathrm{e}})$ grow with t and the normalization $\psi'_{0,t}(\infty)>0$. Moreover, as such, they are given by

$$\begin{aligned}\hat\psi_{0,0}(\zeta)&=\zeta,\\ \hat\psi_{0,1}(\zeta)&=-\zeta\mathbf{H}_{\mathbb{D}_{\mathrm{e}}}\big[(4\varDelta R)^{-\frac12}\big](\zeta),\end{aligned}$$

and, more generally, by

$$\hat\psi_{0,l}(\zeta)=\zeta\mathbf{H}_{\mathbb{D}_{\mathrm{e}}}\big[(4\varDelta R)^{-\frac12}\mathfrak{G}_l\big](\zeta),\qquad l=2,\dots,2\kappa+3,$$

where $\mathfrak{G}_l(\zeta) \in \mathrm{POL}(\Sigma_{0,l-1})$ *is given by the formula* ($\boldsymbol{\mu}_0 = (2,0)$)

$$\mathfrak{G}_l(\zeta) := \frac{1}{(l+1)!}\bigg(4(\Delta R)\,\mathcal{G}^{\circledast}_{\boldsymbol{\mu}_0,(0,l+1)}(\boldsymbol{\Psi}_{0,t})\big|_{t=0} + \sum_{3\le|\boldsymbol{\mu}|\le l+1}(\partial_r^{\mu_1}\partial_\theta^{\mu_2}R)\,\mathcal{G}_{\mu,(0,l+1)}(\boldsymbol{\Psi}_{0,t})\Big|_{t=0} - 2(l+1)(\Delta R)^{\frac12}\mathfrak{g}_l\bigg),$$

where

$$\mathfrak{g}_l := \partial_t^l|\psi_{0,t}|\big|_{t=0} - l!\,\mathrm{Re}(\bar\zeta\hat\psi_{0,l}).$$

The coefficient functions $\hat\psi_{0,l}$ *all extend holomorphically to the domain* $\mathbb{D}_\mathrm{e}(0,\varrho_1)$.

Proof. By Proposition 6.2.3, the conformal mappings $\psi_{0,t}$ are uniquely defined by the given requirements, and $\psi_{0,0}(\zeta) = \zeta$ holds. Moreover, since $t \mapsto \psi_{0,t}$ is smooth, the validity of the indicated expansion follows from Taylor's formula, and the first coefficient then equals $\hat\psi_{0,0}(\zeta) = \psi_{0,0}(\zeta) = \zeta$. In view of Taylor's formula applied to the function $t \mapsto R\circ\psi_{0,t}$, we have that

$$(R\circ\psi_{0,t})(\zeta) = \sum_{l=0}^{2\kappa+1}\frac{t^l}{l!}\partial_t^l(R\circ\psi_{0,t}(\zeta))\big|_{t=0} + \mathrm{O}(|t|^{2\kappa+2}).$$

Since by assumption $R\circ\psi_{0,t}(\zeta) = \frac{t^2}{2}$ holds on $\mathbb{T}$, we find that for $\zeta\in\mathbb{T}$,

$$\partial_t^l(R\circ\psi_{0,t})(\zeta)\big|_{t=0} = \begin{cases}1, & \text{for } l=2,\\ 0, & \text{for } l\neq 2.\end{cases} \tag{6.8.1}$$

It is automatic that (6.8.1) holds for $0\le l\le 1$, since R is quadratically flat on $\mathbb{T}$. We now consider $l=2$. By the multivariate Faà di Bruno formula (6.6.2) with $s=0$ treated as constant, together with the quadratic flatness of R near the unit circle $\mathbb{T}$ a calculation shows that

$$\partial_t^2(R\circ\psi_{0,t})\big|_{t=0} = (\partial_r^2R)(\partial_t|\psi_{0,t}|)^2\big|_{t=0} = 4\Delta R\,[\mathrm{Re}(\bar\zeta\hat\psi_{0,1})]^2 \quad \text{on } \mathbb{T}.$$

Since the left-hand side equals 1 by (6.8.1), we may solve for $\mathrm{Re}(\bar\zeta\hat\psi_{0,1})$ using either the positive or the negative root. We choose the negative square root, which gives that

$$\partial_t|\psi_{0,t}|\big|_{t=0} = \mathrm{Re}(\bar\zeta\hat\psi_{0,1}) = -(4\Delta R)^{-\frac12} \quad \text{on } \mathbb{T}. \tag{6.8.2}$$

This choice is the one which is compatible with the growth of the domains $\psi_{0,t}(\mathbb{D}_\mathrm{e})$ as t increases (so that the loops $\psi_{0,t}(\mathbb{T})$ move inward). Finally, we solve this equation by means of the formula

$$\hat\psi_{0,1}(\zeta) = -\zeta\mathbf{H}_{\mathbb{D}_\mathrm{e}}\big[(4\Delta R)^{-\frac12}\big](\zeta),$$

as in Step 3 of the algorithmic procedure in §6.4. Here, the uniqueness of the solution follows from Remark 6.2.2 (a). Since $(4\Delta R)^{-\frac12}$ has a polarization which is holomorphic in $(z,\bar w)$ for $(z,w)\in\hat{\mathbb{A}}(\varrho_1,\sigma_1)$, the function $\hat\psi_{0,1}$ extends holomorphically to $\mathbb{D}_\mathrm{e}(0,\varrho_1)$, by Proposition 6.1.2 and Remark 6.1.4.

As for the higher order Taylor coefficients, we again apply the multivariate Faà di Bruno formula (6.6.2). As a result, on the circle $\zeta\in\mathbb{T}$ we have for $l\ge 3$ that (apply (6.8.2) in the last step)

$$\begin{aligned}(6.8.3)\quad \partial_t^l(R\circ\psi_{0,t})\big|_{t=0} &= \sum_{2\le|\boldsymbol{\mu}|\le l}(\partial_r^{\mu_1}\partial_\theta^{\mu_2}R)\,\mathcal{G}_{\boldsymbol{\mu},(0,l+1)}(\boldsymbol{\Psi}_{0,t})\Big|_{t=0}\\ &= 4l\,(\mathit{\Delta}R)(\partial_t^{l-1}|\psi_{0,t}|)\,(\partial_t|\psi_{0,t}|)+\mathcal{G}^{\circledast}_{\boldsymbol{\mu}_0,l}(\boldsymbol{\Psi}_{0,t})\Big|_{t=0}\\ &\quad+\sum_{3\le|\boldsymbol{\mu}|\le l}(\partial_r^{\mu_1}\partial_\theta^{\mu_2}R)\,\mathcal{G}_{\boldsymbol{\mu},(0,l)}(\boldsymbol{\Psi}_{0,t})\Big|_{t=0}\\ &= -l!(4\mathit{\Delta}R)^{\frac12}\ \mathrm{Re}(\bar\zeta\hat\psi_{0,l-1})-l(4\mathit{\Delta}R)^{\frac12}\mathfrak{g}_{l-1}+\mathcal{G}^{\circledast}_{\boldsymbol{\mu}_0,(0,l)}(\boldsymbol{\Psi}_{0,t})\Big|_{t=0}\\ &\qquad+\sum_{3\le|\boldsymbol{\mu}|\le l}(\partial_r^{\mu_1}\partial_\theta^{\mu_2}R)\,\mathcal{G}_{\boldsymbol{\mu},(0,l)}(\boldsymbol{\Psi}_{0,t})\Big|_{t=0},\end{aligned}$$

where $\boldsymbol{\mu}_0=(2,0)$ and we recall that

$$\mathfrak{g}_{l-1}=\partial_t^{l-1}|\psi_{0,t}|\big|_{t=0}-(l-1)!\,\mathrm{Re}(\bar\zeta\hat\psi_{0,l-1}).$$

An elementary computation shows that the highest order derivatives cancel out, and it follows that $\mathfrak{g}_{l-1}\in\mathrm{POL}(\Sigma_{0,l-2})$.

We recall that the expression $\mathcal{G}^{\circledast}_{\boldsymbol{\mu}_0,(0,l+1)}(\boldsymbol{\Psi}_{0,t})$ appearing in the above formula is as in (6.6.5). We write

$$\mathfrak{G}_{l-1}=\frac{1}{l!}\bigg(-l(4\mathit{\Delta}R)^{\frac12}\mathfrak{g}_{l-1}+\mathcal{G}^{\circledast}_{\boldsymbol{\mu}_0,(0,l)}(\boldsymbol{\Psi}_{0,t})\big|_{t=0}+\sum_{3\le|\boldsymbol{\mu}|\le l}(\partial_r^{\mu_1}\partial_\theta^{\mu_2}R)\,\mathcal{G}_{\boldsymbol{\mu},(0,l)}(\boldsymbol{\Psi}_t)\Big|_{t=0}\bigg),$$

and claim that $\mathfrak{G}_{l-1}\in\mathrm{POL}(\Sigma_{0,l-2})$. We already saw that $\mathfrak{g}_l$ has this property, and hence $(\mathit{\Delta}R)^{\frac12}\mathfrak{g}_{l-1}=(\mathit{\Delta}R)\,\mathrm{Re}(-\bar\zeta\hat\psi_{0,1})$ does as well. That the same holds for the remaining two terms of the above formula can be seen from Proposition 6.6.1, and hence it follows that $\mathfrak{G}_{l-1}\in\mathrm{POL}(\Sigma_{0,l-2})$. It is a consequence of (6.8.3) that the condition (6.8.1) for $l\ge 3$ may be expressed as

$$-(4\mathit{\Delta}R)^{\frac12}\,\mathrm{Re}(\bar\zeta\hat\psi_{0,l-1})+\mathfrak{G}_{l-1}=0,\qquad l=3,4,5,\ldots.$$

This is an equation of a kind we have met before, and we know that a solution $\hat\psi_{0,l}$ is supplied by the formula (change l by $l+1$ in the previous relation)

$$(6.8.4)\qquad \hat\psi_{0,l}(\zeta)=\zeta\mathbf{H}_{\mathbb{D}_{\mathrm{e}}}\bigg[\frac{\mathfrak{G}_l}{(4\mathit{\Delta}R)^{\frac12}}\bigg](\zeta),\qquad l=2,3,4,\ldots.$$

Let us assume for the moment that the lower order terms $\hat\psi_{0,b}$ with $0\le b\le l-1$ all extend holomorphically to an exterior disk $\mathbb{D}_{\mathrm{e}}(0,\varrho_1)$. Then the entire expression inside brackets in (6.8.4) polarizes to extend to a $2\sigma_1$-fattened diagonal annulus $\hat{\mathbb{A}}(\varrho_1,\sigma_1)$ given that various partial derivatives of R do, as well as $(\mathit{\Delta}R)^{-\frac12}$, which follows from Proposition 6.1.5. Moreover, since ϱ_1 is big enough to guarantee that $\varrho_1\ge(\sqrt{1+\sigma_1^2}+\sigma_1)^{-1}$, then in view of Proposition 6.1.2, the expression on the right-hand side of (6.8.4) will be holomorphic in the same exterior disk $\mathbb{D}_{\mathrm{e}}(0,\varrho_1)$ as well, by Remark 6.1.3. But then we have enough to keep the iteration going, and obtain that all the terms $\hat\psi_{0,l}$ extend holomorphically to a single exterior disk $\mathbb{D}_{\mathrm{e}}(0,\varrho_1)$. □

6.9. **Taylor expansion of the weight term in the master equation.** We continue with the Taylor expansion of the composition $R\circ\psi_{s,t}$ in terms of powers of s and t, where the starting point is the application of the Faà di Bruno formula in (6.6.2). We recall the definition (6.2.1) of the triangular index set $\curlyvee_n$. We work under the assumption that $\psi_{s,t}$ depends sufficiently smoothly on both (s,t) near $(0,0)$. This assumption gets justified in the stepwise proof which we outlined in Subsection 6.8, which retrieves the Taylor coefficients of $\psi_{s,t}$ in (s,t) iteratively. We use the notion of polynomial complexity classes $\mathrm{POL}(j,\Sigma)$ and the index sets $\Sigma_{p,q}$ from §6.7.

Proposition 6.9.1. *On the unit circle $\mathbb{T}$, the function $R\circ\psi_{s,t}$ enjoys the expansion*

$$2R\circ\psi_{s,t}=2R\circ\psi_{0,t}+\sum_{(j,l)\in\curlyvee_{2\kappa}}s^{j+1}t^{l}\mathfrak{R}_{j,l}+\mathrm{O}\big(|s|\big(|s|^{\kappa+\frac12}+|t|^{2\kappa+1}\big)\big),$$

where $\mathfrak{R}_{0,0}=0$, while for the remaining indices $(j,l)\neq(0,0)$, we have

$$\mathfrak{R}_{j,l}=\frac{2}{(j+1)!l!}\Big((4\varDelta R)\,[\mathcal{H}_{\boldsymbol{\mu}_0,(j+1,l)}(\boldsymbol{\Psi}_{s,t})]\Big|_{s=t=0}+\mathfrak{r}_{j,l}\Big),$$

where $\boldsymbol{\mu}_0=(2,0)$. Here, the main term is given in terms of $\mathcal{H}_{\boldsymbol{\mu}_0,(j+1,l)}(\boldsymbol{\Psi}_{s,t})$, defined by

$$\mathcal{H}_{\boldsymbol{\mu}_0,(j+1,l)}(\boldsymbol{\Psi}_{s,t})=\begin{cases}l\,(\partial_t|\psi_{s,t}|)(\partial_s^{j+1}\partial_t^{l-1}|\psi_{s,t}|), & \text{if } j\geq 0,\ l\geq 1,\\ (j+1)(\partial_s|\psi_{s,t}|)(\partial_s^{j}|\psi_{s,t}|), & \text{if } j\geq 2,\ l=0,\\ (\partial_s|\psi_{s,t}|)^2, & \text{if } j=1,\ l=0,\end{cases}$$

while the term $\mathfrak{r}_{j,l}$, considered as a remainder, is given by

$$\mathfrak{r}_{j,l}=(4\varDelta R)\,\mathcal{G}^{\circledast}_{\boldsymbol{\mu}_0,(j+1,l)}(\boldsymbol{\Psi}_{s,t})+\sum_{3\leq|\boldsymbol{\mu}|\leq j+l+1}(D^{\boldsymbol{\mu}}_{r,\theta}R)\,\mathcal{G}_{\boldsymbol{\mu},(j+1,l)}(\boldsymbol{\Psi}_{s,t})\Big|_{s=t=0},$$

where we recall that $\mathcal{G}^{\circledast}_{\boldsymbol{\mu}_0,(j+1,l)}(\boldsymbol{\Psi}_{s,t})$ is given by (6.6.5). For $j\geq 0$ and $l\geq 1$, we have

$$\mathfrak{r}_{j,l}\in\mathrm{POL}(\Sigma),\quad \textit{with}\quad \Sigma=\{(p,q)\in\Sigma_{j+1,l}:\,(p,q)\prec_{\mathrm{L}}(j+1,l-1)\}).$$

In a similar fashion, for $j\geq 1$, we have that the Taylor coefficient $\mathfrak{R}_{j,0}\in\mathrm{POL}(\Sigma_{j,0})$. Moreover, the implied constant in the above expansion of $R\circ\psi_{s,t}$ remains bounded if the weight R is confined to a uniform family in $\mathcal{W}(\varrho_0,\sigma_0)$ for some fixed $0<\varrho_0<1$ and $\sigma_0>0$, while the functions $\psi_{s,t}$ are smooth with bounded norms in $C^{2\kappa+4}$ with respect to (s,t) in a neighborhood of $(0,0)$, uniformly on the circle $\mathbb{T}$.

Proof. The fact that $R\circ\psi_{s,t}$ enjoys an expansion of the indicated form for some coefficients $\mathfrak{R}_{j,l}$ with the given error term is an immediate consequence of the multivariate Taylor's formula. The coefficients $\mathfrak{R}_{j,l}$ are then obtained from the successive partial derivatives (6.6.2). It just remains to calculate them:

$$\begin{aligned}\mathfrak{R}_{j,l}=\frac{2}{(j+1)!l!}\,\partial_s^{j+1}\partial_t^{l}\big(R\circ\psi_{s,t}-R\circ\psi_{0,t}\big)\Big|_{s=t=0}&=\frac{2}{(j+1)!l!}\,\partial_s^{j+1}\partial_t^{l}\big(R\circ\psi_{s,t}\big)\Big|_{s=t=0}\\&=\frac{2}{(j+1)!l!}\sum_{2\leq|\boldsymbol{\mu}|\leq j+l+1}(D^{\boldsymbol{\mu}}_{r,\theta}R)\,\mathcal{G}_{\boldsymbol{\mu},(j+1,l)}(\boldsymbol{\Psi}_{s,t})\Big|_{s=t=0}.\end{aligned}$$

In particular, $\mathfrak{R}_{0,0}=0$, as the sum is over the empty set. In the right-hand side, the sum over $|\boldsymbol{\mu}|=2$ is special as the only nontrivial contribution comes from the index $\boldsymbol{\mu}=\boldsymbol{\mu}_0=$

$(2,0)$, by (6.6.3):

$$\sum_{|\boldsymbol{\mu}|=2}(D^{\boldsymbol{\mu}}_{r,\theta}R)\,\mathcal{G}_{\boldsymbol{\mu},(j+1,l)}(\boldsymbol{\Psi}_{s,t})\Big|_{s=t=0}=(4\Delta R)\,\mathcal{G}_{\boldsymbol{\mu}_0,(j+1,l)}(\boldsymbol{\Psi}_{s,t})\Big|_{s=t=0}\quad\text{on }\ \mathbb{T}.$$

Here, we use the fact that $\partial_r^2 R=4\Delta R$ on $\mathbb{T}$. It follows that for $(j,l)\neq(0,0)$, we have on $\mathbb{T}$ that

$$\mathfrak{R}_{j,l}=\frac{2}{(j+1)!l!}\Big((4\Delta R)\,\mathcal{G}_{\boldsymbol{\mu}_0,(j+1,l)}(\boldsymbol{\Psi}_{s,t})+\sum_{3\le|\boldsymbol{\mu}|\le j+l+1}(D^{\boldsymbol{\mu}}_{r,\theta}R)\,\mathcal{G}_{\boldsymbol{\mu},(j+1,l)}(\boldsymbol{\Psi}_{s,t})\Big)\Big|_{s=t=0}.$$

We write $\boldsymbol{\nu}:=(j+1,l)$, and split the expression $\mathcal{G}_{\boldsymbol{\mu}_0,\boldsymbol{\nu}}(\boldsymbol{\Psi}_{s,t})$ further according to formula (6.6.6):

$$\mathcal{G}_{\boldsymbol{\mu}_0,\boldsymbol{\nu}}(\boldsymbol{\Psi}_{s,t})=\mathcal{G}^{\circledast}_{\boldsymbol{\mu}_0,\boldsymbol{\nu}}(\boldsymbol{\Psi}_{s,t})+\mathcal{H}_{\boldsymbol{\mu}_0,\boldsymbol{\nu}}(\boldsymbol{\Psi}_{s,t}).$$

We turn to the task of expressing

$$\mathcal{H}_{\boldsymbol{\mu}_0,\boldsymbol{\nu}}(\boldsymbol{\Psi}_{s,t})=\boldsymbol{\nu}!\prod_{j=1}^{k_{\boldsymbol{\nu}}}\frac{[\partial^{\boldsymbol{\alpha}_j^{\circledast}}\boldsymbol{\Psi}_{s,t}]^{\boldsymbol{\beta}_j^{\circledast}}}{\boldsymbol{\beta}_j^{\circledast}![\boldsymbol{\alpha}_j^{\circledast}!]^{|\boldsymbol{\beta}_j^{\circledast}|}}.$$

in explicit form in the various cases as outlined in Proposition 6.6.1. First, if $j\ge 0$ and $l\ge 1$, then $k_{\boldsymbol{\nu}}=2$ and

$$\mathcal{H}_{\boldsymbol{\mu}_0,\boldsymbol{\nu}}(\boldsymbol{\Psi}_{s,t})=l(\partial_t|\psi_{s,t}|)(\partial_s^{j+1}\partial_t^{l-1}|\psi_{s,t}|).$$

It remains to consider $j\ge1$ and $l=0$. If $j=1$ and $l=0$, then

$$\mathcal{H}_{\boldsymbol{\mu}_0,\boldsymbol{\nu}}(\boldsymbol{\Psi}_{s,t})=(\partial_s|\psi_{s,t}|)^2,$$

while if instead $j\ge2$ and $l=0$, then

$$\mathcal{H}_{\boldsymbol{\mu}_0,\boldsymbol{\nu}}(\boldsymbol{\Psi}_{s,t})=(j+1)(\partial_s|\psi_{s,t}|)(\partial_s^{j}|\psi_{s,t}|).$$

It remains to discuss the algebraic properties of $\mathfrak{r}_{j,l}$ for $j\ge0,l\ge1$ and those of $\mathfrak{R}_{j,0}$ for $j\ge1$. In view of Proposition 6.6.1, for $j\ge0,l\ge1$ all the indices $\boldsymbol{\alpha}_i$ have

$$0\prec_{\mathrm{L}}\boldsymbol{\alpha}_0\prec_{\mathrm{L}}\cdots\prec_{\mathrm{L}}\boldsymbol{\alpha}_k\prec_{\mathrm{L}}(j+1,l-1),$$

provided that $(\boldsymbol{\alpha};\boldsymbol{\beta})\in\mathfrak{d}^{\mathrm{L}}_k(\boldsymbol{\mu},\boldsymbol{\nu})$ holds for a $k=1,\ldots,|\boldsymbol{\nu}|$, given that $|\boldsymbol{\mu}|\ge3$. In addition, the same conclusion remains valid if $\boldsymbol{\mu}=\boldsymbol{\mu}_0=(2,0)$ provided that it is assumed that $(\boldsymbol{\alpha};\boldsymbol{\beta})\in\mathfrak{d}^{\mathrm{L}}_{k,\circledast}(\boldsymbol{\mu}_0,\boldsymbol{\nu})$, which excludes the extremal multi-index. After some additional simplifications, this shows that $\mathfrak{r}_{j,l}$ has the claimed form. For $j\ge1$ and $l=0$, the assertion about the algebraic properties of $\mathfrak{R}_{j,0}$ follows from the observation that if $(\boldsymbol{\alpha};\boldsymbol{\beta})\in\mathfrak{d}^{\mathrm{L}}_k(\boldsymbol{\mu},\boldsymbol{\nu})$ with $\boldsymbol{\nu}=(j+1,0)$, then for $i=1,\ldots,k$, we have $\boldsymbol{\alpha}_i=(a_i,0)$ with $0<a_i\le j$, by Proposition 6.6.1. The computational aspects are analogous to the case already discussed. This completes the proof. □

6.10. **Taylor expansion of the remaining terms in the master equation.** We recall that h_s and $\psi_{s,t}$ stand for the functions

$$h_s(\zeta)=\sum_{j=0}^{\kappa}s^j b_j(\zeta)\quad\text{and}\quad\psi_{s,t}(\zeta)=\psi_{0,t}(\zeta)+\sum_{\substack{(j,l)\in\mathfrak{r}_{2\kappa+1}\\ j\ge1}}s^jt^l\hat{\psi}_{j,l}(\zeta),$$

where κ is a (big) positive integer. Moreover, the b_j are bounded holomorphic functions in the exterior disk $\mathbb{D}_{\mathrm{e}}(0,\varrho_1)$, and $\psi_{0,t}$ is a conformal mapping of the exterior disk onto the exterior of the level curves Γ_t of R as above, and where $\hat\psi_{j,l}$ are holomorphic functions on $\mathbb{D}_{\mathrm{e}}(0,\varrho_1)$ with bounded derivative. Let us denote by $\mathfrak{H}_{j,l}$, $\mathfrak{R}_{j,l}$ and $\mathfrak{I}_{j,l}$ the corresponding coefficient functions in the following three expansions (for $\zeta\in\mathbb{T}$):

$$2\,\mathrm{Re}(h_s\circ\psi_{s,t})(\zeta)=\sum_{(j,l)\in\curlyvee_{2\kappa}}s^jt^l\mathfrak{H}_{j,l}(\zeta)+\mathrm{O}\big(|s|^{\kappa+\frac12}+|t|^{2\kappa+1}\big), \tag{6.10.1}$$

$$2s^{-1}\big(R\circ\psi_{s,t}(\zeta)-\tfrac12t^2\big)=\sum_{(j,l)\in\curlyvee_{2\kappa}}s^jt^l\mathfrak{R}_{j,l}(\zeta)+\mathrm{O}\big(|s|^{\kappa+\frac12}+|t|^{2\kappa+1}\big), \tag{6.10.2}$$

$$\log\mathrm{Re}\,\big(-\bar\zeta\partial_t\psi_{s,t}(\zeta)\overline{\psi'_{s,t}(\zeta)}\big)=\sum_{(j,l)\in\curlyvee_{2\kappa}}s^jt^l\mathfrak{I}_{j,l}(\zeta)+\mathrm{O}\big(|s|^{\kappa+\frac12}+|t|^{2\kappa+1}\big). \tag{6.10.3}$$

where (6.10.2) holds since $R\circ\psi_{0,t}(\zeta)=\frac{t^2}{2}$ on $\mathbb{T}$ as a matter of definition. Moreover, we recall that by Proposition 6.8.1 we have that

$$\exp(\mathfrak{I}_{0,0})=\mathrm{Re}(-\bar\zeta\partial_t\psi_{s,t}(\zeta)\overline{\psi'_{s,t}(\zeta)})\big|_{s=t=0}=(4\varDelta R(\zeta))^{-\frac12}\quad\text{on }\mathbb{T}.$$

We have already analyzed the coefficient functions $\mathfrak{R}_{j,l}$ for $(j,l)\in\curlyvee_{2\kappa}$ back in Proposition 6.9.1. Here, we refine the analysis and obtain a more convenient splitting of $\mathfrak{R}_{j,l}$ into a main term plus a remainder, and express the coefficients $\mathfrak{H}_{j,l}$ and $\mathfrak{I}_{j,l}$ in terms of the successive partial derivatives of the functions b_j and $\psi_{s,t}$.

Proposition 6.10.1. *In the above context, the Taylor coefficients $\mathfrak{H}_{j,l}$ in (s,t) of the function $2\,\mathrm{Re}\,h_s\circ\psi_{s,t}$ in (s,t) according to* (6.10.1) *have the following properties. For $j\ge0$ we have*

$$\mathfrak{H}_{j,0}=2\,\mathrm{Re}\,b_j+\mathfrak{h}_{j,0}$$

where $\mathfrak{h}_{j,0}\in\mathrm{POL}(j-1,\Sigma_{j,0})$. On the other hand, for $(j,l)\in\curlyvee_{2\kappa}$ with $l\ge1$, we see that $\mathfrak{H}_{j,l}\in\mathrm{POL}(j,\Sigma_{j,l})$.

As for the Taylor coefficients $\mathfrak{R}_{j,l}$ associated with $R\circ\psi_{s,t}$ according to (6.10.2)*, we have for $(j,l)\in\curlyvee_{2\kappa}$ with $j\ge0$ and $l\ge1$ that*

$$\mathfrak{R}_{j,l}=2(4\varDelta R)^{\frac12}\,\mathrm{Re}(\bar\zeta\hat\psi_{j+1,l-1})+\mathfrak{s}_{j,l}$$

where

$$\mathfrak{s}_{j,l}\in\mathrm{POL}\big(\{(p,q)\in\Sigma_{j+1,l}:\,(p,q)\prec_{\mathrm{L}}(j+1,l-1)\}\big).$$

As for the coefficients $\mathfrak{I}_{j,l}$ appearing in (6.10.3)*, $\mathfrak{I}_{0,0}$ is given by*

$$\mathfrak{I}_{0,0}=\log\,\mathrm{Re}(-\bar\zeta\hat\psi_{0,1})=-\frac12\log(4\varDelta R),$$

while for $(j,l)\in\curlyvee_{2\kappa}\setminus\{(0,0)\}$ we see that $\mathfrak{I}_{j,l}\in\mathrm{POL}(\Sigma_{j,l+1})$.

Proof. This follows from an application of the multivariate Taylor's formula, together with the multivariate Faà di Bruno formula (Proposition 6.5.1), and the equation (6.10.2) above. Let us indicate the necessary computations, starting with the coefficients $\mathfrak{H}_{j,l}$. For $(j,l)\in\curlyvee_{2\kappa}\setminus\{(r,0):r\ge0\}$ we have

$$\mathfrak{H}_{j,l}=2\,\mathrm{Re}\sum_{i=0}^{j}\sum_{1\le\mu\le j+l-i}\partial^\mu b_i(\zeta)\sum_{k=1}^{j+l-i}\sum_{(\boldsymbol\alpha,\beta)\in\mathfrak{d}_k^{\mathrm{L}}(\mu,(j-i,l))}\prod_{r=1}^{k}\frac{[\partial_{s,t}^{\boldsymbol\alpha_r}\psi_{s,t}(\zeta)]^{\beta_r}}{(\boldsymbol\alpha_r!)^{\beta_r}\beta_r!}, \tag{6.10.4}$$

and consequently $\mathfrak{H}_{j,l}\in\mathrm{POL}(j,\Sigma_{j,l})$, while for indices $(j,0)$ with $j\ge0$ we have

$$\mathfrak{H}_{j,0}=2\,\mathrm{Re}\,b_j+\mathfrak{h}_{j,0}$$

where $\mathfrak{h}_{j,0} \in \mathrm{POL}(j-1, \Sigma_{j,0})$ is given by

$$\mathfrak{h}_{j,0} = 2\,\mathrm{Re}\sum_{i=0}^{j-1}\sum_{1\le\mu\le j-i}\partial^{\mu} b_i(\zeta)\sum_{k=1}^{j-i}\sum_{(\alpha,\beta)\in\mathfrak{d}_k^{\mathrm{L}}(\mu,(j-i,0))}\prod_{r=1}^{k}\frac{[\partial_{s,t}^{\alpha_r}\psi_{s,t}]^{\beta_r}}{(\alpha_r!)^{\beta_r}\beta_r!}. \tag{6.10.5}$$

Turning to the claim about the coefficient $\mathfrak{R}_{j,l}$, we note that

$$\mathfrak{s}_{j,l} = \frac{2}{(j+1)!l!}\mathfrak{r}_{j,l} + 2(4\varDelta R)^{\frac12}\left(\frac{\partial_s^{j+1}\partial_t^{l-1}|\psi_{s,t}|}{(j+1)!(l-1)!} - \mathrm{Re}(\bar\zeta\hat\psi_{j+1,l-1})\right)\bigg|_{s=t=0}. \tag{6.10.6}$$

The claim in the proposition follows from Proposition 6.9.1 together with the observation that

$$\frac{\partial_s^{j+1}\partial_t^{l-1}|\psi_{s,t}(\zeta)|}{(j+1)!(l-1)!}\bigg|_{s=t=0} - \mathrm{Re}(\bar\zeta\hat\psi_{j+1,l-1}(\zeta)) \\ \in \mathrm{POL}\big(\{(p,q)\in\Sigma_{j+1,l} : (p,q)\prec_{\mathrm{L}} (j+1,l-1)\}\big).$$

In order to see why this claim holds, we simply observe that the first term on the left-hand side is the Taylor coefficient in (s,t) corresponding to the multi-index $(j+1,l-1)$ of the function $|\psi_{s,t}|$. The Taylor expansion of this function may be found as follows. We notice that $\psi_{0,0} = \zeta$, so that if we apply the generalized binomial theorem with exponent $\frac12$, we obtain

(6.10.7)

$$|\psi_{s,t}(\zeta)| = \left|\Big(1+\sum_{(p,q)\ne(0,0)} s^p t^q\bar\zeta\hat\psi_{p,q}\Big)^{\frac12}\right|^2 = \left|1+\sum_{k\ge1}\binom{\frac12}{k}\Big(\sum_{(p,q)\ne(0,0)} s^p t^q\bar\zeta\hat\psi_{p,q}\Big)^k\right|^2 \\ = 1+\sum_{k,k'\ge1}\binom{\frac12}{k}\binom{\frac12}{k'}\Big(\sum_{(p,q)\ne(0,0)} s^p t^q\bar\zeta\hat\psi_{p,q}\Big)^k\Big(\sum_{(p,q)\ne(0,0)} s^p t^q\zeta\overline{\hat\psi}_{p,q}\Big)^{k'} \\ + 2\,\mathrm{Re}\sum_{k\ge1}\binom{\frac12}{k}\Big(\sum_{(p,q)\ne(0,0)} s^p t^q\bar\zeta\hat\psi_{p,q}\Big)^k, \qquad \zeta\in\mathbb{T}.$$

Apart from the contribution from the conformal mapping $\psi_{0,t}$, the series involve a truncation given by the index set $\boldsymbol{\Upsilon}_{2\kappa+1}$, and hence we have no convergence issues. The maximal index (p,q) in the lexicographical ordering for which $\hat\psi_{p,q}$ appears in the Taylor coefficient for $s^{j+1}t^{l-1}$ of the above expression (6.10.7), is easily seen to be $(j+1,l-1)$. The contribution corresponding to the maximal index comes from the last term on the right-hand side of (6.10.7), and equals

$$2\,\mathrm{Re}\binom{\frac12}{1}\bar\zeta\hat\psi_{j+1,l-1} = \mathrm{Re}(\bar\zeta\hat\psi_{j+1,l-1}).$$

As for all the other indices, the contribution in the above sum to the Taylor coefficient lies in the complexity class

$$\mathrm{POL}\big(\{(p,q)\in\Sigma_{j+1,l-1} : (p,q)\prec_{\mathrm{L}} (j+1,l-1)\}\big),$$

and the claim follows.

Finally, we turn to the coefficient $\mathfrak{I}_{j,l}$. We know that $\mathfrak{I}_{0,0} = \log\mathrm{Re}(-\bar\zeta\hat\psi_{0,1})$, while for indices $(j,l)\in\boldsymbol{\Upsilon}_{2\kappa}\setminus\{(0,0)\}$ we apply the Faà di Bruno's formula to the logarithm of the

Jacobian expression to obtain

$$\text{(6.10.8)}\quad \mathfrak{J}_{j,l}=\sum_{1\le\mu\le j+l}(-1)^{\mu}(\mu-1)!(4\Delta R)^{\frac{\mu}{2}}\times\sum_{k=1}^{j+l}\sum_{(\boldsymbol{\alpha},\beta)\in\mathfrak{d}_k^{\mathrm{L}}(\mu;(j,l))}\prod_{r=1}^{k}\frac{[\partial_{s,t}^{\boldsymbol{\alpha}_r}\operatorname{Re}(-\bar\zeta\partial_t\psi_{s,t}\overline{\psi_{s,t}'})]^{\beta_r}}{(\boldsymbol{\alpha}_r!)^{\beta_r}\beta_r!}.$$

As we may eliminate the half-powers of ΔR by writing

$$(4\Delta R)^{\frac{\mu}{2}}=(4\Delta R)^{\mu}(4\Delta R)^{-\frac{\mu}{2}}=(4\Delta R)^{\mu}\operatorname{Re}(-\bar\zeta\hat\psi_{0,1}(\zeta))^{\mu},\qquad \zeta\in\mathbb{T},$$

it follows that $\mathfrak{J}_{j,l}\in\mathrm{POL}(\Sigma_{j,l+1})$. □

6.11. **Taylor expansion of the density in the master equation.** We recall from §6.4 the function

$$\Pi_{s,t}(\zeta)=2\operatorname{Re}h_s\circ\psi_{s,t}(\zeta)-\tfrac{2}{s}\big((R\circ\psi_{s,t})(\zeta)-\tfrac12 t^2\big)+\log\Big(\operatorname{Re}\big(-\bar\zeta\partial_t\psi_{s,t}(\zeta)\overline{\psi_{s,t}'(\zeta)}\big)\Big).$$

We compute the Taylor coefficients $\hat\Pi_{j,l}$ for $(j,l)\in\curlyvee_{2\kappa}$ given implicitly by

$$\Pi_{s,t}(\zeta)=\sum_{(j,l)\in\curlyvee_{2\kappa}}s^jt^l\hat\Pi_{j,l}(\zeta)+\mathrm{O}\big(|s|^{\kappa+\frac12}+|t|^{2\kappa+1}\big).$$

This will determine what the master equation for the Taylor coefficients (6.4.2) entails for the coefficient functions b_k and $\hat\psi_{p,q}$ for $k\le\kappa$ and $(j,l)\in\curlyvee_{2\kappa+1}$.

We recall that the Taylor coefficients $\mathfrak{H}_{j,l}$, $\mathfrak{h}_{j,0}$, $\mathfrak{J}_{j,0}$, $\mathfrak{R}_{j,0}$, and $\mathfrak{s}_{j,l}$ have appeared above in Propositions 6.9.1 and 6.10.1. See in addition the explicit formulæ (6.10.4), (6.10.5), (6.10.6), and (6.10.8).

Proposition 6.11.1. *The coefficients $\hat\Pi_{j,l}(\zeta)$ in the above expansion are given explicitly as follows. For $j=l=0$, we have*

$$\hat\Pi_{0,0}(\zeta)=2\operatorname{Re}b_0(\zeta)+\log\operatorname{Re}(-\bar\zeta\hat\psi_{0,1}(\zeta)),$$

while for $l=0$ and $j=1,2,3,\ldots$ the coefficient function is given by

$$\hat\Pi_{j,0}(\zeta)=2\operatorname{Re}b_j(\zeta)+\mathfrak{T}_{j,0},$$

where $\mathfrak{T}_{j,0}:=\mathfrak{h}_{j,0}-\mathfrak{R}_{j,0}+\mathfrak{J}_{j,0}\in\mathrm{POL}(j-1,\Sigma_{j,1})$. Moreover, for $j=0,1,2,\ldots$ and $l=1,2,3,\ldots$ the coefficient function $\hat\Pi_{j,l}$ is given by

$$\hat\Pi_{j,l}=-2(4\Delta R(\zeta))^{\frac12}\operatorname{Re}(\bar\zeta\hat\psi_{j+1,l-1}(\zeta))+\mathfrak{T}_{j,l},$$

where $\mathfrak{T}_{j,l}:=\mathfrak{H}_{j,l}-\mathfrak{s}_{j,l}+\mathfrak{J}_{j,l}\in\mathrm{POL}(j,\Sigma)$, with Σ as the index set

$$\Sigma=\{(p,q)\in\Sigma_{j+1,l}\cup\Sigma_{j,l+1}:(p,q)\prec_{\mathrm{L}}(j+1,l-1)\}.$$

Proof. The formula for $\hat\Pi_{0,0}=\Pi_{s,t}|_{s=t=0}$ is immediate from the definition (6.4.1). Indeed, $\psi_{0,0}(\zeta)=\zeta$, and the formula (6.4.1) reads, where $h_s=2\operatorname{Re}\log f_s$,

$$\Pi_{0,0}(\zeta)=h_0(\zeta)+\log\big(\operatorname{Re}\big(-\bar\zeta\hat\psi_{0,1}(\zeta)\big)\big),$$

where we use the fact that $\mathfrak{R}_{0,0}=0$ according to Proposition 6.9.1. The conclusion now follows by observing that $h_0(\zeta)=2\operatorname{Re}b_0(\zeta)$.

The coefficients $\hat\Pi_{j,0}$ for $j\ge1$ are given by

$$\hat\Pi_{j,0}=\mathfrak{H}_{j,0}-\mathfrak{R}_{j,0}+\mathfrak{J}_{j,0}.$$

The main contribution will come from the term $\mathfrak{H}_{j,0}$, and we need to prove that the remainder of this term, as well as both terms $\mathfrak{R}_{j,0}$ and $\mathfrak{H}_{j,0}$ belong to the polynomial complexity class $\mathrm{POL}(j-1,\Sigma_{j,1})$. By Proposition 6.10.1, it follows that

$$\mathfrak{H}_{j,l} = 2\,\mathrm{Re}\,b_j + \mathfrak{h}_{j,l}$$

with $\mathfrak{h}_{j,l} \in \mathrm{POL}(j-1,\Sigma_{j,0})$. Moreover, by Proposition 6.9.1 and Proposition 6.10.1, respectively, it follows that

$$\mathfrak{R}_{j,0} \in \mathrm{POL}(\Sigma_{j,0}) \subset \mathrm{POL}(j-1,\Sigma_{j,1}), \quad \text{and} \quad \mathfrak{I}_{j,0} \in \mathrm{POL}(\Sigma_{j,1}) \subset \mathrm{POL}(j-1,\Sigma_{j,1})$$

and hence the claim follows.

We next turn to the coefficients $\hat{\Pi}_{j,l}$ with $(j,l) \in \curlyvee_{2\kappa}$ for which $l \geq 1$. The main term of

$$\hat{\Pi}_{j,l} = \mathfrak{H}_{j,l} - \mathfrak{R}_{j,l} + \mathfrak{I}_{j,l}$$

will come from the term $\mathfrak{R}_{j,l}$, while the total remainder, consisting of the remainder from the term $\mathfrak{R}_{j,l}$ together with the full terms $\mathfrak{H}_{j,l}$ and $\mathfrak{I}_{j,l}$, is supposed to lie in the correct polynomial complexity class $\mathrm{POL}(j,\Sigma)$, where

$$\Sigma = \{(p,q) \in \Sigma_{j+1,l} \cup \Sigma_{j,l+1} : (p,q) \prec_{\mathrm{L}} (j+1,l-1)\}.$$

By Proposition 6.10.1, we have for such indices (j,l) that

$$\mathfrak{R}_{j,l} = -2(4\,\mathrm{Re}\,R)^{\frac{1}{2}}\,\mathrm{Re}(\bar{\zeta}\hat{\psi}_{j+1,l-1}) + \mathfrak{s}_{j,l},$$

where $\mathfrak{s}_{j,l} \in \mathrm{POL}(j,\Sigma)$. By Proposition 6.10.1 it also follows that

$$\mathfrak{H}_{j,l} \in \mathrm{POL}(j,\Sigma_{j,l}) \subset \mathrm{POL}(j,\Sigma), \quad \text{and} \quad \mathfrak{I}_{j,l} \in \mathrm{POL}(\Sigma_{j,l+1}) \subset \mathrm{POL}(j,\Sigma).$$

which proves the claim. □

6.12. **Algorithmic resolution of the master equation.** We are now ready to make the algorithm outlined in §6.4 rigorous. We recall the master equation for the Taylor coefficients (6.4.2)

$$\hat{\Pi}_{j,l} = \begin{cases} -\frac{1}{2}\log(4\pi) & \text{for} \quad \zeta \in \mathbb{T} \ \text{ and } \ (j,l) = (0,0), \\ 0 & \text{for} \quad \zeta \in \mathbb{T} \ \text{ and } \ (j,l) \in \curlyvee_{2\kappa} \setminus \{(0,0)\}. \end{cases}$$

In order to solve this system, we solve for the coefficient functions of h_s and $\psi_{s,t}$ iteratively, according to the algorithm outlined in §6.4.

Proof of Proposition 6.2.1. In view of Propositions 6.2.3 and 6.8.1, the conformal mapping $\psi_{0,t}$ and its Taylor coefficients $\hat{\psi}_{0,l}$ for $l = 0, 1, \ldots, 2\kappa+1$ with respect to the time parameter t of the flow are well-defined, and they satisfy the required smoothness properties: for t near 0, the conformal mapping $\psi_{0,t}$ extends holomorphically across the boundary $\mathbb{T}$ to an exterior disk $\mathbb{D}_{\mathrm{e}}(0,\sqrt{\varrho_1})$ according to Proposition 6.3.1. In addition, the derivative $\psi'_{0,t}$ remains uniformly bounded as long as the weight R is confined to a uniform family in $\mathcal{W}(\varrho_0,\sigma_0)$ for fixed ϱ_0 and σ_0. Moreover, the coefficient functions $\hat{\psi}_{0,l}$ extend holomorphically to $\mathbb{D}_{\mathrm{e}}(0,\varrho_1)$, by Remark 6.1.4. This completes Step 1 of the algorithmic procedure.

Turning our attention to Step 2, we recall from Proposition 6.8.1 that on the circle $\mathbb{T}$, we have $\mathrm{Re}(-\bar{\zeta}\hat{\psi}_{0,1}) = (4\varDelta R(\zeta))^{-\frac{1}{2}}$. Hence, by Proposition 6.11.1 the equation $\hat{\Pi}_{0,0} = -\frac{1}{2}\log(4\pi)$ is equivalent to

$$2\,\mathrm{Re}\,b_0 - \tfrac{1}{2}\log(4\varDelta R) = -\frac{1}{2}\log(4\pi) \quad \text{on } \mathbb{T}.$$

Since we want the function f_s to be zero-free in the exterior disk and real at infinity, this tell us that

$$b_0 = -\tfrac{1}{4}\log(4\pi) + \tfrac{1}{4}\mathbf{H}_{\mathbb{D}_e}\big[\log(4\Delta R)\big].$$

We note that this automatically gives the normalization $\operatorname{Im} b_0(\infty) = 0$. By Proposition 6.1.5 and Remark 6.1.4, it follows that b_0 extends holomorphically to the exterior disk $\mathbb{D}_e(0, \varrho_1)$. Moreover, b_0 clearly remains uniformly bounded provided that R is confined to a uniform family in $\mathcal{W}(\varrho_0, \sigma_0)$. This completes Step 2.

We proceed to Step 3 of the algorithmic procedure. We are now in the following situation. For some $j_0 \geq 1$, our known data set is $\mathrm{POL}(j_0 - 1, \Sigma)$, where

$$\Sigma = \{(j, l) \in \Upsilon_{2\kappa} : (j, l) \prec_{\mathrm{L}} (j_0, 0)\}$$

and all elements of $\mathrm{POL}(j_0 - 1, \Sigma)$ meet the required extension conditions. In particular, all the functions $b_0, \ldots, b_{j_0-1}$ and $\hat{\psi}_{j,l}$ for all $(j, l) \in \Upsilon_{2\kappa+1}$ with $(j, l) \prec_{\mathrm{L}} (j_0, 0)$ are already known. In addition, the relations (6.4.2) are met for all $(j, l) \in \Upsilon_{2\kappa}$ with $(j, l) \prec_{\mathrm{L}} (j_0 - 1, 1)$. We will now show how this allows us to obtain the relations (6.4.2) for all subsequent indices $(j, l) \in \Upsilon_{2\kappa}$ with $(j, l) \prec_{\mathrm{L}} (j_0, 0)$, by making appropriate choices of the functions $\hat{\psi}_{j_0,l}$ for $l \geq 0$ with $(j_0, l) \in \Upsilon_{2\kappa+1}$. The additional indices for which we need to solve (6.4.2) are those $(j, l) \in \Upsilon_{2\kappa}$ of the form $(j, l) = (j_0 - 1, l + 1)$, where $l \geq 0$.

To achieve this, we assume that for all l with $0 \leq l \leq l_0 - 1$, we have obtained the coefficient functions $\hat{\psi}_{j_0,l}$ by solving the equation (6.4.2) for the index pair $(j_0 - 1, l + 1)$, and turn to the next equation. This reads $\hat{\Pi}_{j_0-1,l_0+1} = 0$, as long as $(j_0 - 1, l_0 + 1) \in \Upsilon_{2\kappa}$. At this point, the known data set is $\mathrm{POL}(j_0 - 1, \Sigma')$, where

$$\Sigma' = \{(j, l) \in \Upsilon_{2\kappa+1} : (j, l) \prec_{\mathrm{L}} (j_0, l_0)\}$$

If l_0 is too large for $(j_0, l_0) \in \Upsilon_{2\kappa+1}$ to hold, we are in fact done, we don't need to obtain $\hat{\psi}_{j_0,l_0}$ for such indices. On the other hand, if $(j_0, l_0) \in \Upsilon_{2\kappa+1}$ we proceed as follows. By Proposition 6.11.1, the equation $\hat{\Pi}_{j_0-1,l_0+1} = 0$ may be written in the form

$$\hat{\Pi}_{j_0-1,l_0+1} = -2(4\Delta R)^{\frac{1}{2}} \operatorname{Re}(\bar{\zeta}\hat{\psi}_{j_0,l_0}) + \mathfrak{T}_{j_0-1,l_0+1} = 0 \quad \text{on } \mathbb{T},$$

where $\mathfrak{T}_{j_0-1,l_0+1} \in \mathrm{POL}(j_0 - 1, \Sigma')$. We provide a solution to this equation by the formula

$$\hat{\psi}_{j_0,l_0} = \tfrac{1}{2}\zeta\mathbf{H}_{\mathbb{D}_e}\left[\frac{\mathfrak{T}_{j_0-1,l_0+1}}{(4\Delta R)^{\frac{1}{2}}}\right]. \tag{6.12.1}$$

The function $\mathfrak{T}_{j_0-1,l_0+1}$ has a polarization which is holomorphic in $(z, \bar{w})$ for $(z, w) \in \hat{\mathbb{A}}(\varrho_1, \sigma_1)$, and the same holds for the weight R. As a consequence, it follows that $\hat{\psi}_{j_0,l_0}$ extends holomorphically to the exterior disk $\mathbb{D}_e(0, \varrho_1)$, and that $\hat{\psi}_{j_0,l_0}(\zeta) = \mathrm{O}(|\zeta|)$ with an implicit constant which is uniformly bounded, provided that R is confined to a uniform family in $\mathcal{W}(\varrho_0, \sigma_0)$.

The base step $l_0 = 0$ of the induction procedure of Step 3 is entirely analogous. Indeed, the known data set is $\mathrm{POL}(j_0 - 1, \Sigma)$ with Σ as above, and by Proposition 6.11.1 the relevant equation $\hat{\Pi}_{j_0-1,1} = 0$ takes the form

$$-2(4\Delta R)^{\frac{1}{2}} \operatorname{Re}(\bar{\zeta}\hat{\psi}_{j_0,0}) + \mathfrak{T}_{j_0-1,1} = 0 \quad \text{on } \mathbb{T},$$

where $\mathfrak{T}_{j_0-1,1}$ in particular lies in $\mathrm{POL}(j_0 - 1, \Sigma)$. Hence we obtain $\hat{\psi}_{j_0,0}$ by the formula (6.12.1) with $l = 0$ replaced by 0.

We now turn to Step 4. After the completion of Step 3, the situation is as follows. the known data set is $\mathrm{POL}(j_0-1,\Sigma)$ with

$$\Sigma = \{(j,l) : (j,l) \prec_{\mathrm{L}} (j_0+1,0)\},$$

where every element of $\mathrm{POL}(j_0-1,\Sigma)$ polarizes to $\hat{\mathbb{A}}(\varrho_1,\sigma_1)$. In addition, the relations (6.4.2) are met for all $(j,l)\in \Upsilon_{2\kappa}$ with $(j,l)\prec_{\mathrm{L}}(j_0,0)$. We recall that this in particular this means that the known data set consists of the coefficient functions $b_0,\ldots,b_{j_0-1}$ and $\hat{\psi}_{j,l}$ for $(j,l)\in\Upsilon_{2\kappa+1}$ with $(j,l)\prec_{\mathrm{L}}(j_0+1,0)$ are known. In this step, we need to find the function b_{j_0}, and verify that the relation (6.4.2) is then met with $(j,l)=(j_0,0)$. To this end, we apply Proposition 6.11.1, and observe that the equation (6.4.2) with $(j,l)=(j_0,0)$ is equivalent to having

$$\hat{\Pi}_{j_0,0} = 2\,\mathrm{Re}\, b_{j_0} + \mathfrak{T}_{j_0,0} = 0 \quad \text{on } \mathbb{T},$$

where $\mathfrak{T}_{j_0,0}\in\mathrm{POL}(j_0-1,\Sigma)$, with the same Σ as above. Since $\mathrm{POL}(j_0-1,\Sigma)$ is a collection of known functions, we hence obtain an equation for the unknown function b_{j_0}, with solution

$$b_{j_0} = -\tfrac{1}{2}\mathbf{H}_{\mathbb{D}_{\mathrm{e}}}\big[\mathfrak{T}_{j_0,0}\big].$$

In view of Proposition 6.1.5 and Remark 6.1.4, the function b_{j_0} extends holomorphically to the exterior disk $\mathbb{D}_{\mathrm{e}}(0,\varrho_1)$, and remains uniformly bounded if the weight R is confined to a uniform family in $\mathcal{W}(\varrho_0,\sigma_0)$. Moreover, we observe that b_{j_0} has the required normalization at infinity: $\mathrm{Im}\, b_{j_0}(\infty)=0$.

We finally turn to Step 5. The key observation is that we are now in a position to return to Step 3 followed by Step 4, with j_0 replaced by j_0+1. Since Step 1 and Step 2 combine to form the initial data for Steps 3 and 4 with $j_0=1$, the algorithm produces iteratively the entire set of coefficient functions, and solves in the process all the equations (6.4.2) for $(j,l)\in\Upsilon_{2\kappa}$.

Equipped with the functions b_j for $j=0,\ldots,\kappa$, the conformal mappings $\psi_{0,t}$ and the coefficients $\hat{\psi}_{j,l}$ for $(j,l)\in\Upsilon_{2\kappa+1}\cap\{(j,l):j\ge 1\}$, we observe that the functions h_s and $\psi_{s,t}$ given by

$$h_s(\zeta)=\sum_{j=0}^{\kappa} s^j b_j(\zeta), \qquad \psi_{s,t}(\zeta)=\psi_{0,t}+\sum_{\substack{(j,l)\in\Upsilon_{2\kappa+1}\\ j\ge 1}} s^j t^l \hat{\psi}_{j,l}(\zeta)$$

are well-defined, and have the desired smoothness and mapping properties. By the Becker-Pommerenke criterion of Lemma 6.2.5, it is immediate that $\psi_{s,t}$, as defined, is univalent in a neighborhood of the closed exterior disk $\overline{\mathbb{D}}_e$ for s and t close to 0. As $\psi_{s,t}$ extends holomorphically to the exterior disk $\mathbb{D}_{\mathrm{e}}(0,\sqrt{\varrho_1})$, and since $\psi_{s,t}$ is a smooth perturbation of the identity it follows that $\psi_{s,t}$ is univalent on $\mathbb{D}_{\mathrm{e}}(0,\varrho_2)$ and that

$$\psi_{s,t}(\mathbb{D}_{\mathrm{e}}(0,\varrho_2))\subset\mathbb{D}_{\mathrm{e}}(0,\varrho_1)$$

for s,t close to 0, provided that ϱ_2 is chosen appropriately with $\sqrt{\varrho_1}\le\varrho_2<1$.

The conclusion of Proposition 6.2.1 is now an immediate consequence of the relations (6.4.2) for the Taylor coefficients of the logarithm of the function

$$\exp(\Pi_{s,t}(\zeta)) = |f_s\circ\psi_{s,t}(\zeta)|^2 e^{-2s^{-1}\{(R\circ\psi_{s,t})(\zeta)-\frac{1}{2}t^2\}}\,\mathrm{Re}\big(-\bar{\zeta}\partial_t\psi_{s,t}(\zeta)\overline{\psi'_{s,t}(\zeta)}\big)$$

for $\zeta\in\mathbb{T}$, in the variables (s,t) near $(0,0)$, as verified in the above algorithm. □

6.13. **Implementation of the orthogonal foliation flow for $R = R_\tau$.** It remains only to prove the key lemma (Lemma 3.4.1). The hard work was completed in the previous subsection. The existence of the orthogonal foliation flow now follows if we use $s = 1/m$ as our quantization parameter.

Proof of Lemma 3.4.1. We first claim that $Q \circ \phi_\tau^{-1}$ is uniformly real-analytic in the exterior disk $\mathbb{D}_{\mathrm{e}}(0, \rho_{0,0})$ for $\tau \in I_{\epsilon_0}$. By this we mean that there exists a number $\sigma_{0,0} > 0$ such that $Q \circ \phi_\tau$ has a polarization which is uniformly bounded on the $2\sigma_{0,0}$-fattened diagonal annulus $\hat{\mathbb{A}}(\rho_{0,0}, \sigma_{0,0})$ (see Definition 6.1.1). Let $\rho_{1,0}$ be the number given by $\rho_{1,0} := \max\{\rho_{0,0}, ((1 + \sigma_{0,0}^2)^{\frac{1}{2}} + \sigma_{0,0})^{-1}\}$. Moreover, the function $Q_\tau^{\circledast} \circ \phi_\tau^{-1}$, which is the harmonic extension of $Q \circ \phi_\tau^{-1}|_{\mathbb{T}}$ to the exterior disk $\mathbb{D}_{\mathrm{e}}$, is uniformly bounded on $\hat{\mathbb{A}}(\rho_{1,0}, \sigma_{0,0})$ in view of Proposition 6.1.2 and an elementary decomposition of harmonic functions into holomorphic and conjugate holomorphic functions. In view of (2.1.2), the same holds for $\breve{Q}_\tau \circ \phi_\tau^{-1}$ and consequently also for $R_\tau = (Q - \breve{Q}_\tau) \circ \phi_\tau^{-1}$. In view of the uniform flatness of R_τ near the unit circle, the function $R_{\tau,0}$ defined by the relation $R_\tau(\zeta) = (1 - |\zeta|^2)^2 R_{\tau,0}(\zeta)$ enjoys the same property as well, namely that its polarization is uniformly bounded on $\hat{\mathbb{A}}(\rho_{1,0}, \sigma_{0,0})$. By possibly replacing $\sigma_{0,0}$ by a smaller positive number $\sigma_{1,0}$ we may guarantee that the polarization of $R_{\tau,0}$ remains bounded away from zero in the slightly smaller fattened diagonal annulus $\hat{\mathbb{A}}(\rho_{1,0}, \sigma_{1,0})$. If necessary, we replace $\rho_{1,0}$ by the larger number $\rho_{2,0} = \max\{\rho_{1,0}, ((1 + \sigma_{1,0}^2)^{\frac{1}{2}} + \sigma_{1,0})^{-1}\}$, which is still smaller than 1.

In view of the above considerations and the uniform bounds from Proposition 3.3.1, the family R_τ for $\tau \in I_{\epsilon_0}$ constitute a uniform family in $\mathcal{W}(\varrho_0, \sigma_0)$ where $\varrho_0 = \rho_{2,0}$ and $\sigma_0 = \sigma_{1,0}$. By Proposition 6.1.5 we obtain numbers ϱ_1 and σ_1 with $0 < \varrho_1 < 1$ and $\sigma_1 > 0$. We set $\rho_0 = \varrho_1$ and apply Proposition 6.2.1 to obtain the desired conformal mappings $\psi_{m,n,t} = \psi_{s,t}$ and $f_{m,n}^{\langle\kappa\rangle} = f_s$ with associated asymptotic expansion to precision κ, where $s = m^{-1}$. Here, the function $f_{m,n}^{\langle\kappa\rangle}$ is holomorphic and bounded on $\mathbb{D}_{\mathrm{e}}(0, \rho_0)$, positive at infinity and bounded away from zero in the entire exterior disk $\mathbb{D}_{\mathrm{e}}(0, \rho_0)$. Moreover, the flow equation (3.4.2) of Lemma 3.4.1 holds to the desired accuracy, in view of Proposition 6.2.1 with $s = m^{-1}$. □

7. Connection with soft Riemann-Hilbert problems

7.1. **Matrix $\bar{\partial}$-problems and orthogonal polynomials.** Given the successful application of Riemann-Hilbert problem methods to the study of orthogonal polynomials in the context of the real line and the unit circle, it has been proposed that the planar orthogonal polynomials should be approached in a similar fashion. Following Its and Takhtajan [39], we consider a matrix $\bar{\partial}$-problem (or a *soft Riemann-Hilbert problem*) and see how it fits in with our orthogonal foliation flow.

A polynomial is said to be *monic* if it has leading coefficient equal to 1. So, let $\pi_{m,n}$ denote the monic orthogonal polynomial of degree n with respect to the measure $e^{-2mQ}\,dA$ where Q is assumed 1-admissible. In other words, $\pi_{m,n}$ is given by

$$\pi_{m,n}(z) = \kappa_{m,n}^{-1} P_{m,n}(z),$$

where $\kappa_{m,n}$ is the leading coefficient of the normalized orthogonal polynomial $P_{m,n}$.

If $f \in L^p(\mathbb{C})$ for some $1 < p < 2$, we let $\mathbf{C}f$ be its Cauchy transform, given by

$$\mathbf{C}f(z) = \int_{\mathbb{C}} \frac{f(w)}{z-w} dA(w)$$

which is well-defined almost everywhere and represents a function which is locally in the Sobolev space $W^{1,p}$. The importance of the Cauchy transform comes from the fact that in the sense of distribution theory, $\bar\partial \mathbf{C} f = f$.

In [39], Its and Takhtajan propose to study the asymptotics of $\pi_{m,n}$ starting from the observation that the matrix-valued function

$$Y_{m,n}(z) = \begin{pmatrix} \pi_{m,n}(z) & -\mathbf{C}\big[\bar\pi_{m,n} e^{-2mQ}\big](z) \\ -\kappa_{m,n-1}^2 \pi_{m,n-1}(z) & \kappa_{m,n-1}^2 \mathbf{C}\big[\bar\pi_{m,n-1} e^{-2mQ}\big](z), \end{pmatrix} \tag{7.1.1}$$

solves the $\bar\partial$-problem

$$\begin{cases} \bar\partial Y(z) = -\bar Y(z) W(z), & \text{for } z \in \mathbb{C}, \\ Y(z) = \big(I + \mathrm{O}(z^{-1})\big) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}, & \text{as } |z| \to +\infty, \end{cases} \tag{7.1.2}$$

where $W(z) = W_m(z)$ is the matrix-valued function

$$W(z) = \begin{pmatrix} 0 & e^{-2mQ(z)} \\ 0 & 0 \end{pmatrix}.$$

Moreover, the solution is unique, as shown in [39]. We remark that classical Riemann-Hilbert problems where a jump occurs on a curve Γ may be phrased as $\bar\partial$-problems where $\bar\partial Y(z)$ is understood as a matrix-valued measure supported on Γ, and the above problem is a natural generalization to a more genuinely two-dimensional situation.

The idea that underlies the Its-Takhtajan approach, as well as the classical Riemann-Hilbert approach to orthogonal polynomials, is the expectation that one may constructively obtain an approximate solution $\tilde Y = \tilde Y_{m,n}(z)$ to the problem (7.1.1) (or the corresponding RHP), which should then produce an entry $(\tilde Y_{m,n})_{1,1}$ which is approximately equal to $\pi_{m,n}(z)$.

7.2. **Integration of Riemann-Hilbert problems along curve families.** Unfortunately, it has proven difficult to solve the problem (7.1.1) constructively. The following simple observation shows how our orthogonal foliation flow reduces the $\bar\partial$-problem to a family of more classical Riemann-Hilbert problems along closed curves.

In order to describe this problem, we denote by $J(z)$ a 2×2 jump matrix and let Γ be an oriented smooth simple closed curve in $\mathbb{C}$. We denote by Ω^+ and Ω^- the interior and exterior components of the complement $\mathbb{C} \setminus \Gamma$, respectively. If f is a function defined on $\mathbb{C} \setminus \Gamma$, which is continuous up to the boundary Γ as seen from each component, we define the two boundary value functions f^+ and f^- on Γ by

$$f^{\pm}(\zeta) = \lim_{\substack{z \to \zeta \\ z \in \Omega^{\pm}}} f(z), \qquad \zeta \in \Gamma.$$

We consider the Riemann-Hilbert problem of finding a 2×2 matrix-valued function $Y(z)$ which meets

$$\begin{cases} Y \text{ is holomorphic on } \mathbb{C} \setminus \Gamma, & \\ Y^+(z) = Y^-(z) + \bar{Y}^-(z) J(z), & \text{for } z \in \Gamma, \\ Y(z) = (I + \mathrm{O}(z^{-1})) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}, & \text{as } |z| \to +\infty. \end{cases} \tag{7.2.1}$$

In order to analyze this problem, we need a variant of the Cauchy transform, which applies to functions defined on Γ. For smooth Γ and reasonable f, we write

$$\mathbf{C}_\Gamma f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{w - z} dw, \qquad z \in \mathbb{C} \setminus \Gamma.$$

As is well-known, the classical Plemelj formula is a useful tool in the study of Riemann-Hilbert problems:

$$(\mathbf{C}_\Gamma f)^+(z) = (\mathbf{C}_\Gamma f)^-(z) + f(z). \tag{7.2.2}$$

We now connect the more classical Riemann-Hilbert problem (7.2.1) with the matrix $\bar{\partial}$-problem (7.1.2).

Proposition 7.2.1. *Let $\{\Gamma_t\}_{t \in I}$ be a smooth strictly expanding flow of positively oriented simple closed curves, and denote by $\mathcal{D}$ the union $\mathcal{D} = \bigcup_{t \in I} \Gamma_t$. Let $\omega(z)$ denote a smooth positive function on $\mathcal{D}$, and denote by $\xi : \mathcal{D} \to \mathbb{C}$ the vector field $\nu\bar{\eta}$, where $\eta(z)$ denotes the outward unit normal field to the curve family and ν denotes the scalar normal velocity of the flow. Then, for each $t \in I$, there is a unique solution $Y_t(z)$ to the Riemann-Hilbert problem* (7.2.1) *with jump matrix*

$$J = \begin{pmatrix} 0 & 2\omega\xi \\ 0 & 0 \end{pmatrix}.$$

Moreover, if there exists a continuous positive function $\lambda(t)$ such that $(Y_t)_{1,1}$ and $\lambda(t)(Y_t)_{2,1}$ are independent of t, then the matrix-valued function

$$Y(z) = \Lambda_1^{-1} \int_I \Lambda(t) Y_t(z)\, dt\, \Lambda_2^{-1}$$

is the unique solution to (7.1.2), *with* $W = \begin{pmatrix} 0 & 1_{\mathcal{D}}\,\omega \\ 0 & 0 \end{pmatrix}$, *provided that*

$$\Lambda(t) = \begin{pmatrix} 1 & 0 \\ 0 & \lambda(t) \end{pmatrix}, \quad \Lambda_1 = \begin{pmatrix} 1 & 0 \\ 0 & \int_I \lambda(t)\, dt \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} |I| & 0 \\ 0 & 1 \end{pmatrix}.$$

Proof. We first establish the existence of solutions to the problem (7.2.1) of Γ_t, which may be expressed in terms of a family of t-dependent orthogonal polynomials. We recall that ξ factors as $\nu\bar{\eta}$, where ν denotes the speed of the boundary in the normal direction while η denotes the outward pointing unit normal field. Since arc-length measure $|dz|$ on Γ_t relates to the complex line element dz by $dz = \tau|dz|$ where τ denotes the unit tangent vector field along Γ_t, it follows that

$$\frac{1}{2\pi i} dz = \frac{1}{2\pi}(-i\tau)|dz| = \eta\, ds$$

where we recall the convention $ds = \frac{|dz|}{2\pi}$. From this it follows that $(2\pi i)^{-1}\xi\, dz = \nu\, ds$, and we may consequently define an inner product by

$$\langle f, g\rangle_t := \int_{\Gamma_t} f(z)\bar{g}(z)\nu(z)\,ds(z) = \frac{1}{2\pi i}\int_{\Gamma_t} f(z)\bar{g}(z)\xi(z)\,dz.$$

Let $\{\pi^\star_{n,t}\}_n$ denote the sequence of monic orthogonal polynomials with respect to this inner product, such that $\pi^\star_{n,t}$ has degree n, and denote by $\kappa^\star_{n,t}$ the sequence of leading coefficient of the corresponding normalized orthogonal polynomials $P^\star_{n,t} = \kappa^\star_{n,t}\pi^\star_{n,t}$. It is straightforward to check that the function

$$\begin{pmatrix} \pi^\star_{n,t} & 2\mathbf{C}_{\Gamma_t}[\bar{\pi}^\star_{n,t}\omega\xi] \\ -\frac{1}{2}(\kappa^\star_{n-1,t})^2\pi^\star_{n-1,t} & -(\kappa^\star_{n-1,t})^2\mathbf{C}_{\Gamma_t}[\bar{\pi}^\star_{n-1,t}\omega\xi](z) \end{pmatrix}$$

supplies a solution to the Riemann-Hilbert problem (7.2.1).

As for the uniqueness, it is clear from Plemelj's formula (7.2.2) and the jump condition that any solution $Y_t(z)$ must take the form

$$Y_t(z) = \begin{pmatrix} a_t(z) & u_t(z) + 2\mathbf{C}_{\Gamma_t}[\bar{a}_t\omega\xi](z) \\ b_t(z) & v_t(z) + 2\mathbf{C}_{\Gamma_t}[\bar{b}_t\omega\xi](z) \end{pmatrix},$$

where a_t, b_t, u_t, v_t are entire functions. From the growth constraint at infinity, we see that these four functions are all polynomials. Moreover, $u_t = v_t = 0$ for the same reason. A standard expansion of the Cauchy kernel at infinity shows that a_t is a monic polynomial of degree n which is orthogonal to the lower degree polynomials Pol_n with respect to $\omega\xi\,dz$ on Γ_t. It follows that $a_t = \pi^\star_{n,t}$. Analogously, b_t is given by $b_t = -\frac{1}{2}(\kappa^\star_{n-1,t})^2\pi^\star_{n-1,t}$. We have established the unique solvability of the Riemann-Hilbert problem (7.2.1) with the given jump matrix.

Next, we turn to the connection with the $\bar\partial$-problem (7.1.2). Under the assumption that $(Y_t)_{1,1} = a_t = A$ is independent of t, and that for some t-dependent parameter $\lambda(t)$, the expression $\lambda(t)(Y_t)_{2,1} = \lambda(t)b_t = B$ is also independent of t, we may consequently write

$$\Lambda(t)\,Y_t(z) = \begin{pmatrix} A(z) & 2\mathbf{C}_{\Gamma_t}[\bar{A}\omega\xi](z) \\ B(z) & 2\mathbf{C}_{\Gamma_t}[\bar{B}\omega\xi](z) \end{pmatrix},$$

where we recall that $\Lambda(t)$ is the matrix given in the proposition. Recall that we may integrate over the flow $\{\Gamma_t\}_t$ using the disintegration

$$\int_{t\in I}\left(2\int_{\Gamma_t} u(z)\nu(z)\,ds\right)dt = \int_{\mathcal{D}} u(z)\,dA(z),$$

for functions u such that the indicated integrals have a well-defined meaning. It now follows that if $\langle\lambda\rangle_I = \int_I \lambda(t)\,dt$, the matrix-valued function

$$\begin{aligned}\hat{Y}(z) := \Lambda_1^{-1}\int_I \Lambda(t)Y_t(z)\,dt\,\Lambda_2^{-1} &= \Lambda_1^{-1}\begin{pmatrix} |I|\,A(z) & -\mathbf{C}[\bar{A}\omega\,1_{\mathcal{D}}](z) \\ |I|\,B(z) & -\mathbf{C}[\bar{B}\omega\,1_{\mathcal{D}}](z) \end{pmatrix}\Lambda_2^{-1} \\ &= \begin{pmatrix} A(z) & -\mathbf{C}[\bar{A}\omega\,1_{\mathcal{D}}](z) \\ (\langle\lambda\rangle_I)^{-1}\,B(z) & -(\langle\lambda\rangle_I)^{-1}\,\mathbf{C}[\bar{B}\omega\,1_{\mathcal{D}}](z) \end{pmatrix}\end{aligned}$$

solves

$$\bar\partial\hat{Y}(z) = \begin{pmatrix} 0 & -\bar{A}\omega\,1_{\mathcal{D}} \\ 0 & -(\langle\lambda\rangle_I)^{-1}\bar{B}\omega\,1_{\mathcal{D}} \end{pmatrix} = -\overline{\hat{Y}(z)}\begin{pmatrix} 0 & \omega\,1_{\mathcal{D}} \\ 0 & 0 \end{pmatrix}$$

with asymptotics

$$\hat{Y}(z) = \big(I + \mathrm{O}(z^{-1})\big) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}, \text{ as } |z| \to +\infty,$$

as a consequence of the corresponding asymptotics of Y_t for each $t \in I$. □

Remark 7.2.2. (a) For the orthogonal foliation flow, in the context of a neighborhood of the boundary curve of the droplet $\mathcal{S}_\tau$ with $\tau = \frac{n}{m}$, the (approximate) orthogonal polynomial of degree n is also approximately orthogonal to the lower degree polynomials along the individual flow loops corresponding to $\omega = e^{-2mQ}$. So, in view of Proposition 7.2.1, the conditions

$$\partial_t (Y_t)_{1,1} = 0, \quad \partial_t\big(\lambda(t)(Y_t)_{2,1}\big) = 0 \tag{7.2.3}$$

should be met at least approximately for some appropriate scalar-valued function $\lambda(t)$ (cf. the presentation in §1.6). Alternatively, we could use (7.2.3) as a criterion to define a flow of curves. In the given setting, this should give us back our orthogonal foliation flow. In other words, (7.2.3) should be analogous to the condition (6.4.2), once the Riemann-Hilbert problems of Proposition 7.2.1 are approximately solved in a constructive fashion, and we would expect that in an approximate sense,

$$\Gamma_t \sim \phi_\tau^{-1}(\psi_{m,n,-t}(\mathbb{T})).$$

It is entirely possible that the conditions (7.2.3) would be more stable close to the zeros of the orthogonal polynomial $\pi_{m,n}$. For instance, this might be the case with a highly eccentric ellipse.

(b) In their work, Its and Takhtajan use a bounded domain Ω to address possible convergence issues. Here, the potential Q grows sufficiently rapidly, so there is no need for us to consider such a truncation.

References

[1] Agam, O., Bettelheim, E., Wiegmann, P., Zabrodin, A., *Viscous fingering and a shape of an electronic droplet in the Quantum Hall regime*, Phys. Rev. Lett. **88** no. 236801 (2002).

[2] Ameur, Y., Hedenmalm, H., Makarov, N., *Berezin transform in polynomial Bergman spaces.* Comm. Pure Appl. Math. **63** (2010), no. 12, 1533-1584.

[3] Ameur, Y., Hedenmalm, H., Makarov, N., *Fluctuations of random normal matrices.* Duke Math. J. **159** (2011), 31-81.

[4] Ameur, Y., Hedenmalm, H., Makarov, N., *Random normal matrices and Ward identities.* Ann. Prob. **43** (2015), 1157-1201.

[5] Ameur, Y., Kang, N.-G., Makarov, N., *Rescaling Ward identities in the random normal matrix model.* Constr. Approx. **50** (2019), 63-127.

[6] Ameur, Y., Kang, N.-G., Makarov, N., Wennman, A., *Scaling limits of random normal matrix processes at singular boundary points.* J. Funct. Anal. **278**, (2020).

[7] Becker, J., Pommerenke, Ch, *Schlichtheitskriterien und Jordangebiete*, J. Reine Angew. Math. **354** (1984), 74-94.

[8] Berman, R. J., *Bergman kernels for weighted polynomials and weighted equilibrium measures of $\mathbb{C}^n$*, Indiana Univ. Math. J. **58** (2009) 1921-1946.

[9] Berman, R., Berndtsson, B., Sjöstrand, J., *A direct approach to Bergman kernel asymptotics for positive line bundles.* Ark. Mat. **46** (2008), no. 2, 197-217.

[10] Balogh, F., Bertola, M., Lee., S-Y., McLaughlin, K. D., *Strong asymptotics of orthogonal polynomials with respect to a measure supported on the plane.* Comm. Pure Appl. Math. **68** (2015), no. 1, 112-172.

[11] Bleher, P. M., Kuijlaars, A. B. J., *Orthogonal polynomials in the normal matrix model with a cubic potential*, Adv. Math. **230** (2012), 1272-1321.

[12] Bufetov, A. I., Fan, S., Qui, Y., *Equivalence of Palm measures for determinantal processes governed by Bergman kernels*. Probab. Theory Related Fields **172** (2018), no. 1-2, 31-69.

[13] Carleman, T., *Über die Approximation analytischer Funktionen durch lineare Aggregate von vorgegebenen Potenzen*. Ark. Mat. Astron. Fys. **17** (1923), 1-30.

[14] Carleman, T., *Édition complète des articles de Torsten Carleman*. Edited by Å. Pleijel, in collaboration with L. Lithner and J. Odhnoff. Published by the Mittag-Leffler mathematical institute. Litos Reprotryck, Malmö, 1960.

[15] Constantine, G. M., Savits, T. H., *A multivariate Faà di Bruno formula with applications*. Trans. Amer. Math. Soc. **348** (1996), no. 2, 503-520.

[16] Deift, P. A., *Orthogonal polynomials and random matrices: a Riemann-Hilbert approach*. Courant Lecture Notes in Mathematics, **3**. New York University, Courant Institute of Mathematical Sciences, New York, NY. American Mathematical Society, Providence, RI, 1999.

[17] Deift, P., Zhou, X., *A steepest descent method for oscillatory Riemann-Hilbert problems. Asymptotics for the MKdV equation*. Ann. of Math. (2) **137** (1993), no. 2, 295-368.

[18] Dragnev, P., Miña-Diaz, E., *On a series representation for Carleman orthogonal polynomials*. Proc. Amer. Math. Soc. **138**, (2010) 4271-4279.

[19] Dragnev, P., Miña-Diaz, E., *Asymptotic behavior and zero distribution of Carleman orthogonal polynomials*. J. Approx. Theory **162** (2010) no 11., 1982-2003.

[20] Duren, P., *Polynomials orthogonal over a curve*. Michigan Math. J. **12** (1965), 313-316.

[21] Faddeeva, V. N., Terent'ev, N. M. *Tables of values of the function* $w(z) = e^{-z^2}(1 + 2i\pi^{-\frac{1}{2}}\int_0^z e^{t^2}\,dt)$ *for complex argument*. Mathematical Tables Series, Vol. 11 Pergamon Press, Oxford-London-New York-Paris 1961 v+280 pp.

[22] Fefferman, C., *The Bergman kernel and biholomorphic mappings of pseudoconvex domains*. Invent. Math. **26** (1974), 1-65.

[23] Fokas, A. S., Its, A. R., Kitaev, A. V., *Discrete Painlevé equations and their appearance in quantum gravity*. Comm. Math. Phys. **142** (1991), no. 2, 313-344.

[24] Fokas, A. S., Its, A. R., Kitaev, A. V., *The isomonodromy approach to matrix models in 2D quantum gravity*. Comm. Math. Phys. **147** (1992), no. 2, 395-430.

[25] Garnett, J. B., Marshall, D. E., *Harmonic measure*. New Mathematical Monographs, vol. 2, Cambridge University Press, Cambridge, 2005.

[26] Gaunard, F., Hedenmalm, H., Shimorin, S., *Unpublished manuscript*.

[27] Gustafsson, B., Vasil'ev, A., Teodorescu, R., *Classical and Stochastic Laplacian Growth*. Advances in Mathematical Fluid Mechanics, Birkhäuser, Springer International Publishing, 2014.

[28] Haimi, A., Hedenmalm, H., *Asymptotic expansion of polyanalytic Bergman kernels*. J. Funct. Anal. **267** (2014), no. 12, 4667-4731.

[29] Hedenmalm, H., Makarov, N., *Coulomb gas ensembles and Laplacian growth*. Proc. London Math. Soc. (3) **106** (2013), 859-907.

[30] Hedenmalm, H., Olofsson, A., *Hele-Shaw flow on weakly hyperbolic surfaces*. Indiana Univ. Math. J. **54** (2005), no. 4, 1161-1180.

[31] Hedenmalm, H., Perdomo-G., Y., *Mean value surfaces with prescribed curvature form*. J. Math. Pures Appl. (9) **83** (2004), no. 9, 1075-1107.

[32] Hedenmalm, H., Shimorin, S., *Hele-Shaw flow on hyperbolic surfaces*. J. Math. Pures Appl. **81** (2002), 187-222.

[33] Hedenmalm, H., Jakobsson, S., Shimorin, S., *A biharmonic maximum principle for hyperbolic surfaces*. J. Reine Angew. Math. **550** (2002), 25-75.

[34] Hedenmalm, H., Wennman, A., *Off-spectral analysis of Bergman kernels*. Comm. Math. Phys. **373** (2020), 1049-1083.

[35] Hedenmalm, H., Wennman, A., *Riemann-Hilbert hierarchies for hard edge planar orthogonal polynomials*. Submitted (2020).

[36] Hedenmalm, H., Wennman, A., *A strong planar Szegő limit theorem*. Work in progress (2021).

[37] Hörmander, L., *L^2 estimates and existence theorems for the $\bar{\partial}$ operator.* Acta Math. **113** (1965), 89-152.

[38] Hörmander, L., *The analysis of linear partial differential operators. I. Distribution theory and Fourier analysis.* Second edition. Grundlehren der Mathematischen Wissenschaften, **256**. Springer-Verlag, Berlin, 1990.

[39] Its, A., Takhtajan, L. *Normal matrix models, $\bar{\partial}$-problem, and orthogonal polynomials in the complex plane.* (2007), arXiv:07083867.

[40] Kinderlehrer, D., Stampacchia, G., *An Introduction to Variational Inequalities and Their Applications*, SIAM Classics in Applied Mathematics, 2000.

[41] Lee, S.-Y., Riser, R., *Fine asymptotic behavior for eigenvalues of random normal matrices: Ellipse case.* J. Math. Phys. **57** no. 2, (2016), 29pp.

[42] Lubinsky, D. S., Saff, E. B., *Strong asymptotics for extremal polynomials associated with weights on* $\mathbb{R}$, Lecture Notes in Mathematics, vol. **1305**, Springer, New York, 1988.

[43] Miña-Diaz, E., *An asymptotic integral representation for Carleman orthogonal polynomials.* Int. Math. Res. Not. IMRN **2008**, no. 16, Art. ID rnn065, 38 pp.

[44] Putinar, M., Stylianopoulos, N. *Finite-term relations for planar orthogonal polynomials.* Complex Anal. Oper. Theory **1** (2007), no. 3, 447-456.

[45] Pommerenke, C., *Boundary behaviour of conformal maps.* Grundlehren der Mathematischen Wissenschaften **299.** Springer-Verlag, Berlin, 1992.

[46] Riser, R., *Universality in Gaussian Random Normal Matrices.* arXiv:1312.0068.

[47] Ross, J., Singer, M., *Asymptotics of partial density functions for divisors.* J. Geom. Anal. **27** (2017), no. 3, 1803-1854.

[48] Ross, J., Witt Nyström, D., *The Hele-Shaw flow and moduli of holomorphic discs.* Compos. Math. **151** (2015) no. 12, 2301-2328.

[49] Saff, E. B., Totik, V. *Logarithmic Potentials with External Fields*, Grundlehren der mathematischen Wissenschaften, Springer Verlag, New York-HeidelBerg-Berlin (1997).

[50] Sakai, M., *Regularity of a boundary having a Schwarz function.* Acta Math. **166** (1991), no. 3-4, 263-297.

[51] Simon, B., *Orthogonal polynomials on the unit circle. Part 1. Classical theory.* American Mathematical Society Colloquium Publications, 54, Part 1. American Mathematical Society, Providence, RI, 2005.

[52] Simon, B., *Orthogonal polynomials on the unit circle. Part 2. Spectral theory.* American Mathematical Society Colloquium Publications, 54, Part 2. American Mathematical Society, Providence, RI, 2005.

[53] Stahl, H. and Totik, V., *General orthogonal polynomials*, Encyclopedia of Mathematics and its Applications **43**, Cambridge University Press, 1992.

[54] Suetin, P. K., *Polynomials orthogonal over a region and Bieberbach polynomials.* Translated from the Russian by R. P. Boas. Proceedings of the Steklov Institute of Mathematics, No. 100 (1971). American Mathematical Society, Providence, R.I., 1974.

[55] Szegő, G., *Über orthogonale Polynome die zu einer gegebenen Kurve der komplexen Ebene gehören.* Math. Z. **9** (1921), 218-270.

[56] Szegő, G., *Orthogonal polynomials.* Fourth edition. American Mathematical Society, Colloquium Publications, Vol. XXIII. American Mathematical Society, Providence, R.I., 1975.

[57] Teodorescu, R., Bettelheim, E. Agaom, O. Zabrodin, A., Wiegmann, P., *Semiclassical evolution of the spectral curve in the normal random matrix ensemble as Whitham hierarchy.* Nuclear Phys. B **700** (2004) no. 1-3, 521-532.

[58] Tian, G., *On a set of polarized Kähler metrics on algebraic manifolds.* J. Differential Geom. **32** (1990), 99-130.

[59] Totik, V., *Weighted approximation with varying weights*, Lecture Notes in Mathematics, vol **1559**, Springer Verlag, New York, 1994.

[60] Wiegmann, P., *Aharonov-Bohm Effect in the Quantum Hall Regime and Laplacian Growth Problems.* Statistical field theories (Como, 2001), 337-349, NATO Sci. Ser. II Math. Phys. Chem., 73, Kluwer Acad. Publ., Dordrecht, 2002.

[61] Wiegmann, P., Zabrodin, A., *Large N-expansion for the $2D$ Dyson Gas.* J. Phys. A **39** (2006), no. 28, 8933-8963.

[62] Zelditch, S., Zhou, P., *Interface asymptotics of partial Bergman kernels on S^1-symmetric Kähler manifolds*. J. Symplectic Geom. **17** (2019), no. 3, 793-856,

[63] Zelditch, S., Zhou, P., *Central limit theorem for spectral partial Bergman kernels*. Geom. Topol. **23** (2019), 1961-2004.

Hedenmalm: Department of Mathematics, The Royal Institute of Technology, S – 100 44 Stockholm, SWEDEN

Email address: `haakanh@math.kth.se`

Wennman: School of mathematical sciences, Tel Aviv University, Tel Aviv 69978, ISRAEL

Email address: `aronwennman@tauex.tau.ac.il`