Hyperbolic Fourier series and the Klein-Gordon equation

Haakan Hedenmalm

30 May 2023

Reports on joint work with A. Bakan, (A. Montes-Rodríguez), D. Radchenko, M. Viazovska

Exponential solutions of the Klein-Gordon equation

The Klein-Gordon equation in $1+1$ dimensions can be written in the form

$$u''_{xy} + u = 0.$$

If we try for pure complex exponential solutions

$$u(x,y) = \mathrm{e}^{\mathrm{i}\lambda x + \mathrm{i}\mu y},$$

then

$$u''_{xy} + u = (1 - \lambda\mu)\,\mathrm{e}^{\mathrm{i}\lambda x + \mathrm{i}\mu y} = 0$$

holds if and only if $\lambda\mu = 1$. Such pure complex exponential solutions are bounded if and only if $(\lambda, \mu) \in \mathbb{R}^2$. In other words, the bounded complex exponential solutions are parametrized by $(\lambda, \mu) \in \mathbb{R}^2$ with $\lambda\mu = 1$. Replacing λ by t and μ by $1/t$, such solutions are given by

$$u(x,y) = \mathrm{e}^{\mathrm{i}tx + \mathrm{i}y/t},$$

for $t \in \mathbb{R}^\times := \mathbb{R} \setminus \{0\}$.

L^1-mixed bounded exponential solutions

A function of the form

$$u(x,y) = U_\varphi(x,y) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}tx+\mathrm{i}y/t}\varphi(t)\,\mathrm{d}t$$

where $\varphi \in L^1(\mathbb{R})$, is said to be an *$L^1$-mixed bounded exponential solution*. It is bounded and solves the Klein-Gordon equation $u''_{xy} + u = 0$. The restrictions to the characteristic axes,

$$u(x,0) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}tx}\varphi(t)\,\mathrm{d}t, \quad u(0,y) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}y/t}\varphi(t)\,\mathrm{d}t$$

are inter-connected. For instance, it is possible to see that

$$u(0,y) = u(0,0) - \int_0^{+\infty} \mathrm{J}_1(-y,t)u(t,0)\,\mathrm{d}t, \qquad y \le 0, \qquad (1)$$

where

$$\mathrm{J}_1(x,y) := \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!(k+1)!} x^{k+1} y^k$$

is a two-variable version of the familiar Bessel function.

The Goursat problem for L^1-mixed bounded solutions

The Klein-Gordon equation $u''_{xy} + u = 0$ has as characteristic directions the two axes (or any translate of the axes). Starting from the intersection point of the axes (=the origin), we can split the plane in four quarter-planes:

$$x, y \geq 0 \quad \text{or} \quad x, y \leq 0 \qquad \text{(time-like)},$$

and

$$x \geq 0, y \leq 0 \quad \text{or} \quad x \leq 0, y \geq 0 \qquad \text{(space-like)}.$$

In the space-like quarter-planes, the values on the the boundary lines influence each other strongly, as in the formula with the Bessel function.

Goursat problem (space-like)

For an L^1-mixed bounded solution u, we would like to prescribe

$$u(x,0) = f(x), \quad u(0,y) = g(y), \qquad x \geq 0, y \leq 0,$$

for given continuous functions f, g.

Overdetermination of the Goursat problem

In view of the relationship (1), the Goursat problem cannot be solved for arbitrary, say smooth, functions f and g, given that we look for bounded solutions u of the given form. Of course, one way out is to ask that f, g should be connected, but an alternative idea is to reduce the quantity of boundary data. If we ask that the boundary data should hold on thinned out versions of the two semi-axes, we might stand a chance to have a well-defined problem.

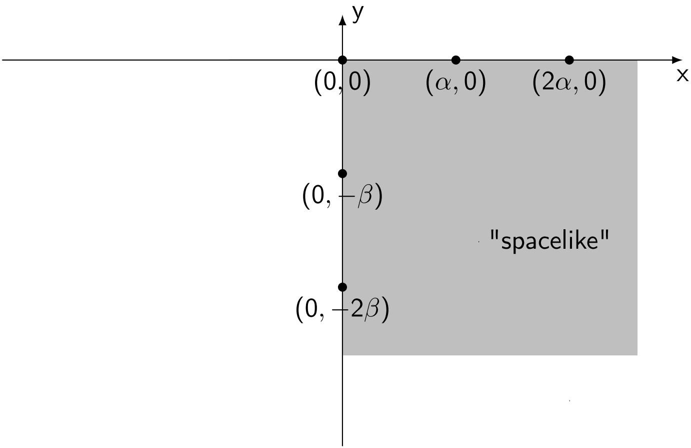
Lattice-cross Goursat problem

For the quarter-plane $x \geq 0, y \leq 0$, we consider lattice-cross data

$$u(\alpha m, 0) = f(\alpha m), \quad u(0, -\beta n) = g(-\beta n), \qquad m, n \in \mathbb{Z}_{\geq 0}.$$

Here, α, β are positive reals which are spacing parameters. The functions f, g are assumed smooth with compact support.

Illustration of the discretized Goursat problem



Critical lattice-cross Goursat problem

Theorem (over/underdetermined)

The lattice-cross Goursat problem is overdetermined if $\alpha\beta < \pi^2$, underdetermined if $\alpha\beta > \pi^2$, and critical if $\alpha\beta = \pi^2$.

What the theorem says is (a) if $\alpha\beta < \pi^2$, the points are too dense in the lattice-cross so that some dependence between the data still remains, and (b) if $\alpha\beta > \pi^2$ the density is too low for the solution u to be unique. However, if $\alpha\beta = \pi^2$, the density is precisely right (critical). In the critical case, we may use scaling invariance to assume $\alpha = \beta = \pi$.

Critical Goursat: uniqueness

(Hedenmalm, Montes-R) If $\alpha = \beta = \pi$, we have that for $u = U_\varphi$ with $\varphi \in L^1(\mathbb{R})$,

$$\forall m, n \in \mathbb{Z}_{\geq 0} : u(\pi m, 0) = u(0, -\pi n) = 0$$
$$\iff \quad \forall x \geq 0, y \leq 0 : u(x, y) = 0.$$

Dual formulation of critical Goursat

Let $H^p_+(\mathbb{R})$ denote the usual Hardy space on the real line consisting of $L^p(\mathbb{R})$ functions whose Poisson formula harmonic extensions to the upper half-plane (=hyperbolic plane)

$$\mathbb{H} := \{\tau \in \mathbb{C} : \operatorname{Im} \tau > 0\}$$

are holomorphic.

THEOREM (critical Goursat)

The system

$$\mathrm{e}^{\mathrm{i}\pi mt}, \ \mathrm{e}^{-\mathrm{i}\pi n/t}, \qquad m, n \in \mathbb{Z}_{\geq 0},$$

is weak-star complete in $H^\infty_+(\mathbb{R})$. In other words, for $\varphi \in L^1(\mathbb{R})$, we have that

$$\forall m, n \in \mathbb{Z}_{\geq 0} : \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\pi mt}\varphi(t)\,\mathrm{d}t = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\pi n/t}\varphi(t)\mathrm{d}t = 0 \Longleftrightarrow \varphi \in H^1_+(\mathbb{R}).$$

Remark

This theorem is due to Hedenmalm and Montes-Rodriguez (Annals 2011), and the proof is based on the dynamics of the Gauss-type map $\mathrm{g}_2(x) = -1/x \bmod 2\mathbb{Z}$ on the interval $[-1, 1]$.

Critical Goursat: the interpolation problem

Problem

Suppose we try to solve for Kronecker delta data:

$$u(\pi m, 0) = \delta_{k,m}, \quad u(0, -\pi n) = 0, \qquad m \in \mathbb{Z},\ n \in \mathbb{Z}_{\neq 0},$$

or

$$u(\pi m, 0) = 0, \quad u(0, -\pi n) = \delta_{k,n}, \qquad m \in \mathbb{Z},\ n \in \mathbb{Z}_{\neq 0}.$$

Here, $k \in \mathbb{Z}_{\geq 0}$ and $u = U_\varphi$ with $\varphi \in L^1(\mathbb{R})$.

Note:

(a) By symmetry, the two instances are equivalent, as we may interchange $x \leftrightarrow y$ in the Klein-Gordon equation and nothing happens. So we may concentrate on the first instance.
(b) Do such solutions $u = U_\varphi$ exist? What do they look like? What is the corresponding φ? If so, are they smooth? How smooth? These functions would be the Klein-Gordon analogue of the cardinal sine functions for the Paley-Wiener space, or the Pehr Beurling functions for H^∞ interpolation.

Critical interpolation in terms of the density φ

Problem

Given $k \in \mathbb{Z}_{\geq 0}$, find $\varphi_k \in L^1(\mathbb{R})$ such that

$$\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\pi m t} \varphi_k(t) \mathrm{d}t = \delta_{k,m}, \quad \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\pi n/t} \varphi_k(t) \mathrm{d}t = 0,$$

for all $m \in \mathbb{Z}$ and $n \in \mathbb{Z}_{\neq 0}$.

Note that if φ_k exists, it must belong to the Hardy space $H^1_-(\mathbb{R})$. It turns out that we find the functions φ_k for free if we study *hyperbolic Fourier series*.

Hyperbolic Fourier series

Definition

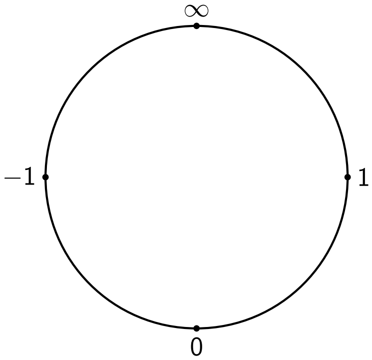
A series of the form
$$\sum_{n\in\mathbb{Z}} a_n \mathrm{e}^{\mathrm{i}\pi n t} + b_n \mathrm{e}^{-\mathrm{i}\pi n/t}$$
is called a *hyperbolic (or bipolar) Fourier series*. To avoid double representation of constants, we usually ask that $b_0 = 0$.

Note:

(a) We have not made any requirement about convergence, so it may be just a formal expression. Moreover, whereas the sum of two hyperbolic Fourier series is another hyperbolic Fourier series (just add the corresponding coefficients), the product of two hyperbolic Fourier series does not appear to be a hyperbolic Fourier series (at least not automatically).
(b) Which functions may be expressed as hyperbolic Fourier series?
(c) Can we even extend the notion to hyperbolic Fourier series expansion to distribution theory?

Why study hyperbolic Fourier series?

Hyperbolic Fourier series may look strange at first glance.



Basic idea

Like in differential geometry, we cover the line with two patches and corresponding coordinates: On the interval $(-1, 1)$ we use t, while on the complement, we use $1/t$ (or $-1/t$) as the basic coordinate.

Hyperbolic Fourier series and critical interpolation

Underlying observation

Using the series

$$\sum_{n\in\mathbb{Z}} a_n \mathrm{e}^{\mathrm{i}\pi nt}$$

we can represent any function on $(-1, 1)$, while using the series

$$\sum_{n\in\mathbb{Z}} b_n \mathrm{e}^{-\mathrm{i}\pi n/t}$$

we can represent any function on the complement $\mathbb{R} \setminus [-1, 1]$. By summing the two we should be able to represent any function on the whole line, although we ought to *worry about interference* (due to the periodicity of each term).

Question

How would hyperbolic Fourier series help us find the function φ_k we mentioned before?

Hyperbolic Fourier series and critical interpolation II

Suppose we are lucky and able to expand rather general functions or even distributions into hyperbolic Fourier series in a stable way, i.e.

$$f(t) \sim \sum_{n\in\mathbb{Z}} a_n(f)\mathrm{e}^{\mathrm{i}\pi nt} + b_n(f)\mathrm{e}^{-\mathrm{i}\pi n/t}$$

with well-defined and unique coefficients $a_n(f)$ and $b_n(f)$ that depend linearly on f, as in Fourier theory.

Biorthogonal system

$f \mapsto a_n(f)$ and $f \mapsto b_n(f)$ are linear functionals. In terms of the pairing

$$\langle f, g\rangle_{\mathbb{R}} = \int_{\mathbb{R}} f(t)g(t)\,\mathrm{d}t,$$

a_n, b_n should correspond to test functions A_n, B_n associated with the space of distributions we are working with. These functions form a biorthogonal system to $\{\mathrm{e}^{\mathrm{i}\pi nt}, \mathrm{e}^{-\mathrm{i}\pi n/t}\}_n$, and $\varphi_n = A_n$ is the function we seek.

The Schrödinger transform

Schrödinger evolution

If $\varphi(x)$ is a function on $\mathbb{R}^d$, the extended function

$$\Phi(x,t) := \mathrm{e}^{-\mathrm{i}t\Delta_x}\varphi(x)$$

solves the potential-free Schrödinger equation with initial datum $\Phi(x,0) = \varphi(x)$:

$$\mathrm{i}\partial_t\Phi - \Delta_x\Phi = 0.$$

Definition of Schrödinger transform

The Schrödinger transform of φ is the function

$$\mathfrak{S}\varphi(t) := \Phi(0,t).$$

Question

Does the Schrödinger transform $\mathfrak{S}\varphi$ determine φ uniquely?

The Schrödinger transform and Fourier theory

Let the Fourier transform $\hat{\varphi}$ be given by

$$\hat{\varphi}(\xi) := \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}\langle x,\xi\rangle}\varphi(x)\,\mathrm{dVol}_d(x), \qquad \xi \in \mathbb{R}^d.$$

Then the Schrödinger evolution of φ may be written in the form

$$\Phi(x,t) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}\langle x,\xi\rangle}\mathrm{e}^{\mathrm{i}t|\xi|^2}\hat{\varphi}(\xi)\,\mathrm{dVol}_d(\xi)$$

so that the Schrödinger transform becomes

$$\mathfrak{S}\varphi(t) = \Phi(0,t) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}t|\xi|^2}\hat{\varphi}(\xi)\,\mathrm{dVol}_d(\xi).$$

If $\hat{\varphi}$ is radial, i.e., if φ is radial, this looks like a variant of the Fourier transform and we expect the Schrödinger transform to be one-to-one on such functions. In fact, if $\hat{\varphi}$ is replaced by its radialization

$$\hat{\varphi}_{\mathrm{rad}}(\xi) = \frac{1}{|\mathbb{S}^{d-1}|}\int_{\mathbb{S}^{d-1}} \hat{\varphi}(|\xi|\eta)\mathrm{dVol}_{d-1}(\eta)$$

and φ_{rad} denotes its inverse Fourier transform, which is radial too, then $\mathfrak{S}\varphi_{\mathrm{rad}} = \mathfrak{S}\varphi$.

The Schrödinger transform of basis functions

Complex exponential φ

If $\varphi(x) = e_\alpha(x) := \mathrm{e}^{\mathrm{i}\langle \alpha, x\rangle}$ for some vector $\alpha \in \mathbb{R}^d$, then its Schrödinger evolution is $\Phi(x,t) = \mathrm{e}^{\mathrm{i}\langle \alpha, x\rangle + \mathrm{i}|\alpha|^2 t}$, and, in particular,

$$\mathfrak{S}\varphi(t) = \mathfrak{S}e_\alpha(t) = \Phi(0,t) = \mathrm{e}^{\mathrm{i}|\alpha|^2 t}.$$

Point mass φ

If $\varphi(x) = \delta_\beta(x)$ for some $\beta \in \mathbb{R}^d$, then its Schrödinger evolution is $\Phi(x,t) = 2^{-d}\pi^{-d/2}(t/\mathrm{i})^{-d/2}\mathrm{e}^{-\mathrm{i}|x-\beta|^2/(4t)}$, and, in particular,

$$\mathfrak{S}\varphi(t) = \mathfrak{S}\delta_\beta(t) = \Phi(0,t) = 2^{-d}\pi^{-d/2}(t/\mathrm{i})^{-d/2}\mathrm{e}^{-\mathrm{i}|\beta|^2/(4t)}.$$

Choice of parameter values

We choose $\alpha, \beta \in \mathbb{R}^d$ such that $|\alpha|^2 = \pi m$ and $|\beta|^2 = 4\pi n$ for integers m, n with $m \geq 0$ and $n > 0$. Then

$$\mathfrak{S}e_\alpha(t) = \mathrm{e}^{\mathrm{i}\pi m t}, \quad \mathfrak{S}\delta_\beta(t) = 2^{-d}\pi^{-d/2}(t/\mathrm{i})^{-d/2}\mathrm{e}^{-\mathrm{i}\pi n/t}.$$

Fourier interpolation

Assumption

Suppose that we have obtained a unique *power skewed holomorphic hyperbolic Fourier series expansion*

$$f(\tau) = \sum_{n=0}^{+\infty} a_{n,d}(f)\,\mathrm{e}^{\mathrm{i}\pi n\tau} + b_{n,d}(f)\,(\tau/\mathrm{i})^{-d/2}\,\mathrm{e}^{-\mathrm{i}\pi n/\tau}, \qquad \tau \in \mathbb{H},$$

for a wide collection of holomorphic functions f in $\mathbb{H}$. Suppose moreover that

$$a_{n,d}(f) = \langle A_{n,d}, f\rangle_{\mathbb{R}}, \quad b_{n,d}(f) = \langle B_{n,d}, f\rangle_{\mathbb{R}},$$

for smooth functions $A_{n,d}$ and $B_{n,d}$.

Consequence: Fourier interpolation formula

Suppose $g : \mathbb{R}^d \to \mathbb{C}$ is a smooth radial test function, and write also g for the function on $\mathbb{R}_{\geq 0}$ given by $g(|x|) = g(x)$. Let $\mathfrak{S}^*$ be the adjoint of $\mathfrak{S}$, which takes functions on $\mathbb{R}$ and makes them radial functions on $\mathbb{R}^d$. We then obtain the Fourier interpolation formula

$$g(y) = \sum_{n=0}^{+\infty} \hat{g}(\sqrt{\pi n})\,\mathfrak{S}^* A_{n,d}(y) + 2^d \pi^{d/2} g(2\sqrt{\pi n})\,\mathfrak{S}^* B_{n,d}(y), \qquad y \in \mathbb{R}^d.$$

Distribution theory and harmonic extensions

Distributions on the unit circle

A distribution on the unit circle $\mathbb{T}$ is a continuous linear functional on $C^\infty(\mathbb{T}) = \mathcal{D}(\mathbb{T})$, written $u(f) = \langle f, u\rangle_{\mathbb{T}}$, where we may think of u as a "generalized function".

Harmonic functions from distributions on $\mathbb{T}$

For $z \in \mathbb{D}$, let

$$P(z, w) := (2\pi)^{-1} \frac{1 - |z|^2}{|1 - z\bar{w}|^2}, \qquad w \in \mathbb{T},$$

be the Poisson kernel, and associate with a distribution $u \in \mathcal{D}'(\mathbb{T})$ the harmonic function

$$\tilde{u}(z) := \langle P(z, \cdot), u\rangle_{\mathbb{T}}, \qquad z \in \mathbb{D}. \tag{2}$$

The harmonic function then has the growth bound ("moderate growth")

$$|\tilde{u}(z)| = \mathrm{O}(1 - |z|^2)^{-N}$$

as $|z| \to 1$, for some finite number N. The number $N + 1$ is then an upper bounds for the order of the distribution.

One-to-one correspondence distribution $\leftrightarrow$ harmonic function

The harmonic function $\tilde{u}$ has the distribution u as "boundary values" (boundary trace). Moreover, the distributions on $\mathbb{T}$ are in a one-to-one correspondence with the harmonic functions on $\mathbb{D}$ with moderate growth. For this reason, we may skip the tilde and denote both by u.

Ultradistribution theory/Hyperfunctions

In distribution theory, all distributions have finite order. For instance, only a finite number of derivatives of delta functions are allowed. But if we reduce the space of test functions from C^∞ to some other class of smooth functions, like the Carleman classes, we can allow for (ultra)distributions of infinite order. The formula (2) defines a harmonic function also for a given ultradistribution, but the growth rate goes up. Still, the one-to-one correspondence between u and $\tilde{u}$ stays in place. The *support* of an ultradistribution is the complement on the circle $\mathbb{T}$ of the open set where the ultradistribution vanishes (in the sense of Schwarz reflection). Given two complementary arcs on the circle (the intersection set is two points only), when can we split the ultradistribution as the sum of two ultradistributions of the same type, one supported on one arc and the other on the other arc? This can be explained in terms of the growth rate of the harmonic function.

Non-quasianalyticity condition

The non-quasianalyticity condition decides when such splitting is possible.

Theorem

Suppose $u : \mathbb{D} \to \mathbb{C}$ is harmonic with

$$|u(z)| = \mathrm{O}(M(|z|)) \quad \text{as} \;\; |z| \to 1^-,$$

where $M : [0, 1[\to \mathbb{R}$ is an increasing function. Given two closed arcs $I_1, I_2 \subset \mathbb{T}$ with $I_1 \cup I_2 = \mathbb{T}$ and $I_1 \cap I_2$ consisting of two points, we may split $u = u_1 + u_2$ where each u_j has the same kind of growth and is supported on I_j ($j = 1, 2$), provided that

$$\int_0^1 \log^+ \log^+ M(t) \, \mathrm{d}t < +\infty.$$

Ultradistributions on the extended real line

The Cayley transform

$$\zeta = \frac{\tau - \mathrm{i}}{\tau + \mathrm{i}}$$

takes the upper half plane $\mathbb{H}$ onto the unit disk $\mathbb{D}$. Moreover, the Cayley transform takes the extended real line $\mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$ onto the unit circle $\mathbb{T}$. A calculation shows that

$$\frac{1}{1 - |\zeta|^2} = \frac{|\tau + \mathrm{i}|^2}{4 \operatorname{Im} \tau} \asymp \frac{|\tau|^2 + 1}{\operatorname{Im} \tau},$$

which tells us that distributions on $\mathbb{R}_\infty$ are in a one-to-one correspondence with the harmonic functions on $\mathbb{H}$ of moderate growth:

$$|u(\tau)| = \mathrm{O}\left(\frac{1 + |\tau|^2}{\operatorname{Im} \tau} \right)^n.$$

Faster growth would be interpreted as ultradistribution theory.

Harmonic extension of hyperbolic Fourier series

The harmonic extension to $\mathbb{H}$ of $\mathrm{e}^{\mathrm{i}\pi nt}$ is the function

$$\begin{cases} \mathrm{e}^{\mathrm{i}\pi n\tau} & \text{if } \ n \geq 0, \\ \mathrm{e}^{\mathrm{i}\pi n\bar{\tau}} & \text{if } \ n < 0. \end{cases}$$

Likewise, the harmonic extension to $\mathbb{H}$ of $\mathrm{e}^{-\mathrm{i}\pi n/t}$ is the function

$$\begin{cases} \mathrm{e}^{-\mathrm{i}\pi n/\tau} & \text{if } \ n \geq 0, \\ \mathrm{e}^{-\mathrm{i}\pi n/\bar{\tau}} & \text{if } \ n < 0. \end{cases}$$

It follows that the harmonic extension of the hyperbolic Fourier series is

$$\sum_{n\geq 0} \left\{ a_n \mathrm{e}^{\mathrm{i}\pi n\tau} + b_n \mathrm{e}^{-\mathrm{i}\pi n/\tau} \right\} \ + \ \sum_{n<0} \left\{ a_n \mathrm{e}^{\mathrm{i}\pi n\bar{\tau}} + b_n \mathrm{e}^{-\mathrm{i}\pi n/\bar{\tau}} \right\}.$$

For this series to converge in $\mathbb{H}$ we need to impose that

$$|a_n|, |b_n| = \mathrm{O}(\exp(\epsilon |n|))$$

as $|n| \to +\infty$, for each positive real ϵ.

Uniqueness of hyperbolic Fourier series

Uniqueness theorem

Suppose $b_0 = 0$ while for each $\epsilon > 0$,

$$|a_n|, |b_n| = \mathrm{O}(\exp(\epsilon|n|))$$

as $|n| \to +\infty$, and that

$$|a_n| = \mathrm{o}\big(|n|^{-3/4} \exp\left(2\pi\sqrt{|n|}\right)\big)$$

as $|n| \to +\infty$. If

$$\sum_{n\geq 0} \left\{ a_n \mathrm{e}^{\mathrm{i}\pi n\tau} + b_n \mathrm{e}^{-\mathrm{i}\pi n/\tau} \right\} \; + \; \sum_{n<0} \left\{ a_n \mathrm{e}^{\mathrm{i}\pi n\bar{\tau}} + b_n \mathrm{e}^{-\mathrm{i}\pi n/\bar{\tau}} \right\} = 0$$

holds for each $\tau \in \mathbb{H}$, then $a_n = b_n = 0$ for all n.

Sharpness considerations

Question

Do we really need to impose any additional restraints on the coefficients a_n, b_n beyond that which guarantees convergence in $\mathbb{H}$?

Answer

It is needed and the condition is sharp. We consider the modular lambda function

$$\lambda(\tau) = \frac{\vartheta_2(\tau)^4}{\vartheta_3(\tau)^4} = \sum_{n=1}^{+\infty} \hat{\lambda}(n)\, \mathrm{e}^{\mathrm{i}\pi n\tau},$$

where $\hat{\lambda}(1) = 16$, $\hat{\lambda}(2) = -128$, $\hat{\lambda}(3) = 704$, etc. It has the functional property that

$$\lambda(\tau) + \lambda(-\tau^{-1}) - 1 = -1 + \sum_{n=1}^{+\infty} \hat{\lambda}(n)\big(\mathrm{e}^{\mathrm{i}\pi n\tau} + \mathrm{e}^{-\mathrm{i}\pi n/\tau}\big) = 0.$$

The coefficient growth is $\hat{\lambda}(n) = \mathrm{O}\big(|n|^{-3/4} \exp\left(2\pi\sqrt{|n|}\right)\big)$.

Beyond uniqueness

Question

What happens when we allow for faster growth of the coefficients? Can we say that the coefficients are unique modulo some small linear space of exceptional functions?

Exceptional coefficient classes $\mathcal{E}_k$

For integers $k \geq 0$, we say that the pair of sequences $\{a_n, b_n\}_{n\geq 0}$ with $b_0 = 0$ belongs to the *exceptional coefficient class* $\mathcal{E}_k^{\mathrm{hol}}$ provided that there exists a polynomial P of degree $\leq k$ with $P(0) = 0$ such that

$$\sum_{n\geq 0} a_n \,\mathrm{e}^{\mathrm{i}\pi n\tau} = P(\lambda(\tau)), \quad \sum_{n>0} b_n \,\mathrm{e}^{\mathrm{i}\pi n\tau} = -P(1-\lambda(\tau)).$$

Moreover, we say that the pair of sequences $\{a_n, b_n\}_{n\in\mathbb{Z}}$ with $b_0 = 0$ belongs to the exceptional coefficient class $\mathcal{E}_k^{\mathrm{harm}}$ provided that both $\{a_n, b_n\}_{n\geq 0}$ and $\{a_{-n}, b_{-n}\}_{n\geq 0}$ are in $\mathcal{E}_k^{\mathrm{hol}}$. The complex dimension of the linear space $\mathcal{E}_k^{\mathrm{hol}}$ equals k, while the dimension of $\mathcal{E}_k^{\mathrm{harm}}$ equals $2k$.

What happens beyond uniqueness

Beyond uniqueness theorem

Suppose $b_0 = 0$ while for each $\epsilon > 0$, $|a_n|, |b_n| = \mathrm{O}(\exp(\epsilon|n|))$ as $|n| \to +\infty$, and that for some integer $k \geq 1$,

$$|a_n| = \mathrm{o}\big(|n|^{-3/4} \exp\left(2\pi\sqrt{k|n|}\right)\big)$$

holds as $|n| \to +\infty$. If

$$\sum_{n\geq 0} \left\{a_n \mathrm{e}^{\mathrm{i}\pi n\tau} + b_n \mathrm{e}^{-\mathrm{i}\pi n/\tau}\right\} \;+\; \sum_{n<0} \left\{a_n \mathrm{e}^{\mathrm{i}\pi n\bar{\tau}} + b_n \mathrm{e}^{-\mathrm{i}\pi n/\bar{\tau}}\right\} = 0$$

holds for each $\tau \in \mathbb{H}$, then $\{a_n, b_n\}_{n\in\mathbb{Z}}$ belongs to the exceptional coefficient class $\mathcal{E}_{k-1}^{\mathrm{harm}}$.

Growth classes of harmonic and holomorphic functions

Growth classes

Let $M : \mathbb{H} \to \mathbb{R}_+$ denote the function

$$M(\tau) := \frac{\max\{1, |\tau|^2\}}{\operatorname{Im}\tau}.$$

For a real $\gamma > 0$, we let $\mathcal{M}^\gamma_{\mathrm{harm}}$ denote the space of harmonic functions $h : \mathbb{H} \to \mathbb{C}$ with the growth bound

$$|h(\tau)| = \mathrm{O}(\exp(\gamma M(\tau))$$

uniformly in $\mathbb{H}$. Likewise, we denote by $\mathcal{M}^\gamma_{\mathrm{hol}}$ the subspace of $\mathcal{M}^\gamma_{\mathrm{harm}}$ consisting of holomorphic functions.

Lemma

Each $h \in \mathcal{M}^\gamma_{\mathrm{harm}}$ decomposes as $h = f + \bar{g}$, where $f, g \in \mathcal{M}^\gamma_{\mathrm{hol}}$.

We know that $\nabla = (\partial, \bar{\partial})$ applied to h gives $\partial h = f'$ and $\bar{\partial} h = \bar{g}'$. We estimate the gradient and integrate backwards.

Harmonic hyperbolic Fourier series: step 1

The setting

We have a harmonic function h in the upper half-plane $\mathbb{H}$. We want to expand it in a convergent hyperbolic Fourier series

$$h(\tau) = \sum_{n\geq 0} a_n \mathrm{e}^{\mathrm{i}\pi n\tau} + b_n \mathrm{e}^{-\mathrm{i}\pi n/\tau} \;+\; \sum_{n<0} a_n \mathrm{e}^{\mathrm{i}\pi n\bar{\tau}} + b_n \mathrm{e}^{-\mathrm{i}\pi n/\bar{\tau}}$$

for $\tau \in \mathbb{H}$. This can be interpreted in terms of expanding the boundary hyperfunction/ultradistribution along the extended real line $\mathbb{R}_\infty$.

First step

We split $h = f + \bar{g}$, where f, g are holomorphic. This is unique up to an additive constant. It is then enough to expand f, if possible, as a holomorphic hyperbolic Fourier series

$$f(\tau) = \sum_{n\geq 0} a_n \mathrm{e}^{\mathrm{i}\pi n\tau} + b_n \mathrm{e}^{-\mathrm{i}\pi n/\tau}.$$

This is because g would then have the analogous expansion.

Why the given growth classes?

Coefficient classes

$(0 < \alpha < +\infty)$ A coefficient sequence $\{a_n\}_n$ is in $\mathcal{G}_\alpha$ if

$$|a_n| = \mathrm{O}(\exp(\alpha\sqrt{|n|}))$$

as $|n| \to +\infty$.

Remark

From the uniqueness theorem we have uniqueness in the representation if one of the sequences $\{a_n\}_n$ or $\{b_n\}_n$ is in $\mathcal{G}_\alpha$ for $\alpha < 2\pi$, and, at the same time we have controlled non-uniqueness for $2\pi \leq \alpha < +\infty$.

Proposition

If $\{a_n\}_n$ and $\{b_n\}_n$ are both in $\mathcal{G}_\alpha$, then the associated hyperbolic Fourier series

$$h(\tau) = \sum_{n \geq 0} \left\{ a_n \, \mathrm{e}^{\mathrm{i}\pi n\tau} + b_n \, \mathrm{e}^{-\mathrm{i}\pi n/\tau} \right\} + \sum_{n<0} \left\{ a_n \, \mathrm{e}^{-\mathrm{i}\pi n\bar{\tau}} + b_n \, \mathrm{e}^{\mathrm{i}\pi n/\bar{\tau}} \right\}$$

is in $\mathcal{M}^\gamma_{\mathrm{harm}}$ for each $\gamma > \alpha^2/4\pi$.

Fundamental HFS theorem

Our fundamental theorem is a converse to the proposition on the previous slide.

HFS theorem

Suppose $h:\mathbb{H}\to\mathbb{C}$ is harmonic. Then h has a (possibly non-unique) hyperbolic Fourier series representation

$$h(\tau)=\sum_{n\geq 0}\left\{a_n\,\mathrm{e}^{\mathrm{i}\pi n\tau}+b_n\,\mathrm{e}^{-\mathrm{i}\pi n/\tau}\right\}+\sum_{n<0}\left\{a_n\,\mathrm{e}^{-\mathrm{i}\pi n\bar{\tau}}+b_n\,\mathrm{e}^{\mathrm{i}\pi n/\bar{\tau}}\right\}$$

where $b_0=0$ and $|a_n|,|b_n|=\mathrm{O}(\exp(\epsilon|n|))$ as $|n|\to+\infty$. If h is in the growth class $\mathcal{M}^{\gamma}_{\mathrm{harm}}$ for some γ, $0<\gamma<+\infty$, then coefficients may be found which belong to the classes classes $\mathcal{G}_\alpha$ for any given real parameter $\alpha>2\sqrt{\pi\gamma}$. The coefficients are *unique* in the interval $0<\gamma<\pi$. For $\pi k\leq\gamma<\pi(k+1)$, however, they are unique up to an exceptional pair of sequences in $\mathcal{E}_k^{\mathrm{harm}}$.

Observations

Remarks

The boundary "values" on $\mathbb{R}_\infty$ exist as an ultradistribution, in a one-to-one correspondence with the harmonic function. In this sense any such ultradistribution has a hyperbolic Fourier series expansion, which is unique with coefficients in the growth class $\mathcal{G}_\alpha$ for $\alpha < 2\pi$ provided that the growth of the corresponding harmonic function is from $\mathcal{M}^\gamma_{\mathrm{harm}}$ with $0 < \gamma < \pi$. For $k\pi \leq \gamma < (k+1)\pi$, we have uniqueness in the expansion up to an additive exceptional sequence in $\mathcal{E}_k^{\mathrm{harm}}$.

The holomorphic HFS theorem

Holomorphic HFS theorem

Suppose $f : \mathbb{H} \to \mathbb{C}$ is holomorphic. Then f has a (possibly non-unique) holomorphic hyperbolic Fourier series representation

$$f(\tau) = \sum_{n \geq 0} \left\{ a_n \, \mathrm{e}^{\mathrm{i}\pi n \tau} + b_n \, \mathrm{e}^{-\mathrm{i}\pi n/\tau} \right\}$$

where $b_0 = 0$ and $a_n, b_n = \mathrm{O}(\exp(\epsilon|n|))$ as $|n| \to +\infty$. If f is in the growth class $\mathcal{M}^{\gamma}_{\mathrm{hol}}$ for some γ, $0 < \gamma < +\infty$, then coefficients $\{a_n, b_n\}_n$ may be found from the coefficient classes $\mathcal{G}_\alpha$ for any given $\alpha > 2\sqrt{\pi\gamma}$. The choice is unique if $\alpha < 2\pi$ is allowed, which is the case for $0 < \gamma < \pi$. For If $k\pi \leq \gamma < (k+1)\pi$, the choice is unique up to an additive exceptional sequence in $\mathcal{E}^{\mathrm{hol}}_k$.

Some notes

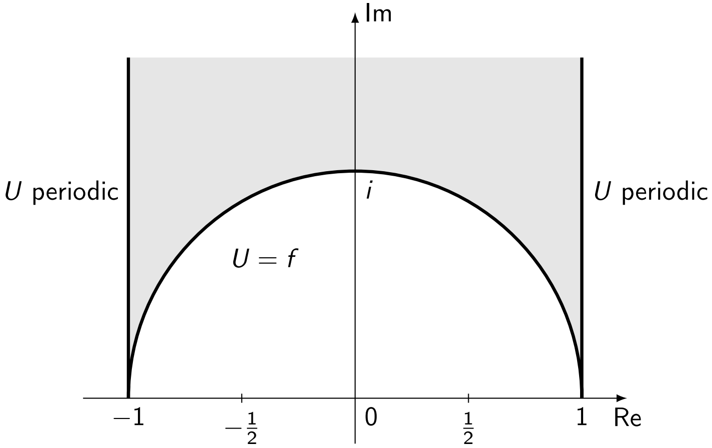
Remark

Clearly, this is a special case of the fundamental HFS theorem. However, the holomorphic HFS theorem implies the general result, as we may split $h \in \mathcal{M}^{\gamma}_{\mathrm{harm}}$ as $h = f + \bar{g}$, with $f, g \in \mathcal{M}^{\gamma}_{\mathrm{hol}}$ and expand f and g separately using the holomorphic HFS theorem.

Remark

In the presentation below, we consider mainly $f \in \mathcal{M}^{\gamma}_{\mathrm{hol}}$ for $0 < \gamma < \pi$.

Solving a Dirichlet problem



The domain $\mathcal{D}_\Theta$ and the corresponding Dirichlet problem.

Observations

Considering that we have periodic conditions along the linear semi-infinite segments, we may think of the domain as glued together there to form a cylindrical domain with boundary points along the circle and the point $+\mathrm{i}\infty$. We fill in the point $+\mathrm{i}\infty$, so that what remains is just the upper half of the unit circle. The function U is the harmonic function on this surface with boundary values equal to those of f. To ensure the existence of U, the boundary values cannot be too wild. This is ensured by the condition that f is in the growth class $\mathcal{M}^{\gamma}_{\mathrm{hol}}$ for $\gamma < \pi$.

Immediate properties of U

The function U is harmonic and 2-periodic: $U(\tau + 2) = U(\tau)$. It is initially defined in the region Ω_1 obtained by all removing circular disks of radius 1 from $\mathbb{H}$, centered at $2\mathbb{Z}$. As a 2-periodic function, U has a Fourier series

$$U(\tau) = \sum_{n \geq 0} \hat{U}(n)\, \mathrm{e}^{\mathrm{i}\pi n\tau} + \sum_{n<0} \hat{U}(n)\, \mathrm{e}^{\mathrm{i}\pi n\bar{\tau}}, \qquad \mathrm{Im}\,\tau > 1.$$

Moreover, $U(\tau) = f(\tau)$ holds for $\tau \in \mathbb{H}$ with $|\tau| = 1$.

From the Dirichlet problem to the HFS

We make the following temporary assumption.

Assumption

Suppose the Fourier series expansion of U actually converges for all $\tau \in \mathbb{H}$.

Conclusion

Then for $\tau \in \mathbb{H}$ with $|\tau| = 1$, $\bar{\tau} = 1/\tau$, so that

$$f(\tau) = U(\tau) = \sum_{n \geq 0} \hat{U}(n)\, \mathrm{e}^{\mathrm{i}\pi n\tau} + \sum_{n<0} \hat{U}(n)\, \mathrm{e}^{\mathrm{i}\pi n/\tau}.$$

Now, since

$$f(\tau) - \sum_{n \geq 0} \hat{U}(n)\, \mathrm{e}^{\mathrm{i}\pi n\tau} + \sum_{n<0} \hat{U}(n)\, \mathrm{e}^{\mathrm{i}\pi n/\tau}$$

is a holomorphic function in $\mathbb{H}$ which vanishes on $|\tau| = 1$, by the uniqueness theorem for holomorphic functions it vanishes identically. Consequently, we obtain the HFS expansion

$$f(\tau) = \sum_{n \geq 0} \hat{U}(n)\, \mathrm{e}^{\mathrm{i}\pi n\tau} + \sum_{n<0} \hat{U}(n)\, \mathrm{e}^{\mathrm{i}\pi n/\tau}, \qquad \tau \in \mathbb{H}.$$

What do we need to know?

Conclusion

It follows that

$$f(\tau) = \sum_{n \geq 0} a_n \, \mathrm{e}^{\mathrm{i}\pi n\tau} + b_n \, \mathrm{e}^{-\mathrm{i}\pi n/\tau}, \qquad \tau \in \mathbb{H},$$

where $a_n = \hat{U}(n)$, $b_n = \hat{U}(-n)$, and $b_0 = 0$.

Question

What else would we need to know?

Answer

We would like to know that the sequences $\{a_n\}_n$ and $\{b_n\}_n$ belong to the growth classes $\mathcal{G}_\alpha$, for each $\alpha > 2\sqrt{\pi\gamma}$. This is then the same as asking that the sequence $\{\hat{U}(n)\}_n$ is in $\mathcal{G}_\alpha$. Morally speaking, this should be the same as getting an appropriate growth bound on the harmonic function U in $\mathbb{H}$. Let us try to see what we can get.

A first growth bound

Growth bound on $\mathcal{D}_\Theta$

Since $f \in \mathcal{M}^{\gamma}_{\mathrm{hol}}$, we know that

$$|f(\tau)| = \mathrm{O}\left(\exp\left(\frac{\gamma}{\operatorname{Im}\tau}\right)\right)$$

as $\operatorname{Im}\tau \to 0^+$ while $\tau \in \mathbb{T}_+ := \mathbb{T} \cap \mathbb{H}$. Since $0 < \gamma < \pi$, the function is integrable with respect to harmonic measure, and, moreover, we may estimate that

$$|U(\tau)| = \mathrm{O}\left(\exp\left(\frac{\gamma}{\operatorname{Im}\tau}\right)\right)$$

holds uniformly on $\mathcal{D}_\Theta$.

Comment

The region $\mathcal{D}_\theta$, with the two linear boundary segments zipped to form a cylinder, may be mapped conformally to the unit disk $\mathbb{D}$. In terms of disk coordinates we have a boundary function with bound $\mathrm{O}(|z-1|^{-\beta})$ where $0 < \beta < 1$, and then the harmonic function on $\mathbb{D}$ with those boundary values is $\mathrm{O}(|z-1|^{-\beta})$ as well.

Schwarz reflection

Our boundary data f for the Dirichlet problem is a holomorphic function in $\mathbb{H}$. This tells us that U must be better as well.

Schwarz reflection

The harmonic function $U(\tau) - f(\tau)$, which is initially well-defined in $\Omega_1 = \mathbb{H} \setminus \cup_j \bar{\mathbb{D}}(2j, 1)$, vanishes along $\mathbb{T}_+$, so it extends across $\mathbb{T}_+$ by Schwarz reflection to

$$-U\left(\frac{1}{\bar{\tau}}\right) + f\left(\frac{1}{\bar{\tau}}\right).$$

Thus extended, the function U enjoys the functional identity

$$U(\tau) + U\left(\frac{1}{\bar{\tau}}\right) = f(\tau) + f\left(\frac{1}{\bar{\tau}}\right).$$

Consequences of Schwarz reflection

Some transformations on $\mathbb{H}$

The transformations $S(\tau) := -1/\tau$, $S^{\circledast}(\tau) := 1/\bar{\tau}$, $R(\tau) := -\bar{\tau}$, and $T(\tau) := \tau + 1$ all act on $\mathbb{H}$. We see that $S \circ R = R \circ S = S^{\circledast}$, and, moreover, that

$$T^n \circ R = R \circ T^{-n},$$

so that

$$T^n \circ S^{\circledast} = T^n \circ R \circ S = R \circ T^{-n} \circ S.$$

Invariance property of U

In terms of $f_{\mathrm{sym}} := f + f \circ S^{\circledast}$, the solution u to the Dirichlet problem has the following two properties:

$$U = U \circ T^2, \quad u = -U \circ S^{\circledast} + f_{\mathrm{sym}}.$$

Combine the two:

$$U = -U \circ T^{2k} \circ S^{\circledast} + f_{\mathrm{sym}}, \qquad k \in \mathbb{Z}.$$

Iterated Schwarz reflection

Iterated invariance property

By iteration:

$$U = (-1)^N U \circ T^{2k_N} \circ S^{\circledast} \circ \cdots \circ T^{2k_1} \circ S^{\circledast} \\ + f_{\mathrm{sym}} + \sum_{j=1}^{N-1} (-1)^j f_{\mathrm{sym}} \circ T^{2k_j} \circ S^{\circledast} \circ \cdots \circ T^{2k_1} \circ S^{\circledast}.$$

In terms of S

For even j,

$$T^{n_j} \circ S^{\circledast} \circ \cdots \circ T^{n_1} \circ S^{\circledast} = T^{n_j} \circ S \circ T^{-n_{j-1}} \circ S \circ \cdots \circ T^{-n_1} \circ S,$$

while for odd j,

$$T^{n_j} \circ S^{\circledast} \circ \cdots \circ T^{n_1} \circ S^{\circledast} = R \circ T^{-n_j} \circ S \circ T^{n_{j-1}} \circ S \circ \cdots \circ T^{-n_1} \circ S.$$

When is this identity useful?

The strip $\mathcal{S}_+$

Let

$$\mathcal{S}_+ := \{\tau \in \mathbb{H} : \ |\mathrm{Re}\,\tau| \le 1\}.$$

Facts about u

1. $U(\tau + 2) = U(\tau)$ so it is enough to estimate $U(\tau)$ for $\tau \in \mathcal{S}_+$.
2. We have a good estimate of $U(\tau)$ for $\tau \in \mathcal{S}_+$ with $|\tau| > 1$, i.e. for $\tau \in \mathcal{D}_\Theta$.

The remaining region

We need to control $U(\tau)$ for $\tau \in \mathcal{S}_+$ with $|\tau| \le 1$, i.e. for $\tau \in \mathcal{S}_+ \cap \bar{\mathbb{D}}$. In so doing, we also extend U harmonically to all of $\mathbb{H}$.

Application of the identity

Suppose $\tau \in \mathcal{S}_+ \cap \bar{\mathbb{D}}$ while $T^{2k_N} \circ S^{\circledast} \circ \cdots \circ T^{2k_1} \circ S^{\circledast}(\tau) \in \mathcal{D}_\Theta$. The identity then allows us to estimate $U(\tau)$ in terms of known quantities.

The lifted Gauss-type map g_2

What we want

For $\tau \in \mathcal{S}_+$, we want $T^{2k_N} \circ S^{\circledast} \circ \cdots \circ T^{2k_1} \circ S^{\circledast}(\tau) \in \mathcal{D}_\Theta$ in a minimal number of steps ($= N$). If we are lucky, we begin with $\tau \in \mathcal{D}_\Theta$, and then there is nothing to do. We should choose $k_1 \in \mathbb{Z}$ such that $\tau_1 := T^{2k_1} \circ S^{\circledast}(\tau) \in \mathcal{S}_+$. If then $|\tau_1| > 1$, i.e. $\tau_1 \in \mathcal{D}_\Theta$, we stop with $N = 1$. If not, we keep going and choose $k_2 \in \mathbb{Z}$ such that $\tau_2 := T^{2k_2} \circ S^{\circledast}(\tau_1) \in \mathcal{S}_+$. Again, if $|\tau_2| > 1$, i.e. $\tau_2 \in \mathcal{D}_\Theta$, we stop with $N = 2$. If not, keep going.

The map mod_2 and the lifted Gauss map g_2

Let $\mathrm{mod}_2 : \mathbb{H} \to \mathcal{S}_+$ be such that $\tau - \mathrm{mod}_2(\tau) \in 2\mathbb{Z}$. Moreover, we put $\mathrm{g}_2 := \mathrm{mod}_2 \circ S$. Then g_2 maps $\mathcal{S}_+ \to \mathcal{S}_+$, and we are interested in the first time the g_2-iterates of $\tau \in \mathcal{S}_+$ hits the subdomain $\mathcal{D}_\Theta$, or at least the closure $\bar{\mathcal{D}}_\Theta$. The required number of iterates defines the *height* (or stopping time) $\mathrm{n}(\tau) \in \mathbb{Z}_{\geq 0}$ of the point $\tau \in \mathcal{S}_+$.

The height and orbits of points in $\mathbb{H}$

Height of points according to $\mathrm{g}_2^{\circledast}$

Let $\mathrm{g}_2^{\circledast} := \mathrm{mod}_2 \circ S^{\circledast}$ and define the $\mathrm{g}_2^{\circledast}$-height of a point $\tau \in \mathcal{S}_+$ to be the number of iterates required to get the point in $\mathcal{S}_+$ to fall in $\mathcal{D}_\Theta$.

Observation

The $\mathrm{g}_2^{\circledast}$-height of a point $\tau \in \mathcal{S}_+$ is the same as the g_2-height $\mathbb{n}(\tau)$.

Extension of the height to $\mathbb{H}$

We extend the height function $\mathbb{n} : \mathcal{S}_+ \to \mathbb{Z}_{\geq 0}$ to $\mathbb{n} : \mathbb{H} \to \mathbb{Z}_{\geq 0}$ by declaring $\mathbb{n}(\tau + 2k) = \mathbb{n}(\tau)$ for $k \in \mathbb{Z}$.

Lemma

We have that for $\tau \in \mathbb{H}$ with $|\tau| < 1$,

$$\mathrm{Im}\, \mathrm{g}_2(\tau) = \mathrm{Im}\, S(\tau) = -\mathrm{Im}\, \frac{1}{\tau} = -\mathrm{Im}\, \frac{\bar{\tau}}{|\tau|^2} = \frac{\mathrm{Im}\, \tau}{|\tau|^2} > \mathrm{Im}\, \tau.$$

In particular, the g_2-iterates (and the $\mathrm{g}_2^{\circledast}$-iterates) of points move upward away from the real line, up until the stopping time (= the height).

Iterated invariance up to the stopping time

Application of the iterated invariance property

For τ in the semistrip $\mathcal{S}_+$, we have

$$U(\tau) = (-1)^{\mathtt{n}(\tau)} U \circ (\mathrm{g}_2^{\circledast})^{\circ \mathtt{n}(\tau)}(\tau) + \sum_{j=0}^{\mathtt{n}(\tau)-1} (-1)^j f_{\mathrm{sym}} \circ (\mathrm{g}_2^{\circledast})^{\circ j}(\tau).$$

Fundamental estimates

(RV = Radchenko-Viazovska, BRS = Bondarenko-Radchenko-Seip) We have that as $y \to 0^+$,

$$\mathtt{n}(t + \mathrm{i}y) = \mathrm{O}(1/y)$$

while

$$\int_{-1}^{1} \mathtt{n}(t + \mathrm{i}y)\,\mathrm{d}t = \frac{2}{\pi} \log^2 y + \mathrm{O}(\log y).$$

Estimation of U for $f \in \mathcal{M}^{\gamma}_{\rm hol}$

We recall that

$$f_{\rm sym}(\tau) = f(\tau) + f \circ S^{\circledast}(\tau) = f(\tau) + f\left(\frac{1}{\bar{\tau}}\right).$$

Estimate of $f_{\rm sym}$

$f \in \mathcal{M}^{\gamma}_{\rm hol} \Longrightarrow f_{\rm sym} \in \mathcal{M}^{\gamma}_{\rm hol}$.

First estimate of U

Suppose that $f \in \mathcal{M}^{\gamma}_{\rm hol}$ for some $0 < \gamma < \pi$. Then by the estimate of U on $\mathcal{D}_{\Theta}$, we have, uniformly on $\mathcal{S}_+$,

$$\left|U \circ (\mathrm{g}_2^{\circledast})^{\circ \mathtt{n}(\tau)}(\tau)\right| = \mathrm{O}\big(\exp(\gamma / \mathrm{Im}\,(\mathrm{g}_2^{\circledast})^{\circ \mathtt{n}(\tau)}(\tau)\big) = \mathrm{O}\big(\exp(\gamma / \mathrm{Im}\,\tau)\big).$$

Estimate of $f_{\rm sym}$ terms

For $\tau \in \mathcal{S}_+$ and $j = 0, 1, \ldots, \mathtt{n}(\tau)$, we have

$$\left|f_{\rm sym} \circ (\mathrm{g}_2^{\circledast})^{\circ j}(\tau)\right| = \mathrm{O}\big(\exp(\gamma / \mathrm{Im}\,(\mathrm{g}_2^{\circledast})^{\circ j}(\tau)\big) = \mathrm{O}\big(\exp(\gamma / \mathrm{Im}\,\tau)\big).$$

Implementation of the estimate of U for $f \in \mathcal{M}^{\gamma}_{\mathrm{hol}}$

Estimate of U

For $0 < \gamma < \pi$ and $f \in \mathcal{M}^{\gamma}_{\mathrm{hol}}$, we have that for $\tau \in \mathcal{S}_+$,

$$|U(\tau)| = \mathrm{O}\big((\mathbb{n}(\tau) + 1)\exp(\gamma/\operatorname{Im}\tau)\big) = \mathrm{O}\big((1 + 1/\operatorname{Im}\tau)\exp(\gamma/\operatorname{Im}\tau)\big),$$

where in the last step, we applied the fundamental estimate of Radchenko-Viazovska. Moreover, as U is by definition 2-periodic, the above estimate extends to all of $\mathbb{H}$.

Splitting U in analytic and conjugate-analytic parts

Decomposing U

We split $U = U_+ + U_-$, where U_+ is holomorphic in $\mathbb{H}$ while U_- is conjugate-holomorphic. Since U is 2-periodic, the functions U_+, U_- are 2-periodic as well. Moreover, as U is bounded in any half-plane $\mathbb{H}_\epsilon = \mathrm{i}\epsilon + \mathbb{H}$ with $\epsilon > 0$, the same goes for U_+, U_-, and we arrange it so that $U_-(+\mathrm{i}\infty) = 0$. The estimate

$$|U(\tau)| = \mathrm{O}\big((1 + 1/\mathrm{Im}\,\tau)\exp(\gamma/\mathrm{Im}\,\tau)\big), \qquad \tau \in \mathbb{H},$$

carries over to U_+, U_-:

$$|U_+(\tau)| = \mathrm{O}\big((1 + 1/\mathrm{Im}\,\tau)\exp(\gamma/\mathrm{Im}\,\tau)\big), \qquad \tau \in \mathbb{H},$$

and

$$|U_-(\tau)| = \mathrm{O}\big((1 + 1/\mathrm{Im}\,\tau)\exp(\gamma/\mathrm{Im}\,\tau)\big), \qquad \tau \in \mathbb{H}.$$

Representation of f

It follows that $f = U_+ + U_- \circ S^{\circledast}$, i.e., $f(\tau) = U_+(\tau) + U_-(1/\bar{\tau})$.

Estimate of Fourier coefficients

Consequences

$U = U_+ + U_-$, where

$$U_+(\tau) = \sum_{n=0} \hat{U}_+(n)\,\mathrm{e}^{\mathrm{i}\pi n\tau}, \qquad \tau \in \mathbb{H},$$

and

$$U_-(\tau) = \sum_{n=1}^{+\infty} \hat{U}_-(-n)\,\mathrm{e}^{-\mathrm{i}\pi n\bar{\tau}}, \qquad \tau \in \mathbb{H}.$$

The growth control on U_+, U_- implies that for each $\alpha > 2\sqrt{\pi\gamma}$,

$$|\hat{U}_+(n)| = \mathrm{O}(\exp(\alpha\sqrt{n})) \quad \text{as} \quad n \to +\infty,$$

and

$$|\hat{U}_-(-n)| = \mathrm{O}(\exp(\alpha\sqrt{n})) \quad \text{as} \quad n \to +\infty.$$

Proof of the analytic HFS theorem

Since

$$f(\tau) = U_+(\tau) + U_-(1/\bar{\tau}) = \sum_{n=0}^{+\infty} \hat{U}(n)\,\mathrm{e}^{\mathrm{i}\pi n\tau} + \sum_{n=1}^{+\infty} \hat{U}(-n)\,\mathrm{e}^{-\mathrm{i}\pi n/\tau}, \qquad \tau \in \mathbb{H},$$

the analytic HFS theorem follows.

Integral representation of the function U

The modular lambda function $z = \lambda(\tau)$ maps $\mathcal{D}_\Theta$ one-to-one and onto the slit half-plane

$$\left\{ z \in \mathbb{C} : \operatorname{Re} z < \frac{1}{2},\;\; z \notin \mathbb{R}_{\le 0} \right\}.$$

The two half-lines where $\operatorname{Re}\tau = \pm 1$ are mapped to $\mathbb{R}_{\le 0} \pm \mathrm{i}0$ (respectively), while the semi-circle $\mathbb{T}_+$ gets mapped to the line $\operatorname{Re} z = \frac{1}{2}$. We denote by $\lambda_\triangle$ the local inverse to λ, which may be expressed in terms of a ratio of the hypergeometric function. Then $U \circ \lambda_\triangle$ becomes harmonic in the above slit half-plane.

Observation

$U(\tau + 2) \equiv U(\tau)$ implies that $U \circ \lambda_\triangle$ extends harmonically across the slit $\mathbb{R}_{\le 0}$.

Poisson representation

$$U(\tau) = \frac{1}{\pi} \int_{\mathbb{T}_+} \frac{\frac{1}{2} - \operatorname{Re}\lambda(\tau)}{|\lambda(\tau) - \lambda(\eta)|^2} f(\eta)\, |\lambda'(\eta)|\, |\mathrm{d}\eta|, \qquad \tau \in \mathcal{D}_\Theta.$$

Consequences of the Poisson representation

Poisson representation, rewritten

If $\mathbb{T}_+$ is given a counter-clockwise orientation, we have $\lambda(\eta) \in \frac{1}{2} + \mathrm{i}\mathbb{R}$ for $\eta \in \mathbb{T}_+$ and $\mathrm{i}\mathrm{d}\lambda(\eta) = \mathrm{i}\lambda'(\eta)\mathrm{d}\eta = |\lambda'(\eta)|\,|\mathrm{d}\eta|$, so that

$$U(\tau) = \frac{1}{\mathrm{i}2\pi}\int_{\mathbb{T}_+}\bigg(\frac{1-\lambda(\tau)}{(1-\lambda(\eta))(\lambda(\tau)-\lambda(\eta))} + \frac{\bar{\lambda}(\tau)}{\bar{\lambda}(\eta)(\bar{\lambda}(\tau)-\bar{\lambda}(\eta))}\bigg) f(\eta)\,\lambda'(\eta)\,\mathrm{d}\eta = U_+(\tau) + U_-(\tau), \qquad \tau \in \mathcal{D}_\Theta,$$

where

$$U_+(\tau) = \frac{1}{\mathrm{i}2\pi}\int_{\mathbb{T}_+}\frac{1-\lambda(\tau)}{(1-\lambda(\eta))(\lambda(\tau)-\lambda(\eta))} f(\eta)\,\lambda'(\eta)\,\mathrm{d}\eta, \qquad \tau \in \mathcal{D}_\Theta,$$

and

$$U_-(\tau) = \frac{1}{\mathrm{i}2\pi}\int_{\mathbb{T}_+}\frac{\bar{\lambda}(\tau)}{\bar{\lambda}(\eta)(\bar{\lambda}(\tau)-\bar{\lambda}(\eta))} f(\eta)\,\lambda'(\eta)\,\mathrm{d}\eta, \qquad \tau \in \mathcal{D}_\Theta.$$

Consequences of the Poisson representation, cont

Since $\lambda(+\mathrm{i}\infty) = 0$, it follows that:

Properties of U_+, U_-

The function U_+ is holomorphic in $\mathbb{H}$ while U_- is conjugate-holomorphic. Moreover, both functions are 2-periodic: $U_+(\tau + 2) = U_+(\tau)$ and $U_-(\tau + 2) = U_-(\tau)$. Finally, $U_-(+\mathrm{i}\infty) = 0$ while $U_+(+\mathrm{i}\infty) = U(+\mathrm{i}\infty)$.

Schwarz reflection on U_-

We apply Schwarz reflection in the semi-circle $\mathbb{T}_+$ to u_-, to get a holomorphic function for $\tau \in S^{\circledast}(\mathcal{D}_\Theta) = S(\mathcal{D}_\Theta)$:

$$U_-\left(\frac{1}{\bar{\tau}}\right) = \frac{1}{\mathrm{i}2\pi} \int_{\mathbb{T}_+} \frac{\bar{\lambda}(\frac{1}{\bar{\tau}})}{\bar{\lambda}(\eta)(\bar{\lambda}(\frac{1}{\bar{\tau}}) - \bar{\lambda}(\eta))} f(\eta)\, \lambda'(\eta)\, \mathrm{d}\eta.$$

Rewriting the Schwarz reflection of U_-

Since $\bar{\lambda}(\frac{1}{\bar{\tau}}) = \lambda(-\frac{1}{\tau}) = 1 - \lambda(\tau)$ and $\bar{\lambda}(\eta) = 1 - \lambda(\eta)$, we obtain

$$U_-\left(\frac{1}{\bar{\tau}}\right) = -\frac{1}{\mathrm{i}2\pi}\int_{\mathbb{T}_+} \frac{1-\lambda(\tau)}{(1-\lambda(\eta))(\lambda(\tau)-\lambda(\eta))} f(\eta)\,\lambda'(\eta)\,\mathrm{d}\eta$$

for $\tau \in S(\mathcal{D}_\Theta)$. Note that except for the sign, this formula looks identical with the formula for U_+, which may make us doubt our calculations! After all, the sum would add up to 0, which cannot be true. However, what makes things work is the fact that the two expressions $U_+(\tau)$ and $U_-(1/\bar{\tau})$ are defined on the disjoint sets $\mathcal{D}_\Theta$ and $S(\mathcal{D}_\Theta)$, respectively.

Movement of contours

Now, the integrand is holomorphic, so we may deform the contour. To this end, let Γ_+, Γ_- be smooth deformations of $\mathbb{T}_+$ such that some (small) neighborhood of each endpoint ± 1 is preserved. The rest of the curve Γ_+ is deformed downward (but not going further down than $S(\mathcal{D}_\Theta)$, while Γ_2 is similarly deformed upward. We find that in the above formulae deining $U_+(\tau)$ and $U_-(1/\bar{\tau})$, we may deform $\mathbb{T}_+$ to Γ_1 and Γ_2, respectively, since the integrand is holomorphic in the relevant region.

Integral HFS representation of f

$$
\begin{aligned}
f(\tau) = U_+(\tau) + U_-\left(\frac{1}{\bar{\tau}}\right) \\
= \frac{1}{\mathrm{i}2\pi} \int_{\Gamma_1 - \Gamma_2} \frac{1-\lambda(\tau)}{(1-\lambda(\eta))(\lambda(\tau)-\lambda(\eta))} \, f(\eta) \, \lambda'(\eta) \, \mathrm{d}\eta
\end{aligned}
$$

holds for $\tau \in \mathbb{T}_+ \cup \mathcal{D}_\Theta \cup S(\mathcal{D}_\Theta)$ in the region between the contours Γ_1 and Γ_2.

A comment

Note on the integral HFS representation

This representation is a straightforward consequence of the Cauchy integral formula. From this point of view, the HFS is a consequence of the remarkable properties of the modular lambda function.

The associated projections

If $\mathcal{M}_{\rm per}^{\gamma *}$ denotes the space of locally bounded continuous functions $F : \mathbb{H} \to \mathbb{C}$ with $F(\tau + 2) = F(\tau)$ and

$$F(\tau) = \mathrm{O}\left(\left(1 + \frac{1}{\mathrm{Im}\,\tau}\right) \exp\left(\frac{\gamma}{\mathrm{Im}\,\tau}\right)\right)$$

uniformly in $\mathbb{H}$, then

$$\max\left\{\|U_+\|_{\mathcal{M}_{\rm per}^{\gamma *}}, \|U_-\|_{\mathcal{M}_{\rm per}^{\gamma *}}\right\}, \leq C_\gamma \|f\|_{\mathcal{M}^\gamma}, \qquad 0 < \gamma < \pi.$$

In other words, $f \mapsto U_+$ and $f \mapsto U_- \circ S^{\circledast}$ defines projections $\mathcal{M}_{\rm hol}^{\gamma} \to \mathcal{M}_{\rm hol,per}^{\gamma *}$ and $\mathcal{M}_{\rm hol}^{\gamma} \to \mathcal{M}_{\rm conj-hol,per}^{\gamma *} \circ\, S^{\circledast}$, which add up to the identity: $U_+ + U_- \circ S^{\circledast} = f$. We write $\mathrm{Q}f = U_+$ and $\mathrm{Q}_{\circledast} f = U_- \circ S^{\circledast}$ to have convenient notation.

The coefficient functionals

For $f \in \mathcal{M}^{\gamma}_{\mathrm{hol}}$, $\mathsf{Q}f = U_+ \in \mathcal{M}^{\gamma *}_{\mathrm{hol,per}}$ and hence may be expressed by a Taylor-Fourier series

$$\mathsf{Q}f(\tau) = \sum_{n=0}^{+\infty} a_n(f)\, \mathrm{e}^{\mathrm{i}n\pi\tau}, \qquad \tau \in \mathbb{H}.$$

We may then recover the coefficients via the formula

$$a_n(f) = \frac{1}{2}\mathrm{e}^{\pi\epsilon n} \int_{[-1,1]} \mathsf{Q}f(t + \mathrm{i}\epsilon)\, \mathrm{e}^{-\mathrm{i}\pi n t} \mathrm{d}t,$$

for any $\epsilon > 0$. The map $f \mapsto a_n(f)$ then defines a bounded linear functional $\mathcal{M}^{\gamma}_{\mathrm{hol}} \to \mathbb{C}$ and it becomes a legitimate question how such linear functionals may be represented. If we fix a γ' with $\gamma < \gamma' < \pi$, we may write

$$a_n(f) = \int_{\mathbb{H}} f(\tau)\, \mathrm{e}^{-\gamma' M(\tau)} \mathrm{d}\nu_n(\tau), \qquad f \in \mathcal{M}^{\gamma}_{\mathrm{hol}},$$

for a finite complex Borel measure ν_n on $\mathbb{H}$, because γ' may replace γ above. How does this translate into a smooth function on $\mathbb{R}$ instead?

Duality formula for the circle

Suppose F is bounded and holomorphic in $\mathbb{D}$, and Φ is C^1-smooth in $\bar{\mathbb{D}}$, then we have the identity

$$\langle \zeta F, \Phi\rangle_{\mathbb{T}} = \int_{\mathbb{T}} \zeta F(\zeta)\Phi(\zeta)\mathrm{d}s(\zeta)$$
$$= \frac{1}{2\pi\mathrm{i}}\int_{\mathbb{T}} F(\zeta)\Phi(\zeta)\,\mathrm{d}\zeta = \int_{\mathbb{D}} F(\zeta)\bar{\partial}_\zeta\Phi(\zeta)\mathrm{d}A(\zeta).$$

Moreover, even if F is unbounded in the disk, we may make sense of the left-hand side in terms of the dilates $F_r(\zeta) = F(r\zeta)$:

$$\langle \zeta F, \Phi\rangle_{\mathbb{T}} := \lim_{r\to 1^-} \langle \zeta F_r, \Phi\rangle_{\mathbb{T}}.$$

From the above formula, we see that it is enough to ask that $F_r\bar{\partial}\Phi$ converges in $L^1(\mathbb{D})$ to $F\bar{\partial}\Phi \in L^1(\mathbb{D})$ as $r \to 1^-$. This is thought as a requirement on Φ. Note that we may minimize the smoothness assumptions on Φ.

Duality formula for the real line

Suppose f is bounded and holomorphic in $\mathbb{H}$, and Φ is C^1-smooth in $\bar{\mathbb{H}}$, with appropriate control at infinity, then we have the analogous identity

$$\langle f, \Phi \rangle_{\mathbb{R}} = \int_{\mathbb{R}} f(t)\Phi(t)\mathrm{d}t = \mathrm{i}2\pi \int_{\mathbb{H}} f(\tau)\bar{\partial}_\tau \Phi(\tau)\mathrm{d}A(\tau).$$

We should thus solve

$$\bar{\partial}_\tau \Phi_n = \frac{1}{\mathrm{i}2\pi}\mu_n,$$

where

$$\mathrm{d}\mu_n = \mathrm{e}^{-\gamma' M}\mathrm{d}\nu_n, \quad M(\tau) = \frac{\max\{1, |\tau|^2\}}{\operatorname{Im}\tau},$$

and we know ν_n is a finite complex Borel measure on $\mathbb{H}$. The natural solution is

$$\Phi_n(\tau) = \frac{1}{\mathrm{i}2\pi^2} \int_{\mathbb{H}} \frac{\mathrm{e}^{-\gamma' M(\xi)}}{\tau - \xi}\,\mathrm{d}\nu_n(\xi), \qquad \tau \in \mathbb{H}.$$

Then

$$\langle f, \Phi_n \rangle_{\mathbb{R}} = \int_{\mathbb{H}} f(\tau)\,\mathrm{e}^{-\gamma' M(\tau)}\mathrm{d}\nu_n(\tau) = a_n(f).$$

Properties of Φ_n

Smoothness of Φ_n

The function Φ_n is holomorphic in $\bar{\mathbb{H}}_-$, and extends C^∞-smoothly to $\bar{\mathbb{H}}_-$ with decay $|\Phi_n(\tau)| = \mathrm{O}(|\tau|^{-1})$ at infinity.

Decay of Φ_n for $n > 0$

For $n = 1, 2, 3, \ldots$, $|\Phi_n(\tau)| = \mathrm{O}(|\tau|^{-2})$ at infinity in $\bar{\mathbb{H}}_-$. Moreover, the function $\tau \mapsto \tau^{-2}\Phi_n(-1/\tau)$ is C^∞-smooth on $\bar{\mathbb{H}}_-$ as well. We have the Gevrey type bound $\|\Phi_n^{(k)}\|_{L^\infty(\mathbb{R})} = \mathrm{O}_n(k^{\frac{1}{2}}(k!)^2\gamma^{-k})$ and the analogous bound holds for for the derivatives of $t^{-2}\Phi_n(-1/t)$ as well.

Remark

This is based on the property that

$$0 = a_n(1) = \int_{\mathbb{H}} \mathrm{e}^{-\gamma' M(\tau)} \mathrm{d}\nu_n(\tau).$$

Connection with the biorthogonal system

Recall that we write

$$a_n(h) = \langle h, A_n \rangle_{\mathbb{R}} = \int_{\mathbb{R}} h(t) A_n(t) \,\mathrm{d}t, \quad b_n(h) = \langle h, B_n \rangle_{\mathbb{R}} = \int_{\mathbb{R}} h(t) B_n(t) \,\mathrm{d}t,$$

in connection with the hyperbolic Fourier series for h.

Theorem

We have $A_n = \Phi_n$ for $n > 0$, so that in particular,

$$\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\pi j t} \Phi_n(t) \mathrm{d}t = 0, \quad \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\pi k/t} \Phi_n(t) \mathrm{d}t = 0,$$

for all integer pairs (j, k) with $j \neq n$.

Symmetry property

We have $A_{-n}(t) = \bar{A}_n(t)$ for all $n \in \mathbb{Z}$, whereas for $n \neq 0$, we have $B_n(t) = t^{-2} A_n(-1/t)$.

The function $\tilde{\Phi}_0$

We put

$$\tilde{\Phi}_0(\tau) := \frac{1}{\mathrm{i}2\pi^2}\int_{\mathbb{H}}\left(\frac{1}{\tau-\xi}-\frac{1}{\tau-\bar{\xi}}\right)\mathrm{e}^{-\gamma' M(\xi)}\mathrm{d}\nu_0(\xi)$$
$$= \frac{1}{\pi^2}\int_{\mathbb{H}}\frac{\mathrm{Im}\,\xi}{(\tau-\xi)(\tau-\bar{\xi})}\,\mathrm{e}^{-\gamma' M(\xi)}\mathrm{d}\nu_0(\xi).$$

Here, it is possible to argue that we may require without loss of generality that ν_0 is real-valued (if not it may be replaced by a real-valued measure which gives rise to the same functional on the subspace of $L^\infty(\mathbb{H})$ of functions $\mathrm{e}^{-\gamma' M}h$ for $h \in \mathcal{M}^{\gamma}_{\mathrm{harm}}$). Consequently, $\tilde{\Phi}_0$ is real-valued along the line $\mathbb{R}$. Moreover,

$$\bar{\partial}\tilde{\Phi}_0(\tau) = \frac{1}{\mathrm{i}2\pi}\, e^{-\gamma' M(\tau)}\mathrm{d}\nu_0(\tau), \qquad \tau \in \mathbb{H},$$

and $\tilde{\Phi}(\tau) = \mathrm{O}(|\tau|^{-2})$ as $|\tau| \to +\infty$, so that

$$a_0(f) = \int_{\mathbb{H}} f\,\mathrm{e}^{-\gamma' M}\mathrm{d}\nu_0 = \int_{\mathbb{R}} f(t)\tilde{\Phi}_0(t)\mathrm{d}t = \langle f, \tilde{\Phi}_0\rangle_{\mathbb{R}}, \qquad f \in \mathcal{M}^{\gamma}_{\mathrm{hol}}.$$

The function A_0

Theorem

We have $A_0 = \tilde{\Phi}_0$, so that in particular,

$$\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\pi j t} \tilde{\Phi}_0(t) \mathrm{d}t = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\pi j/t} \tilde{\Phi}_0(t) \mathrm{d}t = 0,$$

for all integers $j \neq 0$.

Decomposition of a point mass

The harmonic extension of a point mass

We consider $h = h_\xi = \delta_\xi$, the unit point mass at $\xi \in \mathbb{R}$. The corresponding harmonic extension is

$$h(\tau) = \frac{1}{\pi} \frac{\operatorname{Im} \tau}{|\xi - \tau|^2}, \qquad \tau \in \mathbb{H}.$$

If

$$f(\tau) = f_\xi(\tau) = \frac{1}{\mathrm{i}2\pi} \frac{1}{\xi - \tau}, \qquad \tau \in \mathbb{H},$$

then f is holomorphic and $h(\tau) = 2\operatorname{Re} f(\tau) = f(\tau) + \overline{f}(\tau)$.

We now look for the HFS decomposition of $f = f_\xi$, which in its turn gives the HFS decomposition of $h = \delta_\xi$:

$$h(\tau) = 2\operatorname{Re} \left\{ \sum_{n \geq 0} a_n(f)\, \mathrm{e}^{\mathrm{i}\pi n \tau} + b_n(f)\, \mathrm{e}^{-\mathrm{i}\pi n/\tau} \right\},$$

so that $a_0(h) = 2\operatorname{Re} a_0(f)$, while $a_n(h) = a_n(f)$, $b_n(h) = b_n(f)$ for $n > 0$, and $a_n(h) = \overline{a_{-n}(f)}$, $b_n(h) = \overline{b_{-n}(f)}$ for $n < 0$.

Relation with the bilinear system

The bilinear system

For $n \in \mathbb{Z}$ and $m \in \mathbb{Z}_{\neq 0}$ and $\xi \in \mathbb{R}$,

$$a_n(h_\xi) = a_n(\delta_\xi) = A_n(\xi), \quad b_m(h_\xi) = b_m(\delta_\xi) = B_m(\xi).$$

Basic properties

For $\xi \in \mathbb{R}$, and $n \in \mathbb{Z}_{>0}$, we have that

$$A_0(\xi) = 2\mathrm{Re}\, a_0(f_\xi), \quad A_n(\xi) = a_n(f_\xi), \quad B_n(\xi) = b_n(f_\xi),$$

and, in addition, for $n \in \mathbb{Z}_{\neq 0}$,

$$A_n(\xi) = \overline{A_{-n}(\xi)} = A_{-n}(-\xi), \quad B_n(\xi) = \overline{B_{-n}(\xi)} = B_{-n}(-\xi).$$

HFS decomposition of $f = f_\xi$ (1)

The coefficient $a_0(f_\xi)$

We have

$$a_0(f_\xi) = \frac{1}{2\pi}\int_{\mathbb{T}_+} \frac{|\lambda'(\eta)|}{|\lambda(\eta)|^2} f(\eta)\,|\mathrm{d}\eta| = \frac{1}{\mathrm{i}4\pi^2}\int_{\mathbb{T}_+} \frac{|\lambda'(\eta)|}{|\lambda(\eta)|^2}(\xi-\eta)^{-1}\,|\mathrm{d}\eta|,$$

so that with $\lambda_\triangle(\zeta) = \mathrm{i}F_\triangle(1-\zeta)/F_\triangle(\zeta)$ and $F_\triangle(\zeta) = F(\frac{1}{2},\frac{1}{2};1;\zeta)$,

$$\begin{aligned} A_0(\xi) = a_0(h_\xi) = 2\mathrm{Re}\, a_0(f_\xi) &= \frac{1}{2\pi^2}\int_{\mathbb{T}_+} \frac{|\lambda'(\eta)|}{|\lambda(\eta)|^2}\frac{\mathrm{Im}\,\eta}{|\xi-\eta|^2}\,|\mathrm{d}\eta| \\ &= \frac{1}{2\pi^2}\int_{\frac{1}{2}+\mathrm{i}\mathbb{R}} \frac{\mathrm{Im}\,\lambda_\triangle(\zeta)}{|\xi-\lambda_\triangle(\zeta)|^2}\frac{|\mathrm{d}\zeta|}{|\zeta|^2}. \end{aligned}$$

Symmetry property of f_ξ

We have

$$f_\xi \circ S(\tau) = \frac{1}{\mathrm{i}2\pi\xi} + \xi^{-2} f_{-1/\xi}(\tau).$$

Consequences of the symmetry property

Corollary

(a) We have that

$$a_0(f_\xi) - \xi^{-2} a_0(f_{-1/\xi}) = \frac{1}{\mathrm{i}2\pi\xi},$$

so that $A_0(\xi) = \xi^{-2} A_0(-1/\xi)$.
(b) For $n \in \mathbb{Z}_{\neq 0}$, we have

$$B_n(\xi) = b_n(f_\xi) = \xi^{-2} a_n(f_{-1/\xi}) = \xi^{-2} A_n(-1/\xi).$$

It follows that once we have the coefficients $A_n(\xi)$, we automatically have $B_n(\xi)$ as well. Consequently, we may focus on expressing $A_n(\xi)$ for $n > 0$, by symmetry.

The coefficient functions A_n

Since

$$U_\xi(\tau) = \frac{1}{\mathrm{i}2\pi^2} \int_{\mathbb{T}_+} \frac{\frac{1}{2} - \operatorname{Re}\lambda(\tau)}{|\lambda(\tau) - \lambda(\eta)|^2} \frac{1}{\xi - \eta} |\lambda'(\eta)|\,|\mathrm{d}\eta|, \qquad \tau \in \mathcal{D}_\theta,$$

the Fourier coefficient $\hat{U}_\xi(n)$ of this harmonic function is, for $n > 0$, given by

$$A_n(\xi) = \hat{U}_\xi(n) = \frac{1}{2} \int_{-1}^{1} \mathrm{e}^{-\mathrm{i}\pi n(t+\mathrm{i}\beta)} U_\xi(t + \mathrm{i}\beta)\,\mathrm{d}t, \qquad \beta > 1.$$

It is easy to see that in the decomposition $U_\xi = U_+ + U_-$, $\hat{U}_-(n) = 0$ for $n > 0$. Consequently, $\hat{U}_\xi(n) = \hat{U}_+(n)$ for $n > 0$, and we find that if Γ_β denotes the horizontal line segment from $-1 + \mathrm{i}\beta$ to $1 + \mathrm{i}\beta$,

$$\begin{aligned} A_n(\xi) = \hat{U}_\xi(n) = \hat{U}_+(n) &= \frac{1}{2} \int_{\Gamma_\beta} \mathrm{e}^{-\mathrm{i}\pi n\tau} U_+(\tau)\,\mathrm{d}\tau \\ &= \frac{1}{\mathrm{i}4\pi} \int_{\mathbb{T}_+} \frac{f_\xi(\eta)\lambda'(\eta)\,\mathrm{d}\eta}{1 - \lambda(\eta)} \int_{\Gamma_\beta} \frac{1 - \lambda(\tau)}{\lambda(\tau) - \lambda(\eta)}\,\mathrm{e}^{-\mathrm{i}\pi n\tau}\mathrm{d}\tau. \end{aligned}$$

A change of variables

By integration by parts and periodicity, we obtain that

$$\int_{\Gamma_\beta} \frac{1-\lambda(\tau)}{\lambda(\tau)-\lambda(\eta)}\, \mathrm{e}^{-\mathrm{i}\pi n\tau} \mathrm{d}\tau = -\frac{1-\lambda(\eta)}{\mathrm{i}\pi n} \int_{\Gamma_\beta} \mathrm{e}^{-\mathrm{i}\pi n\tau} \frac{\lambda'(\tau)}{(\lambda(\tau)-\lambda(\eta))^2} \mathrm{d}\tau.$$

We apply the change of variables $\tau = \lambda_\triangle(\zeta)$, so that $\zeta = \lambda(\tau)$, where ζ is in $\mathbb{C} \setminus (]-\infty, 0] \cup [1, +\infty[)$. This gives

$$\int_{\Gamma_\beta} \frac{1-\lambda(\tau)}{\lambda(\tau)-\lambda(\eta)}\, \mathrm{e}^{-\mathrm{i}\pi n\tau} \mathrm{d}\tau = -\frac{1-\lambda(\eta)}{\mathrm{i}\pi n} \int_{\tilde{\Gamma}_\beta} \frac{\mathrm{e}^{-\mathrm{i}\pi n\lambda_\triangle(\zeta)}}{(\zeta-\lambda(\eta))^2} \mathrm{d}\zeta$$

where $\tilde{\Gamma}_\beta = \lambda(\Gamma_\beta)$. The function

$$G_n(\zeta) := \exp(-\mathrm{i}\pi n\lambda_\triangle(\zeta))$$

is holomorphic in $\mathbb{C} \setminus (\{0\} \cup [1, +\infty[)$, with a pole of order n at $z = 0$. The curve $\tilde{\Gamma}_\beta$ goes one counterclockwise loop around 0 and does not go around the point $\lambda(\eta) \in \frac{1}{2} + \mathrm{i}\mathbb{R}$.

Evaluating the integral

We split the function G_n in two components:

$$G_n(\zeta) = S_n(1/\zeta) + R_n(\zeta),$$

where S_n is a polynomial of degree n with $S_n(0) = 0$, while R_n is holomorphic in the slit plane $\mathbb{C} \setminus [1, +\infty[$. It now follows that

$$\begin{aligned}
\int_{\Gamma_\beta} \frac{1-\lambda(\tau)}{\lambda(\tau)-\lambda(\eta)}\, \mathrm{e}^{-\mathrm{i}\pi n\tau} \mathrm{d}\tau &= -\frac{1-\lambda(\eta)}{\mathrm{i}\pi n} \int_{\tilde{\Gamma}_\beta} \frac{G_n(\zeta)}{(\zeta-\lambda(\eta))^2}\, \mathrm{d}\zeta \\
&= -\frac{1-\lambda(\eta)}{\mathrm{i}\pi n} \left\{ \int_{\tilde{\Gamma}_\beta} \frac{S_n(1/\zeta)\,\mathrm{d}\zeta}{(\zeta-\lambda(\eta))^2} + \int_{\tilde{\Gamma}_\beta} \frac{R_n(\zeta)\,\mathrm{d}\zeta}{(\zeta-\lambda(\eta))^2} \right\} \\
&= -\frac{1-\lambda(\eta)}{\mathrm{i}\pi n} \int_{\tilde{\Gamma}_\beta} \frac{S_n(1/\zeta)\,\mathrm{d}\zeta}{(\zeta-\lambda(\eta))^2} = \frac{2}{n}(1-\lambda(\eta)) \frac{d}{d\zeta} S_n(1/\zeta)\Big|_{\zeta=\lambda(\eta)}.
\end{aligned}$$

The coefficient functions A_n for $n > 0$

The functions A_n

For $n > 0$, we obtain that

$$A_n(\xi) = \frac{\mathrm{i}}{2\pi n} \int_{\mathbb{T}_+} \frac{f_\xi(\eta) S_n'(1/\lambda(\eta))}{\lambda(\eta)^2} \, \lambda'(\eta) \, \mathrm{d}\eta$$
$$= \frac{1}{\mathrm{i}2\pi n} \int_{\mathbb{T}_+} S_n(1/\lambda(\eta)) f_\xi'(\eta) \, \mathrm{d}\eta = -\frac{1}{4\pi^2 n} \int_{\mathbb{T}_+} \frac{S_n(1/\lambda(\eta))}{(\xi - \eta)^2} \, \mathrm{d}\eta$$
$$= -\frac{1}{4\pi^2 n} \int_{\frac{1}{2}+\mathrm{i}\mathbb{R}} \frac{S_n(1/\zeta) \, \lambda_\triangle'(\zeta) \mathrm{d}\zeta}{(\xi - \lambda_\triangle(\zeta))^2}.$$

Here, we used integration by parts. We should mention that

$$\lambda_\triangle(\zeta) = \mathrm{i} \frac{F_\triangle(1-\zeta)}{F_\triangle(\zeta)}, \quad \lambda_\triangle'(\zeta) = \frac{1}{\mathrm{i}\pi\zeta(1-\zeta) F_\triangle(\zeta)^2},$$

so that

$$\frac{\lambda_\triangle'(\zeta)}{(\xi - \lambda_\triangle(\zeta))^2} = \frac{1}{\mathrm{i}\pi\zeta(1-\zeta)(\xi F_\triangle(\zeta) - \mathrm{i} F_\triangle(1-\zeta))^2}.$$

Distributional identities

Decomposition of the point mass

We have, for $\xi \in \mathbb{R}$, in the sense of distribution theory,

$$\delta_\xi(t) = A_0(\xi) + \sum_{n\in\mathbb{Z}_{\neq 0}} A_n(\xi)\, \mathrm{e}^{\mathrm{i}\pi n t} + B_n(\xi)\, \mathrm{e}^{-\mathrm{i}\pi n/t}.$$

Decomposition of Hilbert kernel

We have, for $\xi \in \mathbb{R}$, in the sense of distribution theory,

$$\mathrm{pv}\, \frac{1}{\pi(t-\xi)} = \tilde{A}_0(\xi) + \sum_{n\in\mathbb{Z}_{\neq 0}} \tilde{A}_n(\xi)\, \mathrm{e}^{\mathrm{i}\pi n t} + \tilde{B}_n(\xi)\, \mathrm{e}^{-\mathrm{i}\pi n/t},$$

where $\tilde{A}_0(\xi) = 2\mathrm{Im}\, a_0(f_\xi)$, and, for $n \in \mathbb{Z}_{\neq 0}$,

$$\tilde{A}_n(\xi) = -\mathrm{i}\,\mathrm{sgn}(n)\, A_n(\xi), \quad \tilde{B}_n(\xi) = -\mathrm{i}\,\mathrm{sgn}(n)\, B_n(\xi).$$

Here, $\tilde{A}_0(\xi) - \xi^{-2}\tilde{A}_0(-1/\xi) = -1/(\pi\xi)$.

A concrete example

The classical theta function

$$\vartheta(\tau) = \vartheta_3(\tau) := \sum_{n\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}\pi n^2\tau}, \qquad \tau \in \mathbb{H},$$

enjoys the functional equation

$$\vartheta(\tau) = (\tau/\mathrm{i})^{-\frac{1}{2}}\vartheta(-1/\tau).$$

We form fourth powers in the functional equation:

$$\vartheta(\tau)^4 = -\tau^{-2}\vartheta(-1/\tau)^4, \quad \vartheta(\tau)^4 = 1 + \sum_{n\in\mathbb{Z}_{>0}} r_4(n)\,\mathrm{e}^{\mathrm{i}\pi n\tau}.$$

Here, $r_4(n)$ stands for the number of ways to represent n as the sum of four squares of integers. After integration, this gives that

$$\sum_{n\in\mathbb{Z}_{>0}} \frac{r_4(n)}{\mathrm{i}\pi n}(\mathrm{e}^{\mathrm{i}\pi n\tau} + \mathrm{e}^{-\mathrm{i}\pi n/\tau}) = \frac{1}{\tau} - \tau + C.$$

Consequence for A_n, B_n, $n > 0$

Since

$$\frac{\mathrm{i}}{2\pi}\,\tau^{-1} = a_0(f_0) + \sum_{n\in\mathbb{Z}_{>0}} A_n(0)\,\mathrm{e}^{\mathrm{i}\pi n\tau} + B_n(0)\,\mathrm{e}^{-\mathrm{i}\pi n/\tau},$$

we obtain that

$$\frac{\mathrm{i}}{2\pi}(\tau^{-1} - \tau) = 2a_0(f_0) + \sum_{n\in\mathbb{Z}_{>0}} (A_n(0) + B_n(0))\,(\mathrm{e}^{\mathrm{i}\pi n\tau} + \mathrm{e}^{-\mathrm{i}\pi n/\tau}).$$

Consequence

It follows that for $n \in \mathbb{Z}_{>0}$,

$$A_n(0) + B_n(0) = \frac{r_4(n)}{2\pi^2 n}.$$

Further consequences

Another identity

A similar argument based on another theta identity gives that for $n > 0$,

$$A_n(0) = \frac{\tilde{r}_4(n)}{2\pi^2 n},$$

where $\tilde{r}_4(n)$ counts the number of ways to write n as the sum of four squares of half-integer numbers (i.e., from $\frac{1}{2} + \mathbb{Z}$).

Consequence for $A_0(0)$

We have that $a_0(f_0) = \frac{1}{2} A_0(0)$, where

$$A_0(0) = \frac{1}{\pi} - \sum_{n=1}^{+\infty} \frac{r_4(n)}{\pi^2 n} \, \mathrm{e}^{-\pi n} = \frac{4}{\pi^2} \log 2.$$

Asymptotics of $a_0(f_\xi)$ as $|\xi| \to +\infty$

$$a_0(f_\xi) = \frac{\xi^{-1}}{\mathrm{i} 2\pi} + \frac{1}{2} A_0(0) \, \xi^{-2} + \mathrm{O}(\xi^{-3}).$$

One more example of a hyperbolic Fourier series

Expansion of the sign function

Let

$$c_k = \sum_{m,n\geq 1: 2mn=k} \frac{1}{n} - 2 \sum_{m,n\geq 1:(2m-1)n=k} \frac{(-1)^{n-1}}{n}.$$

We then have that

$$\mathrm{sgn}(t) = -\frac{2}{\pi} \sum_{k=1}^{+\infty} c_k (\sin \pi kt + \sin \pi k/t)$$

and

$$\mathrm{pv}\ \log |t| = 2 \sum_{k=1}^{+\infty} c_k (\cos \pi kt - \cos \pi k/t).$$

As a consequence, we have

$$\delta_0(t) = -2 \sum_{k=1}^{+\infty} k c_k (\cos \pi kt - t^{-2} \cos \pi k/t).$$

The effect of scaling

We may rescale the distributional identities on the previous slide.

Decomposition of the point mass

We have, for $\xi \in \mathbb{R}$ and $\epsilon > 0$, in the sense of distribution theory,

$$\delta_\xi(t) = \epsilon A_0(\xi) + \epsilon \Big\{ \sum_{n \in \mathbb{Z}_{\neq 0}} A_n(\epsilon\xi)\, \mathrm{e}^{\mathrm{i}\pi n\epsilon t} + B_n(\epsilon\xi)\, \mathrm{e}^{-\mathrm{i}\pi n/(\epsilon t)} \Big\}.$$

We may compare this formula with the Fourier integral identity

$$\delta_\xi(t) = \frac{1}{2} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\pi x t}\, \mathrm{e}^{-\mathrm{i}\pi x\xi} \mathrm{d}x,$$

which suggests that

$$\lim_{n \to +\infty} A_n\Big(\frac{x\xi}{n}\Big) = \frac{1}{2}\, \mathrm{e}^{-\mathrm{i}\pi x\xi}.$$

We note that for $x = 0$, $A_n(0) = \tilde{r}_4(n)/(2\pi^2 n) \to \frac{1}{2}$ holds only *on average* as $n \to +\infty$ (like in QUE).

ℓ^2 theory

There is no good ℓ^2 theory associated with the hyperbolic Fourier series. We can of course obtain a bilinear formula

$$\int_{\mathbb{R}} f(t)g(t)\,\mathrm{d}t = \langle f, g\rangle_{\mathbb{R}} = \langle f, A_0\rangle_{\mathbb{R}}\langle g, 1\rangle_{\mathbb{R}} + \sum_{n\in\mathbb{Z}_{\neq 0}} \langle f, A_n\rangle_{\mathbb{R}}\langle g, e_n\rangle_{\mathbb{R}} + \langle f, B_n\rangle_{\mathbb{R}}\langle g, e_n^{\circledast}\rangle_{\mathbb{R}},$$

where $e_n(t) = \mathrm{e}^{\mathrm{i}\pi nt}$, $e_n^{\circledast}(t) = \mathrm{e}^{-\mathrm{i}\pi n/t}$. But when we have such a formula, we would like to know which f, g we can plug in. A natural question then is for which f we are sure that $a_n(f) = \langle f, A_n\rangle_{\mathbb{R}}$ and $b_n(f) = \langle f, B_n\rangle_{\mathbb{R}}$ are in ℓ^2, and for which g we know that $\langle g, e_n\rangle_{\mathbb{R}}$ and $\langle g, e_n^{\circledast}\rangle_{\mathbb{R}}$ are in ℓ^2.

Theorem

If $f \in W^{\frac{1}{2},2}(\mathbb{R})$, then $(\{a_n(f)\}_n, \{b_n(f)\}_n) \in \ell^2 \oplus \ell^2$. Moreover, if $\int_{\mathbb{R}}(1+t^2)|g(t)|^2\mathrm{d}t < +\infty$, then $\langle g, e_n\rangle_{\mathbb{R}}$ and $\langle g, e_n^{\circledast}\rangle_{\mathbb{R}}$ are both in ℓ^2. On the other hand, if $\int_{\mathbb{R}} |f(t)|^2(1+t^2)^{-1}\mathrm{d}t < +\infty$, then we get $a_n(f), b_n(f) = \mathrm{o}(|n|^{3/2})$ as $|n| \to +\infty$.

Conjugate hyperbolic Fourier series

Conjugate hyperbolic Fourier series

If $g : \mathbb{R} \to \mathbb{C}$ is measurable with $\int_{\mathbb{R}} (1+t^2)|g(t)|^2 \mathrm{d}t < +\infty$, we obtain from the distributional identity the conjugate hyperbolic Fourier series expansion

$$g(\xi) = \alpha_0(g) A_0(\xi) + \sum_{n \in \mathbb{Z}_{\neq 0}} \alpha_n(g) A_n(\xi) + \beta_n(g) B_n(\xi),$$

in the sense of distribution theory, where

$$\alpha_n(g) = \langle g, e_n \rangle_{\mathbb{R}}, \quad \beta_n(g) = \langle g, e_n^{\circledast} \rangle_{\mathbb{R}},$$

form ℓ^2 sequences. Moreover, if in addition

$$\sum_{n \in \mathbb{Z}} (|n|+1)^{3/2} (|\alpha_n(g)| + |\beta_n(g)|) < +\infty,$$

then the series converges in $L^2(\mathbb{R}, (1+t^2)\mathrm{d}t)$.

The distributional identity and Poisson summation

Poisson summation

In the sense of distribution theory, we have that

$$\sum_{k\in\mathbb{Z}} \delta_{\xi+2k}(t) = \frac{1}{2}\sum_{n\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}\pi nt}\mathrm{e}^{-\mathrm{i}\pi n\xi}.$$

We want to relate the decomposition of the point mass to the Poisson summation formula. We obtain from that decomposition that

$$\sum_{k\in\mathbb{Z}} \delta_{\xi+2k}(t) = \sum_{k\in\mathbb{Z}} A_0(\xi+2k) + \sum_{n\in\mathbb{Z}_{\neq 0}} \left\{ \mathrm{e}^{\mathrm{i}\pi nt} \sum_{k\in\mathbb{Z}} A_n(\xi+2k) + \mathrm{e}^{-\mathrm{i}\pi n/t} \sum_{k\in\mathbb{Z}} B_n(\xi+2k) \right\}.$$

The Poisson summation identity follows from this identity once the following properties of the functions A_n, B_n are established.

Summation properties of the biorthogonal system

Summation properties

We have that in the sense of distribution theory,

$$\sum_{k\in\mathbb{Z}} A_n(\xi+2k) = \mathrm{e}^{-\mathrm{i}\pi n\xi}, \qquad n\in\mathbb{Z},$$

and

$$\sum_{k\in\mathbb{Z}} B_n(\xi+2k) = 0, \qquad n\in\mathbb{Z}_{\neq 0}.$$

Moreover, $B_n(\xi) = \xi^{-2}A_n(-1/\xi)$ for $n\in\mathbb{Z}_{\neq 0}$.

Symmetrized summation properties

Symmetrized biorthogonal system

For $n \in \mathbb{Z}_{\neq 0}$, let

$$A_n^+(\xi) := A_n(\xi) + B_n(\xi) = A_n(\xi) + \xi^{-2} A_n(-1/\xi)$$

and

$$A_n^-(\xi) := A_n(\xi) - B_n(\xi) = A_n(\xi) - \xi^{-2} A_n(-1/\xi).$$

These functions enjoy the symmetry properties $\xi^{-2} A_n^+(-1/\xi) = A_n^+(\xi)$ and $\xi^{-2} A_n^-(-1/\xi) = -A_n^-(\xi)$.

Transfer operator T

Let T denote the transfer operator

$$\mathrm{T} f(t) = \sum_{k \in \mathbb{Z}_{\neq 0}} (t + 2k)^{-2} f(-1/(t + 2k)),$$

which acts contractively on $L^1(-1, 1)$.

Transfer operator equation for A_n^+, A_n^-

Transfer operator equation

For $n \in \mathbb{Z}_{\neq 0}$, we have that

$$(\mathsf{I} + \mathsf{T})A_n^+(\xi) = \mathrm{e}^{-\mathrm{i}\pi n\xi}$$

and

$$(\mathsf{I} - \mathsf{T})A_n^-(\xi) = \mathrm{e}^{-\mathrm{i}\pi n\xi}.$$

Remark

This suggests that a careful spectral analysis of the transfer operator T on a space of smooth functions (Lipschitz etc) could give us A_n^+ and A_n^- via calculation of $(\mathsf{I} + \mathsf{T})^{-1}$ and $(\mathsf{I} - \mathsf{T})^{-1}$.

Multiplicative properties of HFS

Question

When can we multiply two hyperbolic Fourier series? How to write the result as a hyperbolic Fourier series?

Possible answer

Since we can express a rather general distribution, even an ultradistribution, as a hyperbolic Fourier series, this question contains with it the subtle issue of multiplying distributions. This is often impossible, but it is possible to multiply holomorphic functions in $\mathbb{H}$ and obtain a holomorphic function. This suggests multiplying two series of the type

$$f(\tau) = a_0 + \sum_{n \in \mathbb{Z}_{\geq 1}} a_n \, \mathrm{e}^{\mathrm{i}\pi n \tau} + b_n \, \mathrm{e}^{-\mathrm{i}\pi n/\tau},$$

which leads to the issue of expanding functions of the type

$$f_{k,l}(\tau) := \mathrm{e}^{\mathrm{i}\pi(k\tau - l/\tau)}, \qquad k, l \in \mathbb{Z}_{\geq 1},$$

in a hyperbolic Fourier series for $\tau \in \mathbb{H}$.

Symmetry property of $f_{k,l}$

Symmetry property

$$f_{k,l} \circ S(\tau) = f_{k,l}(-1/\tau) = f_{l,k}(\tau).$$

Consequences

(a) $a_0(f_{k,l}) = a_0(f_{l,k})$.
(b) $b_n(f_{k,l}) = a_n(f_{l,k})$ for $n \in \mathbb{Z}_{\geq 1}$.

Note

$a_n(f_{k,l}) = b_n(f_{k,l}) = 0$ holds for $n \in \mathbb{Z}_{<0}$ since $f_{k,l}$ is holomorphic.

The constant coefficient $a_0(f_{k,l})$

The constant coefficient

We have that

$$a_0(f_{k,l}) = \frac{1}{\mathrm{i}2\pi} \int_{\frac{1}{2}+\mathrm{i}\mathbb{R}} \exp\left(\mathrm{i}\pi(k\lambda_\triangle(\zeta) - l/\lambda_\triangle(\zeta))\right) \frac{\mathrm{d}\zeta}{\zeta(1-\zeta)}.$$

The remaining coefficients

The coefficients $a_n(f_{k,l})$ for $n \in \mathbb{Z}_{>0}$

$$a_n(f_{k,l}) = \frac{1}{4\pi^2 n} \int_{\frac{1}{2}+\mathrm{i}\mathbb{R}} \zeta^{-2} S_n'(1/\zeta)\, \mathrm{e}^{\mathrm{i}\pi(k\lambda_\triangle(\zeta) - l/\lambda_\triangle(\zeta))} \mathrm{d}\zeta.$$

$$= \frac{1}{4\pi^2 n} \int_{\frac{1}{2}+\mathrm{i}\mathbb{R}} S_n(1/\zeta)\, \frac{\mathrm{d}}{\mathrm{d}\zeta} \big(\mathrm{e}^{\mathrm{i}\pi(k\lambda_\triangle(\zeta) - l/\lambda_\triangle(\zeta))}\big) \mathrm{d}\zeta.$$

Comment

We should study the asymptotic behavior of $a_n(f_{k,l})$ as $n \to +\infty$. It appears we can show that $a_n(f_{k,l}) = \mathrm{O}(\log^2 n)$ as $n \to +\infty$ holds uniformly in $k, l \in \mathbb{Z}_{>0}$. Is it in fact $\mathrm{O}(1)$ or even $\mathrm{o}(1)$?

Growth control on A_n, B_n

We already know the functions A_n, B_n are extremely smooth (Gevrey class). But can we also say something about how big they are, uniformly and in integral sense?

Theorem

We have the control

$$A_n(\xi), B_n(\xi) = \mathrm{O}\left(\frac{n \log^2(n+1)}{1+\xi^2}\right)$$

uniformly in $n > 0$ and $\xi \in \mathbb{R}$.

Theorem

We have the integral control

$$\int_{\mathbb{R}} |A_n(\xi)| \mathrm{d}\xi = \int_{\mathbb{R}} |B_n(\xi)| \mathrm{d}\xi = \mathrm{O}(\log^3(n+1))$$

uniformly in $n > 0$.

Power skewed hyperbolic Fourier series

If $f \in \mathcal{M}^{\gamma}_{\mathrm{hol}}$ with $0 < \gamma < \pi$, we have the hyperbolic Fourier series expansion

$$f(\tau) = \sum_{n=0}^{+\infty} a_n \, \mathrm{e}^{\mathrm{i}\pi n\tau} + b_n \, \mathrm{e}^{-\mathrm{i}\pi n/\tau}, \qquad \tau \in \mathbb{H}.$$

If $\beta \geq 0$, we speak of a power β-skewed expansion if instead

$$f(\tau) = \sum_{n=0}^{+\infty} a_n \, \mathrm{e}^{\mathrm{i}\pi n\tau} + (\tau/\mathrm{i})^{-\beta} b_n \, \mathrm{e}^{-\mathrm{i}\pi n/\tau}, \qquad \tau \in \mathbb{H}.$$

Such power skewed hyperbolic Fourier series appear in the context of *Fourier interpolation pairs of radial functions* (see [5] below). Note that such series appear naturally with $\beta = 2$ when we take the derivative in the hyperbolic Fourier series (the constant term disappears, though). The quantity 2β corresponds to the dimension of the space where the Fourier transform is applied. For other β, we may multiply the hyperbolic Fourier series by a power of the theta function to obtain power skewed expansions.

N. B. : When $\beta < 0$ instead we may use the transformation S to consider $\beta > 0$ instead.

The theta function

We will use the theta function

$$\vartheta(\tau)=\vartheta_3(\tau):=\sum_{n\in\mathbb{Z}}\mathrm{e}^{\mathrm{i}\pi n^2\tau},\qquad \tau\in\mathbb{H},$$

so that $\vartheta\circ T^2(\tau)=\vartheta(\tau+2)=\vartheta(\tau)$ while $\vartheta\circ R(\tau)=\vartheta(-\bar{\tau})=\bar{\vartheta}(\tau)$, which moreover enjoys the functional equation

$$\vartheta(\tau)=(\tau/\mathrm{i})^{-\frac{1}{2}}\vartheta(-1/\tau).$$

This theta function has no zeros in $\mathbb{H}$, and has modest growth:

$$|\vartheta(\tau)|=\mathrm{O}\big(1+(\operatorname{Im}\tau)^{-\frac{1}{2}}\big).$$

The decay however is at times much more drastic:

$$|\vartheta(\tau)|^{-1}=\mathrm{O}\big((1+(\operatorname{Im}\tau)^{-\frac{1}{2}})\exp(\tfrac{\pi}{4}(\operatorname{Im}\tau)^{-1})\big).$$

We may form, for a given real parameter β, the power $(\vartheta(\tau))^{2\beta}$, and get a function obeying the functional identity

$$(\vartheta(\tau))^{2\beta}=(\tau/\mathrm{i})^{-\beta}(\vartheta(-1/\tau))^{2\beta}.$$

Twisting by theta powers

Method

Suppose $f : \mathbb{H} \to \mathbb{C}$ is holomorphic function. We form $g := (\vartheta)^{-2\beta} f$. If f is in the growth class $\mathcal{M}^{\gamma}_{\mathrm{hol}}$, then g is in the growth class $\mathcal{M}^{\gamma'}_{\mathrm{hol}}$ for each $\gamma' > \gamma + \frac{1}{2}\beta\pi$. If $\gamma' < \pi$, we apply the hyperbolic Fourier series decomposition to g:

$$g(\tau) = \sum_{n \geq 0} a_n(g)\, \mathrm{e}^{\mathrm{i}\pi n\tau} + b_n(g)\, \mathrm{e}^{-\mathrm{i}\pi n/\tau},$$

where $b_0(g) = 0$ by definition. Since $f = (\vartheta)^{2\beta} g$ and ϑ enjoys its functional equation, the representation

$$f(\tau) = (\vartheta(\tau))^{2\beta} \sum_{n \geq 0} a_n(g)\, \mathrm{e}^{-\mathrm{i}\pi n\tau} + (\tau/\mathrm{i})^{-\beta} (\vartheta(-1/\tau))^{2\beta} \sum_{n > 0} b_n(g)\, \mathrm{e}^{-\mathrm{i}\pi n/\tau}$$

holds. This is the sought after power skewed decomposition!

Observation

Requirements

We need $0 < \gamma < \pi - \frac{1}{2}\beta\pi$ here.

Growth of coefficients

If we write

$$(\vartheta(\tau))^{2\beta} \sum_{n \geq 0} a_n(g)\, \mathrm{e}^{-\mathrm{i}\pi n \tau} = \sum_{n \geq 0} a_n(f, \beta)\, \mathrm{e}^{\mathrm{i}\pi n \tau}$$

and

$$(\vartheta(-1/\tau))^{2\beta} \sum_{n > 0} b_n(g)\, \mathrm{e}^{-\mathrm{i}\pi n/\tau} = \sum_{n > 0} b_n(f, \beta)\, \mathrm{e}^{-\mathrm{i}\pi n/\tau},$$

we can easily control the growth of the coefficients $a_n(f, \beta), b_n(f, \beta)$:

$$|a_n(f, \beta)|,\ |b_n(f, \beta)| = \mathrm{O}(\exp(\alpha\sqrt{n}))$$

as $n \to +\infty$, for each $\alpha > \sqrt{4\gamma\pi + 2\beta\pi^2}$.

The limit as $\beta \to 2$

Remark

When $\beta = 2$ the above decomposition method does not apply, for a simple reason. By simply differentiating the holomorphic hyperbolic Fourier series expansion, we obtain the $\beta = 2$ case. Moreover, since differentiation kills constants, the expansion now lacks a constant term, while the twisted theta decomposition would have one. The constant term is indeed superfluous, as it holds that

$$-\frac{1}{2\pi^2} = \sum_{n>0} nB_n(0)\,\mathrm{e}^{\mathrm{i}\pi n\tau} + nA_n(0)\,\tau^{-2}\mathrm{e}^{-\mathrm{i}\pi n/\tau}, \qquad \tau \in \mathbb{H}.$$

Exponentially skewed hyperbolic Fourier series

Definition

A representation of the form

$$f(t) \sim \sum_{n\in\mathbb{Z}} a_n \, \mathrm{e}^{\mathrm{i}\pi(n+\alpha)t} + b_n \, \mathrm{e}^{-\mathrm{i}\pi n/t}$$

where $\alpha \in \mathbb{R}$ we call an *exponentially skewed hyperbolic Fourier series*. Without loss of generality, $0 \le \alpha < 1$, and we exclude $\alpha = 0$ as the already known instance of HFS.

Special case

For holomorphic f, we look for

$$f(\tau) = \sum_{n\geq 0} a_n \, \mathrm{e}^{\mathrm{i}\pi(n+\alpha)\tau} + b_n \, \mathrm{e}^{-\mathrm{i}\pi n/\tau}.$$

Twisting by powers of λ

Method

We form $g(\tau) = (\lambda(\tau))^{-\alpha} f(\tau)$. We then expand g in a hyperbolic Fourier series:

$$g(\tau) = a_0(g) + \sum_{n>0} a_n(g)\, \mathrm{e}^{\mathrm{i}\pi n\tau} + b_n(g)\, \mathrm{e}^{-\mathrm{i}\pi n/\tau}.$$

We then recover f, and use that $\lambda(\tau) = 1 - \lambda(-1/\tau)$:

$$f(\tau) = \lambda^{\alpha}(\tau) \sum_{n\geq 0} a_n(g)\, \mathrm{e}^{\mathrm{i}\pi n\tau} + (1 - \lambda(-1/\tau))^{\alpha} \sum_{n\geq 0} b_n(g)\, \mathrm{e}^{-\mathrm{i}\pi n/\tau}.$$

The second term is of the correct form, while the first has a factor $\mathrm{e}^{\mathrm{i}\pi\alpha\tau}$, as required.

Origin of exponentially twisted series

The Klein-Gordon equation and the lattice

Considering instead at the outset the interpolation set of points

$$(\pi(n+\alpha), 0), \quad (0, \pi m), \qquad m, n \in \mathbb{Z},$$

leads naturally to exponentially skewed hyperbolic Fourier series.

Remark

Exponentially skewed hyperbolic Fourier series may arise when we consider certain functional identities for theta functions and form products of four such, and finally take the primitive.

Bibliography

1. Borichev, A., Hedenmalm, H., *Completeness of translates in weighted spaces on the half-line*. Acta Math. 174 (1995), 1-84.
2. Hedenmalm, H., Montes-Rodríguez, A., *Heisenberg uniqueness pairs and the Klein-Gordon equation*. Ann. of Math. 173 (2011), 1507-1527.
3. Hedenmalm, H., Montes-Rodríguez, A., *The Klein-Gordon equation, the Hilbert transform, and dynamics of Gauss-type maps*. J. Eur. Math. Soc. 22 (2020), 1703-1757.
4. Hedenmalm, H., Montes-Rodríguez, A. *The Klein-Gordon equation, the Hilbert transform, and Gauss-type maps: H^∞ approximation*. J. Anal. Math. 144 (2021), 119-190.
5. Bakan, A., Hedenmalm, H., Montes-Rodríguez, A., Radchenko, D., Viazovska, M., *Fourier uniqueness in even dimensions*. Proc. Natl. Acad. Sci. USA 118 (2021), no. 15, Paper No. 2023227118
6. Radchenko, D., Viazovska, M., *Fourier interpolation on the real line*. Publ. Math. IHES 129 (2019), 51-81.
7. Bakan, A., Hedenmalm, H., Montes-Rodríguez, A., Radchenko, D., Viazovska, M., *Hyperbolic Fourier series*. arXiv: 2110.00148

Bibliography, cont

8. Finkelstein, M., Scheinberg, S., *Kernels for solving problems of Dirichlet type in a half-plane*. Advances in Math. 18 (1975), 108-113. 31A25