

Hadamard's variational formula and the Gaussian free field

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Gaussian free field (GFF)

Let $\Omega \subset \mathbb{C}$ be a bounded C^2 -domain, and $H_0^1(\Omega) = W_0^{1,2}(\Omega)$ the Sobolev space with zero boundary values under the Dirichlet inner product

$$\langle f, g \rangle_{H_0^1} = \int_{\Omega} \nabla f \cdot \nabla g \, dA \quad (dA = \text{area measure}).$$

The **Gaussian free field** on Ω can be formally defined as

$$\Psi = \sum_{j=1}^{\infty} \xi_j \psi_j$$

where (ψ_j) is an ON basis of $H_0^1(\Omega)$ and (ξ_j) are independent $N(0, 1)$ variables. More precisely, for each $f \in H_0^1(\Omega)$, may define

$$\langle f, \Psi \rangle_{H_0^1} = \sum_{j=1}^{\infty} \xi_j \langle f, \psi_j \rangle_{H_0^1} \sim N(0, \|f\|_{H_0^1}^2)$$

with the covariance structure

$$E[\langle f, \Psi \rangle_{H_0^1} \langle g, \Psi \rangle_{H_0^1}] = \langle f, g \rangle_{H_0^1}.$$

Gaussian free field (GFF)

Alternatively, Ψ acts on $L^2(\Omega)$ as

$$\begin{aligned} E[\langle f, \Psi \rangle_{L^2} \langle g, \Psi \rangle_{L^2}] &= \langle f, (-\Delta)^{-1} g \rangle_{L^2} \\ &= \int_{\Omega} \int_{\Omega} G(z, w) f(z) g(w) dA(z) dA(w) \end{aligned}$$

where G is the **Dirichlet Green function** of Ω that solves, for each $w \in \Omega$,

$$\begin{cases} -\Delta G(\cdot, w) = \delta_w & \text{on } \Omega, \\ G(z, w) = 0 & \text{for } z \in \partial\Omega. \end{cases}$$

Flow of domains – setup

From now on, suppose $\Omega(t) \subset \mathbb{C}$ ($0 < t \leq 1$) are bounded C^2 domains such that

- (a) $\overline{\Omega(t)} \subset \Omega(t')$ for $t < t'$
- (b) $\partial\Omega(t)$ has C^2 dependence on t
- (c) $\Omega(t)$ has a constant finite connectivity (independent of t)
- (d) the intersection $\bigcap_t \Omega(t)$ is a continuum with zero area and trivial Green function

Hadamard's formula for the Green function

Hadamard's variational formula (1908): the Green function of $\Omega(t)$ can be written as

$$G_t(z, w) = \int_0^t \left\{ \int_{\partial\Omega(\tau)} P_\tau(z, \zeta) P_\tau(w, \zeta) \rho(\zeta) ds(\zeta) \right\} d\tau$$

with $ds = \text{arc-length}$, and

- ▶ $P_\tau(z, \zeta) = \frac{\partial}{\partial \mathbf{n}(\zeta)} G_\tau(z, \zeta)$ Poisson kernel of $\Omega(\tau)$ ($= 0$ outside $\overline{\Omega(t)}$)
- ▶ $\rho(\zeta) = \text{rate at which } \partial\Omega(\tau) \text{ moves at point } \zeta \text{ in the direction of outward normal as } \tau \text{ grows.}$

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- ▶ $\rho(\zeta) =$ rate at which $\partial\Omega(\tau)$ moves at point ζ in the direction of outward normal as τ grows.

Note: For each t , the boundaries $\partial\Omega(\tau)$ ($0 < \tau < t$) almost foliate $\Omega(t)$:

$$\Omega(t) = \bigcup_{0 < \tau < t} \partial\Omega(\tau) \quad \text{a.e.}$$

Moreover, we have “polar coordinates”: $dA(\zeta) = \rho(\zeta) ds(\zeta) d\tau$.

Reinterpretation in terms of operators

We define, for each t , the **Hadamard operator** and its **adjoint**

$$Q_t f(z) = \int_{\Omega(t)} P_{\tau(\zeta)}(z, \zeta) f(\zeta) dA(\zeta)$$

$$Q_t^* f(\zeta) = \chi_{\Omega(t)}(\zeta) \int_{\Omega(t)} P_{\tau(\zeta)}(z, \zeta) f(z) dA(z)$$

where $\tau(\zeta) =$ the unique τ such that $\zeta \in \partial\Omega(\tau)$.

Formally, Q_t maps δ_ζ onto $P_{\tau(\zeta)}(\cdot, \zeta)$.

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Proposition

- (a) $Q_t : L^2(\Omega(t)) \rightarrow H_0^1(\Omega(t))$ is unitary
- (b) $Q_t^* : H^{-1}(\Omega(t)) \rightarrow L^2(\Omega(t))$ is unitary
- (c) $Q_t Q_{t'}^* = (-\Delta_{\min(t, t')})^{-1}$

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Here $(-\Delta_t)^{-1} =$ inverse of minus Laplacian with vanishing Dirichlet boundary values:

$$(-\Delta_t)^{-1} f(z) = \int_{\Omega(t)} G_t(z, w) f(w) dA(w)$$

Application to GFF

Let $\Omega = \Omega(1)$ be the largest domain, and Φ the **white noise** field on Ω :

$$\langle f, \Phi \rangle_{L^2} \sim N(0, \|f\|_{L^2}^2) \quad \text{for } f \in L^2(\Omega)$$

$$E[\langle f, \Phi \rangle_{L^2} \langle g, \Phi \rangle_{L^2}] = \langle f, g \rangle_{L^2}$$

Define for each $0 < t \leq 1$:

$$\Psi_t = Q_t \Phi, \quad \text{i.e.} \quad \langle f, \Psi_t \rangle_{L^2} = \langle Q_t^* f, \Phi \rangle_{L^2}.$$

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Theorem

Above Ψ_t is a GFF on $\Omega(t)$.

Moreover, the process Ψ_t , $0 \leq t \leq 1$ (with $\Psi_0 \equiv 0$) has independent increments: for all $0 = t(0) < t(1) < \dots < t(n) = 1$, the fields

$\Psi_{t(j)} - \Psi_{t(j-1)}$, $j = 1, \dots, n$, are independent.

Application to GFF

In particular, for $0 < t < t' \leq 1$, we have the decomposition

$$\Psi_{t'} = \Psi_t + (\Psi_{t'} - \Psi_t) \quad (\text{independent summands})$$

where

- ▶ $\Psi_t = \text{GFF on } \Omega(t)$
- ▶ $\Psi_{t'} - \Psi_t = \text{random field which is harmonic in } \Omega(t) \text{ and equals } \Psi_{t'}$
on $\Omega(t') \setminus \Omega(t)$.

(Markov property!)

Time-derivative of Ψ_t

Define the **harmonic sweep** operator from $\Omega(t)$ to $\partial\Omega(t)$ by

$$P_t^* f(\zeta) = \int_{\Omega(t)} P_t(z, \zeta) f(z) dA(z) \quad (\zeta \in \partial\Omega(t)).$$

May write

$$\begin{aligned} E[\langle f, \Psi_t \rangle_{L^2} \langle g, \Psi_{t'} \rangle_{L^2}] &= E[\langle Q_t^* f, \Phi \rangle_{L^2} \langle Q_{t'}^* g, \Phi \rangle_{L^2}] \\ &= \langle Q_t^* f, Q_{t'}^* g \rangle_{L^2} \\ &= \int_0^{\min(t, t')} \left\{ \int_{\partial\Omega(\tau)} P_\tau^* f P_\tau^* g \rho ds \right\} d\tau \end{aligned}$$

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Formally: If Ψ'_t is the time-derivative of Ψ_t , then

$$E[\langle f, \Psi'_t \rangle_{L^2} \langle g, \Psi'_{t'} \rangle_{L^2}] = \delta_0(t - t') \int_{\partial\Omega(t)} P_t^* f P_t^* g \rho ds$$

Time-derivative of Ψ_t – precise statement

1. Let Ξ_t be a **weighted white noise** on $\partial\Omega(t)$ with weight $\rho^{1/2}$, i.e., a Gaussian random field acting on $L^2(\partial\Omega(t))$ such that

$$E[\langle \varphi, \Xi_t \rangle_{L^2(\partial\Omega(t))} \langle \psi, \Xi_t \rangle_{L^2(\partial\Omega(t))}] = \int_{\partial\Omega(t)} \varphi \psi \rho \, ds$$

for $\varphi, \psi \in L^2(\partial\Omega(t))$. Let $P_t \Xi_t$ be the Poisson extension of Ξ_t , i.e., a harmonic field on $\Omega(t)$ given by

$$\langle P_t \Xi_t, f \rangle_{L^2(\Omega(t))} = \langle \Xi_t, P_t^* f \rangle_{L^2(\partial\Omega(t))} \quad \text{for } f \in L^2(\Omega(t)).$$

Then the derivative of $t \mapsto \Psi_t$ in the sense of distributions has the same law as $P_t \Xi_t$.

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Then the derivative of $t \mapsto \Psi_t$ in the sense of distributions has the same law as $P_t \Xi_t$.

2. For a fixed f ,

$$\langle f, \Psi_t \rangle_{L^2} \stackrel{d}{=} B\left(\int_0^t \kappa(\tau) \, d\tau\right)$$

where B is a BM and $\kappa(\tau) = \int_{\partial\Omega(\tau)} |P_\tau^* f|^2 \rho \, ds$.