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An invariant subspace of the Bergman space having the codimension two property

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0. Introduction

Let $L_a^2(\mathbb{D})$ denote the standard Bergman space of all holomorphic functions on the open unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$ that satisfy the integrability condition

$$\|f\|_{L^2} = \left(\int_{\mathbb{D}} |f(z)|^2 dS(z)\right)^{1/2} < \infty .$$

Here, dS denotes area measure in $\mathbb{C}$, normalized by a constant factor:

$$dS(z) = dxdy/\pi, \quad z = x + iy .$$

A closed subspace J of $L_a^2(\mathbb{D})$ is said to be z-invariant, or invariant, provided the product zf belongs to J whenever $f \in J$. Here, we use the standard notation z for the coordinate function:

$$z(\lambda) = \lambda, \quad \lambda \in \mathbb{D} .$$

The structure of the lattice of invariant subspaces in $L_a^2(\mathbb{D})$ has attracted a lot of attention from operator theorists as well as function theorists, but most results have been disappointing, in the sense that one has realized that no simple characterization such as is known for the Hardy space $H^2(\mathbb{D})$ is possible for the Bergman space. The famous theorem on the invariant subspaces of $H^2(\mathbb{D})$ is due to Arne Beurling [2], and it asserts that every z-invariant subspace J of $H^2(\mathbb{D})$, analogously defined as for the Bergman space, is either trivial, that is, $J = \{0\}$, or has the form $J = uH^2(\mathbb{D})$, where u an inner function, that is, a bounded analytic function on $\mathbb{D}$ with nontangential boundary values of modulus 1 almost everywhere. Beurling's proof relied heavily on function theory; later streamlined proofs more dependent on the Hilbert space structure have made the assertion seem like a triviality. The Bergman space does share some features with the Hardy space, for instance, the author [3] has found a canonical way to factor out zeros for the Bergman space, using functions analogous to Blaschke products. However, in contrast to the Hardy space situation, it has been discovered by Constantin Apostol, Hari Bercovici, Ciprian Foiaş, and Carl Pearcy [1], that there exist in the Bergman space z-invariant subspaces J having the so-called codimension n property, for any integer $n = 1, 2, 3, \ldots$; n may even assume the value ∞. One says that a z-invariant subspace J in $L_a^2(\mathbb{D})$ has the codimension n property if the subspace zJ has codimension n in J. However, their approach is non-constructive, and very complicated. In fact, it is even necessary for them to use the axiom of choice to show the existence of such subspaces. There is a clear need to increase our insight into this phenomenon. In this paper, we shall present a simple concrete example of an invariant subspace having the codimension 2 property, and mention ways to extend the example to get subspaces having the codimension n property, for arbitrary n. Such "pathology" never occurs in the Hardy space, and the principal reason for this is that intersection of two invariant subspaces is never $\{0\}$, unless one of the invariant subspaces is $\{0\}$ itself. For instance, the union of two zero sequences for the Hardy space is always a zero sequence, but this is not always the case for the Bergman space, as was discovered by Charles Horowitz [5].

The example given in this paper uses in an essential way the recent advances by Kristian Seip [8], in which he gives a complete description of interpolating and sampling sequences for the Bergman space $L_a^2(\mathbb{D})$ in terms of densities. Stefan Richter's paper [7] has been very helpful guidance to see what function theoretic properties to look for.

1. Sampling, interpolating, and zero sets for the Bergman space

A sequence may be considered to be a set provided that its points are distinct. Following Kristian Seip [8], we say that a sequence $A = \{a_j\}_j$ is a *sampling sequence* for $L_a^2(\mathbb{D})$ provided we can find positive constants K_1, K_2 such that

$$K_1 \int_{\mathbb{D}} |f(z)|^2 dS(z) \leqq \sum_j (1 - |a_j|^2)^2 |f(a_j)|^2 \leqq K_2 \int_{\mathbb{D}} |f(z)|^2 dS(z)$$

holds for all $f \in L_a^2(\mathbb{D})$. Furthermore, the sequence A is said to be an *interpolating sequence* for $L_a^2(\mathbb{D})$, provided that to every ℓ^2 sequence $\{w_j\}_j$, there exists a function $f \in L_a^2(\mathbb{D})$ having

$$(1 - |a_j|^2) f(a_j) = w_j \quad \text{for all } j .$$

If f is a holomorphic function on $\mathbb{D}$, we write $Z(f)$ for the sequence of zeros of f, counting multiplicities, provided f does not vanish identically. If f vanishes identically, we write $Z(f) = \mathbb{D}$. A sequence of points in $\mathbb{D}$ is called a *Bergman space zero sequence* provided it coincides with $Z(f)$ for some nonidentically vanishing function $f \in L_a^2(\mathbb{D})$. Clearly, every interpolating sequence A for $L_a^2(\mathbb{D})$ is also a Bergman space zero sequence: just take an interpolant for the sequence $w_1 = 1$, $w_j = 0$ for all other j, and multiply this function by $z - a_1$ to get a nonidentically vanishing function that vanishes on the sequence A. This actually only shows that A must be a subsequence of a Bergman space zero sequence, but it is well known, and not too hard to show, that every subsequence of a Bergman space zero sequence is itself a zero sequence [5], [3]. However, the *union* of two zero sequences need not be a zero sequence [5]; in fact, it may be so far away from being a zero sequence as to be a sampling sequence, as we shall see in Theorem 2.1.

2. Statement of main results

We need the following technical result, which is a simple consequence of Seip's work on sampling and interpolation in Bergman spaces [8]. Seip's results are reviewed in Section 3, and it is therefore natural to postpone the proof of it until Section 4.

Theorem 2.1. *There exists a sampling sequence for $L_a^2(\mathbb{D})$ which is the union of two disjoint zero sequences.*

If $\mathscr{H}_1$ and $\mathscr{H}_2$ are two Hilbert spaces, it is standard to denote by $\mathscr{H}_1 \oplus \mathscr{H}_2$ their direct sum, that is, the linear space of all pairs (x_1, x_2), with $x_1 \in \mathscr{H}_1$ and $x_2 \in \mathscr{H}_2$, supplied with the norm

$$\|(x_1, x_2)\|_{\mathscr{H}_1 \oplus \mathscr{H}_2} = (\|x_1\|^2_{\mathscr{H}_1} + \|x_2\|^2_{\mathscr{H}_2})^{1/2},$$

which makes $\mathscr{H}_1 \oplus \mathscr{H}_2$ a Hilbert space. If $\mathscr{H}_1$ and $\mathscr{H}_2$ are closed subspaces of a bigger Hilbert space $\mathscr{H}$, one can consider their sum $\mathscr{H}_1 + \mathscr{H}_2$, and in case $\mathscr{H}_1$ and $\mathscr{H}_2$ are orthogonal subspaces, one then replaces the plus sign $(+)$ with a direct plus $(\oplus)$ sign. In this paper we shall take the liberty to write $\mathscr{H}_1 \oplus \mathscr{H}_2$ provided the closed subspaces $\mathscr{H}_1$ and $\mathscr{H}_2$ have $\mathscr{H}_1 \cap \mathscr{H}_2 = \{0\}$, and the direct sum norm on $\mathscr{H}_1 \oplus \mathscr{H}_2$ is equivalent to the restriction of the $\mathscr{H}$-norm to $\mathscr{H}_1 + \mathscr{H}_2$; given that $\mathscr{H}_1 \cap \mathscr{H}_2 = \{0\}$, this last property is equivalent to requiring the sum $\mathscr{H}_1 + \mathscr{H}_2$ to be closed in $\mathscr{H}$, by the closed graph theorem.

Given a Bergman space zero sequence A, we denote by $\mathscr{I}(A)$ the invariant subspace of all functions in $L_a^2(\mathbb{D})$ that vanish on A, counting multiplicities.

Theorem 2.1 ensures that the assumptions of the following proposition can be fulfilled.

Proposition 2.2. *Let A and B be two disjoint zero sequences whose union is a sampling sequence. Then $\mathscr{I}(A) \cap \mathscr{I}(B) = \{0\}$, and the subspace $\mathscr{I}(A) + \mathscr{I}(B)$ is closed in $L_a^2(\mathbb{D})$, allowing us to write $\mathscr{I}(A) \oplus \mathscr{I}(B)$ instead of $\mathscr{I}(A) + \mathscr{I}(B)$. The subspace $\mathscr{I}(A) \oplus \mathscr{I}(B)$ is z-invariant, and it has the codimension 2 property.*

Proof. Write $C = A \cup B$, with $C = \{c_j\}_j$, and let $f \in \mathscr{I}(A)$ and $g \in \mathscr{I}(B)$ be arbitrary. Clearly, a sampling sequence cannot be a zero sequence, and hence $\mathscr{I}(A) \cap \mathscr{I}(B) = \{0\}$. Our next job is to check that the subspace $\mathscr{I}(A) + \mathscr{I}(B)$ is closed in $L_a^2(\mathbb{D})$. Denote by K_1, K_2 the positive constants associated with the sampling property of the sequence C:

$$K_1 \int_{\mathbb{D}} |f(z)|^2 dS(z) \leqq \sum_j (1 - |c_j|^2)^2 |f(c_j)|^2 \leqq K_2 \int_{\mathbb{D}} |f(z)|^2 dS(z).$$

Let $f \in \mathscr{I}(A)$ and $g \in \mathscr{I}(B)$ be arbitrary. Then for every point c_j in the sequence C, we have

$$|f(c_j) + g(c_j)|^2 = |f(c_j)|^2 + |g(c_j)|^2,$$

so that if we use the sampling property of the sequence C, we get

$$\|f+g\|_{L^2}^2 = \int_{\mathbb{D}} |f(z)+g(z)|^2 dS(z) \geqq K_2^{-1} \sum_j (1-|c_j|^2)^2 |f(c_j)+g(c_j)|^2$$
$$= K_2^{-1} \sum_j (1-|c_j|^2)^2 (|f(c_j)|^2 + |g(c_j)|^2) \geqq K_1 K_2^{-1} (\|f\|_{L^2}^2 + \|g\|_{L^2}^2).$$

The property we shall focus on is that with $\varepsilon = K_1 K_2^{-1} > 0$,

$$\|f+g\|_{L^2}^2 \geqq \varepsilon(\|f\|_{L^2}^2 + \|g\|_{L^2}^2), \quad f \in \mathscr{I}(A),\ g \in \mathscr{I}(B).$$

It implies the assertion that $\mathscr{I}(A) + \mathscr{I}(B)$ is a closed subspace of $L_a^2(\mathbb{D})$, justifying the change of notation to $\mathscr{I}(A) \oplus \mathscr{I}(B)$. Since the subspaces $\mathscr{I}(A)$ and $\mathscr{I}(B)$ are z-invariant, their (direct) sum $\mathscr{I}(A) \oplus \mathscr{I}(B)$ is z-invariant as well. What remains to be done is to demonstrate that the codimension of $z(\mathscr{I}(A) \oplus \mathscr{I}(B))$ in $\mathscr{I}(A) \oplus \mathscr{I}(B)$ is 2. Note that if $f \in \mathscr{I}(A)$ and $g \in \mathscr{I}(B)$, then

$$z(f+g) = zf + zg \in z\mathscr{I}(A) \oplus z\mathscr{I}(B),$$

so that

$$z(\mathscr{I}(A) \oplus \mathscr{I}(B)) = z\mathscr{I}(A) \oplus z\mathscr{I}(B).$$

The direct sum sign is justified because $z\mathscr{I}(A)$ and $z\mathscr{I}(B)$ are closed subspaces of $\mathscr{I}(A)$ and $\mathscr{I}(B)$, respectively. The subspaces $z\mathscr{I}(A)$ and $z\mathscr{I}(B)$ have codimension 1 in the spaces $\mathscr{I}(A)$ and $\mathscr{I}(B)$ [7], p. 597, respectively, and consequently, their direct sum $z\mathscr{I}(A) \oplus z\mathscr{I}(B)$ must have codimension 2 in $\mathscr{I}(A) \oplus \mathscr{I}(B)$. This completes the proof. □

From Theorem 2.1 and Proposition 2.2 we derive the following corollary.

Corollary 2.3. *There exists a z-invariant subspace of $L_a^2(\mathbb{D})$ which has the codimension 2 property. Moreover, this z-invariant subspace can be of the form $J = \mathscr{I}(A) + \mathscr{I}(B)$, where A and B are two disjoint Bergman space zero sequences.*

Stefan Richter [7] has shown that any intersection of z-invariant subspaces in $L_a^2(\mathbb{D})$ having the codimension 1 property also has the codimension 1 property. That makes it meaningful to speak of the smallest z-invariant subspace of $L_a^2(\mathbb{D})$ having the codimension 1 property which contains a given invariant subspace. In a sense, the following result shows how far away the invariant subspace $\mathscr{I}(A) \oplus \mathscr{I}(B)$ constructed in Proposition 2.2 is from having the codimension 1 property.

Theorem 2.4. *Let A and B be two disjoint Bergman space zero sequences. Then the smallest z-invariant subspace of $L_a^2(\mathbb{D})$ containing both $\mathscr{I}(A)$ and $\mathscr{I}(B)$ having the codimension 1 property is $L_a^2(\mathbb{D})$ itself.*

Proof. Let us for convenience denote by J the smallest invariant subspace of $L_a^2(\mathbb{D})$ containing $\mathscr{I}(A)$ and $\mathscr{I}(B)$ with the codimension 1 property; the assertion we wish to prove is that $J = L_a^2(\mathbb{D})$.

Only one of the sequences A and B can contain the point 0, since they are disjoint. Let $A = \{a_j\}_{j=1}^{\infty}$ be the one that does not contain 0. We may then consider the extremal function G_A for the problem

$$\sup\{\operatorname{Re} f(0) : f = 0 \text{ on } A, \|f\|_{L^2} \leqq 1\},$$

which has the property of vanishing only on the sequence A, among other things, according to [3]. Let A_N be the finite subsequence $A_N = \{a_j\}_{j=1}^{N}$, and let G_{A_N} be the extremal function associated with the zero sequence A_N. We know from [3] that G_{A_N} is a rational function whose poles are located at the reflected points $\{1/\bar{a}_j\}_{j=1}^{N}$; it vanishes precisely at A_N in the unit disk $\mathbb{D}$, has $|G_{A_N}(z)| \geqq 1$ on the circle $\mathbb{T}$, has $\|G_{A_N}\|_{L^2} = 1$, and moreover, it has the expansive multiplier property

$$\|f\|_{L^2} \leqq \|G_{A_N} f\|_{L^2}, \quad f \in L_a^2(\mathbb{D}). \tag{2.1}$$

Denote by $Z(J)$ the common zero set in $\mathbb{D}$ of the functions in J. Since the sequences A and B are disjoint, we have $Z(J) = \emptyset$. Richter [7] has shown that if a z-invariant subspace I has the codimension 1 property, then if $\lambda \in \mathbb{D}$ is any point which does not belong to the common zero set $Z(I)$ of I, and $f \in I$ has $f(\lambda) = 0$, the function $f(z)/(z-\lambda)$ also belongs to I. Let us apply this argument repeatedly to the invariant subspace J, and the function G_A. We then obtain as a conclusion that G_A/G_{A_N} also belongs to J, for every positive integer N. But as $N \to \infty$, $G_A/G_{A_N} \to 1$ pointwise in $\mathbb{D}$, and by (2.1), with $f = G_A/G_{A_N}$, we have

$$\|G_A/G_{A_N}\|_{L^2} \leqq \|G_A\|_{L^2} = 1,$$

so that in fact, by an argument analogous to the one used in [3], pp. 57–58, $G_A/G_{A_N} \to 1$ in norm as $N \to \infty$. Hence the constant function 1 has to belong to J as well. But the constant function 1 generates, as an invariant subspace, the whole Bergman space $L_a^2(\mathbb{D})$, and we arrive at the conclusion $J = L_a^2(\mathbb{D})$. □

Corollary 2.5. *Let A and B be two disjoint Bergman space zero sequences. If one of the sequences A and B does not accumulate at every point of the unit circle $\mathbb{T}$, then $\mathscr{I}(A) + \mathscr{I}(B)$ is dense in $L_a^2(\mathbb{D})$.*

Proof. Let J be the norm closure of the sum $\mathscr{I}(A) + \mathscr{I}(B)$, which is an invariant subspace in $L_a^2(\mathbb{D})$. In [4] (see also [7]), the concept of the weak spectrum $\sigma'(z[I])$ associated with an invariant subspace I was introduced, and by [4], Theorem 1.3 and Corollary 1.5, we see that for our particular choice $I = J$, we have

$$\sigma'(z[J]) \subset \bar{A} \cap \bar{B}.$$

Note that by assumption, the set $\bar{A} \cap \bar{B}$ is a proper closed subset of the unit circle $\mathbb{T}$. But Richter (see [7] and [4], Theorem 1.1) has shown that weak spectra associated with invariant subspaces not having the codimension 1 property must contain the whole unit circle, and therefore, the invariant subspace J must necessarily have the codimension 1 property. Theorem 2.4 now applies, and the assertion follows. □

3. Seip's work on densities

The pseudohyperbolic metric on $\mathbb{D}$ is given by the expression

$$\varrho(z, \zeta) = \left| \frac{\zeta - z}{1 - \bar{\zeta} z} \right|, \quad z, \zeta \in \mathbb{D}.$$

A sequence (or set) of points $\Lambda = \{\lambda_j\}_j$ in $\mathbb{D}$, finite or infinite, is said to be *uniformly discrete* provided that

$$\inf\{\varrho(a_j, a_k) : j \neq k\} > 0.$$

Clearly, a uniformly discrete sequence has to consist of distinct points, and for finite sequences, this is the only restriction.

If Λ is a sequence of points in the unit disk $\mathbb{D}$, and $\zeta \in \mathbb{D}$ an arbitrary point, let Λ_ζ denote the image of Λ under the conformal automorphism of the unit disk

$$\varphi_\zeta(z) = \frac{\zeta - z}{1 - \bar{\zeta} z}, \quad z \in \mathbb{D}.$$

Associate with the sequence Λ_ζ the function $n(r, \Lambda_\zeta)$, which counts the number of points of Λ_ζ contained within the disk

$$\{z \in \mathbb{D} : |z| < r\}.$$

Moreover, we shall need the definite integral

$$N(r, \Lambda_\zeta) = \int_0^r n(t, \Lambda_\zeta)\, dt, \quad 0 < r < 1.$$

If $A(r)$ now stands for the function

$$A(r) = \log \frac{1 + r}{1 - r}, \quad 0 < r < 1,$$

Seip defines his upper density of Λ as

$$D^+(\Lambda) = \limsup_{1 > r \to 1} \sup_{\zeta \in D} \frac{N(r, \Lambda_\zeta)}{A(r)},$$

and his lower density of Λ as

$$D^-(\Lambda) = \liminf_{1 > r \to 1} \inf_{\zeta \in D} \frac{N(r, \Lambda_\zeta)}{A(r)}.$$

For the special case of the unweighted Bergman space, his main results are as follows.

Theorem 3.1 (Seip). *A sequence Λ of distinct points in $\mathbb{D}$ is sampling for $L_a^2(\mathbb{D})$ if and only if it can be expressed as a finite union of uniformly discrete sets and it contains a uniformly discrete subsequence Λ' for which $D^-(\Lambda') > 1/2$.*

Theorem 3.2 (Seip). *A sequence Λ of distinct points in $\mathbb{D}$ is interpolating for $L_a^2(\mathbb{D})$ if and only if it is uniformly discrete and $D^+(\Lambda) < 1/2$.*

4. The critical example

We now prove the existence of two disjoint interpolating sequences, the union of which is a sampling set.

Theorem 4.1. *There exists a sampling sequence C for $L_a^2(\mathbb{D})$ which is the union of two disjoint interpolating sequences A and B for $L_a^2(\mathbb{D})$.*

Proof. The upper half plane

$$\mathbb{U} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$$

is mapped conformally onto the unit disk $\mathbb{D}$ by the Moebius mapping

$$\varphi(z) = (z - i)/(z + i), \quad z \in \mathbb{C} \backslash \{-i\}.$$

We will construct three sequences A', B', and C' *in* $\mathbb{U}$, and then define $A = \varphi(A')$, $B = \varphi(B')$, and $C = \varphi(C')$. Fix two real-valued parameters $\beta, \beta > 1$, and $\gamma, \gamma > 0$, with the property that

$$2\pi < \gamma \log \beta < 4\pi.$$

The sequence C' will consist of all points in the upper half plane $\mathbb{U}$ of the form

$$c_{j,k} = \beta^j (k\gamma + i),$$

where j, k range over the integers, and i, as always, is the square root of -1. The subsequence A' will consist of all points

$$a_{j,k} = \beta^j (2k\gamma + i),$$

with j, k ranging over the integers, and B' will be the sequence of all points

$$b_{j,k} = \beta^j ((2k + 1)\gamma + i),$$

again with j, k ranging over the integers. The sequences A and C are very regular, and Seip [8] has already computed their densities:

$$D^+(C) = D^-(C) = \frac{2\pi}{\gamma \log \beta} > 1/2,$$

and

$$D^+(A) = D^-(A) = \frac{\pi}{\gamma \log \beta} < 1/2.$$

The sequence B is also very regular, although it does not fit the class of regular sequences for which Seip has calculated the densities, and it is in fact possible to verify that

$$D^+(B) = D^-(B) = \frac{\pi}{\gamma \log \beta} < 1/2\,.$$

By Theorems 3.1 and 3.2, A and B are interpolating sequences, and C is a sampling sequence. □

Theorem 4.1 asserts something even stronger that Theorem 2.1, the zero sequences can actually be chosen to be interpolating.

5. Invariant subspaces having the codimension *n* property

To carry out a construction of an invariant subspace I of $L_a^2(\mathbb{D})$ having the codimension n property, that is, zI should have codimension n in I, along the lines of Section 2, we need to find invariant subspaces $I_1, I_2, \ldots, I_n$, all having the codimension 1 property, which are far apart, that is,

$$\|f_1 + \cdots + f_n\|_{L^2} \geqq \varepsilon(\|f_1\|_{L^2} + \cdots + \|f_n\|_{L^2}) \tag{5.1}$$

holds for some $\varepsilon > 0$ and for all $f_1 \in I_1, f_2 \in I_2, \ldots, f_n \in I_n$. Then the (direct) sum

$$I = I_1 \oplus I_2 \oplus \cdots \oplus I_n$$

has the codimension n property. There are several ways to get such a collection of invariant subspaces; the one outlined here was suggested to me by Boris Korenblum [6].

Let the sequence C' be the regular sequence in the upper half plane $\mathbb{U}$ appearing in the proof of Theorem 4.1. Only this time the parameters $\beta, \beta > 1$, and $\gamma, \gamma > 0$, must be chosen such that

$$4\pi(n-1)/n < \gamma \log \beta < 4\pi\,.$$

Let $B'_1, \ldots, B'_n$ be subsequences of C', the sequence B'_m ($m = 1, \ldots, n$) consisting of all the points

$$b^m_{j,k} = \beta^j\big((nk+m)\gamma + i\big)\,,$$

with j, k ranging over the integers. Let $A'_m = C' \backslash B'_m$ for $m = 1, \ldots, n$, and put $A_m = \varphi(A'_m)$, where φ is the Moebius mapping

$$\varphi(z) = (z-i)/(z+i), \quad z \in \mathbb{C}\backslash\{-i\}\,,$$

which sends $\mathbb{U}$ onto $\mathbb{D}$. Note that by the regular nature of the sets A_m, it can in fact be shown that

$$D^+(A_m) = D^-(A_m) = \frac{2(n-1)\pi}{n\gamma \log \beta} < 1/2\,,$$

so that by Seip's Theorem 3.2, A_m is interpolating for $L_a^2(\mathbb{D})$, for each m. Since the sequence $C = \varphi(C')$ is sampling, an argument analogous to the one used in the proof of Proposition 2.2 now shows that the invariant subspaces

$$I_m = \mathscr{I}(A_m), \quad m = 1, \ldots, n,$$

meet condition (5.1) for some constant $\varepsilon > 0$.

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