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# A factorization theorem for square area-integrable analytic functions

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## 0. Introduction

Let $\mathbb{D}$ be the open unit disc in the complex plane $\mathbb{C}$, $\mathbb{T}$ the unit circle, and $L_a^2(\mathbb{D})$ the Bergman space, consisting of those analytic functions on $\mathbb{D}$ that are square integrable on $\mathbb{D}$ with respect to area measure. The Bergman space is a closed subspace of the Hilbert space $L^2(\mathbb{D})$ of all square integrable complex-valued functions on $\mathbb{D}$. The inner product in $L^2(\mathbb{D})$, and hence in $L_a^2(\mathbb{D})$, is given by the formula

$$\langle f, g\rangle_{L^2} = \frac{1}{\pi} \int_{\mathbb{D}} f(z)\bar{g}(z)\,dA(z), \quad f, g \in L^2(\mathbb{D}),$$

where $dA$ denotes planar area measure. The associated norm is denoted by $\|\cdot\|_{L^2}$. Suppose $\boldsymbol{a} = \{a_j\}_1^N$ is a finite sequence of points in $\mathbb{D}\setminus\{0\}$, and let

$$B_{\boldsymbol{a}}(z) = \prod_{j=1}^{N} \frac{\bar{a}_j}{|a_j|} \frac{a_j - z}{1 - \bar{a}_j z}, \quad z \in \mathbb{D},$$

be the Blaschke product associated with $\boldsymbol{a}$. Usually, we will find it convenient to think of $\boldsymbol{a}$ as a set of points lying in the disk $\mathbb{D}$, although strictly speaking it is not, but at times, it will suit us better to think of it as an $N$-tuple lying in the polydisk $\mathbb{D}^N$. This will hopefully not lead to any confusion. Suppose for the moment that $f$ is an analytic function on $\mathbb{D}$ which vanishes on $\boldsymbol{a}$, by which we mean that we count multiplicities should the same number occur more than once in the sequence $\boldsymbol{a}$. If $f$ belongs to the Hardy space $H^2(\mathbb{D})$, $f/B_{\boldsymbol{a}} \in H^2(\mathbb{D})$, and $\|f/B_{\boldsymbol{a}}\|_{H^2} = \|f\|_{H^2}$. However, if $f \in L_a^2(\mathbb{D})$, it is still true that $f/B_{\boldsymbol{a}} \in L_a^2(\mathbb{D})$, but $\|f/B_{\boldsymbol{a}}\|_{L^2} > \|f\|_{L^2}$ unless $\boldsymbol{a}$ is empty or $f$ vanishes entirely. This indicates that Blaschke products are unsuitable as divisors for the Bergman space. In $H^2(\mathbb{D})$, $B_{\boldsymbol{a}}$ appears as the unique extremal function for the problem

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$$\sup\{\operatorname{Re} f(0): f \in H^2(\mathbb{D}), f = 0 \text{ on } \boldsymbol{a}, \|f\|_{H^2} \leqq 1\}.$$

This suggests that we should study the extremal functions for the problem

$$\sup\{\operatorname{Re} f(0): f \in L^2_a(\mathbb{D}), f = 0 \text{ on } \boldsymbol{a}, \|f\|_{L^2} \leqq 1\}. \tag{0.1}$$

Consider for a moment the related extremal problem

$$\inf\{\|f\|_{L^2}: f \in L^2_a(\mathbb{D}), f = 0 \text{ on } \boldsymbol{a}, \operatorname{Re} f(0) = 1\},$$

which has a unique extremal function, by standard Hilbert space theory (see, for instance, [10], p. 83). Since these two problems are essentially the same, we realize that this implies that (0.1) has a unique extremal function, which we will call $G_{\boldsymbol{a}}$. The main result of this paper is that $G_{\boldsymbol{a}}$ is a contractive divisor on the Bergman space, that is, if $f \in L^2_a(\mathbb{D})$ vanishes on $\boldsymbol{a}$, we have $f/G_{\boldsymbol{a}} \in L^2_a(\mathbb{D})$, and $\|f/G_{\boldsymbol{a}}\|_{L^2} \leqq \|f\|_{L^2}$. In case all $a_j$ are distinct, it turns out that $G_{\boldsymbol{a}}$ is a finite linear combination of the functions $1, (1-\bar{a}_1 z)^{-2}, \ldots, (1-\bar{a}_N z)^{-2}$, and as such it is uniquely determined by the conditions $G_{\boldsymbol{a}}(a_j) = 0$ for $j = 1, \ldots, N$, $G_{\boldsymbol{a}}(0) > 0$, and $\|G_{\boldsymbol{a}}\|_{L^2} = 1$. Suppose $\boldsymbol{b} = \{b_j\}_1^\infty$ is an infinite sequence of points in $\mathbb{D}\setminus\{0\}$, and put $\boldsymbol{b}_N = \{b_j\}_1^N$. The functions $G_{\boldsymbol{b}_N}$ converge as $N \to \infty$ to a function $G_{\boldsymbol{b}} \in L^2_a(\mathbb{D})$, which vanishes on $\boldsymbol{b}$, and $\|G_{\boldsymbol{b}}\|_{L^2}$ is either 0 or 1. If $\|G_{\boldsymbol{b}}\|_{L^2} = 0$, $G_{\boldsymbol{b}}$ vanishes identically, and $\boldsymbol{b}$ is not the zero set of an $L^2_a(\mathbb{D})$ function. If $\|G_{\boldsymbol{b}}\|_{L^2} = 1$, $G_{\boldsymbol{b}}$ is the unique extremal function for the problem

$$\sup\{\operatorname{Re} f(0): f \in L^2_a(\mathbb{D}), f = 0 \text{ on } \boldsymbol{b}, \|f\|_{L^2} \leqq 1\},$$

$G_{\boldsymbol{b}}$ vanishes precisely on $\boldsymbol{b}$, and if $f \in \mathrm{L}^2_a(\mathbb{D})$ vanishes on $\boldsymbol{b}$, we have $f/G_{\boldsymbol{b}} \in L^2_a(\mathbb{D})$ and $\|f/G_{\boldsymbol{b}}\|_{L^2} \leqq \|f\|_{L^2}$. In other words, every function $f \in \mathrm{L}^2_a(\mathbb{D})$ *which vanishes on* $\boldsymbol{b}$ *has the form* $f = G_{\boldsymbol{b}} \cdot g$, *where* $g \in \mathrm{L}^2_a(\mathbb{D})$, *and* $\|g\|_{L^2} \leqq \|f\|_{L^2}$. It follows that the subspace

$$\mathscr{I}(\boldsymbol{b}) = \{f \in L^2_a \mathbb{D}): f = 0 \text{ on } \boldsymbol{b}\}$$

has the form

$$\mathscr{I}(\boldsymbol{b}) = L^2_a(\mathbb{D}) \cap G_{\boldsymbol{b}} L^2_a(\mathbb{D}).$$

It turns out that $G_{\boldsymbol{b}}$ is unique, up to unimodular constant multiples, among all functions $G \in \mathscr{I}(\boldsymbol{b})$ with norm $\leqq 1$ admitting the above factorization. One may wonder for which $f \in L^2_a(\mathbb{D})$ we have $G_{\boldsymbol{b}} f \in L^2_a(\mathbb{D})$. We shall see that this holds for all $f \in H^2(\mathbb{D})$; in fact, $G_{\boldsymbol{b}}$ is contractive as a multiplier $H^2(\mathbb{D}) \to L^2_a(\mathbb{D})$, that is,

$$\|G_{\boldsymbol{b}} f\|_{L^2} \leqq \|f\|_{H^2}, \quad f \in H^2(\mathbb{D}).$$

It follows from [13], p. 232, that $G_{\boldsymbol{b}}$ enjoys the estimate

$$|G_{\boldsymbol{b}}(z)| \leqq (1-|z|^2)^{-1/2}, \quad z \in \mathbb{D},$$

so that $G_{\boldsymbol{b}}$ is somewhat better than an arbitrary Bergman space function. This is analogous to the Hardy space situation, where the extremal functions are Blaschke products, which are uniformly bounded on $\mathbb{D}$, and hence multipliers $H^2(\mathbb{D}) \to H^2(\mathbb{D})$. The functions $G_b$ have appeared previously in the literature; they are solutions to certain Akutowicz-Carleson minimum interpolation problems, see [2], [11]. Charles Horowitz [5], [6] and Boris Korenblum [7] have constructed divisors for Bergman space functions, but in general, these do not lie in the Bergman space themselves.

Korenblum [8] has introduced a domination concept for analytic functions on the unit disk. If $g, h \in L^2_a(\mathbb{D})$, we say that $g \prec h$ if

$$\int_{\mathbb{D}} |gf|^2 \, dA \leqq \int_{\mathbb{D}} |hf|^2 \, dA$$

for all (analytic) polynomials $f$. In particular, $h \succ 1$ if $\|hf\|_{L^2} \geqq \|f\|_{L^2}$ for all polynomials $f$.

An invariant subspace of $L^2_a(\mathbb{D})$ is a closed subspace which is invariant under multiplication by the coordinate function $z$. If $I$ is an invariant subspace, and $0 \notin Z(I)$, where

$$Z(I) = \{z \in \mathbb{D}: f(z) = 0 \text{ for all } f \in I\},$$

let $G_I$ be the unique extremal function for the problem

$$\sup \{\operatorname{Re} f(0): f \in I, \|f\|_{L^2} \leqq 1\}.$$

If $\boldsymbol{b}$ is an infinite sequence as above, and $\mathcal{I}(\boldsymbol{b}) = \{f \in L^2_a(\mathbb{D}): f = 0 \text{ on } \boldsymbol{b}\}$ its associated invariant subspace, then $G_{\mathcal{I}(\boldsymbol{b})} = G_{\boldsymbol{b}}$. We will show that for an invariant subspace $I$, $G_I$ is a contractive divisor on the invariant subspace $I(G_I)$ generated by $G_I$, that is, $\|f/G_I\|_{L^2} \leqq \|f\|_{L^2}$ for all $f \in I(G_I)$. In other words, $G_I \succ 1$. On the other hand, $G_I$ is a contractive multiplier $H^2(\mathbb{D}) \to L^2_a(\mathbb{D})$. The functions $G_I$ correspond to inner functions in the Hardy space case. They are described intrinsically as follows: a function $G \in L^2_a(\mathbb{D})$ is a $G_I$ for some invariant subspace $I$ of $L^2_a(\mathbb{D})$ with $0 \notin Z(I)$ if and only if $G(0) > 0$, $\|G\|_{L^2} = 1$, and $\|Gf\|_{L^2} \geqq |f(0)|$ for all polynomials $f$.

Our main tool for proving the above results is a technical result, which is stated first in Theorem 2.4, and later, in a more general setting, in Theorem 4.1. The reason why we first prove the result in a simpler situation is that we hope that this disposition will make the proof more accessible to the reader.

This paper was circulated in preprint form in the spring of 1990. Shortly afterwards, Harold Shapiro [12], Peter Duren, and Dmitry Khavinson found a way to prove my factorization result, based on the fact that the biharmonic Green's function for the unit disk is positive, which had the additional benefit to be applicable to the spaces $L^p_a(\mathbb{D}, dA)$, for $1 < \infty$; this will eventually be published jointly [3]. When $p$ tends to infinity, the extremal functions for finite zero sets converge to Blaschke products, and this is as it should be, because in the limit one would like to have the well-known factorization theory of the space $H^\infty(\mathbb{D})$.

## 1. Preliminaries

Let $H^\infty(\mathbb{D})$ denote the space of bounded analytic functions on $\mathbb{D}$, supplied with the uniform norm. Also, let $I(f)$ denote the invariant subspace generated by $f \in L_a^2(\mathbb{D})$ in $L_a^2(\mathbb{D})$.

**Proposition 1.1.** *Let $g, h \in L_a^2(\mathbb{D})$, and suppose $g \prec h$. We then have $(g/h)\phi \in I(g)$ for every $\phi \in I(h)$, and*

$$\|(g/h)\phi\|_{L^2} \leqq \|\phi\|_{L^2}, \quad \phi \in I(h).$$

*In other words, $g/h$ is a contractive multiplier $I(h) \to I(g)$. In particular,*

$$\int_{\mathbb{D}} |gf|^2 \, dA \leqq \int_{\mathbb{D}} |hf|^2 \, dA \tag{1.1}$$

*holds for all $f \in H^\infty(\mathbb{D})$, and if $g, h \in H^\infty(\mathbb{D})$, (1.1) holds for all $f \in L_a^2(\mathbb{D})$.*

*Proof.* We first establish that $g/h$ is a contractive multiplier $I(h) \to I(g)$. To this end, let $\phi \in I(h)$ be arbitrary. There must then be polynomials $p_n$ such that $hp_n \to \phi$ in $L_a^2(\mathbb{D})$ as $n \to \infty$. The estimate

$$\|gp_n - gp_m\|_{L^2} = \|g(p_n - p_m)\|_{L^2} \leqq \|h(p_n - p_m)\|_{L^2} = \|hp_n - hp_m\|_{L^2}$$

shows that the functions $gp_n$ form a Cauchy sequence in $L_a^2(\mathbb{D})$, and hence converge to some function $G \in I(g)$. Since $p_n \to \phi/h$ pointwise on $\mathbb{D}$, $G$ must equal $(g/h)\phi$. Now $\|gp_n\|_{L^2} \leqq \|hp_n\|_{L^2}$ for all $n$, and $\|gp_n\|_{L^2} \to \|(g/h)\phi\|_{L^2}$ and $\|hp_n\|_{L^2} \to \|\phi\|_{L^2}$ as $n \to \infty$, from which we conclude that

$$\|(g/h)\phi\|_{L^2} \leqq \|\phi\|_{L^2}. \tag{1.2}$$

If we put $\phi = hf$, and use the well-known facts that $I(h) \supset h \cdot H^\infty(\mathbb{D})$ in general and $I(h) \supset h \cdot L_a^2(\mathbb{D})$ if $h \in H^\infty(\mathbb{D})$, (1.1) follows from (1.2). The proof is complete.

**Proposition 1.2.** *Suppose $f_j, g_j \in L_a^2(\mathbb{D})$, $f_j \prec g_j$ for all $j$, $g_j \to g$ as $j \to \infty$ in the norm of $L_a^2(\mathbb{D})$, and $f_j \to f$ as $j \to \infty$ uniformly on compact subsets of $\mathbb{D}$. Then $f \prec g$.*

*Proof.* Let $\varphi$ be a polynomial; then

$$\int_{\mathbb{D}} |f_j\varphi|^2 \, dA \leqq \int_{\mathbb{D}} |g_j\varphi|^2 \, dA.$$

Since $g_j \to g$ as $j \to \infty$ in $L_a^2(\mathbb{D})$, the right hand side converges to $\int_{\mathbb{D}} |g\varphi|^2 \, dA$, and by Fatou's lemma,

$$\int_{\mathbb{D}} |f\varphi|^2 \, dA \leqq \liminf_{j \to \infty} \int_{\mathbb{D}} |f_j\varphi|^2 \, dA.$$

We conclude that

$$\int_{\mathbb{D}} |f\varphi|^2 dA \leqq \int |g\varphi|^2 dA ,$$

which completes the proof.

Let $A(\mathbb{D})$ denote the disk algebra, which consists of those analytic functions in $\mathbb{D}$ that extend continuously to the closed disk $\bar{\mathbb{D}}$,

**Proposition 1.3.** *Let $f \in A(\mathbb{D})$, and suppose $f \succ 1$. Then $|f| \geqq 1$ on $\mathbb{T}$.*

*Proof.* Fix a point $z_0 \in \mathbb{T}$, and let $p$ be the analytic function

$$p(z) = (z + z_0)/2, \quad z \in \bar{\mathbb{D}},$$

which peaks at $z_0$. For $n = 1, 2, 3, \ldots$, consider the functions

$$\phi_n(z) = (p(z))^n / \|p^n\|_{L^2}, \quad z \in \bar{\mathbb{D}};$$

they have $\|\phi_n\|_{L^2} = 1$ and converge to 0 uniformly on compact subsets of $\bar{\mathbb{D}} \setminus \{z_0\}$ as $n \to \infty$. Since $f$ is continuous at $z_0$,

$$\int_{\mathbb{D}} |f(z)\phi_n(z)|^2 dA(z)/\pi \to |f(z_0)|^2$$

as $n \to \infty$. On the other hand, we have

$$\int_{\mathbb{D}} |f(z)\phi_n(z)|^2 dA(z)/\pi \geqq \int_{\mathbb{D}} |\phi_n(z)|^2 dA(z)/\pi = 1$$

because $f \succ 1$. We conclude that $|f(z_0)| \geqq 1$, and the assertion follows.

**Remark.** Using a more sophisticated peaking function, one can show that if $f, g \in H^\infty(\mathbb{D})$ and $f \prec g$, then $|f| \leqq |g|$ almost everywhere on $\mathbb{T}$.

## 2. Extremal functions for finite zero sets

We shall now try to compute the extremal function $G_{\boldsymbol{a}}$ for the problem

$$\sup\{\operatorname{Re} f(0) : f \in \mathscr{I}(\boldsymbol{a}), \|f\|_{L^2} \leqq 1\},$$

where $\mathscr{I}(\boldsymbol{a}) = \{f \in L^2_a(\mathbb{D}) : f = 0 \text{ on } \boldsymbol{a}\}$ and $\boldsymbol{a} = \{a_j\}_{j=1}^N$ is a finite sequence of points in $\mathbb{D} \setminus \{0\}$. If $J$ is a subspace of $L^2_a(\mathbb{D})$, its annihilator is the set

$$J^\perp = \{f \in L^2_a(\mathbb{D}) : \langle f, g \rangle_{L^2} = 0 \text{ for all } g \in J\}.$$

We shall have need for the following general result.

**Proposition 2.1.** *Let $I$ be an invariant subspace of $L_a^2(\mathbb{D})$ with $0 \notin Z(I)$, and let $G_I$ be the extremal function for the problem*

$$\sup\{\operatorname{Re} f(0) : f \in I, \|f\|_{L^2} \leqq 1\}.$$

*Write $\mathscr{I}_0 = \{f \in L_a^2(\mathbb{D}) : f(0) = 0\}$. Then $G_I \in I \cap (I \cap \mathscr{I}_0)^\perp$, and $I \cap (I \cap_0)^\perp$ is a one-dimensional subspace of $L_a^2(\mathbb{D})$. Thus $G_I$ is the unique member of this subspace which has $\|G_I\|_{L^2} = 1$ and $G_I(0) > 0$.*

*Proof.* A moment's thought reveals that $\|G_I\|_{L^2} = 1$ and $G_I(0) > 0$ hold. For $\varepsilon \in \mathbb{C}$ and $h \in I \cap \mathscr{I}_0$, introduce the functions

$$F_\varepsilon = \frac{G_I + \varepsilon h}{\|G_I + \varepsilon h\|_{L^2}},$$

and observe that $F_\varepsilon \in I$, $\|F_\varepsilon\|_{L^2} = 1$, and

$$F_\varepsilon(0) = G_I(0)/\|G_I + \varepsilon h\|_{L^2}.$$

Since $G_I$ is extremal, we must have $F_\varepsilon(0) \leqq G_I(0)$, and consequently, $\|G_I + \varepsilon h\|_{L^2} \geqq 1$ for all $\varepsilon \in \mathbb{C}$. Now

$$\|G_I + \varepsilon h\|_{L^2}^2 = \|G_I\|_{L^2}^2 + |\varepsilon|^2 \|h\|_{L^2}^2 + 2\operatorname{Re}\langle G_I, \varepsilon h\rangle_{L^2},$$

so by varying $\varepsilon$ and observing that $\|G_I\|_{L^2} = 1$, we obtain $\langle G_I, h\rangle_{L^2} = 0$, and the assertion $G_I \in I \cap (I \cap \mathscr{I}_0)^\perp$ follows.

We shall now demonstrate that $I \cap (I \cap \mathscr{I}_0)^\perp$ is one-dimensional. Since $0 \notin Z(I)$, we can find a $\psi \in I$ with $\psi(0) = 1$. If $Q: L_a^2(\mathbb{D}) \to I \cap \mathscr{I}_0$ is the orthogonal projection, then $\varphi = \psi - Q\psi \in I \cap (I \cap \mathscr{I}_0)^\perp$, and $\varphi(0) = 1$. Now if $f \in I \cap (I \cap \mathscr{I}_0)^\perp$ is arbitrary we have

$$f - f(0)\varphi \in (I \cap \mathscr{I}_0) \cap (I \cap \mathscr{I}_0)^\perp = \{0\},$$

so that $f = f(0)\varphi$. We conclude that $I \cap (I \cap \mathscr{I}_0)^\perp$ is spanned by the vector $\varphi$, which completes the proof.

**Remark.** It is an immediate consequence of Proposition 2.1 that $G_I$ equals a constant multiple of the orthogonal projection of 1 onto $I$. For if $f \in I$, $f(0) = 0$, and $P$ is the orthogonal projection onto $I$, then

$$\langle P1, f\rangle_{L^2} = \langle 1, Pf\rangle_{L^2} = \langle 1, f\rangle_{L^2} = \int_{\mathbb{D}} f(z)\,dA(z)/\pi = f(0) = 0,$$

so that $P1$ belongs to $I \cap (I \cap \mathscr{I}_0)^\perp$. Also, $P1 \neq 0$ because 1 does not belong to $I^\perp$.

For $\alpha \in \mathbb{D}$, the annihilator of the invariant subspace $\{f \in L_a^2(\mathbb{D}) : f(\alpha) = 0\}$ is the linear space spanned by the kernel function

$$k_\alpha(z) = (1 - \bar{\alpha}z)^{-2}, \quad z \in \mathbb{D}.$$

From this it is evident that if the points $a_1, \ldots, a_N$ are all distinct, the annihilator $(\mathscr{I}(\boldsymbol{a}) \cap \mathscr{I}_0)^\perp$ is the $(N+1)$-dimensional subspace of $L_a^2(\mathbb{D})$ spanned by the vectors 1, $k_{a_1}, \ldots, k_{a_N}$, so that in this case, Proposition 2.1 takes the following form.

**Corollary 2.2.** *Suppose $\boldsymbol{a} = \{a_j\}_{j=1}^N$ is a finite sequence of distinct points in $\mathbb{D} \setminus \{0\}$, and that $G_{\boldsymbol{a}}$ and $k_\alpha$ are as above. Then $G_{\boldsymbol{a}} \in L_a^2(\mathbb{D})$ is uniquely determined by the following conditions*:

(a) *$G_{\boldsymbol{a}}$ vanishes on $\boldsymbol{a}$,*

(b) *$G_{\boldsymbol{a}}$ is a linear combination of the functions $1, k_{a_1}, \ldots, k_{a_N}$,*

(c) $\|G_{\boldsymbol{a}}\|_{L^2} = 1$,

(d) $G_{\boldsymbol{a}}(0) > 0$.

**Remark.** If the points $a_1, \ldots, a_N$ are not distinct, the situation is a little more complicated. Say, for instance, that the point $\alpha \in \mathbb{D}$ occurs $k$ times in $\boldsymbol{a}$. Then the annihilator $\mathscr{I}(\boldsymbol{a})^\perp$ contains the functions $(1-\bar{\alpha}z)^{-2}, z(1-\bar{\alpha}z)^{-3}, \ldots, z^{k-1}(1-\bar{\alpha}z)^{-k-1}$. If we do this for each point in $Z(\mathscr{I}(\boldsymbol{a}))$, we get a basis for $\mathscr{I}(\boldsymbol{a})^\perp$. If we add the function 1 to this basis afterwards, we get a basis for $(\mathscr{I}(\boldsymbol{a}) \cap \mathscr{I}_0)^\perp$.

For an open set $\Omega \subset \mathbb{C}$, let $\mathscr{O}(\Omega)$ denote the Fréchet space of holomorphic functions on $\Omega$, supplied with the topology of uniform convergence on compact subsets. If $K \subset \mathbb{C}$ is compact, let $\mathscr{O}(K)$ be the space of germs of functions holomorphic on neighborhoods of $K$, supplied with its ordinary inductive limit topology; if $f \in \mathscr{O}(K)$, we say that $f$ is analytic on $K$. For instance, $f \in \mathscr{O}(\bar{\mathbb{D}})$ if and only if it has an analytic extension to some neighborhood of $\mathbb{D}$, and if $f_j$ is a sequence of functions in $\mathscr{O}(\bar{\mathbb{D}})$, $f_j \to f$ in $\mathscr{O}(\bar{\mathbb{D}})$ means that $f_j$ converges to $f$ uniformly on some region $r\mathbb{D} = \{z \in \mathbb{C}: |z| < r\}$ with $r > 1$. We need to understand the continuity aspects of the mapping $\boldsymbol{a} \mapsto G_{\boldsymbol{a}}$.

**Proposition 2.3.** *For each $\boldsymbol{a} = \{a_j\}_{j=1}^N \in (\mathbb{D} \setminus \{0\})^N$, the function $G_{\boldsymbol{a}}$ is analytic on the set $\mathbb{C} \cup \{\infty\} \setminus \bigcup_{j=1}^N \{1/\bar{a}_j\}$, and, moreover, the mapping $\boldsymbol{a} = \{a_j\}_{j=1}^N \mapsto G_{\boldsymbol{a}}$ is continuous $(\mathbb{D} \setminus \{0\})^N \to \mathscr{O}(\bar{\mathbb{D}})$.*

*Proof.* From Corollary 2.2 and the remark thereafter, we realize that $G_{\boldsymbol{a}}$ is analytic on $\mathbb{C} \cup \{\infty\} \setminus \bigcup_{j=1}^N \{1/\bar{a}_j\}$.

If the points in $\boldsymbol{a}$ are distinct, $\mathscr{I}(\boldsymbol{a})^\perp$ is spanned by the kernel functions $k_{a_1}, \ldots, k_{a_N}$. We need another set of spanning vectors which works for all $\boldsymbol{a} \in (\mathbb{D} \setminus \{0\})^N$. For $j = 1, \ldots, N$, put

$$K_j^1(z, \boldsymbol{a}) = k_{a_j}(z),$$

and define inductively

$$K_j^{n+1}(z, \boldsymbol{a}) = \big(K_j^n(z, \boldsymbol{a}) - K_n^n(z, \boldsymbol{a})\big) / (\bar{a}_j - \bar{a}_n) \tag{2.1}$$

for $j \geqq n+1$. By the construction, the function $\boldsymbol{a} \mapsto K_j^n(z, \boldsymbol{a})$, which is defined for $j \geqq n$, is an antiholomorphic rational function which is antiholomorphic in the region $(\mathbb{C} \setminus \{1/\bar{z}\})^N$; the division in (2.1) produces no additional singularities because $K_j^n(z, \boldsymbol{a}) = K_n^n(z, \boldsymbol{a})$ for a $j \geqq n+1$ if $a_j = a_n$. Also, observe that for $j \geqq n$, $K_j^n(z, \boldsymbol{a})$ is independent of the variables $a_{j+1}, \ldots, a_N$. If we look at the two variables $z$ and $\boldsymbol{a}$ jointly, we see that with the notation $\boldsymbol{a}^* = \{\bar{a}_j\}_{j=1}^N$, the mapping $(z, \boldsymbol{a}) \mapsto K_j^n(z, \boldsymbol{a}^*)$ is holomorphic in the region

$$\mathbb{C}^{N+1} \setminus \{(z, \boldsymbol{a}) \in \mathbb{C}^{N+1} \colon z a_k = 1 \text{ for some } 1 \leqq k \leqq j\}$$

for $j \geqq n$. It follows that for $j \geqq n$, $\boldsymbol{a} \mapsto K_j^n(\cdot, \boldsymbol{a})$ is a continuous mapping $\mathbb{D}^N \to \mathcal{O}(\bar{\mathbb{D}})$.

*Claim.* The functions $K_1^1(z, \boldsymbol{a}), K_2^2(z, \boldsymbol{a}), \ldots, K_N^N(z, \boldsymbol{a})$ form a basis for $\mathscr{I}(\boldsymbol{a})^\perp$.

First, observe that by the remark following Corollary 2.2, the dimension of $\mathscr{I}(\boldsymbol{a})^\perp$ is $N$, so at least the number of spanning vectors is correct. By the construction of $K_j^n(z, \boldsymbol{a})$,

$$K_j^1(z, \boldsymbol{a}) = \sum_{m=1}^{j} K_m^m(z, \boldsymbol{a}) \cdot \sum_{n=1}^{m-1} (\bar{a}_j - \bar{a}_n) .$$

Now suppose the point $a_j$ is repeated $l$ times in the sequence $\boldsymbol{a}$, where $l \geqq 1$. Since $K_m^m(z, \boldsymbol{a})$ is independent of the variable $a_j$ if $m < j$, we have

$$\begin{aligned} \frac{\partial}{\partial \bar{a}_j} K_j^1(z, \boldsymbol{a}) &= \sum_{m=1}^{j} K_m^m(z, \boldsymbol{a}) \cdot \frac{\partial}{\partial \bar{a}_j} \prod_{n=1}^{m-1} (\bar{a}_j - \bar{a}_n) + \frac{\partial}{\partial \bar{a}_j} K_j^j(z, \boldsymbol{a}) \cdot \sum_{n=1}^{m-1} (\bar{a}_j - \bar{a}_n) \\ &= \sum_{m=1}^{j} K_m^m(z, \boldsymbol{a}) \cdot \frac{\partial}{\partial \bar{a}_j} \prod_{n=1}^{m-1} (\bar{a}_j - \bar{a}_n) \end{aligned}$$

if $l \geqq 2$, so that $\partial / \partial \bar{a}_j K_j^j(z, \boldsymbol{a})$ belongs to the linear span of the vectors $K_1^1(z, \boldsymbol{a}), \ldots, K_N^N(z, \boldsymbol{a})$. If we continue like this, we see that $(\partial^n / \partial \bar{a}_j^n) K_j^1(z, \boldsymbol{a})$ belong to the linear span of the vectors $K_1^1(z, \boldsymbol{a}), \ldots, K_N^N(z, \boldsymbol{a})$ for all $n \leqq l-1$. However, we have

$$\frac{\partial^n}{\partial \bar{a}_j^n} K_j^1(z, \boldsymbol{a}) = (n+1)!\, z^n (1 - \bar{a}_j z)^{-n-2} ,$$

so by the remark following Corollary 2.2, all the spanning vectors of $\mathscr{I}(\boldsymbol{a})^\perp$ belong to the linear span of $K_1^1(z, \boldsymbol{a}), \ldots, K_N^N(z, \boldsymbol{a})$; on the other hand, $\mathscr{I}(\boldsymbol{a})^\perp$ has dimension $N$, and the possibly bigger linear span of $K_1^1(z, \boldsymbol{a}), \ldots, K_N^N(z, \boldsymbol{a})$ has dimension $\leqq N$, so these spaces must coincide. The claim has been verified.

It follows that $1, K_1^1(z, \boldsymbol{a}), K_2^2(z, \boldsymbol{a}), \ldots, K_N^N(z, \boldsymbol{a})$ form a basis for $(\mathscr{I}(\boldsymbol{a}) \cap \mathscr{I}_0)^\perp$. Let $H_{\boldsymbol{a}} = G_{\boldsymbol{a}} / G_{\boldsymbol{a}}(0)$; then $H_{\boldsymbol{a}}$ has the form

$$H_{\boldsymbol{a}}(z) = \sum_{j=0}^{N} \lambda_j K_j^j(z, \boldsymbol{a}) ,$$

where $K_0^0(z, \boldsymbol{a}) \equiv 1$, and the coefficients $\lambda_j$ are uniquely defined by the conditions $H_{\boldsymbol{a}}(0) = 1$ and

$$\sum_{j=0}^{N} \lambda_j \langle K_j^j(\cdot, \boldsymbol{a}), K_n^n(\cdot, \boldsymbol{a}) \rangle_{L^2} = 0, \quad n = 1, \ldots, N.$$

Let $A(z, \boldsymbol{a})$ be the matrix $A(z, \boldsymbol{a}) = (A_{j,k}(z, \boldsymbol{a}))_{j,k=0}^N$, where

$$A_{j,0}(z, \boldsymbol{a}) = K_j^j(z, \boldsymbol{a}), \quad 0 \leqq j \leqq N,$$

and

$$A_{j,k}(z, \boldsymbol{a}) = \langle K_j^j(\cdot, \boldsymbol{a}), K_k^k(\cdot, \boldsymbol{a}) \rangle_{L^2}, \quad 0 \leqq j \leqq N, \quad 1 \leqq k \leqq N,$$

and let $B(\boldsymbol{a})$ be the matrix $B(\boldsymbol{a}) = (B_{j,k}(\boldsymbol{a}))_{j,k=0}^N$, where

$$B_{j,k}(\boldsymbol{a}) = \langle K_j^j(\cdot, \boldsymbol{a}), K_k^k(\cdot, \boldsymbol{a}) \rangle_{L^2}, \quad 0 \leqq j, k \leqq N.$$

Since the $K_j^j(\cdot, \boldsymbol{a})$ are linearly independent for $j = 0, \ldots, N$, $B(\boldsymbol{a})$ is invertible for all $\boldsymbol{a} \in (\mathbb{D} \setminus \{0\})^N$. One easily checks that

$$H_{\boldsymbol{a}}(z) = \det A(z, \boldsymbol{a}) / \det B(\boldsymbol{a}),$$

where "det" means taking the determinant. This formula, together with the fact that $G_{\boldsymbol{a}} = H_{\boldsymbol{a}} / \| H_{\boldsymbol{a}} \|_{L^2}$, clearly demonstrates the assertion.

To simplify our notation, let us write $\partial$ and $\bar{\partial}$ instead of $\partial / \partial z$ and $\partial / \partial \bar{z}$, respectively; recall that $4 \partial \bar{\partial} = \Delta$, the Laplace operator. We now present the first version of the main technical result of this paper.

**Theorem 2.4.** *Suppose $\boldsymbol{a} = \{a_j\}_1^N$ is a finite sequence of distinct points in $\mathbb{D} \setminus \{0\}$, and let $G_{\boldsymbol{a}}$ be the extremal function for the problem*

$$\sup \{ \operatorname{Re} f(0) : f \in L_a^2(\mathbb{D}), f = 0 \text{ on } \boldsymbol{a}, \| f \|_{L^2} \leqq 1 \}.$$

*There exists a unique function $\Phi \in C^2(\bar{\mathbb{D}})$ such that $\Phi = 0$ on $\mathbb{T}$ and $\partial \bar{\partial} \Phi = |G_{\boldsymbol{a}}|^2 - 1$ on $\mathbb{D}$. This solution $\Phi$ is infinitely differentiable on $\bar{\mathbb{D}}$, and enjoys the properties*

(a) *$\partial \Phi / \partial n = 0$ on $\mathbb{T}$, where $\partial / \partial n$ is differentiation in the outward normal direction, and*

(b) *$0 \leqq \Phi(z) \leqq 1 - |z|^2$ for all $z \in \bar{\mathbb{D}}$.*

**Remark.** We make the assumption that the points in $\boldsymbol{a}$ are distinct for technical reasons only. The assertion remains valid without it, as can be seen from the proof of Theorem 4.1.

**Corollary 2.5.** *For all finite sequences $\boldsymbol{a} = \{a_j\}_1^N$ of points in $\mathbb{D} \setminus \{0\}$, we have $G_{\boldsymbol{a}} \succ 1$, that is,*

$$\int_{\mathbb{D}} |G_{\boldsymbol{a}} f|^2 \, dA \geqq \int_{\mathbb{D}} |f|^2 \, dA$$

*holds for all* $f \in L_a^2(\mathbb{D})$.

*Proof of the corollary.* If $u$ and $v$ are two $C^2$ functions on $\bar{\mathbb{D}}$, Green's formula [4], p. 236, states that

$$\int_{\mathbb{D}} (v\Delta u - u\Delta v)\,dA = \int_{\mathbb{T}} (v(\partial u/\partial n) - u(\partial v/\partial n))\,ds, \tag{2.2}$$

where $ds$ is arc length measure on $\mathbb{T}$. Let $f$ be an arbitrary (analytic) polynomial. First, we assume the points in $\boldsymbol{a}$ to be distinct. If we apply (2.2) to the case $u = \Phi$ and $v = |f|^2$, we get

$$\int_{\mathbb{D}} (|f|^2 \partial\bar{\partial}\Phi - \Phi|f'|^2)\,dA = 0,$$

because both $\Phi$ and $\partial\Phi/\partial n$ vanish on $\mathbb{T}$. We obtain

$$\int_{\mathbb{D}} (|G_{\boldsymbol{a}}|^2 - 1)|f|^2\,dA = \int_{\mathbb{D}} |f|^2 \partial\bar{\partial}\Phi\,dA = \int_{\mathbb{D}} \Phi|f'|^2\,dA \geqq 0,$$

because $\Phi \geqq 0$ on $\bar{\mathbb{D}}$, and so

$$\int_{\mathbb{D}} |G_{\boldsymbol{a}} f|^2\,dA \geqq \int_{\mathbb{D}} |f|^2\,dA \tag{2.3}$$

holds for all polynomials $f$, or in other words, $G_{\boldsymbol{a}} \succ 1$. Those finite sequences $\boldsymbol{a} \in (\mathbb{D}\setminus\{0\})^N$ whose points $a_1, \dots, a_N$ are all distinct, form a dense open subset of $(\mathbb{D}\setminus\{0\})^N$. By Propositions 1.2 and 2.3, we get $G_{\boldsymbol{a}} \succ 1$ for all $\boldsymbol{a} \in (\mathbb{D}\setminus\{0\})^N$. Since $G_{\boldsymbol{a}} \in H^\infty(\mathbb{D})$, Proposition 1.1 extends (2.3) to all $f \in L_a^2(\mathbb{D})$, which completes the proof of the corollary.

*Proof of Theorem* 2.4. The uniqueness of $\Phi$ is obvious, since any harmonic function on $\mathbb{D}$ which extends continuously to $\bar{\mathbb{D}}$ and vanishes on $\mathbb{T}$ must vanish identically. By Corollary 2.2, $G_{\boldsymbol{a}}$ has the form

$$G_{\boldsymbol{a}}(z) = \lambda_0 + \sum_{j=1}^{N} \lambda_j (1 - \bar{a}_j z)^{-2}, \quad z \in \mathbb{D},$$

for some scalars $\lambda_0, \dots, \lambda_N \in \mathbb{C}$, which we can compress to

$$G_{\boldsymbol{a}}(z) = \sum_{j=0}^{N} \lambda_j (1 - \bar{a}_j z)^{-2}, \quad z \in \mathbb{D}$$

by putting $a_0 = 0$. Observe that we have for $n = 0, 1, 2, \dots,$

$$\langle z^n G_{\boldsymbol{a}}, G_{\boldsymbol{a}} \rangle_{L^2} = \langle \sum_{j=0}^{N} \lambda_j z^n (1 - \bar{a}_j z)^{-2}, \sum_{k=0}^{N} \lambda_k (1 - \bar{a}_k z)^{-2} \rangle_{L^2} \tag{2.4}$$

$$= \sum_{j,k=0}^{N} \lambda_j \bar{\lambda}_k \langle z^n (1 - \bar{a}_j z)^{-2}, (1 - \bar{a}_k z)^{-2} \rangle_{L^2} = \sum_{j,k=0}^{N} \lambda_j \bar{\lambda}_k a_k^n (1 - \bar{a}_j a_k)^{-2}.$$

Introduce the smooth function

$$\Psi(z)=\sum_{j,k=0}^{N}\lambda_j\bar{\lambda}_k(1-\bar{a}_j a_k)^{-1}(1-\bar{a}_j z)^{-1}(1-a_k\bar{z})^{-1},\quad z\in \mathbb{C}\setminus\bigcup_{j=1}^{N}\{1/\bar{a}_j\};$$

the formula

$$\begin{aligned}\Psi(z)&=\sum_{j,k=0}^{N}\lambda_j\bar{\lambda}_k\Big(\sum_{n=0}^{\infty}(\bar{a}_j a_k)^n\Big)(1-\bar{a}_j z)^{-1}(1-a_k\bar{z})^{-1}\\&=\sum_{n=0}^{\infty}\sum_{j,k=0}^{N}\lambda_j\bar{\lambda}_k\bar{a}_j^n a_k^n(1-\bar{a}_j z)^{-1}(1-a_k\bar{z})^{-1}\\&=\sum_{n=0}^{\infty}\left|\sum_{j=0}^{N}\lambda_j\bar{a}_j^n(1-\bar{a}_j z)^{-1}\right|^2,\quad z\in \mathbb{C}\setminus\bigcup_{j=1}^{N}\{1/\bar{a}_j\}\end{aligned}$$

shows that $\Psi(z)\geqq 0$ on $\mathbb{C}\setminus\bigcup_{j=1}^{N}\{1/\bar{a}_j\}$. Moreover, $\Psi$ is subharmonic on the region $\mathbb{C}\setminus\bigcup_{j=1}^{N}\{1/\bar{a}_j\}$, as the following computation shows:

$$\begin{aligned}\partial\bar{\partial}\Psi(z)&=\sum_{j,k=0}^{N}\lambda_j\bar{\lambda}_k\bar{a}_j a_k(1-\bar{a}_j a_k)^{-1}(1-\bar{a}_j z)^{-2}(1-a_k\bar{z})^{-2}\\&=\sum_{n=0}^{\infty}\sum_{j,k=0}^{N}\lambda_j\bar{\lambda}_k\bar{a}_j^{n+1}a_k^{n+1}(1-\bar{a}_j z)^{-2}(1-a_k\bar{z})^{-2}\\&=\sum_{n=0}^{\infty}\left|\sum_{j=0}^{N}\lambda_j\bar{a}_j^{n+1}(1-\bar{a}_j z)^{-2}\right|^2,\quad z\in \mathbb{C}\setminus\bigcup_{j=1}^{N}\{1/\bar{a}_j\}.\end{aligned}$$

One easily checks that for $n\geqq 0$,

$$\int_{\mathbb{T}} z^n(1-\bar{a}_j z)^{-1}(1-a_k\bar{z})^{-1}ds(z)/2\pi=a_k^n(1-\bar{a}_j a_k)^{-1},$$

so that by (2.4) we get

$$\hat{\Psi}(-n)=\int_{\mathbb{T}} z^n\Psi(z)ds(z)/2\pi=\sum_{j,k=0}^{N}\lambda_j\bar{\lambda}_k a_k^n(1-\bar{a}_j a_k)^{-2}=\langle z^nG_{\boldsymbol{a}},G_{\boldsymbol{a}}\rangle_{L^2}$$

for all integers $n\geqq 0$. For $n\geqq 1$, $z^nG_{\boldsymbol{a}}\in\mathscr{I}(\boldsymbol{a})\cap\mathscr{I}_0$, so by Proposition 2.1, $G_{\boldsymbol{a}}\perp z^nG_{\boldsymbol{a}}$. We conclude that $\hat{\Psi}(n)=0$ for all $n\leqq -1$, and the function $\Psi$ being real-valued, its positive Fourier coefficients must vanish as well, by the formula $\hat{\Psi}(-n)=\overline{\hat{\Psi}(n)}$. Consequently, $\Psi$ must be constant on $\mathbb{T}$, and this constant must be $\hat{\Psi}(0)=\|G_{\boldsymbol{a}}\|_{L^2}^2=1$. Also, since $\Psi$ is subharmonic, we have $\Psi\leqq 1$ on $\bar{\mathbb{D}}$. The desired function $\Phi$ is now given by the formula

$$\Phi(z)=(1-|z|^2)\big(1-\Psi(z)\big),\quad z\in\bar{\mathbb{D}}.$$

By the formula defining $\Psi$, it is clear that this $\Phi$ is real analytic on $\bar{\mathbb{D}}$, and in particular, infinitely differentiable. Since $\Psi=1$ on $\mathbb{T}$, it is easy to check that this $\Phi$ vanishes along with its normal derivative $\partial\Phi/\partial n$ on $\mathbb{T}$. Also, $0\leqq\Phi(z)\leqq 1-|z|^2$ on $\bar{\mathbb{D}}$ because $0\leqq\Psi\leqq 1$ on $\bar{\mathbb{D}}$.

What remains for us to do is to check that it solves the differential equation $\partial\bar{\partial}\Phi = |G_{\boldsymbol{a}}|^2 - 1$. A computation shows that

$$\partial\bar{\partial}((1-|z|^2)(1-\bar{a}_j z)^{-1}(1-a_k\bar{z})^{-1}) = -(1-\bar{a}_j a_k)(1-\bar{a}_j z)^{-2}(1-a_k\bar{z})^{-2},$$

so that

$$\begin{aligned}\partial\bar{\partial}\Phi(z) &= \partial\bar{\partial}\Big(1-|z|^2 - \sum_{j,k=0}^{N} \lambda_j\bar{\lambda}_k(1-\bar{a}_j a_k)^{-1}(1-|z|^2)(1-\bar{a}_j z)^{-1}(1-a_k\bar{z})^{-1}\Big)\\ &= -1 + \sum_{j,k=0}^{N} \lambda_j\bar{\lambda}_k(1-\bar{a}_j z)^{-2}(1-a_k\bar{z})^{-2}\\ &= -1 + \Big|\sum_{j=0}^{N} \lambda_j(1-\bar{a}_j z)^{-2}\Big|^2 = |G_{\boldsymbol{a}}|^2 - 1, \quad z \in \mathbb{D},\end{aligned}$$

which completes the proof of the theorem.

If a sequence consists of a single point $\beta \in \mathbb{D}\setminus\{0\}$ only, let us write $G_\beta$ instead of $G_{\{\beta\}}$.In the introduction, we promised to prove that for a sequence $\boldsymbol{a} = \{a_j\}_{j=1}^N$ of points in $\mathbb{D}\setminus\{0\}$, $G_{\boldsymbol{a}}$ has no more zeros in $\bar{\mathbb{D}}$, counted with respect to multiplicity, than those in the sequence $\boldsymbol{a}$. In order to do that, we shall have need for the following formula:

$$G_\beta(z) = \frac{|\beta|}{\sqrt{2-|\beta|^2}} \frac{(1-z/\beta)(2-|\beta|^2-\bar{\beta}z)}{(1-\bar{\beta}z)^2}, \quad z \in \mathbb{D}.$$

One easily verifies that this $G_\beta$ is a linear combination of the functions 1 and $k_\beta$, has norm 1, and satisfies $G_\beta(0) > 0$, as prescribed by Corollary 2.2. Observe that $G_\beta$ only vanishes at $\beta$ inside $\bar{\mathbb{D}}$.

**Theorem 2.6.** *Let $\boldsymbol{a} = \{a_j\}_{j=1}^N$ be a finite sequence of points in $\mathbb{D}\setminus\{0\}$. Then the extremal function $G_{\boldsymbol{a}}$, which is analytic in a neighborhood of $\bar{\mathbb{D}}$, vanishes precisely on $\boldsymbol{a}$ in $\bar{\mathbb{D}}$, counting multiplicities. Moreover, $|G_{\boldsymbol{a}}| \geqq 1$ on $\mathbb{T}$.*

*Proof.* Suppose first that $G_{\boldsymbol{a}}$ vanishes at the point $\beta \in \mathbb{D}$ at a higher multiplicity than what is prescribed by the sequence $\boldsymbol{a}$; $\beta$ cannot be 0, because $G_{\boldsymbol{a}}(0) > 0$. Consider the function

$$\tilde{G}_{\boldsymbol{a}}(z) = G_{\boldsymbol{a}}(z)/G_\beta(z), \quad z \in \bar{\mathbb{D}},$$

where $G_\beta$ is as above. The function $\tilde{G}_{\boldsymbol{a}}$ belongs to $L_a^2(\mathbb{D})$, vanishes on $\boldsymbol{a}$, has

$$\tilde{G}_{\boldsymbol{a}}(0) = G_{\boldsymbol{a}}(0)/G_\beta(0) = G_{\boldsymbol{a}}(0)/(|\beta|\sqrt{2-|\beta|^2}) > G_{\boldsymbol{a}}(0),$$

and since by Corollary 2.5, multiplication by $G_\beta$ increases the norm, we have $\|\tilde{G}_{\boldsymbol{a}}\|_{L^2} \leqq \|G_{\boldsymbol{a}}\|_{L^2} = 1$. These properties contradict the extremality of $G_{\boldsymbol{a}}$, so we conclude that $G_{\boldsymbol{a}}$ cannot have any zeros in $\mathbb{D}$ other than the points in $\boldsymbol{a}$. By Proposition 1.3, $|G_{\boldsymbol{a}}| \geqq 1$ on $\mathbb{T}$, so $G_{\boldsymbol{a}}$ cannot have any zeros on $\mathbb{T}$ either. The proof is complete.

**Corollary 2.7.** *Let $\boldsymbol{a} = \{a_j\}_{j=1}^N$ be a finite sequence of points in $\mathbb{D}\setminus\{0\}$, and put*

$$\mathscr{I}(\boldsymbol{a}) = \{f \in L_a^2(\mathbb{D}) : f = 0 \text{ on } \boldsymbol{a}\}.$$

*Then the extremal function $G_{\boldsymbol{a}}$ is a contractive divisor $\mathscr{I}(\boldsymbol{a}) \to L_a^2(\mathbb{D})$, that is,*

$$\|f/G_{\boldsymbol{a}}\|_{L^2} \leqq \|f\|_{L^2}, \quad f \in \mathscr{I}(\boldsymbol{a}).$$

*In other words, every $f \in \mathscr{I}(\boldsymbol{a})$ has a factorization $f = G_{\boldsymbol{a}} \cdot g$, where $g \in L_a^2(\mathbb{D})$ has $\|g\|_{L^2} \leqq \|f\|_{L^2}$.*

*Proof.* If $f \in \mathscr{I}(\boldsymbol{a})$, $f/G_{\boldsymbol{a}}$ is holomorphic on $\mathbb{D}$, by Theorem 2.6, and it must also be square integrable on $\mathbb{D}$, because $f$ is, and $G_{\boldsymbol{a}}$ is bounded away from zero near the circle $\mathbb{T}$. The rest follows from Corollary 2.5.

## 3. Extremal functions for infinite zero sets

In what follows, $\boldsymbol{b} = \{b_j\}_{j=1}^\infty$ is an infinite sequence of points in $\mathbb{D}\setminus\{0\}$, and $\mathrm{b}_N$ its finite subsequence $\{b_j\}_{j=1}^N$. As in the introduction, $G_{\boldsymbol{b}}$ is the extremal function for the problem

$$\sup\{\operatorname{Re} f(0) : f \in L_a^2(\mathbb{D}), f = 0 \text{ on } \boldsymbol{b},\ \|f\|_{L^2} \leqq 1\},$$

which is unique by the same argument which was applied to finite sequences in the introduction, or simply by Proposition 2.1. The sequence $\boldsymbol{b}$ is a Bergman space zero sequence if there exists a function $f \in L_a^2(\mathbb{D})$ which vanishes precisely on $\boldsymbol{b}$ in $\mathbb{D}$ (counting multiplicities). If there is a function $f \in L_a^2(\mathbb{D})$, other than 0, which vanishes on $\boldsymbol{b}$, we say that $\boldsymbol{b}$ is a subsequence of a Bergman space zero sequence; observe that in that case, $G_{\boldsymbol{b}}(0) > 0$ and $\|G_{\boldsymbol{b}}\|_{L^2} = 1$. It is a consequence of the following result that every subsequence of a Bergman space zero sequence is in fact itself a Bergman space zero sequence; this fact has been noted earlier by Charles Horowitz [5].

**Proposition 3.1.** *Let $\boldsymbol{b}$ be as above. If $\boldsymbol{b}$ is a subsequence of a Bergman space zero sequence, $G_{\boldsymbol{b}}$ vanishes precisely on $\boldsymbol{b}$ in $\mathbb{D}$, and $G_{\boldsymbol{b}_N} \to G_{\boldsymbol{b}}$ in $L_a^2(\mathbb{D})$ as $N \to \infty$. If $\boldsymbol{b}$ is not a subsequence of any Bergman space zero sequence, $G_{\boldsymbol{b}_N} \to 0$ uniformly on compact subsets of $\mathbb{D}$.*

*Proof.* Let us first deal with the case when $\boldsymbol{b}$ is not a subsequence of a Bergman space zero sequence. Let $G$ be a normal limit to the sequence of $G_{\boldsymbol{b}_N}$'s, which then has $\|G\|_{L^2} \leqq 1$, because $\|G_{\boldsymbol{b}_N}\|_{L^2} = 1$ for all $N$, and since $G$ is analytic and vanishes on $\boldsymbol{b}$, $G$ must vanish identically. If every normal limit is 0, the sequence must converge to 0 uniformly on compact subsets of $\mathbb{D}$.

We now look at the remaining case when $\boldsymbol{b}$ is the zero sequence of a Bergman space function. Again, let $G$ be a normal limit to the sequence of $G_{\boldsymbol{b}_N}$'s, which has $\|G\|_{L^2} \leqq 1$ and $G(0) \geqq G_{\boldsymbol{b}}(0)$, because $G_{\boldsymbol{b}_N} \geqq G_{\boldsymbol{b}}(0)$ for all $N$. By the extremality of $G_{\boldsymbol{b}}$, $G$ must coincide with $G_{\boldsymbol{b}}$, and we obtain that $G_{\boldsymbol{b}_N}$ converges to $G_b$ uniformly on compact subsets of $\mathbb{D}$. It follows that for $r$, $0 < r < 1$, we have

$$\int_{rD} |G_{b_N}|^2 dA \to \int_{rD} |G_b|^2 dA \quad \text{as} \quad N \to \infty ,$$

and since $\|G_{b_N}\|_{L^2} = \|G_b\|_{L^2} = 1$, we get

$$\lim_{N\to\infty} \int_{D\setminus rD} |G_{b_N}|^2 dA \to 0 \quad \text{as} \quad r \to 1 ,$$

from which the assertion $\|G_{b_N} - G_b\|_{L^2} \to 0 \to 0 \to \infty$ easily follows. The fact that $G_b$ vanishes precisely on $b$ in $\mathbb{D}$ follows by duplicating the argument in the proof of Theorem 2.6, or by using the fact that each $G_{b_N}$ vanishes precisely on $b_N$, and that in the limit no extra zeros can appear unless the limit function collapses.

We are now ready to present the general factorization theorem for Bergman space functions.

**Theorem 3.2.** *Let $b = \{b_j\}_{j=1}^{\infty} \subset \mathbb{D}\setminus\{0\}$ be a Bergman space zero sequence, and suppose $f \in L_a^2(\mathbb{D})$ vanishes on $b$. Then $f = G_b \cdot g$, where $g \in L_a^2(\mathbb{D})$ and $\|g\|_{L^2} \leqq \|f\|_{L^2}$.*

*Proof.* Let $g = f/G_b$, which is analytic on the disk $\mathbb{D}$, by Proposition 3.1, and let $b_N$ be the cutoff sequence $\{b_j\}_1^N$, as above. By Corollary 2.7, $f$ has a factorization $f = G_{b_N} \cdot g_N$, where $g_N \in L_a^2(\mathbb{D})$ has $\|g_N\|_{L^2} \leqq \|f\|_{L^2}$. By Proposition 3.1, we must have $g_N \to g$ uniformly on compact subsets of $\mathbb{D}$ as $N \to \infty$, so by Fatou's lemma,

$$\|g\|_{L^2} \leqq \liminf_{N\to\infty} \|g_N\|_{L^2} \leqq \|f\|_{L^2} .$$

The proof is complete.

The following result emphasizes the uniqueness of the function $G_b$ in the formulation of Theorem 3.2.

**Theorem 3.3.** *Let $b = \{b_j\}_{j=1}^{\infty} \subset \mathbb{D}\setminus\{0\}$ be a Bergman space zero sequence, and suppose $G \in L_a^2(\mathbb{D})$ vanishes on $b$, $G(0) \geqq 0$, $\|G\|_{L^2} \leqq 1$, and that for every $f \in L_a^2(\mathbb{D})$ that vanishes on $b$, $G$ allows to be factored $f = G \cdot g$, where $g \in L_a^2(\mathbb{D})$ has $\|g\|_{L^2} \leqq \|f\|_{L^2}$. Then $G = G_b$.*

*Proof.* The function $G_b \in L_a^2(\mathbb{D})$ vanishes on b, so it must have a factorization $G_b = G \cdot g$, where $g \in L_a^2(\mathbb{D})$ has $\|g\|_{L^2} \leqq \|G_b\|_{L^2} = 1$. Since $G_b(0) > 0$, we must have $G(0) > 0$ and therefore $g(0) > 0$. The estimate $|g(0)|^2 \leqq \|g\|_{L^2}^2 \leqq 1$ shows that $G(0) \geqq G_b(0)$. The extremality of $G_b$ now forces $G$ to coincide with $G_b$.

Inner functions, which are the extremal functions in the Hardy space case, and in particular Blaschke products, have analytic pseudo-extensions to the region outside the closed unit disk, with the exception of a countable number of isolated poles there [4], pp. 75–76. It would be of interest to know if anything similar can be said about our extremal functions $G_I$. If $a = \{a_j\}_1^N$ is a finite sequence of points in $\mathbb{D}\setminus\{0\}$, and $G_a$ is the corresponding extremal function, let

$$H_a(z) = \int_0^z G_a(\zeta)d\zeta\,, \quad z \in \mathbb{D}\,.$$

It is a consequence of Corollary 2.2 and the remark thereafter that $H_a$ extends to an analytic function on $\mathbb{C} \cup \{\infty\} \setminus \bigcup_{j=1}^{N} \{1/\bar{a}_j\}$. Outside the closed disk $\bar{\mathbb{D}}$, the extension is given by the formula

$$(3.1) \qquad H_a(z) = \frac{z}{\bar{G}_a(1/\bar{z})} \langle G_a, w \mapsto \frac{G_a(1/\bar{z}) - \bar{z}wG_a(w)}{1 - \bar{z}w} \rangle_{L^2}, \quad z \in \mathbb{C} \setminus \bar{\mathbb{D}} \setminus \bigcup_{j=1}^{N} \{1/\bar{a}_j\}\,.$$

If $I$ is a closed invariant subspace of $L_a^2(\mathbb{D})$ with $0 \notin Z(I)$ and $G_I$ is the corresponding extremal function, let

$$H_I(z) = \int_0^z G_I(\zeta)d\zeta, \quad z \in \mathbb{D}\,.$$

The formula

$$(3.2) \qquad H_I(z) = \frac{z}{\bar{G}_I(1/\bar{z})} \langle G_I, w \mapsto \frac{G_I(1/\bar{z}) - \bar{z}wG_I(w)}{1 - \bar{z}w} \rangle_{L^2}$$

defines an analytic function on $\mathbb{C} \setminus \bar{\mathbb{D}} \setminus Z_*(G_I)$, where $Z_*(G_I) = \{1/\bar{z} \colon z \in \mathbb{D}, G_I(z) = 0\}$.

**Question 3.4.** In what sense does (3.2) define an extension of $H_I|_{\mathbb{D}}$?

We have obtained the following.

**Theorem 3.5.** *Let $\boldsymbol{b}$ and $\boldsymbol{b}_N$ be as before. If $\boldsymbol{b}$ is a Bergman space zero sequence in $\mathbb{D} \setminus \{0\}$ and $E \subset \mathbb{T}$ is its cluster set, then the function $H_b = H_{\mathcal{I}(b)}$, and hence $G_b = H_b'$, extends analytically across $\mathbb{T} \setminus E$; the extension is given by* (3.2) *outside the closed unit disk. Moreover, the functions $G_{b_N}$ converge to $G_b$ as $N \to \infty$, uniformly on compact subsets of $\mathbb{C} \cup \{\infty\} \setminus E \setminus \bigcup_{j=1}^{\infty} \{1/\bar{b}_j\}$.*

*Proof sketch.* The fact that $G_{b_N}$ converges in the norm of $L_a^2(\mathbb{D})$ to $G_b$ as $N \to \infty$ can be used to show that the expression in (3.1), with $\boldsymbol{a} = \boldsymbol{b}_N$, converges to the expression in (3.2), with $I = \mathcal{I}(\boldsymbol{b})$. Each function $G_{b_N}$ is holomorphic on $\mathbb{C} \cup \{\infty\} \setminus \bigcup_{j=1}^{N} \{1/\bar{b}_j\}$, the extension being given by (3.1) outside $\bar{\mathbb{D}}$. By (3.1), we have the estimate (14) in [11], which permits us to carry through the proof of Theorem 2 in [11], to conclude that the functions $H_{b_N}$ form a locally bounded family in the region $\mathbb{C} \setminus E \setminus \bigcup_{j=1}^{\infty} \{1/\bar{b}_j\}$. Since we have convergence toward $H_b$ outside the unit circle, we see that $H_{b_N} \to H_b$, and hence $H_{b_N} \to G_b$, uniformly on compact subsets of $\mathbb{C} \setminus E \setminus \bigcup_{j=1}^{\infty} \{1/\bar{b}_j\}$, by a normal families argument. A more careful analysis near the point at infinity shows that $G_{b_N} \to G_b$ uniformly near $\infty$.

**Remark.** If the points in $b$ are all distinct, Theorem 3.5 is more or less a direct consequence of the results in [11].

## 4. Extremal functions for general invariant subspaces

This section is devoted to generalizing Theorem 2.5 to the extremal functions $G_I$ for arbitrary invariant subspaces $I$ in $L_a^2(\mathbb{D})$. This time multiple zeros will not be a problem. The result we have obtained is the following.

**Theorem 4.1.** *Suppose $I$ is an invariant subspace with $0 \notin Z(I)$, where*

$$Z(I) = \{z \in \mathbb{D}: f(z) = 0 \text{ for all } f \in I\},$$

*and let $G_I$ be the extremal function for the problem*

$$\sup\{\operatorname{Re} f(0): f \in I, \|f\|_{L^2} \leqq 1\}.$$

*There exists a unique function $\Phi \in C(\overline{\mathbb{D}}) \cap C^\infty(\mathbb{D})$ such that $\Phi = 0$ on $\mathbb{T}$ and $\partial\bar{\partial}\Phi = |G_I|^2 - 1$ on $\mathbb{D}$. This solution enjoys the properties*

(a) *for every polynomial $f$ in the variables $z$ and $\bar{z}$, $\int_{r\mathbb{T}} (\partial\Phi/\partial n)(z) f(z)\, ds(z) \to 0$ as $r \to 1^-$, where $\partial/\partial n$ is differentiation in the outward normal direction, and $ds$ is arc length measure, and*

(b) $0 \leqq \Phi(z) \leqq 1 - |z|^2$ *for all $z \in \overline{\mathbb{D}}$.*

*Proof.* The uniqueness of $\Phi$ is obvious. For $n = 0, 1, 2, \ldots$, introduce the functions

$$F_n(z) = \int_{\mathbb{D}} \bar{w}^n (1 - \bar{w}z)^{-1} G_I(w)\, dA(w)/\pi,$$

which are all analytic on $\mathbb{D}$. If $G_I$ has the power series expansion

$$G_I(z) = \sum_{j=0}^{\infty} \alpha_j z^j, \quad z \in \mathbb{D},$$

a computation shows that

$$F_n(z) = \sum_{j=0}^{\infty} \frac{\alpha_{j+n}}{j+n+1} z^j, \quad z \in \mathbb{D},$$

so that $F_n$ belongs to the Dirichlet space, meaning that $F_n' \in L_a^2(\mathbb{D})$, and in particular $F_n \in H^2(\mathbb{D})$. Observe that

$$\text{(4.1)} \qquad \sum_{n=0}^{\infty} \|F_n\|_{H^2}^2 = \sum_{j,n=0}^{\infty} |\alpha_{j+n}|^2/(j+n+1)^2 = \sum_{k=0}^{\infty} |\alpha_k|^2/(k+1) = \|G_I\|_{L^2}^2 = 1.$$

Extend the functions $F_n$ to the circle $\mathbb{T}$ via non-tangential boundary values; then each $F_n$ is defined almost everywhere (with respect to arc length measure) on $\mathbb{T}$, and belongs to $L^2(\mathbb{T})$ there. For $N = 0, 1, 2, \ldots,$ introduce the functions

$$\Psi_N(z) = \sum_{n=0}^{N} |F_n(z)|^2, \quad z \in \overline{\mathbb{D}},$$

and their limit case

$$\Psi(z) = \sum_{n=0}^{\infty} |F_n(z)|^2, \quad z \in \overline{\mathbb{D}};$$

these functions are defined everywhere on $\mathbb{D}$, and almost everywhere on $\mathbb{T}$. Clearly, $\Psi_N$ belongs to $C^\infty(\mathbb{D})$, but we do not know much about the function $\Psi$. We shall see, however, that $\Psi$, too, is infinitely differentiable on $\mathbb{D}$. If $f \in H^2(\mathbb{D})$, a simple calculation yields the following two estimates:

$$(4.2) \qquad |f(z)| \leqq \|f\|_{H^2}(1 - |z|^2)^{-1/2}, \qquad z \in \mathbb{D},$$

$$(4.3) \qquad |f'(z)| \leqq 2\|f\|_{H^2}(1 - |z|^2)^{-3/2}, \qquad z \in \mathbb{D}.$$

Using (4.1), we arrive at the estimates

$$\Psi_N(z) \leqq \sum_{n=0}^{N} \|F_n\|_{H^2}^2 (1 - |z|^2)^{-1} \leqq (1 - |z|^2)^{-1}, \quad z \in \mathbb{D},$$

and

$$|\partial \Psi_N(z)| = |\bar{\partial} \Psi_N(z)| \leqq \sum_{n=0}^{N} |F_n'(z) F_n(z)|$$

$$\leqq 2(1 - |z|^2)^{-2} \sum_{n=0}^{N} \|F_n\|_{H^2}^2 \leqq 2(1 - |z|^2)^{-2}, \quad z \in \mathbb{D},$$

which tell us that the sequence $\{\Psi_N\}_{N=0}^\infty$ forms an equicontinuous family in $C(\mathbb{D})$. Since any normal limit of the $\Psi_N$'s must equal $\Psi$, Arzela-Ascoli's theorem [1], p. 222, shows that $\Psi$ is continuous on $\mathbb{D}$, and that $\Psi_N$ converges to $\Psi$ uniformly on compact subsets of $\mathbb{D}$ as $N \to \infty$. It is not difficult to write down estimates of the kind (4.2), (4.3) for higher order derivatives of $H^2(\mathbb{D})$ functions, which can be used to show that $\{\partial \Psi_N\}_{N=0}^\infty$ and $\{\bar{\partial} \Psi_N\}_{N=0}^\infty$ form equicontinuous families of functions in $C(\mathbb{D})$, and a simple argument using line integrals shows that they must converge to $\partial \Psi$ and $\bar{\partial} \Psi$, respectively. If we continue this way, we can show that $\Psi \in C^\infty(\mathbb{D})$ and that $\Psi_N$ converges to $\Psi$ in $C^\infty(\mathbb{D})$ as $N \to \infty$.

Observe that by (4.1),

$$\int_{\mathbb{T}} \Psi(z) ds(z)/2\pi = \sum_{n=0}^{\infty} \int_{\mathbb{T}} |F_n(z)|^2 ds(z)/2\pi = \sum_{n=0}^{\infty} \|F_n\|_{H^2}^2 = 1,$$

so that $\Psi|_{\mathbb{T}} \in L^1(\mathbb{T})$. For $f \in L^1(\mathbb{T})$, let $P[f]$ be its Poisson integral:

$$P[f](z)=\int_{T}\frac{1-|z|^2}{|\zeta-z|^2}f(\zeta)ds(\zeta)/2\pi,\quad z\in D.$$

Since $|F_n|^2$ is subharmonic and has a harmonic majorant, we have

$$|F_n(z)|^2\leqq P[|F_n|^2](z),\quad z\in D,$$

where we, for reasons of convenience, write $P[|F_n|^2]$ instead of $P[|F_n|^2|_T]$. It follows that

$$\Psi_N(z)\leqq P[\Psi_N](z),\quad z\in D,$$

and since $\Psi_N|_T\to\Psi|_T$ in $L^1(T)$ as $N\to\infty$, we get

$$\Psi(z)\leqq P[\Psi](z),\quad z\in D. \tag{4.4}$$

Our next step is to show that $\Psi=1$ almost everywhere on $T$, so that $\Psi\leqq 1$ on $D$. We will do this by computing the Fourier coefficients of $\Psi$. Let $m\geqq 0$ be an integer. Since the Taylor polynomials of $F_n$ converge to $F_n$ in the norm of $H^2(D)$, an approximation argument shows that

$$\int_{-\pi}^{\pi}e^{-im\theta}|F_n(e^{i\theta})|^2d\theta/2\pi=\sum_{k=0}^{\infty}\frac{\alpha_{k+m+n}\bar{\alpha}_{k+n}}{(k+m+n+1)(k+n+1)},$$

where the right hand side is absolutely convergent, because $\sum_{n=0}^{\infty}|\alpha_n|^2/(n+1)=1<\infty$. Forming finite sums, we obtain

$$\int_{-\pi}^{\pi}e^{-im\theta}\Psi_N(e^{i\theta})d\theta/2\pi=\sum_{n=0}^{N}\sum_{k=0}^{\infty}\frac{\alpha_{k+m+n}\bar{\alpha}_{k+n}}{(k+m+n+1)(k+n+1)}. \tag{4.5}$$

Observe that

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{|\alpha_{k+m+n}\bar{\alpha}_{k+n}|}{(k+m+n+1)(k+n+1)}=\sum_{j=0}^{\infty}|\alpha_{j+m}\alpha_j|/(j+m+1)\\&\leqq\Big(\sum_{j=0}^{\infty}|\alpha_{j+m}|^2/(j+m+1)\Big)^{1/2}\Big(\sum_{j=0}^{\infty}|\alpha_j|^2/(j+m+1)\Big)^{1/2}\\&\leqq\sum_{j=0}^{\infty}|\alpha_j|^2/(j+1)=1<\infty,\end{aligned}$$

so that since $\Psi_N|_T\to\Psi|_T$ in $L^1(T)$ as $N\to\infty$, we get

$$\begin{aligned}\int_{-\pi}^{\pi}e^{-im\theta}\Psi(e^{i\theta})d\theta/2\pi&=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\alpha_{k+m+n}\bar{\alpha}_{k+n}}{(k+m+n+1)(k+n+1)}\\&=\sum_{j=0}^{\infty}\alpha_{j+m}\bar{\alpha}_j/(j+m+1)\end{aligned} \tag{4.6}$$

by letting $N$ tend to infinity in (4.5). On the other hand,

$$(4.7) \qquad \langle G_I, z^m G_I \rangle_{L^2} = \sum_{j=0}^{\infty} \alpha_{j+m}\bar{\alpha}_j/(j+m+1)$$

as well, so that (4.6) boils down to

$$\hat{\Psi}(m) = \int_{-\pi}^{\pi} e^{-im\theta}\Psi(e^{i\theta})\,d\theta/2\pi = \langle G_I, z^m G_I \rangle_{L^2}$$

for all integers $m \geqq 0$. For $m \geqq 1$, $z^m G_I \in I \cap \mathscr{I}_0$, so by Proposition 2.1, $G_I \perp z^n G_I$. We conclude that $\hat{\Psi}(n) = 0$ for all $m \geqq 1$, and the function $\Psi$ being real-valued, its negative Fourier coefficients must vanish as well, by the formula $\hat{\Psi}(-m) = \overline{\hat{\Psi}(m)}$. Consequently, $\Psi$ must be constant almost everywhere on $\mathbb{T}$, and this constant must be $\hat{\Psi}(0) = \|G_I\|_{L^2}^2 = 1$. We conclude that $\Psi \leqq 1$ on $\mathbb{D}$. The desired function $\Phi$ is now given by the formula

$$\Phi(z) = (1-|z|^2)(1-\Psi(z)), \quad z \in \mathbb{D}.$$

The function $\Phi$ is infinitely differentiable on $\mathbb{D}$, and since $0 \leqq \Psi \leqq 1$ on $\mathbb{D}$, we have

$$0 \leqq \Phi(z) \leqq 1-|z|^2, \quad z \in \mathbb{D},$$

so that $\Phi$ has a continuous extension to $\overline{\mathbb{D}}$, which vanishes on $\mathbb{T}$. Let us now check that $\Phi$ solves the differential equation $\partial\bar{\partial}\Phi = |G_I|^2 - 1$. Clearly,

$$\Phi(z) = 1-|z|^2 - \sum_{n=0}^{\infty} |F_n(z)|^2 + \sum_{n=0}^{\infty} |zF_n(z)|^2, \quad z \in \mathbb{D},$$

and since these sums converge in $C^2(\mathbb{D})$ because $\Psi_N \to \Psi$ in $C^2(\mathbb{D})$ as $N \to \infty$, and we have $d/dz(zF_{n+1}(z)) = F_n'(z)$ and $d/dz(zF_0(z)) = G_I(z)$, we get

$$\partial\bar{\partial}\Phi(z) = -1 - \sum_{n=0}^{\infty} |F_n'(z)|^2 + \sum_{n=0}^{\infty} |d/dz(zF_n(z))|^2 = |G_I(z)|^2 - 1, \quad z \in \mathbb{D}.$$

What remains for us to do is to check that (a) holds. To this end, let $H_I$ be the function

$$H_I(z) = \int_0^z G_I(\zeta)\,d\zeta = \sum_{n=0}^{\infty} (\alpha_n/(n+1))z^{n+1} = \sum_{n=1}^{\infty} (\alpha_{n-1}/n)z^n, \quad z \in \mathbb{D},$$

which belongs to the Dirichlet space, and in particular, to the Hardy space $H^2(\mathbb{D})$. Consider the function

$$U_I(z) = |H_I(z)|^2 - \Phi(z) + 1 - |z|^2, \quad z \in \mathbb{D}.$$

A computation shows that $\partial\bar{\partial}U_I = 0$ on $\mathbb{D}$, so that $U_I$ is harmonic, and since $0 \leqq \Phi(z) \leqq 1-|z|^2$ on $\mathbb{D}$, we have $U_I(z) \geqq |H_I(z)|^2$ on $\mathbb{D}$. On the other hand, the functions

$z \mapsto U_I(rz)$, $z \in \mathbb{T}$, converge to $|H_I|^2|_{\mathbb{T}}$ in $L^1(\mathbb{T})$ as $r \to 1^-$, and so $U_I$ must be the smallest harmonic majorant to $|H_I|^2$, that is, $U_I = P[|H_I|^2]$. This shows that $\Phi$ has the representation

$$\Phi(z) = 1 - |z|^2 + |H_I(z)|^2 - P[|H_I|^2](z), \quad z \in \mathbb{D}.$$

If we expand $H_I$ and $P[|H_I|^2]$, we get

$$\Phi(z) = 1 - |z|^2 + \sum_{n,m \geq 0} \frac{\alpha_n \bar{\alpha}_m}{(n+1)(m+1)} z^{n+1} \bar{z}^{m+1} - \sum_{n \geq m \geq} \frac{\alpha_n \bar{\alpha}_m}{(n+1)(m+1)} z^{n-m}$$

$$- \sum_{m > n \geq 0} \frac{\alpha_n \bar{\alpha}_m}{(n+1)(m+1)} \bar{z}^{m-n}, \quad z \in \mathbb{D},$$

so that

$$\partial\Phi(z) = -\bar{z} + \sum_{n,m \geq 0} \frac{\alpha_n \bar{\alpha}_m}{m+1} z^n \bar{z}^{m+1} - \sum_{n > m \geq 0} (n-m) \frac{\alpha_n \bar{\alpha}_m}{(n+1)(m+1)} z^{n-m-1}, \quad z \in \mathbb{D},$$

or, in polar coordinates,

$$\begin{aligned} \partial\Phi(re^{i\theta}) &= -re^{-i\theta} + \sum_{n,m} \frac{\alpha_n \bar{\alpha}_m}{m+1} r^{n+m+1} e^{i(n-m-1)\theta} \\ &\quad - \sum_{n,m:n>m} (n-m) \frac{\alpha_n \bar{\alpha}_m}{(n+1)(m+1)} r^{n-m-1} e^{i(n-m-1)\theta} \\ &= -re^{-i\theta} + \sum_k e^{i(k-1)\theta} \sum_m \frac{\alpha_{m+k} \bar{\alpha}_m}{m+1} r^{2m+k+1} \\ &\quad - \sum_{k>0} e^{i(k-1)\theta} \sum_m \frac{\alpha_{m+k} \bar{\alpha}_m}{(m+k+1)(m+1)} kr^{k-1}, \end{aligned}$$

where we declare $\alpha_j = 0$ for $j > 0$. Let $l$ be an integer. We shall try to evaluate

$$\lim_{r \to 1^-} \int_{-\pi}^{\pi} e^{il\theta} \partial\Phi(re^{i\theta}) d\theta / 2\pi.$$

If $l \geq 2$, we see that

$$\begin{aligned} \lim_{r \to 1^-} \int_{-\pi}^{\pi} e^{il\theta} \partial\Phi(re^{i\theta}) d\theta / 2\pi &= \sum_m \alpha_{m-l+1} \bar{\alpha}_m / (m+1) \\ &= \sum_j \alpha_j \bar{\alpha}_{j+l-1} / (j+l) = \langle z^{l-1} G_I, G_I \rangle_{L^2} = 0, \end{aligned}$$

by (4.7) and the fact that $G_I \perp z^{l-1} G_I$, which follows from Proposition 2.1. For $l = 1$, we have

$$\lim_{r\to 1^-}\int_{-\pi}^{\pi} e^{i\theta}\partial\Phi(re^{i\theta})d\theta/2\pi = -1+\sum_m \alpha_m\bar{\alpha}_m/(m+1) = \|G_I\|_{L^2}^2 - 1 = 0\,,$$

and for $l \leqq 0$, we have

$$\lim_{r\to 1^-}\int_{-\pi}^{\pi} e^{il\theta}\partial\Phi(re^{i\theta})d\theta/2\pi = \sum_m \alpha_{m-l+1}\bar{\alpha}_m/(m+1) - \sum_m(-l+1)\frac{\alpha_{m-l+1}\bar{\alpha}_m}{(m-l+2)(m+1)}$$
$$= \sum_m \alpha_{m-l+1}\bar{\alpha}_m/(m-l+2) = \langle G_I, z^{-l+1}G_I\rangle_{L^2} = 0\,,$$

again by (4.7) and Proposition 2.1. We conclude that for any pair of integers $n, m \geqq 0$,

$$\lim_{r\to 1^-}\int_{r\mathbb{T}} z^n\bar{z}^m\partial\Phi(z)ds(z) = 0\,.$$

If we shift $n$ an $m$ and take complex conjugates in the above relation, we get

$$\lim_{r\to 1^-}\int_{r\mathbb{T}} z^n\bar{z}^m\bar{\partial}\Phi(z)ds(z) = 0\,,$$

and since

$$\partial/\partial n = (z/t)\partial/\partial\partial z + (\bar{z}/r)\partial/\partial\bar{z}$$

on $r\mathbb{T}$, we arrive at

$$\lim_{r\to 1^-}\int_{r\mathbb{T}} z^n\bar{z}^m(\partial\Phi/\partial n)(z)ds(z) = 0\,,$$

from which (a) follows by forming finite sums. The proof of the theorem is complete.

**Corollary 4.2.** *Suppose $I$ is an invariant subspace in $L_a^2(\mathbb{D})$ with $0\notin Z(I)$, and let $G_I$ be the corresponding extremal function. Let $\mathscr{A}$ be the space*

$$\mathscr{A} = \{f\in L_a^2(\mathbb{D})\colon \int_{\mathbb{D}}\Phi|f'|^2dA<\infty\}\,,$$

*supplied with the norm*

$$\|f\|_{\mathscr{A}}^2 = \|f\|_{L^2}^2 + \int_{\mathbb{D}}\Phi|f'|^2dA/\pi\,,$$

*where $\Phi$ is as in Theorem 4.1. Moreover, let $\mathscr{A}_0$ be the closure of the polynomials in $\mathscr{A}$. Then multiplication by $G_I$ is an isometry $\mathscr{A}_0\to L_a^2(\mathbb{D})$.*

*Proof.* If $u$ and $v$ are two $C^2$ functions on $\mathbb{D}$, Green's formula [4], p. 236, states that

$$\int_{r\mathbb{D}}(v\Delta u - u\Delta v)dA = \int_{r\mathbb{T}}\big(v(\partial u/\vartheta n) - u(\partial v/\partial n)\big)ds\,, \tag{4.8}$$

where $ds$ is arc length measure on $r\mathbb{T}$. Let $f$ be an arbitrary (analytic) polynomial. If we apply (4.8) to the case $u = \Phi$ and $v = |f|^2$, we get

$$\int_{rD} (|f|^2 \partial\bar{\partial}\Phi - \Phi|f'|^2)\, dA = (1/4) \int_{rT} (|f|^2(\partial\Phi/\partial n) - (\partial(|f|^2)/\partial n)\Phi)\, ds\,,$$

and by (a) and (b) of Theorem 4.1, we get

$$\int_{D} (|f|^2 \partial\bar{\partial}\Phi - \Phi|f'|^2)\, dA = 0$$

if we let $r$ tend to 1, because $|f|^2$ is a polynomial in $z$ and $\bar{z}$. We obtain

$$\int_{D} (|G_I|^2 - 1)|f|^2\, dA = \int_{D} |f|^2 \partial\bar{\partial}\Phi\, dA = \int_{D} \Phi|f'|^2\, dA\,,$$

so that

$$\int_{D} |G_I f|^2\, dA = \int_{D} |f|^2\, dA + \int_{D} |f|^2 \Phi\, dA\,.$$

In other words, we have the isometry $\|G_I f\|_{L^2} = \|f\|_{\mathscr{A}}$ for all polynomials $f$. An approximation argument now extends the isometry to all $f \in \mathscr{A}_0$.

**Remark.** It is an easy consequence of Corollary 4.2 that $G_I \cdot \mathscr{A}_0 = I(G_I)$.

The space $\mathscr{A}$ introduced in Corollary 4.2, is contained in $L^2_a(\mathbb{D})$, and the injection mapping $\mathscr{A} \to L^2_a(\mathbb{D})$ is contractive. On the other hand, the norm on $H^2(\mathbb{D})$ can be written in the form

$$\|f\|^2_{L^2} + \int_{D} (1 - |z|^2)|f'(z)|^2\, dA(z)/\pi\,,$$

so by (b) of Theorem 4.1 and the fact that the polynomials are dense in $H^2(\mathbb{D})$, $H^2(\mathbb{D})$ is contained within $\mathscr{A}_0$, and the injection mapping $H^2(\mathbb{D}) \to \mathscr{A}_0$ is contractive. We arrive at the following result.

**Corollary 4.3.** *Suppose I is an invariant subspace in $L^2_a(\mathbb{D})$ with $0 \notin Z(I)$, and let $G_I$ be the corresponding extremal function. Then $G_I \succ 1$, so that $G_I$ is a contractive divisor $I(G_I) \to L^2_a(\mathbb{D})$:*

$$\|f/G_I\|_{L^2} \leqq \|f\|_{L^2}\,, \quad f \in I(G_I)\,.$$

*Moreover, $G_I$ is contractive multiplier $H^2(\mathbb{D}) \to L^2_a(\mathbb{D})$.*

**Remark.** We cannot expect $G_I$ to be a multiplier $L^2_a(\mathbb{D}) \to L^2_a(\mathbb{D})$ in general, for it would then be bounded, and its zeros would satisfy the Blaschke condition, which violates Proposition 3.1, because not all Bergman space zero sequences are Blaschke sequences.

**Corollary 4.4.** *Suppose $I$ is an invariant subspace in $L_a^2(\mathbb{D})$ with $0 \notin Z(I)$, and let $G_I$ be the corresponding extremal function. Then*

$$|G_I(z)| \leqq (1 - |z|^2)^{-1/2}, \quad z \in \mathbb{D}.$$

*Proof.* This follows from [13], p. 232, since $G_I$ is a contractive multiplier $H^2(\mathbb{D}) \to L_a^2(\mathbb{D})$, by Corollary 4.3.

**Corollary 4.5.** *Suppose $\boldsymbol{b}$ is an infinite sequence of points in $\mathbb{D} \setminus \{0\}$, and that $\boldsymbol{b}'$ is a subsequence of $\boldsymbol{b}$. If $\boldsymbol{b}$ is a Bergman space zero sequence, and $G_{\boldsymbol{b}}$ and $G_{\boldsymbol{b}'}$ are the extremal functions associated with $\boldsymbol{b}$ and $\boldsymbol{b}'$, respectively, then*

$$|G_{\boldsymbol{b}}(z) / G_{\boldsymbol{b}'}(z)| \leqq (1 - |z|^2)^{-1/2}, \quad z \in \mathbb{D}.$$

*Proof.* First, observe that by Proposition 3.1, the function $G_{\boldsymbol{b}} / G_{\boldsymbol{b}'}$ is holomorphic on $\mathbb{D}$. By Theorem 3.2 and Corollary 4.3, we have for polynomials $f$

$$\|(G_{\boldsymbol{b}} / G_{\boldsymbol{b}'}) f\|_{L^2} = \|G_{\boldsymbol{b}} f / G_{\boldsymbol{b}'}\|_{L^2} \leqq \|G_{\boldsymbol{b}} f\|_{L^2} \leqq \|f\|_{H^2},$$

which makes $G_{\boldsymbol{b}} / G_{\boldsymbol{b}'}$ a contractive multiplier $H^2(\mathbb{D}) \to L_a^2(\mathbb{D})$. The assertion is now immediate from [13], p. 232.

It is tempting to suggest that the following is true; the notation is as in Corollary 4.2.

**Conjecture 4.6.** *Multiplication by $G_I$ is an isometry $\mathscr{A} \to L_a^2(\mathbb{D})$.*

In this context, the following question is fundamental.

**Question 4.7.** For which extremal functions $G_I$ do we have $\mathscr{A}_0 = \mathscr{A}$?

If the invariant subspace $I$ has the so-called codimension one property (see, for instance, [9]), we see by mimicking the proof of Theorem 2.6 that $G_I$ has no more zeros than do the functions in $I$, so that $f / G_I$ is holomorphic in $\mathbb{D}$ for all $f \in I$; the author does not know if this remains true for general invariant subspaces. All singly generated invariant subspaces have the codimension one property [9], p. 596. We are inclined to make the following conjecture; if we want to be cautious, we could add the requirement that $I$ be singly generated to our assumptions.

**Conjecture 4.8.** *Suppose $I$ is an invariant subspace in $L_a^2(\mathbb{D})$ with $0 \notin Z(I)$, having the codimension one property, and let $G_I$ be the corresponding extremal function. Then $G_I$ is a contractive divisor $I \to L_a^2(\mathbb{D})$, that is, $f / G_I \in L_a^2(\mathbb{D})$ and $\|f / G_I\|_{L^2} \leqq \|f\|_{L^2}$ for all $f \in I$.*

By the results in the previous section, this is true for invariant subspaces associated with zero sets.

If Conjectures 4.6 and 4.8 both have affirmative answers, we could expect the following structure theorem to hold.

**Conjecture 4.9.** *Every invariant subspace $I$ of $L_a^2(\mathbb{D})$ with $0 \notin Z(I)$, which has the codimension one property, has the form $I = G_I \cdot \mathscr{B}$, where $\mathscr{B}$ is a closed z-invariant subspace of $\mathscr{A}$ containing $\mathscr{A}_0$.*

**Remark.** It is possible to define extremal functions $G_I$ also for invariant subspaces $I$ with $0 \in Z(I)$. Say, for instance, that the functions in $I$ have a common zero of order $n$ at 0. We can define $G_I$ to be the extremal function for the problem

$$\sup \{\operatorname{Re} f^{(n)}(0) : f \in I, \|f\|_{L^2} \leqq 1\}.$$

Most of the results presented in this paper hold for these extremal functions as well.

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