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Interpolating sequences and invariant subspaces of given index in the Bergman spaces

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1. Introduction

For $0 < \alpha,\ p < +\infty$, let $A_p^{-\alpha} = A_p^{-\alpha}(\mathbb{D})$ be the space of all complex-valued holomorphic functions f on the open unit disk $\mathbb{D}$ that are subject to the boundedness condition

$$\|f\|_{A_p^{-\alpha}} = \Big(\int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^{-1+\alpha p}\, dS(z)\Big)^{1/p} < +\infty\,;$$

here, $dS(z) = dx\,dy$ is area measure in the plane ($z = x + iy$). These spaces are commonly referred to as the *standard weighted Bergman spaces*. We also consider the case $p = \infty$: $f \in A_\infty^{-\alpha}$ if f is holomorphic on $\mathbb{D}$ and

$$\|f\|_{A_\infty^{-\alpha}} = \sup\big\{(1-|z|^2)^\alpha |f(z)| : z \in \mathbb{D}\big\} < +\infty\,.$$

The reason why we use this somewhat nonstandard indexation is that for fixed α, many function theoretic properties of $A_p^{-\alpha}$ remain almost constant in p, such as the collection of zero sets. A closed subspace $\mathcal{M}$ of $A_p^{-\alpha}$ is said to be *invariant* (or z-invariant) if $z\mathcal{M}$ is contained in $\mathcal{M}$. Since the operator of multiplication by z is bounded below on $A_p^{-\alpha}$, $z\mathcal{M}$ is a closed subspace of $\mathcal{M}$. We define the *index* of the invariant subspace $\mathcal{M}$ to be the dimension of the quotient space $\mathcal{M}/z\mathcal{M}$, with values in the set $\{0, 1, 2, \ldots, +\infty\}$. We will at times refer to this number as $\mathrm{ind}(\mathcal{M})$. The index of $\mathcal{M}$ can only equal 0 if $\mathcal{M}$ is the zero subspace.

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It is known that in $A_p^{-\alpha}$ there are invariant subspaces of arbitrary index, finite or infinite. The simplest nontrivial example of an invariant subspace that comes to mind is that of a zero-based one: given a zero sequence Λ in $\mathbb{D}$ for $A_p^{-\alpha}$, let

$$I(\Lambda, A_p^{-\alpha}) = \{f \in A_p^{-\alpha} : f = 0 \text{ on } \Lambda\},$$

counting multiplicities whenever necessary. When it is clear what space we are working with, we write $I(\Lambda)$ in place of $I(\Lambda, A_p^{-\alpha})$. Zero-based invariant subspaces have index 1. The following characterization of general index 1 invariant subspaces is interesting: an invariant subspace $\mathscr{M}$ has index 1 if and only if whenever $f \in \mathscr{M}$ has an extraneous zero $\lambda \in \mathbb{D}$, we may divide f through by $\lambda - z$ and remain in $\mathscr{M}$ [10]; in [10], this was carried out in the Banach space case $1 \leqq p < +\infty$, but the result holds for all $0 < p < +\infty$. As an application, it is possible to show that the collection of all index 1 invariant subspaces is closed under any of the various topologies suggested in [9]. In the closure of the zero-based invariant subspaces we find the Beurling type invariant subspaces [7], where boundary measures play a role. However, to obtain invariant subspaces of higher index, we must resort to other methods of constructing invariant subspaces. Given two invariant subspaces $\mathscr{M}$ and $\mathscr{N}$, we may form the smallest invariant subspace containing them, $\mathscr{M} \vee \mathscr{N}$, by taking the closure of the linear subspace $\mathscr{M} + \mathscr{N}$. The notation naturally extends to larger collections of invariant subspaces. In [5], it was shown that for $p = 2$ and $\alpha = 1/2$, there are zero-based invariant subspaces $\mathscr{M}$ and $\mathscr{N}$ in $A_p^{-\alpha}$ which are at a positive angle from one another, so that $\mathscr{M} \cap \mathscr{N} = \{0\}$ and $\mathscr{M} \vee \mathscr{N} = \mathscr{M} + \mathscr{N}$. It then follows that $\mathscr{M} \vee \mathscr{N}$ has index 2. It was also noted that $\mathscr{M}$ and $\mathscr{N}$ could be chosen so that

$$Z(\mathscr{M}) \cap Z(\mathscr{N}) = \emptyset,$$

where for a general invariant subspace,

$$Z(\mathscr{M}) = \{\lambda \in \mathbb{D} : f(\lambda) = 0 \text{ for all } f \in \mathscr{M}\}.$$

It was also indicated how to build zero-based invariant subspaces $\mathscr{M}_1, \ldots, \mathscr{M}_n$ such that $\mathscr{M}_1 \vee \cdots \vee \mathscr{M}_n$ has index n, for arbitrary integers n, $0 < n < +\infty$. The methods of [5] are of a general nature, and after some technical work they extend to all the spaces $A_p^{-\alpha}$, with $0 < p$, $\alpha < \infty$ (compare with Theorem 6.1 and the remark after it).

Let $\mathscr{M}_1, \ldots, \mathscr{M}_n$ be invariant subspaces in $A_p^{-\alpha}$ with index 1, and assume $1 \leqq p < +\infty$, so that $A_p^{-\alpha}$ is a Banach space. In this paper, we study, first in the context of general Banach spaces of analytic functions, what conditions the $\mathscr{M}_j$ need to satisfy in order that $\mathscr{M}_1 \vee \cdots \vee \mathscr{M}_n$ has maximal index n. We also construct a sequence of points in the disk which is on the "edge" between interpolating and sampling sequences, which enables us to construct an explicit invariant subspace in $A_p^{-\alpha}$ with infinite index. It is explained how to modify the construction to incorporate the case $0 < p < 1$. It is then indicated how the existence of such an object (for $p = 2$) makes the lattice of invariant subspaces on $A_2^{-\alpha}$ so complicated that if we could answer a certain question about it, then we would also have the answer to the invariant subspace problem in Hilbert space.

2. Some general observations about the index of an invariant subspace

In [5] it was shown that if Λ is a zero sequence for $L_a^2 = A_2^{-1/2}$ and if $\mathcal{M}$ is any invariant subspace of $A_2^{-1/2}$ with $Z(\mathcal{M}) \cap \Lambda = \emptyset$, then $\mathcal{M} + I(\Lambda)$ is dense in $A_2^{-1/2}$ if and only if $\operatorname{ind}(\mathcal{M} \vee I(\Lambda)) = 1$. Furthermore, it thus follows from the results of [6] that $\mathcal{M} + I(\Lambda)$ must be dense in $A_2^{-1/2}$ for any $\mathcal{M}$ with $Z(\mathcal{M}) \cap \Lambda = \emptyset$, whenever Λ does not accumulate at every point of the boundary of $\mathbb{D}$. We shall now investigate the situation more closely (see also Section 4).

Let Ω be a region in the complex plane, and let $\mathcal{O}(\Omega)$ be the space of all holomorphic functions on Ω, supplied with the topology of uniform convergence on compact subsets. We say that a Banach space $\mathcal{B}$ is a *Banach space of analytic functions on* Ω, where Ω is a region in $\mathbb{C}$, provided that it is a linear subspace of $\mathcal{O}(\Omega)$, and the injection mapping $\mathcal{B} \to \mathcal{O}(\Omega)$ is continuous.

Nota bene. (a) The term "Banach space of analytic functions" thus requires more of the space than being a Banach space with elements that are analytic functions.

(b) By appealing to the Banach-Steinhaus theorem, that additional condition can be weakened to the following requirement: every point evaluation functional corresponding to points of Ω is continuous.

In the following, we assume that the region Ω is bounded, and let $\mathcal{B}$ be a Banach space of analytic functions on Ω which satisfies the following axioms:

(a) Multiplication by z operates on $\mathcal{B}$, that is, $zf \in \mathcal{B}$ whenever $f \in \mathcal{B}$.

(b) $\mathcal{B}$ has the division property, that is, $f/(z-\lambda) \in \mathcal{B}$ whenever $\lambda \in \Omega$ and $f \in \mathcal{B}$ with $f(\lambda) = 0$.

These are the same axioms that were used in [10]. By use of the closed graph theorem it is easy to check that multiplication by z is a bounded linear transformation on $\mathcal{B}$. We shall denote this operator by $(M_z, \mathcal{B})$ or simply by M_z. Furthermore, for each $\lambda \in \Omega$ the operator $M_z - \lambda$ is bounded below and the codimension of the range of $M_z - \lambda$ in $\mathcal{B}$ is one (see [10]).

We modify the definition of $Z(\mathcal{M})$ in this setting:

$$Z(\mathcal{M}) = \{\lambda \in \Omega : f(\lambda) = 0 \text{ for all } f \in \mathcal{M}\}.$$

It was shown in [10] that for $\lambda \in \Omega$, the dimension of the quotient space $\mathcal{M}/(z-\lambda)\mathcal{M}$ does not depend on λ, that is,

$$\operatorname{ind}(\mathcal{M}) = \dim \mathcal{M}/(z-\lambda)\mathcal{M}, \quad \lambda \in \Omega. \tag{2.1}$$

In [10], Theorem 3.10, a condition on two invariant subspaces $\mathcal{L}$ and $\mathcal{N}$ with index one was found that was equivalent to $\mathcal{L} \vee \mathcal{N}$ having index one. In the following theorem we shall generalize the condition to the span of several invariant subspaces and also modify it.

Thus, let $S \neq \emptyset$ be a finite or countably infinite index set and for each $j \in S$, let $\mathcal{M}_j$ be a nonzero invariant subspace of $(M_z, \mathcal{B})$ with index one. Let $\operatorname{card} S$ be the cardinality of the set S (that is, the number of points in S), taking values in $\{1, 2, \ldots, +\infty\}$. We set, for $i \in S$,

$$\mathcal{M} = \bigvee \{\mathcal{M}_j : j \in S\} \quad \text{and} \quad \hat{\mathcal{M}}_i = \bigvee \{\mathcal{M}_j : j \in S, j \neq i\}$$

(here the big $\vee$ is used to denote the closed linear span of a family of subspaces). Note that one always has $\operatorname{ind}(\mathcal{M}) \leqq \operatorname{card} S$ (see [10], Proposition 2.16). We shall start out with a condition which implies that $\operatorname{ind}(\mathcal{M}) = \operatorname{card} S$. This condition will be used in Section 6 to construct invariant subspaces with infinite index.

Theorem 2.1. *Suppose there is a* $\lambda \in \Omega \backslash (\bigcup_{j \in S} Z(\mathcal{M}_j))$ *such that for all* $i \in S$

$$|f(\lambda)| \leqq C_i \|f\|_{\mathcal{B}/\hat{\mathcal{M}}_i}, \quad f \in \mathcal{M}_i,$$

holds for some C_i, $0 < C_i < +\infty$. *Then* $\operatorname{ind}(\mathcal{M}) = \operatorname{card} S$.

Proof. We shall prove the theorem by contraposition. Thus, suppose that $\operatorname{ind}(\mathcal{M}) < \operatorname{card} S$, and fix $\lambda \in \Omega \backslash (\bigcup_{j \in S} Z(\mathcal{M}_i))$. We must show that there is an index $i \in S$ and sequences $\{f_n\}_n \subset \mathcal{M}_i$, $\{g_n\}_n \subset \hat{\mathcal{M}}_i$ such that $|f_n(\lambda)| = 1$ and $\|f_n - g_n\|_{\mathcal{B}} \to 0$.

Let Q denote the quotient map $Q : \mathcal{M} \to \mathcal{M}/(z - \lambda)\mathcal{M}$, and for each $j \in S$, pick an $h_j \in \mathcal{M}_j$ with $h_j(\lambda) = 1$.

Notice that $Q\mathcal{M} = \operatorname{span}\{Qh_j : j \in S\}$, because for each $j \in S$, $\operatorname{ind}(\mathcal{M}_j) = 1$ and $Q(z - \lambda)\mathcal{M}_j = (0)$. Thus, for each finite subset S' of S with $\operatorname{ind}(\mathcal{M}) = \dim Q\mathcal{M} < \operatorname{card} S'$, we have that the set $\{Qh_j : j \in S'\}$ is linearly dependent, so there must be an $i \in S' \subset S$ such that $Qh_i \in \operatorname{span}\{Qh_j : j \in S', j \neq i\} \subset Q\hat{\mathcal{M}}_i$. We fix such i and a function $g \in \hat{\mathcal{M}}_i$ such that $h_i - g \in (z - \lambda)\mathcal{M}$. Now note that $\mathcal{M} = \mathcal{M}_i \vee \hat{\mathcal{M}}_i$, thus there must be sequences $\{h_{i,n}\}_n \subset \mathcal{M}_i$ and $\{\tilde{g}_n\}_n \subset \hat{\mathcal{M}}_i$ such that

$$h_i + (z - \lambda) h_{i,n} - g - (z - \lambda)\tilde{g}_n \to 0 \quad \text{as } n \to +\infty .$$

This concludes the proof because the sequences

$$\{f_n\}_n, \quad f_n = h_i + (z - \lambda) h_{i,n}, \quad \text{and} \quad \{g_n\}_n, \quad g_n = g + (z - \lambda)\tilde{g}_n,$$

have the required properties. □

It is clear that the converse to Theorem 2.1 cannot be true if $\operatorname{card} S$ is infinite. Indeed, suppose $S = \mathbb{Z}_+ = \{1, 2, \ldots\}$ and $\operatorname{ind}(\mathcal{M}) = +\infty$. Let $\mathcal{M}_0 = \mathcal{M}_1$ and set

$$\mathcal{N} = \operatorname{span}\{\mathcal{M}_j : j \in \mathbb{Z}_+ \cup \{0\}\} .$$

Then, of course, $\mathcal{N} = \mathcal{M} = \hat{\mathcal{M}}_0$, $\operatorname{ind}(\mathcal{N}) = \operatorname{card}(\mathbb{Z}_+ \cup \{0\}) = +\infty$, and $\|f\|_{\mathcal{B}/\hat{\mathcal{M}}_0} = 0$ for all $f \in \mathcal{M}_0 \subset \hat{\mathcal{M}}_0$. Thus, since $\mathcal{M}_0$ was assumed to be nonzero the condition in the theorem cannot be satisfied. However, if card S is finite, then Theorem 2.1 has a converse. In fact, the argument in the proof of Theorem 2.1 can be reversed. Note that

$$\operatorname{ind}(\mathcal{M}) = \dim(\mathcal{M}/(z-\lambda)\mathcal{M})$$

does not depend on $\lambda \in \Omega$, so the condition in question should be satisfied either for no $\lambda \in \Omega \backslash (\bigcup_{i \in S} Z(\mathcal{M}_i))$ or for all $\lambda \in \Omega$. With a little extra effort we can make the constants independent of λ as long as λ stays in some compact set $K \subset \Omega$.

Lemma 2.2. *Let $\mathcal{L}$ and $\mathcal{N}$ be invariant subspaces of $(M_z, \mathcal{B})$ with $\operatorname{ind}(\mathcal{L}) = 1$, $\operatorname{ind}(\mathcal{N}) = n < +\infty$, and $\operatorname{ind}(\mathcal{L} \vee \mathcal{N}) = n+1$. Then for each compact subset K of Ω, there exists a constant $C(K)$, $0 < C(K) < +\infty$, such that*

$$|f(\lambda)| \leqq C(K) \|f\|_{\mathcal{B}/\mathcal{N}}, \quad f \in \mathcal{L}, \ \lambda \in K.$$

Proof. It suffices to check the bound locally near each fixed point $\lambda_0 \in \Omega$. Since the index of $M_z - \lambda$ does not change with $\lambda \in \Omega$ [10], we can assume without loss of generality that $\lambda_0 = 0$. Set $\mathcal{M} = \mathcal{L} \vee \mathcal{N}$, and let $Q_{\mathcal{M}}$ be the quotient map $\mathcal{M} \to \mathcal{M}/z\mathcal{M}$. We pick $h_0 \in \mathcal{L} \backslash z\mathcal{L}$, and let $h_1, \dots, h_n \in \mathcal{N}$ be such that the images of these vectors under the quotient map $\mathcal{N} \to \mathcal{N}/z\mathcal{N}$ are linearly independent, that is, if $\sum_{j=1}^{n} \alpha_j h_j \in z\mathcal{N}$ for complex α_j, then $\sum_{j=1}^{n} |\alpha_j| = 0$. Since $\operatorname{ind}(\mathcal{N}) = n$, every $g \in \mathcal{N}$ can be written as $g = \sum_{j=1}^{n} \alpha_j h_j + z g_1$, with $g_1 \in \mathcal{N}$. Note that it follows from the hypothesis that the vectors

$$Q_{\mathcal{M}} h_0, Q_{\mathcal{M}} h_1, \dots, Q_{\mathcal{M}} h_n$$

form a basis for $\mathcal{M}/z\mathcal{M}$. The dual of $\mathcal{M}/z\mathcal{M}$ is isometrically isomorphic to the annihilator of $z\mathcal{M}$ in $\mathcal{M}^*$, thus we can find a dual basis $\{e_0, e_1, \dots, e_n\}$ in $(z\mathcal{M})^\perp$, so that

$$\langle h_i, e_j \rangle = \delta_{ij}, \quad 0 \leqq i, j \leqq n.$$

For $h \in \mathcal{M}$, the function $h - \sum_{i=0}^{n} \langle h, e_i \rangle h_i$ is in $z\mathcal{M}$, because it is annihilated by every e_j, $j = 0, \dots, n$. Thus, the map L,

$$Lh(z) = z^{-1}\left(h(z) - \sum_{i=0}^{n} \langle h, e_i \rangle h_i(z)\right), \quad z \in \Omega \backslash \{0\},$$

is a bounded linear transformation from $\mathcal{M}$ into itself. Furthermore, L leaves $\mathcal{L}$ invariant, that is, $L\mathcal{L} \subset \mathcal{L}$. In fact, any $f \in \mathcal{L}$ is of the form $f = \alpha h_0 + z f_1$, for some $\alpha \in \mathbb{C}$ and $f_1 \in \mathcal{L}$, and thus $Lf = f_1 \in \mathcal{L}$, since $z f_1 \perp e_i$, $i = 0, \dots, n$. Similarly, one checks that L leaves $\mathcal{N}$ invariant.

For small $\lambda \in \Omega$, $|\lambda| < \delta = \|L\|^{-1}$, the operator $1 - \lambda L$ is invertible as an operator $\mathcal{M} \to \mathcal{M}$, and on a smaller disk there is a uniform norm bound,

$$\|(1 - \lambda L)^{-1}\| \leqq A, \quad |\lambda| < \varepsilon,$$

for some constant A, $0 < A < +\infty$. Here, we have fixed an ε, $0 < \varepsilon < \delta$, such that the closed disk $|\lambda| \leqq \varepsilon$ is contained in Ω. For $|\lambda| < \varepsilon$, set $L_\lambda = (I - \lambda L)^{-1} L$, which is then also uniformly norm bounded. One checks that for $h \in \mathcal{M}$,

$$(M_z - \lambda) L_\lambda h = (M_z L - \lambda L)(1 - \lambda L)^{-1} h = h - \sum_{i=0}^{n} \langle (1 - \lambda L)^{-1} h, e_i \rangle h_i .$$

Moreover, for $f \in \mathcal{L}$, we have $(1 - \lambda L)^{-1} f \in \mathcal{L}$, so that

$$\begin{aligned} (2.2) \qquad (M_z - \lambda) L_\lambda f &= f - \sum_{i=0}^{n} \langle (1 - \lambda L)^{-1} f, e_i \rangle h_i \\ &= f - \langle (1 - \lambda L)^{-1} f, e_0 \rangle h_0 = f - \alpha(\lambda) h_0 , \end{aligned}$$

where

$$\alpha(\lambda) = \langle (1 - \lambda L)^{-1} f, e_0 \rangle .$$

Similarly, for $g \in \mathcal{N}$, we have

$$(2.3) \qquad (M_z - \lambda) L_\lambda g = g - \sum_{i=1}^{n} \alpha_i(\lambda) h_i, \quad \alpha_i(\lambda) = \langle (I - \lambda L)^{-1} g, e_i \rangle .$$

Let d_0 be the distance of h_0 to the linear span of $h_1, \dots, h_n$:

$$d_0 = \inf \left\{ \left\| h_0 - \sum_{i=1}^{n} \beta_i h_i \right\| : \beta_1, \dots, \beta_n \in \mathbb{C} \right\} > 0 .$$

Then for each $\lambda \in \Omega$ with $|\lambda| < \varepsilon$, $f \in \mathcal{L}$, and $g \in \mathcal{N}$, we have, by (2.2)–(2.3),

$$d_0 |\alpha(\lambda)| \leqq \left\| \alpha(\lambda) h_0 - \sum_{i=1}^{n} \alpha_i(\lambda) h_i \right\| = \| (1 - (z - \lambda) L_\lambda)(f - g) \| \leqq C \| f - g \| ,$$

where $C = \sup\{ \| 1 - (M_z - \lambda) L_\lambda \| : |\lambda| < \varepsilon \} < +\infty$. By the observation that the left hand side of (2.2) vanishes at the point $z = \lambda$, we get $f(\lambda) = \alpha(\lambda) h_0(\lambda)$, so that

$$(2.4) \qquad |f(\lambda)| \leqq C d_0^{-1} |h_0(\lambda)| \| f - g \|, \quad f \in \mathcal{L}, g \in \mathcal{N} .$$

The function $h_0(\lambda)$ is analytic on Ω, and hence uniformly bounded on $|\lambda| < \varepsilon$. Thus, by (2.4), we get

$$|f(\lambda)| \leqq C' \| f \|_{\mathcal{B}/\mathcal{N}}, \quad |\lambda| < \varepsilon, f \in \mathcal{L} ,$$

with $C' = C d_0^{-1} \sup\{ |h_0(\lambda)| : |\lambda| < \varepsilon \}$. This concludes the proof of Lemma 2.2. □

We now return to the situation that we considered in and before Theorem 2.1, only this time we assume that $\operatorname{card} S = n < +\infty$.

Theorem 2.3. *Let* $n \in \mathbb{Z}_+ = \{1, 2, 3, \dots\}$, *and let* $\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_n$ *be invariant subspaces of* $(M_z, \mathcal{B})$, *each with index one. Set* $\mathcal{M} = \bigvee \{ \mathcal{M}_i : i = 1, \dots, n \}$, *and consider the seminorm on* $\mathcal{B}$ *given by*

$$\|f\|_* = \sum_{j=1}^{n} \|f\|_{\mathscr{B}/\hat{\mathscr{M}}_j}, \quad f \in \mathscr{B}.$$

Then the following are equivalent:

(a) $\operatorname{ind}(\mathscr{M}) = n$,

(b) *there is a* $\lambda \in \Omega \setminus \left(\bigcup_{i=1}^{n} Z(\mathscr{M}_i) \right)$ *such that for some constant* C, *we have, for all* $i = 1, \ldots, n$, $|f(\lambda)| \leqq C \|f\|_{\mathscr{B}/\hat{\mathscr{M}}_i}$, $f \in \mathscr{M}_i$,

(c) *there is a* $\lambda \in \Omega \setminus \left(\bigcup_{i=1}^{n} Z(\mathscr{M}_i) \right)$ *such that the point evaluation at* λ *is a bounded functional on* $\mathscr{M}$ *with respect to the seminorm* $\|\cdot\|_*$: *for some constant* C, $|f(\lambda)| \leqq C \|f\|_*$ *holds for all* $f \in \mathscr{M}$,

(d) *the seminorm* $\|\cdot\|_*$ *is a norm on* $\mathscr{M}$, *and the set* $\{h \in \mathscr{M} : \|h\|_* \leqq 1\}$ *is a normal family on* Ω.

Proof. By the definition of what constitutes a Banach space of analytic functions on Ω, (d) implies (c). It is trivial that (c) implies (b), and that (b) implies (a) is Theorem 2.1, so we only need to show the implication (a) $\Rightarrow$ (d).

We assume that $\operatorname{ind}(\mathscr{M}) = n$, and let K be an arbitrary compact subset of Ω.

Fix i, $1 \leqq i \leqq n$, and note that $\operatorname{ind}(\hat{\mathscr{M}}_i) \leqq n - 1$, because $\hat{\mathscr{M}}_i$ is the closed linear pan of $n - 1$ index one invariant subspaces [10]. Since $\mathscr{M} = \mathscr{M}_i \vee \hat{\mathscr{M}}_i$, we have $n = \operatorname{ind}(\mathscr{M}) \leqq \operatorname{ind}(\mathscr{M}_i) + \operatorname{ind}(\hat{\mathscr{M}}_i) = 1 + \operatorname{ind}(\hat{\mathscr{M}}_i) \leqq 1 + (n-1) = n$, so that

$$\operatorname{ind}(\hat{\mathscr{M}}_i) = n - 1 .$$

By Lemma 2.2, and the fact that we have only finitely many invariant subspaces $\mathscr{M}_i$, there is a constant $C(K)$, $0 < C(K) < +\infty$, such that

$$|f(\lambda)| \leqq C(K) \|f\|_{\mathscr{B}/\hat{\mathscr{M}}_i}, \quad f \in \mathscr{M}_i, \ \lambda \in K.$$

For every f of the form $f = \sum_{i=1}^{n} f_i$, $f_i \in \mathscr{M}_i$, we have

$$|f(\lambda)| \leqq \sum_{i=1}^{n} |f_i(\lambda)| \leqq C(K) \sum_{i=1}^{n} \|f_i\|_{\mathscr{B}/\hat{\mathscr{M}}_i} = C(K) \sum_{i=1}^{n} \|f\|_{\mathscr{B}/\hat{\mathscr{M}}_i}, \quad \lambda \in K.$$

However, elements of the form $f = \sum_{i=1}^{n} f_i$ are dense in $\mathscr{M}$, so (d) follows. □

If the subspace $\mathscr{M}$ is complemented in $\mathscr{B}$, then one can substitute $\mathscr{B}$ for $\mathscr{M}$ in part (d) of Theorem 2.3.

Corollary 2.4. *Let $n \in \mathbb{Z}_+$ and let $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_n$ be invariant subspaces of $(M_z, \mathcal{B})$, each with index one. Set $\mathcal{M} = \bigvee \{\mathcal{M}_i : i = 1, \ldots, n\}$, and suppose that $\mathcal{M}$ is complemented in $\mathcal{B}$, that is, there is a norm closed subspace $\mathcal{N}$ of $\mathcal{B}$ such that $\mathcal{M} \cap \mathcal{N} = (0)$ and $\mathcal{M} + \mathcal{N} = \mathcal{B}$. Then $\operatorname{ind}(\mathcal{M}) = n$ if and only if the seminorm*

$$\|f\|_* = \sum_{j=1}^{n} \|f\|_{\mathcal{B}/\hat{\mathcal{M}}_j}, \quad f \in \mathcal{B},$$

is a norm, and the set $\{h \in \mathcal{B} : \|h\|_* \leqq 1\}$ is a normal family on Ω.

One notes that if $\mathcal{B}$ is a Hilbert space, then all subspaces are complemented.

Proof. Clearly, if point evaluations are continuous on $\mathcal{B}_0$, then condition (b) of Theorem 2.3 is satisfied, so $\operatorname{ind}(\mathcal{M}) = n$.

Conversely, assume that $\operatorname{ind}(\mathcal{M}) = n$. Let $K \subset \Omega$ be compact. We shall show that every $\lambda \in K$ is a uniformly bounded point evaluation for $\mathcal{B}$ with the norm $\|\cdot\|_*$. This then finishes the proof by a normal families argument.

Let $\mathcal{N}$ be a complement of $\mathcal{M}$. Then any $h \in \mathcal{B}$ has a unique representation $h = f + g$, $f \in \mathcal{M}$ and $g \in \mathcal{N}$, and there is a constant $\delta > 0$ such that

$$\delta(\|f\| + \|g\|) \leqq \|f + g\| \leqq \|f\| + \|g\|$$

for all $f \in \mathcal{M}$ and $g \in \mathcal{N}$. Now let $g_i \in \hat{\mathcal{M}}_i$; then by Theorem 2.3,

$$\begin{aligned} |h(\lambda)| &\leqq |f(\lambda)| + |g(\lambda)| \leqq C(K)\left(\sum_{i=1}^{n} \|f\|_{\mathcal{B}/\hat{\mathcal{M}}_i} + \|g\|_{\mathcal{B}}\right) \\ &\leqq C(K) \sum_{i=1}^{n} (\|f - g_i\|_{\mathcal{B}} + \|g\|_{\mathcal{B}}) \leqq C(K)\delta^{-1} \sum_{i=1}^{n} \|f + g - g_i\|_{\mathcal{B}} \\ &= C(K)\delta^{-1} \sum_{i=1}^{n} \|h - g_i\|_{\mathcal{B}}, \quad \lambda \in K. \end{aligned}$$

Taking the infimum over all possible choices of $g_i \in \hat{\mathcal{M}}_i$, we obtain $|h(\lambda)| \leqq C(K)\delta^{-1}\|h\|_*$ on $\lambda \in K$, for every $h \in \mathcal{B}$. □

Remark. For $n = 2$, condition (b) of Theorem 2.3 can be relaxed a little. If $\mathcal{L}$ and $\mathcal{N}$ are two invariant subspaces with index one, and if $\lambda \in \Omega \setminus (Z(\mathcal{L}) \cup Z(\mathcal{N}))$ such that $|f(\lambda)| \leqq C\|f\|_{\mathcal{B}/\mathcal{N}}$ for all $f \in \mathcal{L}$, then $|g(\lambda)| \leqq C'\|g\|_{\mathcal{B}/\mathcal{L}}$ for all $g \in \mathcal{N}$. Indeed, for $f \in \mathcal{L}$ and $g \in \mathcal{N}$ we have

$$\begin{aligned} |g(\lambda)| &\leqq |g(\lambda) - f(\lambda)| + |f(\lambda)| \leqq C(\|g - f\| + \|f\|_{\mathcal{B}/\mathcal{N}}) \\ &= C(\|g - f\| + \|g - f\|_{\mathcal{B}/\mathcal{N}}) \leqq C'\|g - f\|. \end{aligned}$$

Corollary 2.5. *Let $\mathscr{M}$ be an invariant subspace of $(M_z, \mathscr{B})$ with* $\operatorname{ind}(\mathscr{M}) = 1$. *Then the following are equivalent*:

(a) *there exists an invariant subspace $\mathscr{N}$ of $(M_z, \mathscr{B})$ which contains $\mathscr{M}$ and has index two*,

(b) *there exists an $f \in \mathscr{B}$, a $\lambda \in \Omega \backslash (Z(\mathscr{M}) \cup Z(f))$, and a constant C, $0 < C < +\infty$, such that*

$$|p(\lambda)| \leqq C \|pf\|_{\mathscr{B}/\mathscr{M}} \text{ for every polynomial } p\,,$$

(c) *there exists an $f \in \mathscr{B}$ such that the seminorm on polynomials $\|p\|_1 = \|pf\|_{\mathscr{B}/\mathscr{M}}$ is a norm, and the set $\{p : p$ polynomial, $\|p\|_1 \leqq 1\}$ is a normal family on Ω.*

Proof. The implication (c) $\Rightarrow$ (b) is clear. (b) $\Rightarrow$ (a) follows from Theorem 2.3 together with the above remark, just take $\mathscr{N} = [f] \vee \mathscr{M}$.

(a) $\Rightarrow$ (c): We have $\mathscr{N} = \bigvee \{[f] : f \in \mathscr{N}\} \vee \mathscr{M}$. Thus, if $\operatorname{ind}(\mathscr{N}) = 2$, then by Theorem 3.13 (b) of [10] there must be an $f \in \mathscr{N}$ such that $\operatorname{ind}(\mathscr{M} \vee [f]) = 2$. For polynomials p, the isometry $\|p\|_1 = \|fp\|_*$ holds, where

$$\|g\|_* = \|g\|_{\mathscr{B}/[f]} + \|g\|_{\mathscr{B}/\mathscr{M}}, \quad g \in \mathscr{N} = \mathscr{M} \vee [f]\,.$$

Thus, (c) folows from Theorem 2.3. □

3. Sequences: interpolation, separation, and sampling

Let $0 < \alpha, p < +\infty$. For a fixed point $\lambda \in \mathbb{D}$, consider the extremal problem

$$\sup \{|f(\lambda)| : \|f\|_{A_p^{-\alpha}} = 1\}\,.$$

By direct inspection of the solution, we see that the norm of the point evaluation functional at λ is comparable to $(1 - |\lambda|^2)^{-\alpha - 1/p}$. Hence, for a sequence $\Lambda = \{\lambda_n\}_n$ of points in $\mathbb{D}$, the operator

$$T_{p,\alpha} : f \mapsto \{(1 - |\lambda_n|^2)^{\alpha + 1/p} f(\lambda_n)\}_n$$

maps $A_p^{-\alpha}$ into ℓ^∞. The sequence Λ is said to be *interpolating* for $A_p^{-\alpha}$ if $T_{p,\alpha}$ maps $A_p^{-\alpha}$ into and onto ℓ^p. Closely related to the concept of interpolating sequences is the notion of separation: the sequence Λ is *separated* if

$$\inf \left\{ \left| \frac{\lambda_i - \lambda_j}{1 - \bar{\lambda}_i \lambda_j} \right| : i, j, i \neq j \right\} > 0\,.$$

Separated sequences are sometimes referred to as being *uniformly discrete*. All interpolating sequences are separated. Moreover, if Λ is interpolating, then

$$\|f\|_{A_p^{-\alpha}/I(\Lambda)} \asymp \|T_{p,\alpha} f\|_{\ell^p} = \left(\sum_{n=1}^{\infty} |f(\lambda_n)|^p (1 - |\lambda_n|^2)^{\alpha p + 1} \right)^{1/p}, \quad f \in A_p^{-\alpha},$$

(the $\asymp$ sign means that the two quantities are comparable in size) where $I(\Lambda)$ is the zero-based invariant subspace associated with Λ. For separated Λ, we just know that

$$\|T_{p,\alpha} f\|_{\ell^p} \leqq C\|f\|_{A_p^{-\alpha}/I(\Lambda)}, \quad f \in A_p^{-\alpha}.$$

The sequence Λ is *sampling* if

$$\|T_{p,\alpha} f\|_{\ell^p} \asymp \|f\|_{A_p^{-\alpha}}, \quad f \in A_p^{-\alpha}.$$

To describe more concretely the sampling and interpolating sequences, we need the concept of density. For separated Λ, let

$$D(\Lambda, r) = \frac{\sum_{j:|\lambda_j|<r} (1 - |\lambda_j|)}{\log \frac{1}{1-r}}, \quad 0 < r < 1.$$

For every $z \in \mathbb{D}$, we form a new sequence

$$\Lambda_z = \left\{ \frac{\lambda_j - z}{1 - \bar{z}\lambda_j} \right\}.$$

The lower uniform density of Λ is

$$D^-(\Lambda) = \liminf_{r \to 1^-} \inf \{D(\Lambda_z, r) : z \in \mathbb{D}\},$$

and the upper uniform density of Λ is

$$D^+(\Lambda) = \limsup_{r \to 1^-} \sup \{D(\Lambda_z, r) : z \in \mathbb{D}\}.$$

The following two theorems from [12] describe completely the interpolating and sampling sequences. Strictly speaking, they were stated explicitly in [12] only for $p = 2$ and $p = \infty$; however, only minor modifications are needed for the general case.

Theorem 3.1. *A sequence $\Lambda = \{\lambda_n\}_n$ of points in $\mathbb{D}$ is interpolating for $A_p^{-\alpha}$ if and only if it is separated and $D^+(\Lambda) < \alpha$.*

Theorem 3.2. *A sequence $\Lambda = \{\lambda_n\}_n$ of points in $\mathbb{D}$ is sampling for $A_p^{-\alpha}$ if and only if it can be expressed as a finite union of separated sequences and there exists a separated subsequence $\Lambda' \subset \Lambda$ for which $D^-(\Lambda') > \alpha$.*

4. Applications to the Bergman spaces

We now apply the general results of Section 2 to the Bergman spaces $\mathscr{B} = A_p^{-\alpha}$ in the Banach space case, that is, $0 < \alpha < +\infty$ and $1 \leqq p < +\infty$. For some zero-based invariant subspaces, we shall be able to make the conditions (b) and (v) of Corollary 2.5, which involve quotient norms, more concrete.

Given a discrete sequence $\Lambda = \{\lambda_n\}_n$ of points in $\mathbb{D}$ and a real parameter γ, $1 < \gamma < +\infty$, we associate a Borel measure $\mu_{\Lambda,\gamma}$ on $\mathbb{D}$,

$$d\mu_{\Lambda,\gamma} = \sum_{n=1}^{\infty} (1 - |\lambda_n|^2)^\gamma d\delta_{\lambda_n},$$

where $d\delta_\lambda$ is the unit point mass at the point $\lambda \in \mathbb{D}$. For a finite positive Borel measure μ on $\mathbb{D}$, we let $P^p(\mu)$ denote the closure of the polynomials in $L^p(\mathbb{D}, \mu)$.

Corollary 4.1. ($1 \leqq p < +\infty$, $0 < \alpha < +\infty$) *Let $\Lambda = \{\lambda_n\}_n$ be interpolating for $A_p^{-\alpha}$. Then the following are equivalent*:

(a) *there is an invariant subspace $\mathcal{N}$ of $A_p^{-\alpha}$ with $I(\Lambda) \subset \mathcal{N}$ and* $\operatorname{ind}(\mathcal{N}) = 2$,

(b) *there are $f \in A_p^{-\alpha}$, $\lambda \in \mathbb{D} \setminus (Z(f) \cup \Lambda)$, and a constant C, $0 < C < +\infty$, such that for all polynomials q,*

$$|q(\lambda)|^p \leqq C \int_{\mathbb{D}} |q(z)|^p |f(z)|^p d\mu_{\Lambda,\alpha p+1},$$

(c) *there is an $f \in A_p^{-\alpha}$ such that with $d\mu = |f(z)|^p d\mu_{G,\alpha p+1}$, $P^p(\mu)$ is a Banach space of analytic functions on $\mathbb{D}$.*

Proof. The equivalence of (a) and (b) follows from Corollary 2.5 and the remarks in Section 3. Furthermore, by the definition of Banach space of analytic functions, (c) implies (b). That (b) implies (c) follows from Corollary 2.5 together with a normal families argument and an easy exercise using the fact that μ is a finite measure which is concentrated in the open unit disk. □

In the case of an interpolating sequence Λ, this corollary explains the earlier observation that if Λ has the property (a) in Corollary 4.1, then Λ has to accumulate at every point of the unit circle. We also note that by the equivalence of norms statements in Section 3, it follows that for an interpolating (in the space $A_p^{-\alpha}$) Λ, property (c) of Corollary 4.1 implies that Λ is dominating for H^∞. In other words, if an interpolating Λ fails to be dominating for H^∞, then any invariant subspace $\mathcal{N}$ that contains $I(\Lambda)$ has index one. Thus, for interpolating Λ which are not dominating for H^∞, it follows from the argument in the proof of [5], Theorem 2.4, that if $\mathcal{N}$ is an invariant subspace of $A_2^{-1/2}$ and if $\mathcal{N} \supset I(\Lambda)$ has no common zeros, then $\mathcal{N} = A_2^{-1/2}$; the proof in [5] generalizes to certain other values of p, α: modulo a few technical points that require checking, it should be all right for $0 < \alpha \leqq 2/p$ (this is definitely so for $p = 2$, and we expect it to generalize to $1 < p < +\infty$).

In the other direction we have the following corollary.

Corollary 4.2. ($1 \leqq p < +\infty$, $0 < \alpha < +\infty$) *Let Λ_1, $\Lambda_2 \subset \mathbb{D}$ be two zero sequences for $A_p^{-\alpha}$ with the following property; there are interpolating subsequences $\Lambda_1' \subset \Lambda_1$, $\Lambda_2' \subset \Lambda_2$, such that for some $\lambda \in \mathbb{D} \setminus (\Lambda_1' \cup \Lambda_2')$, there is a C, $0 < C < +\infty$, such that*

$$|q(\lambda)|^p \leqq C \int_{\mathbb{D}} |q|^p d\mu_{\Lambda_1' \cup \Lambda_2', \alpha p+1}$$

holds for all polynomials q. Then $\operatorname{ind}(I(\Lambda_1) \vee I(\Lambda_2)) = 2$.

Proof. This follows immediately from the results of Sections 2 and 3. □

5. Special sets of uniqueness for $A_p^{-\alpha}$

We shall now construct special sets of uniqueness for $A_p^{-\alpha}$. The crucial point here is that in a certain sense these sequences live on the "edge" between sampling and interpolating sequences. The main result of this section is the following. The notation is as in the previous sections.

Proposition 5.1. $(0 < \alpha, p < +\infty)$ *There exists a separated sequence* $\Lambda \subset \mathbb{D} \setminus \{0\}$ *with* $D^+(\Lambda) = D^-(\Lambda) = \alpha$, *such that*

$$(5.1) \quad |f(0)| \leqq C \|T_{\alpha,p} f\|_{\ell^p} = C\Big(\int_{\mathbb{D}} |f(z)|^p d\mu_{\Lambda, \alpha p+1}(z)\Big)^{1/p}, \quad f \in A_p^{-\alpha},$$

where C a positive constant independent of f.

We shall prove the proposition by modifying the approach of [11]. For given real parameters a, b, and c, with $1 < a < +\infty$ and $0 < b < +\infty$, let

$$w_{m,n} = a^m(b_m n + i), \quad m, n \in \mathbb{Z},$$

which is a sequence of points in the upper half plane. Here,

$$b_m = \begin{cases} b & \text{if } m \geqq 0, \\ b - c/|m| & \text{if } m < 0. \end{cases}$$

In the strip $0 < \Im z < 1$, the sequence W is slightly fatter for $c > 0$ and slightly thinner for $c < 0$ than the hyperbolically equi-distributed sequence which is obtained for $c = 0$. The Cayley transform $z \mapsto (z-i)/(z+i)$ maps the open upper half plane $\mathbb{C}_+$ onto the unit disk $\mathbb{D}$; let $\lambda_{m,n}$ denote the image of $w_{m,n}$ under it, and put $\beta = 2\pi/(b \log a)$. Let Λ' be the sequence $\{\lambda_{m,n}\}_{m,n}$, and set $\Lambda = \Lambda' \setminus \{0\}$. The pseudohyperbolic metric in $\mathbb{D}$ is denoted by ϱ,

$$\varrho(z, \zeta) = \left| \frac{z - \zeta}{1 - \bar{\zeta} z} \right|, \quad z, \zeta \in \mathbb{D},$$

and in $\mathbb{C}_+$ by δ,

$$\delta(z, \zeta) = \left| \frac{z - \zeta}{z - \bar{\zeta}} \right|, \quad z, \zeta \in \mathbb{C}_+.$$

The metrics extend to set-valued entries by taking infima. Clearly, the sequence W is separated, that is,

$$\inf\{\delta(z, w) : z, w \in W, z \neq w\} > 0.$$

Let D be an arbitrary disk in $\mathbb{C}_+$, and write $A(D)$ for its hyperbolic area. A computation then shows that the quotient $\operatorname{card}(W \cap D)/A(D)$ tends to a certain constant value as $A(D) \to +\infty$. The value of that constant, and the slightly different equivalent definition of densities in [12] (see also [11]), give

$$D^+(\Lambda) = D^-(\Lambda) = D^+(\Lambda') = D^-(\Lambda') = \beta .$$

The key to the proof of Proposition 5.1 is the following lemma.

Lemma 5.2. *Let Φ be the function*

$$\Phi(z) = (1 - |z|^2)^{-\beta} \left(\max\left(1, \log \frac{|1-z|^2}{1-|z|^2}\right)\right)^{2\pi c/b^2}, \quad z \in \mathbb{D} .$$

Then the above-defined sequence Λ is the zero set of a function g, analytic in $\mathbb{D}$, which satisfies the estimates

$$C_1 \varrho(z, \Lambda)\, \Phi(z) \leqq |g(z)| \leqq C_2 \Phi(z), \quad z \in \mathbb{D}, \tag{5.2}$$

$$C_1 (1 - |\lambda|^2)^{-1} \Phi(\lambda) \leqq |g'(\lambda)|, \quad \lambda \in \Lambda, \tag{5.3}$$

for some positive constants C_1, C_2.

Proof. To the above-defined sequence $W = \{w_{m,n}\}_{m,n}$, we associate the expression

$$h(z) = \left(\prod_{k=0}^{\infty} \frac{\sin(\pi b_k^{-1}(i - a^{-k} z))}{\sin(\pi b_k^{-1}(i + a^{-k} z))}\right) \left(\prod_{l=-\infty}^{-1} e^{2\pi/b_l} \frac{\sin(\pi b_l^{-1}(a^{-l} z - i))}{\sin(\pi b_l^{-1}(a^{-l} z + i))}\right).$$

By the usual convergence test for infinite products, the two products converge absolutely and for every $z \in \mathbb{C}_+$, and the convergence is uniform on compact subsets. Hence $h(z)$ is analytic in the upper half-plane $\mathbb{C}_+$, and W is its zero set. For integers j, let

$$h_j(z) = \left(\prod_{k=0}^{\infty} \frac{\sin(\pi b_{k-j}^{-1}(i - a^{-k} z))}{\sin(\pi b_{k-j}^{-1}(i + a^{-k} z))}\right) \left(\prod_{l=-\infty}^{-1} e^{2\pi/b_{l-j}} \frac{\sin(\pi b_{l-j}^{-1}(a^{-l} z - i))}{\sin(\pi b_{l-j}^{-1}(a^{-l} z + i))}\right).$$

where, just as with $h(z)$, the infinite products converge absolutely and uniformly on compacts. One checks that

$$h(a^{-j} z) = (-1)^j h_j(z) \exp\left(2\pi \sum_{k=1}^{j} \frac{1}{b_{-k}}\right), \quad z \in \mathbb{C}_+ ; \tag{5.4}$$

should j be negative, the sum $\sum_{k=1}^{j}$ is to be identified with $-\sum_{k=j+1}^{0}$. We use this relation to estimate the growth of h. A computation reveals that there exist positive constants K_1 and K_2, independent of j, such that

$$K_1 \delta(z, a^j W) \leqq |h_j(z)| \leqq K_2, \quad a^{-\frac{1}{2}} \leqq \Im z \leqq a^{\frac{1}{2}} .$$

Combining these inequalities with (5.4) and the fact that $\delta(az, aW) = \delta(z, W)$, we obtain the global estimates ($\beta = 2\pi/(b \log a)$)

$$(5.5) \qquad |h(z)| \leqq C y^{-\beta} \left(\max\left(1, \log \frac{1}{y}\right)\right)^{2\pi c/b^2}, \quad z = x + iy \in \mathbb{C}_+,$$

and

$$(5.6) \quad |h(z)| \geqq C\delta(z, W) y^{-\beta} \left(\max\left(1, \log \frac{1}{y}\right)\right)^{2\pi c/b^2}, \quad z = x + iy \in \mathbb{C}_+.$$

We pass to $\mathbb{D}$ and define

$$g(z) = (1 - z)^{-2\beta} \frac{1}{z} h\left(i \frac{1+z}{1-z}\right), \quad z \in \mathbb{D} \setminus \{0\};$$

the singularity at 0 is removable. The desired estimate (5.2) is obtained from (5.5) and (5.6), and (5.3) follows from (5.2). □

Proof of Proposition 5.1. We assume first $1 < p < \infty$ and that in the construction of Λ, $c \geqq b^2/(\pi p')$, where $p' = p/(p-1)$. Let a and b be such that the associated β equals α. We claim that

$$(5.7) \qquad f(0) = -\sum_{\lambda \in \Lambda} \frac{f(\lambda)}{g'(\lambda)} \frac{g(0)}{\lambda}, \quad f \in A_p^{-\alpha},$$

where g is as in Lemma 5.2. The assertion then follows, since by Hölder's inequality and (5.3), we have

$$(5.8) \qquad \left| \sum_{\lambda \in \Lambda} \frac{f(\lambda)}{g'(\lambda)} \frac{g(0)}{\lambda} \right| \leqq K \left(\sum_{\lambda \in \Lambda} |f(\lambda)|^p (1 - |\lambda|^2)^{\alpha p + 1} \right)^{1/p},$$

where

$$K = C \left(\sum_{\lambda \in \Lambda} (1 - |\lambda|^2) \left| \max\left(1, \log \frac{|1-\lambda|^2}{1 - |\lambda|^2}\right) \right|^{-2\pi c p'/b^2} \right)^{1/p'} < +\infty.$$

To see that $K < +\infty$, let

$$\Omega = \{\lambda \in \mathbb{D} : |1 - \lambda|^2 < 2(1 - |\lambda|^2)\},$$

which is a smaller disk tangential to $\mathbb{D}$ at 1, and observe that since the sequence Λ is separated, we can estimate the sum defining K with an integral (on an appropriate class of functions on $\mathbb{D}$, the point mass $(1 - |\lambda|^2)^2 d\delta_\lambda$ at $\lambda \in \mathbb{D}$ may be estimated by the area measure restricted to the disk with fixed pseudohyperbolic radius r, $0 < r < 1$, around λ)

$$K \leqq C \left(\int_\Omega \frac{dS(z)}{1 - |z|^2} + \int_{\mathbb{D} \setminus \Omega} \left(\log \frac{|1-z|^2}{1 - |z|^2} \right)^{-2} \frac{dS(z)}{1 - |z|^2} \right)^{1/p'};$$

here, we used the assumption $c \geqq b^2/(\pi p')$.

It suffices to verify (5.7) for f that extend holomorphically to a larger disk $r\mathbb{D}$, $r > 1$, because it then extends, by approximation and (5.8), to all $f \in A_p^{-\alpha}$. To this end, we use the separation of Λ to find a small positive real number d such that the circles $\varrho(z, \lambda) = d$, with $\lambda \in \Lambda$, are nonintersecting. For $0 < t < 1$, let $S(t)$ denote the positively oriented closed path around the origin which is the union of the set of $z \in \mathbb{D}$ on the circle $|z| = t$ for which $\varrho(z, \Lambda) \geqq d$ and the shorter arcs of those circles $\varrho(z, \lambda) = d$, with $\lambda \in \Lambda$, which intersect the circle $|z| = t$. It is clear that the length of $S(t)$ is uniformly bounded. The path $S(t)$ encloses a domain $D(t)$. By the calculus of residues, we have

$$\frac{1}{2\pi i} \int_{S(t)} \frac{f(\zeta)}{g(\zeta)\zeta} d\zeta = \frac{f(0)}{g(0)} + \sum_{\lambda \in \Lambda \cap D(t)} \frac{f(\lambda)}{g'(\lambda)\lambda}.$$

In view of (5.2), the integral on the left tends to 0 as $t \mapsto 1$ because $\varrho(S(t), \Lambda) \geqq d$ and $|f(\zeta)|$ is bounded throughout $\mathbb{D}$.

For $0 < p \leqq 1$, we choose $c = 0$ and proceed similarly. □

6. Invariant subspaces with infinite index

In this section, we shall use the results of Sections 2, 3, and 5 to construct an invariant subspace $\mathcal{M}$ of $A_p^{-\alpha}$ with infinite index in the Banach space case

$$0 < \alpha < +\infty, \quad 1 \leqq p < +\infty,$$

and indicate how to extend the construction to $0 < p < 1$.

The existence of such subspaces has been known for some time in the Hilbert space case $p = 2$. It follows from a theorem of Apostol, Bercovici, Foiaş, and Pearcy (see [1]), which asserts that the shift M_z on $A_2^{-\alpha}$ is in the class $\mathbb{A}_{\aleph_0}$ of universal dilations. This theorem also implies that one could answer the invariant subspace problem for separable Hilbert space affirmatively, if, for a fixed α, $0 < \alpha < +\infty$, one were able to show that between any two shift invariant subspaces $\mathcal{N} \subset \mathcal{M}$ of the Bergman space $A_2^{-\alpha}$ with $\dim(\mathcal{M} \ominus \mathcal{N}) = +\infty$, there is an invariant subspace $\mathcal{K}$, $\mathcal{N} \subsetneqq \mathcal{K} \subsetneqq \mathcal{M}$. For the benefit of the reader who is not familiar with the machinery of [1], [2], we indicate in a corollary how that result can be obtained from Theorem 6.1 and two dilation theorems, whose (short) proofs are based on elementary Hilbert space theory.

The existence of invariant subspaces in $A_p^{-\alpha}$ of arbitrary index was settled by Eschmeier [3] in the Banach space case $1 \leqq p < \infty$, with $\alpha = 1/p$. The machinery of [1], [2], [3] provides an existence proof of these invariant subspaces; we carry out an actual construction.

Theorem 6.1. $(1 \leqq p < +\infty, 0 < \alpha < +\infty)$ *There exists an invariant subspace* $\mathcal{M}$ *of* $A_p^{-\alpha}$ *with* $\operatorname{ind}(\mathcal{M}) = +\infty$.

Proof. Let $\Lambda = \{\lambda_n\}_n \subset \mathbb{D}$ be the separated sequence that was constructed in Section 5 (there, Λ was indexed by two parameters n and m, but here it is more convenient to have a single index parameter n). Recall that it satisfies $\Lambda \subset \mathbb{D}\setminus\{0\}$,

$$D^-(\Lambda) = D^+(\Lambda) = \alpha\,,$$

and

$$|f(0)| \leqq C\Big(\int_{\mathbb{D}} |f|^p d\mu_{\Lambda,1+\alpha p}\Big)^{1/p} \quad \text{for all } f \in A_p^{-\alpha},$$

where $d\mu_{\Lambda,1+\alpha p}$ is as in Section 4. Let $\Lambda_0 = \emptyset$, and define inductively Λ_k by saying that it should be formed by taking every other point of $\Lambda \setminus \sum_{j=0}^{k-1} \Lambda_j$, for $k = 1, 2, 3, \ldots$; this sounds vague, but can be made quite specific by looking at the sequence W in $\mathbb{C}_+$ that Λ comes from, and at each horizontal level line that contains points from W, we pick every other point in the sequence corresponding to $\Lambda \setminus \bigcup_{j=0}^{k-1} \Lambda_j$. Form the complementary sequences $B_k = \Lambda\setminus\Lambda_k$. Then, by an argument similar to the one that gave the upper and lower uniform densities of Λ in Section 5, we have, for $j = 1, 2, 3, \ldots$,

$$D^+(\Lambda_i) = D^-(\Lambda_j) = 2^{-j}\alpha < \alpha\,,$$

$$D^+(B_j) = D^-(B_j) = (1-2^{-j})\alpha < \alpha\,,$$

so that by Theorem 3.1, Λ_j and B_j are interpolating sequences for $A_p^{-\alpha}$, and, in particular, $I(B_j) \neq (0)$. We claim that $\mathcal{M} = \bigvee_{j=1}^{\infty} I(B_j)$ has index $+\infty$. By Theorem 2.1, it suffices to verify that for each $j \in \mathbb{N}$,

$$|f(0)| \leqq C_j \|f\|_{A_p^{-\alpha}/\hat{\mathcal{M}}_j}, \quad f \in I(B_j)\,,$$

where $\hat{\mathcal{M}}_j = \bigvee_{i:i\neq j} I(B_i)$. Note that $\hat{\mathcal{M}}_j \subset I(\Lambda_j)$. Thus, for $f \in I(B_j)$, we have

$$\begin{aligned}|f(0)| &\leqq C\Big(\int_{\mathbb{D}} |f|^p d\mu_{\Lambda,1+\alpha p}\Big)^{1/p} = C\Big(\int_{\mathbb{D}} |f|^p d\mu_{\Lambda_j,1+\alpha p}\Big)^{1/p}\\ &\leqq C'\|f\|_{A_p^{-\alpha}/I(\Lambda_j)} \leqq C'\|f\|_{A_p^{-\alpha}/\hat{\mathcal{M}}_j}\,.\end{aligned}$$

Here we used that the sequence Λ_j is separated (see Section 3). The proof is complete. □

Remark. The assertion of Theorem 6.1 holds also for $0 < p < 1$, though to see that this is so requires extra work, because $A_p^{-\alpha}$ is not a Banach space then. Note that for $0 < p < 1$, $A_p^{-\alpha}$ is a homogeneous invariant F-space: it has a complete invariant metric

$$d(f,g) = \|f-g\|_{A_p^{-\alpha}}^p, \quad f, g \in A_p^{-\alpha},$$

which is homogeneous of degree p,

$$d(\tau f, 0) = |\tau|^p d(f,0), \quad f \in A_p^{-\alpha},\ \tau \in \mathbb{C}\,.$$

One checks that these properties are for some purposes sufficient to obtain desired results. For instance, if L is a continuous linear operator on $A_p^{-\alpha}$, then $1-\beta L$ is invertible for $\beta \in \mathbb{C}$ near 0. As a consequence, one can prove that (2.1) holds for $A_p^{-\alpha}$, by writing down a direct proof of it in the Banach space situation which does not appeal to the Fredholm theory. It follows that Theorem 2.1, suitably modified, holds as well (the proof just uses some elementary arguments from finite-dimensional linear algebra). The rest of the construction in the proof of Theorem 6.1 works without modifications for $0 < p < 1$.

Let T be an operator on a Hilbert space $\mathscr{H}$. A subspace $\mathscr{K}$ of $\mathscr{H}$ is called semi-invariant for T, if there are two invariant subspaces $\mathscr{M}$ and $\mathscr{N}$ of T with $\mathscr{N} \subset \mathscr{M}$, such that $\mathscr{K} = \mathscr{M} \ominus \mathscr{N}$. Let P be the orthogonal projection operator $\mathscr{H} \to \mathscr{K}$. Then the compression of T to the semi-invariant subspace $\mathscr{K}$ is $T_{\mathscr{K}} = PT|_{\mathscr{K}}$. The following corollary (see [1]) shows the connection to the invariant subspace problem that was mentioned above.

Corollary 6.2. $(0 < \alpha < +\infty)$ *Let $\mathscr{H}$ be a separable Hilbert space and let A be a strict contraction on $\mathscr{H}$, that is, $\|A\| < 1$. Then there is a semi-invariant subspace $\mathscr{K} \subset A_2^{-\alpha}$ of the Bergman shift M_z such that the compression of M_z to $\mathscr{K}$ is unitarily equivalent to A.*

Proof. Choose β, $\|A\| < \beta < 1$, and let $\mathscr{M}$ be an invariant subspace of M_z with $\operatorname{ind}(\mathscr{M}) = +\infty$. Then for each $n \in \mathbb{N}$, $z^n\mathscr{M}$ has index $+\infty$, and we may choose $n_0 \in \mathbb{N}$ such that if $\mathscr{N} = z^{n_0}\mathscr{M}$, then the operator $M_z|_{\mathscr{N}}$ is bounded below by β. Thus, the adjoint $(M_z|_{\mathscr{N}})^*$ satisfies the hypothesis of Theorem 5.9 of [2] and we may conclude that $M_z|_{\mathscr{N}}$ has a semi-invariant subspace $\mathscr{K}$, so that the compression of $M_z|_{\mathscr{N}}$ to $\mathscr{K}$ is unitarily equivalent to βU, where U is the (forward unweighted) shift of infinite multiplicity. Moreover, it is well known (see [13], Theorems I.4.1 and II.2.1) that the strict contraction A with $\|A\| < \beta$ is unitarily equivalent to the compression of βU to some semi-invariant subspace. It follows that A is unitarily equivalent to the compression of M_z to some semi-invariant subspace. □

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