Bergman spaces and differential geometry

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Geometries induced by the Bergman kernel

Bergman's geometric idea

Stefan Bergman's idea here was to let the Bergman kernel $K(z, w)$ on a given planar domain induce a metric on the domain, analogous to the Poincaré metric for the unit disk [which models the hyperbolic plane].

Two distinct possibilities:

(i) $ds_I^2 := K(z, z)|dz|^2$

(ii) $ds_{II}^2 := [\Delta \log K(z, z)]|dz|^2$.

Here, we write Δ for the normalized Laplacian

$$\Delta := \frac{1}{4}\left(\partial_x^2 + \partial_y^2\right).$$

Remark 1

Up to a constant multiple, these variants both equip the unit disk $\mathbb{D}$ with the Poincaré metric, as a calculation gives that $K(z, w) = (1 - z\bar{w})^{-2}$, and so $ds_I^2 = (1 - |z|^2)^{-2}|dz|^2$ while
$ds_{II}^2 = -2[\Delta \log(1 - |z|^2)]|dz|^2 = 2(1 - |z|^2)^{-2}|dz|^2$.

More on induced geometries

Remark 2

The variant (i) was the first geometry proposed by Bergman. The variant (ii) is now used comonly in the several complex variables setting. Note that it is natural to connect the Laplacian with a 2-form (more specifically, a $(1,1)$-form) so the definition of the Bergman metric may be thought as a choice of switch from exterior (tensor) products to interior (tensor) products.

Remark 3

In the recent paper [HH], Bergman's geometries were carried over to the polyanalytic setting (solutions to $\bar{\partial}^q f = 0$).

We will consider domains in the plane which are already equipped with a metric. We may think of the surface as an abstract Riemannian surface, if we like. We consider a surface $\mathbf{\Omega}$, which is diffeomorphic to the unit disk $\mathbb{D}$ and modelled by the pair $\langle D, \omega \rangle$, with metric $ds^2 = \omega(z)|dz|^2$. Here, ω is assumed strictly positive a smooth up to boundary of $\mathbb{D}$. The curvature form is then

$$\mathbf{K} = -2[\Delta \log \omega(z)] dA(z),$$

where $dA(z) = \pi^{-1} dxdy$ is normalized area measure.

An optimization problem

Problem 4

Minimize the area of $\mathbf{\Omega} = \langle \mathbb{D}, \omega \rangle$ under given curvature form $\mathbf{K}$.

Problem 1 has in general no solution! This is because we may find surfaces with given smooth curvature form whose area is arbitrarily close to 0. To remedy this, we should look for a suitable normalization! What might work? We suggest the following.

Proposed normalization

Assume that $ds^2 \approx |dz|^2$ near $z = 0$, i.e., $\omega(0) = 1$. [Another point $z_0 \in \mathbb{D}$ could be used in place of 0.]

Problem 5 (normalized)

Minimize the area of $\mathbf{\Omega} = \langle \mathbb{D}, \omega \rangle$ under given curvature form $\mathbf{K}$, subject to the condition $\omega(0) = 1$.

Concrete version of the optimization problem

Let us suppose that the curvature form is $\mathbf{K} = 2\mu(z)dA(z)$, where μ is assumed smooth up to the boundary.

Problem 5' (concrete)

Minimize the area

$$\int_{\mathbb{D}} \omega dA,$$

given that $\omega(0) = 1$ and $\Delta \log \omega = -\mu$.

Question 6

Is the above minimal area attained by a strictly positive weight ω which is smooth in the closed unit disk $\bar{\mathbb{D}}$?

Potential theory

We let $G(z, w)$ be the Green function:

$$G(z, w) := 2 \log \left| \frac{z - w}{1 - z\bar{w}} \right|,$$

and form the corresponding potential

$$G[\mu](z) := \int_{\mathbb{D}} G(z, w)\mu(w) dA(w).$$

Then $\Delta G[\mu] = \mu$ automatically. Next, we form the weight ω_1:

$$\omega_1(z) := \exp\{-G[\mu](z)\},$$

so that $\Delta \log \omega_1 = -\mu$, and ω_1 is smooth and strictly positive throughout $\bar{\mathbb{D}}$, with boundary value 1. But ω_1 does not solve our problem, for two reasons: (a) $\omega_1(0) \neq 1$ in general, and (b) it need not be an area minimizer. But we feel that we have achieved something. We may look for an area-minimizer of the form

$$\omega_0(z) := e^{2H(z)}\omega_1(z),$$

where H is real-valued, harmonic in $\mathbb{D}$ and smooth in $\bar{\mathbb{D}}$.

A related optimization problem

It is useful to write

$$F := e^{H+i\tilde{H}},$$

where $\tilde{H}$ is the harmonic conjugate. This is because $|F|^2 = e^{2H}$, so that

$$\omega_0(z) = |F(z)|^2 \omega_1(z).$$

New optimization problem 7

To minimize

$$\int_{\mathbb{D}} |F|^2 \omega_1 dA,$$

over all functions F holomorphic in $\mathbb{D}$ and smooth up to the boundary, subject to the normalization

$$F(0) = \omega_1(0)^{-1/2}.$$

Solution to the related optimization problem

It is quickly understood that the solution to the above *new optimization problem 7* is

$$F(z) = \omega_1(0)^{-1/2} \frac{K_{\omega_1}(z,0)}{K_{\omega_1}(0,0)}.$$

Here, K_{ω_1} is the Bergman kernel for the space with norm

$$\|f\|^2 = \int_{\mathbb{D}} |f|^2 \omega_1 dA.$$

Then we have indeed found the solution to Problem 5', provided that $K_{\omega_1}(z,0) \neq 0$ in $\bar{\mathbb{D}}$.

Elliptic regularity

The function $z \mapsto K_{\omega_1}(z,0)$ is smooth in $\bar{\mathbb{D}}$.

The first theorem

Theorem 8

[Hedenmalm, Perdomo-G.] The original minimization problem has a smooth solution (up to the boundary) if and only if $K_{\omega_1}(z,0) \neq 0$ on $\bar{\mathbb{D}}$.

The second theorem

Theorem 9

[Hedenmalm, Perdomo-G.] Suppose μ is real-valued and smooth on $\bar{\mathbb{D}}$. If, in addition,

$$\mu(z) \leq \frac{1}{(1-|z|^2)^2}, \qquad z \in \mathbb{D},$$

then $K_{\omega_{\mathbf{1}}}(z, w) \neq 0$ if $z, w \in \bar{\mathbb{D}}$ and at least one of these two points lies in the interior $\mathbb{D}$.

Sharpness of the second theorem

Theorem 10

[Hedenmalm, Perdomo-G.] There exists a real-valued function μ, smooth in $\bar{\mathbb{D}}$, with

$$\mu(z) \leq \frac{1.04}{(1-|z|^2)^2}, \qquad z \in \mathbb{D},$$

such that the function $z \mapsto K_{\omega_1}(z, 0)$ has a zero in $\mathbb{D}$.

Bibliography

HH. Haimi, A., Hedenmalm, H., *Asymptotic expansion of polyanalytic Bergman kernels*. Submitted.

HO. Hedenmalm, H., Olofsson, A., *Hele-Shaw flow on weakly hyperbolic surfaces*. Indiana Univ. Math. J. 54 (2005), 1161-1180.

HP. Hedenmalm, H., Perdomo-G., Y., *Mean value surfaces with prescribed curvature form*. J. Math. Pures Appl. 83 (2004), 1075-1107.

HS. Hedenmalm, H., Shimorin, S., *Hele-Shaw flow on hyperbolic surfaces*. J. Math. Pures Appl. 81 (2002), 187-222.