$\boxed{\overset{\text{ㅐ}}{\text{O}}9}$ 1

7.1 3, 15, 37, 50

7.2 5, 15, 27, 33, 37, 39

7.3 3, 15, 27, 39, 49-54, 57,

7.4 7, 21, 25, 29, 39, 53

7.5 5, 11

7.6 1, 7

7.1

3)
$$f(t) = \begin{cases} t, & 0 \le t < 1 \\ 1, & t \ge 1 \end{cases}$$

$$\mathcal{L}\{f\}(s) = ?$$

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\,dt = \int_0^1 e^{-st} \overbrace{f(t)}^{t}\,dt +$$

$$+ \int_1^\infty e^{-st} \overbrace{f(t)}^{1}\,dt =$$

$$= \int_0^1 e^{-st} \cdot t\,dt + \int_1^\infty e^{-st}\,dt =$$ ②

$$= \left[-\frac{1}{s}\left(t+\frac{1}{s}\right)e^{-st}\right]_{t=0}^{t=1} + \left[\frac{e^{-st}}{-s}\right]_{t=1}^{\infty} =$$

$$\underset{\text{an } s>0}{=} -\frac{1}{s}\left(1+\frac{1}{s}\right)e^{-s} + \frac{1}{s^2} + \frac{e^{-s}}{s}$$

---

15) $f(t) = e^{-t}\sin t$

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} e^{-t} \sin t\,dt =$$

$$= \int_0^\infty e^{-(s+1)t} \sin t\,dt = \mathcal{L}\{\sin t\}(s+1).$$

$$\mathcal{L}\{\sin t\}(s) = \int_0^\infty \overset{\downarrow}{e^{-st}} \overset{\uparrow}{\sin t}\,dt \overset{(s>0)}{=} \left[-\cos t \cdot e^{-st}\right]_0^\infty$$

$$- \int_0^\infty (-\cos t)(-s)e^{-st}\,dt \overset{s>0}{\underset{\downarrow}{=}} 1 - s\int_0^\infty \cos t \cdot e^{-st}\,dt$$

$$= 1 - s\,\mathcal{L}\{\cos t\}(s)$$

$$\mathcal{L}\{\cos t\}(s) = \int_0^\infty e^{-st}\cos t\, dt \overset{(s>0)}{=} \Big[\sin t \cdot e^{-st}\Big]_{t=0}^{\infty}$$

$$-\int_0^\infty \sin t \cdot (-s)e^{-st}\, dt = s\int_0^\infty \sin t \cdot e^{-st}\, dt =$$

$$= s\,\mathcal{L}\{\sin t\}(s)$$

Samla ihop:

$$\begin{cases} \mathcal{L}\{\sin t\}(s) = 1 - s\mathcal{L}\{\cos t\}(s) \\ \mathcal{L}\{\cos t\}(s) = s\,\mathcal{L}\{\sin t\}(s) \end{cases}$$

Ekvationssystem som vi kan lösa!

Lösningen blir

$$\begin{cases} \mathcal{L}\{\sin t\}(s) = \dfrac{1}{s^2+1} \\ \mathcal{L}\{\cos t\}(s) = \dfrac{s}{s^2+1} \end{cases}$$

så att

$$\mathcal{L}\{f\}(s) = \mathcal{L}(\sin t)(s+1) = \frac{1}{(s+1)^2+1}.$$

37)

$$\mathcal{L}\{\sin 2t \cos 2t\}(s) = ?$$

$$\begin{cases} \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ \sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta \end{cases} \quad +$$

$$\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin\alpha\cos\beta$$

Så att:

$$\sin 2t \cos 2t = \frac{1}{2}\left(\sin(2t+2t) + \sin 0\right) =$$

$$= \frac{1}{2}\sin 4t$$

och vi får att

$$\mathcal{L}\{\sin 2t\cos 2t\}(s) = \mathcal{L}\{\tfrac{1}{2}\sin 4t\}(s) =$$

$$= \frac{1}{2}\int_0^\infty e^{-st}\sin 4t \, dt \overset{(s>0)}{=} \left[t = \frac{u}{4}\right] =$$

$$= \frac{1}{2}\int_0^\infty e^{-su/4}\sin u \, \frac{du}{4} = \frac{1}{8}\mathcal{L}\{\sin t\}(s/4)$$

$$= \frac{1}{8}\cdot\frac{1}{\left(\frac{s}{4}\right)^2+1} = \frac{2}{s^2+16}$$

7.2 5)

$$\mathcal{L}^{-1}\left\{\frac{(s+1)^3}{s^4}\right\}(t) = ?$$

$$\frac{(s+1)^3}{s^4} = \frac{s^3+3s^2+3s+1}{s^4} = \frac{1}{s} + \frac{3}{s^2} + \frac{3}{s^3} + \frac{1}{s^4}$$

~~Sats 7.21~~ så enligt Sats 7.21 är

$$\mathcal{L}^{-1}\left\{\frac{(s+1)^3}{s^4}\right\}(t) = 1 + 3t + \frac{3}{2}t^2 + \frac{1}{6}t^3$$

---

15) $$\mathcal{L}^{-1}\left\{\frac{2s-6}{s^2+9}\right\} = 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} - 2\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}$$

$$= 2\cos 3t - 2\sin 3t.$$

---

27) $$\mathcal{L}^{-1}\left\{\frac{2s-4}{(s^2+s)(s^2+1)}\right\} = ?$$

$$(s^2+s)(s^2+1) = s(s+1)(s^2+1)$$
så att

$$\frac{2s-4}{(s^2+s)(s^2+1)} = \frac{A}{s} + \frac{B}{s+1} + \frac{Cs+D}{s^2+1}$$

$$2s-4 = A(s+1)(s^2+1) + Bs(s^2+1) + (Cs+D)s(s+1)$$

$s=0$: $-4 = A + 0 \Rightarrow A = -4$

$s=-1$: $-6 = 0 - B\cdot 2 + 0 \Rightarrow B = 3$

Högstagradstermer (s^3):

$$0 = A + B + C = -4+3+C \Rightarrow C = 1$$

s^2-termer:

$$0 = A + D + C = -4 + D + 1 \Rightarrow D = 3$$

$$\frac{2s-4}{(s^2+s)(s^2+1)} = -\frac{4}{s} + \frac{3}{s+1} + \frac{s+3}{s^2+1}$$

och därför blir

$$\mathcal{L}^{-1}\left\{\frac{2s-4}{(s^2+s)(s^2+1)}\right\}(t) = -4 + 3e^{-t} + \cos t + 3\sin t$$

33)

$$\begin{cases} y' + 6y = e^{4t} \\ y(0) = 2 \end{cases}$$

$\mathcal{L}$-transformera ekvationen:

$$\underbrace{\mathcal{L}\{y'+6y\}}_{} = \mathcal{L}\{e^{4t}\} = \frac{1}{s-4}$$

$\mathcal{L}\{y'\} + 6\mathcal{L}\{y\}$

Vi skriver $Y(s) = \mathcal{L}\{y\}(s)$. Eftersom

$$\mathcal{L}\{y'\} = sY(s) - y(0) \underset{\substack{\uparrow \\ \text{begdata}}}{=} sY(s) - 2$$

så har vi

$$sY(s) - 2 + 6Y(s) = \frac{1}{s-4}$$

$$(s+6)Y(s) = 2 + \frac{1}{s-4} \Rightarrow Y(s) = \frac{2}{s+6} + \frac{1}{(s+6)(s-4)} =$$

$$= \frac{2}{s+6} + \frac{1}{10}\left(\frac{1}{s-4} - \frac{1}{s+6}\right) = \frac{19}{10}\cdot\frac{1}{s+6} + \frac{1}{10}\frac{1}{s-4}$$

dvs $y = \mathcal{L}^{-1}\{Y\} = \frac{19}{10}e^{-6t} + \frac{1}{10}e^{4t}$.

7.3

3) $\mathcal{L}\{t^3 e^{-2t}\}(s) = \int_0^\infty e^{-(s+2)t} t^3\, dt \overset{(s>0)}{=}$

$= \dfrac{3!}{(s+2)^4} = \dfrac{6}{(s+2)^4}.$

15) $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+4s+5}\right\} = ?$

$s^2+4s+5 \overset{\text{kv. kompl.}}{=} s^2+4s+4+1 = (s+2)^2+1.$

så $\dfrac{s}{s^2+4s+5} = \dfrac{s}{(s+2)^2+1} = \dfrac{s+2-2}{(s+2)^2+1} = \dfrac{s+2}{(s+2)^2+1} - \dfrac{2}{(s+2)^2+1}$

~~Så att~~

$$\begin{cases} \mathcal{L}^{-1}\left\{\dfrac{s+2}{(s+2)^2+1}\right\} \overset{\text{Sats 7.3.1}}{=} e^{-2t}\cdot \cos t \\ \mathcal{L}^{-1}\left\{\dfrac{2}{(s+2)^2+1}\right\} \overset{\text{Sats 7.3.1}}{=} 2e^{-2t}\sin t \end{cases}$$

och därför är

$\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+4s+5}\right\} = e^{-2t}\cos t - 2e^{-2t}\sin t.$

27) $\begin{cases} y'' - 6y' + 13y = 0 \\ y(0) = 0 \\ y'(0) = -3 \end{cases}$ lös med $\mathcal{L}$-transf.

$$\boxed{\mathcal{L}\{y''\} - 6\mathcal{L}\{y'\} + 13\mathcal{L}\{y\} = 0}$$

$$\begin{cases} \mathcal{L}\{y\} = Y \text{ skriver vi} \\ \mathcal{L}\{y'\} = sY - \underbrace{y(0)}_{=0} = sY \\ \mathcal{L}\{y''\} = s^2Y - s\overset{=0}{y(0)} - \overset{=-3}{y'(0)} = s^2Y + 3 \end{cases}$$

så att

$$s^2Y + 3 - 6(sY) + 13Y = 0.$$

$$\underbrace{(s^2 - 6s + 13)}_{(s-3)^2+4} Y = -3$$

$$Y = -\frac{3}{(s-3)^2+4} \Rightarrow \boxed{y = -\frac{3}{2}e^{3t}\sin(2t)}$$

SVARET

$\mathcal{U}$ = Heaviside-funktionen!

39)

$$\mathcal{L}\{t\,\mathcal{U}(t-2)\} = \mathcal{L}\{(t-2)\mathcal{U}(t-2) + 2\cdot\mathcal{U}(t-2)\} =$$

$$= e^{-2s}\mathcal{L}\{t\}(s) + 2e^{-2s}\mathcal{L}\{1\}(s) =$$

$$= e^{-2s}\cdot\frac{1}{s^2} + 2e^{-2s}\cdot\frac{1}{s}.$$

---

57)

$$f(t) = \begin{cases} 0, & 0 \le t < 1 \\ t^2, & t \ge 1 \end{cases} = t^2\,\mathcal{U}(t-1).$$

7.4

SATS 7.4.1 $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}F(s)$

7)

$$\mathcal{L}\{t e^{2t}\sin 6t\} = (-1)^1\frac{d}{ds}\mathcal{L}\{e^{2t}\sin 6t\} =$$

$$= -\frac{d}{ds}\int_0^\infty e^{-st}e^{2t}\sin 6t\,dt =$$

$$= -\frac{d}{ds}\int_0^\infty e^{-(s-2)t}\sin(6t)\,dt \overset{(s>2)}{=} -\frac{d}{ds}\left[\left.\frac{6}{s^2+36}\right|_{s\to s-2}\right] =$$

$$= -\frac{d}{ds}\frac{6}{(s-2)^2+36} = \frac{6(2(s-2))}{((s-2)^2+36)^2} = \frac{12s-24}{[(s-2)^2+36]^2}$$

SATS 7.4.2

21) $\mathcal{L}\{e^{-t} * e^{t}\cos t\} \overset{\downarrow}{=} \mathcal{L}\{e^{-t}\} \cdot \mathcal{L}\{e^{t}\cos t\} =$

$= \frac{1}{s+1} \cdot \frac{s-1}{(s-1)^2+1}$

25) $\mathcal{L}\{\int_0^t e^{-\tau}\cos\tau\, d\tau\} = \mathcal{L}\{1 * e^{-t}\cos t\} =$

$= \mathcal{L}\{1\} \cdot \mathcal{L}\{e^{-t}\cos t\} = \frac{1}{s} \cdot \frac{s+1}{(s+1)^2+1}$

29) $\mathcal{L}\{t\int_0^t \sin\tau\, d\tau\} = -\frac{d}{ds}\mathcal{L}\{\int_0^t \sin\tau\, d\tau\} =$

$= -\frac{d}{ds}\mathcal{L}\{1 * \sin t\} = -\frac{d}{ds}\left[\mathcal{L}\{1\} \cdot \mathcal{L}\{\sin t\}\right] =$

$= -\frac{d}{ds}\left[\frac{1}{s} \cdot \frac{1}{s^2+1}\right] = -\frac{d}{ds}\left(\frac{1}{s} \cdot \frac{1}{s^2+1}\right) =$

$= -\left(-\frac{1}{s^2} \frac{1}{s^2+1} - \frac{1}{s} \frac{2s}{(s^2+1)^2}\right) = \frac{1}{s^2(s^2+1)} + \frac{2}{(s^2+1)^2} =$

$= \frac{s^2+1+2s^2}{s^2(s^2+1)^2} = \frac{3s^2+1}{s^2(s^2+1)^2}$

59)

(12)

$$f(t) = te^t + \underbrace{\int_0^t \tau f(t-\tau)\,d\tau}_{t * f(t)}$$

$$\mathcal{L}\{f\} = \mathcal{L}\{te^t\} + \underbrace{\mathcal{L}\{t * f(t)\}}_{\mathcal{L}\{t\}\cdot\mathcal{L}\{f\}}$$

$$F = \mathcal{L}\{f\}$$

$$F = \frac{1}{(s-1)^2} + \frac{1}{s^2}F$$

$$F\left(1 - \frac{1}{s^2}\right) = \frac{1}{(s-1)^2}$$

$$F = \frac{1}{(s-1)^2}\cdot\frac{1}{1-s^{-2}} = \frac{s^2}{(s-1)^2(s^2-1)} = \frac{s^2}{(s-1)^3(s+1)}$$

$$\frac{s^2}{(s-1)^3(s+1)} \overset{PBU}{=} \frac{A}{s+1} + \frac{B}{s-1} + \frac{C}{(s-1)^2} + \frac{D}{(s-1)^3}$$

$$s^2 = A(s-1)^3 + B(s-1)^2(s+1) + C(s-1)(s+1) + D(s+1)$$

$\underline{s=1}$: $\quad 1^2 = 0 + D\cdot 2 \Rightarrow D = 1/2$

$\underline{s=-1}$: $\quad (-1)^2 = A(-2)^3 + 0 \Rightarrow A = -1/8$

Högstagradstermen: $\quad 0 = A + B + 0 \Rightarrow B = 1/8$

$\underline{s=0}$ $\quad 0^2 = A(-1)^3 + B(-1)^2(1) + C(-1)(1) + D$

$$0 = -A + B - C + D$$

$$C = -A + B + D = \frac{1}{8} + \frac{1}{8} + \frac{1}{2} = \frac{3}{4}.$$

$$F = -\frac{1}{8}\cdot\frac{1}{s+1} + \frac{1}{8}\cdot\frac{1}{s-1} + \frac{3}{4}\cdot\frac{1}{(s-1)^2} + \frac{1}{2}\cdot\frac{1}{(s-1)^3}$$

$$\therefore \boxed{f(t) = -\frac{1}{8}e^{-t} + \frac{1}{8}e^{t} + \frac{3}{4}te^{t} + \frac{1}{4}t^2e^{t}}$$

7.5 5)

$$\begin{cases} y'' + y = \delta\left(t - \frac{\pi}{2}\right) + \delta\left(t - \frac{3\pi}{2}\right) \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$$

$$\mathcal{L}\{y'' + y\} = \mathcal{L}\left\{\delta\left(t - \tfrac{\pi}{2}\right) + \delta\left(t - \tfrac{3\pi}{2}\right)\right\} = e^{-\frac{\pi}{2}s} + e^{-\frac{3\pi}{2}s}$$

$$\|$$

$$\underbrace{\mathcal{L}\{y''\} + \underbrace{\mathcal{L}\{y\}}_{=Y}}$$

$$\|$$

$$s^2 Y - s\underbrace{y(0)}_{=0} - \underbrace{y'(0)}_{=0} + Y$$

så att

$$\boxed{(s^2+1)Y = e^{-\frac{\pi}{2}s} + e^{-\frac{3\pi}{2}s}}$$ och därför

$$Y = \frac{e^{-\frac{\pi}{2}s}}{s^2+1} + \frac{e^{-\frac{3\pi}{2}s}}{s^2+1} \Rightarrow$$

$$y = \underbrace{\sin\left(t - \tfrac{\pi}{2}\right)}_{-\cos t} \mathcal{U}\left(t - \tfrac{\pi}{2}\right) + \overbrace{\sin\left(t - \tfrac{3\pi}{2}\right)}^{\cos t} \mathcal{U}\left(t - \tfrac{3\pi}{2}\right) =$$

$$= (\cos t)\left[\underbrace{-\mathcal{U}\left(t - \tfrac{\pi}{2}\right) + \mathcal{U}\left(t - \tfrac{3\pi}{2}\right)}_{\text{fyrkantsvåg}}\right]$$

7.6 1) $\begin{cases} \frac{dx}{dt} = -x+y \\ \frac{dy}{dx} = 2x \end{cases}$ BV. $\begin{cases} x(0)=0 \\ y(0)=1 \end{cases}$

$X = \mathcal{L}\{x\}$, $Y = \mathcal{L}\{y\}$,

$\mathcal{L}\{f'\} = sF - f(0)$ generellt så:

$$\begin{cases} \mathcal{L}\{\frac{dx}{dt}\} = sX - x(0) = sX \\ \mathcal{L}\{\frac{dy}{dt}\} = sY - y(0) = sY - 1. \end{cases}$$

$$\begin{cases} sX = -X + Y \\ sY - 1 = 2X \end{cases} \Leftrightarrow \begin{cases} (s+1)X = Y \\ 2X - sY = -1 \end{cases}$$ elw. system

Lösn. $\begin{cases} X = \frac{1}{(s-1)(s+2)} = \frac{1}{3}\left(\frac{1}{s-1} - \frac{1}{s+2}\right) \\ Y = \frac{s+1}{(s-1)(s+2)} = \frac{1}{3}\left(\frac{2}{s-1} + \frac{1}{s+2}\right) \end{cases}$

så att

$$\begin{cases} x(t) = \frac{1}{3}\left(e^{t} - e^{-2t}\right) \\ y(t) = \frac{1}{3}\left(2e^{t} + e^{-2t}\right) \end{cases}$$