Ö7

11.1 11, 21

11.2 5+19, 15

11.3 23, 27, 41

11) $\left\{1, \cos\left(\frac{n\pi}{p}x\right)\right\}$, $n=1,2,3,\ldots$ på $[0,p]$.

Visa att ~~funktionsföljden~~ är ortogonala!

Vi ska alltså visa att

$$\int_0^p 1\cdot\cos\left(\frac{n\pi}{p}x\right)dx = 0. \qquad (n=1,2,3,\ldots)$$

Men

$$\int_0^p 1\cdot\cos\left(\frac{n\pi}{p}x\right)dx = \int_0^p \cos\left(\frac{n\pi}{p}x\right)dx =$$

$$= \left[\left(\sin\left(\frac{n\pi}{p}x\right)\right)\cdot\frac{p}{n\pi}\right]_0^p = \frac{p}{n\pi}\left[\underbrace{\sin n\pi}_{=0} - \underbrace{\sin 0}_{=0}\right]$$

$= 0.$

21)

(a) $f(x) = \cos 2\pi x$. Fundamentala perioden är $T = 1$.

(b) $f(x) = \sin\left(\frac{4}{L}x\right)$ fundamentala perioden är $T = \frac{L}{4} \cdot 2\pi = \frac{\pi L}{2}$.

(c) $f(x) = \sin x + \sin(2x)$. $T = 2\pi$

(d) $f(x) = \sin(2x) + \cos(4x)$. $T = \pi$

(e) $f(x) = \sin(3x) + \cos(2x)$. $T = 2\pi$.

(f) $f(x) = A_0 + \sum_{n=1}^{\infty} \left\{ A_n \cos\left(\frac{n\pi}{p} x\right) + B_n \sin\left(\frac{n\pi}{p} x\right) \right\}$.

$T = 2p$ om det inte är så att

$A_n = B_n = 0$ för alla n odelbara med k, för något $k = 2, 3, \ldots$

5) $$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x^2, & 0 \le x < \pi \end{cases}$$

Vi ska utveckla f i Fourierserie.

$$f(x) \underset{\substack{\uparrow \\ p=\pi}}{\approx} \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos(nx) + b_n \sin(nx) \right\}$$

där
$$\begin{cases} a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx \\ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\,dx \\ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\,dx. \end{cases}$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi} \int_0^{\pi} x^2\,dx = \frac{1}{\pi} \left[\frac{x^3}{3} \right]_0^{\pi} = \frac{\pi^2}{3}.$$

$\underline{n \ge 1}$: $$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\,dx = \frac{1}{\pi} \int_0^{\pi} x^2 \cos(nx)\,dx.$$

Partiell integration:

$$\int_0^{\pi} x^2 \cos(nx)\,dx = \left[\frac{\sin nx}{n} \cdot x^2 \right]_0^{\pi} - \int_0^{\pi} 2x \frac{\sin nx}{n}\,dx =$$

$$= -\frac{2}{n}\int_0^{\pi} x\sin(nx)\,dx =$$

$$= -\frac{2}{n}\left\{\left[-\frac{\cos(nx)}{n}\cdot x\right]_0^{\pi} - \int_0^{\pi}\left(\frac{-\cos(nx)}{n}\right)\cdot 1\,dx\right\}$$

$$= -\frac{2}{n}\left\{-\frac{\overbrace{\cos n\pi}^{(-1)^n}}{n}\cdot\pi - (-0) + \frac{1}{n}\underbrace{\int_0^{\pi}\cos(nx)\,dx}_{0}\right\}$$

$$\left[\frac{\sin nx}{n}\right]_0^{\pi} = 0$$

$$= \frac{2(-1)^n}{n^2}\cdot\pi \Rightarrow a_n = \frac{2(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_0^{\pi} x^2\sin(nx)\,dx.$$

Partiell integration:

$$\int_0^{\pi} \overset{\downarrow}{x^2}\,\overset{\uparrow}{\sin(nx)}\,dx = \left[\frac{-\cos(nx)}{n}\,x^2\right]_0^{\pi} - \int_0^{\pi}\left(\frac{-\cos(nx)}{n}\right)2x\,dx$$

$$= \frac{(-1)^{n-1}}{n}\pi^2 + \frac{2}{n}\int_0^{\pi} x\cos(nx)\,dx = \frac{(-1)^{n-1}}{n}\pi^2 +$$

$$+\frac{2}{n}\left\{\underbrace{\left[\frac{\sin(nx)}{n}\cdot x\right]_0^{\pi}}_{=0} - \int_0^{\pi}\frac{\sin(nx)}{n}\cdot 1\,dx\right\}$$

(5)

Så att

$$\int_0^{\pi} x^2 \sin(nx)\,dx = \frac{(-1)^{n-1}}{n}\pi^2 - \frac{2}{n^2}\underbrace{\int_0^{\pi} \sin(nx)\,dx}_{\left[-\frac{\cos(nx)}{n}\right]_0^{\pi}} =$$

$$= \frac{(-1)^{n-1}}{n}\pi^2 + \frac{2}{n^2}\left[\frac{\cos(nx)}{n}\right]_0^{\pi} = \frac{(-1)^{n-1}}{n}\pi^2 + \frac{2}{n^3}\left[(-1)^n - 1\right]$$

$$\Rightarrow \quad b_n = \frac{(-1)^{n-1}}{n}\pi + \frac{2}{\pi n^3}\left[(-1)^n - 1\right].$$

Detta ger oss Fourierserien

$$f(x) \sim \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^2}\cos(nx) + \left[\frac{(-1)^{n-1}}{n}\pi + \frac{2}{\pi n^3}\overbrace{((-1)^n - 1)}^{=\begin{cases}0 \text{ jämt} \\ -2 \text{ udda}\end{cases}}\right] \times \sin(nx)$$

19) Vi provar $x = \pi$. Av teorin vet vi att Fourierserien konvergerar mot

$$\frac{f(\pi^-) + f(\overbrace{-\pi^+}^{\sim \pi^+})}{2} = \frac{\pi^2 + 0}{2} = \frac{\pi^2}{2}.$$

Så

$$\frac{\pi^2}{2} = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2}{n^2} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{2}\Big[\underbrace{\frac{\pi^2}{2} - \frac{\pi^2}{6}}_{\pi^2/3}\Big] = \frac{\pi^2}{6}$$

⑥

Vi provar nu $x=0$. Av teorin vet vi att Fourierserien konvergerar mot 0 i $x=0$, så att

$$0=\frac{\pi^2}{6}+2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$ och vi får att

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}=\frac{\pi^2}{12}.$$

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15) $f(x)=e^x, \quad -\pi<x<\pi$

ska utvecklas i Fourierserie

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\int_{-\pi}^{\pi}e^x dx=\frac{1}{\pi}\left[e^x\right]_{-\pi}^{\pi}=$$

$$=\frac{e^{\pi}-e^{-\pi}}{\pi}$$ och för $n\geq 1$:

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos(nx)dx=\frac{1}{\pi}\int_{-\pi}^{\pi}e^x\cos(nx)dx$$

$$=\frac{1}{\pi}\left\{\left[e^x\cos(nx)\right]_{-\pi}^{\pi}+\int_{-\pi}^{\pi}e^x\cdot n\cdot\sin(nx)\,dx\right\}$$

$$=\frac{1}{\pi}(-1)^n\left(e^{\pi}-e^{-\pi}\right)+\frac{n}{\pi}\left\{\left[e^x\sin(nx)\right]_{-\pi}^{\pi}-\int_{-\pi}^{\pi}e^x\cdot n\cos(nx)dx\right\}$$

$$a_n = (-1)^n \frac{e^{\pi}-e^{-\pi}}{\pi} - \frac{n^2}{\pi} \underbrace{\int_{-\pi}^{\pi} e^x \cos(nx)\,dx}_{= \pi a_n}$$

så att

$$\overbrace{a_n + n^2 a_n}^{(1+n^2)a_n} = (-1)^n \frac{e^{\pi}-e^{-\pi}}{\pi}$$

och alltså

$$a_n = (-1)^n \frac{e^{\pi}-e^{-\pi}}{\pi} \cdot \frac{1}{1+n^2}.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \sin(nx)\,dx =$$

$$= \frac{1}{\pi}\left\{ \left[e^x \cdot \sin(nx) \right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} e^x \cdot n\cos(nx)\,dx \right\}$$

$$= -\frac{n}{\pi}\int_{-\pi}^{\pi} e^x \cos(nx)\,dx = -n\,a_n = (-1)^{n-1} \frac{e^{\pi}-e^{-\pi}}{\pi} \frac{n}{n^2+1}$$

Fourierserien blir:

$$\frac{e^{\pi}-e^{-\pi}}{2\pi} + \sum_{n=1}^{\infty} (-1)^n \frac{e^{\pi}-e^{-\pi}}{\pi} \frac{1}{1+n^2}\cos(nx) + (-1)^{n-1} \frac{e^{\pi}-e^{-\pi}}{\pi} \frac{n}{n^2+1}\sin(nx)$$

23) $f(x) = |\sin x|$; $-\pi < x < \pi$

$f(x) = f(-x)$ jämn funktion så cosinusserie

$$a_0 = \frac{2}{\pi}\int_0^{\pi} \overbrace{\sin x}^{\geq 0}\, dx = \frac{2}{\pi}\left[-\cos x\right]_0^{\pi} = \frac{2}{\pi}\left[1+1\right] = \frac{4}{\pi}$$

och för $n \geq 1$:

$$a_n = \frac{2}{\pi}\int_0^{\pi} \underbrace{\sin x \cos(nx)}_{\frac{1}{2}\left(\sin(x+nx)+\sin(x-nx)\right)} dx =$$

$$= \frac{1}{\pi}\int_0^{\pi}\left(\sin[(n+1)x] - \sin[(n-1)x]\right)dx \overset{n \neq 1}{=}$$

$$= \frac{1}{\pi}\left[-\frac{\cos[(n+1)x]}{n+1} + \frac{\cos[(n-1)x]}{n-1}\right]_0^{\pi} =$$

$$= \frac{1}{\pi}\left[-\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right] =$$

$$= \frac{1}{\pi}\left\{\frac{1-(-1)^{n-1}}{n+1} - \frac{1-(-1)^{n-1}}{n-1}\right\} = \frac{1-(-1)^{n-1}}{\pi}\left(-\frac{2}{n^2-1}\right) =$$

$$= -\frac{2\left(1-(-1)^{n-1}\right)}{\pi\left(n^2-1\right)}$$

medan

$$a_1 = \frac{1}{\pi}\int_0^{\pi} \sin(2x)\,dx = \frac{1}{\pi}\left[-\frac{\cos 2x}{2}\right]_0^{\pi} = 0.$$

Alla $b_n = 0$ eftersom vi har en cosinusserie!

Alltså får vi:

$$f(x) = |\sin x| \sim \frac{2}{\pi}(+1)\sum_{n=2}^{\infty} \frac{1-(-1)^{n-1}}{\pi(n^2-1)}\cos(nx).$$

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27)
$$f(x) = \cos x, \quad 0 < x < \frac{\pi}{2}.$$

$$\boxed{p = \frac{\pi}{2}.}$$

$$f(x) \overset{\text{cosinusserie}}{=} \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(2nx)$$

$$a_0 = \frac{2}{\pi/2}\int_0^{\pi/2} \cos x\,dx = \frac{4}{\pi}\left[\sin x\right]_0^{\pi/2} = \frac{4}{\pi}$$

$$n \geq 1: \quad a_n = \frac{4}{\pi}\int_0^{\pi/2} \underbrace{\cos x \cos(2nx)}_{\frac{1}{2}(\cos(x+2nx)+\cos(2nx-x))}\,dx$$

Så att

$$a_n = \frac{2}{\pi}\int_0^{\pi/2}\Big[\cos\big((2n+1)x\big) + \cos\big((2n-1)x\big)\Big]dx =$$

$$= \frac{2}{\pi}\left[\frac{\sin[(2n+1)x]}{2n+1} + \frac{\sin[(2n-1)x]}{2n-1}\right]_0^{\pi/2} =$$

$$= \frac{2}{\pi}\left[\frac{\overbrace{\sin\left(n\pi + \frac{\pi}{2}\right)}^{(-1)^n}}{2n+1} + \frac{\overbrace{\sin\left(n\pi - \frac{\pi}{2}\right)}^{(-1)^{n-1}}}{2n-1}\right] =$$

$$= \frac{2\cdot(-1)^n}{\pi}\cdot\frac{\cancel{2n}-1-(\cancel{2n}+1)}{(2n+1)(2n-1)} = \frac{4(-1)^{n-1}}{\pi(4n^2-1)}.$$

PSS

$$f(x) = \overset{\text{sinusserie}}{\sum_{n=1}^{\infty}} b_n \sin(2nx) \quad \text{där}$$

$$b_n = \frac{4}{\pi}\int_0^{\pi/2} \underbrace{\cos x \,\sin(2nx)}_{\frac{1}{2}\left[\sin(2nx+x)+\sin(2nx-x)\right]}\,dx =$$

$$= \frac{2}{\pi}\int_0^{\pi/2}\Big(\sin[(2n+1)x] + \sin[(2n-1)x]\Big)dx =$$

$$= \frac{2}{\pi}\left[-\frac{\cos[(2n+1)x]}{2n+1} - \frac{\cos[(2n-1)x]}{2n-1}\right]_0^{\pi/2} =$$

$$= \frac{2}{\pi}\left[-\frac{\overbrace{\cos\left[n\pi+\frac{\pi}{2}\right]}^{=0}}{2n+1} - \frac{\overbrace{\cos\left[n\pi-\frac{\pi}{2}\right]}^{=0}}{2n-1} + \frac{1}{2n+1} + \frac{1}{2n-1}\right]$$

$$= \frac{2}{\pi}\left[\frac{1}{2n+1} + \frac{1}{2n-1}\right] = \frac{2}{\pi}\,\frac{2n-1+2n+1}{4n^2-1} = \frac{2}{\pi}\,\frac{4n}{4n^2-1} =$$

$$= \frac{8}{\pi}\cdot\frac{n}{4n^2-1}.$$