Computational Game Theory

Lecture 9

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Coalitional games

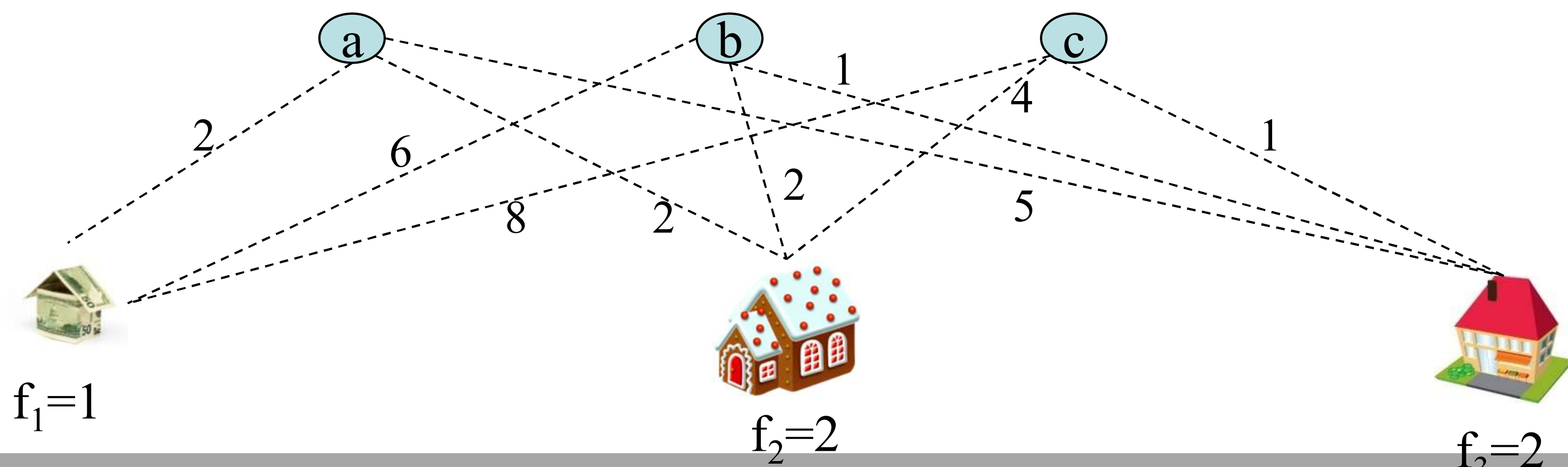
- *N* player non-cooperative games
  - Actions, strategies, payoffs
  - Individual payoff maximization
    - Actions chosen individually
  - No negotiation allowed
    - Correlated NE…

- *N* player coalitional games
  - Players seek to maximize their payoffs (or minimize cost)
  - Players form coalitions
    - If they benefit from it
    - Payoff assigned to the members of the coalition
  - Negotiation is the game

- What coalitions will form?
  - What are the plausible payoff allocations between players

Example: Facility location game

- Set of cities *N*
- Set of facilities *F*
  - Opening facility *j* costs f_j
  - City *i* can produce goods worth *k* using the facility
- Distance of facility *j* from city *i*
  - $d_{ij}: N \times F \rightarrow R$
- Cities can cooperate to decrease costs
  - $S \subseteq N$
  - $c(S) = \min_{F' \subseteq F} \{\sum_{j \in F'} f_j + \sum_{i \in S} \min_{j \in F'} d_{ij}\}$



Example: Majority game

- Three player majority game

- Payoffs to the players
  - Payoff to single player 0
  - Payoff to coalition of two players: 0.6
  - Payoff to grand coalition: 1

- Formal model of the problem?
- What would be a reasonable solution?

Coalitional game with transferable payoff

- Coalitional game with transferable payoff $<N,v>$
  - Finite set N of players
  - Function $v: S \rightarrow R \ \forall S \subseteq N, S \neq \emptyset$ (characteristic function)
- $v(S)$ payoff to coalition S
  - arbitrary division possible between coalition members
  - independent of the decisions of $N \backslash S$
- Coalitional game with transferable payoff $<N,V>$ is cohesive

$$v(N) \geq \sum_{k=1}^{S_k} v(S_k) \quad \forall \text{partition} \{S_1,\ldots,S_k\} \text{ of } N$$

  - Grand coalition has the highest payoff

Example: Majority game

- Three player majority game ($|N|=3$)
- Payoffs

$$v(S)\,|_{|S|=1} = 0$$
$$v(S)\,|_{|S|=2} = 0.6$$
$$v(N) = 1$$

- What would be a reasonable solution?
  - Pareto efficient?

$$x = (0.38, 0.31, 0.31)$$
$$x = (0.4, 0.3, 0.3)$$
$$x = (0.3, 0.3, 0.4)$$
$$x = (0.2, 0.3, 0.5)$$
$$x = (0, 0.5, 0.5)$$

Feasible payoff profile

- <N,v> coalitional game with TP
- Payoff vector and its value for *S*

$$(x_i)_{i \in N} \qquad x(S) = \sum_{i \in S} x_i$$

- S-feasible payoff vector

$$x(S) = v(S)$$

- Feasible payoff profile is an *N*-feasible payoff vector

$$x(N) = v(N)$$

  - Allocation of payoff to the players
  - Feasible payoff profiles for majority game

$$x = (0.3, 0.3, 0.4)$$

$$x = (0.38, 0.31, 0.31)$$

$$x = (0.4, 0.3, 0.3)$$

Individual rationality, imputation

- Payoff profile $(x_i)_{i \in N}$ is individually rational if

$$x_i \geq v(\{i\})$$

- An imputation is a feasible payoff profile that is individually rational

- Payoff profile $(x_i)_{i \in N}$ is coalitionally rational for $S \subseteq N$ if

$$x(S) \geq v(S)$$

- Example for the majority game – are these rational?

$$x = (0, 0.5, 0.5)$$

$$x = (0.1, 0.5, 0.4)$$

$$x = (0.2, 0.4, 0.4)$$

Dummy and Symmetry

- Player *i* is a dummy player if
$$v(S \cup \{i\}) - v(S) = v(\{i\}) \quad \forall S \subseteq N, i \notin S$$
- A payoff profile *x* satisfies the dummy property if for every dummy player *i*
$$x(\{i\}) = v(\{i\})$$
- Players *i* and *j* are interchangeable if
$$v(S \setminus \{i\} \cup \{j\}) = v(S) \quad \forall S \ni i, j \notin S$$
- A payoff profile *x* satisfies the symmetry property if for every pair of interchangeable players *i* and *j*
$$x(\{i\}) = x(\{j\})$$

Desirable solution properties

- Existence
- Uniqueness
  - Might be several plausible solutions
- Feasibility (Efficiency)
- Individual rationality
  - Coalitional rationality?
- Symmetry
  - All players do not have to be equal
- Dummy player
  - Non-contributors should get something?

The Core

- The core of a coalitional game with transferable payoff <*N*,*v*> is the set of feasible payoff profiles $(x_i)_{i\in N}$ for which

$$\neg(\exists S,(y_i)_{i\in S})\sum_{i\in S} y_i = v(S), \quad y_i > x_i \quad \forall i \in S$$

  - no *S*-feasible payoff profile offers more

⇕

- The core of a coalitional game with transferable payoff <*N*,*v*> is the set of feasible payoff profiles $(x_i)_{i\in N}$ for which

$$v(S) \le x(S) \quad \forall S \subseteq N$$

  - Coalitionally rational for all coalitions

- The core is a convex, closed set
  - Nonempty?

D.B. Gillies, "Solutions to General Non-zero Sum Games", in Contributions to the Theory of Games, Vol 4, Annals of Math. Studies, 40, pp. 47-85, 1959

Example: Majority game

- Three player majority game (|*N*|=*3*)
- Payoffs

$$v(S)|_{|S|=1} = 0$$
$$v(S)|_{|S|=2} = \alpha$$
$$v(N) = 1$$

- What is the core of the game?

$$x_1 + x_2 + x_3 = 1$$
$$x_1 + x_2 \geq \alpha$$
$$x_2 + x_3 \geq \alpha$$
$$x_1 + x_3 \geq \alpha$$

$$2(x_1 + x_2 + x_3) \geq 3\alpha$$

$$\frac{2}{3} \geq \alpha$$

$$\alpha < \frac{2}{3} \Rightarrow \text{continuum of solutions}$$

$$\alpha = \frac{2}{3} \Rightarrow x = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$$

$$\alpha > \frac{2}{3} \Rightarrow \text{core is empty}$$

Non-emptiness of the core

- Characteristic vector of *S*

$$(1_S)_i = \begin{cases} 1 & i \in S \\ 0 & i \notin S \end{cases}$$

- $(\lambda_S)_{S\in 2^N}$ is a balanced collection of weights if
  - $\sum_{S\in 2^N} \lambda_S 1_S = 1_N$

- Example for majority game
  - $\lambda_S \,|_{|S|=2} = 0.5 \quad \lambda_S \,|_{|S|\neq 2} = 0$
  - $\lambda_S \,|_{|S|=1} = 1 \quad \lambda_S \,|_{|S|\neq 1} = 0$
  - Players allocate λ_S portion of their time to coalition S
    - They have unit amount of time to spend

- Coalitional game TP <*N,v*> is balanced if for *every* balanced collection of weights

$$\sum_{S\in C} \lambda_S v(S) \leq v(N)$$

Bondareva-Shapley Theorem

- A coalitional game TP <N,v> has a nonempty core iff it is balanced
  - Necessary
    - $(x_i)_{i \in N}$ feasible payoff profile in the core
    - We show $(\lambda_S)_{S \in 2^N}$ balanced

$$\sum_{S \in 2^N} \lambda_S \, v(S) \leq \sum_{S \in 2^N} \lambda_S x(S) = \sum_{i \in N} x_i \sum_{S \ni i} \lambda_S = \sum_{i \in N} x_i = v(N)$$

  - Sufficient
    - Separating hyperplane theorem

- Example for majority game

$$\alpha > \frac{2}{3} \quad \lambda_S \,|_{|S|=2} = \frac{1}{3} \quad \lambda_N = \frac{1}{3} \Rightarrow \sum_{S \in C} \lambda_S v(S) > 1$$

Shapley value

- Marginal contribution of player *i* to $S \subseteq N$ $(i \notin S)$ in the game <*N*,*v*>

$$\Delta_i(S) = v(S \cup \{i\}) - v(S)$$

- Θ set of all orderings of players
- $S_i(R)$ players preceding player *i* in ordering $R \in \Theta$

- Shapley value of player *i* in the game <*N*,*v*> is

$$\varphi_i(N,v) = \frac{1}{|N|!} \sum_{R \in \Theta} \Delta_i(S_i(R)) = \frac{1}{|N|!} \sum_{S \subseteq N \setminus \{i\}} (|N| - |S| - 1)!|S|!\Delta_i(S)$$

  - Average marginal contribution of player *i*
  - Assigns unique feasible payoff profile to every game <*N*,*v*>
  - Note that

$$\sum_{i \in N} \Delta_i(S_i(R)) = v(N) \quad \forall R \in \Theta$$

Example

- Three player majority game N={A,B,C}
  - α=1 ($\rightarrow$core empty)
- Possible permutations

$$\Theta = \{(A,B,C),(A,C,B),(B,A,C),(B,C,A),(C,A,B),(C,B,A)\}$$

- Shapley value

$$\varphi_i(N,v) = \varphi_A(N,v) = \frac{1}{6}\left[0+0+1+0+1+0\right] = \frac{1}{3}$$

Properties of the Shapley value

- A value ψ is additive if for any two games <*N,v*> and <*N,w*> we have

$$\psi_i(v+w)=\psi_i(v)+\psi_i(w) \quad (v+w)(S)=v(S)+w(S)\ \forall S\subseteq N$$

- The Shapley value is the only value that satisfies
  - Symmetry
  - Dummy player
  - Additivity
  - Efficiency

- A value ψ satisfies the balanced contributions property if for every coalitional game with TP <*N,v*> and $i,j\in N$

$$\psi_i(N,v)-\psi_i(N\setminus\{j\},v^{N\setminus\{j\}})=\psi_j(N,v)-\psi_j(N\setminus\{i\},v^{N\setminus\{i\}})$$

  - Reciprocity of loss if other player leaves
- The unique value that satisfies the balanced contributions property is the Shapley value

R.B. Myerson, "Conference structures and fair allocation rules," International Journal of Game Theory vol. 9, pp. 169–182, 1980

Convex games

- A coalitional game with TP is convex if
  $$v(S)+v(T)\leq v(S\cup T)+v(S\cap T)\quad \forall S,T\subseteq N$$
  - Supermodularity of *v(S)*

- Practical consequence
  $$v(S'\cup\{i\})-v(S')\geq v(S''\cup\{i\})-v(S'')\quad \forall i\in N, S''\subset S'\subset N\setminus\{i\}$$
  - Marginal contribution of the player increases with |*S*|
  - Proof:
    - *S=S'* and *T=S''*$\cup$*{i}*

L.S. Shapley, "Cores of Convex Games", International Journal of Game Theory 1(1), 11-26, 1971
M. Maschler, B. Peleg, L.S. Shapley, "The Kernel and Bargaining Set for Convex Games", International Journal of Game Theory 1(1), pp. 73-93, 1971

Example: Markets w. TU

- Set of players N={1,2,3,4}
- Two kinds of resources owned by players (endowment)

$$e^1 = e^2 = (1,0) \quad e^3 = e^4 = (0,1)$$

- Player *i*'s utility function of the resources

$$U^i(x^i) = x_1^i x_2^i$$

- Players can form coalitions to increase their utilities
  - Combine available resources to produce goods
- Coalitional game with TP

$$v(S) = \max_{(x^i)_{i\in S}} \left\{ \sum_{i\in S} x_1^i x_2^i : x^i \in R_+^2, \sum_{i\in S} x^i = \sum_{i\in S} e^i \right\}$$

$$v(\{i\}) = 0 \qquad v(\{i, i+2\}) = 1 \qquad v(\{i, i+3\}) = 1$$

$$v(S)|_{|S|=3} = 2 \quad v(N) = 4$$

- Solution?

Properties of convex games

- The core of a convex coalitional game with TP <*N*,*v*> is non-empty
  - Greedy allocation: arbitrary ordering of players

$$x_i = v(S_i \cup i) - v(S_i) \quad S_i = \{1, \ldots, i-1\}$$

- The Shapley value of a convex game is within its core, it is the center of gravity of the vertices
- The stable set of a convex game is unique and coincides with the core
- The kernel is a singleton $\Rightarrow$ coincides with the nucleolus

L.S. Shapley, "Cores of Convex Games", International Journal of Game Theory 1(1), 11-26, 1971
M. Maschler, B. Peleg, L.S. Shapley, "The Kernel and Bargaining Set for Convex Games", International Journal of Game Theory 1(1), pp. 73-93, 1971

Cost Games

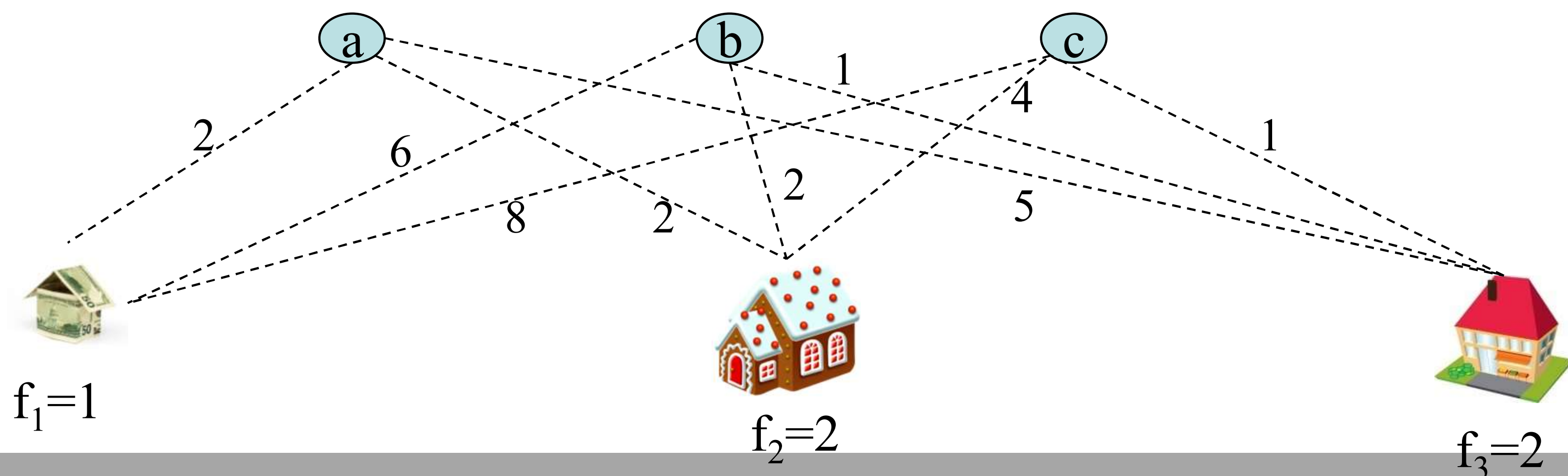
- Coalitional game with divisible cost <*N*,*c*>
  - Finite set *N* of players
  - Function $c:S \rightarrow R \quad \forall S \subseteq N,\ S \neq \emptyset$
- $c(S)$ cost of coalition S
  - Arbitrary division possible between coalition members
  - Independent of the decisions of $N \setminus S$
- Cohesive

$$c(N) \leq \sum_{k=1}^{S_k} c(S_k) \quad \forall \text{partition}\{S_1,\ldots,S_k\} \text{ of } N$$

  - Grand coalition has the smallest cost

Example: Facility location

- Given set of cities *N*
- Set of facilities *F*
  - Opening facility *j* costs f_j
  - City *i* can produce goods worth *k* using the facility
- Distance of facility *j* from city *i*
  - $d_{ij}: N \times F \rightarrow R$ cost function
- Cities can cooperate to decrease costs
  - $S \subseteq N$
  - $c(S) = \min_{F' \subseteq F} \{ \sum_{j \in F'} f_j + \sum_{i \in S} \min_{j \in F'} d_{ij} \}$



Duality of cost and profit games

- Payoff function defined by

$$v(S) = c(N) - c(N \setminus S)$$

$$\begin{aligned} c(\{a\}) &= 3 \\ c(\{b\}) &= 3 \\ c(\{c\}) &= 3 \\ c(\{a,b\}) &= 6 \\ c(\{b,c\}) &= 4 \\ c(\{a,c\}) &= 6 \\ c(\{a,b,c\}) &= 7 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} v(\{a\}) &= 3 \\ v(\{b\}) &= 1 \\ v(\{c\}) &= 1 \\ v(\{a,b\}) &= 4 \\ v(\{b,c\}) &= 4 \\ v(\{a,c\}) &= 4 \\ v(\{a,b,c\}) &= 7 \end{aligned}$$

- The core of dual games is equivalent

$$x(S) \le c(S) \Leftrightarrow x(S) \ge v(S)$$

- "Submodular" cost games vs. "supermodular" profit games
- "Supermodular" profit games vs. "submodular" cost games

Example: Minimum Spanning Tree

- Multicast cost sharing
  - Graph *G(N,E)*
  - Service provided by one node (server)
  - Nodes on the graph want to use the service
    - Either directly connected
    - Or connected via another node

- Coalitional game <*N*,*v*>
  - Set of players *N* the customers
  - *v(S)=c(*grand coalition*)*
    - *c(*minimum spanning tree for S and server*)*

Coalitional game with non-transferable payoffs

- Coalitional game with non-transferable payoff $<N,X,V,\succcurlyeq_i>$
  - Finite set of players N
  - Set of consequences X
  - Function $V{:}S \rightarrow 2^X$, $\forall S \subseteq N$, $S \neq \emptyset$
  - Preference relation $\succcurlyeq_i$ on X

- Coalition
  - Subset $S \subseteq N$

- From consequences to utilities
  - $V(S)=(x_i) \in R^{|S|}$ assigns utility to each member of the coalition S

- Example
  - Matching problems (stable marriage/roommates, etc)

Example: House allocation

- Set *N* of agents, $|N|=n$
  - Each agent owns a house
    - Agent *i* owns house *i*
  - Strict preference ordering over houses
- Find an allocation of the houses to agents
  - All agents should agree with the allocation

  - Feasible allocation

$$a=(a_1,a_2,a_3),\quad a_i\neq a_j\ i\neq j$$

    - All permutations of (1,2,3)

  - Is there an allocation that no one would like to deviate from?

Final words

- Large number of solution concepts for TP games
  - Core, stable sets, kernel, nucleolus, Shapley value
  - Aumann-Shapley value
  - Bargaining set
  - Prenucleolus
  - Prekernel
  - ε-core
  - Owen–Banzhaf coalitional value

- Solution concepts can be adapted to games with non-transferable payoffs

Literature

- M. Osborne, A. Rubinstein, “A Course in Game Theory”, MIT press, 1994
- Nisan, Roughgarden, Tardos, Vazirani (eds.), “Algorithmic Game Theory”, Cambridge UP, 2007

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- J. von Neumann, O. Morgenstern, “Theory of Games and Economic Behavior,” Princeton University Press, 1944
- W. F. Lucas, M. Rabie, “Games with No Solutions and Empty Cores”, Mathematics of Operations Research 7(4), 1982, pp. 491-500
- M. Davis, M. Maschler, "The kernel of a cooperative game", Naval Research Logistics Quarterly, vol 12, pp. 223–259, 1965
- D. Schmeidler,"The nucleolus of a characteristic function game,” SIAM Journal of Applied Math. vol. 17, pp. 1163–1170,1969
- M.D. Davis, “Game Theory: A Nontechnical Introduction”, Dover, 1997

Other solution concepts

- Stable sets (Von-Neumann and Morgenstern)
- Kernel (Davis and Maschler)
- Nucleolus (Schmeidler)

Stable sets and objections

- Imputation *y* dominates the imputation *x* for some coalition *S*

  $$y \succ_S x \quad \equiv \quad \mathrm{y_i} > \mathrm{x_i}\ \forall i \in S, \quad y(S) \le v(S)$$

  - *y* dominates *x* via *S*
- Set of imputations dominated by *Y*

  $$D(Y) = \{z \in X \mid \exists S \subseteq N, y \in Y\ y \succ_S z\}$$

- The subset *Y* of the imputations *X* of the coalitional game with TP <*N*,*v*> is a stable set if
  - *Y* is internally stable

    $$\forall y \in Y \quad \neg\exists S \subseteq N \quad z \succ_S y,\ z \in Y \longrightarrow Y \subseteq X \setminus D(Y)$$

  - *Y* is externally stable

    $$\forall z \in X \setminus Y \quad \exists S \subseteq N, \exists y \in Y \quad y \succ_S z \longrightarrow Y \supseteq X \setminus D(Y)$$

  $$\left.\begin{matrix} Y \subseteq X \setminus D(Y) \\ Y \supseteq X \setminus D(Y) \end{matrix}\right\} Y = X \setminus D(Y)$$

J. von Neumann, O. Morgenstern, "Theory of Games and Economic Behavior," Princeton University Press, 1944

Example: Majority game

- Set of Players $N=\{1,2,3\}$
- Payoff function $v(S)|_{|S|\geq 2}=1 \quad v(S)|_{|S|<2}=0$
  - Core is empty (α>2/3)
- Stable sets

$$Y=\{(1/2,1/2,0),(1/2,0,1/2),(0,1/2,1/2)\}$$

  - Internally stable
    - No two players prefer any profile to the other
  - Externally stable
    - For any $z \notin Y$ there are two players with $z_i<0.5$

$$Y_{i,c}=\{x\in X : x_i=c\} \qquad c\in[0,0.5)$$

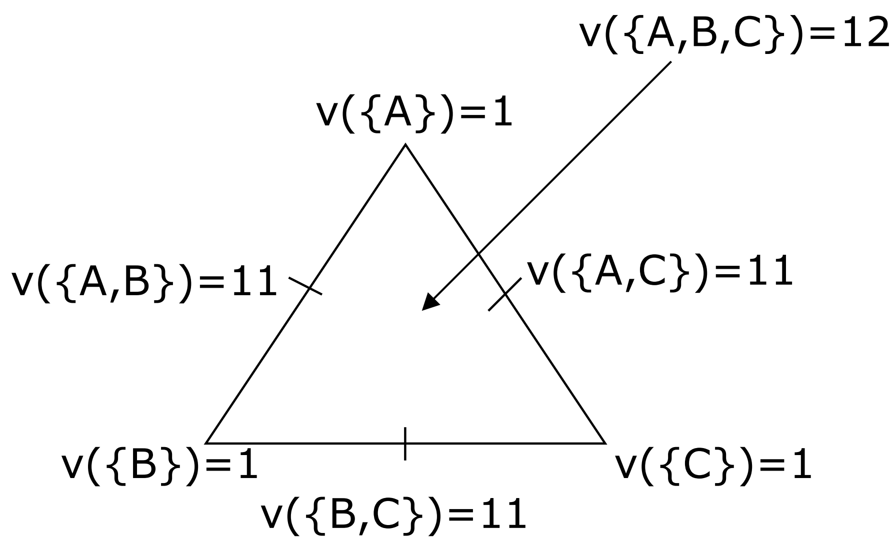
  - Internally stable
    - No two players prefer any profile to the other
  - Externally stable

$$z\in X\setminus Y_{3,c} \quad \begin{cases} z_3>c \Rightarrow z_1+z_2<1-c \Rightarrow \exists y\in Y_{3,c}\ y \succ_{\{1,2\}} z \\ z_3<c \Rightarrow z_i \leq z_j \Rightarrow y=\{y_i=1-c, y_3=c\} \succ_{\{i,3\}} z \end{cases}$$

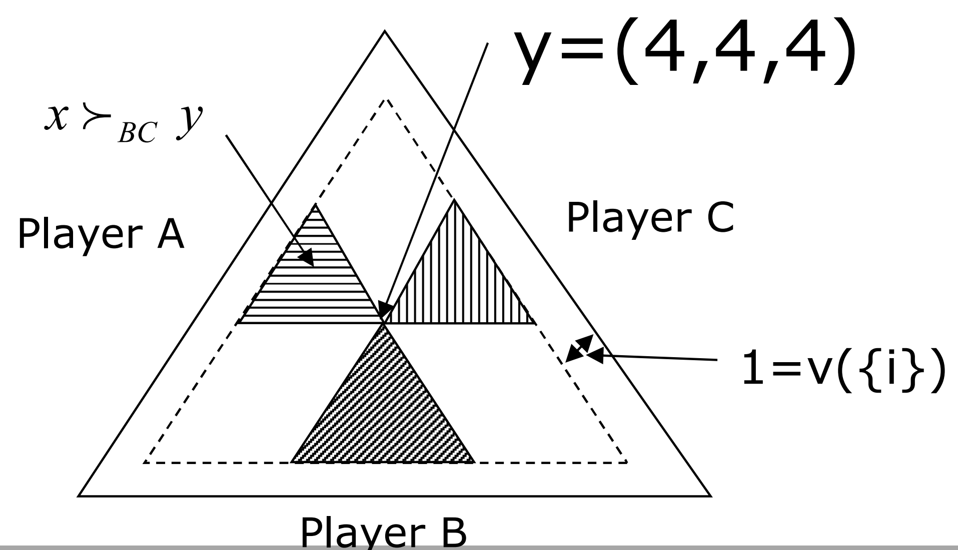
Example

- Three players: A,B,C
- Core empty

$$\left.\begin{array}{l} x_i + x_j \geq 11 \\ x(N) = 12 \end{array}\right\} \Rightarrow 24 \geq 33$$



- Stable set
  - Sum of distances from edges of interior point is constant
  - Y={(5.5,5.5,1),
    (1,5.5,5.5),
    (5.5,1,5.5)}



M.D. Davis, "Game Theory: A Nontechnical Introduction", Dover, 1997

Properties of the stable set

- The core is the set of imputations *x* that is not dominated

$$\{y \in X : \neg \exists S \subseteq N, x \in X \quad x \succ_S y\}$$

- The core is a subset of every stable set
  - Consequence of external stability: every member is an imputation, and is not dominated by any other imputation
- No stable set is a proper subset of any other
  - Consequence of external stability: if $Y_1 \subset Y_2$ then for some $y \in Y_2 \setminus Y_1$, $z \in Y_1$ z dominates y for some S. But $z \in Y_2$, so this is a contradiction
- If the core is a stable set then it is the only stable set
  - Follows from the above two
- The stable set satisfies the dummy property
- Not all games have stable sets
  - There exist games with 14 or more players…

W. F. Lucas, M. Rabie, "Games with No Solutions and Empty Cores", Mathematics of Operations Research 7(4), 1982, pp. 491-500

The Kernel

- The excess of coalition S under imputation *x* is

$$e(S,x)=v(S)-x(S)$$

  - Sacrifice of coalition members
- Maximum excess of player *i* compared to player *j*

$$s_{ij}(x)=\max_{S\in C}\{e(S,x): i\in S, j\in N\setminus S\}$$

  - $s_{ij}(x)-s_{ji}(x)$ bargaining power of *i* against *j*

- The kernel of the coalitional game with TP is the set of imputations $x\in X$ such that for every pair *(i,j)* of players

$$[s_{ij}(x)-s_{ji}(x)][x(\{j\})-v(\{j\})]\leq 0$$
$$[s_{ji}(x)-s_{ij}(x)]\underbrace{[x(\{i\})-v(\{i\})]}_{\geq 0}\leq 0$$

M. Davis, M. Maschler, "The kernel of a cooperative game", Naval Research Logistics Quarterly, vol 12, pp. 223–259, 1965

Example

- Three player majority game
- Kernel

$$x = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$$

  - Assume kernel $x=(x_1,x_2,x_3)$

$$s_{31}(x) = 1-(x_2+x_3)$$
$$s_{13}(x) = 1-(x_2+x_1)$$

  - Assume $x_1 \geq x_2 \geq x_3$ with at least one strict inequality ($x_1>x_3$)

$$s_{31}(x) > s_{13}(x) \quad \Rightarrow s_{31}(x) - s_{13}(x) > 0$$

  - Furthermore $x_1>0$

$$x(\{1\}) > v(\{1\}) \quad \Rightarrow x(\{1\}) - v(\{1\}) > 0$$

  - x is not in the kernel

The Nucleolus

- Ordering of the coalitions *S* for imputation *x* wrt. *e(S,x)*

$$e(S_l, x) \geq e(S_{l+1}, x) \quad l = 1, \ldots, 2^{|N|} - 2$$

- Vector of excesses

$$E(x) = [E_1(x), \ldots, E_{2^{|N|}-2}(x)], \quad E_l(x) = e(S_l, x)$$

  - Excesses in decreasing order

- Partition $B(x)=(B_1(x),\ldots,B_K(x))$ of the coalitions $S \subseteq N$
  - *Coalitions with equal excess*

$$S, S' \in B_k(x) \Leftrightarrow e(S, x) = e(S', x) = e_k(x)$$
$$e_1(x) > \ldots > e_K(x)$$

The Nucleolus

- Lexicographical ordering on *E(x)*

$$E(x) < E(y) \Leftrightarrow E_l(x) < E_l(y) \quad for\ \ l = \min[k \in \{1,\ldots,2^{|N|}-2\} : E_k(x) \neq E_k(y)]$$

$$\Updownarrow$$

$$\exists k^* \mid |B_k(x)| = |B_k(y)|, e_k(x) = e_k(y)\ \forall k < k^*$$

and

$$\mathrm{e}_{k^*}(x) < \mathrm{e}_{k^*}(y) \quad \text{or} \quad \mathrm{e}_{k^*}(x) = \mathrm{e}_{k^*}(y) \text{ and } |B_k(x)| < |B_k(y)|$$

- The nucleolus is the set $x \in X$ of imputations for which

$$\mathrm{E(x)} \leq E(y) \quad \forall y \in X$$

  - Lexicographical minimization (of maximum excess)

D. Schmeidler,"The nucleolus of a characteristic function game,"
SIAM Journal of Applied Math. vol. 17, pp. 1163–1170,1969

Example

- Three player majority game
- Nucleolus

$$x = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$$

  - Excess

$$e(S,x)|_{|S|=1} = -\frac{1}{3} \quad e(S,x)|_{|S|=2} = \frac{1}{3}$$

  - Ordered coalitions

$$[\{A,B\},\{A,C\},\{B,C\},\{A\},\{B\},\{C\}]$$

  - Excess vector

$$E(x) = [\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}]$$

  - Partitions

$$B_1(x) = \{\{A,B\},\{A,C\},\{B,C\}\} \quad e_1(x) = \frac{1}{3}$$

$$B_2(x) = \{\{A\},\{B\},\{C\}\} \quad e_2(x) = -\frac{1}{3}$$

Properties of the nucleolus

- The nucleolus of any coalitional game with TP is
  - non-empty

$$E_k(x) = \min_{T \subseteq 2^{|N|}, |T|=k-1} \{\max_{S \in C \setminus T} \{e(S,x)\}\}$$

  - a singleton
  - a subset of the kernel
    - Consequence: the kernel is non-empty
- If the core is non-empty, the nucleolus is in the core

D. Schmeidler,"The nucleolus of a characteristic function game," SIAM Journal of Applied Math. vol. 17, pp. 1163–1170,1969