Computational Game Theory

Lecture 4

P2/2025

György Dán

Division of Network and Systems Engineering

Today’s Topics

- Approximate equilibria
- Refinements of the Nash Equilibrium
- Correlated equilibrium
- Games with incomplete information (Bayesian)

Equilibria cont'd

- Find the pure NE of the game.

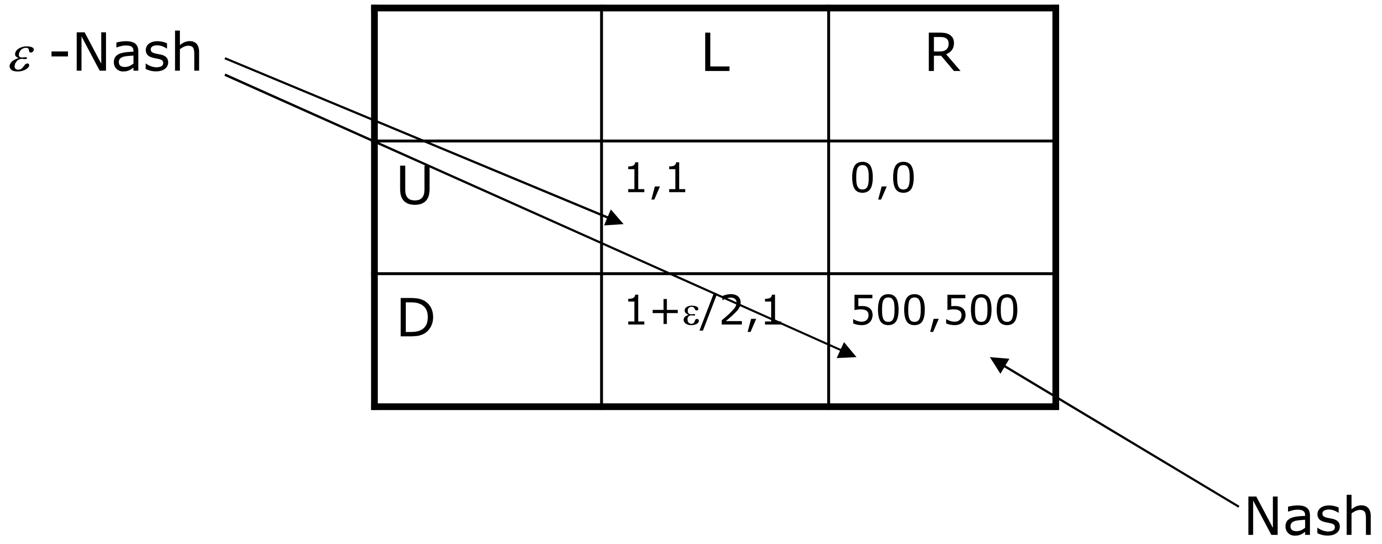
	L	R
T	$-\varepsilon/2, \varepsilon/2$	0,0
B	0,0	-1,1

ε-Nash equilibrium

- In a strategic game $G=<N,(A_i),(u_i)>$ a mixed strategy α is an ε-Nash equilibrium (ε>0) if
  - $u_i(\alpha_{-i},\alpha_i) \geq u_i(\alpha_{-i},\alpha'_i) - \varepsilon$ for $i \in N$, $\alpha'_i \in \Delta(A_i)$
- Every finite strategic game has an ε-Nash equilibrium
  - Every NE is surrounded by ε-Nash equilibria for ε>0
  - The contrary is not true!
- Convenient from a computational point of view
  - Floating point precision limits numerical accuracy

Example

- Find the Nash equilibria and the ε-Nash equilibria



ε -Nash

	L	R
U	1,1	0,0
D	$1+\varepsilon/2,1$	500,500

Nash

- Payoff can be far from the NE payoff
- Can be unlikely to arise in play

Wilson’s theorem

- Let G be a *regular* and *quasi-strong* finite strategic game. Then the number of its equilibria is finite and odd.
    - Based on the topology of the solution graph for the logarithmic game
- Almost all finite games are quasi-strong.
    - The set of extra-weak games is a set of measure zero in the set of strategic games of a particular size.
        - within the set of games that have at least one NE with the same support
- Almost all finite games are regular.

- *Theorem:* In "almost all" finite strategic games, the number of equilibrium points is *finite* and *odd*.

- R. Wilson, “Computing Equilibria in N-person Games,” SIAM Journal of Applied Mathematics, 21(1), pp. 80-87, 1971
- J.C. Harsányi,” Oddness of the number of equilibrium points: A new proof”, International Journal of Game Theory, 2(1), pp. 235-250, 1973

Slightly modified example

- Consider the following games

	L	R
T	1,1	0,0
B	0,0	0,0

	L	R
T	1,1	0,0
B	0,0	-η,-η

- What happens with the NE?

Robustness

- Consider a game $G=<N,(A_i),(u_i)>$
  - Assume that it has some NE
- What if u_i is inaccurate?
  - Inaccurate modeling assumption
  - The payoffs are not common knowledge
- How and when does inaccuracy influence the equilibria?

Proximity of Games

- Distance between payoff vectors (u_i) and $(\tilde{u}_i)$

$$D(u,\tilde{u}) = \max_{i\in N, a\in A} |u_i(a) - \tilde{u}_i(a)|$$

- Distance between mixed strategy profiles (α_i) and $(\tilde{\alpha}_i)$

$$d(\alpha,\tilde{\alpha}) = \max_{i\in N, a\in A} |\alpha_i(a_i) - \tilde{\alpha}_i(a_i)|$$

Essential (Robust) Games

- Let G be strategic game <$N,(A_i),(u_i)$>. A Nash equilibrium (α_i) of G is **essential** (or robust) if

$$\forall \varepsilon > 0 \; \exists \eta > 0 \text{ s.t. if } D(u,\tilde{u}) < \eta \rightarrow d(\alpha,\tilde{\alpha}) < \varepsilon$$

where $(\tilde{\alpha}_i)$ is a NE of the strategic game $\tilde{G} =< N,(A_i),(\tilde{u}_i) >$

- Intuition
There is a nearby Nash equilibrium for nearby games

Example revisited

- Is this game essential?

	L	R
T	1,1	0,0
B	0,0	0,0

NE=(T,L), (R,B)

	L	R
T	1,1	0,0
B	0,0	$-\eta,-\eta$

- Are all games essential?

Essential Games

- Almost all finite strategic games are essential
- Proof idea
  - Essential fixed point theorem (Fort)
    - Compact metric space Σ with distance *d*
    - Continuous mapping $f{:}\,\Sigma\to\Sigma$
    - σ^* essential fixed point of *f* if $\forall\varepsilon>0\ \exists\eta>0$ s.t.

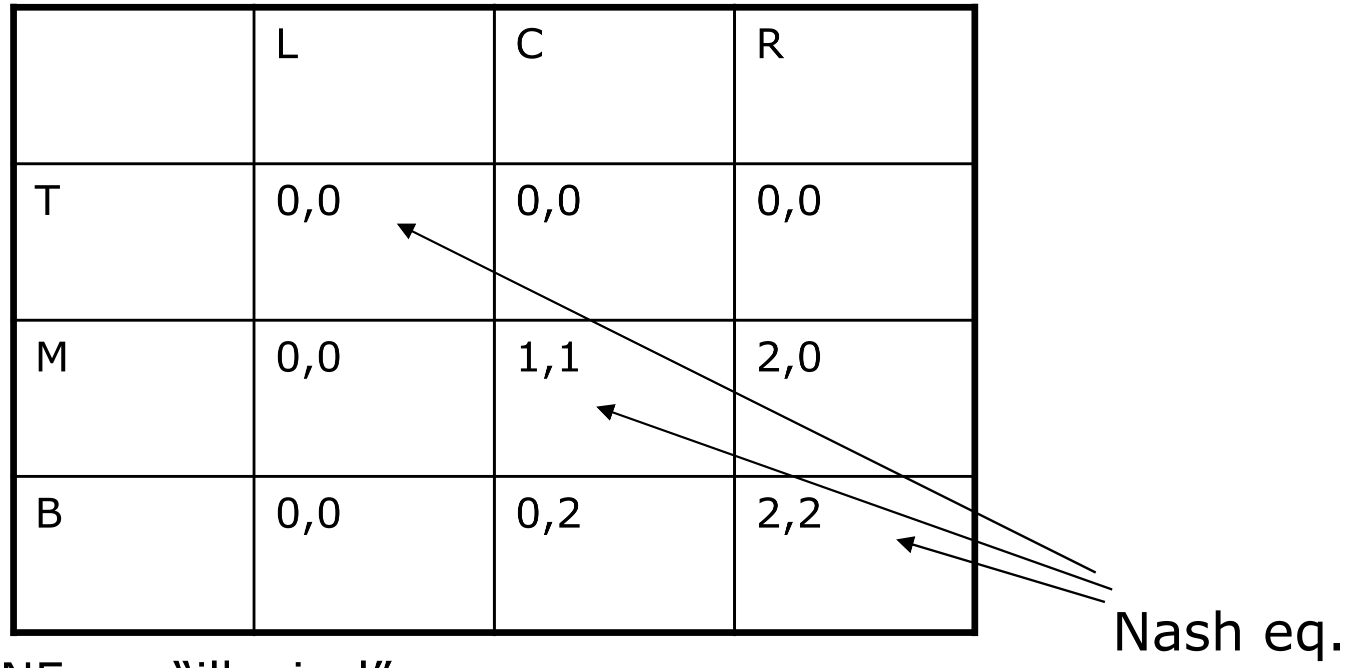
      $$d(f,\tilde{f}) = \max_{\sigma\in\Sigma} d(f(\sigma),\tilde{f}(\sigma)) < \eta \rightarrow \exists\tilde{\sigma}^*\ s.t.\ d(\sigma^*,\tilde{\sigma}^*) < \varepsilon$$
    - Essential mapping: all fixed points essential
    - Set of essential mappings is dense on the set of continuous mappings
  - Identify every game with a corresponding mapping
    - Nash mapping
    - Apply Fort's theorem

M. K. Fort, "Essential and non essential fixed points", Amer. J. Math. vol. 72, pp. 315-322, 1950

W.T. Wu and J.H. Jiang, "Essential equilibrium points of n-person non-cooperative games", Scientia Sinica vol. 11, pp. 1307–1322, 1962

(Trembling hand) Perfect equilibrium

- Find the Nash equilibria



	L	C	R
T	0,0	0,0	0,0
M	0,0	1,1	2,0
B	0,0	0,2	2,2

Nash eq.

- Some NE are "illogical"

R. Selten, "Reexamination of the Perfectness Concept for Equilibrium Points in Extensive Games", International Journal of Game Theory, 4(1), pp. 25-55, 1975

Perfect equilibrium

- A totally mixed strategy in a strategic game $<N,(A_i),(u_i)>$ is a mixed strategy α such that $\alpha_i(a_i)>0$ for $a_i \in A_i$
- ε-perfect equilibrium of a strategic game $<N,(A_i),(u_i)>$ is a totally mixed strategy α such that
  - if $U_i(\alpha_{-i}, e(a'_i)) < U_i(\alpha_{-i}, e(a_i))$ then $\alpha_i(a'_i)<\varepsilon$ for all $a_i \in A_i, a'_i \in A_i$
- A perfect equilibrium of a strategic game $<N,(A_i),(u_i)>$ is a mixed strategy α iff there exist sequences $(\varepsilon_k)_{k=1}^{\infty}$ and $(\alpha^k)_{k=1}^{\infty}$ s.t.

$$\varepsilon_k > 0 \quad \text{and} \quad \lim_{k \to \infty} \varepsilon_k = 0$$

$$\alpha^k \quad \text{are } \varepsilon - \text{perfect} \quad \text{equilibria}$$

$$\lim_{k \to \infty} \alpha_i^k(a_i) = \alpha_i(a_i) \quad \forall i, \forall a_i \in A_i$$

$$\Rightarrow \quad \alpha_i \text{ is a best response to } \alpha_{-i}^k$$

Example

- Find the Nash equilibria and the perfect equilibria

	L	C	R
T	0,0	0,0	0,0
M	0,0	1,1	2,0
B	0,0	0,2	2,2

Perfect NE

Properties of perfect equilibria

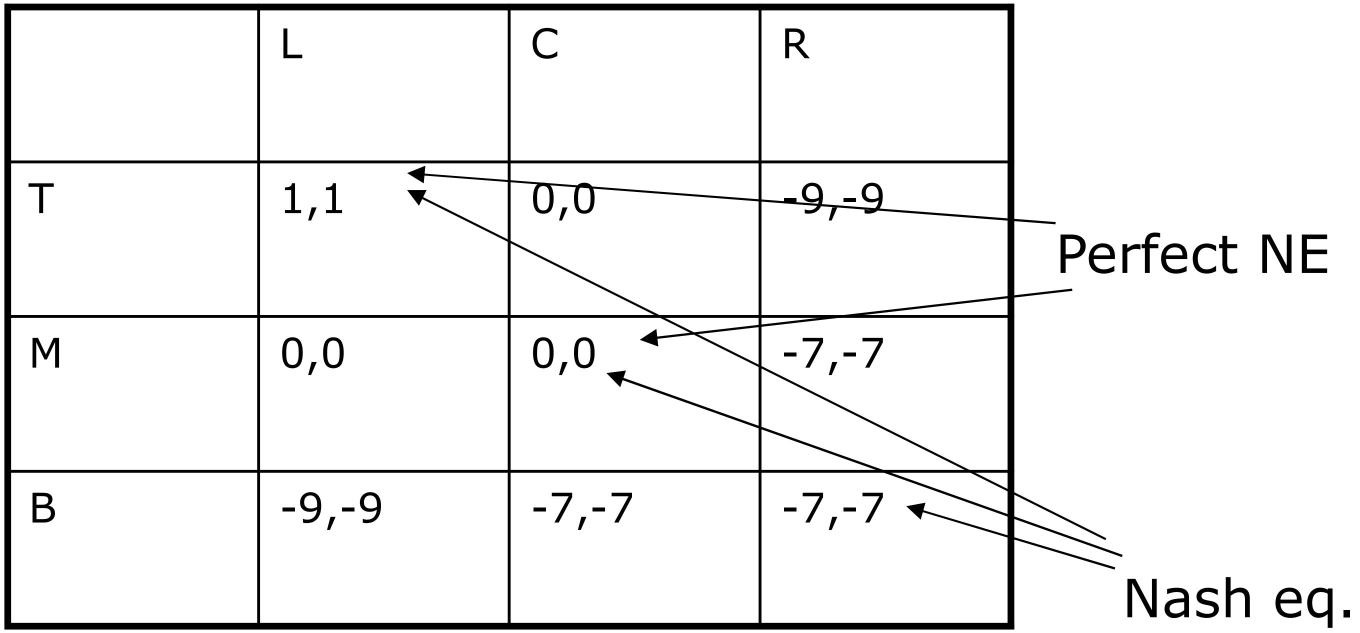
- Every perfect equilibrium is a Nash equilibrium
  - Follows from the continuity of $u_i(\alpha)$ in α

- A strategy profile in a finite two-player strategic game is a perfect equilibrium iff it is a mixed strategy NE and the strategy of neither player is weakly dominated.
  - Not true for $|N|>2$

- Every finite strategic game has a perfect equilibrium
  - Every game has an ε-perfect equilibrium
  - The NE are in a compact subset of a Euclidean space
  - Sequence of NE has convergent subsequence
    - Bolzano-Weierstrass theorem
  - Limit of suitable subsequence is a perfect equilibrium

Perfect equilibrium example

- Find the Nash equilibria and the perfect equilibria



	L	C	R
T	1,1	0,0	-9,-9
M	0,0	0,0	-7,-7
B	-9,-9	-7,-7	-7,-7

Perfect NE

Nash eq.

$$\alpha_1^{\varepsilon}(T)=\varepsilon,\alpha_1^{\varepsilon}(M)=1-2\varepsilon,\alpha_1^{\varepsilon}(B)=\varepsilon$$

$$\alpha_2^{\varepsilon}(L)=\varepsilon,\alpha_2^{\varepsilon}(C)=1-2\varepsilon,\alpha_2^{\varepsilon}(R)=\varepsilon$$

$$U_1(\alpha_2^{\varepsilon},T)=-8\varepsilon,U_1(\alpha_2^{\varepsilon},M)=-7\varepsilon,U_1(\alpha_2^{\varepsilon},B)=-7-2\varepsilon$$

R. B. Myerson, ”Refinements of the Nash equilibrium concept,” International Journal of Game Theory, 7(2) pp. 133-154, 1978.

Proper equilibrium

- ε-proper equilibrium of a strategic game $<N,(A_i),(u_i)>$ is a totally mixed strategy α such that
  - if $u_i(\alpha_{-i}, e(a'_i)) < u_i(\alpha_{-i}, e(a_i))$ then $\alpha_i(a'_i)<\varepsilon\alpha_i(a_i)$ for all $a_i\in A_i, a'_i\in A_i$

- A proper equilibrium of a strategic game $<N,(A_i),(u_i)>$ is a mixed strategy α iff there exist sequences $(\varepsilon_k)_{k=1}^{\infty}$ and $(\alpha^k)_{k=1}^{\infty}$ such that

$$\varepsilon_k > 0 \quad \text{and} \quad \lim_{k\to\infty} \varepsilon_k = 0$$

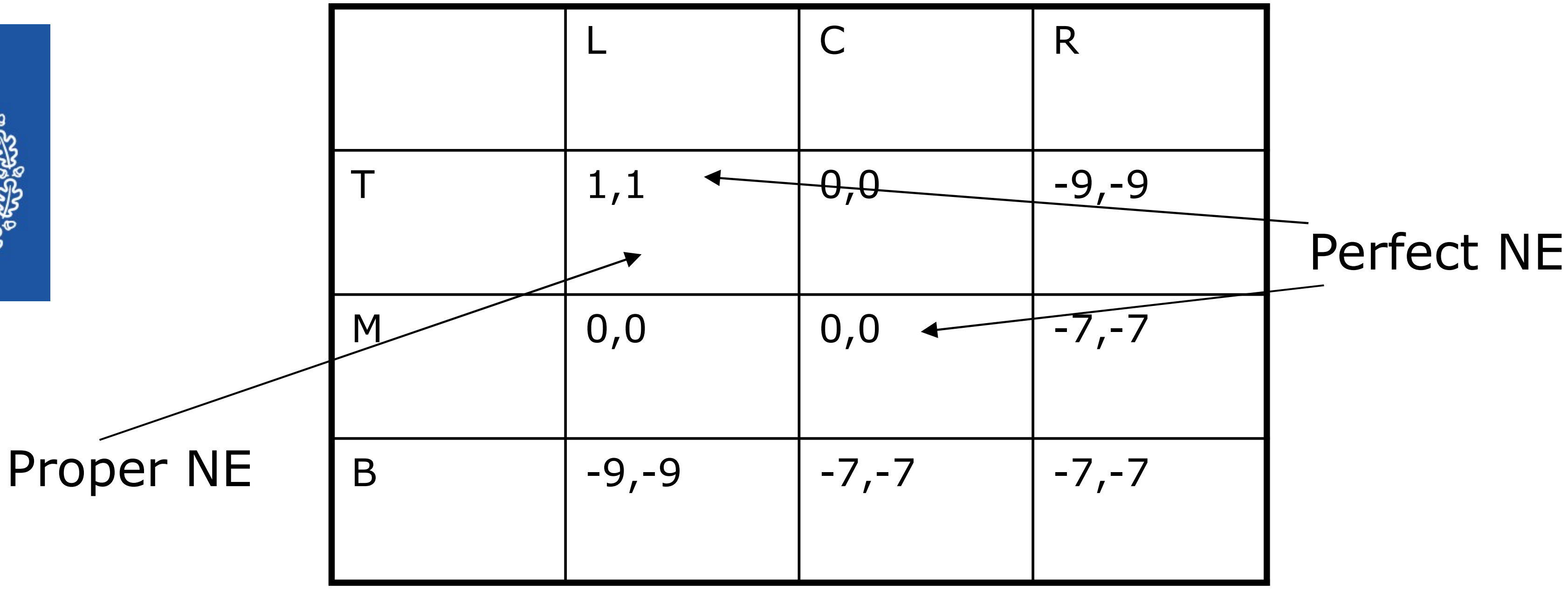
$$\alpha^k \quad \text{are } \varepsilon_k - \text{proper equilibria}$$

$$\lim_{k\to\infty} \alpha_i^k(a_i) = \alpha_i(a_i) \quad \forall i, \forall a_i \in A_i$$

$$\Longrightarrow \quad \alpha_i \text{ is a best response to } \alpha_{-i}^k$$

Example revisited

- Find the proper equilibria



	L	C	R
T	1,1	0,0	-9,-9
M	0,0	0,0	-7,-7
B	-9,-9	-7,-7	-7,-7

Perfect NE

Proper NE

Properties of proper equilibria

- Every proper equilibrium is a perfect equilibrium
  - ε-proper equilibrium is ε-perfect
  - Follows from the continuity of $u_i(\alpha)$ in α
- Every finite strategic game has a proper equilibrium
  - There is always a mixed strategy NE that is a proper equilibrium
    - Proof by Kakutani’s theorem
  - Same convergence argument as for perfect equilibrium
- Proper equilibria $\subseteq$ Perfect equilibria $\subseteq$ Nash equilibria
  - the inclusion can be strict

R. B. Myerson, ”Refinements of the Nash equilibrium concept,” International Journal of Game Theory, 7(2) pp. 133-154, 1978.

Correlated equilibria

- Recall the Nash equilibria of BoS

	Sports	Theatre
Sports	3,2	0,0
Theatre	0,0	2,3

- Payoff profiles are: (3,2),(2,3),(1.2,1.2)
- Assume that there is additional information available
  - r.v. $\Omega\in\{0,1\}$, $\pi(0)=\pi(1)=1/2$
  - Players 1 and 2 observe ω
    - choose action depending on the realization of the r.v.
  - Payoff profile (2.5,2.5) is possible

Another example

- Consider a finite strategic game $<\{1,2\},(\{a_1,b_1\},\{a_2,b_2\}),(u_i)>$
  - r.v. $\Omega\in\{0,1,2\}$, $\pi(0)=1-\zeta-\eta$, $\pi(1)=\eta$, $\pi(2)=\zeta$
  - Player 1 observes whether $\omega=0$ or $\omega\in\{1,2\}$
  - Player 2 observes whether $\omega\in\{0,1\}$ or $\omega=2$
- Assume player 2's strategy is
  - a_2 if $\omega\in\{0,1\}$
  - b_2 if $\omega=2$
- What is player 1's optimal strategy?
  - If $\omega=0$
    - chose action optimal for a_2
  - If $\omega\in\{1,2\}$
    - chose action optimal for a_2 with probability $\eta/(\eta+\zeta)$
    - chose action optimal for b_2 with probability $\zeta/(\eta+\zeta)$

Correlated equilibrium

- Correlated equilibrium of a strategic game $<N,(A_i),(u_i)>$ consists of
  - a finite probability space (Ω,π)
  - for each player $i \in N$ a partition $\mathcal{P}_i$ of Ω (information partition)
  - for each player $i \in N$ a function $\sigma_i: \Omega \to A_i$ for which $\sigma_i(\omega) = \sigma_i(\omega')$ whenever $\omega \in P_i$ and $\omega' \in P_i$ *for some* $P_i \in \mathcal{P}_i$ *(strategy)*

  such that
  - for every $i \in N$ and every function $\tau_i: \Omega \to A_i$ for which $\tau_i(\omega) = \tau_i(\omega')$ whenever $\omega \in P_i$ and $\omega' \in P_i$ *for some* $P_i \in \mathcal{P}_i$ *we have*

$$\sum_{\omega \in \Omega} \pi(\omega) u_i(\sigma_{-i}(\omega), \sigma_i(\omega)) \geq \sum_{\omega \in \Omega} \pi(\omega) u_i(\sigma_{-i}(\omega), \tau_i(\omega))$$

- Player *i*'s strategy is optimal given the other players' strategies and player *i*'s knowledge about ω
- Can be extended to asymmetric beliefs (π_i)

R.J. Aumann, "Subjectivity and Correlation in Randomized Strategies", in Journal of Math. Econ, vol 1.pp.67-96, 1974

Example revisited

- Correlated equilibrium for BoS
  - Set of states
    - r.v. $\Omega \in \{0,1\}$, $\pi(0)=\pi(1)=1/2$
  - Information partitions
    - $P_1=P_2=\{\{0\},\{1\}\}$
  - Strategies
    - $\sigma_i(0)$=‘Theatre’
    - $\sigma_i(1)$=‘Sports’
  - Payoff profile (2.5,2.5) is possible

	Sports	Theatre
Sports	3,2	0,0
Theatre	0,0	2,3

- Needs some interpretation
  - Tossing coins?

Properties of correlated equilibria

- For every mixed strategy NE of a finite strategic game $<N, (A_i),(u_i)>$ there is a correlated equilibrium $<(\Omega,\pi),(\mathcal{P}_i),(\sigma_i)>$ in which for each player $i \in N$ the distribution on A_i induced by σ_i is α_i.
  - take $\Omega=A$, $\pi(a)=\alpha(a)$, $\mathcal{P}_i(b_i)=\{a \in A | a_i=b_i\}$
  - $\sigma_i(a)=a_i$

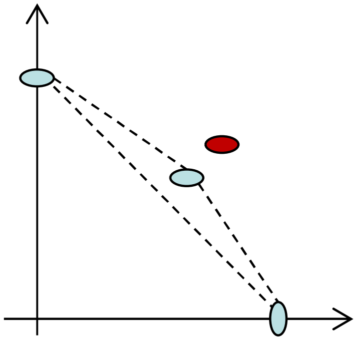
- Let $G=<N,(A_i),(u_i)>$ be a strategic game. Any convex combination of correlated equilibrium payoff profiles of G is a correlated equilibrium payoff profile of G.
  - $\Omega=U_k\Omega^k$, $\mathcal{P}_i=U_k \mathrm{P}_i^k$
  - for $\omega \in \Omega^k$ let $\pi(\omega)=\lambda_k\pi^k(\omega)$ and $\sigma_i(\omega)=\sigma_i^k(\omega)$
  - then

$$\mathrm{u}_i=\sum_{k=1}^{K} \lambda^k u_i^k$$

  - Play the k^{th} correlated equilibrium with probability λ^k

Example

	L	R
T	6,6	2,7
B	7,2	0,0



- NE payoff profiles (7,2),(2,7),(14/3,14/3)
- Payoff outside of the convex hull of these payoffs
  - Set of states
    - r.v. $\Omega \in \{x,y,z\}$, $\pi(x)=\pi(y)=\pi(z)=1/3$
  - Information partitions
    - $P_1=\{\{x\},\{y,z\}\}$, $P_2=\{\{x,y\},\{z\}\}$
  - Strategies
    - $\sigma_1(x)=B$, $\sigma_1(y)=\sigma_1(z)=T$
    - $\sigma_2(x)=\sigma_2(y)=L$, $\sigma_2(z)=R$
- The strategies are optimal with respect to each other
  - payoff profile (5,5)

	L	R
T	y	z
B	x	-

Construction of correlated equilibria

- Let $G=<N,(A_i),(u_i)>$ be a finite strategic game. Every probability distribution over outcomes that can be obtained in a correlated equilibrium of G can be obtained in a correlated equilibrium in which
  - the set of states is A and
  - for each $i \in N$ player i's information partition $\mathcal{P}_i(b_i)$ consists of all sets of the form $\{a \in A | a_i = b_i\}$ for some action $b_i \in A_i$.

- It is enough to consider correlated equilibria in which $\Omega = A$.

Games with incomplete information (Bayesian games)

- Strategic game with complete information
  - Players know each others' preferences
  - Players know what the others know
    - Rationalizability

- Strategic game with incomplete information
  - Players are not certain of the properties of other players
  - Players do not have to know what the others know

  - Uncertainty modeled by the "state of nature"
    - Prior belief of each player
    - Each player observes a signal
      - Determines the type of the player
    - Posterior belief of each player about the state of nature
      - Calculated using Bayes' theorem

Bayesian game

- A Bayesian game consists of
  - a finite set N of players
  - a finite set Ω of states

  and for each player i
  - a set A_i of actions
  - a finite set T_i and a function $\tau_i:\Omega\rightarrow T_i$ (set of signals and signal function)
  - a probability measure p_i on Ω such that $p_i(\tau^{-1}(t_i))>0$ for all $t_i\in T_i$ (prior belief)
  - a preference relation $\succcurlyeq_i$ on the set of probability measures over $Ax\Omega$, where $A=x_{j\in N}A_j$ (preference relation)
- The preference relation is taken over lotteries

John C. Harsányi "Games with incomplete information played by Bayesian players," Management Science, vol. 14, pp. 159-182, pp. 320-334, pp. 486-502, 1967-1968

Simple Examples

- Example 1
  - Let Ω be the set of states of nature
  - $\tau_i(\omega)=\omega$
    - Complete information

- Example 2
  - Let $\Omega=x_{i\in N}T_i$ be the set of states of nature
  - $\tau_i(\omega)=\omega_i$
    - No information about other players

Another example

- Consider a Bayesian game
  - $N=\{1,2\}$
  - $\Omega=\{\omega_1,\omega_2,\omega_3\}$, $p_i(\omega_j)=1/3$
  - Signal functions
    - $\tau_1(\omega_1)= \tau_1(\omega_2)=t'_1$, $\tau_1(\omega_3)=t''_1$
    - $\tau_2(\omega_1)= t'_2$, $\tau_2(\omega_2)=\tau_2(\omega_3)=t''_2$
  - Preference relations
    - $(b,\omega_j) \succ_1 (c,\omega_j)$ for $j=1,2;\quad (c,\omega_3) \succ_1 (b,\omega_3)\quad$ for some b,c
    - Player 2 indifferent for all (a,ω)
- Knowledge about each other depends on the state
  - In state ω_1
    - Player 2 knows that Player 1 prefers b to c
    - Player 1 does not know
      - if player 2 knows that she prefers *b* to *c*
      - if player 2 believes that she prefers *c* to *b*
  - In state ω_2
    - Player 2 does not know if player 1 prefers *b* to *c* or *c* to *b*

BoS with uncertainty

	Sports	Theatre
Sports	2,1	0,0
Theatre	0,0	1,2

	Sports	Theatre
Sports	2,0	0,2
Theatre	0,1	1,0

- $\Omega=\{\omega_a, \omega_b\}$, $\tau_2(\omega)=\omega$, $\tau_1(\omega)=\Omega$
- $p_1(\omega)=0.5$

Nash equilibrium of a Bayesian game (agent form)

- A Nash equilibrium of a Bayesian game $\langle N,\Omega,(A_i),(T_i),(\tau_i),(p_i),(\succcurlyeq_i)\rangle$ is a Nash equilibrium of the strategic game defined as
  - The set of players is (i,t_i) for all $i \in N$, $t_i \in T_i$
  - The set of actions of each player (i,t_i) is A_i
  - The preference relation $\succcurlyeq_{i,t_i}$ of player $(\mathrm{i},\mathrm{t_i})$ is defined as

$$a^* \succcurlyeq_{i,t_i} b^* \Leftrightarrow \quad L_i(a^*,t_i) \succcurlyeq_i L_i(b^*,t_i),$$

where $L_i(a^*,t_i)$ is the lottery over $Ax\Omega$ that assigns the posterior probability given t_i to every $((a^*(j,\tau_j(\omega)))_{j\in N},\omega)$

The posterior probability is

$$\begin{array}{ll} p_i(\omega)/p_i(\tau^{-1}(t_i)) & if\ \omega \in \tau^{-1}(t_i) \\ 0 & otherwise \end{array}$$

Bayesian Nash Equilibrium

- A simplified way of thinking of this
  - Expected utility of strategy in Bayesian game for given strategy profile $s_{-i}(t_{-i})$ of other players

$$E[u_i(s_i \mid s_{-i}, t_i)] = \sum_{t_{-i} \in T_{-i}} u_i(s_i, s_{-i}(t_{-i}), t_i, t_{-i}) p(t_{-i} \mid t_i)$$

  - BNE is NE of the Bayesian game

$$E[u_i(s_i \mid s_{-i}, t_i)] \geq E[u_i(s_i' \mid s_{-i}, t_i)] \qquad \forall s_i(t_i), \forall t_i$$

Bayesian BoS Continued

- Induced normal form (expected payoffs)
- Strategies of Player 2
  - (S,S), (S,T), (T,S), (T,T)

	SS	ST	TS	TT
Sports	2,0.5	1,1.5	1,0	0,1
Theatre	0,0.5	0.5,0	0.5,1.5	1,1

	Sports	Theatre
Sports	2,1	0,0
Theatre	0,0	1,2

	Sports	Theatre
Sports	2,0	0,2
Theatre	0,1	1,0

Literature

- M.Osborne, A Rubinstein, “A course in game theory”, MIT press, 1994
- D. Fudenberg, J. Tirole, “Game Theory”, MIT press, 1991

- M. K. Fort, “Essential and non essential fixed points”, Amer. J. Math. vol. 72, pp. 315-322, 1950
- W.T. Wu and J.H. Jiang, “Essential equilibrium points of n-person non-cooperative games”, Scientia Sinica vol. 11, pp. 1307–1322, 1962
- R. Wilson, “Computing Equilibria in N-person Games,” SIAM Journal of Applied Mathematics, 21(1), pp. 80-87, 1971
- J.C. Harsányi, ”Oddness of the number of equilibrium points: A new proof”, International Journal of Game Theory, 2(1), pp. 235-250, 1973
- R. Selten, “Reexamination of the Perfectness Concept for Equilibrium Points in Extensive Games”, International Journal of Game Theory, 4(1), pp. 25-55, 1975
- R. B. Myerson, ”Refinements of the Nash equilibrium concept,” International Journal of Game Theory, 7(2) pp. 133-154, 1978.