Computational Game Theory

Lecture 2

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Today's Topics

- Pure NE existence proof(s)
- Mixed strategies
  - Actions of equal values
  - Nash's theorem
- Zero-sum games
  - Maxminimization

Existence of Nash equilibria

- The strategic game $<N,(A_i),(\succcurlyeq_i)>$ has a Nash equilibrium if for all $i \in N$
  - the set A_i of actions of player i is a nonempty compact convex subset of a Euclidean space

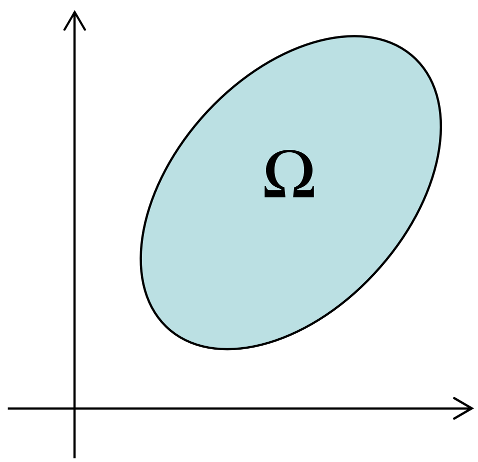
  and the preference relation $\succcurlyeq_i$ is
  - continuous
  - convex on A_i.

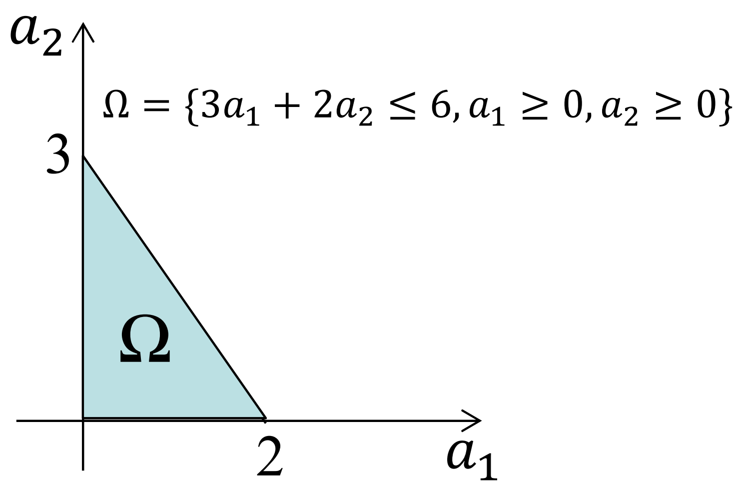
- Proof
  - based on Kakutani's fixed point theorem

(Debreu '52,Glicksberg '52, Fan '52)

Existence for coupled constraints

- Let Ω be a coupled constraint set, convex, closed, bounded. Let $u_i(a_i,a_{-i})$ be concave in a_i for each a_{-i} and continuous in a. Then there exists a pure NE.





J.B. Rosen, "Existence and uniqueness of equilibrium points for concave N-person games", Econometrica, 33(3), Jul. 1965

More existence results

- The strategic game $<N,(A_i),(u_i)>$ has a Nash equilibrium if for all $i \in N$
  - the set A_i of actions of player i is a nonempty compact convex subset of a finite dimensional Euclidean space

  and every payoff function u_i
  - is quasi-concave in a_i.
  - is upper semi-continuous in $a \in A$

    $$\limsup_{n\to\infty} \sum_{i=1}^{N} u_i(a^n) \leq \sum_{i=1}^{N} u_i(a), \quad a^n \to a$$
  - has a continuous maximum

P. Dasgupta and E. Maskin, "The Existence of Equilibrium in Discontinuous Economic Games, I: Theory," Review of Economic Studies, 53(1), pp. 1-26, 1986

Notes on the existence results

- The equilibrium is not necessarily unique
  - Which equilibrium is an appropriate solution?
- The existence is not guaranteed for finite games!
  - For none of the examples considered so far…
- Best response functions could be used to find equilibria
  - Not very efficient

What if no pure NE?

- Matching pennies

	Head	Tail
Head	1,-1	-1,1
Tail	-1,1	1,-1

Games with mixed strategies

- Pure strategy Nash equilibria do not always exist
  - Certain classes of games (later)
    - special structure
    - specific utility function

- Extension of the model of a game
  - Allow players to randomize between actions

- Von Neumann – Morgenstern assumption
  - Utility of a randomized strategy = expected value of the utilities of the action profiles

Axioms of preference

Let $A \subseteq R^n$, and $\Delta(A)$ the set of probability distributions over A. The binary relation $\succ$ is a VNM *rational* preference relation over $\Delta(A)$

- (i) Complete
  $\alpha, \beta \in \Delta(A)$ then $\alpha \succ \beta$ *or* $\beta \succ \alpha$ *or* $\alpha \sim \beta$
- (ii) Transitive
  $\alpha, \beta, \gamma \in \Delta(A)$ then if $\alpha \succ \beta$ and $\beta \succ \gamma$ then $\alpha \succ \gamma$
- (iii) Continuous
  $\alpha, \beta, \gamma \in \Delta(A)$ *s.t.* $\alpha \succ \beta \succ \gamma$, then $\exists a \in [0,1]$ s.t.
  $a\alpha + (1-a)\gamma \sim \beta$
- (iiii) Independent
  $\alpha, \beta, \gamma \in \Delta(A)$ *and* $a \in (0,1]$ then
  $\alpha \succ \beta \Leftrightarrow a\alpha + (1-a)\gamma \succ a\beta + (1-a)\gamma$

J. von Neumann, O. Morgenstern, "Theory of Games and Economic Behavior," Princeton University Press, 1944

von Neumann-Morgenstern theorem

- Let $\Delta(A)$ be a convex subset of a linear space. Let $\succ$ be a binary relation on $\Delta(A)$. Then $\succ$ satisfies axioms (1),(2),(3),(4) iff. $\exists U:\Delta(A)\rightarrow R$ such that
  - U represents $\succ$

    $$\alpha,\beta \in \Delta(A) \;\; U(\alpha) > U(\beta) \Leftrightarrow \alpha \succ \beta$$
  - Utility is in expected value (VNM utility)

    $$U(\alpha) = \sum_{a\in A} \alpha(a) U(a) \quad \alpha \in \Delta(A)$$

- Moreover, if $V: \Delta(A)\rightarrow R$ also represents preferences, then

  $$\exists b > 0, c \in R \; s.t. \; V = bU + c$$

  (U is unique up to a positive linear transformation)

J. von Neumann, O. Morgenstern, ”Theory of Games and Economic Behavior,” Princeton University Press, 1944

Randomizing actions

- A_i finite set
- Let $\Delta(A_i)$ be the set of probability distributions over A_i
- Let $\alpha_i \in \Delta(A_i)$,
  - α_i is a mixed strategy
  - Support of α_i : $\{a_i \in A_i : \alpha_i(a_i) > 0\}$
- Evaluation of a profile of mixed strategies
  - $(\alpha_j)_{j \in N}$ profile of mixed strategies
  - Probability of action profile $a=(a_j)_{j \in N}$

$$\Pi_{j \in N} \alpha_j(a_j)$$

  - Utility of the strategy profile $\alpha=(\alpha_j)_{j \in N}$ for player $i \in N$

$$U_i(\alpha) = \sum_{a \in A} u_i(a) \Pi_{j \in N} \alpha_j(a_j)$$

- Looks trivial but is not necessarily reasonable
  - e.g., risk aversion

Mixed extension of a strategic game

- Mixed extension of a strategic game $<N,(A_i),(u_i)>$ *is the strategic game* $<N,(\Delta(A_i)),(U_i)>$
  - $\Delta(A_i)$ set of probability distributions over A_i
  - $U_i: \times_{j\in N}\Delta(A_j)\rightarrow R$ expected value under u_i *of the lottery over A induced by* $\alpha\in\times_{j\in N}\Delta(A_j)$

$$U_i(\alpha) = \sum_{a\in A} u_i(a)\Pi_{j\in N}\alpha_j(a_j)$$

  - Alternative using degenerate distribution $\alpha_i\ (e(a_i))$

$$U_i(\alpha) = \sum_{a_i\in A_i} \alpha_i(a_i)U_i(\alpha_{-i}, e(a_i))$$

- Utility is multilinear
  - for mixed strategies β_i and γ_i

$$U_i(\alpha_{-i}, \lambda\beta_i + (1-\lambda)\gamma_i) = \lambda U_i(\alpha_{-i}, \beta_i) + (1-\lambda)U_i(\alpha_{-i}, \gamma_i)$$

Mixed strategy Nash equilibrium

- Definition: Mixed strategy Nash equilibrium of a strategic game is the Nash equilibrium of its mixed extension
- Consequence: The set of Nash equilibria of a strategic game is a subset of the Nash equilibria of its mixed extension
  - degenerate α_i $(e(a_i))$

Example

- Matching pennies

	Head	Tail
Head	1,-1	-1,1
Tail	-1,1	1,-1

- No pure strategy equilibria
- Are there mixed strategy equilibria?

Existence of equilibria

- Every finite strategic game has a mixed strategy Nash equilibrium.

  Proof. The mixed extension of the strategic game has a pure strategy Nash equilibrium.

  Nash (1950,1951)

- The result applies if $A_i \subseteq R^n$ compact non-empty and the payoff functions are continuous.
  - Holds for coupled action sets as well
  - Idea: (i) discretize the action set → mixed NE exists

    (ii) as discretization gets finer show that mixed NE converges to a NE of the original game

  Glicksberg (1952)

  Owen (1974)

More existence results

- Let A_i be a closed interval of R. Suppose that
  - u_i is continuous except on a subset $A^{**}(i)$ of $A^*(i)$, where $A^*(i)$ is defined as
    $$A^{**}(i) \subseteq A^*(i) = \{a \in A \mid \exists j \neq i, \exists d \text{ such that } a_j = f_{ij}^d(a_i)\}$$
    ($f_{ij}^d : A_i \to A_j$ are one-to-one, continuous functions)
  - $\sum_{i=1}^{N} u_i(a)$ is upper semi-continuous
    $$\limsup_{n \to \infty} \sum_{i=1}^{N} u_i(a^n) \leq \sum_{i=1}^{N} u_i(a)$$
  - $u_i(a_{-i},a_i)$ is bounded and weakly lower semi-continous in a_i for all $a_{-i} \in A_{-i}^{**}(a)$
    $$\lambda \liminf_{a' \uparrow a_i} u_i(a_{-i}, a_i') + \lambda \liminf_{a' \downarrow a_i} u_i(a_{-i}, a_i') \geq u_i(a_{-i}, a_i) \text{ for some } \lambda \in [0,1]$$

Then the game has a mixed strategy Nash equilibrium

P. Dasgupta and E. Maskin, “The Existence of Equilibrium in Discontinuous Economic Games, I: Theory," Review of Economic Studies, 53(1), pp. 1-26, 1986

Example

- Matching pennies

	Head	Tail
Head	1,-1	-1,1
Tail	-1,1	1,-1

- No pure strategy equilibria
- Are there mixed strategy equilibria?
- What is the best action in equilibrium?

Mixed strategy of best responses

- Let $G=<N, (A_i), (u_i)>$ finite strategic game

  $\alpha^* \in \times_{j\in N}\Delta(A_i)$ is a mixed strategy equilibrium of G

  $\Leftrightarrow$ for all $i\in N$ $\{a_i\in A_i: \alpha_i^*(a_i)>0\}$ are best responses to α^*_{-i}

- The actions used by a player in a mixed strategy are best responses to the equilibrium mixed strategy profile
  - but: not all best responses have to have $\alpha_i^*(a_i)>0$

Actions of equal values

- Let $G=\langle N, (A_i), (u_i)\rangle$ finite strategic game
  - $\alpha^* \in \times_{j\in N}\Delta(A_i)$ is a mixed strategy equilibrium of G $\rightarrow$
    $\alpha_i^*(a_i)>0 \Rightarrow U_i(\alpha^*_{-i},a_i)=U_i(\alpha^*)$
  - If for an $\alpha^* \in \Delta(A)$ and for every player *i* there is a constant c_i such that we have

$$\begin{cases} \alpha_i^*(a_i) > 0 \Rightarrow U_i(\alpha_{-i}^*, a_i) = c_i \\ \alpha_i^*(a_i) = 0 \Rightarrow U_i(\alpha_{-i}^*, a_i) \le c_i \end{cases}$$

    then α^* is a mixed strategy equilibrium

Example

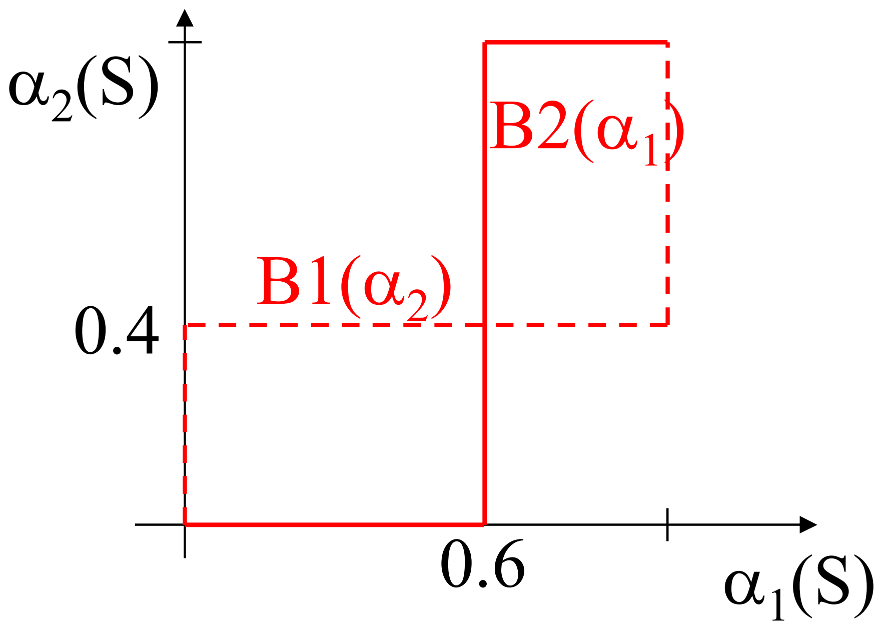
- Battle of the Sexes

	Sports	Theatre
Sports	3,2	0,0
Theatre	0,0	2,3

- How many NE are there?
  - Pure NE
  - Mixed NE

Example continued

- Mixed strategy NE
  - *(0.6,0.4) (0.4,0.6)*
- Best response functions of players *1* and *2*



Why mixed strategies?

- Drawn from a large population
  - Individuals meet at random
  - Pick actions according to a distribution

- Harsányi's model of a disturbed game
  - small perturbation in players' payoffs u_i
    - uncertainty or ignorance
  - perturbation is a r.v. $\varepsilon_i \in [-\varepsilon, \varepsilon]$
  - ε_i known to player i only
  - pure NE of the disturbed game converge to mixed NE of the ordinary game as $\varepsilon \to 0$

J. C. Harsányi, "Games with Randomly Disturbed Payoffs: A New Rationale for Mixed Strategy Equilibrium Points," International Journal of Game Theory, vol. 2, pp.1–23, 1973

Strictly competitive games

- The strategic game $<\{1,2\},(A_i),(\succcurlyeq_i)>$ is strictly competitive if for any $a \in A$ and $b \in A$ we have
  - $a \succcurlyeq_2 b \Leftrightarrow b \succcurlyeq_1 a$

- Equivalent definitions
  - Zero-sum game
    - $u_1(a)=-u_2(b)$
  - Constant-sum game
    - $u_1(a)+u_2(b)=c$

An example

	L	M	R
T	7,-7	-3,3	-5,5
M	2,-2	-1,1	4,-4
B	-5,5	-2,2	9,-9

- The paranoid's approach
  - Take the highest payoff that you can guarantee

Maxminimization

- Let <$\{1,2\},(A_i),(u_i)$> be a strictly competitive strategic game

  - The action $x^* \in A_1$ is called a maxminimizer for player 1 if

    $$\min_{y \in A_2} u_1(x^*, y) \geq \min_{y \in A_2} u_1(x, y) \quad \text{for all } x \in A_1$$

  - The action $y^* \in A_2$ is called a maxminimizer for player 2 if

    $$\min_{x \in A_1} u_2(x, y^*) \geq \min_{x \in A_1} u_2(x, y) \quad \text{for all } y \in A_2$$

	L	M	R
T	7,-7	-3,3	-5,5
M	2,-2	-1,1	4,-4
B	-5,5	-2,2	9,-9

NE and Maxminimization

- *Let G=<{1,2},(A$_i$),(u$_i$)>* be a strictly competitive strategic game
  - If *(x*,y*)* is a NE of *G* then x^* is a maxminimizer for player 1 and y^* for player 2
  - If *(x*,y*)* is a NE of *G* then
    $$\max_{x\in A_1}\min_{y\in A_2} u_1(x,y)=\min_{y\in A_2}\max_{x\in A_1} u_1(x,y)=u_1(x^*,y^*)$$
    and thus all NE of *G* yield the same payoff
  - If $\max_{x\in A_1}\min_{y\in A_2} u_1(x,y)=\min_{y\in A_2}\max_{x\in A_1} u_1(x,y)$ and x^* is a maxminimizer for player 1 and y^* is a maxminimizer for player 2, then (x^*,y^*) is a Nash equilibrium of *G*

J. von Neumann, "Zur Theorie der Gesellschaftsspiele", Mathematische Annalen, 100, pp. 295–300, 1928

Consequences

- In strictly competitive games
  - Can use maxminimization to find Nash equilibria (pure or mixed)
  - Nash equilibria are interchangable

    if (x,y) and (x',y') are NE
      then so are (x,y') and (x',y)

- If $\max_{x\in A_1}\min_{y\in A_2} u_1(x,y) = \min_{y\in A_2}\max_{x\in A_1} u_1(x,y)$

  then this is the value v^* of the game
  - v^* is the minimum payoff of player 1
  - $-v^*$ is the minimum payoff of player 2

The example again

	L	M	R
T	7,-7	-3,3	-5,5
M	2,-2	-1,1	4,-4
B	-5,5	-2,2	9,-9

- Find the Nash equilibrium using maxminimization
- What is the value of the game?

Literature

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