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1.1. Irreducibility and dimension of topological spaces.

→ **(1.1.1) Setup.** [G](EGA 20, 0.14.3) Let X be a topological space. A non-empty subset is *irreducible* if it can not be written as a union of two proper closed subset, or equivalently, if any two open non empty subsets intersect.

(1.1.2) Definition. The *combinatorial dimension* , denoted $\dim X$, of a topological space X is the supremum of the length n of chains

$$X_0 \subset X_1 \subset \cdots \subset X_n$$

of irreducible closed subsets of X.

Given a closed irreducible subset Y of X. The *combinatorial codimension* , denoted $\operatorname{codim}(Y, X)$, is the supremum of the length n of chains

$$Y = X_0 \subset X_1 \subset \cdots \subset X_n$$

of irreducible closed subsets X_i of X.

(1.1.3) Definition. A chain $Z_0 \subset Z_1 \subset \cdots \subset Z_n$ of irreducible closed subsets of a topological space X is *saturated* if there is no irreducible closed subset Z of X such that $Z_i \subset Z \subset Z_{i+1}$ for some i.

A topological space X is *catenary* if the codimension $\operatorname{codim}(Z, Y)$ is finite for all pairs $Z \subset Y$ of closed irreducible subsets and every saturated chain

$$Z = Z_0 \subset Z_1 \subset \cdots \subset Z_n = Y$$

of irreducible closed subsets have the same length.

(1.1.4) Proposition. *A topological space X is catenary if and only if the codimension* $\operatorname{codim}(Z, Y)$ *is finite for all pairs $Z \subset Y$ of irreducible closed subsets, and for every irreducible closed subset T of Z we have that*

$$\operatorname{codim}(T, Y) = \operatorname{codim}(T, Z) + \operatorname{codim}(Z, Y).$$

Proof. It is clear that the formula of the Proposition holds when X is catenary.

Conversely, assume that the formula holds, and that we have two chains between X and Y with lenghts m and n, and with $m \leq n$. If $m = 1$ we must have that $m = n$. We prove the assertion by induction on m and assume that the Proposition holds for all chains of length m. Assume that $m > 1$ and $m < n$. Let $Z = Z_0 \subset \cdots \subset Z_m = Y$ be a saturated chain of length m. We have that $\operatorname{codim}(Z_0, Z_m) \geq n > m$ and $\operatorname{codim}(Z_0, Z_1) = 1$. Hence it follows from the formula of the Proposition that

$$\operatorname{codim}(Z_1, Z_m) = \operatorname{codim}(Z_0, Z_m) - \operatorname{codim}(Z_0, Z_1) > m - 1.$$

However, it follows from the induction assumption that we have $\operatorname{codim}(Z_1, Z_m) = m - 1$. Hence we obtain a contradiction to the assumption that $m < n$, and we must have that $m = n$.

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(1.1.5) Definition. A topolgical space X is *noetherian* if every descending chain

$$X_1 \supseteq X_2 \supseteq \cdots$$

of closed subsets of X is stationary.

(1.1.6) Remark. Every subspace of a noetherian space is noetherian.

Moreover, every noetherian space X is compact. Indeed, consider the family of closed subsets of X that are not compact. If this family is not empty it has a minimal element Y. Given an open covering $(U_i)_{i \in I}$ of Y. For any i such that U_i is non empty we have that $Y \setminus U_i$ is a proper closed subset of Y and thus can be covered by a finite number of the members of $(U_i)_{i \in I}$. However, then Y is covered by these members and U_i, and consequently Y is compact, contrary to the assumption. Hence the family is empty.

(1.1.7) Proposition. *Every noetherian topological space X can be uniquely written as a union $X = \cup_{i=1}^n X_i$ of irreducible closed subsets X_i of X such that $X_i \not\subseteq_{i \neq j} X_j$.*

Proof. Consider the family of closed subsets of X that can not be written as a finite union of closed irreducible subsets. If the family is empty it has a minimal member Y. Then Y is not irreducible so $Y = Y_1 \cup Y_2$ where Y_1 and Y_2 are proper closed subsets of Y. However, then Y_1 and Y_2, and thus Y, can be written as a finite union of closed irreducible subsets, contary to the assumption on Y. The family is thus empty and we have proved the first part of the Proposition.

For the second part, assume that $X = \cup_{i=1}^m X'_i$. For each i we have that $X_i \subseteq \cup_{j=1}^m X'_j$, and thus $X_i \subseteq X'_j$ for some j. A similar reasoning shows that $X_j \subseteq X_k$ for some k. Thus $i = k$ and we have that $X_i = X'_j$. In this way we see that the members of the sets $\{X_1, \ldots, X_n\}$ and $\{X'_1, \ldots, X'_m\}$ are pairwise equal and we have proved the Proposition. ∎

(1.1.8) Definition. Given a noetherian space X. The irreducible sets X_i of X
→ in Proposition (1.1.7) are called the *irreducible components* of X.

We say that X has *pure dimension* if all the components X_i have the same finite dimension.

For each point x in X we denote by $\dim_x X$ the maximum of the dimensions of the irreducible components passing by x.

(1.1.9) Remark. We have that

$$\dim X = \max_{i=1}^{n} \dim X_i$$

because every irreducible subset in X is contained in one of the irreducble components X_i.

1.2. Constructible sets.

(1.2.1) Setup. [G] [EGA 11, III_0, 9.1, 9.2, 9.3]

Let X be a topological space.

(1.2.2) Definition. Given a topological space X and two points x and y. We say that y is a *specialization* of x or that x *specializes* to y if y is in the closure $\overline{\{x\}}$ of x in X. We also say that x is a *generization* of y. When $X = \overline{\{x\}}$, that is, when all points of X are specializations of x, we say that x is a *genric point* for X.

(1.2.3) Remark. Given a point x in a topological space X. Then $\overline{\{x\}}$ is a closed irreducible subset of X with generic point x.

(1.2.4) Definition. A subset of a topological space is *locally closed* if it is the intersection of a closed and an open subset.

(1.2.5) Definition. Given a noetherian topological space X. A subset is *constructible* if it belongs to the smallest family of subsets of X that contain all closed subsets and that are closed under finite intersections and passing to the complement.

(1.2.6) Remark. The constructible sets in a noetherian topological space can equivalently be defined as the smallest family of subsets of X that contain all open subsets and is closed under finite unions and passing to the complement.

(1.2.7) Proposition. *Given a subset Z of a noetherian topological space X. The following three assertions are equivalent:*

(1) *The set Z is constructible.*
(2) *The set Z is a finite union of locally closed subsets of X.*
(3) *The set Z is a finite disjoint union of locally closed subsets of X.*

Proof. It is clear that (3) implies (2) and that (2) implies (1).

We show that (1) implies (2). Since the set of locally closed subsets contains all open sets it suffices to show that the family of sets that consists of all finite unions of locally closed sets is closed under finite intersections and passing to the complement. It is clear that it is closed under finite unions. Write Z as a union $Z = \cup_{i=1}^m (U_i \cap V_i^c)$ of locally closed sets, where U_i and V_i are open subsets of X. Then we have that

$$Z^c = \bigcap_{i=1}^m (U_i \cap V_i^c)^c = \bigcap_{i=1}^m (U_i^c \cup V_i) = \bigcap_{i=1}^m U_i^c \bigcup (U_i \bigcap V_i).$$

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Consequently we have that Z^c is the disjoint union of the sets

$$U_{i_1}^c \cap \cdots \cap U_{i_r}^c \cap (U_{i_{r+1}} \cap V_{i_{r+1}}) \cap \cdots \cap (U_{i_n} \cap V_{i_m}),$$

where $\{i_1, \dots, i_n\} = \{1, \dots, n\}$.

Next we show that (2) implies (3). Let $Z = \cup_{i=1}^m Z_i$ be the union of locally closed sets Z_i. Then Z is the disjoint union of the sets

$$\begin{aligned} Z_{i_1} \cap \cdots \cap Z_{i_r} \setminus (Z_{i_{r+1}} \cup \cdots \cup Z_{i_n} &= Z_{i_1} \cap \cdots \cap Z_{i_r} \cap (Z_{i_{r+1}} \cup \cdots \cup Z_{i_n})^c) \\ &= Z_{i_1} \cap \cdots \cap Z_{i_r} \cap Z_{i_{r+1}}^c \cap \cdots \cap Z_{i_n}^c, \end{aligned}$$

where $\{i_1, \dots, i_m\} = \{1, \dots, m\}$. We saw above that Z_j^c is a disjont union of locally closed subsets. Consequently we have that all the above sets are locally closed and that (3) holds for Z.

(1.2.8) Lemma. *Given a noetherian topological space X. A subset Z of X is constructible if and only if, for every closed irreducible subset Y of X such that $Z \cap Y$ is dense in Y, we have that $Z \cap Y$ contains an open non empty subset of Y.*

Proof. Assume that Z is constructible and write $Z = \cup_{i=1}^m (V_i \cap Z_i)$, where V_i is open in X and Z_i is closed in X. Let Y be a closed irreducible subset of X such that $Z \cap Y$ is dense in Y. We have that $Z \cap Y = \cup_{i=1}^m ((V_i \cap Y) \cap (Z_i \cap Y))$. We obtain that $Y = \overline{(Z \cap Y)} \subseteq \cup_{i=1}^m (Z_i \cap Y)$. However Y is irreducible so that $Y \subseteq Z_i \cap Y$ for some i. It follows that $Y = Z_i \cap Y$. Then we have that $V_i \cap Y = (V_i \cap Y) \cap (Z_i \cap Y) \subseteq Z \cap Y$.

Conversely, assume that $Z \cap Y$ contains an open non empty subset of Y for all closed irreducible subsets of X such that $Z \cap Y$ is dense in Y. Consider the family $\mathcal{F}$ consisting of closed subsets Y of X such that $Z \cap Y$ is not constructible. If this family is non empty it has a minimal element. We replace X by this set, and can assume that $Z \cap Y$ is constructible for all proper closed subsets Y of X.

If we have that $X = X_1 \cup X_2$, where X_1 and X_2 are proper closed subsets of X, then we have that $Z \cap X_1$ and $Z \cap X_2$ are constructible and consequently we have that $Z = (Z \cap X_1) \cup (Z \cap X_2)$ is constructible.

On the other hand, if X is irreducible and the closure $\overline{Z}$ is properly contained in X we have that $Z = Z \cap \overline{Z}$ is constructible. Finally, if X is irreducible and $X = \overline{Z}$ then, by assumption, Z contains an open non empty subset U of X. Then $Y = X \setminus U$ is a proper closed subset of X and we have that $Z = U \cap (Y \cap Z)$ is constructible.

In all cases we have that $Z = Z \cap X$ is constructible. Consequently we have that $\mathcal{F}$ is empty and we have proved the Lemma.

(1.2.9) Proposition. *Given a noetherian topological space such that every closed irreducible subset has a generic point. Let Z be a constructible subset of X and let*

x be a point of Z. Then Z is a neighbourhood of x if and only if every generization of x is in Z.

Proof. It is clear that if Z is a neighbourhood of x, then every generization of x is in Z.

Conversely, assume that every generization of x is in Z. Let $\mathcal{F}$ be the family of closed subsets Y of X that contain x and are such that $Y \cap Z$ is not a neighbourhood of x in Y. If $\mathcal{F}$ is not empty it contains a minimal element. We can replace this subset with X and assume that $Z \cap Y$ is a neighbourhood of x for all proper closed subsets Y of X.

Assume that $X = X_1 \cup X_2$, where X_1 and X_2 are proper closed subsets. If $x \in X_i$ there is an open subset U_i of X_i such that $x \in U_i \subseteq Z \cap X_i$. On the other hand, if $x \notin X_i$ then we let $U_i = \emptyset$. Let $Y_i = X_i \setminus U_i$ and let $Y = Y_1 \cup Y_2$. Then Y is closed in X and $U = X \setminus Y$ is a neighbourhood of x. Moreover we have that $U \subseteq U_1 \cup U_2 \subseteq Z$, and consequently we have that Z is a neighbourhood of x.

If X is irreducible with generic point x', then x' is in Z. Consequently we have that X is the closure of Z, and it follows from Lemma (1.2.8) that there is an open subset U of X contained in Z. If x is in U we have that Z is a neighbourhood of x. If not we have that $Y = X \setminus U$ is a proper closed subset of X that contains x. Consequently $Z \cap Y$ is a neighbourhood of x in Y. Let F be the closure of $X \setminus Z$ in X. Then we have that F is the closure of $X \setminus Z$ in $X \setminus U = Y$. Consequently we have that x is not in F. Consequently we have that $X \setminus F$ is a neighbourhood of x which is contained in Z.

In both the cases we have that Z is a neighbourhood of x in X. This contradicts the assumption on X. Consequently the family $\mathcal{F}$ is empty and we have proved the Proposition.

(1.2.10) Proposition. *Given a noetherian topological space where all closed irreducible subsets have a generic point. A subset U is open if and only if, for every point x in U, every generization of x is contained in U, and $U \cap \overline{\{x\}}$ is a neighbourhood of x in $\overline{\{x\}}$.*

Proof. It is clear that every open subset satisfies the conditions of the Proposition.

Conversely, assume that the conditions are satisfied. It follows from Lemma (1.2.8) that U is constructible and from Proposition (1.2.9) that U is open.

(1.2.11) Lemma. *Given an integral domain B and a subring A of B such that B is a finitely generated A-algebra. Then there is a non zero element a in A such that* $\operatorname{Spec} A_a$ *is contained in the image of the map* $\operatorname{Spec} B \to \operatorname{Spec} A$.

Proof. Write $B = A[x_1, \dots, x_n]$. Denote by K and L the quotient field of A respectively B. If necessary, renumber the generators $x_1, \dots, x_n$ such that $x_1, \dots, x_r$

is a transcendence basis for L over K. We have relations

$$b_{i0}x_i^{n_0} + \cdots b_{in_0} = 0 \quad \text{for } i = r+1, \dots, n$$

with the b_{kj} in $A[x_1, \dots, x_r]$. Let a in $A[x_1, \dots, x_r]$ be one of the non zero coefficients of the polynomial $b = \prod_{i=r+1}^n b_{i0}$.

Let P be a prime ideal of A such that $a \notin P$. Let $Q = P[x_1, \dots, x_r]$. Then Q is a prime ideal in $A[x_1, \dots, x_r]$ and we have that $b \notin Q$ and consequently we have that B_Q is integral over $A[x_1, \dots, x_r]_Q$. We obtain that there is a prime ideal R in B_Q which contracts to $QA[x_1, \dots, x_r]_Q$. Then the contraction of R to $A[x_1, \dots, x_r]$ is Q and the contraction of R to A is consequently A. Hence the contraction of R ot B contracts to P in A. Consequently P is in the image of $\operatorname{Spec} B$ and we have proved the Lemma.

(1.2.12) Proposition. (Chevalley) *Given a morphism of finite type $f\colon X \to Y$ of noetherian schemes. For each constructible subset Z of X the subset $f(Z)$ of Y is constructible.*

Proof. Write $Z = \cup_{i=1}^n Z_i$ where each Z_i is locally closed. We give each Z_i the reduced structure. The immersion of Z_i in X is of finite type since X is noetherian. It follows that we can replace X by the disjoint union of the Z_i and consequently can assume that $X = Z$ and that X is reduced.

Given a closed irreducible subset T of Y such that $T \cap f(X)$ is dense in T. It follows from Lemma (1.2.8) that it suffices to show that $T \cap f(X)$ contains an open subset of T. Since $T \cap f(X) = ff^{-1}(T)$ we can replace Y by T and X by $f^{-1}(T)$, both with their reduced structure. Consequently we can assume that X is reduced that Y is integral, and that $f(X)$ is dense in Y.

We shall prove that $f(X)$ contains an open non empty subset of Y, and can assume that X and Y are affine. Write $X = \cup_{i=1}^m X_i$ as a union of irreducible sets X_i. Since Y is irreducible we have that at least one of the $f(X_i)$ is dense in Y. Consequently we can assume that X is affine and integral. Let $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$. Then A and B are integral domains and A is contained in B. Since f is of finite type the same is true for the A-algebra B. Consequently the Proposition follows from Lemma (1.2.11).

1.3. Constructible functions.

(1.3.1) Setup. [2] [EGA 11, III_0, 9.3] Given a noetherian topological space X.

(1.3.2) Definition. A map $h: X \to T$ from a noetherian topological space X to a set T is *constructible* if $h^{-1}(t)$ is a constructible subset of X for all t in T and $h^{-1}(t)$ is empty except for a finite number of elements t of T.

(1.3.3) Proposition. *Given a map $h: X \to T$ from a noetherian toplogical space X to a set T. Then h is constructible if and only if, for every closed irreducible subset Y of X, there is an open non empty subset U of Y, such that h is constant on U.*

Proof. Assume that h is constructible and let Y be a closed irreducible subset. Then we have that $h^{-1}(t)$ is empty for all but a finite number $t_1, \dots, t_n$ of points in T, and Y is the union of the closures of $Y \cap h^{-1}(t_i)$ for $i = 1, \dots, n$. Since Y is irreducible we have that Y is contained in the closure of $Y \cap h^{-1}(t_i)$ for some i. Consequently Y is the closure of the constructible subset $Y \cap h^{-1}(t_i)$. It follows
→ from Lemma (?) that $Y \cap h^{-1}(t_i)$ contains an open non empty subset U of Y. Moreover, we have that h takes the value t_i on U.

In order to show the converse statement we consider the family $\mathcal{F}$ of closed subsets Y of X such that $h|Y$ is not constructible. If $\mathcal{F}$ is non empty it has a miminal element Y.

Assume that $Y = \cup_{i=1}^n Y_i$ is a union of proper closed subsets Y_i. Then $h|Y_i$ is constructible for $i = 1, \dots, n$. Consequently we have that $h|Y$ is constructible, contrary to the assumption that Y is in $\mathcal{F}$.

If Y is irreducible it follows from the assumption of the Proposition that there is an open non empty subset U of Y where h is constant. However, then $h|(Y \setminus U)$ is constructible because Y is minimal in $\mathcal{F}$. Since constructible subsets of $Y \setminus U$ are constructible in Y it follows that $h|Y$ is constructible, contrary to the assumption that Y is in $\mathcal{F}$.

It follows that $\mathcal{F}$ is empty and that the Proposition holds.

(1.3.4) Corollary. *Given a noetherian topological space X where every closed irreducible subset has a generic point. A map $h: X \to T$ into a set T is constructible if $h^{-1}(t)$ is constructible for all t in T.*

Proof. Given an irreducible closed subset Y of X and let y be the generic point of Y. Then $Y \cap h^{-1}h(y)$ is constructible in Y and contains y. In particular we
→ have that $Y \cap h^{-1}h(y)$ is dense in Y. It follows from Lemma (?) that there is an open non empty subset U of Y contained in $Y \cap h^{-1}h(y)$. However $h(t) = h(y)$ for $t \in U$. It follows from the Proposition that h is constructible.
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(1.3.5) Definition. A map $h\colon X \to T$ from a topological space X to an ordered set T is *upper semi continous* if the set $\{x\colon \quad h(x) < t\}$ is open in X for all $t \in T$.

(1.3.6) Proposition. *Given a noetherian topological space X where all the closed irreducible subsets have a generic point. Let $h\colon X \to T$ be a constructible function from X to an ordered set T. Then h is upper semi continous if and only if we have, for every point x of X and every generization x' of x that $h(x') \leq h(x)$.*

Proof. The function h has only a finite number of values since it is constructible. Consequently h is upper semi contiuous if and only if the set

$$Z_x = \{x' \in X\colon \quad h(x') \leq h(x)\}$$

is a neighbourhood of x for all x in X. We have that Z_x is constructible because
→ it is a finite union of constuctible sets. It follows from Proposition (?) that Z_z is a neighbourhood of x if and only if we have, for every irreducible closed subset Y of X that contains x, that the generic point y of Y lies in Z_x. However an irreducible closed subset Y contains x if and only if the generic point of Y is a generization of x. Consequently we have that Z_x is a neighbourhood of x if and only if every generization of x lies in Z_x. Consequently the Proposition follows
→ from Proposition (?).

1.4. The ring of constructible functions.

(1.4.1) Definition. We let X be a noetherian scheme. For every subset Y of X we denote by χ_Y the characteristic function on Y. The set of all constructible functions $h: X \to \mathbf{Z}$ we denote by $C(X)$. Sending an integer to the constant fuction with the ineteger as value denines an injective map $\mathbf{Z} \to C(X)$.

(1.4.2) Remark. We have that $C(X)$ is a $\mathbf{Z}$-algebra. Indeed, let f and g be in $C(X)$. Then fg and $f+g$ only take a finite number of values. Moreover the sets

$$X_t = \{x: (f+g)(x) = t\}$$

and

$$Y_t = \{x: (fg)(x) = t\}$$

are constructible because X_t is the disjoint union of the constructible sets

$$\{x: f(x) = u\} \cap \{x: g(x)t - u\},$$

for a finite number of $u \in \mathbf{Z}$, and Y_t is the disjoint union of the constructible sets

$$\{x: g(x) = u\} \cap \{x: g(x) = t/u\},$$

for a finite number of $u \in \mathbf{Z}$ when $t \neq 0$ and $Y_0 = \{x: f(x) = 0\} \cup \{x: g(x) = 0\}$.

(1.4.3) Remark. For every constructible set $Z \subseteq X$ we have a direct sum decomposition

$$C(X) = C(Z) \oplus C(X \setminus Z)$$

given by

$$f = f\chi_Z + (1 - \chi_Z)f.$$

(1.4.3) Proposition. *As a $\mathbf{Z}$-module $C(X)$ is free. A basis is given by the characteristic functions of the closed irreducible subsets of X.*

Proof. The characteristic functions of constructible closed subsets of X generate the $\mathbf{Z}$-module $C(X)$ because, if h is constructible we have that

$$h = \sum_{t \in \mathbf{Z}} h(t)\chi_{X_t},$$

→ where $X_t = \{x: h(x) = t\}$. It follows from Proposition (?) that the constructible sets are disjoint unions of locally closed sets. Consequently we have that $C(X)$ is generated as a $\mathbf{Z}$-module, by the characteristic function of locally closed sets. snitt

Given a locally closed set $U \cap Z$, where U is open and Z is closed in X. Then we have that

$$\chi_{U \cap Z} = \chi_Z - \chi_{U^c \cap Z}.$$

Hence the characteristic functions of closed sets generate $C(X)$ as a $\mathbf{Z}$-module. We shall show, by noetherian induction, that che characteristic functions of closed sets can be written as sums, with integer coefficients, of the characteristic functions of closed irreducible sets. Let $\mathcal{F}$ be the family of all non empty closed subsets of X whose characteristic function is not in the group generated by the characteristic functions of closed irreducible sets. If $\mathcal{F}$ is non empty it contains a smallest member Y. Then Y can not be irreducible.

If $Y = Y_1 \cup Y_2$ is a union of two proper closed subsets we have that

$$\chi_Y = \chi_{Y_1} + \chi_{Y_2} - \chi_{Y_1 \cap Y_2}.$$

However the sets Y_1, Y_2 and $Y_1 \cap Y_2$ are all in the family $\mathcal{F}$. It follows that Y is also in the family, contrary to the assumption. Consequently the set $\mathcal{F}$ is empty and we have shown that the characteristic functions of closed irreducible subsets generate $C(X)$.

Finally we have to show that the characteristic functions of closed irreducible subsets are linearly independent over $\mathbf{Z}$. Assume that

$$\sum_{i=1}^{m} n_i \chi_{X_i} = 0$$

where $X_1, \dots, X_m$ are different closed irreducible sets and where $n_1, \dots, n_m$ are non zero integers. Let X_i be a maximal element among the sets $X_1, \dots, X_m$. We can not have that $X_i \subseteq \cup_{j \neq i} X_j$ because then the irreducibility of X_i would imply that $X_i \subseteq X_j$ for some $j \neq i$ which contradicts the maximality of X_i. We can therefore choose an point $x \in X_i \setminus \cup_{j \neq i} X_j$. We obtain that

$$0 = \sum_{i=1}^{m} n_i \chi_{X_i}(x) = n_i,$$

which contradicts the assumption that all the n_i are non zero. Consequently we have no relation of the form $\sum_{i=1}^{m} n_i \chi_{X_i} = 0$, unless all the n_i are zero.

1.5. The mapping cone.

(1.5.1) Setup. The objects that we shall treat will be modules over a fixed ring, or more generally, $\mathcal{O}_X$-modules on a fixed scheme X.

We shall write a complex of modules $(\mathcal{F}, d)$ on the form

$$\cdots \to \mathcal{F}^{-1} \xrightarrow{d^{-1}} \mathcal{F}^0 \xrightarrow{d^0} \mathcal{F}^1 \to \cdots,$$

where $\mathcal{F}^i$ sits in degree i. The *homology module in degree i* of the complex we denote by $H^i(\mathcal{F})$.

(1.5.2) Definition. Given an integer i. The complex $(\mathcal{F}[i], d[i])$ *translated i times* is defined by

$$\mathcal{F}[i]^j = \mathcal{F}^{i+j} \quad \text{and} \quad d[i]^j = (-1)^i d^{i+j}.$$

A map $\varphi\colon \mathcal{F} \to \mathcal{G}$ of complexes is a *quasi isomorphism* if it induces an isomorphism $H^i(\varphi)\colon H^i(\mathcal{F}) \to H^i(\mathcal{G})$ of homology groups for all i.

Given two complexes $(\mathcal{F}, d_{\mathcal{F}})$ and $(\mathcal{G}, d_{\mathcal{G}})$. We denote by $(\mathcal{F} \oplus \mathcal{G}, d_{\mathcal{F}} \oplus d_{\mathcal{G}})$ the complex given by

$$(\mathcal{F} \oplus \mathcal{G})^i = \mathcal{F}^i \oplus \mathcal{G}^i, \quad \text{and} \quad (d_{\mathcal{F}} \oplus d_{\mathcal{G}})^i = \begin{pmatrix} d^i_{\mathcal{F}} & 0 \\ 0 & d_{\mathcal{G}^i} \end{pmatrix}.$$

(1.5.3) Definition. Given a map $\varphi\colon \mathcal{F} \to \mathcal{G}$ of complexes. The *mapping cone* $(\mathcal{K}(\varphi), d(\varphi))$ of φ is the complex defined by

$$\mathcal{K}(\varphi)^i = \mathcal{F}^{i+1} \oplus \mathcal{G}^i, \quad \text{and} \quad d(\varphi)^i = \begin{pmatrix} -d^{i+1} & 0 \\ \varphi^{i+1} & d^i \end{pmatrix}.$$

In other words, if (f, g) are local sections of $\mathcal{F}^{i+1} \oplus \mathcal{G}^i$ then we have that

$$d(\varphi)^i(f, g) = (-d^{i+1}f, \varphi^{i+1}f + d^i g).$$

(1.5.4) Remark. Given an $\mathcal{O}_X$-module $\mathcal{M}$ we associate to $\mathcal{M}$ the complex

$$\cdots \to 0 \to \mathcal{M} \to 0 \to \cdots,$$

where $\mathcal{M}$ sists in degree 0. Given a map of $\mathcal{O}_X$-modules $\mathcal{M} \to \mathcal{N}$. We obtain a map $\varphi\colon \mathcal{F} \to \mathcal{G}$ of the corresponding complexes. Then the mapping cone $\mathcal{K}(\varphi)$ is the complex

$$\cdots \to 0 \to \mathcal{M} \xrightarrow{\varphi} \mathcal{N} \to 0 \to \cdots,$$

where $\mathcal{M}$ sits in degree -1 and $\mathcal{N}$ in degree 0.
snitt

(1.5.5) Remark. Given a complex $\mathcal{F}$ of $\mathcal{O}_X$-modules and let $\varphi\colon 0 \to \mathcal{F}$ be the map from the zero complex. Then $\mathcal{K}(\varphi) = \mathcal{F}$.

(1.5.6) Lemma. *There is an exact sequence of complexes*

$$0 \to \mathcal{G} \xrightarrow{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} \mathcal{K}(\varphi) \xrightarrow{(1\ 0)} \mathcal{F}[1] \to 0.$$

In particular we have a long exact sequence

$$\cdots \to H^i(\mathcal{F}) \xrightarrow{\delta^i} H^i(\mathcal{G}) \to H^i(\mathcal{K}(\varphi)) \to H^{i+1}(\mathcal{F}) \xrightarrow{\delta^{i+1}} \cdots.$$

Proof. It is easy to check that the maps in the short sequence are maps of complexes. For the right map it is important that $d[1]^j = -d^j$. It is clear that we obtain a short exact sequence.

→ **(1.5.6) Proposition.** *In the long exact sequence of Lemma* (1.5.5) *we have that* $\delta^i = H^i(\varphi)$.

In particular, the map φ is a quasi isomorphism if and only if $\mathcal{K}(\varphi)$ is acyclic.

Proof. It suffices to check the Proposition locally. Take an element f in $\mathcal{F}[1]^{i+1} = \mathcal{F}^i$ such that $-d^i f = d[1]^{i-1} f = 0$. The element f is the image of the element $(f, 0)$ in $\mathcal{K}^{i-1}(\varphi) = \mathcal{F}^i \oplus \mathcal{G}^{i-1}$ and the image of $(f, 0)$ by the differential in $\mathcal{K}(\varphi)$ is $(-d^i f, \varphi^i f + d^{i-1} 0) = (0, \varphi^i(f))$. Consequently the image of the class of f by δ^i is equal to the class of $\varphi^i(f)$ in $H^i(\mathcal{G})$. Hence we have proved the first part of the Proposition. The second part follows immediately from the first part.

(1.5.8) Proposition. *Given a commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F}' \arrow[r, "\alpha'"] \arrow[d, "\varphi'"'] & \mathcal{F} \arrow[r, "\alpha''"] \arrow[d, "\varphi"'] & \mathcal{F}'' \arrow[r] \arrow[d, "\varphi''"'] & 0 \\ 0 \arrow[r] & \mathcal{G}' \arrow[r, "\beta'"] & \mathcal{G} \arrow[r, "\beta''"] & \mathcal{G}'' \arrow[r] & 0 \end{tikzcd}$$

of complexes of $\mathcal{O}_X$-modules, where the horizontal sequences are exact. Then we have an exact sequence of complexes

$$0 \to \mathcal{K}(\varphi') \xrightarrow{\begin{pmatrix} \alpha' & 0 \\ 0 & \beta' \end{pmatrix}} \mathcal{K}(\varphi) \xrightarrow{\begin{pmatrix} \alpha'' & 0 \\ 0 & \beta'' \end{pmatrix}} \mathcal{K}(\varphi'') \to 0.$$

Proof. The proof is an easy computation.

(1.5.9) Lemma. *Given maps of complexes $\varphi\colon \mathcal{F} \to \mathcal{G}$ and $\psi\colon \mathcal{G} \to \mathcal{H}$. We obtain a map of complexes*

$$\eta = \begin{pmatrix} \varphi & -1 \\ 0 & \psi \end{pmatrix} \colon \mathcal{F} \oplus \mathcal{G} \to \mathcal{G} \oplus \mathcal{H}.$$

Moreover, we obtain an exact sequence of complexes

$$0 \to \mathcal{K}(\varphi) \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}} \mathcal{K}(\eta) \xrightarrow{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}} \mathcal{K}(\psi) \to 0.$$

Proof. It is clear that η is a map of complexes. Moreover, it is clear that the short sequence is exact in each degree. To check that $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\beta = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ are maps of complexes it suffices to verify from the equalities $A\alpha = \alpha \begin{pmatrix} -d & 0 \\ \varphi & d \end{pmatrix}$ and $\beta A = \begin{pmatrix} -d & 0 \\ \varphi & d \end{pmatrix}$ where $A = \begin{pmatrix} -d & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ \varphi & -1 & 0 & 0 \\ 0 & \psi & 0 & d \end{pmatrix}$.

(1.5.10) Lemma. *Given maps of complexes $\varphi\colon \mathcal{F} \to \mathcal{G}$ and $\psi\colon \mathcal{G} \to \mathcal{H}$ of $\mathcal{O}_X$-modules, and let $\eta = \begin{pmatrix} \varphi & -1 \\ 0 & \psi \end{pmatrix} \colon \mathcal{F} \oplus \mathcal{G} \to \mathcal{G} \oplus \mathcal{H}$ be the resulting maps. Then we obtain an exact sequence*

$$0 \to \mathcal{K}(\mathrm{id}_{\mathcal{G}}) \xrightarrow{\begin{pmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -\psi \end{pmatrix}} \mathcal{K}(\eta) \xrightarrow{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \psi & 1 \end{pmatrix}} \mathcal{K}(\psi\varphi) \to 0.$$

Proof. It is easy to check, locally, that the sequence is exact in each degree. To show that $\alpha = \begin{pmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -\psi \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \psi & 1 \end{pmatrix}$ are maps of complexes we only have to check that $\alpha \begin{pmatrix} -d & 0 \\ 1 & d \end{pmatrix} = A\alpha$ and $\beta A = \begin{pmatrix} -d & 0 \\ \psi\varphi & d \end{pmatrix} \beta$ where $A = \begin{pmatrix} -d & 0 & 0 & 0 \\ 0 & -d & 0 & 0 \\ \varphi & -1 & d & 0 \\ 0 & \psi & 0 & d \end{pmatrix}$.

(1.5.11) Proposition. *Given maps of complexes $\varphi\colon \mathcal{F} \to \mathcal{G}$ and $\psi\colon \mathcal{G} \to \mathcal{H}$ of $\mathcal{O}_X$-modules, and let $\eta = \begin{pmatrix} \varphi & -1 \\ 0 & \psi \end{pmatrix} \colon \mathcal{F} \oplus \mathcal{G} \to \mathcal{G} \oplus \mathcal{H}$ be the resulting maps. Then there is a long exact sequence*

$$\cdots \to H^{i-1}(\mathcal{K}(\psi)) \xrightarrow{\delta^i} H^i(\mathcal{K}(\varphi)) \to H^i(\mathcal{K}(\psi\varphi)) \to H^i(\mathcal{K}(\psi)) \to \cdots.$$

Proof. It follows from Lemma (1.5.8) that we have a long exact sequence

$$\cdots \to H^{i-1}(\mathcal{K}(\psi)) \xrightarrow{\delta^i} H^i(\mathcal{K}(\varphi)) \to H^i(\mathcal{K}(\eta)) \to H^i(\mathcal{K}(\psi)) \to \cdots,$$

From Lemma (1.5.6) it follows that $H^i(\mathcal{K}(\mathrm{id}_{\mathcal{G}})) = 0$. Consequently it follows from Lemma (1.5.8) that $H^i(\mathcal{K}(\eta)) = H^i(\mathcal{K}(\psi\varphi))$, for all i.

1.5. Additive functions on complexes.

(1.5.1) Setup. The objects that we shall treat in this sections are $\mathcal{O}_X$-modules over a fixed scheme X. We give a class $\mathcal{C}'$ of $\mathcal{O}_X$ modules and an additive function

$$\lambda': \mathcal{C}' \to A$$

into an abelian group A. That is, for every exact sequence

$$0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$$

of modules in $\mathcal{C}'$ we have that

$$\lambda'(\mathcal{M}) = \lambda'(\mathcal{M}') + \lambda'(\mathcal{M}'').$$

We denote by $\mathcal{C}$ the class consisting of all complexes $(\mathcal{F}, d)$, where the cohomology modules $H^i(\mathcal{F})$ are all in $\mathcal{C}'$ and where all except a finite number of the $H^i(\mathcal{F})$ are zero. The family $\mathcal{C}'$ will, as usual, be considered as a subfamily of $\mathcal{C}$, by identification of a module $\mathcal{M}$ with the complex $\cdots \to 0 \to \mathcal{M} \to 0 \to \cdots$, where $\mathcal{M}$ sits in degree zero. We can extend λ' to a function

$$\lambda: \mathcal{C} \to A,$$

by

$$\lambda(\mathcal{F}) = \sum_{i=0}^{\infty} (-1)^i \lambda(H^i(\mathcal{F})).$$

It follows from the cohomology sequence associated to a short exact sequence of members of $\mathcal{C}$ that λ is *additive* . That is, given a short exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to CalF'' \to 0$$

of sequences in $\mathcal{C}$, then we have that

$$\lambda(\mathcal{F}) = \lambda(\mathcal{F}') + \lambda(\mathcal{F}'').$$

More generally, given a homomorphism $\varphi: \mathcal{F} \to \mathcal{G}$ of complexes such that the mapping cone $\mathcal{K}(\varphi)$ is in $\mathcal{C}$, we write

$$\lambda(\varphi) = \lambda(\mathcal{K}(\varphi)).$$

(1.5.2) Remark. Given a homomorphism $\varphi: \mathcal{M} \to \mathcal{N}$ of $\mathcal{O}_X$-modules. We
$\to$ consider the modules as complexes. It follows from Remark (?) that the mapping cone $\mathcal{K}(\varphi)$ of the resulting map of complexes is the complex $\cdots \to 0 \to \mathcal{M} \xrightarrow{\varphi} \mathcal{N} \to 0 \to \cdots$, with $\mathcal{N}$ in degree -1 and $\mathcal{M}$ in degree 0. Consequently we have that $\mathcal{K}(\varphi)$ is in $\mathcal{C}$ if and only if $\ker\varphi$ and $\operatorname{coker}\varphi$ are in $\mathcal{C}$. When this is true we have that

$$\lambda(\varphi) = \lambda(\operatorname{coker}\varphi) - \lambda(\ker\varphi).$$

In this case $\lambda(\varphi)$ is often called the *Herbrand quotient* of φ.
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(1.5.3) Remark. Given a complex $\mathcal{F}$ and let $\varphi\colon 0 \to \mathcal{F}$ be the map from the null complex. The we have that $\mathcal{K}(\varphi) = \mathcal{F}$, and if $\mathcal{F}$ is in $\mathcal{C}$ we have that $\lambda(\varphi) = \lambda(\mathcal{F})$.

(1.5.4) Proposition. *Given a map $\varphi\colon \mathcal{F} \to \mathcal{G}$ of complexes that are in $\mathcal{C}$ and such that $\mathcal{K}(\varphi)$ is in $\mathcal{C}$. Then we have that*

$$\lambda(\varphi) = \lambda(\mathcal{G}) - \lambda(\mathcal{F}).$$

Proof. The Proposition immediately follows from the exact sequence of Lemma (?).

(1.5.5) Proposition. *Given a commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}' \arrow[r] \arrow[d, "\varphi'"'] & \mathcal{F} \arrow[r] \arrow[d, "\varphi"'] & \mathcal{F}'' \arrow[r] \arrow[d, "\varphi''"'] & 0 \\
0 \arrow[r] & \mathcal{G}' \arrow[r] & \mathcal{G} \arrow[r] & \mathcal{G}' \arrow[r] & 0
\end{tikzcd}$$

of complexes, where the horizontal sequences are exact. If the sequences $\mathcal{K}(\varphi')$, $\mathcal{K}(\varphi)$ and $\mathcal{K}(\varphi'')$ are all in $\mathcal{C}$, we have that

$$\lambda(\varphi) = \lambda(\varphi') + \lambda(\varphi'').$$

Proof. The Proposition is an immediate consequence of Proposition (?).

(1.5.6) Proposition. *Given maps $\varphi\colon \mathcal{F} \to \mathcal{G}$ and $\psi\colon \mathcal{G} \to \mathcal{H}$ of complexes such that $\mathcal{K}(\varphi)$, $\mathcal{K}(\psi)$ and $\mathcal{K}(\psi\varphi)$ are in $\mathcal{C}$. Then we have that*

$$\lambda(\psi\varphi) = \lambda(\varphi) + \lambda(\psi).$$

Proof. The Proposition is an immediate consequence of Proposition (?).

2.1. The dimension of algebras.

(2.1.1) Setup. The *Krull dimension* , denoted $\dim A$, of a ring is the combinatorial dimension of $\operatorname{Spec} A$ [33] [A-M Ch. 8]. The *height* , denoted $\operatorname{ht} A$, of a prime ideal P of A is the dimension $\dim A_P$ of the localization of A in P.

A chain $P_0 \subset P_1 \subset \cdots \subset P_n$ of prime ideals in A are *saturated* if the corresponding chain of irreducible sets in $\operatorname{Spec} A$ is saturated.

(2.1.2) Remark. For all prime ideals P in A we have that $\operatorname{ht} P = \dim A_P = \operatorname{codim}(\operatorname{Spec} A/P, \operatorname{Spec} A)$. When A is noetherian we have that $\operatorname{ht} P = \dim A_P$ is finite [33] [A-M 11.14].

(2.1.3) Definition. A ring A is *catenary* if $\operatorname{Spec} A$ is catenary, and A is *universally catenary* if every finitely generated A-algebra is catenary.

(2.1.4) Remark. If A is catenary, every residue is catenary. Hence A is universally catenary if and only if polynomial rings $A[x_1, \dots, x_n]$ over A are catenary. Moreover, if A is catenary then the quotient of A in any multiplicatively closed system is catenary.

→ It follows from Proposition (top, 1.4) that a noetherian domain A is catenary if and only if, for every pair of prime ideals $P \subset Q$ we have that

$$\operatorname{ht} Q = \operatorname{ht} P + \operatorname{ht} Q/P$$

or, for every pair of primes $P \subset Q$ in A such that $\operatorname{ht} Q/P = 1$ we have that

$$\operatorname{ht} Q = \operatorname{ht} P + 1.$$

(2.1.5) Theorem. (Noethers normalization lemma). *Given an algebra A of finite type over a field k and let*

$$I_1 \subseteq I_2 \subseteq \cdots \subseteq I_p$$

be a chain of ideals in A with $I_p \neq A$. Then there are algebraically independent elements $x_1, \dots, x_n$ of A such that:

(1) *The ring A is integral over $k[x_1, \dots, x_n]$.* [33] [A-M Ch. 5 p. 60].
(2) *There are integers $h(1) \leq h(2) \leq \cdots \leq h(p)$ such that*

$$I_i \cap k[x_1, \dots, x_n] = (x_1, \dots, x_{h(i)}), \quad \textit{for } i = 1, \dots, p.$$

Proof. It suffices to prove the Theorem when A is a quotient of a ring of polynomials $k[y_1, \dots, y_m]$ over k. Indeed, writing $A = k[y_1, \dots, y_m]/I_0'$, the inverse image of the chain of ideals of the theorem by the residue map gives a chain

$$I_0' \subseteq I_1' \subseteq \cdots \subseteq I_p'$$

snitt

of ideals in the polynomial ring $k[y_1, \dots, y_m]$. If the Theorem holds for the polynomial ring $k[y_1, \dots, y_m]$ there are algebraically independent elements $x'_1, \dots, x'_{n'}$ and a sequence of integers $h'(0), \dots, h'(p)$ that satisfy (1) and (2) for $k[y_1, \dots, y_m]$ and the chain $I'_0 \subseteq \cdots \subseteq I'_p$ of ideals. We have that $I'_0 \cap k[x'_1, \dots, x'_{n'}] = (x'_1, \dots, x'_{h'(0)})$, and consequently the map $k[y_1, \dots, y_m] \to A$ induces an isomorphism between $k[x'_{h'(o)+1}, \dots, x'_{n'}]$ and $k[x_1, \dots, x_n]$, where x_i is the image of $x'_{h'(0)+i}$ and $n = n' = h'(0)$. Hence the Theorem holds for A with $x_1, \dots, x_n$ and $h(i) = h'(i) - h'(0)$ for $i = 1, \dots, p$.

We first assume that $p = 1$ and that I_1 is a principal ideal generated by an element $x_1 \notin k$. Then we have that

$$x_1 = p(y_1, \dots, y_m) = \sum_{(i_1, \dots, i_m)} a_{(i_1, \dots, i_m)} y_1^{i_1} \cdots y_m^{i_m},$$

for some elements $a_{(i_1, \dots, i_m)} \in k$. Choose positive integers $r_2, \dots, r_m$ and let $x_i = y_i - y_1^{r_i}$, for $i = 2, \dots, m$. We have that y_1 satisfies the polynomial equation

$$\sum_{(i_1, \dots, i_m)} a_{(i_1, \dots, i_m)} y_1^{i_1} (x_2 + y_1^{r_2})_2^{i} \cdots (x_m + y_1^{r_m})^{i_m} - x_1 = 0$$

with coefficients in $k[x_1, \dots, x_m]$. Let $f(i_1, \dots, i_m) = i_1 + r_2 i_2 + \cdots + r_m i_m$. We see that, choosing $r_i = l^i$ where l is greater than the total degree of p, we can obtain that the degree $f(i_1, \dots, i_m)$ of the highest power of y_1 in each of the summands above, are all different. With such a choise of the r_i we have that y_1 is integral over $k[x_1, \dots, x_m]$. It follows from the transitivity of integral dependence [33] [A-M 5.4] that $y_1, \dots, y_m$ are all integral over $k[x_1, \dots, x_m]$. Consequently we have that A in integral over $k[x_1, \dots, x_m]$ [33] [A-M 5.3].

We have that $I_1 \cap k[x_1, \dots, x_m] = x_1 k[x_1, \dots, x_m]$ because every element in $I_1 \cap k[x_1, \dots, x_m]$ can be written as $a = x_1 a'$ with a in $k[x_1, \dots, x_m]$. Consequently we have that a' is in $A \cap k(x_1, \dots, x_m)$. However, we have that $k[x_1, \dots, x_m]$
$\to$ is algebraically closed in $k(x_1, \dots, x_m)$ ([AM?]), and consequently that a' is in $k[x_1, \dots, x_m]$. Hence we have that $a \in x_1 k[x_1, \dots, x_m]$. We have finished the case $p = 1$ and I_1 principal.

Next assume that $p = 1$ and I_1 arbitrary. We prove this case by induction on m. If $m = 1$ the Theorem is trivially true. We can assume that $I_1 \neq 0$. Let $x_1 \in I_1$ be a non zero element. Then $x_1 \notin k$. By the case when I'_1 is principal we can find algebraically independent elements $x_1, t_2, \dots, t_m$ in A such that A is integral over $k[x_1, t_2, \dots, t_m]$ and $x_1 A \cap k[x_1, t_2, \dots, t_m] = x_1 k[x_1, t_2, \dots, t_m]$. It follows from the induction assumption that there are algebraically independent elements $x_2, \dots, x_m$ in $k[t_2, \dots, t_m]$ and an integer $h'(1)$ such that the Theorem holds for the ring $k[t_2, \ldots, t_m]$ and the ideal $I_1 \cap k[t_2, \ldots, t_m]$. We have that $x_1, x_2, \ldots, x_m$ are algebraically independent, and, since $x_1 \in I_1$, we have that

$$I_1 \cap k[x_1, \ldots, x_m] = x_1 k[x_1, \ldots, x_m] + I_1 \cap k[x_2, \ldots, x_m].$$

It follows that the elements $x_1, \ldots, x_m$ and the integer $h(1) = h'(1) + 1$ have the required properties with respect to $k[x_1, \ldots, x_m]$ and the ideal I_1. Hence we have proved the Theorem for $p = 1$.

To prove the Theorem for arbitrary p we use induction on p. Assume that the Theorem holds for $p-1$. Then there are elements $t_1, \ldots, t_m$ and integers $h(1) \leq \cdots \leq h(p-1)$ that satisfy the Theorem for the chain $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_{p-1}$. Let $r = h(p-1)$. It follows from the case $p = 1$ applied to the algebra $k[t_{r+1}, \ldots, t_m]$ and the ideal $I_p \cap k[t_{r+1}, \ldots, t_m]$ that there are elements $x_{r+1}, \ldots, x_m$ in $k[t_{r+1}, \ldots, t_m]$ and an integer $h'(1)$ that satisfy the Theorem for the ring $k[t_{r+1}, \cdots, t_m]$ and the ideal $I_p \cap k[t_{r+1}, \ldots, t_m]$. Let $x_i = t_i$ for $i = 1, \ldots, r$. Then the elements $x_1, \ldots, x_m$ and the integers $h(1) \leq \cdots \leq h(p-1) \leq h(p) = r + h'(1)$ satisfy the assertions of the Theorem because the elements $x_1, \ldots, x_r$ are in I_p, and thus

$$I_p \cap k[x_1, \ldots, x_m] = (x_1, \ldots, x_r) k[x_1, \ldots, x_m] + I_p \cap k[x_{r+1}, \ldots, x_m].$$

□

(2.1.6) Proposition. *Given an integral domain A which is a finitely generated algebra over the field k. Then the following three assertions hold:*

(1) *For every saturated chain of prime ideals $P_0 \subset P_1 \subset \cdots \subset P_r$ in A we have that $r = \operatorname{td.deg.}_k A$.*
(2) $\dim A = \operatorname{td.deg.}_k A$.
(3) *For every prime ideal P in A we have that*

$$\dim A = \dim A_P + \dim A/P.$$

Proof. We have that (1) implies (2). Moreover, from (1) applied to chains that contain P we see that (1) implies (3),.

In order to prove (1) we take algebraically independent elements $x_1, \ldots, x_n$ in A such that A is integral over $k[x_1, \ldots, x_n]$ and such that $P_i \cap k[x_1, \ldots, x_n] = (x_1, \ldots, x_{h(i)})$ for some integers $0 \leq h(0) \leq h(1) \leq \cdots \leq h(r) \leq n$. Then $n = \operatorname{td.deg.}_k A$ and we must have that $r \leq n$ and $h(0) = 0$. Since the chain $P_0 \subset P_1 \subset \cdots \subset P_r$ is saturated we must have that $r = n$. Indeed, if $h(i+1) > h(i) + 1$ we can extend the ideal $(x_1, \ldots, x_{h(i)+1})$ to a prime ideal P such that $P_i \subset P \subset P_{i+1}$, using going down [33] [A-M, 5.16] on the algebra A/P_i and the subalgebra $k[x_1, \ldots, x_n]/k[x_1, \ldots, x_n] \cap P_i = k[x_{h(i)+1}, \ldots, x_n]$. □

(2.1.7) Corollary. *The field k is universally catenary.*

Proof. It suffices to show that the polynomial ring $A = k[x_1, \ldots, x_n]$ is catenary. Let $P \subset Q$ be prime ideals in A. It follows from assertion (3) of the Proposition that

$$\operatorname{ht} P = n - \dim A/P$$

and

$$\operatorname{ht} Q = n - \dim A/Q.$$

Using the same assertion on A/P we obtain that

$$\operatorname{ht} Q/P + \dim A/Q = \dim A/P.$$

From the above three equations we obtain that

$$\operatorname{ht} Q/P = \operatorname{ht} Q - \operatorname{ht} P.$$

Using the last equation for the chain $P \subset Q \subset R$ of prime ideals we obtain the equation

$$\operatorname{ht} R/P = \operatorname{ht} R/Q + \operatorname{ht} Q/P.$$

→ We have seen in Remark (2.1.4) that the last equation implies that A is catenary. ∎

2.2. Associated ideals.

(2.2.1) Setup. Most of the material from [44] [L X.2, X.3]

(2.2.2) Definition. Given a ring A and an A-module M. The *annihilator* , denoted $\operatorname{ann} x$, of an element x of M is the ideal $\{a \in A\colon ax = 0\}$ in A. The ideal $\cap_{x\in M} \operatorname{ann} x$ is denoted by $\operatorname{ann} M$ and called the *annihilator* of M.

A prime ideal P of A is *associated* to M if it is the annihilator of a non zero element of M. We denote the set of associated primes of M by $\operatorname{ass} M = \operatorname{ass}_A M$.

An element a in A i called *locally nilpotent* if there, for every element x in M, is a non negative integer n_x such that $a^{n_x} x = 0$.

The prime ideals P such that $M_P \neq 0$ we call the *support* of M. We denote the support of M by $\operatorname{supp} M$.

(2.2.3) Lemma. *Given A-module M, and let x be an element in M. For each prime ideal P in A we have that $(Ax)_P \neq 0$ if and only if $\operatorname{ann} x \subseteq P$.*

Proof. Assume that $(Ax)_P \neq 0$. Then $tx \neq 0$ for all $t \notin P$. Consequently we have that $\operatorname{ann} x \subseteq P$.

Conversely, assume that $(Ax)_P = 0$. Then there is a $t \notin P$ such that $tx = 0$. Consequently we have that $\operatorname{ann} x \notin P$.

(2.2.4) Proposition. *Given an A-module M and an element a in A. The following assertions are equivalent:*

(1) *The element a is locally nilpotent.*
(2) *The element a is contained in every prime ideal P such that $M_P \neq 0$.*

Proof. Assume that a is locally nilpotent and let P be a prime ideal such that $M_P \neq 0$. Then there is an element $x \in M$ such that $tx \neq 0$ for all $t \notin P$, and an n such that $a^n x = 0$. It follows that $a^n \in P$ and thus $a \in P$

Conversely, assume that a is not locally nilpotent. Then there is an element $x \in M$ such that $a^n x \neq 0$ for all $n \geq 0$. Choose an ideal P which is maximal among the ideals that contain $\operatorname{ann} x$ and are disjoint from $\{1, a, a^2, \dots\}$. Then P is a prime ideal and $(Ax)_P \neq 0$. Consequently we have that $M_P \neq 0$.

(2.2.5) Proposition. *Given a non zero A-module M, and let P be an ideal that is maximal among the annihilators $\operatorname{ann} x$ of elements x in M. Then P is a prime ideal.*

Proof. Let $P = \operatorname{ann} x$, and let $ab \in P$ with $a \notin P$. Then $ax \neq 0$ and $\operatorname{ann}(ax) \supseteq bA + P$. It follows from the maximality of P that $b \in P$.
snitt

(2.2.6) Corollary. *Given a noetherian ring A and an A-module M. Then M has an associated ideal.*

When M is finitely generated there is a chain

$$M = M_0 \supset M_1 \supset \cdots \supset M_n = 0$$

of submodules such that M_{i-1}/M_i is isomorphic to A/P_i, where P_i is a prime associated to M.

Proof. Since A is noetherian there is a maximal element among the annihilators to elements in M. It follows from the Proposition that such a maximal element is associated to M. Hence we have proved the first part of the Corollary.

To prove the second part we let N be a maximal element among the submodules of M for which the Corollary holds. If $N \neq M$ we have that $M/N \neq 0$. Consequently, there is an associated prime of M/N. Let $P = \operatorname{ann} y$ for some $y \in M/N$, and denote by x an element of M that maps to y. We have that $Ay \cong A/P \subseteq M/N$ and $N + Ax/N \cong A/P$. This contradicts the maximality of N. Consequently we must have that $N = M$. $\square$

(2.2.7) Proposition. *Given a noetherian ring A and an A-module M. Let a be an element in A. Then $ax = 0$ for some non zero element x in M, if and only if a lies in a prime which is associated to M.*

Proof. It is clear that if a is contained in some associated ideal then $ax = 0$, for some x in M.

Conversely, assume that $ax = 0$ for some $x \neq 0$. It follows from Prop (2.2.5) that Ax has an associated prime ideal P. It is clear that $a \in P$. However, P is also associated to M. $\square$

(2.2.8) Proposition. *Given a noetherian ring and an A-module M. Let a be an element in A. The following assertions are equivalent:*

(1) *The element a is locally nilpotent.*
(2) *The element a is contained in all associated primes of M.*
(3) *The element a lies in all the prime ideals P such that $M_P \neq 0$.*

If P is a prime ideal such that $M_P \neq 0$ then P contains a prime which is associated to M.

Conversely, if P is prime ideal which contains an associated prime of M, then $M_P \neq 0$.

Proof. We have already seen in Proposition (2.2.4) that (1) and (3) are equivalent. Moreover it is clear that (1) implies (2).

To show that (2) implies (3) it suffices to prove that if $M_P \neq 0$ for some prime ideal P, then P contains an associated prime. When $M_P \neq 0$ we have an element

$x \in M$ such that $(Ax)_P \neq 0$. Consequently there is an element $y/t \neq 0$ in $(Ax)_P$, with $y \in Ax$ and $t \notin P$, with annihilator a prime ideal Q. Then we have that $Q \subseteq P$ because, otherwise there would be a $b \in Q \setminus P$ such that $by/t = 0$ in $(Ax)_P$, which contradicts that $y/t \neq 0$. Let $b_1, \dots, b_n$ be generators for Q. Then there are elements $s_1, \dots, s_n$ in $A \setminus P$ such that $s_i b_i y = 0$. We have that A is the annihilator of the element $s_1 \cdots s_n y$ because, it is contained in the annihilator of this element, and, if $as_1 \cdots s_n y = 0$ we have that $ay = 0$ in $(Ax)_P$, and consequently $a \in Q$. ▯

(2.2.9) Corollary. *Given a noetherian ring A and an A-module M. The following assertions are equivalent:*

(1) *There is exactly one associated prime ideal for M.*
(2) *We have that $M \neq 0$ and for every element a in A we have that a is locally nilpotent or $ax = 0$ holds only for the element $x = 0$.*

When the assertions hold the associated ideal consists of the locally nilpotent elements.

→ *Proof.* It follows from Proposition (2.2.7) that $ax = 0$ only for $x = 0$ if and only if a is not contained in an associated prime.

If there is only one associated prime ideal it follows from the Proposition that all the elements in the associated prime are locally nilpotent.

Conversely, if (2) holds, the locally nilpotent elements will be those that are contained in some associated prime ideal. Hence it follows from the Proposition that the union of the associated prime ideals will be equal to their intersection. It follows that (1) holds. ▯

→ **(2.2.10) Remark.** It follows from Proposition (2.2.8) that when A is noetherian and M is finitely generated we have that

$$\operatorname{rad}(\operatorname{ann} M) = \bigcap_{P \in \operatorname{supp} M} P = \bigcap_{P \text{associated}} P.$$

Moreover, it follows that

$$\operatorname{supp} M = \{P \in \operatorname{Spec} A \colon \operatorname{ann} M \subseteq M\}.$$

Indeed, we just say that each prime belonging to $\operatorname{supp} M$ contains $\operatorname{ann} M$. Conversely, if $M_P = 0$ we choose generators $m_1, \dots m_n$ of M and obtain $t_1, \dots, t_n$ such that $t_i m_i = 0$, for $i = 1, \dots, n$. Then $t_1 \cdots t_n$ is contained in $\operatorname{ann} M$, and consequently $\operatorname{ann} M \not\subseteq P$.

(2.2.11) Proposition. *Given an A-module M. Let N be a submodule of M. Every prime associated to N is associated to M. The associated primes of M are associated to N or to M/N.*

Proof. The first assertion is clear.

Let P be associated to M. Then $P = \operatorname{ann} x$ for some $x \in M$. If $Ax \cap N = 0$ we have that P is associated to M/N. If $Ax \cap N \neq 0$ we choose an non zero element $y = ax$ with $a \in A$. Then $P = \operatorname{ann} y$ because $P \subseteq \operatorname{ann} y$, and if $by = 0$ for some $b \in A$ we have that $ba \in P$, and since $a \notin P$ we have that $b \in P$. $\square$

(2.2.12) Proposition. *Given a finitely generated module M over a noetherian ring A. The every submodule N can be written as an intersection $N = N_1 \cap \cdots \cap N_n$ of submodules N_i such that each M/N_i only has one associated prime ideal.*

Proof. Consider the set of submodules of M for which the Proposition does not hold. If this set is non empty there is a maximal element N. Then M/N can not have only one associated prime. It follows from Corollary (2.2.9) that there is an element $a \in A$ such that the homomorphism $\varphi \colon M/N \to M/N$ given by $\varphi(x) = ax$ is neither injective nor nilpotent. We therefore obtain a sequence

$$\ker \varphi \subseteq \ker \varphi^2 \subseteq \cdots$$

of proper submodules of M. This sequence must stop. Assume that $\ker \varphi^r = \ker \varphi^{r+1} = \cdots$ and let $\psi = \varphi^r$. We have that $\ker \psi$ and $\operatorname{Im} \psi$ are proper submodules of M and we have that $\ker \psi = \ker \psi^2$, and consequently that $\ker \psi \cap \operatorname{Im} \psi = 0$. Let N_1 and N_2 be the inverse images of $\ker \phi$ respectively $\operatorname{Im} \phi$ in M. Then N_1 and N_2 contain N properly and $N = N_1 \cap N_2$. By the maximality of N we have that the Proposition holds for N_1 and N_2. Consequently the Proposition holds for N, which contradicts the assumption on N. It follows that the set of submodules of M, for which the Proposition does not hold, is empty. $\square$

(2.2.13) Corollary. *Let M be a finitely generated module over a noetherian ring A. Then the associated prime primes of M coincide with the associated prime ideals of M/N_i for any minimal decomposition $0 = N_1 \cap \cdots \cap N_n$ in modules N_i such that M/N_i has only one associated prime ideal.*

Proof. It follows from the Proposition that we can write $0 = N_1 \cap \cdots \cap N_n$ where M/N_i has only one associated prime ideal. We obtain an injection

$$M \to M/N_1 \oplus \cdots \oplus \cdots M/N_n.$$

It follows from Proposition (2.2.11) that the associated ideals of M are among the associated primes of $M/N_1, \ldots, M/N_n$.

Conversely, assume that n is minimal and let $N = N_2 \cap \cdots \cap N_n$. Then $N \neq 0$ because the intersection is assumed to be minimal. Then $N = N/N \cap N_1 \cong N + N_1/N_1 \subseteq M/N_1$. Consequently the module N has only one associated prime, the prime associated to M/N_1. It follows that this prime is also associated to M. $\square$

(2.2.14) Proposition. *Let A be a noetherian ring. If A is reduced the associated primes are the minimal prime ideals.*

Proof. Every prime ideal contains an associated prime so every minimal prime is associated.

Conversely, let $P_1, \cdots, P_n$ be the minimal associated prime ideals. If P is an associated prime ideal and $t \in P \setminus P_1 \cup \cdots \cup P_n$ there is a non zero a in A such that $aP = 0$, and consequently $ta = 0$. However, then $a \in P_1 \cap \cdots \cap P_n$ and hence $a = 0$ because A is an integral domain. This contradicts the assumption that $a \neq 0$ so we must have that $P \subseteq P_1 \cup \cdots \cup P_n$. Hence, [41] [A-M 1.11], we have that P is equal to one of the $P_1, \ldots, P_n$, and we have proved that all associated primes are minimal. $\square$

(2.2.15) Lemma. *Let I be an ideal in a noetherian ring A such that $aI = 0$ implies that $a = 0$. Then I contains a non zero divisor in A.*

Proof. It follows from Proposition (2.2.7) that the zero divisors in A are the union of the associated prime ideals and it follows from Proposition (2.2.13) that there are only a finite number of associated primes. Hence, if I consists entirely of zero divisors it follows from [41] [A-M 1.11] that I is contained in an associated prime. However, then there is a non zero element a in A such that $aI = 0$.

2.3. Length.

(2.3.1) Setup.

→ **(2.3.2) Definition.** ([A-M p. 76]) Given a ring A and an A-module M. A sequence

$$0 = M_n \subset M_{n-1} \subset \cdots \subset M_0 = M$$

of modules is called a *chain of length* n . The chain is a *composition series* for M it there is no submodule N of M such that $M_{i+1} \subset N \subset M_i$ for some i.

→ **(2.3.3) Proposition.** ([A-M, 6.7]) *Given an A-module M that has a composition series of length n. Then every composition series has length n and every chain can be extended to a composition series.*

Proof. Let $M = M_0 \supset M_1 \supset M_1 \supset \cdots \supset M_n = 0$ be a composition series for M of shortest possible length. Given a submodule N of M. We obtain a chain $N = N_0 \supseteq N_1 \supseteq \cdots \supseteq N_n = 0$, where $N_i = M_i \cap N$. We have that $N_{i-1}/N_i \subseteq M_{i-1}/M_i$, and consequently that either $N_{i-1}/N_i = M_{i-1}/M_i$ or N_{i-1}/N_i. Hence we obtain a composition series for N. This series has length n if and only if $N_{i-1}/N_i = M_{i-1}/M_i$, for all i. By induction on n we see that N has length n if and only if $M = N$. Thus a proper submodule of M has length strictly less than n.

Denote by $\ell'(N)$ the length of a shortest composition series for N. We have that $\ell'(M) = n$, by assumption. Given a chain $M = M'_0 \supset M'_1 \supset \cdots \supset M'_h = 0$ of lenght h in M. We then have that $n = \ell'(M) > \ell'(M') > \cdots > \ell'(M'_h) = 0$, and thus that $h \leq n$. In other words, every chain has length at most n. When the chain is not a composition series we have that at least one quotient M'_{i-1}/M'_i has a proper submodule. Consequently we can find a stricly longer chain by inserting a module $M'_{i-1} \supset N \supset M'_i$. Consequently we either have $h = n$, and then $M'_0 \supset M'_1 \supset \cdots \supset M'_n$ is a composition series, or $h < n$ and the chain can be refined to a chain of length $h + 1$. We can continue to refine the chain until we obtain a chain of length n, which is then a composition series.

→ **(2.3.3) Definition.** ([A-M, p. 77])We say that an A-module M has *finite length* if it has a composition series. The common length $\ell(M) = \ell_A(M)$ of the composition series is called the *length* of the module.

→ **(2.3.4) Lemma.** ([A-M, 6.9])*Let*

$$0 \to M' \to M \to M'' \to 0$$

be a short exact sequence of A-modules. Then M has finite length, if and only if M' and M'' have finite length. When the lengths are finite we have that

$$\ell(M) = \ell(M') + \ell(M'').$$

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Proof. Given a composition series for M. The intersections of the modules in the chain with M' and the images in M'' of the modules give chains that clearly give rise to composition series in M' and M''.

Conversely, a composition series in M' together with the pull back to M of a composition series of M'', give a composition series in M.

(2.3.4) Lemma. *Let $A \to B$ be a homomorphism of a ring A into a integral domain B. If B has finite length as an A-module, then B is a field.*

Proof. Let $b \neq 0$ in B. We have a sequence $(b) \supseteq (b^2) \supseteq \cdots$ of ideals in B. This is, in particular, a sequence of A-modules and must stop since B has finite length as an A-module, and consequently $(b^r) = (b^{r+1}) = \ldots$ for some r. Hence we have that $cb^{r+1} = b^r$ for some c in B. Since B is an integral domain we must have that $bc = 1$. Consequently b has an iverse and B is a field. ∎

(2.3.5) Proposition. *Let A be a noetherian ring and M a finitely generated A-module. Given a chain*

$$M = M_0 \supset M_1 \supset \cdots \supset M_r = 0$$

of A-modules such that $M_{i-1}/M_i \cong A/P_i$, for $i = 1, \ldots, r$, where P_i is a prime ideal in A. Then M has finite length if and only if all primes P_i are maximal.

When M has finite length we have that

$$\ell(M) = \sum_{P \in \operatorname{Spec} A} \ell_{A_P}(M_P).$$

Proof. When M is of finite length if follows by induction on i that each of the
→ rings A/P_i have finite length. Thus it follows from Lemma (2.3.4) that the ideals P_i are maximal.

Conversely, if all the P_i are maximal, we have that the rings A/P_i have finite length. By descending induction on i we obtain that M has finite length.

Assume that M has finite length. Then the chain $M_0 \supset \cdots \supset M_n$ is a composition series. We see from the series that the localization of M in a prime P is non zero if and only if $P = P_i$ for some i. It is clear that M_{P_i} has a composition series where all the quotients are of the form A/P_i and that each such quotient appears in the composition series for M_{P_i} as many times as it appears in the composition series for M. We therefore obtain that $\ell_A(M) = \sum_{i=1}^n \ell_A(A/P_i) = \sum_{P \in \operatorname{Spec} A} \ell_{A_P}(M_P)$. ∎

(2.3.6) Corollary. *Given a noetherian ring a finitely generated A-module A and M. Then M has finite length if and only if the support,* $\operatorname{supp} M$, *consists of maximal ideals. The support is then finite.*

(2.3.7) Proposition. *Given a local homomorphism $A \to B$ of local rings A and B with maximal ideals P respectively Q. Denote by $d = [B/Q \colon A/P]$ the degree of the residue field extension. A non zero B-module M has finite length over A if and only if $d < \infty$ and M has finite length over B.*

When M has finite length over A we have that

$$\ell_A(M) = d\ell_B(M).$$

Proof. If M is of finite length over A or B it is a finitely generated B-module. Since the length is additive we can therefore assume that $M = B/Q'$, where Q' is a prime ideal in B. It follows from Lemma (2.3.4) that if M has finite length, either as an A-module, or as a B-module,we have that $Q' = Q$.

If B/Q has finite length over A it is a finitely generated A-module and thus $[B/Q \colon A/P] < \infty$, and B/Q has finite length as a B-module because Q is maximal. We then have that $\ell_A(B/Q) = \ell_{A/P}(B/Q) = d = d\ell_B(B/Q)$, and we have proved the Lemma. ∎

2.4. Herbrand indices.

(2.4.1) Setup. Given a ring A. For every A-module M of finite length we denote the length by $\ell_A(M)$. Given a map $\varphi\colon \mathcal{F} \to \mathcal{G}$ of A-modules such that the cohomology groups $H^i(\mathcal{K}(\varphi))$ of the mapping cone of φ have finite length for all i and are zero except for a finite number of i's we obtain, as i Section (1.6) used on the associated scheme $X = \operatorname{Spec} A$ and $\mathcal{O}_X$-module $\widetilde{M}$, a *length* $\ell_A(\varphi)$. We say that under the given condition the *length is defined* .

(2.4.2) Remark. Given a map $\varphi\colon M \to N$ of A-modules whose kernel and cokernel have finite length. Then it follows from Remark (?) that

$$\ell_A(\varphi) = \ell_A(N) - \ell_A(M),$$

which is the usual *Herbrand quotient* of φ.

(2.4.3) Remark. Assume that A is noetherian and that $\mathcal{F}$ is a complex consisting of finite generated A-modules such that $H^i(\mathcal{F}) = 0$ for all but a finite number of i's. Then all the $H^i(\mathcal{F})$ are of finite length if and only if, for all prime ideals P of A, the localized complex $\mathcal{F}_P$ given by

$$\cdots \to \mathcal{F}_P^{-1} \xrightarrow{d_P{}^{-1}} \mathcal{F}_P^0 \xrightarrow{d_P^0} \mathcal{F}_P 1 \to \cdots$$

is acyclic, except possibly when P is maximal. Then there is only a finite number of maximal ideals such that $\mathcal{F}_P$ is not acyclic.

(2.4.4) Proposition. *Given a map $\varphi\colon \mathcal{F} \to \mathcal{G}$ of complexes of A-modules such that $\ell_A(\varphi)$ is defined. Then the length $\ell_{A_P}(\varphi_P)$ is defined for all prime ideals P of A and we have that*

$$\ell_A(\varphi) = \sum_{P \in \operatorname{Spec} A} \ell_{A_P}(\varphi_P).$$

Proof. By definition we have that $\ell_A(\varphi) = \ell_A(\mathcal{K}(\varphi))$, where $\mathcal{K}(\varphi)$ is the mapping cone of φ. It follows from Lemma (?) that we have

$$\ell_A(H^i(\mathcal{K}(\varphi)) = \sum_{P \in \operatorname{Spec} A} \ell_{A_P}(H^i(\mathcal{K}(\varphi))_P)$$

for all i. However, localization is exact so that $H^i(\mathcal{K}(\varphi))_P = H^i(\mathcal{K}(\varphi_P))$. Consequently we have that

$$\ell_A(\mathcal{K}(\varphi)) = \sum_{P \in \operatorname{Spec} A} \ell_{A_P}((\varphi)_P) = \sum_{P \in \operatorname{Spec} A} \ell_{A_P}(\mathcal{K}(\varphi_P)) = \sum_{P \in \operatorname{Spec} A} \ell_{A_P}(\varphi_P).$$

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(2.4.5) Proposition. *Given a map $\varphi: \mathcal{F} \to \mathcal{G}$ of finite complexes of A-modules of finite length. Then we have that*

$$\ell_A(\varphi) = \ell_A(\mathcal{G}) - \ell_A(\mathcal{F}).$$

Proof. The Proposition is an immediate consequence of Proposition (?).

(2.4.6) Proposition. *Given a commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}' \arrow[r] \arrow[d, "\varphi'"'] & \mathcal{F} \arrow[r] \arrow[d, "\varphi"'] & \mathcal{F}'' \arrow[r] \arrow[d, "\varphi''"'] & 0 \\
0 \arrow[r] & \mathcal{G}' \arrow[r] & \mathcal{G} \arrow[r] & \mathcal{G}' \arrow[r] & 0
\end{tikzcd}$$

of complexes of A-modules. If two of the lengths $\ell_A(\varphi')$, $\ell_A(\varphi)$ and $\ell_A(\varphi'')$ are defined, then the third is, and we have that

$$\ell_A(\varphi) = \ell_A(\varphi') + \ell_A(\varphi'').$$

Proof. That the third length is defined when the two others are is an immediate consequence of the properties of length of modules. The rest of the Proposition follows immediately from Proposition (?).

(2.4.7) Proposition. *Given maps $\varphi: \mathcal{F} \to \mathcal{G}$ and $\psi: \mathcal{G} \to \mathcal{H}$ of complexes of A-modules. It two of the lengths $\ell_A(\varphi)$, $\ell_A(\psi)$ and $\ell_A(\psi\varphi)$ are defined, then the third is, and we have that*

$$\ell_A(\psi\varphi) = \ell_A(\varphi) + \ell_A(\psi).$$

Proof. That the third length is defined when the two others are follows from the properties of the length of modules. The remaining part follows from Proposition (?).

2.5. Noetherian rings.

(2.5.1) Lemma. *(Artin-Rees) Given a noetherian ring A and an ideal I. Let M be a finitely generated A-module and N a submodule. Then there exists a positive integer m such that*

$$I^n M \cap N = I^{n-m}(I^m M \cap N)$$

for all $n \geq m$.

Proof. It is clear that we have an inclusion $I^{n-m}(I^m M \cap N) \subseteq I^n M \cap N$. In order to prove the opposite inclusion we choose generators $a_1, \dots, a_r$ for I and $m_1, \dots, m_s$ for M. Let $A[x_1, \dots, x_r]$ be the ring of polynomials in the variables $x_1, \dots, x_r$ with coefficients in A., and let

$$J_n = \{(f_1, \dots, f_s) | f_i \in A[x_1, \dots, x_r] \text{ homogeneous of degree } n$$
$$\text{and } \sum_{i=1}^{s} f_i(a_1, \dots, a_r) m_i \in N\}.$$

Denote by P the $A[x_1, \dots, x_r]$–submodule of $A[x_1, \dots, x_r]^s$ generated by $\cup_{i=1}^{\infty} J_n$. Since $A[x_1, \dots, x_r]$ is noetherian we can find a finite number og generators

$$(p_{1,1}, \dots, p_{1,s}), \dots, (p_{t,1}, \dots, p_{t,s})$$

for P. We can choose the p_i such that the $p_{i,j}$ have the same degree d_j for $i = 1, \dots, s$. Let m be the maximum of $d_1, \dots, d_t$.

Given an element $l \in I^n M \cap N$. We can write $l = \sum_{i=1}^{s} f_i(a_1, \dots, a_r) m_i$, with $(f_1, \dots, f_s) \in J_n$. consequently we get

$$(f_1, \dots, f_s) = \sum_{j=1}^{t} g_j(x_1, \dots, x_r)(p_{j,1}, \dots, p_{j,s})$$

with $g_j \in A[x_1, \dots, x_r]$. On the left hand side we have homogeneious polynomials of degree n. Consequently, we may, after possibly removing terms on the right hand side, assume that $\deg g_j + d_j = n$ for $j = 1, \dots, t$ and $i = 1, \dots, s$. Then we have that

$$l = \sum_{i=1}^{s} f_i(a_1, \dots, a_r) m_i = \sum_{j=1}^{t} g_j(a_1, \dots, a_r) \sum_{i=1}^{s} p_{i,j}(a_1, \dots, a_r) m_i$$

where $\sum_{i=1}^{s} p_{i,j}(a_1, \dots, a_r) m_i \in I_j^d M \cap N$, since $(p_{j,1}, \dots, p_{j,s}) \in J_{d_i}$. for $n \geq m$ we have that $g_J(a_1, \dots, a_r) \in I^{n-d_j} = I^{n-m} I^{m-d_j}$. Consequently we have that $l \in \sum_{j=1}^{t} I^{n-m} I^{m-d_j}(I^{d_j} M \cap N) \subseteq I^{n-m}(I^m M \cap N)$.
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(2.5.2) Theorem. *(Krull) Given a noetherian ring A and an ideal I in A. Let M be a finitely generated A–module and let $N = \cap_{i=1}^{\infty} I^n M$. Then there is an element $a \in A$ such that $(1 + a)N = 0$.*

In particular, when $I \subseteq \operatorname{rad} A$ we have that $\cap_{i=1}^{\infty} I^n M = 0$.

→ *Proof.* It follows from the Artin–Rees Lemma (2.5.1) that we have an inclusion $I^n M \cap N \subseteq I^{n-m}(I^m M \cap N) \subseteq IN$ for big n. However we have that $N \in I^n M$ for all n. Thus we have that $N \subseteq I^n M \cap N \subseteq IN$, and thus that $IN = N$. It follows from Nakayamas Lemma that there is an element $a \in I$ such that $(1 + a)N = 0$.

When $I \subseteq \operatorname{rad} A$ we have that $1 + a$ is a unit in A and consequently that $n = 0$.

2.6. Flatness.

Definert i [73][A-M, Ch.2 p. 29]. Se [76][L, XVI, §3]

(2.6.1) Setup. Given a ring A and a prime ideal P. We shall write $\kappa(P) = A_P/PA_P$. Moreover, given an A-algebra B and a prime ideal Q in B. We denote by $Q \cap A$ the *contraction* of Q to A, that is $Q \cap A = \varphi^{-1}(Q)$, where $\varphi: A \to B$ is the map defining the algebra structure.

(2.6.2) Definition. Given an A-module M. The module M is *flat* over A if every short exact sequence

$$0 \to N' \to N \to N'' \to 0$$

gives rise to a short exact sequence

$$0 \to M \otimes_A N' \to M \otimes_A N \to M \otimes_A N'' \to 0.$$

(2.6.3) Remark.

(1) (*Long exact sequences*) We can break long exact sequences into short exact sequences. Hence M is flat over A if and only if every exact sequence

$$\cdots \to N' \to N \to N'' \to \cdots$$

of A-modules gives rise to an exact sequence

$$\cdots \to M \otimes_A N' \to N \otimes_A N \to M \otimes_A N'' \to \cdots .$$

→ (2) (*Left exactness*) Since the tensor product is right exact (73) [A-M, 2.18] we have that M is flat over A if every injective map $N' \to N$ of A-modules gives rise to an injective map $M \otimes_A N' \to M \otimes_A N''$.

(3) (*Localization*)Let S be a multiplicatively closed subset of A. It follows from the definition of localization $S^{-1}A$ of A in S that $S^{-1}A$ is a flat A-module.

(4) (*Base change*) Given a flat A-module N, and let B be an A-algebra. Then $B \otimes_A N$ is a flat B-module. Indeed, for every B-module P we have an isomorphism $P \otimes_B (B \otimes_A N) \cong P \otimes_A N$.

(5) (*Direct sums*) For every set $(N_i)_{i\in I}$ of A-modules and every A-module P we have that $F \otimes_A (\oplus_{i\in I} N_i) \cong \oplus_{i\in I}(P \otimes_A N_i)$. We have that $\oplus_{i\in I} N_i$ is exact in $(N_i)_{i\in I}$ if and only if it is exact in every factor N_i. Consequently $\oplus_{i\in I} N_i$ is flat over A if and only if each summand N_i is flat over A. It follows in particular that every free A-module is flat. Moreover, projective A-modules are flat because they are direct summands of free modules.

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(2.6.4) Lemma. *Given an exact sequence*

$$0 \to M \to N \to F \to 0$$

of A-modules, where F is flat. Then the sequence

$$0 \to P \otimes_A M \to P \otimes_A N \to P \otimes_A F \to 0$$

is exact for all A-modules P.

Proof. Write P as a quotient of a free A-module L,

$$0 \to K \to L \to P \to 0.$$

We obtain a commutative diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] \\
 & K \otimes_A M \arrow[r] \arrow[d] & K \otimes_A N \arrow[r] \arrow[d] & K \otimes_A F \arrow[d] \\
0 \arrow[r] & L \otimes_A M \arrow[r] \arrow[d] & L \otimes_A N \arrow[r] \arrow[d] & L \otimes_A F \\
 & P \otimes_A M \arrow[r] \arrow[d] & P \otimes_A N \arrow[d] & \\
 & 0 & 0 &
\end{tikzcd}$$

where the upper right vertical map is injective because F is flat, and the middle left horizontal map is injective because L is free. A diagram chase gives that $P \otimes_A M \to P \otimes_A N$ is injective.

(2.6.5) Proposition. *Given an exact sequence*

$$0 \to F' \to F \to F'' \to 0$$

of A-modules with F'' flat. Then F is flat if and only if F' is flat.

Proof. Given an injective map $M' \to M$. We obtain a commutative diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
0 \arrow[r] & M' \otimes_A F' \arrow[r] \arrow[d] & M' \otimes_A F \arrow[r] \arrow[d] & M' \otimes_A F'' \arrow[r] \arrow[d] & 0. \\
0 \arrow[r] & M \otimes_A F' \arrow[r] & M \otimes_A F \arrow[r] & M \otimes_A F'' \arrow[r] & 0
\end{tikzcd}$$

The rows are exact to the left by Proposition (2.6.4), and we have injectivity of top vertical map since F'' is flat. The Proposition follows from a diagram chase.

(2.6.6) Lemma. *Given an A-module M such that the map*

$$I \otimes_A M \to IM$$

is an isomorphism for all ideals I in A. For every free A-module F and every injective map $K \to F$ of A-modules we have that

$$K \otimes_A M \to F \otimes_A M$$

is injective.

Proof. Since every element in $K \otimes_A M$ is mapped into $F' \otimes_A M$ where F' is a finitely generated free submodule of F we can assume that F is finitely generated.

When the rank of F is 1 the Lemma follows from the assumption. We prove the Lemma by induction on the rank r of F. We have an exact sequence $0 \to F_1 \to F \to A \to 0$, where F_1 is a free rank $r-1$ module. Let $K_1 = K \cap F_1$ and let K_2 be the image of K in A. We obtain a diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & \\
 & K_1 \otimes_A M \arrow[r] \arrow[d] & K \otimes_A M \arrow[r] \arrow[d] & K_2 \otimes_A M \arrow[r] \arrow[d] & 0. \\
0 \arrow[r] & F_1 \otimes_A M \arrow[r] & F \otimes_A M \arrow[r] & A \otimes_A M &
\end{tikzcd}$$

where the right and left vertical maps are injective by the induction assumption and it follows from Lemma (2.6.6) that the lower left map is injective because A is free. A diagram chase proves that the middle vertical map is injective.

(2.6.7) Proposition. *An A-module M is flat if and only if the map*

$$I \otimes_A M \to IM$$

is an isomorphism for all ideals finitely generated ideals I of A.

Proof. If M is flat the tensor product $I \otimes_A M \to M$ of the map $I \to A$ is injective so $I \otimes_A M \to IM$ is an isomorphism.

Conversely, we can assume that $I \otimes_A M \to IM$ is an isomorphism for all ideals I of A. Indeed, every element of $I \otimes_A M$ is contained in $J \otimes_A M$, where J is a finitely generated ideal, and if $J \otimes_A M \to M$ is injective and the element is not zero then it is not mapped to zero by the map $I \otimes_A M \to M$.

Let $N' \to N$ be an injective map and write N as a quotient $0 \to K \to F \to N \to 0$ of a free A-module F. Let F' be the inverse image of N' in F. Then we have an exact sequence $0 \to K \to F' \to N' \to 0$ and we obtain a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & & \\
K \otimes_A M \arrow[r] \arrow[d] & F' \otimes_A M \arrow[r] \arrow[d] & N' \otimes_A M \arrow[r] \arrow[d] & 0, \\
K \otimes_A M \arrow[r] & F \otimes_A M \arrow[r] & N \otimes_A M &
\end{tikzcd}$$

where it follows from Lemma (2.6.6) that the middle vertical map is injective. A diagram chase shows that the right vertical map is injective. Consequently M is flat over A.

(2.6.8) Remark. It follows from Proposition (2.6.7) that a module over a principal ideal domain is flat if and only if it does not have torsion.

(2.6.9) Lemma. *Given a ring A and an A-module M. The following two assertions are equivalent:*

(1) *The module M is flat over A.*
(2) *For every A module homomorphism $u\colon F \to M$ from a finitely generated free module F, and for every element e in the kernel of u, there is a factorization of u via an A-module homomorphism $f\colon F \to G$ into a finitely generated free A-module G such that $f(e) = 0$, and an A-module homomorphism $v\colon G \to M$.*

Proof. Assume that M is flat over A and that we have a homomorphism $u: F \to M$ such that $u(e) = 0$ for some element e in F. Let $f_1, \dots, f_m$ be a basis for F and write $e = \sum_{i=1}^m a_i f_i$ and $x_i = u(f_i)$. Then $u(e) = \sum_{i=1}^m a_i x_i = 0$.

We define an A-module homomorphism $a: F \to A$ by $a(f_i) = a_i$ for $i = 1, \dots, m$ and we denote the kernel of a by E. From the exact sequence $0 \to E \xrightarrow{i} F \xrightarrow{a} A \to 0$ we obtain an exact sequentce $0 \to E \otimes_A M \xrightarrow{i_M} M^m \xrightarrow{a_M} M \to 0$. We have that $a_M(x_1, \dots, x_m) = \sum_{i=1}^m a_i x_i = 0$. Consequently $(x_1, \dots, x_m) = i_M(\sum_{j=1}^n a_{i,j} f_i \otimes y_j) = (\sum_{j=1}^n a_{1,j} y_j, \dots, \sum_{j=1}^n a_{m,j} y_j)$ for elements $\sum_{i=1}^m a_{i,j} f_i$ in E and y_j in M, for $j = 1, \dots, n$. We have that $0 = a(\sum_{i-1}^m a_{i,j} f_i) = \sum_{i=1}^m a_{i,j} a_i$ for $j = 1, \dots, m$. Let G be the free A-module with basis $g_1, \dots, g_n$ and define A-module homomorphisms $f: F \to G$ and $v: G \to M$ by $f(f_i) = \sum_{j=1}^n a_{i,j} g_j$ for $i = 1, \dots, m$ respectively $v(g_j) = y_j$ for $j = 1, \dots, n$. We then have that $vf(f_i) = v(\sum_{j=1}^n a_{i,j} g_j) = \sum_{j=1}^m a_{i,j} y_j = x_i = u(f_i)$ for $i = 1, \dots, m$ and $f(e) = f(\sum_{i=1}^m a_i f_i) = \sum_{i-1}^m \sum_{j=1}^n a_i a_{i,j} g_j = \sum_{j=1}^n (\sum_{i=1}^m a_i a_{i,j}) g_j = 0$. Consequently assertion (2) holds.

Conversely, assume that assertion (2) holds. We shall show that M is flat over A by showing that the map $I \otimes_A M \to M$ is injective for all ideals I in A. Assume that we have an element $\sum_{i=1}^m a_i \otimes x_i$ in $I \otimes_A M$ that maps to zero in M, that is $\sum_{i=1}^m a_i x_i = 0$. We shall show that $\sum_{i=1} a_i \otimes x_i = 0$. Let F be a free A–module with basis $f_1, \dots, f_m$ and define an A-module homomorphism $u: F \to M$ by $u(f_i) = x_i$ for $i = 1, \dots, m$. Then we have that $e = \sum_{i=1}^m a_i f_k$ is in the kernel of u. By assumption there is a factorization of u via homomorphisms $f: F \to G$ and $v: G \to M$, where G is free with a basis $g_1, \dots, g_n$ and $f(e) = 0$. Write $f(f_i) = \sum_{j=1}^n a_{i,j} g_j$ for $i = 1, \dots, m$ and $v(g_j) = y_j$ for $j = 1, \dots, n$. Then we have that $0 = f(e) = f(\sum_{i=1}^m a_i f_i) = \sum_{i=1}^m \sum_{j=1}^n a_i a_{i,j} g_j$, and consequently that $\sum_{i=1}^m a_i a_{i,j} = 0$ for $j = 1, \dots, n$. We have that $x_i = u(f_i) = vf(f_i) = v(\sum_{j=1}^n a_{i,j} g_j) = \sum_{j=1}^n a_{i,j} y_j$, for $i = 1, \dots, n$. Hence we have that $\sum_{i=1}^m a_i \otimes x_i = \sum_{i=1}^m \sum_{j=1}^n a_i \otimes a_{i,j} y_j = \sum_{i=1}^m \sum_{j=1}^n a_i a_{i,j} \otimes y_j = 0 = \sum_{j=1}^n (\sum_{i=1}^m a_i a_{i,j}) \otimes y_j$. We have proved that $I \otimes_A M \to M$ is injective and consequently that M is flat over A.

(2.6.10) Lemma. *Given a map $\varphi: A \to B$ of rings and let F be a B-module. Then F is flat over A if and only if F_Q is flat over A_P for all prime ideals P in A and Q in B such that $\varphi^{-1}(Q) = P$.*

Proof. Assume that F is flat over A. Since B_Q is flat over B the functor that sends an A_P-module N to $B_Q \otimes_B (N \otimes_A F)$ is exact. However $B_Q \otimes_B (F \otimes_A N) = F_Q \otimes_A N = F_Q \otimes_{A_P} N$. Consequently the functor that sends N to $F_Q \otimes_{A_P} N$ is exact, that is, the A_P-module F_Q is flat.

Conversely, assume that F_Q is a flat A_P module for all prime ideals Q in B

with $P = \varphi^{-1}(Q)$. The functor that sends an A-module N ot the A_P-module N_P is exact. Consequently the functor that sends N to $F_Q \otimes_{A_P} N_P$ is exact. However, we have that $F_Q \otimes_{A_P} N_P = F_Q \otimes_{A_P} (A_P \otimes_A N) = F_Q \otimes_A N$. Hence the functor that sends an A–module N to $F_Q \otimes_A N$ is exact. However, the functor that sends an A–module N to the B–module $F \otimes_A N$ is exact if and oly if the functor that sends the A–module N to the B_Q–module $F_Q \otimes_A N$ is exact for all prime ideal Q of B. We thus have that F is a flat A–module.

(2.6.11) Proposition. *Given a local homomorphism $A \to B$ of local rings, such that B is flat over A. Then the resulting map $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective.*

Proof. Given a prime ideal P in A. Since B/PB is flat over A/P we have that $B/PB \neq 0$ and since B is flat over A we have that the image S in B/PB of the non zero elements in A/P consists of non zero divisors in B/PB. Let Q be an ideal which is maximal among the ideals in B/PB that are disjoint from S. Then Q is a prime ideal and $Q \cap A/P = 0$. The inverse image R of Q by the residue map $B \to B/PB$ consequently satisfies $\varphi^{-1}(Q) = P$.

(2.6.12) Corollary. *Given a homomorphism $\varphi\colon A \to B$ of rings, such that B is flat over A. Let P be a prime ideal in A, and Q a prime ideal in B containing P. Then there is a prime ideal R in B such that $R \subseteq Q$ and $\varphi^{-1}(R) = P$.*

Proof. It follows from Proposition (2.6.9) that the map $A_{\varphi^{-1}(Q)} \to B_Q$ is flat. The Proposition asserts that there is a prime ideal R' in B_Q such that the contraction of R' to $A_{\varphi^{-1}(Q)}$ is equal to $PA_{\varphi^{-1}(Q)}$. Then we have that the contraction R of R' to B is contained in Q and that $\varphi^{-1}(R)$ is equal to the contraction P of $PA_{\varphi^{-1}(Q)}$ to A.

(2.6.13) Remark. The Corollary has many interesting reformulations and variations.

(1) The property of the Corollary can be restated by saying that flat maps satisfy the *going down property* .
(2) The property of the Corollary can be stated geometrically as:
    Let $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ be the map corresponding to the map $A \to B$. Given a point x in $\operatorname{Spec} X$ and assume that $f(x)$ is a specialization of a point η in $\operatorname{Spec} Y$. Then there is a point ξ in $\operatorname{Spec} X$ such that $f(\xi) = \eta$ and x is a specialization of ξ.
(3) It follows from the Corollary that the contraction $\varphi^{-1}(Q)$ of a minimal prime Q in B to A is a minimal prime in A.
(4) A flat morphism of finite type to a noetherian scheme is open. Indeed, it follows from Theorem (1.2.10) that $f(\operatorname{Spec} B)$ is constructible. Thus it follows from Proposition (?) that $f(\operatorname{Spec} B)$ is open.

→ **(2.6.14) Proposition.** *Given a regular* (73) [A-M, 11.22] *one dimensional ring A and a homomorphism $\varphi\colon A \to B$ into a noetherian ring B. Then B is flat over A if and only if $\varphi^{-1}(Q) = 0$ for all associated prime ideals Q in B.*

In particular we have that if B is reduced then B is flat over A if and only if $\varphi^{-1}(Q) = 0$ for all minimal primes Q of B.

Proof. Assume that B is flat over A and let Q be a prime ideal in B. If $P = \varphi^{-1}(Q)$ is maximal we have that A_P is a discrete valutation ring ([73] 9.2 and 11.23). Let $t \in PA_P$ be a generator for the maximal ideal. Since t is not a zero divisor in A_P and B_Q is a flat A_P module it follows that t is not a zero divisor in B_Q. Consequently Q is not an associated prime in B.

Coversely, assume that $\varphi^{-1}(Q)$ is zero for all associated primes Q of B. It → follows from Proposition (?) that we must prove that B_R is flat over $A_{\varphi^{-1}(R)}$ for all prime ideals R in B. If $\varphi^{-1}(R) = 0$ we have that $A_{\varphi^{-1}(R)}$ is a field and consequently that B_R is flat. On the other hand, if $P = \varphi^{-1}(R)$ is a maximal ideal we choose a $t \in \varphi^{-1}(R)$ that generates the ideal PA_P. Since A_P is a principal → ideal domain it follows from Remark (?) that it suffices to show that B_R is a torsion free A_P-module. Since all elements of A_P can be written as a power of t times a unit, this means that it suffices to prove that t is not a zero divisor → in B_R. However, if t were a zero divisor in B_R it follows from Proposition (ass) that it is contained in an associated prime ideal Q of B. However, by assumption $\varphi^{-1}(Q) = 0$. This is impossible because $t \neq 0$. Hence t is not zero divisor and we have proved the Proposition.

2.7. Flatness and associated primes.

(2.7.15*) Lemma. *Given a homomorphism $A \to B$ of noetherian rings and a B-module G which is flat over A. Let P be a prime ideal in A such that $G/PG \neq 0$. Then we have that:*

(1) $\operatorname{ass}_B(G/PG) = \operatorname{ass}_B(G \otimes_A A_P/PA_P)$.
(2) $\{P\} = \operatorname{ass}_A(G/PG) = \operatorname{ass}_A(G \otimes_A A_P/PA_P)$.
(3) *The contraction of the ideals in* $\operatorname{ass}_B(G/PG)$ *to* A *is* $\{P\}$.

Proof. (1) Since G is flat over A we have that G_P is flat over A_P. Consequently $G/PG = G \otimes_A A/P$ and $G \otimes_A A_P/PA_P = G \otimes_A A/P \otimes_A A_P$ is flat over A/P. Since the ring A/P is a domain it follows that the non zero element in A/P are non zero divisors in G/PG and $G \otimes_A A_P/PA_P$. It follows that $\{P\} = \operatorname{ass}_A(G/PG) = \operatorname{ass}_A(G \otimes_A A_P/PA_P)$.

(2) Since the elements in $A \setminus P$ are not zero divisors in G/PG we obtain an injection $G/PG \to G/PG \otimes_A A_P$ and consequently $\operatorname{ass}_B(G/PG) \subseteq \operatorname{ass}_P(G/PG \otimes_A A_P) = \operatorname{ass}_B(G \otimes_A A_P/PA_P)$. Conversely, if $Q = \operatorname{ann}(x/t)$ is in $\operatorname{ass}_B(G \otimes_A A_P/PA_P)$ with $a \in G/PG$ and $t \in A/P$ we have that $Q = \operatorname{ass} x$ because t is invertible in A_P. Consequently we have that $Q \in \operatorname{ass}_B(G/PG)$.

(3) If $Q \in \operatorname{ass}_B(G/PG)$ there is an element $x \in G/PG$ such that $Q = \operatorname{ann} x$ in B. Then we have that the contraction of Q to A is the annihilator of x in A. Consequently we have that the contraction of Q to A is in $\operatorname{ass}_A(G/PG) = \{P\}$, and thus equal to P.

□

A more general result than the following can be found in the written notes.
→ References are (82), (85), (86) [Ma, Ch 3, §9], [Mb, Ch. 8, §23] [EGA IV$_2$, 24, 3.3.1].

(2.7.16*) Proposition. *Given a flat A-algebra B. Then we have that*

$$\operatorname{ass} B = \cup_{P \in \operatorname{ass} A} \operatorname{ass}_B(B/PB) = \cup_{P \in \operatorname{ass} A} \operatorname{ass}_B(B \otimes_A A_P/PA_P).$$

Proof. When P is in $\operatorname{ass} A$ we have an injection $A/P \to A$. Since B is flat over A we obtain an injection $A/P \to A$. It follows that $\operatorname{ass}_B(B/PB) \subseteq \operatorname{ass}(B)$.

Conversely, let $Q \in \operatorname{ass} B$ and let P be the contraction of Q to A.

→ We first show that $P \in \operatorname{ass}(A)$. Choose, as in Proposition (?), a minimal decomposition $0 = N_1 \cap \cdots \cap N_r$ of zero in A, such that A/N_i has only one associated
→ prime P_i. Then it follows from Proposition (?) that the primes $P_1, \dots, P_r$ are the associated primes of A. We have an injective homomorphism $A \to \oplus_{i=1}^r A/N_i$ and, since B is flat over A, we get an injective homomorphism $B \to \oplus_{i=1}^r B/N_iB$. Consequently we have that $Q \in \operatorname{ass} B/N_iB$ for some i. However, we have that snitt

$P_i^n \subseteq N_i$, and thus $P_i^n B/N_i B = 0$ for some integer n. In particular we have that the elements of P_i are nilpotent for the module B/N_iB, and consequently it follows from Proposition (?) that $P_i \subseteq P$. On the other hand, the elements in $A \setminus P_i$ are not zero divisors in A/N_i, and, since B is flat over A, they are not zero divisors in $B/N_iB = A/N_i \otimes_A B$. Hence $A \setminus P_i$ is disjoint from P. It follows that $P_i = P$, and thus that $P \in \operatorname{ass} A$.

Next we show that $Q \in \operatorname{ass}_B(B/PB)$. It follows from Proposition (?) that there is a chain of ideals $A = I_0 \supset I_1 \supset \cdots \supset I_t = 0$ in A such that $I_{i-1}/I_i \cong A/P_i$ for a prime ideal P_i in A. Since B is flat over A we obtain a chain $B = I_0 \supset I_1B \supset \cdots \supset I_tB = 0$, such that $I_{i-1}B/I_iB \cong B/P_iB$. It follows that $Q \in \operatorname{ass}_B(B/P_iB)$ for some i. However, it follows from Lemma (?) that the contraction of Q to A is P_i. Consequently we have that $P_i = P$ and $Q \in \operatorname{ass}_B(B/PB)$. We have proved the first equality of the Proposition.

The last equality of the Proposition follows from Lemma (2.7.16).

(2.7.17*) Proposition. *Given A-algebras B and C and assume that C is flat over A. Then we have that*

$$\operatorname{ass}(B \otimes_A C) = \bigcup_{Q \in \operatorname{ass} B} \ \bigcup_{R' \in \operatorname{ass}(\kappa(P) \otimes_A C)} \operatorname{ass}_{B \otimes_A C}(\kappa(Q) \otimes_{\kappa(P)} \kappa(R))$$

where R is the contraction of R' by the map $C \to C_P$ and the contraction of Q and R to A is P.

Proof. Proposition (2.7.16) applied to the flat B-algebra $B \otimes_A C$ yields

$$\operatorname{ass}(B \otimes_A C) = \bigcup_{Q \in \operatorname{ass} B} \operatorname{ass}_{B \otimes_A C}(B_Q/QB_Q \otimes (B \otimes_A C)).$$

On the other hand the same Proposition applied to the flat $\kappa(P) \otimes_A C$-algebra $\kappa(Q) \otimes_A C$ gives the formula

$$\begin{aligned} &\operatorname{ass}_{B \otimes_A C}(\kappa(Q) \otimes_A C) \\ &= \bigcup_{R' \in \operatorname{ass}(\kappa(P) \otimes_A C)} \operatorname{ass}((\kappa(Q) \otimes_A C) \otimes_{\kappa(P) \otimes_A C} (\kappa(P) \otimes_A C)_{R'}/R'(\kappa(P) \otimes_A C)). \end{aligned}$$

It follows from Lemma (2.7.15) that the contraction of R' to A is P, and hence the contraction of R to A is also P.

There is a canonical isomorphism

$$\kappa(Q) \otimes_A C \to \kappa(Q) \otimes_A C \otimes_{\kappa(P) \otimes_A C} (\kappa(P) \otimes_A C),$$

where the ideal $\kappa(Q) \otimes_A R$ corresponds to $\kappa(Q) \otimes_A C \otimes_{\kappa(P) \otimes_A C} R'$. It follows that we have an isomorphism

$$\kappa(Q) \otimes_A \kappa(R) \to (\kappa(Q) \otimes_A C) \otimes_{\kappa(P) \otimes_A C} (\kappa(P) \otimes \otimes_A C)_{R'} / R'(\kappa(P) \otimes_A C)_{R'}.$$

Finally we notice that we have a canonical isomorphism

$$\kappa(Q) \otimes_A \kappa(R) \to \kappa(Q) \otimes_{\kappa(P)} \kappa(R),$$

and we have proved the Proposition.

2.8. Local criteria of flatness.

(2.8.1) Theorem. (Local criterion of flatness). *Given a ring A, an ideal I in A, and an A-module M. Consider the following assertions*

(1) *The module M is flat over A.*
(2) *The module M/IM is flat over A/I and the homomorphism $I \otimes_A M \to M$ is injective.*
(3) *The module M/IM is flat over A/I and the homomorphism $I^n/I^{n+1} \otimes_A M \to I^n M/I^{n+1}M$ is bijective for all n.*
(4) *The module $M/I^n M$ is flat over A/I^n for all n.*

Then we have that (1) *implies* (2), *that implies* (3), *and that implies* (4).

If A is noetherian, B is a finitely generated A-algebra such that $I \subseteq \mathrm{rad}(B)$, and M is a finitely generated B-module. Then we have that (4) *implies* (1).

→ *Proof.* We proved in (?) that (1) implies (2).

To prove that (2) implies (3) we first show that when (2) holds it follows that when $0 \to N' \to N \to N'' \to 0$ is an exact sequence of A-modules with $IN'' = 0$, that is the A-module structure on N' induces an A/I-module structure on N''. then $0 \to N' \otimes_A M \to N \otimes_A M \to N'' \otimes_A M \to 0$ is exact. To prove this we choose surjective maps $\alpha\colon F' \to N'$ and $\beta\colon F \to N$ of A-modules with F' fnad F'' free. Consider the natural commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & F' \oplus IF \arrow[r] \arrow[d, "(\alpha,\beta|IF)"'] & F' \oplus F \arrow[r] \arrow[d, "(i\alpha,\beta)"'] & F/IF \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N' \arrow[r] & N \arrow[r] & N'' \arrow[r] & p
\end{tikzcd}$$

with exact rows. Let $0 \to G' \to G \to G'' \to 0$ be the exact sequence of A-modules induced on the kernels of the three vertical maps of the diagram. We obtain a snitt commutative diagram

$$
\begin{tikzcd}
 & & & 0 \arrow[d] \\
 & G' \otimes_A M \arrow[r] \arrow[d] & G \otimes_A M \arrow[r] \arrow[d] & G'' \otimes_A M \arrow[d] \\
0 \arrow[r] & (F' \oplus IF) \otimes_A M \arrow[r] \arrow[d] & (F' \oplus F) \otimes_A M \arrow[r] \arrow[d] & F/IF \otimes_A M \arrow[d] \\
 & N' \otimes_A M \arrow[r] \arrow[d] & N \otimes_A M \arrow[r] \arrow[d] & N'' \otimes_A M \\
 & 0 & 0 &
\end{tikzcd}
$$

with exact rows and columns. The right vertical column is exact since $Q \otimes_A M = Q \otimes_{A/I} M/IM$ for all A/I-modules Q because M/IM is flat over A/I, and the middle horizontal row is exact since $I \otimes_A M \to M$ is injective. It follows from the diagram that $N' \otimes_A M \to N \otimes_A M$ is injective as we wanted to prove.

To prove that (2) implies (3) we consider the commutative diagram

$$
\begin{tikzcd}
0 \arrow[r] & I^{n+1} \otimes_A M \arrow[r] \arrow[d, "\alpha_{n+1}"'] & I^n \otimes_A M \arrow[r] \arrow[d, "\alpha_n"'] & I^n/I^{n+1} \otimes_A M \arrow[r] \arrow[d, "\gamma_n"'] & 0 \\
0 \arrow[r] & I^{n+1}M \arrow[r] & I^n M \arrow[r] & I^n M/I^{n+1}M \arrow[r] & 0
\end{tikzcd}
$$

with exact rows. The top row is exact by the observation made immediately above. We have that α_1 is injective by assumption. consequently it follows by induction on n that α_n is injective, and thus bijective, for $n = 1, 2, \dots$. Consequently we have a bijection $I^n/I^{n+1} \otimes_A M = (I^n \otimes_A M)/(I^{n+1} \otimes_A M) \to I^n M/I^{n+1}M$, and we have proved that assertion (3) holds.

To prove that (3) implies (4) we fix $n \geq 0$. We shall show that $M/I^n M$ is flat over A/I^n. For $0 \leq i \leq n-1$ we have a commutative diagram

$$
\begin{tikzcd}
 & I^{i+1}/I^n \otimes_A M \arrow[r] \arrow[d, "\alpha_{i+1}"'] & I^i/I^n \otimes_A M \arrow[r] \arrow[d, "\alpha_i"'] & I^i/I^{i+1} \otimes_A M \arrow[r] \arrow[d, "\gamma_i"'] & 0 \\
0 \arrow[r] & I^{i+1}M/I^n M \arrow[r] & I^i M/I^n M \arrow[r] & I^i M/I^{i+1}M \arrow[r] & 0.
\end{tikzcd}
$$

with exact rows. It follows from the assumptions that γ_i is an isomorphism for $i = 0, 1, \ldots$. By descending induction on i, starting with $\alpha_n = 0$, it follows that α_i is an isomorphism for $i = 0, \ldots, n$. In particular we have that

$$\alpha_1 \colon I/I^n \otimes_A M = I/I^n \otimes_{A/I^n} M/I^n M \to IM/I^n M$$

is an isomorphism. It follows that (2) holds for the A/I^n-module $M/I^n M$, and the ideal I/I^n. The proof that (2) implies (3) used for the A/I^n-module $M/I^n M$ shows that given an exact sequence $0 \to N' \to N \to N'' \to 0$ of A/I^n-modules, with $IN'' = 0$ we obtain an injection

$$N' \otimes_{A/I^n} M/I^n M = N' \otimes_A M \to N \otimes_A M = N \otimes_{A/I^n} M/I^n M.$$

Consequently we have that $M/I^n M$ is flat over A/I^n.

To prove that assertion (4) implies assertion (1) under the conditions of the last part of the Theorem, we shall show that $j \colon J \otimes_A M \to M$ is injective for all finitely generated ideals J of A. Since B is a noetherian A-albebra and M is a finitely generated B-module we have that $J \otimes_A M$ is a finitely generated B-module. Since $IB \subseteq \operatorname{rad}(B)$ by assumption it follows from (2.5.2) that $\cap_{i=0}^{\infty} I^n(J \otimes_A M) = 0$. Consequently it suffices to show that the kernel of j is contained in $I^n(J \otimes_A M)$ for all n. For fixed n it follows from the Artin-Rees Lemma (2.5.2) that $I^k \cap J \subseteq I^n J$ for big k. We have maps

$$J \otimes_A M \xrightarrow{f} J/(I^k \cap J) \otimes_A M \xrightarrow{g} (J/I^n J) \otimes_A M = (J \otimes_A M)/I^n(J \otimes_A M).$$

Since $M/I^k M$ is a flat A/I^k-module we have that the map

$$J/(I^k \cap J) \otimes_A M = J/(I^k \cap J) \otimes_{A/I^k} M/I^k M \to M/I^k M$$

is injective. It follows from the diagram

$$\begin{tikzcd}
J \otimes_A M \arrow[r, "f"] \arrow[d, "j"'] & J/(I^k \cap J) \otimes_A M \arrow[d] \\
M \arrow[r] & M/I^k M
\end{tikzcd}$$

that $\ker(j) \subseteq \ker(f) \subseteq \ker(gf) = I^n(J \otimes_A M)$, which is the inclusion that we wanted to prove.

(2.8.4) Lemma. *Given a local homomorphism $A \to B$ of local noetherian rings, and let P be the maximal ideal in A. Let $u\colon M \to N$ be a homomorphism of finitely generated B-modules where N is flat over A. The following two assertions are equivalent:*

(1) *The homomorphism $u\colon M \to N$ is injective and the cokernel is flat.*
(2) *The homomorphism $\mathrm{id}_{A/P} \otimes_A u\colon A/P \otimes_A M \to A/P \otimes_A N$ is injective.*

Proof. We have seen in (2.6.4) that (1) implies (2). To show that (2) implies (1) we denote by C the cokernel of u. We have a commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & P \otimes_A u(M) \arrow[r] \arrow[d] & P \otimes_A N \arrow[r] \arrow[d] & P \otimes_A C \arrow[r] \arrow[d] & 0 \\
 & u(M) \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & C & \\
0 \arrow[r] & A/P \otimes_A u(M) \arrow[r] & A/P \otimes_A N & &
\end{tikzcd}$$

of A-modules with exact rows and columns, where the middle vertical sequence is exact because N is flat over A, and the bottom left map is injective by the assumption that $\mathrm{id}_{A/P} \otimes_A u$ is injective. It follows from the diagram that $P \otimes_A C \to C$ is injective. Since C/PC is flat over the field A/P it follows from the local criterion of flatness that C is flat over A. Hence it follows from (2.6.5) that the kernel $u(M)$ of the map $N \to C$ is flat over A. Denote by K the kernel of the map $M \to u(M)$. We obtain an exact sequence $0 \to A/P \otimes_A K \to A/P \otimes_A M \to A/P \otimes_A u(M) \to 0$. However, we have that $A/P \otimes_A M \to A/P \otimes_A u(M)$ is injective by the assumptions of the Lemma. Consequently we have that $A/P \otimes_A K = 0$, that is $K = PK$, and hence $K = QK$ where Q is the maximal ideal of B. It follows from Nakayamas Lemma that $K = 0$.

2.9. Generic flatness.

(2.9.14) Lemma. *Given an integral domain A and an A-algebra B of finite type. Moreover, given a finitely generated B-module N. Then there is an element $f \in A$ such that N_f is free over A_f.*

Proof. Write $B = A[u_1, \dots, u_h]$. We shall prove the Lemma by induction on h. → When $h = 0$ we have that $A = B$. It follows from Lemma (2.6) that we can choose a filtration $N = N_n \supset N_{n-1} \supset \cdots \supset N_0 = 0$ by A-modules such that $N_i/N_{i-1} = A/P_i$, where P_i is a prime ideal in A. Since A is an integral domain we have that the intersection of the non zero primes P_i is not zero. Choose a non zero $f \in A$ in this intersection if there is one non zero prime P_i and let $f = 1$ otherwise. Then $(N_i/N_{i-1})_f$ is zero if P_i is a non zero prime and isomorphic to A_f when $P_i = 0$. Consequently we have that N_f is a free A_f–module.

Assume that $h > 0$ and that the Lemma holds for $h - 1$. Choose generators $n_1, \dots, n_s$ for the B-module N and write $B' = A[u_1, \dots, u_{h-1}]$. Then $B = B'[u_h]$. Moreover, let $N' = B'n_1 + \cdots B'n_s$. We have that N' is a finitely generated B'-module such that $BN' = N$. It follows from the induction assumption used to the A-algebra B' and the B'-module N' that we can find an element $f' \in A$ such that $N'_{f'}$ is a free $A_{f'}$-module. It therefore remains to prove that we can find an element $f'' \in A$ such that $(N/N')_{f''}$ is a free $A_{f''}$-module. To this end we write

$$N'_i = N' + u_h N' + \cdots + u_h^i N'$$

and

$$P_i = \{n \in N' \colon u_h^{i+1} n \in N'_i\}.$$

Clearly N'_i is a B'-submodule of N and P_i a B'-submodule of N'. We obtain a filtration

$$N'_1/N' \subseteq N'_2/N' \subseteq \cdots \subseteq N/N'$$

of N/N' by B'-modules N'_i/N' such that $\cup_i N'_i/N' = N/N'$. The B'-linear homomorphism $N' \to N'_{i+1}$ which sends n to $u_h^{i+1} n$ defines an isomorphism $N'/P_i \to N'_{i+1}/N'_i$ for all i. Since B' is noetherian, the sequence $P_0 \subseteq P_1 \subseteq \cdots \subseteq N'$ must stabilize. That is, among the quotients N'_{i+1}/N'_i there appears only a finite number of B'-modules. It follows from the induction assumption that we can find an element $f'' \in A$ such that all the modules $(N'_{i+1}/N'_i)_{f''}$ are free $A_{f''}$-modules. Hence $(N/N')_{f''}$ is a free $A_{f''}$-module, as we wanted to prove. ▯

We shall give another proof of the algebraic Lemma of generic flatness.

(2.9.14*) Lemma. *Let A be an integral domain and B an A-algebra of finite type. Moreover, let N be a finitely generated B-module. Then there is an element $f \in A$ such that N_f is free over A_f.*
snitt

Proof. Let K be the quotient field of A. Then $B \otimes_A K$ is a K-algebra of finite type and $N \otimes_A K$ is a $B \otimes_A K$-module of finite type.

Let $s = \dim \operatorname{supp}(N \otimes_A K)$ be the Krull dimension of the support of $N \otimes_A K$ in $\operatorname{Spec}(B \otimes_A K)$. We shall prove the Lemma by induction on s. When $s < 0$ we have that $N \otimes_A K = 0$. Since K is flat over A we have that $N \otimes_A K = 0$ implies that each element in N has A torsion, and since N is a finitely generated B-module there is an element $f \in A$ such that $fN = 0$.

Fix an $s \geq 0$ and assume that the Lemma holds for all modules with support of lower dimension than s. Since $s > 0$ there is a sequence $N = N_1 \supset N_2 \supset \cdots \supset N_s$ of B-modules such that $N_i/N_{i+1} = B/P_i$, for some prime ideal P_i in B. It suffices to prove the Lemma for the quotients N_i/N_{i+1}, because if N_i/N_{i+1} and N_{i+1} are free then, N_i is free, and we can conclude that N is free by descending induction on i. Hence we can assume that $N = B/P$, where P is a prime ideal in B. If $P \cap A \neq 0$ we can take $f \in P \cap A$ and get $N_f = 0$. Hence we can also assume that $P \cap A = 0$. We have that the support of $B/P \otimes_A K$ in $\operatorname{Spec}(B \otimes_A K)$ is the same as the support in the closed subset $\operatorname{Spec}(B/P \otimes_A K)$. Hence it suffices to prove the Lemma for B/P. Hence we can assume that B is an integral domain that contains A.

It follows from the Noether normalization Lemma that there are elements $x_1, \dots, x_s$ in $B \otimes_A K$ that are algebraically independent over K and such that $B \otimes_A K$ is integral over $K[x_1, \dots, x_s]$. Let $g \in A$ be a common multiple of all the denominators that appear in the integral relatons, with coefficients in $K[x_1, \dots, x_s]$, for the generators of B as an A-algebra. Then B_g is integral over $C = A_g[x_1, \dots, x_s]$ and B_g is a finitely generated C-module. We can therefore find a C-submodule of B isomorphic to $C^{\oplus t}$ for some t such that all the elements in the quotient module N' have C-torsion. We have that $s = \dim \operatorname{supp}(N \otimes_A K) = \operatorname{tr.deg}_K(B \otimes_A K) = \operatorname{tr.deg}_K(C \otimes_A K)$, and since N' has C-torsion we have that $\dim \operatorname{supp}(N' \otimes_A K) < \operatorname{tr.deg}_K(C \otimes_A K)$. It follows from the induction hypothesis that we can find an element $h \in A_g$ such that N'_h is free over A_{gh}. However C_h is a free A_{gh}-module. Hence, it follows from the exact sequence $0 \to C_k^{\oplus t} \to B_{gh} \to N'_k \to 0$ that B_{gh} is a free A_{gh}-module, and we have proved the Lemma.

(2.9.3) Theorem. *Given a noetherian ring A, a finitely generated A–algebra B, and a finitely generated B–module M. The the set*

$$U = \{Q \in \operatorname{Spec} B | M_Q \text{ flat over } A\}$$

is open in $\operatorname{Spec} B$.

→ *Proof.* It follows from (Proposition .?) that it suffices to prove that every generalization of a point in U is contained in U, and that $U \cap \overline{\{x\}}$ is a neighbourhood of x in $\overline{\{x\}}$.

to prove the first condition we observe that if $R \supseteq Q$ are prime ideals of B then we have that $N \otimes_A M_Q = (N \otimes_A M) \otimes_B B_Q = (N \otimes_A M) \otimes_B G_R \otimes_{B_R} B_Q = (N \otimes_A M_R) \otimes_{B_R} B_Q$. Consequently we have that M_Q is a flat A–module when M_R is.

To prove the second condition we take a prime ideal Q of B such that M_Q is flat over A. Let P be the trace of Q in A. For every prime ideal R in B that contains Q we have that $PB_R \subseteq \operatorname{rad} B_R$. It follows from the local criterion for
→ flatness (?) that M_R is flat over A if and only if M_R/PM_R is glat over A/P and $P \otimes_A M_R \to M_R$ is injective.

We have an exact sequence

$$0 \to K \to P \otimes_A M \to M \to A/P \otimes_A M \to 0$$

of B–modules, that defines K. Since B is noetherian and M is finitely generated over B we have that K is a finitely generated B–module. Since M_Q is flat over A we obtain, localizing the sequence at Q, that $K_Q = 0$. Consequently there is an $b \in B$ such that $K_R = 0$ for all primes R in B that do not contain b.

→ By generic flatness (?) there is an element $a \in A \backslash ?$ such that M_Q/PM_Q is free over A_Q/PA_Q. For eacn prime R of B not containing a we then have that M_R/PM_R is flat over A/P. Consequently we have that the open subset $\{R \in \operatorname{Spec} B | ab \notin ab\}$ is contained in U. Thus the second condition holds and we have proved that U is open.

2.10. The dimension of rings.

(2.10.1) Setup. Given a ring A and a prime ideal P. We write $\kappa(P) = A_P/PA_P$.

(2.10.2) Theorem. *Given a homomorphism $\varphi: A \to B$ of noetherian rings and Q a prime ideal in B. Let $P = \varphi^{-1}(Q)$ be the contraction of Q to A. Then we have that:*

(1) $\operatorname{ht} Q \leq \operatorname{ht} P + \dim B_Q/PB_Q$.
(2) *If going down holds between A and B we have equality in* (1).

Proof. (1) We can replace A and B with A_P and B_Q and consequently assume that φ is a local homomorphism between local rings. Then we can write (1) as

$$\dim B \leq \dim A + \dim B/PB.$$

→ Let $a_1, \dots, a_r$ be a system of parameters in A (?) [97] [A-M, Ch. 11, p. 122] and choose $b_1, \dots, b_s$ in B such that the image of these elements in B/PB for a parameter system for B/PB. Then we have, for some integers m and n, that $Q^n \subseteq PB + \sum_{i=1}^s b_iB$ and $P^m \subseteq \sum_{i=1}^r a_iA$. Consequently we have that $Q^{mn} \subseteq \sum_{i=1}^r a_iA + \sum_{i=1}^s b_iB$. Thus we have that $\dim B \leq r + s = \dim A + \dim B/PB$.

(2) Let $\dim B/PB = s$ and let $Q = Q_0 \supset Q_1 \supset \cdots \supset Q_s \supseteq PB$ be a chain of prime ideals in B. We haver that $\varphi^{-1}(Q_i) = P$, for $i = 0, \dots, s$, since $\varphi^{-1}(Q) = P$. Let $\dim A = r$ and let $P = P_0 \supset P_1 \supset \cdots \supset P_r$ be a chain of prime ideals in A. Since we assume that going down holds between A and B there is a descending chain

$$Q_s \supset Q_{s+1} \supset \cdots \supset Q_{s+r}$$

of prime ideals in B such that $\varphi^{-1}(Q_{s+i}) = P_i$. Consequently we have that $\dim B \geq r + s = \dim A + \dim B/PB$. Together with formula (1) we obtain assertion (2).

(2.10.3) Theorem. (The dimension formula) [EGA 24, IV, 5.58], [Ma, Ch. 5, 14.C], [Mb, Ch. 5, 15.5] *Given a noetherian ring A and let B be an integral domain that contains A. Let Q be a prime ideal in B, and let $P = Q \cap A$. Then we have that*

$$\operatorname{ht} Q + \operatorname{td.deg.}_{\kappa(P)} \kappa(Q) \leq \operatorname{ht} P + \operatorname{td.deg.}_A B,$$

where $\operatorname{td.deg.}_A B$ *is the transcendence degree of the quotient field of B over the quotient field of A.*

When B is a polynomial ring over A, or when A is universally caternary, we have equality in the formula.

Proof. We can assume that B is a finitely generated A algebra. Indeed, if the right hand side is finite and m and t are non negative integers such that $m \leq \operatorname{ht} Q$ and snitt

$t \leq \operatorname{td.deg.}_{\kappa(P)} \kappa(Q)$, then we have a chain of prime ideals $Q = Q_0 \supset Q_1 \cdots \supset Q_m$ in B and elements and let $c_1, \dots, c_t$ in B whose images in B/Q are algebraically independent over A/P. Pick an element $a_i \in Q_{i-1} \setminus Q_i$, for $i = 1, \dots, m$. Let $C = A[a_1, \dots, a_m, c_1, \dots, c_t]$. If the Theorem holds for C we have that

$$m+t \leq \operatorname{ht}(Q \cap C) + \operatorname{td.deg.}_{\kappa(P)} \kappa(Q \cap C) \leq \operatorname{ht} P + \operatorname{td.deg.}_A C \leq \operatorname{ht} P + \operatorname{td.deg.}_A B.$$

Consequently the Theorem holds for B.

Assume that B is finitely generated as a A-algebra. We use induction on the number of generators. Assume that B is generated by one element. If $B = A[x]$ is a polynomial ring over A we can replace A with the localization A_P and B with the localization $B_P = A_P[x]$, and therefore assume that A is local with maximal ideal P. Since B is flat over A, and consequently going down holds between A and B by Remark (?). It follows from Theorem (2.10.2) that we have

$$\operatorname{ht} Q = \operatorname{ht} P + \operatorname{ht}(Q/PB).$$

Since $B/PB = \kappa(P)[x]$ is a polynomial ring in one variable over $\kappa(P)$ we have that $Q = PB$ or $\operatorname{ht}(Q/PB) = 1$. If $Q = PB$ we have that

$$\operatorname{ht} Q + \operatorname{td.deg.}_{\kappa(P)} \kappa(Q) = \operatorname{ht} P + \operatorname{ht}(Q/PB) + 1 = \operatorname{ht} P + 1,$$

and if $\operatorname{ht}(Q/PB) = 1$ we obtain that

$$\operatorname{ht} Q + \operatorname{td.deg.}_{\kappa(P)} \kappa(Q) = \operatorname{ht} P + \operatorname{ht}(Q/PB) + 0 = \operatorname{ht} P + 1.$$

Consequently the formula holds with equality in both cases.

Next assume that B is generated by one element x over A, but that it is not a polynomial ring. We can then write $B = A[x]/I$, where I is a non zero prime ideal in $A[x]$. We have that $\operatorname{td.deg.}_A B = 0$. Since $A \subseteq B$ we have that $I \cap A = 0$. Consequently, if we denote by K the quotient field of A we have that $\operatorname{ht} I = \operatorname{ht} IK[t] = 1$. Let Q' be the inverse image of Q by the canonical surjection $A[x] \to B$. Then we have that $Q = Q'/I$ and $\kappa(Q) = \kappa(Q')$. We obtain, using the case when B is a polynomial ring over A, that

$$\operatorname{ht} Q \leq \operatorname{ht} Q' - \operatorname{ht} I = \operatorname{ht} Q' - 1 = \operatorname{ht} P + 1 + \operatorname{td.deg.}_{\kappa(P)} \kappa(Q') - 1.$$

If A is universally catenary it follows from Remark (?) that $\operatorname{ht} Q = \operatorname{ht} Q' - \operatorname{ht} I$ and we obtain equality.

We have proved the case when B is generated by one element over A. However, if $A \subseteq C \subseteq B$ and the formula of the Theorem holds bestween A and C and C and B, it clearly holds between A and B. Consequently the Theorem follows by induction.

(2.10.4) Definition. When two rings A and B satisfy the conditions of the Theorem and the formula of the Theorem holds with equality we say that the *dimesion formula* holds between A and B.

(2.10.5) Theorem. *A noetherian ring A is universally catenary if and only if the dimension theorem holds between A/P and B for all prime ideals P of A and all integral domains B containing A/P, such that B is finitely generated as an A-algebra.*

Proof. Assume that A is universally catenary. Then A/P is universally catenary so we can assume that A is an integral domain and that B contains A. We have that $B = A[x_1, \dots, x_n]/I$ where $A[x_1, \dots, x_n]$ is a polynomial ring over A and I a prime ideal in $A[x_1, \dots, x_n]$. Let Q be a prime ideal in B. Then we have that $Q = R/I$, where R is a prime ideal in $A[x_1, \dots, x_n]$. However the ring $A[x_1, \dots, x_n]$ is catenary, and consequently it follows from Proposition (?) that we have $\operatorname{ht} Q = \operatorname{ht} R - \operatorname{ht} I$. Since we have equality in the formula of Theorem (2.10.4) for polynomial rings we have that

$$\operatorname{ht} R + \operatorname{td.deg.}_{\kappa(P)} \kappa(R) = \operatorname{ht} P + \operatorname{td.deg.}_A A[x_1, \dots, x_r]$$

and

$$\operatorname{ht} I + \operatorname{td.deg.}_{\kappa(0)} \kappa(I) = \operatorname{ht} 0 + \operatorname{td.deg.}_A A[x_1, \dots, x_r].$$

Consequently we have that

$$\operatorname{ht} Q = \operatorname{ht} R - \operatorname{ht} I = \operatorname{ht} P + \operatorname{td.deg.}_{\kappa(0)} \kappa(I) - \operatorname{td.deg.}_{\kappa(P)} \kappa(R).$$

However, we have that $\operatorname{td.deg.}_{\kappa(0)} \kappa(I) = \operatorname{td.deg.}_A B$ and $\operatorname{td.deg.}_{\kappa(P)} \kappa(R) = \operatorname{td.deg.}_{\kappa(P)} \kappa(Q)$, and thus we obtain equality in the formula of Theorem (2.10.3).

Conversely, assume that the dimension formula holds between A/P and B for all prime ideals P in A, and all integral domains B that are finitely generated over A/P. In order to show that A is universally catenary it sufficies to show that all finitely generated A-algebras that are integral domains, are catenary. Given prime ideals $Q \subseteq Q'$ in B. We must show that

$$\dim B_{Q'}/QB_{Q'} + \dim B_Q = \dim B_{Q'}.$$

Denote by P and P' the contraction of Q respectively Q' to A, and denote by K the kernel of the homomorphism $A_{P'} \to B_{Q'}$. The image of $A_{P'}$ in $B_{Q'}$ is isomorphic to $A_{P'}/K$ and we have that the dimension formula holds between $A_{P'}/K$ and $B_{Q'}$. Hence we obtain that

$$\dim A_{P'}/K + \operatorname{td.deg.}_{A_{P'}/K}(B_{Q'}) = \dim B_{Q'} + \operatorname{td.deg.}_{\kappa(P')} \kappa(Q').$$

On the other hand, the kernel of the homomorphism $A_P \to B_Q$ is equal to KA_P since $B'_Q \to B_Q$ is injective. The dimenasion formula holds between A_P/KA_P and B_Q so that we obtain

$$\dim A_P/KA_P + \operatorname{td.deg.}_{A_P/KA_P}(B_Q) = \dim B_Q + \operatorname{td.deg.}_{\kappa(P)} \kappa(Q).$$

Finally, we use that B/Q is an integral domain that is finitely generated over A, that the contraction of Q'/Q to A is P', and that the kernel of the homomorphism $A_{P'} \to B_{Q'}/QB_{Q'}$ is $PA_{P'}$. Thus the dimension formula holds between $A_{P'}/PA_{P'}$ and $B_{Q'}/QB_{Q'}$ and we obtain that

$$\begin{aligned}\dim A_{P'}/PA_{P'} &+ \operatorname{td.deg.}_{A_{P'}/PA_{P'}}(B_{Q'}/QB_{Q'}) \\ &= \dim B_{Q'}/QB_{Q'} + \operatorname{td.deg.}_{\kappa(P')} \kappa(Q').\end{aligned}$$

We add the last two formuas and use that $\kappa(P)$ and $\kappa(Q)$ are the fraction fields of $A_{P'}/PA_{P'}$ respectively $B_{Q'}/QB_{Q'}$, that A_P/KA_P and $A_{P'}/K$ ave the same fraction field, and that B_Q and $B_{Q'}$ ave the same fraction field. We obtain the formula

$$\begin{aligned}\dim A_P/KA_P &+ \dim A_{P'}/PA_{P'} + \operatorname{td.deg.}_{A_{P'}/K}(B_{Q'}) \\ &= \dim B_Q + \dim B_{Q'}/QB_{Q'} + \operatorname{td.deg.}_{\kappa(P')} \kappa(Q').\end{aligned}$$

Since A is catenary we have that $A_{P'}/K$ is catenary and from the chain $P'A_{P'} \supseteq PA_{P'}/K$ we get the formula

$$\dim A_{P'}/PA_{P'} + \dim A_P/KA_P = \dim A_{P'}/K.$$

Consequently we have that

$$\dim A_{P'}/K + \operatorname{td.deg.}_{A_{P'}/K}(B_{Q'}) = \dim B_Q + \dim B_{Q'}/QB_{Q'} + \operatorname{td.deg.}_{\kappa(P')} \kappa(Q').$$

From the first formula we proved we thus obtain that

$$\dim B_{Q'} = \dim B_Q + \dim B_{Q'}/QB_{Q'},$$

which is the formula we wanted to prove.

(2.10.6) Remark. We saw in Proposition (dim.alg.) that fields are universally catenary. Consequently, it follows from Proposition (2.10.5) that the ring of integers $\mathbf{Z}$ is universally catenary.

2.11. Regular sequences.

(2.11.1) Lemma. *Given a ring A and a noetherian algebra B. Let M be a finitely generated B-module, and J an ideal in B that is contained in the Jacobsson radical of B.*

If M/J^nM is a flat A-module for all $n > 0$, we have that M is flat over A.

In particular, if b is an element in the Jacobsson radical of B which is regular for M, and such that M/bM is flat over A. Then M is flat over A.

→ *Proof.* Let $I \subseteq A$ be a finitely generated ideal in A. It follows from Lemma (?) that it suffices to show that the map $u: I \otimes_A M \to M$ is injective.

For $n \geq 1$ we have that

$$(I \otimes_A M)/J^n(I \otimes_A M) = (I \otimes_A M) \otimes_B B/J^n = I \otimes_A M/J^nM$$

and $I \otimes_A M/J^nM \to M/J^nM$ is injective since M/J^nM is flat over A. It follows from the commutative diagram

$$\begin{array}{ccc} I \otimes_A M & \xrightarrow{u} & M \\ \big\downarrow & & \big\downarrow \\ (I \otimes_A M)/J^n(I \otimes_A M) & \longrightarrow & M/J^nM \end{array}$$

that the kernel of u is contained in $J^n(I \otimes_A M)$. We have that $I \otimes_A M$ is a finitely generated B-module. Since B is noetherian it follows that $\cap_{n=1}^{\infty} J^n(I \otimes_A M) = 0$, and consequently we have that the kernel of u is 0.

Given b as in the Lemma. Since b is regular for M we have an exact sequence

$$0 \to M/b^iM \xrightarrow{b} M/b^{i+1}M \to M/bM \to 0.$$

It follows by induction on i, starting with $i = 1$, that M/b^iM is flat for all i. Hence, it follows from the first part of the Lemma that M is flat.

(2.11.2) Lemma. *Given a ring A and an ideal I. Let M be an A-module. If an element $x \in A$ is regular for $I^iM/I^{i+1}M$ for $i = 0, 1, \dots,$ then x is regular for M/I^iM for $i = 1, 2, \dots.$*

Moreover, if x is regular for M/I^iM for $i = 1, 2, \dots$ and $\cap_{i=1}^{\infty} I^iM = 0$, we have that x is regular for M.

Proof. We show the first assertion by induction on i. For $i = 1$ the assertion holds by assumption. Assume that x is regular for M/I^iM and that there is an $m \in M$ such that $xm \in I^{i+1}M$. Since x is regular for M/I^iM, , we have that $m \in I^iM$, snitt and since x is regular for $I^iM/I^{i+1}M$ we obtain that $m \in I^{i+1}M$, as we wanted to show.

To prove the second assertion we take an $m \in M$. If $m \neq 0$ we can, since $\cap_{i=1}^{\infty} I^iM = 0$, choose an i such that $m \in I^iM \setminus I^{i+1}M$. If $xm \in I^{i+1}M$ we must have that $m \in I^{i+1}M$. Consequently we have that $xm \notin I^{i+1}$, and we conclude that x is regular for M.

(2.11.3) Proposition. ([G] 0, 10, 15.1.1.6) *Given noetherian local rings A and B, and a local homomorphism $A \to B$. Let P be the maximal ideal of A and let M be a finitely generated B-module. Given elements $f_1, \dots, f_n$ in the maximal ideal of B. The following assertions are equivalent:*

(1) *The sequence $f_1, \dots, f_n$ is M-regular and we have that the residue modules $M_i = M/(\sum_{j=1}^{i} f_jM)$ are flat A modules for $j = 1, 2, \dots, n$.*
(2) *The sequence $f_1, \dots, f_n$ is M-regular, and we hae that the module $M_n = M/(\sum_{j=1}^{n} f_jM)$ is flat over A.*
(3) *The module M is flat over A and the images $g_1, \dots, g_n$ in B/PB of the elements $f_1, \dots, f_n$ are $A/P \otimes_A M$-regular.*
(4) *The module M is flat over A, and for every homomorphism $\rho\colon A \to A'$ the sequence $1 \otimes_A f_1, \cdots, 1 \otimes_A f_n$ in $A' \otimes_A B$ is $A' \otimes_A M$-regular.*

Proof. It is clear that (1) implies (2) and that (4) implies (3).

We prove first that (2) implies (4). Note that $M_{i+1} = M_i/f_{i+1}M_i$. In order to prove that M is flat over A it therefore suffices, by induction on i, starting with $M_0 = M$, to prove that for b regular in B and M/bM flat over A, we have that M is flat over A. However, this follows by Lemma (?). Since M is flat over A it follows by induction on i that every regular sequence $f_1, \dots, f_n$ for M gives a regular sequence $1 \otimes_A f_1, \dots, 1 \otimes_A f_n$ for $A' \otimes_A M$.

To prove that (3) implies (1) we note that it follows from Lemma (?) that the multiplication $M \xrightarrow{f_1} M$ is injective and that the cokernel M/f_1M is flat over A. We show by induction on i that the multiplication $M_{i-1} \xrightarrow{f_i} M_{i-1}$ is injective and that M_i is flat over A. The case $i = 1$ we just proved. Assume that the assertion holds for $i - 1$. Since $A/P \otimes M_{i-1} \xrightarrow{f_i} A/P \otimes_A M_{i-1}$ is injective it follows from Lemma (?) that f_i is injective and that $M_{i-1}/f_iM_{i-1} = M_i$ is flat over A.

(2.11.5) Lemma. *Given a ring A and an A-module M. Let x be an element of A and J and ideal in A. Write $I = J + xA$. If x is regular for $\sum_{i=9}^{\infty} J^iM/J^{i+1}M$ the map*

$$\varphi\colon \sum_{j=0}^{\infty} J^iM/J^{i+1}M \otimes_A (A/xA)[t] \to \sum_{j=0}^{\infty} I^iM/I^{i+1}M$$

that sends t to x is an isomorphism. In other words, the maps

$$\varphi_i\colon \sum_{j=0}^{i} J^j M/J^{j+1}M \otimes_A (A/xA)t^{i-j} \to I^i M/I^{i+1}M$$

induced by the injection $J \subseteq I$ is an isomorphism for $i = 0, 1, \dots$.

Conversely, if we have that $\cap_{n=0}^{\infty} I^n(M/JM) = 0$ and φ is an isomorphism, then x is M/JM-regular.

Proof. It is clear that φ is surjective. Fix a non-negative number h. Let

$$P = \sum_{j=0}^{h} J^j/J^{j+1} \otimes_A (A/xA)t^{h-j} \text{ and } Q = I^h M/I^{h+1}M.$$

We have that the A module P is filtered by the modules

$$P_i = \sum_{j=i}^{h} J^j M/J^{j+1}M \otimes_A (A/xA)t^{h-j}$$

and Q by $Q_i = \varphi(P_i)$, for $i = 0, \dots, h$. In order to prove that φ_i is injective, it suffices, since $P_0 = P$ and $P_{h+1} = 0$, to show that the induced maps

$$P_i/P_{i+1} = J^i M/J^i Mx + J^{i+1}M \to Q_i/Q_{i+1}$$

is injective, where Q_{i+1} is the image of $R = J^{i+1}Mx^{h-i-1} + J^{i+2}Mx^{h-i-2} + \cdots + J^h M$ in $I^h M/I^{h+1}M$. Hence we must show that if $y \in J^i M$ and $x^{h-i}y \in R + I^{h+1}M$, we have that $y \in xJ^i M + J^{i+1}M$

→ Since x is regular for $J^i M/J^{i+1}M$ for all i it follows from Lemma (?) that x is regular for $M/J^i M$ for all i. We have that $x^{h-i}y \in J^{i+1}M + I^{h+1}M \subseteq J^{i+1}M + x^{h-i+1}M$. Consequently there is an $z \in M$ such that $y - xz \in J^{i+1}M$. Since $y \in J^i M$ and thus $xz \in J^i M$ we have that $z \in J^i M$. Consequently we have that $y \in xJ^i M + J^{i+1}M$, as we wanted to show.

→ To prove the converse it follows from Lemma (?) that it suffices to prove that x is regular for $I^i(M/JM)/I^{i+1}(M/JM) = I^i M + JM/I^{i+1}M + JM$ for all i. To this end, let $m \in I^i M = x^i M + JM$. We must show that if $xm \in I^{i+2}M + JM^i$, the we have that $m \in I^{i+1}M + JM$. Write $m = x^i m_1 + p$ with $p \in JM$ and assume that $xm = x^{i+2}m_2 + q$ with $q \in M$. Then we have that $x^{i+1}m_1 - x^{i+2}m_1 \in JM$. However we have that $M/JM \otimes_A (A/xA)t^{i+1} \to I^{i+1}M/I^{i+2}M$ is injective and $m_1 \otimes_A t^{i+1}$ maps to zero. Thus we have that $m_1 \in xM + JM$. However, then we have that $m \in x^{i+1}M + JM = I^{i+1} + JM$ and we have proved the converse assertion.

(2.11.6) Theorem. *Given a ring A and an A-module M. Let $x_1, \dots, x_r$ be an M-regular sequence and write $I = (x_1, \dots, x_r)$. Then the following assertions hold:*

(1) *We have that the homomorphism*

$$\psi_r \colon M/IM \otimes_A A[t_1, \dots, t_r] \to \sum_{i=0}^{\infty} I^i M/I^{i+1}M$$

that sends t_i to x_i is an A-module isomorphism.

(2) *If ψ_R is an A-module isomorphism and we have that*

$$\cap_{j=0}^{\infty} I^j (M/(x_1, \dots, x_i)M) = 0 \textit{ for } i = 1, \dots, r$$

we have that $x_1, \dots, x_r$ is an M-regular sequence.

Proof. Write $J_s = (x_1, \dots, x_s)$. We show by induction on s that the map

$$\psi_s \colon M/J_sM \otimes_A A[t_1, \dots, t_s] \to \sum_{i=0}^{\infty} J_s^i M/J_s^{i+1}M,$$

is an isomorphism. For $s = 1$, with $J = 0$, we obtain from Lemma (?) that ψ_1 is an isomorphism Assume that ψ_s with $s \leq p$ is an isomorphism. We have that x_s is regular for $M/J_{s-1}M$ and consequently regular for $\sum_{i=o}^{\infty} J_{s-1}^i M/J_{s-1}^{i+1}M$. From Lemma (?) we conclude that the map

$$\varphi \colon \sum_{i=0}^{\infty} J_{s-1}^i M/J_{s-1}^{i+1}M \otimes_A (A/x_sA)[t_s] \to \sum_{i=0}^{\infty} J_s^i M/J_s^{i+1}M$$

is an isomorphism. However, it is clear that $\psi_s = \varphi(\psi_{s-1} \otimes_A \mathrm{id}_{(A/x_sA)[t_s]}$. Thus ψ_s is an isomorphism and (1) holds.

To prove (2) we shall show that if ψ_s is an isomorphism and the condition of the converse holds for $i = 1, \dots, s$, then we have that $x_1, \dots, x_s$ is an M-regular sequence. The assertion holds for $s = 1$ by Lemma (?) with $J = 0$. Assume that ψ_{s-1} is an isomorphism and that (2) holds for $i = 1, \dots, s-1$. By assumption we have that $x_1, \dots, x_{s-1}$ is an M-regular sequence. We have seen that $\psi_s = \varphi(\psi_{s-1} \otimes_A \mathrm{id}_{(A/x_sA)[t_s]})$. Since ψ_{s-1} is surjective we have that if ψ_s is an isomorphism, then φ is an isomorphism. It follows from Lemma (?) used on $M/J_{s-1}M$ that x_s is $M/J_{s-1}M$-regular, and we have proved the Theorem.

(2.11.7) Proposition. *Given a local homomorphism $A \to B$ of local noetherian rings, and let M be a finitely generated B-module. Let $x_1, \dots, x_r$ be an A-regular sequence and write $I = (x_1, \dots, x_r)$. Then M is flat over A if and only if M/IM is flat over A/I, and $x_1, \dots, x_r$ is M-regular.*

Proof. It is clear that if M is flat over A then M/IM is flat over A/I, and it follows by induction on i that $x_1, \dots, x_i$ is M regular for $i = 1, \dots, r$.

Conversely, assume that M/IM is flat over A/I and that $x_1, \dots, x_r$ is M-regular. It follows from the local criterion of flatness (?) that it suffices to prove that $I^i/I^{i+1} \otimes_A M \to I^i M/I^{i+1}M$ is an isomorphism for $i = 0, 1, \dots$. Since the sequence $x_1, \dots, x_r$ is M-regular it follows from Theorem (?) that we have an isomorphism

$$(M/IM) \otimes_A A[t_1, \dots, t_r] \to \sum_{i=0}^{\infty} I^i M/I^{i+1}M$$

and since the sequence $x_1, \dots, x_r$ is A regular we have an isomorphism

$$(A/IA) \otimes_A A[t_1, \dots, t_r] \to \sum_{i=0}^{\infty} I^i/I^{i+1}.$$

Consequently we have isomorphisms

$$M/IM \otimes_A F \to I^i M/I^{i+1}M \text{ and } A/I \otimes_A F \to I^i/I^{i+1}$$

where F is the free A-module generated by the monomials of degree i in $t_1, \dots, t_r$. We tensor the last isomorphism by M over A and obtain an isomorphism

$$M/IM \otimes_A F \to I^i/I^{i+1} \otimes_A M.$$

However, the inverse of the latter map composed with the above isomorphism $M/IM \otimes_A F \to I^i M/I^{i+1}M$ is the map $I^i/I^{i+1} \otimes_A M \to I^i M/I^{i+1}M$ and we have proved the Proposition.

2.12. Reduction to noetherian rings.

(2.12.1) Lemma. ([G] IV, 28, 11.2.4) *Given a ring A and an ideal I of A. Let A' be an A–algebra and M an A–module. We write $M' = A' \otimes_A M$. Then there is a commutative diagram with exact rows*

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d] & I \otimes_A M \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M/IM \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K' \arrow[r] & IA' \otimes_{A'} M' \arrow[r] & M' \arrow[r] & M'/IM' \arrow[r] & 0
\end{tikzcd}$$

that defines the A–module K and the A'–module K'. If the A/I–module M/IM is flat, we have that the left vertical map defines a surjection $A' \otimes_A K \to K'$ of A'–modules.

Proof. We first note that K and K' are A/I, respectively A'/IA'-modules. Consequently we have that $K = A/I \otimes_A K$ and $K' = A'/IA' \otimes_{A'} K'$. Consequently we have that $A' \otimes_A K \to K'$ is surjective if and only if the map $A'/IA' \otimes_{A/I} K \to K'$ is surjective.

Let $L_1 \to L_0 \to M \to 0$ be an exact sequence with L_1 and L_0 free A-modules. Write $L_i' = A' \otimes_A L_i$ for $i = 0, 1$. Then we have an exact sequence $L_1' \to L_0' \to M' \to 0$ is an exact sequence of A'-modules. We obtain a commuative diagram

$$\begin{tikzcd}
 & & & & 0 \arrow[d] & \\
 & & & & T \arrow[d] & \\
 & & I \otimes_A L_1 \arrow[r] \arrow[d] & L_1 \arrow[r] \arrow[d] & A/I \otimes_A L_1 \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] \arrow[d] & I \otimes_A L_0 \arrow[r] \arrow[d] & L_0 \arrow[r] \arrow[d] & A/I \otimes_A L_0 \arrow[d] & \\
0 \arrow[r] & K \arrow[r] & I \otimes_A M \arrow[r] \arrow[d] & M \arrow[r] & A/I \otimes_A M & \\
 & & 0 & & &
\end{tikzcd}$$

snitt

of A-modules, where the right vertical column defines T. Similarly we obtain a commutative diagram

$$
\begin{tikzcd}
 & & & & 0 \arrow[d] & \\
 & & & & T' \arrow[d] & \\
 & & IA' \otimes_{A'} L_1' \arrow[r] \arrow[d] & L_1' \arrow[r] \arrow[d] & A'/IA' \otimes_{A'} L_1' \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] \arrow[d] & IA' \otimes_{A'} L_0' \arrow[r] \arrow[d] & L_0' \arrow[r] \arrow[d] & A'/IA' \otimes_{A'} L_0' \arrow[d] & \\
0 \arrow[r] & K' \arrow[r] & IA' \otimes_{A'} M' \arrow[r] \arrow[d] & M' \arrow[r] & A'/IA' \otimes_{A'} M' & \\
 & & 0 & & &
\end{tikzcd}
$$

of A'-modules where the right vertical column defines T'.

It follows from the two above diagrams that we have a surjection $T \to K$ of A-modules, respectively a surjection $T' \to K'$ of A'-modules. We have a commutative diagram

$$
\begin{tikzcd}
0 \arrow[r] & T \arrow[r] \arrow[d] & A/I \otimes_A L_1 \arrow[r] \arrow[d] & A/I \otimes_A L_0 \arrow[r] \arrow[d] & A/I \otimes_A M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T' \arrow[r] & A'/IA' \otimes_{A'} L_1' \arrow[r] & A'/IA' \otimes_{A'} L_0' \arrow[r] & A'/IA' \otimes_{A'} M' \arrow[r] & 0
\end{tikzcd}
$$

with exact rows, where the upper row consists of A/I-modules and the bottom of $A'/IA/$-modules, and the vertical maps are maps of A-modules. To prove the Lemma it suffices to show that the map $A'/IA' \otimes_{A/I} T \to T'$ obtained from the left vertical map of the latter diagram is surjective. We break the upper horizontal sequence into two exact sequences

$$0 \to Q \to A/I \otimes_A L_0 \to A/I \otimes_A M \to 0$$

and

$$0 \to T \to A/I \otimes_A L_1 \to Q \to 0$$

of A/I-modules. Tensorize the last two sequences with A'/IA' over A/I. We obtain two exact sequences

$$0 \to A'/IA' \otimes_{A/I} Q \to A'/IA' \otimes_A L_0 \to A'/IA' \otimes_A M \to 0$$

and

$$A'/IA' \otimes_{A/I} T \to A'/IA' \otimes_A L_1 \to A'/IA' \otimes_A Q \to 0$$

of A'/IA'-modules, where ethe first is exact since M/IM is a flat A/I-module by assumption. Consequently we obtain a commutative diagram

$$\begin{tikzcd}
 & B \otimes_{A/I} T \arrow[r] \arrow[d] & B \otimes_A L_1 \arrow[r] \arrow[d] & B \otimes_A L_0 \arrow[r] \arrow[d] & B \otimes_A M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T' \arrow[r] & B \otimes_{A'} L_1' \arrow[r] & B \otimes_{A'} L_o' \arrow[r] & B \otimes_{A'} M' \arrow[r] & 0
\end{tikzcd}$$

of $B = A'/IA'$-modules, where the three right vertical maps are isomorphisms. Consequently the left vertical map is surjective.

(2.12.2) Lemma. ([G] IV, 28, 2.5) *Given a ring A and an ideal I of A. Let B be an A-algebra, and M a B-module. Moreover, let A' be a noetherian A-algebra. Write $B' = A' \otimes_A B$ and $M' = A' \otimes_A M = B' \otimes_B M$. We assume that B' is a finitely generated A'-algebra and that M' is a finitely generated B'-module. Given a prime ideal Q' in B' that contains IB'. We have a commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d] & I \otimes_A M \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M/IM \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K' \arrow[r] \arrow[d] & IA' \otimes_{A'} M' \arrow[r] \arrow[d] & M' \arrow[r] \arrow[d] & M'/IM' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K'_{Q'} \arrow[r] & IA' \otimes_{A'} M'_{Q'} \arrow[r] & M'_{Q'} \arrow[r] & M'_{Q'}/IM'_{Q'} \arrow[r] & 0
\end{tikzcd}$$

with exact rows and columns, and where the two first rows define the A-module K respectively the A'-module K'. If M/IM is a flat A/I-module, and the composite map $K \to K' \to K'_{Q'}$ of the diagram is zero, we have that $M'_{Q'}$ is a flat A'-module.

Proof. It follows from the assumptions of the Lemma that the $A'/IA' = A' \otimes_A A/I$-module $M'/IM' = A' \otimes_A M/IM$ is flat. Consequently we have that the A'/IA'-module $M'_{Q'}/IM'_{Q'}$ is flat. It follows from the local criteron of flatness (?) that it suffices to prove that $K'_{Q'} = 0$.

It follows from Lemma (?) that the map $A' \otimes_A K \to K'$ is surjective. Since the composite map $K \to K' \to K'_{Q'}$ is zero by assumption it follows that the map $K' \to K'_{Q'}$ is zero. Since B' is noetherian, and M' is a finitely generated B'-module by assumption we have that K' is a finitely generated B'-module. Hence, since the map $K' \to K'_{Q'}$ is zero, we have that $K'_{Q'} = 0$.

(2.12.3) Lemma. *Given a ring A. Let B be a finitely presented A-algebra and M a finitely presented B-module. Then there is a finitely generated $\mathbf{Z}$-subalgebra A_0 of A, a finitely generated A_0-algebra B_0, and a finitely generated B_0-module M_0, with canonical maps $B_0 \to B$ and $M_0 \to M$ of A_0-algebras, respectively B_0-modules, such that the resulting maps $A \otimes_{A_0} B_0 \to B$ and $B \otimes_{B_0} M_0 = A \otimes_{A_0} M_0 \to M$ are isomorphisms.*

Proof. Write $B = A[x_1, \dots, x_r]/I$ as the residue ring of the polynomial ring in r variables $x_1, \dots, x_r$ over A , by the finitely generated ideal $I = (f_1, \dots, f_n)$, and let $B^m \xrightarrow{(b_{i,j})} B^n \to M \to 0$ be a presentation of the B-module M. Choose polynomials $g_{i,j}$ in $A[x_1, \dots, x_r]$ whose classes in B are $b_{i,j}$.

We define A_0 to be the $\mathbf{Z}$-algebra generated by the coefficients of the polynomials $f_1, \dots, f_n$ and $g_{1,1}, \dots, g_{m,n}$. Moreover let $B_0 = A_0[x_1, \dots, x_r]/(f_1, \dots, f_n)$ and let M_0 be the cokernel of the map

$$B_0^m \xrightarrow{(b_{i,j})} B_0^n.$$

The inclusion of $A_0[x_1, \dots, x_r]$ in $A[x_1, \dots, x_r]$ induces a homomorphism $B_0 \to B$ and we have an isomorphism

$$A \otimes_{A_0} B_0 = A \otimes_{A_0} A_0[x_1, \dots, x_r]/(f_1, \dots, f_n) = A[x_1, \dots, x_r]/(f_1, \dots, f_n) = B.$$

Moreover, we have a commutative diagram

$$\begin{tikzcd}
B_0^m \arrow[r, "(b_{i,j})"] \arrow[d] & B_0^n \arrow[r] \arrow[d] & M_0 \arrow[r] \arrow[d] & 0 \\
B^m \arrow[r, "(b_{i,j})"] & B^n \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

where the upper row is an exact sequence of B_0-modules and the vertical maps are B_0-module homomorphisms. Since the map $A \otimes_{A_0} B_0^i \to B^i$ is an isomorphism for $i = m, n$ it follows that we get a canonical isomorphism $A \otimes_{A_0} B_0 \to M$.

→ **(2.12.4) Remark.** Given A, B and M as in Lemma (?), and let A_0, B_0 and M_0 satisfy the conditions of the conclusions of the Lemma. It is clear that we can find finitely generated $\mathbf{Z}$-algebras A_λ, with λ in some totally ordered index set I with first element 0, such that $A_\lambda \subseteq A_\mu$ whenever $\lambda \le \mu$, and with $A = \cup_{\lambda \in I} A_\lambda$. We write $B_\lambda = A_\lambda \otimes_{A_0} B_0$ and $M_\lambda = A_\lambda \otimes_{A_0} M_0$. Then we have that B_λ is a finitely generated A_λ-algebra, and M_λ is a finitely generated B_λ-module.

We obtain natural maps $B_\lambda \to B_\mu$ and $M_\lambda \to M_\mu$ when $\lambda \le \mu$ such that $B_\mu = A_\mu \otimes_{A_\lambda} B_\lambda$ and $M_\mu = A_\mu \otimes_{A_\lambda} M_\lambda$. Moreover we have natural maps $B_\lambda \to B$ and $M_\lambda \to M$ that induce isomorphisms $A \otimes_{A_\lambda} B_\lambda \to B$ and $A \otimes_{A_\lambda} M_\lambda = B \otimes_{B_\lambda} M_\lambda \to M$, for all indices λ, and these maps are compatible with the maps $A \otimes_{A_\lambda} B_\lambda \to A \otimes_{A_\mu} B_\mu$ and $A \otimes_{A_\lambda} M_\lambda \to A \otimes_{A_\mu} M_\mu$ when $\lambda \le \mu$.

Since A is the union of the rings A_λ we have that B is the union of the images of the rings B_λ and that M is the union of the images of the modules M_λ.

(2.12.5) Theorem. *Given a ring A, and let B be a finitely presented A-algebra and M a finitely presented B-module which is flat over A. Then there is a finitely generated $\mathbf{Z}$-subalgebra A_0 of A, a finitely generated A_0-algebra B_0, and a finitely generated B_0-module M_0 which is flat over A_0, together with canonical maps $B_0 \to B$ and $M_0 \to M$ of A_0-algebras respectively B_0-modules, such that the resulting maps $A \otimes_{A_0} B_0 \to B$ and $B \otimes_{B_0} M_0 = A \otimes_{A_0} M \to M$ are isomorphisms.*

Proof. It follows from Lemma (?) that we can find a finitely generated $\mathbf{Z}$-subalgebra A_0 of A, a finitely generated A_0-algebra B_0 and a finitely generated B_0-module M_0 together with canonical maps $B_0 \to B$ and $M_0 \to M$ such that the resulting maps $A \otimes_{A_0} B_0 \to B$ and $A \otimes_{A_0} M_0 = B \otimes_{B_0} M_0 \to M$ are isomorphisms. We define A_λ, B_λ and M_λ for λ in some totally ordered index set as in Remark (?). In order to prove the Theorem it suffices to prove that for every prime ideal Q in B there is a λ, dependent on Q, such that if Q_λ is the trace of Q in B_λ, we have that $(M_\lambda)_{Q_\lambda}$ is a flat A_λ-module. Indeed, then it follows from (?) ([M] 24.3, [G] 28, 11.1.1) that there is an open subset U_λ of $\operatorname{Spec} B_\lambda$ such that $(M_\lambda)_{R_\lambda}$ is a flat A_λ-module for all $R_\lambda \in U_\lambda$, and by our assumptions we have that if V_λ is the inverse image of U_λ by the map $\operatorname{Spec} B \to \operatorname{Spec} B_\lambda$ coming frm the canonical map $B_\lambda \to B$, we have that $\operatorname{Spec} B$ is the union of the V_λ's. Moreover we have that $V_\lambda \subseteq V_\mu$ whenever $\lambda \le \mu$, because if M_λ is a flat A_λ-module, and $R_\lambda \subseteq B_\lambda$ comes from a prime $R \subseteq B$, we have that $(M_\mu)_{R_\mu} = (A_\mu \otimes_{A_\lambda} M_\lambda)_{R_\mu}$ is a flat A_μ-module for every prime ideal R'_μ in B_μ that restricts to R_λ and thus, in particular, to R_μ. Consequently we have that R is in V_μ.

We have seen that $\{V_\lambda\}_\lambda$ is a family of stricly open subsets of $\operatorname{Spec} B$ whose union is $\operatorname{Spec} B$. Since $\operatorname{Spec} B$ is compact we have that $\operatorname{Spec} B = V_\lambda$ for some λ. Consequently it suffices to fix a prime ideal Q in B and show that there is an index λ such that $(M_\lambda)_{Q_\lambda}$ is a flat A_λ-module, where Q_λ is the trace of Q in B_λ. Let P_0 be the trace of Q in A_0. By basis extension we can clearly replace A_0 with A_{P_0}, and thus assume that A_0 is local with maximal ideal P_0, which is the trace of Q.

We have a commutative diagram

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & K_0 & \longrightarrow & P_0 \otimes_{A_0} M_o & \longrightarrow & M_0 & \longrightarrow & A_0/P_0 \otimes_A M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_\lambda & \longrightarrow & P_0 A_\lambda \otimes_{A_\lambda} M_\lambda & \longrightarrow & M_\lambda & \longrightarrow & A_\lambda/P_0 A_\lambda \otimes_{A_\lambda} M_\lambda & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & P_0 A \otimes_A M & \longrightarrow & M & \longrightarrow & A/P_0 A \otimes_A M & \longrightarrow & 0
\end{array}$$

with exact rows. By assumption we have that $K_Q = 0$. Since B_0 is noetherian and M_0 is a finitely generated B_0-module, we have that K_0 is a finitely generated B_0-module. Choose generators $k_1^0, \dots, k_n^0$ for K_0, and let $k_1^\lambda, \dots, k_n^\lambda$, and $k_1, \dots, k_n$ be the image of these generators in K_λ respectively K. The map $P_0 A_\lambda \otimes_{A_\lambda} M_\lambda \to P_0 A \otimes_A M$ of the diagram is the same as the map $P_0 A_\lambda \otimes_{A_0} M_0 \to P_0 A \otimes_{A_0} M$. Since $P_0 A$ is the union of the modules $P_0 A_\lambda$, it follows from the definition of a tensor product that , if an element $h_\lambda \in K_\lambda$ lies in the kernel of $K_\lambda \to K$, then there is an index $\mu \geq \lambda$ such that h_λ lies in the kernel of $K_\lambda \to K_\mu$. We conclude that, since $K_Q = 0$, there is an element $t \in B \setminus Q$ such that $t k_i = 0$ for $i = 1, \dots, n$. However, the ring B is the union of the images of $B_\lambda \to B$. Hence there is an index λ such that t is the image of $t_\lambda \in B_\lambda \setminus Q_\lambda$. Consequently we have that $t_\lambda k_i^\lambda$ is in the kernel of $K_\lambda \to K$. Thus there is an index $\mu \geq \lambda$ such that $t_\mu k_i^\mu = 0$ for $i = 1, \dots, n$, where t_μ is the image t_λ by the map $B_\lambda \to B_\mu$. In particular we have that $t_\mu \in B_\mu \setminus Q_\mu$. Consequently we have that $(K_\mu)_{Q_\mu} = 0$. It follows
$\rightarrow$ from Lemma (?) applied to A_0, $I = P_0$, B_0, M_0, and $A' = A_\mu$, that $(M_\mu)_{Q_\mu}$ is a flat A_μ-module. The condition that $M_0 / P_0 M_0$ is flat over A_0 / P_0, is automatically fulfilled because P_0 is a maximal ideal.

2.13. Fitting ideals.

(2.13.1) Setup. Let M be a finitely generated A-module, and fix a non-negative integer r. Choose generators $m_1, \dots, m_s$ for M and let

$$N = \{(a_1, \dots, a_s) \in A^n : a_1 m_1 + \cdots + a_s m_s = 0\}.$$

Moreover, choose generators $\{n_\alpha = (a_{\alpha,1}, \dots, a_{\alpha,s})\}_{\alpha \in \mathcal{I}}$ for the A-module N. We denote by I_r the ideal in A generated by the $(s-r)$-minors of the $(\#\mathcal{I} \times s)$-matrix $A = (a_{\alpha,1}, \dots, a_{\alpha,s})_{\alpha \in \mathcal{I}}$. When $r < 0$ we let $I_r = (0)$ and when r is at least equal to s or the number of elements in $\mathcal{I}$ we let $I_r = A$. We have that $0 = I_{-1} \subseteq I_0 \subseteq \cdots \subseteq I_s = A = I_{s+1} = \cdots$.

(2.13.2) Note. Given an element $n = (a_1, \dots, a_s)$ in N. Let J be the ideal in A generated by the $s-r$ minors of the $((\#\mathcal{I}+1) \times s)$-matrix B obtained from A by adding $(a_1, \dots, a_s)$ as the first row. Then $J = I_r$. Indeed, it is clear that $I_r \subseteq J$ because the matrix A is formed from the rows $2, 3, \dots$ of B.

To prove the opposite inclusion we only have to show that the $s-r$-minors containing the first row of B are contained in I_r. However, we have that $n = b_1 n_{\alpha_1} + \cdots b_s n_{\alpha_s}$, for some b_i in A, and α_i in $\mathcal{I}$. Hence, the first row of B is a sum of rows $\alpha_1 + 1, \cdots, \alpha_s + 1$ multiplied with $b_1, \dots, b_s$ respectively. Hence the $(s-r)$-minors containing the first row can be expanded as a sum of the $(s-r)$-minors containing rows $\alpha_1 + 1, \dots, \alpha_s + 1$ multiplied by $b_1, \dots, b_s$. We consequently have that $J \subseteq I_r$.

By (transfinite, if necessary) induction, we obtain that the ideal in A obtained from the $(s-r)$-minors of the matrix obtained by adding to A rows coming from any set of elements of N, is equal to I_r. In particular we obtain that the ideal I_r is independent of the choise of generators n_α. Indeed, if we chose another set of generators for M, we have that the ideal obtained from the union of the two sets of generators is equal to the ideal obtained from each set.

(2.13.3) Note. Let m be an element of M. Moreover, let

$$P = \{(a, a_1, \dots, a_s) \in A^{s+1} : am + a_1 m_1 + \cdots + a_s m_s = 0\}.$$

Then, if we write $m = -b_1 m_1 + \cdots + b_s m_s$, with b_i in A, we have that P contains the element $p = (1, b_1, \dots, b_s)$, and that P is generated by the element p and elements $\{p_\alpha = (0, a_{\alpha,1}, \dots, a_{\alpha,s})\}_{\alpha \in \mathcal{I}}$, where $\{(a_{\alpha,1}, \dots, a_{\alpha,s})\}_{\alpha \in \mathcal{I}}$ are generators for N. Let J be the ideal in A generated by the $(s-r+1)$-minors of the $((\#\mathcal{I}+1) \times (s+1))$-matrix whose first row is the element p and whose $(\alpha+1)$'st row is the elements
→ p_α. It follows from Note (2.13.2) that J is independent of the choise of generators of P. It is clear that we have an equality $J = I_r$. By induction on t we obtain that snitt

the ideal in A generated by the $(s-r+t)$-minors of the $(\#I+t)\times(s+t)$-matrix obtained from $m_1,\dots,m_s$ and t additional elements, is equal to I_r.

In particular we have that the ideal I_r is independent of the choise of generators $m_1,\dots,m_s$ of A. Indeed, if we had another set of generators we have that the ideal obtained from the union of the two sets of generators is equal to the ideal obtained from each set.

(2.13.4) Definition. Let M be a finitely generated A-module and r a non-negative integer. In Setup (2.13.1) we chose generators $m_1,\dots,m_s$ of M and defined I_r to be the ideal generated by the $(s-r)$-minors of the matrix whose rows are generators for the A-module $N=\{(a_1,\dots,a_s)\in A^s\colon \sum_{i=0}^s a_i m_i=0\}$. In Note (2.13.2) we proved that I_r is independent of the choise of generators for N, and in Note (2.13.3) we showed that it is independent of the choise of generators of M. The ideal therefore depends on M only. We denote it by $F_r(M)$ and call it the *r-th Fitting ideal of the module M* .

(2.13.5) Remark. We have an inclusion $F_{r-1}(M)\subseteq F_r(M)$.

(2.13.6) Note. Given generators $m_1,\dots,m_s$ for the A-module M. We obtain a surjection

$$A^s\to N$$

and it is clear that N of Setup (2.13.1) is the kernel to this map. The choise of generators $\{n_\alpha\}_{\alpha\in\mathcal{I}}$ for N gives an exact sequence

$$A^{\oplus\mathcal{I}}\to B^{\oplus s}\to M\to 0$$

of A-modules. It follows from Definition (2.13.4) that $F_r(M)$ is the ideal of A generated by the $(s-r)$-minors to the $(\#\mathcal{I})\times s$-matrix $A^{\oplus\mathcal{I}}\to A^{\oplus s}$.

(2.13.7) Lemma. *Let B be an A-algebra and let M be a finitely generated A-module. Then we have an equality*

$$F_r(M)B=F_r(M\otimes_A B)$$

of ideals in B.

Proof. It follows from Remark (2.13.5) that we have a presentation

$$A^{\oplus\mathcal{I}}\xrightarrow{\alpha}A^{\otimes s}\to M\to 0$$

of M. We obtain a presentation

$$A^{\oplus\mathcal{I}}\otimes_A B=B^{\oplus\mathcal{I}}\xrightarrow{\alpha\otimes\mathrm{id}_B}A^{\oplus s}\otimes_A B=B^s\to M\otimes_A B\to 0$$

of $M\otimes_A B$. It follows from Remark (2.13.5) that $F_r(M)$ and $F_r(M\otimes_A B)$ are generated by the $(s-r)$-minors of α respectively $\alpha\otimes\mathrm{id}_B$.

(2.13.8) Proposition. *Assume that A is a local noetherian ring and M a finitely generated module. Then M is free of rank r if and only if $F_r(M) = A$ and $F_{r-1}(M) = 0$.*

Proof. When M is free of rank r we have a presentation $0 \to A^r \to M \to 0$ and we obtain that $F_{r-1}(M) = 0$ and that $F_r(M) = A$ by definition.

Conversely assume that $F_r(M) = A$ and that $F_{r-1}(M) = 0$. Choose a presentation $A^t \xrightarrow{\alpha} A^s \to M \to 0$. If $F_r(M) = A$ there is an $(s-r)$-minor of the matrix α which is invertible. We can reorder the basis vectors for A^s and A^t such that this minor is the one of the upper left corner of α. The $(s-r) \times (s-r)$-matrix in the upper left corner then defines an isomorphism between the A-modules spanned by the first $(s-r)$ basis vectors in A^s respectively A^t. By choosing new bases for these A-modules we may assume that the $(s-r) \times (s-r)$ matrix in the upper left corner is the identity matrix. We can now use row and column operatins on α to put α in a form where the lower left $r \times (s-r)$ corner, respectively the upper right $(s-r) \times (t-s+r)$ corners are equal to zero. Since we have assumed that $F_{r-1}(M) = 0$ we have that the left $r \times (t-s+r)$ corner also is zero. If follows immediately from the form of the matrix α that M is free of rank r.

2.14. Formal smoothness.

(2.14.1) Definition. ([G] 20, 0.19.9.1) Given a ring k and a k-algebra A. Let B be an A-algebra. We say that B is *formally smooth over A relative to k* , if, for every A-algebra C, for every nilpotent ideal I in C, and for every A-algebra homomorphism $u_0: B \to C/I$ that factors via C as $B \xrightarrow{u} C \xrightarrow{\varphi} C/I$, where u is a k-algebra homomorphism and φ is the residue map, we also have a factorization $B \xrightarrow{v} C \xrightarrow{\varphi} C/I$, where v is an A-algebra homomorphism. When $A = k$ we say that B is *formally smooth over* A if, for every A-algebra C, for every nilpotent ideal I in C, and for every A-algebra homomorphism $u_0: B \to C/I$, we have a factorization $B@ > v >> C \xrightarrow{\varphi} C/I$, where w is an A-algebra homomorphism. When the latter factorization is *unique* we say that B is *étale* over A.

→ **(2.14.2) Remark.** It suffices to assume in Definition (?) that every homomorphism $u_0: B \to C/I$ that factors via $B \xrightarrow{u} C \xrightarrow{\varphi} C/I$, also factors via $B \xrightarrow{v} C \xrightarrow{\varphi} C/I$ for all ideals I in C such that $I^2 = 0$. Indeed we can sucessively lift $A \to B/I^{i-1}$ to $A \to B/I^i$, and reason by induction.

(2.14.3) Lemma. *We have that an A-algebra B is formally étale over A if and only if B is formally smooth over A and $\Omega^1_{B/A} = 0$.*

Proof. If $\Omega^1_{B/A} = 0$ we can not have two A-algebra homomorphism $u, v: B \to C$ which give the same map under composition with $\varphi: C \to B$. Indeed, then $u - v: B \to C$ would be a non-trivial A-derivation when we consider C as a B-algebra via φ or ψ.

Conversely, if B is formally étale we have that the isomorphism $B \to B \otimes_A B/I$ factors via $B \to B \otimes_A B/I^2$ in a unique way, where I is the ideal in $B \otimes_A B$ that defines the diagonal. However, then the two A-algebra homomorphisms that send an element to the first, respectively the second, factor give factorizations. Consequently they are equal, and thus $1 \otimes_A b - b \otimes_A 1$ is in I^2 for all b in B. In others words we have that $I = I^2$, and consequently that $\Omega^1_{B/A} = 0$.

(2.14.4) Proposition. *Given a ring k. Then:*

(1) *The ring k is a formally smooth k-algebra.*
(2) *Given a formally smooth k-algebra, and a formally smooth A algebra B. Then B is a formally smooth k-algebra.*
(3) *Given a formally smooth k-algebra A, and a k-algebra k'. Then we have that $k' \otimes_k A$ is a formally smooth k'-algebra.*
(4) *Given a formally smooth k-algebra A, and S and T multiplicatively closed subsets of k respectively A, such that the image of S in A is contained in T. Then we have that $T^{-1}A$ is a formally smooth $S^{-1}k$-algebra.*

snitt

(5) *Given k-algebras A_i for $i = 1, \dots, n$. Then we have that $\prod_{i=1}^{n} A_i$ is a formally smooth k-algebra if and only if all the A_i are formally smooth k-algebras.*

Proof. All the properties are easy to check.

(2.14.5) Proposition. *Given a formally smooth k-algebra A and let I be an ideal in A. Then we have that*

$$\Omega^1_{A/k} \otimes_k A/I$$

is a projective A/I-module.

Proof. Given a surjection $u\colon L \to M$ of A/I-modules. It suffices to prove that the map

$$\mathrm{Hom}_A(\Omega^1_{A/k}, L) \xrightarrow{\mathrm{Hom}(\mathrm{id}_{\Omega^1_{A/k}}, u)} \mathrm{Hom}_A(\Omega^1_{A/k}, M)$$

is surjective. In other words it suffices to prove that the map

$$\mathrm{Der}_k(A, L) \xrightarrow{Der_k(\mathrm{id}_A, u)} \mathrm{Der}_k(A, M)$$

is surjective. Let $D\colon B \to M$ be a k-derivation. Consider the homomorphism of k-algebras

$$v\colon A \to A/I \oplus M$$

defined by $v(a) = (u(a), Da)$, where $u\colon A \to A/I$ is the residue map, and where we consider $A/I \oplus M$ as an A/I-algebra by $(0, m)(0, m') = (0, 0)$, for all m and m' in M. Since A is formally smooth over k we have a lifting

$$w\colon A \to A/I \oplus L$$

of $A/I \oplus L \xrightarrow{\mathrm{id}_A \oplus u} A/I \oplus M$. The map w gives a k-derivation $E\colon A \to L$ defined by $w(a) = (u(a), E(a))$ and, since $(\mathrm{id}_A \oplus u)w = v$, we have that $uE(a) = D(a)$. Consequently D' is a lifting of D.

(2.14.6) Theorem. *Given a separable field extension L of a field K. Then L is K-étale. If k is a subfield of K we have that $\Omega^1_{L/k} = \Omega^1_{K/k} \otimes_K L$.*

Proof. Let C be a K-algebra and I and ideal in C such that $I^2 = 0$. Moreover given a map $\varphi\colon L \to C/I$ of K-algebras. Given a field L' between K and L which is finitely generated over K. Then we have that $L' = K(\alpha)$ where α has a minimal polynomial $f(x)$ such that $f'(\alpha) \neq 0$. We want to lift $\varphi|L'$ to C. To this end we have to find an element c of C such that $f(c) = 0$ and $c \equiv \varphi(\alpha) \pmod{I}$. Choose a c' in C that maps to $\varphi(\alpha)$ by the residue map $C \to C/I$. Then we have that $f(c') \equiv \varphi(f(\alpha)) \equiv 0 \pmod{I}$. We have that $I^2 = 0$ for each $d \in I$. Consequently

$$f(c'+d) = f(c') + f'(c)d$$

in C. However, we have that $f'(\alpha)$ is a unit in L. Consequently $\varphi(f'(\alpha)) \equiv f'(c') \pmod{I}$ is a unit in C/I, and thus $f'(c)$ is a unit in C. With $d = -f(c')/f'(c)$ we have that $f(c'+d) = 0$. We choose $c = c' + d$ and the K-algebra map $L' = K(\alpha) \to C$ which sends α to c becomes a lifting. We see that this lifting is unique. Consequently we can lift the map to the whole of L.

The second assertion of the Theorem follows from the first part and from the $\to$ equality $\Omega^1_{L/K} = 0$ of Lemma (?).

(2.14.7) Theorem. ([G] 20, 0.22.6.1) *Given a formally smooth k-algebra A and an ideal I in A. The following two assertions are equivalent:*

(1) *The algebra A/I is formally smooth over k.*
(2) *The canonical map*

$$\delta\colon I/I^2 \to \Omega^1_{A/k} \otimes_A A/I,$$

which sends the class of an element a in I to $da \otimes_A 1$, is left invertible.

Proof. To prove that (1) implies (2) we note that δ is left invertible if and only if $\mathrm{Der}_k(A, M) \to \mathrm{Hom}_{A/I}(I/I^2, M)$, which sends a derivation $D\colon A \to N$ to the induced A/I-homomorphism $I/I^2 \to M$ is surjective for all A/I-modules M. We fix an A/I-module homomorphism $u\colon I/I^2 \to M$. Let $C = A/I^2 \oplus M/\{(a, m) | a \text{ is the class of an element in I and } u(a) = m\}$. Then C is an A/I-algebra when the multiplication is defined by $(0, m)(0, m') = (0, 0)$ for all m and m' in M. We have a canonical exact sequence

$$0 \to M \to C \xrightarrow{\sigma} A/I \to 0.$$

The identity on A/I factors via a map $v\colon A/I \to C$ such that $\sigma v\varphi = \varphi$, where $\varphi\colon A \to A/I$ is the residue map. Moreover, we have a canonical homomorphism $\psi\colon A \to A/I^2 \to C$ such that $\sigma\psi = \varphi$. We obtain a map $\psi - v\varphi\colon A \to C$ such that $\sigma(\psi - v\varphi) = 0$. Consequently we have that the image of $\psi - v\varphi$ is in M and thus $\psi - v\varphi$ induces a k-derivation $w\colon A \to M$, and thus an A/I-linear map $w\colon I/I^2 \to M$. If $\overline{a}$ in I/I^2 is the image of a in I we have that $w(\overline{a})\psi(a) - v\varphi(a) = \psi(a) = u(\overline{a})$, where the last inequality holds since $\psi(a) = (\overline{a}, 0) = (0, u(\overline{a}))$ considered as a subset of C for all $a \in I$.

To prove that (2) implies (1) we note that it follows from assertion (2) that the map $\mathrm{Der}_k(A, M) \to \mathrm{Hom}_{A/I}(I/I^2, M)$ is surjective for all A/I-modules M. Given a k-algebra B and an ideal J in B such that $J^2 = 0$. Assume that we have a map $u: A/I \to B/J$ of k-algebras. Since A is formally smooth over k we have that the composite map $A \xrightarrow{\varphi} A/I \xrightarrow{u} B/J$ lifts to a map $v: A \to B$ of k-algebras. We have that $\psi v = u\varphi$ where $\psi: B \to B/J$ is the residue map. If $a \in I$ we have that $v(a) \in J$. The map v induces a map $w: I/I^2 \to B$ since $J^2 = 0$ and the image of w is in J. We consequently have a k-derivation $D: A \to B$ which induces w. Since D has image in J it follows that $D(a) = v(a)$ for all $a \in I$. We therefore obtain a k-algebra homomorphism $v - D: A \to B$ which factors via A/I, and for $a \in A$ we have that $\psi(v - D)(a) = \psi v(a) - \psi D(a) = \psi v(a) = u\varphi(a)$. Thus the map $A/I \to B$ induced by $v - D$ lifts $u: A/I \to B/J$.

(2.14.8) Corollary. *Assume that A and A/I are formally smooth over k. Then we have that I/I^2 is a projective A/I-module.*

Proof. We have that I/I^2 is a direct summand of the module $\Omega^1_{A/k} \otimes_k A/I$, which
$\to$ is projective by Lemma (?).

(2.14.9) Theorem. ([G] 20, 0.20.5.7, [M] 28.4) *Given a k-algebra A and let $u: A \to B$ be a map of k-algebras. The following two assertions are equivalent:*

(1) *The algebra B is formally smooth over A relative a k.*
(2) *The B-module homomorphism*

$$\Omega^1_{A/k} \otimes_A B \to \Omega^1_{B/k}$$

is left invertible.

Proof. We first prove that (1) implies (2). To show that the map in (2) is left invertible we must prove that the map $\mathrm{Der}_k(B, M) \to \mathrm{Der}_k(A, M)$ is surjective for all B-modules M. Fix a k-derivation $D: A \to M$. Consider the A-algebra $B \oplus M$, where M is an ideal in $B \oplus M$ with $M^2 = 0$, and where the A-algebra structure is given by $\varphi: A \to B \oplus M$, where $\varphi(a) = (u(a), D(a))$. The identity map on B factors via the k algebra homomorphism $B \to B \oplus M$, which sends B to the first factor. Since B is formally smooth over A relative to k the identity on B factors via an A-algebra homomorphism $v: B \to B \oplus M$. The map $E: B \to M$ into the second factor is a k-derivation, and since w is an A-algebra homomorphism we obtain that $E(u(a)b) = E(ab) = bD(a) + u(a)E(b)$, which for $b = 1$ gives that $D = Eu$. Consequently we can lift D to E.

To show that (2) implies (1) we let C be an A-algebra via the homomorphism $\psi: A \to C$, and let $I \subseteq C$ be an ideal with $I^2 = 0$.

Given an A-algebra homomorphism $v_0: B \to C/I$ that factors via a k-algebra homomorphism $v: B \to C/I$. Then the A-algebra structure gives that $v_0 u = \varphi\psi$ and the lifting gives that $\varphi v = v_0$. Since we have that $\varphi(\psi - vu)(a) = \varphi\psi -$

$\varphi vu(a) = v_0u - v_0u(a) = 0$, for all $a \in A$, we obtain a k-derivation $D: A \to I$ defined by $D(a) = \psi(a) - vu(a)$, for all $a \in A$.

We can consider I as a B-module via v, and for $c \in I$ and $a \in A$ we have that $vu(a)c = (vu-\psi)(a)c+\psi(a)c = \psi(a)c$, such that this B-module structure induces an A-module structure that coincides with that induced by ψ. It follows from (2) that D can be lifted to a k-derivation $E: B \to I$, that is $Eu = D_0$. We define the homomorphism $w: B \to C$ by $w(b) = v(b) + E(b)$ for $b \in B$. Since we have that $E(b) \in I$ and $I^2 = 0$ this is a k-algebra homomorphism. Moreover we have that $w(u(a)b) = w(u(a))w(b) = (vu(a) + Eu(a))w(b) = (\psi(a) - Da + Da)w(b) = \psi(a)w(b)$. Consequently w is an A-algebra homomorphism.

Finally we note that $\varphi w(b) = \varphi v(b) + \varphi E(b) = \varphi v(b) = v_0(b)$ since $E(b) \in I$. Consequently we have that w lifts v_0.

(2.14.10) Theorem. ([G] 20, 0.19.5.4) *Given a formally smooth k-algebra A and an ideal $I \subseteq A$ such that A/I is smooth over k. The following two assertions are equivalent:*

(1) *The k-algebra A is formally smooth.*
(2) *The A/I-module I/I^2 is projective and the canonical homomorphism*

$$\varphi: \operatorname{Sym}_{A/I}(I/I^2) \to \sum_{i=0}^{\infty} I^i/I^{i+1}$$

is bijective.

→ *Proof.* We first show that (1) implies (2). It follows from Proposition (?) that I/I^2 is a projective A/I-module. Let

$$E_n = A/I^{n+1} \text{ and } F_n = \operatorname{Sym}_{A/I}(I/I^2)/I^{n+1}\operatorname{Sym}_{A/I}(I/I^2).$$

We have that the ideal I/I^{n+1} is nilpotent in E_n. Consequently the identity of A/I factors as $A/I \xrightarrow{f} E_n \to A/I$, where f is a k-algebra homomorphism. We have seen that I/I^2 is a projective A/I-module. Consequently the identity on I/I^2 factors as $I/I^2 \xrightarrow{g} I/I^{n+1} \to I/I^2$, where g is A/I-linear. From the homomorphisms f and g we obtain a homomorphism $\operatorname{Sym}_{A/I}(I/I^2) \to E_n$ of A/I-algebras, where E_n is an A/I-algebra via f. It follows from the definition of g that the homomorphism is zero on $I^{n+1}\operatorname{Sym}_{A/I}(I/I^2)$. Thus we obtain a k-algebra homomorphism $v: F_n \to E_n$ The latter map is surjective since $f(\bar{a}) - \bar{a} \in I/I^{n+1}$, for all $a \in I$, and $g(\bar{a}) - \bar{a} \in I^2/I^{n+1}$ for all $a \in I$. From the definition of v if also follows that $\operatorname{gr}^0 v$ and $\operatorname{gr}^1 v$ are the identities on A/I respectively I/I^2. We first conclude that the kernel N of v lies in IF_n, and consequently is nilpotent, and then that $\operatorname{gr}^i v = \varphi^i$ for $i \leq n$.

Since A is formally smooth over k the canonical map $p_n\colon A \to E_n = A/I^{n+1}$ factors via $A \xrightarrow{w} F_n \xrightarrow{v} E_n$, where w is a k-algebra homomorphism. Since p_n is an A-algebra homomorphism and $\mathrm{gr}^0\, v$ is the identity, we have that $w(I) \subseteq IF_n$, and consequently that $w(I^{n+1}) = 0$. Thus w factors as $A \xrightarrow{p_n} A/I^{n+1} = E_n \xrightarrow{w'} F_n$, where the composite map $E_n \xrightarrow{w'} F_n \xrightarrow{v} E_n$ is the identity, since $vw = p_n$. Consequently we have that $\mathrm{gr}^0(w')$ and $\mathrm{gr}^1(w')$ are the identity on A/I respectively I/I^2. However we have that $\mathrm{gr}(E_n)$ is generated by I/I^2 as an A/I-algebra. We obtain that the composite map

$$\mathrm{gr}^i(F_n) \xrightarrow{\varphi_i} \mathrm{gr}^i(E_n) \xrightarrow{\mathrm{gr}^i(w')} \mathrm{gr}^i(F_n)$$

is the identity for $i \leq n$. In particular we have that φ_i is injective for $i \leq n$.

(2) $\Rightarrow$ (1). Her må vi sannsynligvis bruke at Grothendiecks reduksjon fra ikke noetherske til noetherske, som er gjort i eget kapittel.

(2.14.1) Lemma. *Given a ring A and a finitely generated A-module M. Moreover, let F be a projective A-module and let $\varphi\colon M \to F$ be an A-linear map. Given a prime ideal P in A. The following three assertions are equivalent:*

(1) *The map $\varphi_P\colon M_P \to F_P$ is left invertible.*
(2) *There are elements $x_1, \dots, x_m$ in M and $v_1, \dots, v_m$ in $\mathrm{Hom}_A(F, A)$ such that $M_P = \sum_{i=1}^m A_P x_i$ and $\det(v_i(\varphi(x_j))) \notin P$.*
(3) *There is an element $f \in A \setminus P$ such that $\varphi_f\colon M_f \to F_f$ is left invertible.*

The set of prime ideals P in $\mathrm{Spec}\, A$ that satisfy the conditions of the Lemma is open.

Proof. We have that F is a direct summand of a free A-module. Since M is finitely generated we have that $\varphi(M)$ is contained in a fintely generated free module. The conditions (1), (2), (3) become the same if we replace F with the free submodule containing M. Consequently we may assume that F is free and finitely generated.

We first prove that (1) implies (2). It follows from (1) that M_P is a free A_P-module. Choose $x_1, \dots, x_n$ in M that give a basis of M_P. Then we have that $\varphi_P(x_1), \dots, \varphi_P(x_m)$ is part of a basis of F_P. Consequently there are linear maps $v_i'\colon M_P \to A_P$ such that $v_i'(\varphi_P(x_j)) = \delta_{i,j}$. Since F_P is a free finitely generated A-module we can write $v_i' = s_i^{-1} v$ for some $s_i \in A \setminus P$, where $v_i \in \mathrm{Hom}_A(F, A)$. It is clear that $\det(v_i(\varphi(x_j)) \notin P$.

We next show that (2) implies (3). Since M is a finitely generated A-module and $M_P = \sum_{i=1}^m A_P x_i$ there is an element $g \in A \setminus P$ such that $M_g = \sum_{i=1}^m A_g x_i$. Let $d = \det(v_i(\varphi(x_j))$ and $f = gd$. Then we have that $M_f = \sum_{i=1}^m A_f x_i$ and that d is a unit in A_f. The maps v_i give a map $B\colon F_f \to A_f^m$. Since we have that $\det(v_i(\varphi(x_j))$ is invertible in A_f we can find a matrix $B' = (b_{i,j}')$ such that $B'B = \mathrm{id}_{A_f}$. Then

$B' \colon A_f^m \to F_f$ is a map that sends $(a_1, \dots, a_m)$ to $(\sum_{i=1}^m a_k b'_{1,i}, \dots, \sum_{i=1}^m a_k b_{m,i})$. We obtain that the composite map $M_f \xrightarrow{\varphi_f} F_f \xrightarrow{B} A_f^m \to M_f$, where the right hand map sends the vector $(0, \dots, 0, a, 0, \dots, 0)$ with the 1 in the i'th coordinate, to x_i, is the identity. We have thus found a left inverse to φ_f.

It is clear that (3) implies (1).

Finally we notice that the prime ideals that satisfy (3) is open.

(2.14.12) Lemma. *Given a ring A and a finitely generated A-module M. Then every surjective A-linear homomorphism $f \colon M \to M$ is an automorphism.*

Proof. We consider M as an $A[t]$-module via the action $tm = \alpha(m)$ of t on M. By assumption we have that $M = tM$. It follows from Nakayama's Lemma that there is an element $\varphi(t) \in A[t]$ such that $(1 + t\varphi(t))M = 0$. However, then we have that $tm = 0$, for some $m \in M$, implies that $m = 0$. Hence f is injective.

(2.14.13) Lemma. *Given a ring A and an ideal I in A. Moreover, let $\varphi \colon M \to F$ be an A-linear homomorphism between A-modules, where F is projective. Assume that one of the following conditions hold:*

(1) *The ideal I is nilpotent.*
(2) *The A-module M is finitely generated and $I \subseteq \operatorname{rad}(A)$.*

Then the map φ is left invertible if and only if the induced map

$$\psi \colon M/IM \to F/IF,$$

of A/I-modules, is left invertible.

Proof. It is clear that if φ is left invertible, then ψ is, even without the conditions of the Lemma.

Assume conversely that ψ is left invertible with inverse $\xi \colon F/IF \to M/IM$. Since F is projective we can lift ξ to $\zeta \colon F \to M$. We obtain a map $\alpha \colon M \xrightarrow{\varphi} F \xrightarrow{\zeta} M$. Then we have that $M = \alpha(M) + IM$ since α induces the identity modulo I. It follows from Nakayama's Lemma that $M = \alpha(M)$. Consequently it follows from
$\to$ Lemma (?) that α is an isomorphism. Hence $\alpha^{-1}\zeta$ is a left inverse to φ.

(2.14.14) Theorem. ([M], 29.E, Theorem 64) *Given a formally smooth k-algebra A and an ideal I of A. Let R be a prime ideal in A/I and Q the inverse image of R in A. Denote by P the restriction of Q to k and $\kappa(Q)$ the residue field of A_Q, or equivalently of $(A/I)_R$. The following assertions are equivalent:*

(1) *The ring $(A/I)_R$ is formally smooth over k, or k_P.*
(2) *The map*
$$(I/I^2) \otimes_{A/I} \kappa(Q) \to \Omega^1_{A/k} \otimes_A \kappa(Q)$$

is left invertible

(3) *The map*
$$(I/I^2) \otimes_{A/I} A/I_R \to \Omega^1_{A/k} \otimes_A (A/I)_R$$
is left invertible.

(4) *There are elements* $F_1, \dots, F_r$ *in* I *and* $D_1, \dots, D_r$ *in* $\mathrm{Der}_k(A, A/I)$ *such that* $IA_Q = \sum_{i=1}^r A_Q F_i$ *and* $\det(D_i F_j) \notin R$.

(5) *There is an element* $f \in (A/I) \setminus R$ *such that* $(A/I)_f$ *is formally smooth over* k.

When the above conditions hold the set
$$\{R \in \mathrm{Spec}(A/I) \colon (A/I)_R \textit{ is smooth over } k\},$$
is open in $\mathrm{Spec}(A/I)$.

Proof. We first prove that (1) implies (3). We know that A_Q is formally smooth over k, and we have that $(A/I)_R = A_Q/IA_Q$, and $\Omega^1_{A_Q/k} = \Omega^1_{A/k} \otimes_A A_Q$. Hence it follows from Theorem (?) that (1) implies (3).

It is clear that (3) implies (2).

It follows from Proposition (?) that $\Omega^1_{A/k} \otimes_A (A/I)_R$ is a projective $(A/I)_R$-module. Consequently the it follows from Lemma (?) that (2) implies (3).

That (3) implies (4) follows from Lemma (?) applied to the A/I-linear map $I/I^2 \to \Omega^1_{A/k} \otimes_A A/I$.

We next prove that (4) implies (5). It follows from Lemma (?) applied to the A/I-linear map $I/I^2 \to \Omega^1_{A/k} \otimes_A A/I$ that there is an element $f \in (A/I) \setminus R$ such that $I/I^2 \otimes_{A/I} (A/I)_f \to \Omega^1_{A/k} \otimes_A (A/I)_f$ is left invertible. Consequently it follows from Theorem (?) that $(A/I)_f$ is formally smooth over A.

It is clear that (5) implies (1).

3.1. Products of algebraic schemes.

(3.1.1) Setup. We fix a field k.

(3.1.2) Lemma. [EGA 24, IV, 4.2.1] *Given a finitely generated field extension K of k and a field extension L of k. Then all the associated prime ideals in $K \otimes_k L$ are minimal.*

Denote by E be the residue field in an associated prime ideal of $K \otimes_k L$. Then we have that

$$\operatorname{td.deg.}_L(E) = \operatorname{td.deg.}_k K.$$

Proof. Note that the ring $K \otimes_k L$ is noetherian because K is the quotient field to the residue ring of a polynomial ring $k[x_1, \dots, x_n]$, and consequently $K \otimes_k L$ is the quotient ring, in a multiplicative system, of a residue ring of the polynomial ring $L[x_1, \dots, x_n]$.

Since K is a finitely generated field extension of k we have that K is a finitely generated field extension of the quotient field of a polynomial ring $k[x_1, \dots, x_n]$. The ring $k[x_1, \dots, x_n] \otimes_k L = L[x_1, \dots, x_n]$ is an integral domain. Consequently the ring $k(x_1, \dots, x_n) \otimes_k L$ is an integral domain. The quotient field of $k(x_1, \dots, x_n) \otimes_k L$ is $L(x_1, \dots, x_n)$.

The ring $k(x_1, \dots, x_n) \otimes_k L$ is a subring of $K \otimes_k L$, since $k(x_1, \dots, x_n)$ is a subring of K. Moreover, we have that K is flat over $k(x_1, \dots, x_n)$, and thus $K \otimes_k L$ is flat over $k(x_1, \dots, x_n) \otimes_k L$. Hence the associated primes in $K \otimes_k L$ intersect the subring $k(x_1, \dots, x_n) \otimes_k L$ in (0) because the non zero elements of $k(x_1, \dots, x_n) \otimes_k L$ are not zero divisors in $K \otimes_k L$.

We have that $K \otimes_k L$ is an integral extension of $k(x_1, \dots, x_n) \otimes_k L$ since K is
$\rightarrow$ an algebraic extension of $k(x_1, \dots, x_n)$. Consequently (113) [A-M, Cor. 5.9] the associated prime ideals in $K \otimes_k L$ are minimal. We have proved the first part of the Lemma.

To prove the last part we note that the residue field E of $K \otimes_k L$ in an associated prime is algebraic over the residue field $L(x_1, \dots, x_n)$ of $k(x_1, \dots, x_n) \otimes_k L$ in the zero ideal. Consequently we have that $\operatorname{td.deg.}_L E = \operatorname{td.deg.}_L L(x_1, \dots, x_n) = n = \operatorname{td.deg.}_k K$.

(3.1.3) Lemma. *Given a morphism $f\colon X \to Y$ of schemes X and Y where Y is irreducible. Denote by η the generic point of Y. Then there is a bijection between the irreducible components of the fiber $f^{-1}(\eta)$ and the components of X that dominate Y.*

In particular, when X is irreducible with generic point ξ, there is a bijection between the components of $X \times_k Y$ and the components of $\kappa(\xi) \otimes_k \kappa(\eta)$ that dominate both factors.
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Proof. Let Z be a component of $X \times_k Y$ that dominates Y and denote by ζ the generic point of Z. Then $f(\zeta) = \eta$ and consequently $\zeta \in f^{-1}(\eta)$. The closure of ζ in $f^{-1}(\eta)$ is a component of $f^{-1}(\eta)$, because otherwise it would be a subset of a component whose generic point would be a generic point for a component of X that strictly contains Z.

Conversely, let ζ be the generic point for a compoent of $f^{-1}(\eta)$. Let Z be the closure of ζ in Z. Then Z will dominate Y, and Z is a component of X, beacuse otherwise it would be contained in a component whose generic point would be in $f^{-1}(\eta)$ and the closure of this generic point in $f^{-1}(\eta)$ will be an irreducible set that strictly contains the given component of $f^{-1}(\eta)$.

The components of $X \times_k Y$ that dominate the second factor correspond, as we have seen, to the components of $X \times_k \kappa(\eta)$. Such a component Z will dominate the first factor if and only if the generic point ζ of Z is in $\kappa(\xi) \times_k \kappa(\eta)$. The second part therefore follows using the first part to the morphism $X \otimes_k \kappa(\eta) \to X$.

(3.1.4) Lemma. *Let A be a ring and S a multiplicatively closed system of A. Let Q be a prime ideal in S^{-1} and P the contraction of Q to A. Then the canonical map $A_P \to (S^{-1}A)_Q$ is an isomprhism.*

Proof. We have a map $S^{-1}A \to A_P$ since $P \cap S = \emptyset$. The elemnts in $S^{-1}A \setminus Q$ can be written in the form a/s with $a \in A \setminus P$ and $s \in S$. Then $s \notin P$ and the element a/s in $S^{-1}A \setminus Q$ maps to the element a/s in $A_P \setminus PA_P$. It follows that the above map induces a map $(S^{-1}A)_Q \to A_P$. It is clear that this map is the inverse of the map of the Lemma.

(3.1.5) Proposition. [EGA 24, IV$_2$, 4.2.4] *Given a variety X and an integral noetherian scheme Y over the field k. Then the following three assertions hold:*

(1) *The irreducible components of $X \times_k Y$ correspond bijectively to the minimal prime ideals in $R(X) \otimes_k R(Y)$.*

(2) *The local ring $\mathcal{O}_{X \times_k Y, \zeta}$ to $X \times_k Y$ in a generic point ζ of an irreducible component of $X \times_k Y$ is isomorphic to the fraction ring of $R(X) \otimes_k R(Y)$ in the corresponding minimal prime.*

(3) *Let E be the residue field of $X \times_k Y$ in the generic point ζ. Then we have that*

$$\dim X = \operatorname{td.deg.}_{R(Y)} E.$$

In particular, if Y is a variety all the components of $X \times_k Y$ have dimension $\dim X + \dim Y$.

Proof. Note that since X is a variety and Y is noetherian we have that $X \times_k Y$ is noetherian.

(1) Let ξ and η be the generic points for X respectively Y. Then we have that $R(X) = \kappa(\xi)$ and $R(Y) = \kappa(\eta)$. Since the projections to the factors are flat it

follows from Remark (?) that the components Z of $X \times_k Y$ dominate X and Y. It follows from Lemma (3.1.3) that the irreducible components of $X \times_k Y$ correspond to the components of $\kappa(\xi) \times_k \kappa(\eta)$, and these correspond to the minimal primes of $\kappa(\xi) \times_k \kappa(\eta)$ by Lemma (3.1.2).

(2) It suffices to show that (2) holds when X and Y are affine. Let $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. It follows from assertion (1) that each minimal prime P in $A \otimes_k B$ is the contraction of a minimal prime ideal Q in $R(X) \otimes_k R(Y)$. However, $R(X)$ and $R(Y)$ are the localizations of A respectively B in the sets consisting of the non zero elements. Consequently we have that $R(X) \otimes_k R(Y)$ is the fraction ring of $A \otimes_k B$ in a multiplicative set. Hence assertion (2) follows from Lemma (3.1.4).

(3) It follows from assertions (1) and (2) that the dimension of every irreducible component Z of $X \times_k Y$ is equal to the transcendece degree of the residue field of $R(X) \otimes_k R(Y)$ in a prime ideal that corresponds to the generic point for Z. It follws from Lemma (3.1.2) that $\operatorname{td.deg.}_{R(Y)} E = \operatorname{td.deg.}_k R(X) = \dim X$. When Y is a variety we get, reasoning the same way, that $\operatorname{td.deg.}_{R(X)} E = \operatorname{td.deg.}_k R(Y) = \dim Y$. Hence the dimension of each irreducible component is

$$\operatorname{td.deg.}_k E = \operatorname{td.deg.}_k R(X) + \operatorname{td.deg.}_{R(X)} E = \dim X + \dim Y.$$

(3.1.6) Remark. It follows from Proposition (?) that in assertion (1) above we have that $\operatorname{ass}(X \times_k Y) = \operatorname{ass}(\kappa(\xi) \otimes_k \kappa(\eta)) = \operatorname{ass}(R(X) \otimes_k R(Y))$, where ξ and η are the generic point of X regaspectively Y. It follows from Lemma (?) that $X \times_k Y$ does not have imbedded components.

(3.1.7) Proposition. [EGA 24, IV$_2$, 4.2.6] *Given an algebraic scheme X and a scheme Y that is noetherian and defined over the field k. The irreducible components of $X \times_k Y$ are exactly the irreducible components of the closed subsets $X' \times_k Y'$, where X' and Y' are irreducible components in X respectively Y, given their reduced structure. Moreover, the irreducible components of $X' \times_k Y'$ dominate both factors.*

Proof. Let X' and Y' be irreducible compoents in X respectively Y and let Z' be an irreducible component of $X' \times_k Y'$. Then Z' is contained in a component Z of $X \times_k Y$. However, it follows from assertion (2) of Proposition (3.1.5) that Z' dominates X' and Y'. Consequently we have that Z maps into X' and Y' by the two projections. Consequently we have that $Z \subseteq X' \times_k Y'$. Thus we must have that $Z = Z'$.

Conversely, let Z be a component of $X \times_k Y$. The images of Z in X and Y by the two projections are irreducible and therefore contained in irreducible components X' and Y'. Consequently we have that $Z \subseteq X' \times_k Y'$ and Z must be an irreducible component of $X' \times_k Y'$.

(3.1.8) Proposition. [EGA 24, IV_2, 4.2.8] *Given an algebraic scheme X and a field extension K of k. Then the components of X and $X \times_k K$ have the same dimensions.*

→ *Proof.* Given a component Z of $X \times_k K$. It follows from Proposition (3.1.7) that Z is a components $X' \times_k K$ for some irreducible component X' of X. Moreover
→ it follows from Proposition (3.1.5) that if E is the residue field in a generic point of Z we have that $\dim Z = \operatorname{td.deg.}_K E = \dim X'$.

3.2. Relative dimension.

(3.2.1) Setup. A morphism $f\colon X \to Y$ of schemes is *flat* if the ring $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{Y,f(x)}$-module, via the homomorphism induced by f, for all points x of X. It follows from Proposition (?) that f is flat if and only if we have that $\mathcal{O}_X(U)$ is flat over $\mathcal{O}_Y(V)$, via the homomorphism induced by f, for all affine open subsets U of X and V of Y such that $f(U) \subseteq V$.

(3.2.2) Definition. A morphism $f\colon X \to Y$ of noetherian schemes has *relative dimension* n if, for all integral closed subschemes Z of Y we have that all components of $f^{-1}(Z)$ dominate Z, and $f^{-1}f(x)$ is equidimensional of dimension n for all points x in X.

(3.2.3) Proposition. [123] [H, 9.6] *Given a flat morphism $f\colon X \to Y$ of finite type between algebraic schemes X and Y, where Y is irreducible. The following three assertions are equivalent:*

(1) *The morphism f has relative dimension n.*
(2) *Every irreducible component of X has dimension $\dim Y + n$.*
(3) *We have that $f^{-1}f(x)$ has pure dimension n for all points x in X.*

Proof. The induced morphism $f^{-1}Z \to Z$ is flat. Hence it follows from Remark (?) that every component of $f^{-1}(Z)$ dominates Z, for every closed subvariety Z of Y. Consequently we have that assertions (1) and (3) are equivalent.

We will show that assertion (2) implies assertion (3). Given a point y in $f(X)$ and let Z be an irreducible components of $f^{-1}(y)$. Choose a closed point x in Z which is not in any other component of $f^{-1}(y)$. Then we have that $\dim \mathcal{O}_{f^{-1}(y),x} = \dim \mathcal{O}_{Z,x}$. Since going down holds for flat morphisms it follows from Remark (?) and Proposition (?) that

$$\dim \mathcal{O}_{X,x} = \dim \mathcal{O}_{Y,y} + \dim \mathcal{O}_{X,x} \times_{\mathcal{O}_{Y,y}} \kappa(x).$$

We have that $\mathcal{O}_{X,x} \times_{\mathcal{O}_{Y,y}} \kappa(x) = \mathcal{O}_{f^{-1}(y)x}$. Moreover, since x is closed in Z we have that $\dim \mathcal{O}_{Z,x} = \dim Z$. We have proved that

$$\dim \mathcal{O}_{X,x} - \dim \mathcal{O}_{Y,y} = \dim Z.$$

Since Y is irreducible and X is equidimensional it follows from Proposition (?) that

$$\dim \mathcal{O}_{X,x} + \dim \overline{\{x\}} = \dim X$$

and

$$\dim \mathcal{O}_{Y,y} + \dim \overline{\{y\}} = \dim Y.$$

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However, we have that x is closed in $f^{-1}(y)$. If follows from the Hilbert Nullstellensatz that $\kappa(x)$ is algebraic over $\kappa(y)$. Consequently we have that $\dim \overline{\{x\}} = \dim \overline{\{y\}}$ and we have proved that

$$n = \dim X - \dim Y = \dim Z.$$

Finally we prove that assertion (3) implies assertion (2). Let Z be an irreducible component of X and let x be a closed point of Z which is not in any other component of X. Then we have that $\dim \mathcal{O}_{X,x} = \dim \mathcal{O}_{Z,x} = \dim Z$. Since x is closed in Z we have that x is closed in $f^{-1}(y)$, and consequently that $\dim \mathcal{O}_{f^{-1}(y),x} = n$, where $y = f(x)$. Moreover we have that $y = f(x)$ is closed in Y because $\kappa(x)$, and consequently $\kappa(y) \subseteq \kappa(x)$ are algebraic over k. We obtain that $\dim \mathcal{O}_{Y,y} = \dim Y$. Finally, since f is flat we have, as we observed above the formula

$$\dim \mathcal{O}_{X,x} = \dim \mathcal{O}_{Y,y} + \dim \mathcal{O}_{f^{-1}(y),x}.$$

This prove that $\dim Z = \dim Y + n$ and we have proved the Proposition.

(3.2.4) Proposition. *Given a flat morphism $f\colon X \to Y$ of noetherian schemes of relative dimension n. For every morphism $g\colon Y' \to Y$ we have that the base extension $f'\colon X' = X \times_Y Y' \to Y'$ is of relative dimension n.*

Proof. Since f is flat, and consequently the induced morphism $f^{-1}Z' \to Z'$ is flat for all subschemes Z' of Y' it suffices to check that the fiber ${f'}^{-1}(y')$ has pure dimension n for all points y' of Z'. However we have that ${f'}^{-1}(y') = X \times_Y \kappa(y') = X \times_Y \kappa(g(y')) \times_{\kappa(g(y'))} \times \kappa(y')$. Since f is of relative dimension n we have that that $X \times_Y \kappa(y)$ is of pure dimension n. Consequently it follows from Proposition
→ (?) that $X \times_Y \kappa(y')$ is of pure dimension n.

(3.2.5) Lemma. *Given a dominating morphism $f\colon X \to Y$ of finite type between irreducible noetherian schemes. Let ξ be the generic point of X and let $\eta = f(\xi)$. For every point x of X the irreducible components of $f^{-1}f(x)$ have dimension at least* $\dim f^{-1}(\eta)$.

Proof. Assume first that $\mathcal{O}_{Y,f(x)}$ is universally catenary for all points x of X. Let x be a generic point for an irreducible component Z of $f^{-1}(y)$. It follows from the dimension formula that we have

$$\begin{aligned}\dim \mathcal{O}_{X,x} &= \dim \mathcal{O}_{Y,y} + \operatorname{td.deg.}_{\kappa(\eta)} \kappa(\xi) - \operatorname{td.deg.}_{\kappa(y)} \kappa(x)\\ &= \dim \mathcal{O}_{Y,y} + \dim f^{-1}(\eta) - \dim Z.\end{aligned}$$

→ On the other hand, it follows from Proposition (?) that $\dim \mathcal{O}_{X,x} \leq \dim \mathcal{O}_{Y,y} + \dim \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} \kappa(y)$. Since x is a generic point of $f^{-1}(y)$ and $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} \kappa(y)$ is the local ring of $f^{-1}(y)$ in x we obtain that $\dim \mathcal{O}_{X,x} \leq \dim \mathcal{O}_{Y,y}$. It follows that $\dim Z \geq \dim f^{-1}(\eta)$ as we wanted to show.

We shall reduce the general case to the situation when $\mathcal{O}_{Y,f(x)}$ is catenary for all x in X.

It is clear that the assertion of the Lemma is local on Y and thus local on X. We can also replace X and Y by their reduced subschemes. Consequently we can assume that $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$, where A and B are ingegral domains, and A is a subring of B in such a way that B becomes a finitely generated A-algebra. We can write B as a residue ring $B = A[x_1, \dots, x_n]/I$ of the polynomial ring $k[x_1, \dots, x_n]$ by the prime ideal I. Let $A_0 \subseteq A$ be a finitely generated $\mathbf{Z}$-algebra that contains all the coefficients of a set of generators $f_1, \dots, f_m$ of I. Let $B_0 = A_0[x_1, \dots, x_n]/(f_1, \dots, f_m)$. We have that the natural map $B_0 \otimes_{A_0} A \to B$ is an isomorphism. Moreover we have that A_0 is a subring of B_0 because A is a subring of B. Denote by P the kernel of the map $B_0 \to B$. Then P is a prime ideal and $P \cap A_0 = 0$ because P is the inverse image of (0) by $B_0 \to B$. We consequently obtain a factorization

$$B_0 \otimes_{A_0} A \to B_0/P \otimes_{A_0} A \to B$$

of the map $B_0 \otimes_{A_0} A \to B$. We can thus assume that we have an integral domain B_0 that contains A_0, such that B_0 is a finitely generate algebra over A_0 and such that $B_0 \otimes_{A_0} A \to B$ is an isomorphism. It follows from Remark (?) that $\mathbf{Z}$ is universally catenary, and consequently that A_0 is universally catenary. Let $X_0 = \operatorname{Spec} B_0$ and $Y_0 = \operatorname{Spec} A_0$, and let $f_0 \colon X_0 \to Y_0$ be the morphism corresponding to the inclusion of A_0 in B_0. Moreover, let $g \colon Y \to Y_0$ be the morphism corresponding to the inclusion of A_0 in A. Denote by η the generic point of Y. Then $\eta_0 = g(\eta)$ is the generic point of Y_0. It follows from Proposition (?) that for every point x of X we have that the dimensions of the components of $f_0^{-1} g f(x)$ are the same as the dimension of the components of $f^{-1} f(x)$. It follows from the universally catenary case that the dimensions of the components of $f^{-1} f(x)$ are at least equal to $\dim f_0^{-1}(\eta_0)$. However, again using Proposition (?), we have that $\dim f^{-1}(\eta) = \dim f_0^{-1}(\eta_0)$, and we have proved the Lemma.

(3.2.6) Lemma. *Given a morphism $f \colon X \to Y$ of finite type between noetherian schemes, where Y is irreducible. Let Z be an irreducible component of X that dominates Y and denote by ζ the generic point of Z. Moreover, let z be a point in Z such that*

$$\dim \mathcal{O}_{X,x} = \dim \mathcal{O}_{Y,y} + \dim \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} \kappa(y),$$

with $y = f(x)$. Then we have that

$$\dim_x f^{-1} f(x) \leq \dim(f^{-1}(\eta) \cap Z) + \dim \mathcal{O}_{X,x} - \dim \mathcal{O}_{Z,x}.$$

Proof. It follows from the dimension formula (?) applied to the morphism $g\colon Z \to Y$ induced by f that we have

$$\dim \mathcal{O}_{Z,x} \le \dim \mathcal{O}_{Y,y} - \operatorname{td.deg.}_{\kappa(y)} \kappa(x) + \operatorname{td.deg.}_{\kappa(\eta)} \kappa(\zeta),$$

where $\eta = f(\zeta)$. Moreover, it follows from Remark (?) that

$$\dim_x f^{-1}f(x) = \dim \mathcal{O}_{f^{-1}(y),x} + \operatorname{td.deg.}_{\kappa(y)} \kappa(x),$$

and from the same Remark we we obtain that

$$\dim g^{-1}(\eta) = \operatorname{td.deg.}_{\kappa(\eta)} \kappa(\zeta).$$

Hence we obtain the formula

$$\dim \mathcal{O}_{Z,x} \le \dim \mathcal{O}_{Y,y} + \dim \mathcal{O}_{f^{-1}(y),x} - \dim f^{-1}f(x) + \dim g^{-1}(Z).$$

However, we have $\mathcal{O}_{f^{-1}(y),x} = \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} \kappa(y)$ and $g^{-1}(\eta) = f^{-1}(\eta) \cap Z$, and consequently,

$$\dim_x f^{-1}f(x) \le \dim(f^{-1}(\eta) \cap Z) \dim \mathcal{O}_{Y,y} + \dim \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} \kappa(y) - \dim \mathcal{O}_{Z,x}.$$

The Lemma thus follows from the assumptions.

(3.2.7) Proposition. *Given a dominant morphism $f\colon X \to Y$ of finite type between integral noetherian schemes. Denote by ξ the generic point of X, and let x be a point of X. Let $\eta = f(\xi)$ and $y = f(x)$. Assume that the formula*

$$\dim \mathcal{O}_{X,x} = \dim \mathcal{O}_{Y,y} + \dim \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} \kappa(y)$$

holds. Then all the components of $f^{-1}f(x)$ have dimension $\dim f^{-1}(\eta)$.

Proof. It follows from Lemma (3.2.5) that all the irreducible components of the fiber $f^{-1}f(x)$ have dimension at least $\dim f^{-1}(\eta)$, and it follows from Lemma (3.2.6), with $X = Z$ that all the components of $f^{-1}f(x)$ has dimension at most $\dim f^{-1}(\eta)$.

(3.2.8) Proposition. *Given a flat morphism $f\colon X \to Y$ of finite type beween noetherian schemes, where Y is irreducible. Assume that $\mathcal{O}_{X,x}$ is equidimensional for all x in X and that $f^{-1}(\eta)$ is equidimensional. Then we have that $f^{-1}f(x)$ is equidimentional of dimension $\dim f^{-1}(\eta)$ for all x in X.*

Proof. Since f is flat it follows from Remark (?) that the assumption of Lemma (3.2.6) holds for all irreducible components Z of X and all x in Z. Moreover, since

$\mathcal{O}_{X,x}$ is equidimensional we have that $\dim \mathcal{O}_{X,x} = \dim \mathcal{O}_{Z,z}$. We therefore obtain that

$$\dim_x f^{-1}f(x) \leq \dim(f^{-1}(\eta) \cap Z).$$

Since $f^{-1}(\eta)$ is equidimensional it follows from Remark (?) that $\dim(f^{-1}(\eta) \cap Z) = \dim f^{-1}(\eta)$. All the components of $f^{-1}f(x)$ therefore have dimension at most equal to $\dim f^{-1}(\eta)$. Denote by $g\colon Z \to Y$ the morphism induced by f. It follows from Lemma (3.2.5) that all the components of $g^{-1}g(x)$ have dimensions that are at least equal to $\dim(f^{-1}(\eta) \cap Z) = \dim f^{-1}(\eta)$. However, we have that $g^{-1}g(x)$ is a closed subscheme of $f^{-1}f(x)$,. thus the dimensions of the components of $f^{-1}f(x)$ that contain a component of $g^{-1}g(x)$ is at least equal to $\dim f^{-1}(\eta)$. Since $f^{-1}f(x)$ is the union of $Z \cap f^{-1}f(x) = g^{-1}g(x)$ for the irreducible components Z of X we have that all the components of $f^{-1}f(x)$ have dimension at least equal to $f^{-1}(\eta)$.

(3.2.9) Theorem. (Chevalley) *Given a morphism $f\colon X \to Y$ of finite type to a noetherian scheme Y. For every integer n the set $F_n(X)$ consisting of the points x in X such that $\dim_x f^{-1}f(x) \geq n$ is closed.*

Proof. We can assume that X and Y are reduced. Let $\mathcal{F}$ be the family consisting of closed subschemes of X for which the Theorem does not hold. It $\mathcal{F}$ is not empty it contains a minimal element Y'. We can clearly assume that $Y = Y'$ and thus assume that the Theorem holds for all proper closed subshcemes of Y but not for Y.

Let $X_1, \dots, X_n$ be the irreducible components of X with their reduced structure. We have that $F_n(X) = \cup_{i=1}^n F_n(X_i)$ because every irreducible component of $f^{-1}(y)$ is contained in an $X_i \cap f^{-1}(y)$ and therefore in an irreducible component of $X_i \cap f^{-1}(y)$, and conversely every irreducible component of $X_i \cap f^{-1}(y)$ is contained in a component of $f^{-1}(y)$ and, by the preceeding argument, equal to that component. We can therefore assume that X is irreducible.

Denote by Z the integral subscheme of Y that has $\overline{f(X)}$ as underlying set. The morphism $X \to Z$ induced by f is of finite type and the fibers are the same as the fibers of f. We can therefore assume that $Z = Y$, and thus that Y is integral and f dominating.

Let η be the generic point of Y. It follows from Lemma (3.2.5) that for $n \leq \dim f^{-1}(\eta)$ we have that $F_n(X) = X$.

Assume that $n > \dim f^{-1}(\eta)$. It follows from the Lemma (?) of generic flatness that there is an open non empty subset U of Y such that the morphism $f^{-1}(U) \to U$ induced by f, is flat. From Lemma (?) and Proposition (3.2.77) we obtain, since going down holds for f, that $\dim_x f^{-1}f(x) = \dim f^{-1}(\eta)$ for all $x \in f^{-1}(U)$. Consequently we have that $F_n(X) \subseteq f^{-1}(Y \setminus U)$. By the assumption that the Theorem holds for all proper irreducible subsets of Y we have that the Theorem holds for $Y \setminus U$. Consequently $F_n(X)$ is closed in $Y \setminus U$, and hence in Y. This contradicts the assumption that the Theorem holds for Y. Consequently the family $\mathcal{F}$ is empty, and the Theorem holds.

3.3. Total quotients.

(3.3.1) Setup. Given a ring A. We denote by $R(A)$ the *total quotien ring* of A, that is the localization of A in the multiplicatively closed set consisting of non zero divisors in A.

(3.3.2) Remark. The natural map $A \to R(A)$ is injective.

(3.3.3) Lemma. *Given a ring A. An element of A is not a zero divisor in A if and only if the image of the element in A_P is not a zero divisor in A_P for all primes P of A.*

Proof. Let a be an element in A which is not a zero divisor. Then for any prime P of A the image of a in A_P is is not a zero divisor in A_P because, if $ab/s = 0$ in A_P we have that $tab = 0$ for some b in A and t in $A \setminus P$. However, then we have that $tb = 0$ in A and consequently $b/s = 0$ in A_P.

Conversely, if the image of a in A_P is not a zero divisor in A_P for all primes P in A we have that the annihilator of a can not be contained in any prime ideal of A. Consequently the annihilator contains 1 and we must have that $a = 0$.

(3.3.4) Remark. Given a scheme X. For every open subset U of X we have an injective map $\mathcal{O}_X(U) \to \prod_{x\in U} \mathcal{O}_{X,x}$. It follows from Lemma (3.3.3) that, when U is affine, the non zero divisors in $\mathcal{O}_X(U)$ are exactly those elements that map to non zero divisors in $\mathcal{O}_{X,x}$ for all points x in U. In particular we obtain an injective map

$$R(\mathcal{O}_X(U)) \to \prod_{x\in U} R(\mathcal{O}_{X,x}),$$

and, for every open affine subset V of U, we obtain a natural map

$$R(\mathcal{O}_X(U)) \to R(\mathcal{O}_X(V)).$$

(3.3.5) Lemma. *Given a ring A and a prime ideal P in A. Let c be an element in the kernel of $R(A) \to R(A_P)$. Then there is an element s in $A \setminus P$ such that c is in the kernel of $R(A) \to R(A_s)$.*

Proof. Write $c = a/u$ with a and u in A, and where u is a non zero divisor. Then a is in the kernel of the composite map $A \to R(A) \to R(A_P)$, and consequently in the kernel of $A \to A_P \to R(A_P)$. Since the map $A_P \to R(A_P)$ is injective we obtain that a is in the kernel of $A \to A_P$. Consequently there is an $s \in A \setminus P$ such that a is in the kernel of $A \to A_s$. However, then we have that $c = a/u$ is in the kernel of $R(A) \to R(A_s)$.
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(3.3.6) Lemma. *Given a noetherian ring A, a prime ideal P in A, and an element a in A whose image in A_P is not a zero divisor. Then there is an element s in $A \setminus P$ such that the image of a in A_s is not a zero divisor.*

Proof. If a is not a zero divisor in A we can take $s = 1$.

Assume that a is a zero divisor and let $a_1, \dots, a_n$ be the generators for the annihilator of a in A. Since a is not a zero divisor in A_P there are elements $s_1, \dots, s_n$ in $A \setminus P$ such that $s_i a_i = 0$ for $i = 1, \dots, n$. Let $s = s_1 \cdots s_n$. Then we have that a is not a zero divisor in A_s. Indeed, if $ab/s^m = 0$ for some $b \in A$ and some non negative integer m we have that $s^p ba = 0$ in A, for some non negative integer p. In other words we have that $s^p b$ is in the annihilator of a and consequently that $s^{p+1} b = 0$, or equivalently that $b/s^m = 0$ in A_s.

(3.3.7) Lemma. *Given a noetherian ring A, a prime ideal P in A and an element c in $R(A_P)$. Then there is an element $s \in A \setminus P$ and an element in $R(A_s)$ that maps to c by the map $R(A_s) \to R(A_P)$.*

Proof. We have that c is the quotient of $a/u \in A_P$ by $b/v \in A_P$, with u, v in $A \setminus P$, and a, b in A, and where the image of b in A_P is not a zero divisor. It follows from Lemma (3.3.6) that there is an element t in $A \setminus P$ such that the image of b in A_t is not a zero divisor. However, then b/v is not a zero divisor in A_{tuv} because, if it were a zero divisor we would have $(tuv)^m db = 0$ in A for some non negative integer m and an element d in A. Then we have that the image in A_t of $(uv)^m d$ is zero and consequently that $(tuv)^p d = 0$ in A for some non negative integer p. Consequently the image of d in A_{tuv} is zero. Let $s = tuv$. Then we have that $s \in A \setminus P$ and it is clear that the quotient of a/u by b/v in $R(A_t)$ is mapped to c by the map $R(A_t) \to R(A_P)$.

(3.3.8) Definition. For every open subset U of a scheme X we let

$$\begin{aligned} \mathcal{R}_X(U) = \{(s_x) \in \prod_{x \in U} R(\mathcal{O}_{X,x}) | \ & \text{for every } x \in U \text{ there is an affine open neigh-} \\ & \text{bourhood } V \text{ of } x \text{ contained in } U, \text{ and an element in } R(\mathcal{O}_X(V)) \text{ that} \\ & \text{is mapped to } s_y \text{ by the map } R(\mathcal{O}_X(V)) \to R(\mathcal{O}_{X,y}) \text{ for every } y \in V\}. \end{aligned}$$

(3.3.9) Remark. It is clear that $\mathcal{R}_X$ defines a sheaf on X. We have an injection $\mathcal{O}_X(U) \to \mathcal{R}_X(U)$ that sends an element $s \in \mathcal{O}_X(U)$ to the fiber $s_x \in \mathcal{O}_{X,x} \subseteq R(\mathcal{O}_{X,x})$. This gives an injection of sheaves

$$\mathcal{O}_X \to \mathcal{R}_X.$$

We consider $\mathcal{R}_X$ as an $\mathcal{O}_X$-algebra via this map. For all points x in X and every neighbourhood U of x there is a natural map

$$\mathcal{R}_X(U) \to R(\mathcal{O}_{X,x}),$$

that sends the element $(s_x)_{x\in U}$ to s_x. Consequently there is a natural map

$$\mathcal{R}_{X,x} \to R(\mathcal{O}_{X,x}).$$

→ It follows from Lemma (3.3.5) that this map is injective. Moreover, it follows
→ from Lemma (3.3.7) that, when X is noetherian, the map is also surjective, and consequently an isomorphism.

(3.3.10) Proposition. *Given a noetherian scheme X and an open affine subset $U = \operatorname{Spec} A$. Then the natural map*

$$R(A) \to \mathcal{R}_X(U)$$

is an isomorphism.

Proof. From the injective map $A \to \prod_{P\in\operatorname{Spec} A} A_P$ we obtain an injective map $R(A) \to \prod_{P\in\operatorname{Spec} A} R(A_P)$. Consequently we have that the map of the Proposition is injective.

To prove that the map of the Proposition is surjective we take an element $s = (s_P)_{P\in\operatorname{Spec} U}$ in $\mathcal{R}_X(U)$. We can find open affine subsets $\operatorname{Spec} A_{f_i}$, for $i = 1, \dots, n$, that cover U and elements $s_i \in R(A_{f_i})$ such that the image of s_i in A_P is s_P for all prime ideals P in A that do not contain f_i. Consequently we have that s_i and s_j have the same image in $R(A_{f_if_j})$ for all i and j. For all i we write $s_i = a_i/t_i$ where a_i and t_i are in A and where the image of t_i in A_{f_i} is not a zero divisor. Then we have that $(f_if_j)^{n_{ij}}(a_it_j - a_jt_i) = a_if_i^{n_{ij}}t_jf_j^{n_{ij}} - a_jf_j^{n_{ij}}t_if_i^{n_{ij}} = 0$ in A. Multiplying a_i and t_i with a high power of f_i, for all i, we may therefore assume that $a_it_j = a_jt_i$ in A, for all i and j.

Let

$$I = \{a \in A | aa_i \in (t_i) \text{ in } A_{f_i} \text{ for all } i\}.$$

then we have that $t_1, \dots, t_n$ are all in I because $t_ja_i = t_ia_j$, for all i and j.

If $aI = 0$ for some $a \in A$ we have that $at_i = 0$ for all i. However, the image of the element t_i in A_{f_i} is not a zero divisor. Consequently we can find a non negative integer m such that $f_i^m a = 0$ in A, for all i. However we have that the sets $\operatorname{Spec} A_{f_i} = \operatorname{Spec} A_{f_i^m}$ cover U. Consequently we have that the ideal $(f_1^m, \dots, f_n^m)$ is all of A and consequently that $a = 0$.

We have proved that if $aI = 0$ then $a = 0$. Consequently it follows from Lemma
→ (?) that I contains a non zero divisor t. We have that $ta_i \in (t_i)$ in A_{f_i} for all i. Hence we have that $ta_i = t_ic_i/f_i^p$ in A_{f_i}, for some non negative integer p and elements $c_1, \dots, c_n$ in A, for $i = 1, \dots, n$. Then $ta_i/t_i = c_i/f_i^p$ in A_{f_i} considered as a subring of $R(A_{f_i})$. However we have that a_i/t_i and a_j/t_j are equal considered as elements in $R(A_{f_if_j})$. Consequently we have that c_i/f_i^p and c_j/f_j^p are equal in the subring $A_{f_if_j}$. Therefore there is an element b in A such that $ta_i/t_i = b$ in $R(A_{f_i})$, for all i. We have that the element b/t is in $R(A)$ and it maps to a_i/t_i in $R(A_{f_i})$ for $i = 1, \dots, n$. Consequently we have that b/t maps to s in $\mathcal{R}_X(U)$.

(3.3.11) Example. For all open affine subsets $U = \operatorname{Spec} A$ of a scheme X there is an isomorphism $\mathcal{R}_X(U) \cong R(A)$. It follows therefore, from the definition of $\mathcal{R}_X$ that, when X is integral, we have that $\mathcal{R}_X$ is the constant sheaf associated to the field $\mathcal{R}_X(X) = \mathcal{O}_{X,\xi}$, where ξ is the generic point of X. In particular we have that $\mathcal{R}_X$ is quasi coherent when X is integral.

To give an example of a non integral scheme where the sheaf $\mathcal{R}_X$ is not necessarily quasi coherent we let A be the localization of the ring $k[x,y,z]/(x^2, xy, xz)$ in the maximal ideal $(x,y,z)/(x^2, xy, xz)$, where x, y and z are independent variables over k. Then we have that $A = R(A)$, and if b is the class of y in A, we have that $A_b = \left(k[y,z]_{(y,z)}\right)_y$. Let $X = \operatorname{Spec} A$. It follows from Proposition (3.3.10) that $\mathcal{R}_X(X_b) = R(A_b) = k(y,z)$. However, the ring $\mathcal{R}_X(X)_b = R(A)_b$ is different from $k(y,z)$ because (z) is a maximal ideal in $\subseteq \left(k[y,z]_{(y,z)}\right)_y$ with residue field $k(y)$. Consequently we have that $\mathcal{R}_X$ is different from $\widetilde{R(A)}$, and $\mathcal{R}_X$ is not quasi coherent.

3.4. Normalization.

(3.4.1) Setup. Given a scheme X. For every open subset U of X we let

$$\begin{aligned}\mathcal{O}'_X(U) = \{s \in \mathcal{R}_X(U) | &\text{for all } x \in U \text{ there is a neighbourhood } V \text{ of } x \\ &\text{in } U \text{ such that } s|V \text{ is integral over the subring } \mathcal{O}_X(V) \text{ of } \mathcal{R}_X(V)\}\end{aligned}$$

It follows from the definition that $\mathcal{O}'_X$ is a subsheaf of $\mathcal{R}_X$ that contains $\mathcal{O}_X$ and that $\mathcal{O}'_X$ is a subalgebra of $\mathcal{R}_X$ via the inclusions $\mathcal{O}_X \subseteq \mathcal{O}'_X \subseteq \mathcal{R}_X$. For every point x in X we have inclusions

$$\mathcal{O}_{X,x} \subseteq \mathcal{O}'_{X,x} \subseteq \mathcal{R}_{X,x}.$$

(3.4.2) Remark. We have that

$$\begin{aligned}\mathcal{O}'_X(U) = \{s \in \mathcal{R}_X(U) | &\text{the image } s_x \text{ of } s \text{ in the fiber } \mathcal{R}_{X,x} \text{ is integral over } \mathcal{O}_{X,x} \\ &\text{for all } x \in U\}.\end{aligned}$$

Indeed, if $s|V$ is integral over $\mathcal{O}_X(V)$ we clearly have that s_x is integral over $\mathcal{O}_{X,x}$ for all x in V.

Conversely, if $s_x \in \mathcal{R}_{X,x}$ is integral over $\mathcal{O}_{X,x}$ we have that $s_x^n + a_{1,x}s_x^{n-1} + \cdots + a_{n,x} = 0$, for some elements $a_1, \dots, a_n$ of $\mathcal{O}_X(V)$, where V is a neighbourhood of x. However, then we have that $(s|W)^n + (a_1|W)(s|W)^{n-1} + \cdots + (a_n|W) = 0$, for some neighbourhood W of x contained in V.

We also see that $\mathcal{O}'_{X,x}$ consists of the elements in $\mathcal{R}_{X,x}$ that are integral over $\mathcal{O}_{X,x}$.

(3.4.3) Proposition. *If $\mathcal{R}_X$ is a quasi coherent $\mathcal{O}_X$-module we have that $\mathcal{O}'_X$ is a quasi coherent $\mathcal{O}_X$-module, and for every open affine subset U of X we have that $\mathcal{O}'_X(U)$ is the integral closure of $\mathcal{O}_X(U)$ in $\mathcal{R}_X(U)$.*

Proof. We can assume that $X = \operatorname{Spec} A$ is affine and that $\mathcal{R}_X(X) = \widetilde{B}$ where $B = R(A)$. Let A' be the integral closure of A in B. We must show that $\widetilde{A'} = \mathcal{O}'_X$. It is clear that $\widetilde{A'} \subseteq \mathcal{O}'_X$. To show the opposite inclusion we only have to show that, for every prime ideal P in A the elements in B_P that are integral over A_P lie in $(A')_P$. Hence it suffices to prove the equality $(A')_P = (A_P)'$. To show the latter equality we first note that we have an inclusion $(A'_P) \subseteq (A_P)'$. To show the other inclusion we let $b/t \in (A_P)'$, with $b \in B$ and $t \in A \setminus P$. Then we have that $(b/t)^n + (a_{n-1}/t_{n-1})(b/t)^{n-1} + \cdots + (a_0/t_0) = 0$ in B_P, for some elements $a_0, \dots, a_n$ in A and $t_0, \dots, t_{n-1}$ in $A \setminus P$. Multiplying the equation by $(tt_0 \cdots t_{n-1})^n$ we see that $t_0 \cdots t_{n-1}b \in A'$. Consequently we have that $b/t = (t_0 \cdots t_{n-1}b)/(t_0 \cdots t_{n-1}t) \in (A')_P$.
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(3.4.4) Definition. Given a scheme X and let $X' = \operatorname{Spec} \mathcal{O}'_X$. We call the scheme X' the *integral closure* of X. When X is integral we call X' the *normalization* of X. We say that X is *normal* if the structure morphism $X' \to X$ is an isomorphism.

(3.4.5) Remark. The normalization of an integral scheme is integral.

(3.4.6) Remark. Given a scheme X and let $\varphi: X' \to X$ be the structure morphism of the integral closure of X. Let U be an open subset of X. Then we have that the induced morphism $\varphi^{-1}(U) \to U$ is the integral closure of U.

(3.4.7) Remark. Given a scheme X and let $\varphi: X' \to X$ be the structure morphism of the integral closure of X. Then φ is affine. When $\mathcal{R}_X = \operatorname{Spec} A$ is quasi coherent we have, for every open affine subset $U = \operatorname{Spec} A$ of X, that $\varphi^{-1}(U) = \operatorname{Spec} A'$, where A' is the integral closure of A in $R(A)$.

(3.4.8) Proposition. *Given an integral noetherian scheme Y and let $\psi: Y' \to Y$ be the normalization. Let $\varphi: X \to Y$ be a dominant morphism from a normal scheme X. Then there is a unique morphism $\varphi': X \to Y'$ such that $\varphi = \psi\varphi'$.*

Proof. The morphism φ gives rise to an inclusion $\mathcal{R}_Y(Y) \to \mathcal{R}_X(X)$, that for every open affine subset U of Y induces an inclusion $\mathcal{O}_Y(U) \to \mathcal{O}_X(\varphi^{-1}(U))$. Since X is normal it follows from Remark (3.4.7) that $\mathcal{O}_X(\varphi^{-1}(U))$ is integrally closed in $\mathcal{R}_X(X)$, and since $\mathcal{O}'_Y(U)$ consists of elements that are integral over $\mathcal{O}_Y(U)$ we have that $\mathcal{O}_Y(U) \subseteq \mathcal{O}'_Y(U) \subseteq \mathcal{O}_X(\varphi^{-1}(U))$ in $\mathcal{R}_X(X)$. It follows from these inclusions that there is a unique morphism $\varphi': X \to Y'$ such that $\varphi = \psi\varphi'$.

(3.4.9) Remark. Given an algebraic variety X. Then the structure morphism $\varphi: X' \to X$ of the normalization is a finite morphism. To see this we need, since φ is affine, to show that the integral closure of $\mathcal{O}_X(U)$ in $\mathcal{R}_X(U)$ is a finitely generated $\mathcal{O}_X(U)$-module for all affine subsets U of X. However, we have that R is an integral domain that is a finitely generated algebra over a field. Then the integral closure of R in its quotient field is a finitely generated R-module [144] [Z-S, vol. 1, p. 267].

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Reserved results

(00.0.1) Lemma. *Given a local map $A \to B$ of local zero dimensional rings, such that B is flat over A. Then we have that*

$$\ell(B) = \ell(A)\ell(B/PB),$$

where P is the maximal ideal of A.

Proof. Since A is zero dimensional it is of finite length and has a composition series

$$A = I_0 \supset I_1 \supset \cdots \supset I_r = 0.$$

We have that B is flat over A so we obtain a chain

$$B = I_0B \supset I_1B \supset \cdots \supset I_rB = 0$$

of ideals in B such that $I_{i-1}B/I_iB \cong I_{i-1}/I_i \otimes_A B = A/P \otimes_A B = B/PB$. From the additivity of length we obtain that $\ell(B) = r\ell_B(B/PB)$.

(00.0.8) Proposition. *Given a finitely generated field extension K of the field k and let A be a local noetherian k-algebra with maximal ideal P. Let $B = K \otimes_k A$. Then we have that:*

(1) *If $[K\colon k]\ell(A) < \infty$ then*

$$[K\colon k]\ell(A) = \sum_{\substack{Q \text{ minimal prime of } B \\ Q\cap A = P}} [B/Q\colon A/P]\ell(B_Q).$$

(2) *If $[K\colon k]\ell(A) = \infty$ then*

$$\ell(B_Q)[B/Q\colon A/P] = \infty$$

for all primes Q in B.

Proof. Assume first that $[K\colon k]\ell(A) < \infty$. Then B is a free A-module of rank $[K\colon k]$. Consequently B has finite length as an A-module and $\ell_A(B) = [K\colon k]\ell(A)$. We filter B by B-modules such that the quotients are of the form B/Q where Q is a prime ideal in B. Then B/Q has finite length as an A-module and consequently
→ Q is a maximal ideal by Lemma (lenght). Hence B has finite length. The quotient B/Q appears $\ell(B_Q)$ times as a quotient in the filtaration. However B/Q is a finite field extension of A/P since B is a finitely generated A-module and $\ell_A(B/Q) =$ snitt

$\ell_{A/P}(B/Q) = [B/Q\colon A/P]$. Consequently the length of B as an A-module is also equal to $\sum [B/Q\colon A/P]\ell(B_Q)$, where the sum is over all minimal primes Q in B that contracts to P.

Assume secondly that $[K\colon k]\ell(A) = \infty$. If $[B/Q\colon A/P]\ell(B_Q) < \infty$ for some prime ideal Q in B, it follows from $\ell(B_Q) < \infty$ that Q is a minimal prime ideal of B. We have that B is flat over A. Consequently it follows from Proposition (?) that P is minimal in A. It follows from Proposition (?) that $\ell(A) < \infty$ and we must have that $[K\colon k] = \infty$. Since K is a finitely generated field extension we can find an element t in K that is trancendent over k. The image of t in B/Q is algebraic over A/P since $[B/Q\colon A/P] < \infty$. Consequently we can find elements $a_0, \dots, a_n$ in A such that

$$a_0 t^n + a_1 t^{n-1} + \cdots + a_n \in Q$$

with $a_0 \notin P$. The field K is flat over $k[t]$ since $k[t]$ is a principal ideal domain and K is without torsion. It follows that B is flat over the subring $A[t]$. Hence it follows from Proposition (?) that $Q \cap A[t]$ is a minimal ideal in $A[t]$. However, the ideal $Q \cap A[t]$ contains the prime ideal $P[t]$ such that $Q \cap A[t] = P[t]$. Hence we have that $a_0 t^n + a_1 + \cdots + a_n \in Q \cap A[t] = P[t]$, which contradict the assumption that $a_0 \notin P$. Consequently we must have that $[B/Q\colon A/P]\ell(B_Q) = \infty$ for all prime ideals Q in B.

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