Hilbert schemes

0. Contents.

1. Cohomology of sheaves on schemes.

(1.1) Setup. Given a noetherian scheme S and a $f\colon X \to S$ separated morphism of finite type. Moreover, given a quasi–coherent $\mathcal{O}_X$–module $\mathcal{F}$. Let $g\colon T \to S$ be a morphism from a scheme T. We write $X_T = T \times_S X$ and the maps of the resulting cartesian diagram we denote as follows:

$$\begin{array}{ccc} X_T & \xrightarrow{g_X} & X \\ {\scriptstyle f_T}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ T & \xrightarrow[g]{} & S. \end{array}$$

Moreover, we write $\mathcal{F}_T = g_X^* \mathcal{F}$.

We choose an affine open covering $\mathcal{U} = \{U_0, \dots, U_r\}$ of X.

(1.2) Definition. Assume that $S = \operatorname{Spec} A$ is affine. We have a sequence of A–modules

$$\mathcal{F}_{\mathcal{U}}\colon 0 \to \bigoplus_{0 \le i_0 \le r} \mathcal{F}(U_{i_0}) \xrightarrow{d^0} \bigoplus_{0 \le i_0 < i_1 \le r} \mathcal{F}(U_{i_0} \cap U_{i_1}) \xrightarrow{d^1}$$
$$\cdots \xrightarrow{d^{r-1}} \bigoplus_{0 \le i_0 < \cdots < i_{r+1} \le r} \mathcal{F}(U_0 \cap \cdots \cap U_r) \to 0,$$

where the A–linear maps d^i are given by

$$d^p(f)_{i_0 \dots i_{p+1}} = \sum_{q=0}^{p+1} (-1)^q f_{i_0 \dots \widehat{i_q} \dots i_{p+1}} | U_{i_0} \cap \cdots \cap U_{i_{p+1}},$$

where $\widehat{i_q}$ means that i_q has been deleted. It is easy to check that the sequence $\mathcal{F}_{\mathcal{U}}$ is a complex. The cohomology of the sequence is independent of the choice
→ of the covering $U_0, \dots, U_r$, and thus also of r ([H], (III, §4, Theorem 4.5)). We denote the i'th cohomology group of the complex by $H^i(X, \mathcal{F})$, and call it the *i'th cohomolgy group of $\mathcal{F}$* .

→ **(1.3) Note.** It follows from Definition (1.2) that $H^i(X, \mathcal{F}) = 0$ for $i > r$ and $i < 0$.

(1.4) Note. Assume that $S = \operatorname{Spec} A$. The map which sends a quasi–coherent $\mathcal{O}_X$–module $\mathcal{F}$ to the A–module $H^i(X, \mathcal{F})$ is a covariant functor from quasi–coherent $\mathcal{O}_X$–modules to A–modules. Indeed, given a homomorphism $\mathcal{F} \to \mathcal{G}$ of quasi–coherent $\mathcal{O}_X$–modules. We obtain a map

$$\mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_p}) \to \mathcal{G}(U_{i_0} \cap \cdots \cap U_{i_p}),$$

for each $i_0, \dots, i_p$, and consequently a map

$$\mathcal{F}_{\mathcal{U}} \to \mathcal{G}_{\mathcal{U}}$$

of complexes of A–modules. Thus there is an A–linear map

$$H^i(X, \mathcal{F}) \to H^i(X, \mathcal{G})$$

of cohomology modules, for each i. It is clear from the construction of the latter map that the map from quasi–coherent $\mathcal{O}_X$–modules to A–modules that sends $\mathcal{F}$ to $H^i(X, \mathcal{F})$ is a functor.

(1.5) Note. Assume that $S = \operatorname{Spec} A$. From a short exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

of quasi–coherent $\mathcal{O}_X$–modules, we obtain a long exact sequence

$$\cdots \to H^i(X, \mathcal{F}') \to H^i(X, \mathcal{F}) \to H^i(X, \mathcal{F}'') \to H^{i+1}(X, \mathcal{F}') \to \cdots .$$

Indeed, we have an exact sequence

$$0 \to \mathcal{F}'(U_{i_0} \cap \cdots \cap U_{i_p}) \to \mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_p}) \to \mathcal{F}''(U_{i_0} \cap \cdots \cap U_{i_p}) \to 0,$$

for each $0 \le i_0 < \cdots < i_p \le r$. Hence we obtain a short exact sequence

$$0 \to \mathcal{F}'_{\mathcal{U}} \to \mathcal{F}_{\mathcal{U}} \to \mathcal{F}''_{\mathcal{U}} \to 0$$

of complexes that gives rise to the long exact sequence.

(1.6) Note. Assume that $S = \operatorname{Spec} A$. Let $\iota\colon Y \subseteq X$ be a closed immersion of schemes, and let $\mathcal{G}$ be a quasi–coherent $\mathcal{O}_Y$–module. The map i induces an equality

$$H^i(Y, \mathcal{G}) = H^i(X, \iota_* \mathcal{G})$$

of A–modules. Indeed, let $V_i = U_i \cap Y = i^{-1}(U_i)$. Then $\mathcal{V} = \{V_0, \dots, V_r\}$ is an affine open covering of Y and we have that $(i_*\mathcal{G})(U_{i_0} \cap \cdots \cap U_{i_p}) = \mathcal{G}(V_{i_0} \cap \cdots \cap V_{i_p})$. Consequently $(i_*\mathcal{G})_{\mathcal{U}} = \mathcal{G}_{\mathcal{V}}$ and we obtain the equality.

(1.7) Definition. Given a morphism $g\colon T \to S$ from a noetherian scheme T. Given an open affine subset $\operatorname{Spec} A$ of S and let $\mathcal{U} = \{U_0, \dots, U_r\}$ be an affine open affine covering of $f^{-1}(\operatorname{Spec} A)$. Moreover, let $\operatorname{Spec} B$ be an open affine subset of T that maps to $\operatorname{Spec} A$. For every open affine subset U of X that maps into

$\operatorname{Spec} A$ we have that $V = X_{\operatorname{Spec} B} \cap g_X^{-1} U = \operatorname{Spec} B \times_{\operatorname{Spec} A} U$ is an affine open subset of $X_{\operatorname{Spec} B}$ and we have that

$$B \otimes_A \mathcal{F}(U) = (\mathcal{F} \otimes_{\mathcal{O}_{\operatorname{Spec} A}} \mathcal{O}_{\operatorname{Spec} B})(V) = g_X^* \mathcal{F}(V) = \mathcal{F}_{\operatorname{Spec} B}(V).$$

Hence, if we let $V_i = X_{\operatorname{Spec} B} \cap g_X^{-1} U_i$, we obtain an open affine covering $\mathcal{V} = \{V_0, \dots, V_r\}$ of $X_{\operatorname{Spec} B}$, and we have an isomorphism

$$B \otimes_A \mathcal{F}(U_{i_0} \cap \dots \cap U_{i_p}) \to \mathcal{F}_{\operatorname{Spec} B}(V_{i_0} \cap \dots \cap V_{i_p})$$

for each $0 \le i_0 < \dots < i_p \le r$ of B–modules. Consequently we obtain an isomorphism

$$B \otimes_A \mathcal{F}_{\mathcal{U}} \to (\mathcal{F}_{\operatorname{Spec} B})_{\mathcal{V}} \tag{1.7.1}$$

of complexes of B–modules. Thus we obtain an A–B–linear map

$$\mathcal{F}_{\mathcal{U}} \to B \otimes_A \mathcal{F}_{\mathcal{U}} \to (\mathcal{F}_{\operatorname{Spec} B})_{\mathcal{V}} \tag{1.7.2}$$

where the left map sends f to $1 \otimes_A f$.

We obtain a *restriction map*

$$H^i(X_{\operatorname{Spec} A}, \mathcal{F}_{\operatorname{Spec} A}) \to H^i(X_{\operatorname{Spec} B}, \mathcal{F}_{\operatorname{Spec} B}) \tag{1.7.3}$$

of $H^0(X_{\operatorname{Spec} A}, \mathcal{O}_{X_{\operatorname{Spec} A}})$–$H^0(\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B})$–modules.

In particular, when we associate to each open affine subscheme $\operatorname{Spec} A$ of S the A–module $H^i(X_{\operatorname{Spec} A}, \mathcal{F}_{\operatorname{Spec} A})$, we obtain a pre–sheaf of $\mathcal{O}_S$–modules. The associated $\mathcal{O}_S$–module we denote by $R^i f_* \mathcal{F}$. We have that

$$R^i f_* \mathcal{F} | \operatorname{Spec} A = \widetilde{H^i(X_{\operatorname{Spec} A}, \mathcal{F}_{\operatorname{Spec} A})}, \tag{1.7.4}$$

for all open affine subsets $\operatorname{Spec} A$ of S ([H], (III §8, Proposition 8.6)).

(1.8) Note. From (1.7.4) it follows that the sheaves $R^i f_* \mathcal{F}$ are quasi–coherent $\mathcal{O}_S$–modules. Moreover, it follows from the Notes (1.3)–(1.6), applied to an affine open covering of S, that:

(1) We have $R^i f_* \mathcal{F} = 0$ for $i > r$ and $i < 0$, when X, and thus all affine open subsets of X, can be covered by $r + 1$ open affine subsets.

(2) The correspondence that sends a quasi–coherent $\mathcal{O}_X$–module $\mathcal{F}$ to the quasi–coherent $\mathcal{O}_X$–module $R^i f_* \mathcal{F}$ is functorial in $\mathcal{F}$.

(3) Given a short exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ of quasi–coherent $\mathcal{O}_X$–modules, we obtain a long exact sequence

$$\cdots \to R^i f_* \mathcal{F}' \to R^i f_* \mathcal{F} \to R^i f_* \mathcal{F}'' \to R^{i+1} f_* \mathcal{F} \to \cdots$$

of $\mathcal{O}_S$–modules.

(4) Given a closed immersion $\iota \colon Y \subseteq X$ and a quasi–coherent sheaf $\mathcal{G}$ on Y we have that $(R^i f_*) \iota_* \mathcal{G} = R^i (f_* \iota_*) \mathcal{G} = R^i ((f\iota)_*) \mathcal{G}$.

(1.9) Definition. Given a complex

$$F\colon 0 \to F^0 \xrightarrow{d^0} F^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{r-1}} F^r \to 0$$

of A–modules. We write $Z^i = Z^i(F) = \operatorname{Ker} d^i$ and $B^i = B^i(F) = \operatorname{Im} d^{i-1}$. Then $H^i(F) = Z^i(F)/B^i(F)$ is the *cohomology* of the sequence F. There are exact sequences

$$0 \to Z^i(F) \to F^i \to B^{i+1}(F) \to 0, \tag{1.9.1}$$

and

$$0 \to B^i(F) \to Z^i(F) \to H^i(F) \to 0 \tag{1.9.2}$$

of A–modules for $i = 0, \dots, r$.

Given an A–algebra B. We obtain a complex

$$B \otimes_A F\colon 0 \to B \otimes_A F^0 \xrightarrow{\mathrm{id}_B \otimes_A d^0} B \otimes_A F^1 \xrightarrow{\mathrm{id} \otimes_A d^1} \cdots \xrightarrow{\mathrm{id} \otimes_A d^{r-1}} B \otimes_A F^r \to 0$$

of B–modules, and a map of complexes

$$F \to B \otimes_A F,$$

which sends an element m in F^i to $1 \otimes_A m$ in $B \otimes_A F^i$. For each i we get a map $H^i(F) \to H^i(B \otimes_A F)$ of cohomology, which is a map of A–B–modules. We extend this map to a map

$$B \otimes_A H^i(F) \to H^i(B \otimes_A F) \tag{1.9.3}$$

of B–modules which is called the *map obtained by changing the base from A to B* , or simply the *base change map* .

(1.10) Note. The natural map $B \otimes_A B^i(F) \to B^i(B \otimes_A F)$ of B–modules is a surjection because $B \otimes_A F^i = F^i(B \otimes_A F)$ for all i, and $d^i_{B \otimes_A F}(b \otimes_A m) = b \otimes_A d^i_F(m)$ where $b \in B$ and $m \in F^{i-1}$.

(1.11) Definition. Given a morphism $g\colon T \to S$ from a noetherian scheme T. Let $\operatorname{Spec} A$ of S be an affine subscheme and $\operatorname{Spec} B$ an open affine subscheme of T which maps to $\operatorname{Spec} A$. We obtain from the maps (1.7.1) and (1.9.3) a base change map $B \otimes_A H^i(\mathcal{F}_{\mathcal{U}}) \to H^i(B \otimes_A \mathcal{F}_{\mathcal{U}}) = H^i((\mathcal{F}_{\operatorname{Spec} B})_{\mathcal{V}})$, that is, a B–linear (base change map)

$$B \otimes_A H^i(X_{\operatorname{Spec} A}, \mathcal{F}_{\operatorname{Spec} A}) \to H^i(X_{\operatorname{Spec} B}, \mathcal{F}_{X_{\operatorname{Spec} B}}). \tag{1.11.1}$$

We apply this map to each member S_i of an affine open cover of S, and to each member of an affine open cover of $g^{-1}(S_i)$. It follows from the Definitions of (1.7) that we obtain a *base change map*

$$\mathcal{O}_T \otimes_{\mathcal{O}_S} R^i f_* \mathcal{F} = g^* R^i f_* \mathcal{F} \to R^i f_{T*}(g_X^* \mathcal{F}) = R^i f_{T*} \mathcal{F}_T. \tag{1.11.2}$$

When $S = \operatorname{Spec} A$ we obtain a (base change) map

$$\mathcal{O}_T \otimes_{\mathcal{O}_{\operatorname{Spec} A}} \widetilde{H^i(X, \mathcal{F})} \to R^i f_{T*} \mathcal{F}_T. \tag{1.11.3}$$

2. Cohomology of sheaves on projective spaces.

(2.1) Setup. Given a noetherian ring A and a free A–module E of rank $r+1$. We choose an A–basis $e_0, e_1, \dots, e_r$ of E. Denote by $R = \mathrm{Sym}_A(E)$ the symmetric algebra of E over A and write $\mathbf{P}(E) = \mathrm{Proj}(R)$ for the r–dimensional *projective space over* $\mathrm{Spec}\, A$. The choice of basis $e_0, \dots, e_r$ defines an isomorphism between R and the polynomial ring $A[x_0, x_1, \dots, x_r]$ in the variables $x_0, \dots, x_r$ with coefficients in the ring A. In this way we obtain an isomorphism $\mathbf{P}(E) \cong \mathbf{P}^r_A$. The $r+1$ open affine sets $D_+(e_i)$ cover $\mathbf{P}(E)$.

Denote by $p\colon \mathbf{P}(E) \to \mathrm{Spec}\, A$ the structure map of the projective space, and by $\mathcal{O}_{\mathbf{P}(E)}(1)$ the *tautological invertible sheaf on* $\mathbf{P}(E)$. There is a canonical surjection $p^*E \to \mathcal{O}_{\mathbf{P}(E)}(1)$ of $\mathcal{O}_{\mathbf{P}(E)}$–modules.

A standard calculation ([H], (III, Theorem 5.1)) gives:

(1) The canonical map $R_m \to H^0(\mathbf{P}(E), \mathcal{O}_{\mathbf{P}(E)}(m))$ is an isomorphism.

(2) We have that $H^i(\mathbf{P}(E), \mathcal{O}_{\mathbf{P}(E)}(m)) = 0$ for $i > 0$ and $m \geq 0$.

Given an ideal I in R. Let $X = \mathrm{Proj}(R/I)$, and let $\iota\colon X \to \mathbf{P}(E)$ be the corresponding closed immersion. The $r+1$ open affine sets $U_i = X \cap D_+(e_i)$ cover X.

Given a coherent $\mathcal{O}_X$–module $\mathcal{F}$ on X. For each integer n we write $\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_X} i^*\mathcal{O}_{\mathbf{P}(E)}(n)$. Then we have that $i_*(\mathcal{F}(n)) = i_*(\mathcal{F} \otimes_{\mathcal{O}_X} i^*\mathcal{O}_{\mathbf{P}(E)}(n)) = (i_*\mathcal{F})(n)$, and $i^*i_*\mathcal{F}(n) \to \mathcal{F}(n)$ is an isomorphism for all n.

Write $K = \bigoplus_{m \in \mathbf{Z}} \Gamma(X, \mathcal{F}(m))$. Then we have a canonical isomorphism ([H], (II §5, Proposition 5.15))

$$\beta\colon \widetilde{K} \to \mathcal{F}.$$

Hence $\mathcal{F}$ is the sheaf associated to a graded R/I–module K. We can take this R/I–module to be finitely generated. Indeed, we can choose a finite number of homogeneous elements m of K of degree d such that the elements m/y_i^d, where y_i is the class of e_i in R/I, generate $\mathcal{F}(U_i)$, for $i = 0, \dots, r$. The submodule of K generated by these elements for $i = 0, 1, \dots, r$ defines $\mathcal{F}$. We choose a finitely generated R/I–module $M_{\mathcal{F}}$ such that $\mathcal{F} = \widetilde{M_{\mathcal{F}}}$.

(2.2) Theorem. (Serre) *There is an m_0 such that for $m \geq m_0$ we have:*

(1) *The canonical map*

$$(M_{\mathcal{F}})_m \to H^0(X, \mathcal{F}(m))$$

is an isomorphism.

(2) *There is an equality $H^i(X, \mathcal{F}(m)) = 0$ for $i > 0$*

(3) *The canonical map $\mathcal{O}_X \otimes_{\mathcal{O}_{\mathrm{Spec}\, A}} H^0(X, \mathcal{F}(m)) = f^*f_*\mathcal{F}(m) \to \mathcal{F}(m)$ of $\mathcal{O}_X$–modules is surjective.*

Proof. To simplify the notation we first show that it suffices to prove the Theorem when $X = \mathbf{P}(E)$. It follows from Note (1.6) and the equality $i_*(\mathcal{F}(m)) = (i_*\mathcal{F})(m)$ of Setup (2.1) that $H^i(X, \mathcal{F}(m)) = H^i(\mathbf{P}(E), (\iota_*\mathcal{F})(m))$. Let $M_\mathcal{F}$ be the R/I–submodule of $\oplus_{m\in\mathbf{Z}} H^i(X, \mathcal{F}(m))$ chosen in Setup (2.1). Denote by M the module $M_\mathcal{F}$ considered as a R–submodule of $\oplus_{m\in\mathbf{Z}} H^i(\mathbf{P}(E), (\iota_*\mathcal{F})(m))$. Since $\widetilde{(M_\mathcal{F})} = \mathcal{F}$ on X, we obtain that $\widetilde{(M)} = i_*\mathcal{F}$ on $\mathbf{P}(E)$. Hence we can choose the module M for the module $M_{i_*\mathcal{F}}$ of Setup (2.1). It follows that it suffices to prove assertions (1) and (2) of the Theorem in the case when $X = \mathbf{P}(E)$. Since $i^*i_*\mathcal{F} \to \mathcal{F}$ is an isomorphism it also follows that it suffices to prove assertion (3) in this case.

When $M = R(d)$ is R with gradind translated by d we have that $\mathcal{F} = \mathcal{O}_{\mathbf{P}(E)}(d)$, and, as we noted in (2.1), we have

$$M_m = R_{d+m} \cong H^0(\mathbf{P}(E), \mathcal{O}_{\mathbf{P}(E)}(d+m)), \text{ and } H^i(\mathbf{P}(E), \mathcal{O}_{\mathbf{P}(E)}(d+m)) = 0$$

for $i > 0$ and $d + m \geq 0$. Hence assertions (1) and (2) of the Theorem hold for the modules $\mathcal{O}_{\mathbf{P}(E)}(d)$.

In general, choose a short exact sequence of graded R–modules

$$0 \to K \to L \to M \to 0, \tag{2.2.1}$$

where L is the direct sum of finitely many modules of the form $R(d)$. Since A is noetherian we have that K is a finitely generated A–module. We shall prove, by descending induction on i, that the second assertion of the Theorem holds. Since $\mathbf{P}(E)$ can be covered by $r+1$ open affines it follows from Note (1.3) that the assertion holds for $i > r$. Assume that we have proved that $H^{i+1}(\mathbf{P}(E), \mathcal{F}(m)) = 0$ for all coherent $\mathcal{O}_{\mathbf{P}(E)}$–modules $\mathcal{F}$ for sufficiently big m depending on $\mathcal{F}$. From the short exact sequence sequence (2.2.1) we obtain a long exact sequence

$$\cdots \to H^i(\mathbf{P}(E), \widetilde{K}(m)) \to H^i(\mathbf{P}(E), \widetilde{L}(m)) \to \\ H^i(\mathbf{P}(E), \mathcal{F}(m)) \to H^{i+1}(\mathbf{P}(E), \widetilde{K}(m)) \to \cdots .$$

As we already observed assertion (2) of the Theorem holds for $\widetilde{L}$ by Note (2.1), and by the induction assumption $H^{i+1}(\mathbf{P}(E), \widetilde{K}(m)) = 0$ for big m. Consequently we have that $H^i(\mathbf{P}(E), \mathcal{F}(m)) = 0$ for big m. Hence we have proved the second part of the Theorem. In particular we have that $H^1(\mathbf{P}(E), \widetilde{K}(m)) = 0$ Thus the map $H^0(\mathbf{P}(E), \widetilde{L}(m)) \to H^0(\mathbf{P}(E), \mathcal{F}(m))$ is surjective when m is sufficiently big. We obtain a commutative diagram of A–modules

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_m & \longrightarrow & L_m & \longrightarrow & M_m & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^0(\mathbf{P}(E), \widetilde{K}(m)) & \longrightarrow & H^0(\mathbf{P}(E), \widetilde{L}(m)) & \longrightarrow & H^0(\mathbf{P}(E), \mathcal{F}(m)) & \longrightarrow & 0,
\end{array}$$

with exact rows, where the middle vertical map is an isomorphism since we observed that assertion (1) of the Theorem holds for L. Consequently the right vertical map is surjective for big m. Since this holds for all finitely generated R–modules the left vertical map is also surjective for big m. Consequently we have that the right vertical map is an isomorphism for big m, and we have proved the first part of the Theorem.

The third part of the Theorem holds for the modules $\mathcal{O}_{\mathbf{P}(E)}(d)$ because of the surjection $f^* S^{m+d}(E) = S^{m+d}(E) \otimes_A \mathcal{O}_{\mathbf{P}(E)} \to \mathcal{O}_{\mathbf{P}(E)}(m+d)$, and the isomorphism $S^{m+d}(E) \to R_{m+d} \to H^0(\mathbf{P}(E), \mathcal{O}_{\mathbf{P}(E)}(m+d))$. Hence the left vertical map of the commutative diagram

$$\begin{tikzcd}
\mathcal{O}_{\mathbf{P}(E)} \otimes_{\mathcal{O}_{\operatorname{Spec} A}} H^0(\mathbf{P}(E), \tilde{L}(m)) \arrow[r] \arrow[d] & \mathcal{O}_{\mathbf{P}(E)} \otimes_{\mathcal{O}_{\operatorname{Spec} A}} H^0(\mathbf{P}(E), \mathcal{F}(m)) \arrow[d] & \\
\tilde{L}(m) \arrow[r] & \mathcal{F}(m) \arrow[r] & 0
\end{tikzcd}$$

is surjective for big m. It follows that the right vertical map is surjective, and we have proved the third part of the Theorem.

(2.3) Note. There is an m_0 such that for each $m \geq m_0$ there is a surjection

$$\mathcal{O}_X^n \to \mathcal{F}(m)$$

of $\mathcal{O}_X$–modules, where n depends on m. Indeed, it follows from the first part of Theorem (2.2) that we can find a surjection $A^n \to H^0(X, \mathcal{F}(m))$, for fixed big m, and from the third part of Theorem (2.2) that we have a surjection $\mathcal{O}_X \otimes_{\operatorname{Spec} A} \widetilde{H^0(X, \mathcal{F}(m))} \to \mathcal{F}(m)$ for big m.

(2.4) Note. For every integer m we have a map

$$\beta_m \colon f_* \mathcal{F}(m) \otimes_{\mathcal{O}_{\operatorname{Spec} A}} f_* \mathcal{O}_X(1) \to f_* \mathcal{F}(m+1) \tag{2.4.1}$$

of $\mathcal{O}_{\operatorname{Spec} A}$–modules induced by the isomorphism $\mathcal{F}(m) \otimes_{\mathcal{O}_X} \mathcal{O}_X(1) \to \mathcal{F}(m+1)$. Equivalently we have a map

$$\beta_m(\operatorname{Spec} A) \colon H^0(X, \mathcal{F}(m)) \otimes_A H^0(X, \mathcal{O}_X(1)) \to H^0(X, \mathcal{F}(m+1)), \tag{2.4.2}$$

of A–modules. There is an m_0 such that for $m \geq m_0$ this map is surjective. This can be seen from the commutative diagram

$$\begin{tikzcd}
(M_{\mathcal{F}})_m \otimes_A (R/I)_1 \arrow[r] \arrow[d] & (M_{\mathcal{F}})_{m+1} \arrow[d] \\
H^0(X, \mathcal{F}(m)) \otimes_A H^0(X, \mathcal{O}_X(1)) \arrow[r] & H^0(X, \mathcal{F}(m+1)),
\end{tikzcd}$$

where the upper row is multiplication. Since $M_{\mathcal{F}}$ is a finitely generated (R/I)–module the multiplication map is surjective for big m. It follows from Theorem (2.2) that the right vertical map is an isomorphism for big m. Thus there is an m_0 such that the bottom row is surjective for $m \geq m_0$. That is, the map β_m is surjective for big m.

We also note that if (2.4.1) is surjective for $m \geq m_0$, then

$$\alpha_m \colon f^* f_* \mathcal{F}(m) \to \mathcal{F}(m)$$

is surjective. To see this we note that from the maps β_m we obtain maps

$$\beta_{m,d} \colon f_* \mathcal{F}(m) \otimes_{\mathcal{O}_A} f_* \mathcal{O}_X(d) \to f_* \mathcal{F}(m+d)$$

for each integer d. If β_m is surjective for $n \geq m$ we have that $\beta_{m,d}$ is surjective. We obtain a commutative diagram

$$\begin{tikzcd}
f^* f_* \mathcal{F}(m) \otimes_{\mathcal{O}_{\operatorname{Spec} A}} f^* f_* \mathcal{O}_X(d) \arrow[r, "f^*\beta_{m,d}"] \arrow[d, "\alpha_m \otimes \gamma_d"'] & f^* f_* \mathcal{F}(m+d) \arrow[d] \\
\mathcal{F}(m) \otimes_{\mathcal{O}_{\operatorname{Spec} A}} \mathcal{O}_X(d) \arrow[r] & \mathcal{F}(m+d)
\end{tikzcd}$$

for each d, where $f^*\beta_{m,d}$ is surjective. It follows from Theorem (2.2) that the right vertical map is surjective for d sufficiently big. Since the bottom horizontal map is an isomorphism we have that $\alpha_m \otimes \gamma_d$ is surjective for big d. However we have that $\gamma_d \colon f^* f_* \mathcal{O}_X(d) = f^* \operatorname{Sym}^d(E) \to \mathcal{O}_X(d)$ is surjective for $d \geq 0$. Hence α_m is surjective, as asserted.

(2.5) Definition. Let A be a noetherian ring. A graded A–algebra $S = \oplus_{i=0}^{\infty} S_i$ is called *standard* if $S_0 = A$ and S is generated, as an A–algebra, by the elements S_1 of degree 1.

(2.6) Lemma. *Let S be a standard A–algebra and N a finitely generated graded S–module such that $N_m \neq 0$ for big m. Then N has a filtration $0 = N_0 \subset N_1 \subset \cdots \subset N_n = N$ by graded submodules such that N_i/N_{i-1} is isomorphic to $(S/P_i)(m_i)$, where P_i is a prime ideal of S, and m_i is an integer. In particular the support of $\widetilde{N}$ on $\operatorname{Proj}(S)$ consists of the homogeneous prime ideals in S that contain one of the ideals P_i.*

Proof. See [H] (I §7 Proposition 7.4).

(2.7) Theorem. *The A–module $H^i(X, \mathcal{F})$ is finitely generated for all i.*

Proof. To simplify the notation we note that from the equality $H^0(X, \mathcal{F}) = H^0(\mathbf{P}(E), \iota_* \mathcal{F})$ it follows that we only have to prove the Theorem when $X = \mathbf{P}(E)$.

We shall prove the Theorem when $X = \mathbf{P}(E)$ by induction on the dimension s of the support $\operatorname{Supp} \mathcal{F}$ of $\mathcal{F} = \widetilde{M}$. When $s < 0$ we have that $\mathcal{F} = 0$ and the statement is true. Assume that $s \geq 0$. It follows from Lemma (2.6) that M has a finite filtration whose quotients are isomorphic to $(R/P)(d)$, where P is a prime ideal in R. Since $s \geq 0$ we have that P does not contain the ideal $(e_0, \dots, e_r)$, and the support of $\mathcal{F}$ is the union of the irreducible varieties $Z(P)$ in $\mathbf{P}(E)$. Consequently we can assume that $\mathcal{F}$ is the sheaf associated to $L = (R/P)(d)$. Choose a homogeneous element f of degree m in R not contained in P. We have an exact sequence

$$0 \to L \xrightarrow{f} L(m) \to N \to 0. \tag{2.7.1}$$

The dimension of $\operatorname{Supp} N$ is strictly less than s because $\operatorname{Supp} \mathcal{F} = Z(P)$ and f is an isomorphism at the generic point of $Z(P)$. It follows from Theorem (2.2) that we can choose m so big that $H^0(\mathbf{P}(E), \mathcal{F}(m))$ is a finitely generated A–module, and $H^i(\mathbf{P}(E), \mathcal{F}(m)) = 0$ for $i > 0$. From the short exact sequence (2.7.1) we obtain a long exact sequence,

$$\cdots \to H^{i-1}(\mathbf{P}(E), \widetilde{N}) \to H^i(\mathbf{P}(E), \mathcal{F}) \to \\ H^i(\mathbf{P}(E), \mathcal{F}(m)) \to H^i(\mathbf{P}(E), \widetilde{N}) \to \cdots .$$

Since the A–module $H^i(\mathbf{P}(E), \widetilde{N})$ is finitely generated for all i, by the induction assumption, it follows that $H^i(\mathbf{P}(E), \mathcal{F})$ is a finitely generated A–module.

3. Flat maps.

(3.1) Setup. Given a ring A and an A–module M. For each prime ideal P of A we write $\kappa(P) = A_P/PA_P$. Let E be a free A–module of rank $r+1$ and $e_0, \dots, e_r$ a basis of E. Denote by $R = \mathrm{Sym}_A(E)$ the symmetric algebra of E over A and write $\mathbf{P}(E) = \mathrm{Proj}(R)$ for the r–dimensional projective space over $\operatorname{Spec} A$.

The particular quotient $A[x]/(x^2)$ we denote by $A[\varepsilon]$ where ε is the class of the variable x over A. Moreover we let $M[\varepsilon] = A[\varepsilon] \otimes_A M$.

(3.2) Definition. Given an A–module M. The module M is *flat* over A if every short exact sequence

$$0 \to N' \to N \to N'' \to 0$$

gives rise to a short exact sequence

$$0 \to M \otimes_A N' \to M \otimes_A N \to M \otimes_A N'' \to 0.$$

(3.3) Definition. Given a morphism $f \colon X \to S$ of schemes and an $\mathcal{O}_X$–module $\mathcal{F}$. We say that $\mathcal{F}$ *is flat over* S if, for every point x of X, we have that $\mathcal{F}_x$ is a flat $\mathcal{O}_{S,f(x)}$–module, where the module structure comes from the map $f^{-1}\mathcal{O}_{S,f(x)} \to \mathcal{O}_{X,x}$, or equivalently from the composite map $\mathcal{O}_{S,f(x)} \to (f_*\mathcal{O}_X)_{f(x)} \to \mathcal{O}_{X,x}$. The morphism f *is flat* if $\mathcal{O}_X$ is flat over S.

When f is the identity we say that $\mathcal{F}$ *is a flat* $\mathcal{O}_S$*–module* .

(3.4) Remark. Flatness has the following fundamental properties:

(1) (*Long exact sequences*) We can break long exact sequences into short exact sequences. Hence M is flat over A if and only if every exact sequence

$$\cdots \to N' \to N \to N'' \to \cdots$$

of A–modules gives rise to an exact sequence

$$\cdots \to M \otimes_A N' \to N \otimes_A N \to M \otimes_A N'' \to \cdots .$$

→ (2) (*Left exactness*) Since the tensor product is right exact ([A-M], (2.18)) we have that M is flat over A if every injective map $N' \to N$ of A–modules gives rise to an injective map $M \otimes_A N' \to M \otimes_A N''$.

(3) (*Localization*) Let S be a multiplicatively closed subset of A. It follows from the definition of localization that the localization $S^{-1}A$ of A in S, that $S^{-1}A$ is a flat A–module.

(4) (*Base change*) Given a flat A–module N, and let B be an A–algebra. Then $B \otimes_A N$ is a flat B–module. Indeed, for every B–module P we have an isomorphism $P \otimes_B (B \otimes_A N) \cong P \otimes_A N$.

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(5) (*Direct sums*) For every set $(N_i)_{i\in I}$ of A–modules and every A–module P we have an isomorphism $P \otimes_A (\oplus_{i\in I} N_i) \cong \oplus_{i\in I}(P \otimes_A N_i)$. Hence $\oplus_{i\in I} N_i$ is exact if and only if it is exact in every factor N_i. We conclude that $\oplus_{i\in I} N_i$ is flat over A if and only if each summand N_i is flat over A. It follows in particular that every free A–module is flat. Moreover, projective A–modules are flat because they are direct summands of free modules.

(3.5) Lemma. *Given an exact sequence*

$$0 \to M \to N \to F \to 0$$

of A–modules, where F is flat. Then the sequence

$$0 \to P \otimes_A M \to P \otimes_A N \to P \otimes_A F \to 0$$

is exact for all A–modules P.

Proof. Write P as a quotient of a free A–module L,

$$0 \to K \to L \to P \to 0.$$

We obtain a commutative diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] \\
 & K \otimes_A M \arrow[r] \arrow[d] & K \otimes_A N \arrow[r] \arrow[d] & K \otimes_A F \arrow[d] \\
0 \arrow[r] & L \otimes_A M \arrow[r] \arrow[d] & L \otimes_A N \arrow[r] \arrow[d] & L \otimes_A F \\
 & P \otimes_A M \arrow[r] \arrow[d] & P \otimes_A N & \\
 & 0 & &
\end{tikzcd}$$

where the upper right vertical map is injective because F is flat, and the middle left horizontal map is injective because L is free. A diagram chase gives that $P \otimes_A M \to P \otimes_A N$ is injective.

(3.6) Proposition. *Given an exact sequence*

$$0 \to F' \to F \to F'' \to 0$$

of A–modules with F'' flat. Then F is flat if and only if F' is flat.

Proof. Given an injective map $M' \to M$. We obtain a commutative diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
0 \arrow[r] & M' \otimes_A F' \arrow[r] \arrow[d] & M' \otimes_A F \arrow[r] \arrow[d] & M' \otimes_A F'' \arrow[r] \arrow[d] & 0. \\
0 \arrow[r] & M \otimes_A F' \arrow[r] & M \otimes_A F \arrow[r] & M \otimes_A F'' \arrow[r] & 0
\end{tikzcd}$$

The rows are exact to the left by Lemma (3.5), and we have injectivity of the top vertical map since F'' is flat. The Proposition follows from a diagram chase.

(3.7) Lemma. *Given an A–module M such that the map*

$$I \otimes_A M \to IM$$

is an isomorphism for all ideals I in A. For every free A–module F and every injective map $K \to F$ of A–modules we have that

$$K \otimes_A M \to F \otimes_A M$$

is injective.

Proof. Since every element in $K \otimes_A M$ is mapped into $F' \otimes_A M$ where F' is a finitely generated free submodule of F we can assume that F is finitely generated.

When the rank of F is 1 the Lemma follows from the assumption. We prove the Lemma by induction on the rank r of F. We have an exact sequence $0 \to F_1 \to F \to A \to 0$, where F_1 is a free rank $r-1$ module. Let $K_1 = K \cap F_1$ and let K_2 be the image of K in A. We obtain a diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & \\
 & K_1 \otimes_A M \arrow[r] \arrow[d] & K \otimes_A M \arrow[r] \arrow[d] & K_2 \otimes_A M \arrow[r] \arrow[d] & 0. \\
0 \arrow[r] & F_1 \otimes_A M \arrow[r] & F \otimes_A M \arrow[r] & A \otimes_A M &
\end{tikzcd}$$

where the right and left top vertical maps are injective by the induction assumption and it follows from Lemma (3.5) that the lower left map is injective because A is free. A diagram chase proves that the middle vertical map is injective.

(3.8) Proposition. *An A–module M is flat if and only if the map*

$$I \otimes_A M \to IM$$

is an isomorphism for all finitely generated ideals I of A.

Proof. If M is flat the tensor product $I \otimes_A M \to M$ of the map $I \to A$ is injective so $I \otimes_A M \to IM$ is an isomorphism.

Conversely, we can assume that $I \otimes_A M \to IM$ is an isomorphism for all ideals I of A. Indeed, every element of $I \otimes_A M$ is contained in $J \otimes_A M$, where J is a finitely generated ideal, and if $J \otimes_A M \to M$ is injective and the element is not zero then it is not mapped to zero by $I \otimes_A M \to M$.

Let $N' \to N$ be an injective map and write N as a quotient $0 \to K \to F \to N \to 0$ of a free A–module F. Let F' be the inverse image of N' in F. Then we have an exact sequence $0 \to K \to F' \to N' \to 0$, and we obtain a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & & \\
K \otimes_A M \arrow[r] \arrow[d] & F' \otimes_A M \arrow[r] \arrow[d] & N' \otimes_A M \arrow[r] \arrow[d] & 0. \\
K \otimes_A M \arrow[r] & F \otimes_A M \arrow[r] & N \otimes_A M &
\end{tikzcd}$$

It follows from Lemma (3.7) that the top vertical map is injective. A diagram chase shows that the right vertical map is injective. Consequently M is flat over A.

(3.9) Remark. It follows from Proposition (3.8) that a module over a principal ideal domain is flat if and only if it does not have torsion.

(3.10) Lemma. *Given a map $\varphi: A \to B$ of rings and let N be a B–module. Then N is flat over A if and only if N_Q is flat over A_P for all prime ideals P in A and Q in B such that $\varphi^{-1}(Q) = P$.*

Proof. Assume that N is flat over A. Since B_Q is flat over B the functor that sends an A_P–module F to $B_Q \otimes_B (N \otimes_A F)$ is exact. However $B_Q \otimes_B (N \otimes_A F) = N_Q \otimes_A F = N_Q \otimes_{A_P} F$. Consequently the functor that sends the A_P–module F to the A_P–module $N_Q \otimes_{A_P} F$ is exact, that is, the A_P–module N_Q is flat.

Conversely, assume that N_Q is a flat A_P module for all prime ideals Q in B with $P = \varphi^{-1}(Q)$. The functor that sends an A–module F to the A_P–module F_P

is exact by Note (3.4(3)). Consequently the functor that sends the A–module F to the B_Q–module $N_Q \otimes_{A_P} F_P$ is exact. However, we have that $N_Q \otimes_{A_P} F_P = N_Q \otimes_{A_P} (A_P \otimes_A F) = N_Q \otimes_A F$. Hence the functor that sends an A–module F to $N_Q \otimes_A F$ is exact. However, the functor that sends an A–module F to the B–module $N \otimes_A F$ is exact if and only if the functor that sends the A–module F to the B_Q–module $N_Q \otimes_A F$ is exact for all prime ideal Q of B. We thus have that N is a flat A–module.

(3.11) Note. Given a morphism $f\colon X \to S$ of schemes and a quasi–coherent $\mathcal{O}_X$–module $\mathcal{F}$. It follows from Lemma (3.10) that $\mathcal{F}$ is flat over $\operatorname{Spec} A$ if and only if $\mathcal{F}(U)$ is a flat A–module for all open affine subsets U of X.

In particular, if $\mathcal{F}$ is flat over $\operatorname{Spec} A$, and $U_0, \dots, U_r$ is an open affine covering of X, the module $\mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_p})$ is flat over A for all $0 \le i_0 < \cdots < i_p \le r$, and $\mathcal{F}_{\mathcal{U}}$ is a complex of flat A–modules.

(3.12) Lemma. *Given a regular* ([A-M], (Theorem 11.22)) *one dimensional ring A and a homomorphism $\varphi\colon A \to B$ into a noetherian ring B. Then B is flat over A if and only if $\varphi^{-1}(Q) = 0$ for all associated prime ideals Q in B.*

In particular, when B is reduced, we have that B is flat over A if and only if $\varphi^{-1}(Q) = 0$ for all minimal primes Q of B.

Proof. Assume that B is flat over A and let Q be a prime ideal in B. If $P = \varphi^{-1}(Q)$ is maximal we have that A_P is a discrete valutation ring ([A-M] (Proposition 9.2 and Lemma 11.23)). Let $t \in PA_P$ be a generator for the maximal ideal. Since t is not a zero divisor in A_P and B_Q is a flat A_P–module it follows that t is not a zero divisor in B_Q. Consequently Q is not an associated prime in B.

Conversely, assume that $\varphi^{-1}(Q)$ is zero for all associated primes Q of B. It follows from Lemma (3.10) that we must prove that B_R is flat over $A_{\varphi^{-1}(R)}$ for all prime ideals R in B. If $\varphi^{-1}(R) = 0$ we have that $A_{\varphi^{-1}(R)}$ is a field and consequently that B_R is flat. On the other hand, if $P = \varphi^{-1}(R)$ is a maximal ideal we choose a $t \in \varphi^{-1}(R)$ that generates the ideal PA_P. Since A_P is a principal ideal domain it follows from Remark (3.9) that it suffices to show that B_R is a torsion free A_P–module. Since all elements of A_P can be written as a power of t times a unit, this means that it suffices to prove that t is not a zero divisor in B_R. However, if t were a zero divisor in B_R it would be contained in an associated prime ideal Q of B since B is noetherian. This is impossible because $t \neq 0$ and, by assumption, $\varphi^{-1}(Q) = 0$. Hence t is not zero divisor and we have proved the first part of the Proposition.

The last part of the Proposition follows since in a reduced ring the associated primes are the minimal primes. Indeed, on the one hand every prime ideal contains an associated prime so that the minimal primes are associated. Conversely, let Q be an associated prime and $Q_1, \dots, Q_n$ be the minimal primes. Choose a non zero element a such that $aQ = 0$. We have that $Q \subseteq Q_1 \cup \cdots \cup Q_n$ because if $b \in Q \setminus Q_1 \cup \cdots \cup Q_n$ then $ab = 0$ and thus $a \in Q_1 \cap \cdots \cap Q_n = 0$, contrary to the assumption that a is not zero. Hence $Q \subseteq Q_1 \cup \cdots \cup Q_n$ and thus $Q \subseteq Q_i$ for some i ([A-M] (Proposition 1.11)). Hence $Q \subseteq Q_i$ and Q is minimal.

(3.13) Proposition. *Assume that A is a regular ring of dimension one. Given a morphism $f\colon X \to \operatorname{Spec} A$ from a noetherian scheme X. Then f is flat if and only if the associated points of X are mapped to the generic point of* $\operatorname{Spec} A$.

In particular, if X is reduced we have that f is flat if and only if the components of X all dominate $\operatorname{Spec} A$.

Proof. The Proposition is an immediate consequence of Lemma (3.12).

(3.14) Lemma. *Assume that A is noetherian and that M is a finitely generated A–module. Then M is flat if and only if M_P is a free A_P–module for all prime ideals P of A.*

Proof. It follows from Lemma (3.12) that M is flat over A if and only if M_P is flat over A_P for all primes P of A. Since M_P is flat over A_P if M_P is free over A_P it follows that when M_P is a free A_P–module for all prime ideals P of A, we have that M is a flat A–module.

Coversely, assume that M is a flat A–module. Given a prime ideal P of A. The M_P is a flat A_P–module. Since M is finitely generated it follows from Nakayama's Lemma that we can choose a surjection $A_P^n \to M_P$ such that $(\kappa(P))^n \to \kappa(P) \otimes_{A_P} M_P$ is an isomorphism of $\kappa(P)$–vectorspaces. Denote by L the kernel of $A_P^n \to M_P$. Since A is noetherian we have that L a is finitely generated A–module. However, since M is flat, we have that $\kappa(P) \otimes_{A_P} L = 0$. It follows by Nakayamas Lemma that $L = 0$. Consequently we have that the map $A_P^n \to M_P$ is an isomorphism, and that M_P is a free A_P–module.

(3.15) Lemma. *With the notation of Definition* (1.9)*, assume that the A–modules $F^0, F^1, \ldots$ of the complex F are flat and that $H^i(F)$ is a flat A–module for $i \geq p$. Then the A–modules $B^i(F)$ and $Z^{i-1}(F)$ are flat for $i \geq p$.*

Proof. We prove the Lemma by descending induction on p. The Lemma holds for $p > r$ since $Z^r = F^r$. Assume that the Lemma holds for $p+1$. By the induction assumption we have that B^{p+1} and Z^p are flat. From the sequence (1.9.2) with $i = p$ and Proposition (3.6) it follows that B^p is flat. Then, from the sequence (1.9.1) with $i = p-1$ and Proposition (3.6) it follows that Z^{p-1} is flat.

(3.16) Theorem. *Given a noetherian scheme S and a morphism $f\colon X \to S$ which is separated of finite type. Let $\mathcal{F}$ be a (kvasi?) coherent $\mathcal{O}_X$–module. Then:*

(1) *Assume that $\mathcal{F}$ is flat over S and that $R^i f_* \mathcal{F} = 0$ for $i > 0$. Then $f_*\mathcal{F}$ is a flat $\mathcal{O}_S$–module.*

In particular, if $f_*\mathcal{F}$ is coherent, we have that $f_*\mathcal{F}$ is locally free.

(2) *Assume that $S = \operatorname{Spec} A$ and that X is a closed subscheme of $\mathbf{P}(E)$. If there is an m_0 such that $f_*\mathcal{F}(m)$ is locally free for $m \geq m_0$, we have that $\mathcal{F}$ is flat over* $\operatorname{Spec} A$.

Proof. Both assertions are local on S. Hence we can assume that $S = \operatorname{Spec} A$ in both cases. Then it follows from the equality (1.7.4) that $f_*\mathcal{F} = H^0(X, \mathcal{F})$. Hence $f_*\mathcal{F}$ is a flat $\mathcal{O}_S$–module if and only if $H^0(X, \mathcal{F})$ is flat over A. The last part of (1) consequently follows from the first part of Lemma (3.14).

If $\mathcal{F}$ is flat over $\operatorname{Spec} A$ it follows from Note (3.11) that $\mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_p})$ is flat over A, and thus that the complex $\mathcal{F}_{\mathcal{U}}$ consists of flat modules. From the assumption of the Theorem we have that $H^i(\mathcal{F}_{\mathcal{U}}) = H^i(X, \mathcal{F}) = 0$ for $i > 0$. It follows from Lemma (3.15) with $p = 1$ that $Z^0(\mathcal{F}_{\mathcal{U}}) = H^0(X, \mathcal{F})$ is flat, and we have proved the first assertion.

By Assumption we have that $H^0(X, \mathcal{F}(m)) = f_*\mathcal{F}(m)(\operatorname{Spec} A)$ is flat for $m \geq m_0$. Let $N = \oplus_{m \geq m_0} H^0(X, \mathcal{F}(m))$. Then it follows from Setup (2.1) that N is an R/I–module such that $\mathcal{F} = \widetilde{N}$, where $I \subseteq R$ is an ideal defining X in $\mathbf{P}(E)$. We have, with the notation of Setup (2.1) that $\mathcal{F}(U_i) = \widetilde{N}(U_i) = \widetilde{N_{(y_i)}}$, where y_i is the class of e_i in R/I. It therefore suffices to prove that $N_{(y_i)}$ is flat over A. However, the module N is a direct sum of flat A–modules, and thus flat over A. Hence the functor which sends an A–module L to the A–module $N \otimes_A L$ is exact. We consider $N \otimes_A L$ as an R/I–module, via the action of R/I on N. Since $(R/I)_{y_i}$ is flat over R/I for all i we have that the functor that sends an A–module L to the A–module $(R/I)_{y_i} \otimes_{(R/I)} N \otimes_A L$ is exact. Hence $(R/I)_{y_i} \otimes_{(R/I)} N = N_{y_i}$ is a flat A–module. The same is therefore true for the direct summand $N_{(y_i)}$ of degree zero.

(3.17) Lemma. *Given a noetherian integral domain A and an A–algebra B of finite type. Moreover, given a finitely generated B–module N. Then there is a non–zero element $f \in A$ such that N_f is free over A_f.*

Proof. Write $B = A[u_1, \dots, u_h]$. We shall prove the Lemma by induction on h. When $h = 0$ we have that $A = B$. It follows from Lemma (2.6) in the non graded case that we can choose a filtration $N = N_n \supset N_{n-1} \supset \cdots \supset N_0 = 0$ by A–modules such that $N_i/N_{i-1} = A/P_i$, where P_i is a prime ideal in A. Since A is an integral domain we have that the intersection of the non zero primes P_i is not zero. Choose a non zero $f \in A$ in this intersection if there is one non zero prime P_i and let $f = 1$ otherwise. Then $(N_i/N_{i-1})_f$ is zero if P_i is a non zero prime and isomorphic to A_f when $P_i = 0$. Consequently we have that N_f is a free A_f–module.

Assume that $h > 0$ and that the Lemma holds for $h - 1$. Choose generators

$n_1, \dots, n_s$ for the B–module N and write $B' = A[u_1, \dots, u_{h-1}]$. Then $B = B'[u_h]$. Moreover, let $N' = B'n_1 + \cdots B'n_s$. We have that N' is a finitely generated B'–module such that $BN' = N$. It follows from the induction assumption used to the A–algebra B' and the B'–module N' that we can find an element $f' \in A$ such that $N'_{f'}$ is a free $A_{f'}$–module. It therefore remains to prove that we can find an element $f'' \in A$ such that $(N/N')_{f''}$ is a free $A_{f''}$–module. To this end we write

$$N'_i = N' + u_h N' + \cdots + u_h^i N'$$

and

$$P_i = \{n \in N' : u_h^{i+1} n \in N'_i\}.$$

Clearly N'_i is a B'–submodule of N and P_i a B'–submodule of N'. We obtain a filtration

$$N'_1/N' \subseteq N'_2/N' \subseteq \cdots \subseteq N/N'$$

of N/N' by B'–modules N'_i/N' such that $\cup_i N'_i/N' = N/N'$. The B'–linear homomorphism $N' \to N'_{i+1}$ which sends n to $u_h^{i+1} n$ defines an isomorphism $N'/P_i \to N'_{i+1}/N'_i$ for all i. Since B' is noetherian, the sequence $P_0 \subseteq P_1 \subseteq \cdots \subseteq N'$ must stabilize. That is, among the quotients N'_{i+1}/N'_i there appears only a finite number of B'–modules. It follows from the induction assumption that we can find an element $f'' \in A$ such that all the modules $(N'_{i+1}/N'_i)_{f''}$ are free $A_{f''}$–modules. Hence $(N/N')_{f''}$ is a free $A_{f''}$–module, as we wanted to prove.

(3.18) Proposition. (Generic flatness) *Given a morphism $f: X \to S$ of finite type to a noetherian integral scheme S, and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$–module. Then there is an open dense subset U of S such that $\mathcal{F}_U$ is flat over U.*

Proof. We clearly can assume that S is affine. Since f is of finite type we can cover
→ X with a finite number of open affine subschemes X_i. It follows from Lemma (3.17) that, for each i, there is an open dense affine subset U_i of S such that $(\mathcal{F}|X_i)_{U_i}$ is flat over U_i. We can take U to be the intersection of the sets U_i.

(3.19) Proposition. *Given a morphism $f: X \to S$ finite type to a noetherian scheme S and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$–module. Then S is a finite set theoretic union of locally closed reduced and disjoint subschemes S_i such that $\mathcal{F}_{S_i}$ is flat over S_i.*

Proof. Assume that the Proposition does not hold. Since S is noetherian there is a closed subscheme T of X which is minimal among the closed subschemes for which the Proposition does not hold. Let T' be an irreducible component of T with the reduced scheme structure and let V' be an open subset of T' that does not intersect the other components of T. Then V' is also open in T. It follows

→ from Proposition (3.18) that there is an open non–empty subset V of V' such that $\mathcal{F}_V$ is flat over V. By the induction assumption the complement of V in T has a stratification, and together with V this gives a stratification of T. This contradicts the assumption that T has no stratification and we have proved the Proposition.

(3.20) Proposition. *Assume that A is a regular ring of dimension one. Let x be a closed point in* $\operatorname{Spec} A$ *and Y a closed subscheme of $p^{-1}(\operatorname{Spec} A \setminus \{x\})$ which is flat over* $\operatorname{Spec} A \setminus x$ *and $\overline{Y}$ the scheme theoretic closure of Y in $\mathbf{P}(E)$ Then $\overline{Y}$ is the unique closed subscheme of $\mathbf{P}(E)$ which is flat over* $\operatorname{Spec} A$ *and whose restriction to $p^{-1}(\operatorname{Spec} A \setminus x)$ is equal to Y.*

Proof. Let P be the prime ideal in A corresponding to the point x of $\operatorname{Spec} A$. It clearly suffices to prove the Proposition for an open affine subset $\operatorname{Spec} C$ of $\mathbf{P}(E)$. Let $\varphi\colon A \to C$ be the homomorphism induced by the projection of $\mathbf{P}(E)$.

We have that $\operatorname{Spec} A \setminus x = \operatorname{Spec} A_t$ where t in P is the generator of PA_P. We have that $\operatorname{Spec} C \cap f^{-1}(\operatorname{Spec} A \setminus \{x\}) = \operatorname{Spec} C_{\varphi(t)}$. Let $C_{\varphi(t)} \to B$ define the closed subscheme $Y \cap \operatorname{Spec} C_{\varphi(t)}$ of $\operatorname{Spec} C_{\varphi(t)}$. The closure of $Y \cap \operatorname{Spec} C_{\varphi(t)}$ in $\operatorname{Spec} C$ is defined by the kernel I of the composite map $C \to C_{\varphi(t)} \to B$.

Since A is a principal ideal domain and B is flat, we have that B has no torsion over A. Hence the submodule C/I of B has no torsion, and thus C/I is flat over A. We have proved that the scheme theoretic closure $\overline{Y}$ of Y is flat over $\operatorname{Spec} A$. Hence $C_{\varphi(t)}/IC_{\varphi(t)}$ is flat over A_t(?).

To prove that Y is unique with the given properties we let J be an ideal in C that defines a closed subset which is flat over $\operatorname{Spec} A$ and whose restriction to $\operatorname{Spec} C_{\varphi(t)}$ is Y. That is, the ring C/J is flat over A and has the same image in $C_{\varphi(t)}$ as I. Then $J \subseteq I$. It remains to show that $I \subseteq J$. Let $c \in I$. Since I and J have the same image in $C_{\varphi(t)}$ we have that $t^n c \in J$ for some n. Since C/J is flat over A we have that C/J has no A–torsion. Hence $c \in J$ and we have that $I = J$.

(3.21) Lemma. *Let $A \to B$ be an A-algebra and F a B-module. Moreover let $H \subseteq F$ be a sumodule such that F/H is flat over A. For every homomorphism of B-modules*

$$u : H \to F/H$$

we define

$$H_u = \{x + \varepsilon y \in F[\varepsilon] : x \in H \text{ and } u(x) = u_{F/H}(y)\}.$$

Then:

(1) *The group H_u is a $B[\varepsilon]$-submodule of $F[\varepsilon]$ with image by the canonical map $u_{F[\varepsilon]/\varepsilon F[\varepsilon]} : F[\varepsilon] \to F$ equal to H, and where $F[\varepsilon]/H_u$ a flat $A[\varepsilon]$-module.*
(2) *The correspondence that sends the homomorphism u to H_u gives a bijection between $\operatorname{Hom}_B(H, F/H)$ and $B[\varepsilon]$-submodules H' of $F[\varepsilon]$ with image by $u_{F[\varepsilon]/\varepsilon F[\varepsilon]}$ equal to H, and where $F[\varepsilon]/H'$ is flat over $A[\varepsilon]$.*

Proof. It is clear that H_u is a $B[\varepsilon]$-module of $F[\varepsilon]$ and that the image by $u_{F[\varepsilon]/\varepsilon F[\varepsilon]}$ is H. In order to verify that $F[\varepsilon]/H_u$ is flat over $A[\varepsilon]$ it suffices by Proposition (3.8) to verify that the map

$$F[\varepsilon]/H_u \otimes_{A[\varepsilon]} (\varepsilon) \to F[\varepsilon]/H_u \tag{3.21.1}$$

is injective. Let $x + \varepsilon y \in F[\varepsilon]$ be an elements such that $u_{F[\varepsilon]/H_u}(x+\varepsilon y) \otimes_{A[\varepsilon]} \varepsilon$ is in the kernel of the map (3.21.1). Then we have that $x\varepsilon \in H_u$ and consequently that $u_{F/H}(x) = 0$. Hence we have that $x \in H$. Choose an element $y' \in F$ such that $u(x) = u_{F/H}(y')$. Then we have that $x + \varepsilon y' \in H_u$ and consequently that $u_{F[\varepsilon]/H_u}(x+\varepsilon y) \otimes_{A[\varepsilon]} \varepsilon = u_{F[\varepsilon]/H_u}(x) \otimes_{A[\varepsilon]} \varepsilon = u_{F[\varepsilon]/H_u}(x+\varepsilon y')_{A[\varepsilon]}\varepsilon = 0$. Hence we have proved that (3.21.1) is injective.

Conversely let $H' \in F[\varepsilon]$ be a $B[\varepsilon]$-submodule with image H by $u_{F[\varepsilon]/\varepsilon F[\varepsilon]}$ and where $F[\varepsilon]/H'$ is flat over $A[\varepsilon]$. It follows from lemma (3.5) that the sequence

$$0 \to H' \otimes_{A[\varepsilon]} A \to F[\varepsilon] \otimes_{A[\varepsilon]} A \to F[\varepsilon]/H' \otimes_{A[\varepsilon]} A \to 0 \tag{3.21.2}$$

is exact. The image of $H' \otimes_{A[\varepsilon]} A$ in $F[\varepsilon] \otimes_{A[\varepsilon]} A = F$ by (3.21.2) is H by assumption. The mid right map in (3.21.2) consequently induced an isomorphism

$$F/H \to F[\varepsilon]/H' \otimes_{A[\varepsilon]} A. \tag{3.21.3}$$

Tensor the exact sequence

$$0 \to A \xrightarrow{\varepsilon} A[\varepsilon] \to A \to 0$$

with $F[\varepsilon]/H'$ over $A[\varepsilon]$. We obtain an exact sequence

$$0 \to F[\varepsilon]/H' \otimes_{A[\varepsilon]} A \to F[\varepsilon]/H' \otimes_{A[\varepsilon]} A[\varepsilon] \to F[\varepsilon]/H' \otimes_{A[\varepsilon]} A \to 0. \tag{3.21.3}$$

From the sequence (3.21.3) we obtain an exact sequence

$$0 \to F/H \xrightarrow{\delta} F[\varepsilon]/H' \xrightarrow{\eta} F/H \to 0. \tag{3.21.4}$$

We have that $\eta(u_{F[\varepsilon]/H'}|F) = u_{F/H}$ and $\delta u_{F/H} = \varepsilon(u_{F[\varepsilon]/H'}|F)$. For $x \in H$ we have that $\eta u_{F[\varepsilon]/H'}(x) = u_{F/H}(x) = 0$. Consequently it follows from (3.21.4) that there is a unique element $u_{F/H}(y)$ in F/H such that $\delta u_{F/H}(y) = u_{F[\varepsilon]/H'}(x)$. Write $u(x) = u_{F/H}(y)$. In this way we define a B-module homomorphism

$$u : H \to F/H.$$

It remains to prove that $H' = H_u$.

Let $x - \varepsilon y \in H' \subseteq F[\varepsilon]$. Then we have that $x \in H$ because $u_{F[\varepsilon]/\varepsilon F[\varepsilon]}(H') = H$. We obtain that $0 = u_{F[\varepsilon]/H'}(x - \varepsilon y) = u_{F[\varepsilon]/H'}(x) - \varepsilon u_{F[\varepsilon]/H'}(y)$ and consequently we have that $u_{F[\varepsilon]/H'}(x) = \varepsilon u_{F[\varepsilon]/H'}(y) = \delta u_{F/H}(y)$. We obtain from the definition of u that $u(x) = u_{F/H}(y)$, and consequently that $x - \varepsilon y \in H_u$.

Conversely let $x - \varepsilon y \in H_u$ with $x \in H$ and $u(x) = u_{F/H}(y)$. By the definition of u we then have that $u_{F[\varepsilon]/H'}(x) = \delta u_{F/H}(y)$. We obtain that $u_{F[\varepsilon]/H'}(x - \varepsilon y) = u_{F[\varepsilon]/H'}(x) - \varepsilon u_{F[\varepsilon]/H'}(y) = \delta u_{F/H}(y) - \varepsilon u_{F/H}(y) = 0$. Hence we have proved that $H' = H_u$.

→ **(3.21') Lemma.** (Generalisering av Lemma (3.21)) *La $\varphi : A \to B$ be an A-algebra and let I be an ideal in A such that $I^2 = 0$. For each B-module H we let $H_0 = H \otimes_A A_0 = H/IH$ where H is considered as an A-module by restriction of scalars. For every B-module F and submodule H we let $u_{F/H} : F \to F/H$ be the canonical residue map.*

Let F be a B-module and H a submodule such that the module $G = F/H$ is a flat A-module. We have an exact sequence of A_0-modules

$$0 \to H \otimes_A A_0 \to F \otimes_A A_0 \to F/H \otimes_A A_0 \to 0,$$

that is the exact sequence of A_0-modules

$$0 \to H_0 \to F_0 \to (F/H)_0 \to 0.$$

In particular we have a canonical isomorphism $F_0/H_0 \to (F/H)_0$. We also have an exact sequence of B-modules

$$0 \to F/H \otimes_A I \to F/H \otimes_A A \to F/H \otimes_A A_0 \to 0,$$

that is the exact sequence of B-modules

$$0 \to F_0/H_0 \otimes_{A_0} I \to F/H \to F_0/H_0 \to 0.$$

We shall identify the B-module $F_0/H_0 \otimes_{A_0} I$ with its image $I(F/H)$ in F/H.

Let

$$u : H_0 \to F_0/H_0 \otimes_{A_0} I$$

be a B_0-module homomorphism, that we with the above identification consider as a B-module homomorphism

$$u : H_0 \to F/H$$

with image in the kernel $I(F/H)$ of the map $F/H \to F_0/H_0$. Let

$$H_u = \{x + y : x \in H, y \in IF, u(u_{H/IH}(x)) = u_{F/H}(y)\}.$$

Then we have that:

(1) *The group H_u is a B-module such that the image of H_u by the map $u_{F/IF}: F \to F_0$ is H_0 and F/H_u is a flat A-module.*
(2) *The correspondence that sends u to H_u defines an operation of the module $\mathrm{Hom}_{B_0}(H_0, F_0/H_0 \otimes_{A_0} I)$ on the set $\mathcal{Q}$ of all B-submodules H' of F such that the image of H' by the map $u_{F/IF}: F \to F_0$ is H_0 and F/H' is a flat A-module. This action makes $\mathcal{Q}$ into a principal homogeneous space under $\mathrm{Hom}_{B_0}(H_0, F_0/H_0 \otimes_{A_0} I)$.*

Proof. We have that H_u is a B-module since $u_{H/IH}$, $u_{F/H}$ and u are B-module homomorphisms. Moreover the image of H_u by the homomorphism $u_{F/IF}: F \to F_0$ is H_0. It is clear that the image contains H_0. Conversely, when $x_0 \in H_0$ we choose $x \in H$ such that $u_{H/IH}(x) = x_0$. We have that $u(u_{H/IH}(x)) = u(x_0)$ lies in the kernel $I(F/H)$ of $F/H \to F_0/H_0$. Consequently we can find $y \in IF$ such that $u_{F/H}(y) = u(u_{H/IH}(x))$. It follows that $x + y \in H_u$, and thus that x_0 lies in the image of H_u by the homomorphism $u_{F/IF}: F \to F_0$.

We notice that H and H_u have the same image H_0 by the map $u_{F/IF}: F \to F_0$ if and only if $H \subseteq H_u + IF$ and $H_u \subseteq H + IF$.

Next we shall show that F/H_u is flat over A. It follows from the Local Criterion
$\rightarrow$ of Flatness (3.?) that when A is noetherian it is necessary and sufficient that the homomorphism

$$F/H_u \otimes_A I \to F/H_u$$

is injective. Let $\sum_{\alpha \in J} u_{F/H_u}(x_\alpha) \otimes_A i_\alpha$ with $x_\alpha \in F$ and $i_\alpha \in I$ be in the kernel. That is, we have $\sum_{\alpha \in J} i_\alpha x_\alpha \in H_u$. Since $\sum_{\alpha \in J} i_\alpha x_\alpha \in IF$ it follows from the definition of H_u that we have $0 = u(u_{H/IH}(0)) = u_{F/H}(\sum_{\alpha \in J} i_\alpha x_\alpha)$. Consequently we have that $\sum_{\alpha \in J} i_\alpha x_\alpha \in H$. Then $\sum_{\alpha \in J} u_{F/H}(x_\alpha) \otimes_A i_\alpha$ is in the kernel of $F/H \otimes_A I \to F/H$, and since F/H is flat over A by assumption we have that $\sum_{\alpha \in J} u_{F/H}(x_\alpha) \otimes_A i_\alpha = 0$ in $F/H \otimes_A I$.

We have a B-linear map $F/H \otimes_A I \to F/H_u \otimes_A I$ that is uniquely determined by mapping $u_{F/H}(x) \otimes_A i$ with $x \in F$ and $i \in I$ to $u_{F/H_u}(x) \otimes_A i$. This map is well defined because from the equality $u_{F/H}(x_1) = u_{F/H}(x)$ we obtain that $x_1 - x \in H$ so we can find elements $x' \in H_u$ and $y \in IF$ such that $x_1 - x = x' + y$. Then we have that $u_{F/H_u}(x_1) \otimes_A i = u_{F/H_u}(x) \otimes_A i + u_{F/H_u}(x') \otimes_A i + u_{F/H_u}(y) \otimes_A i = u_{F/H_u}(x) \otimes_A i$. In particular we have that $0 = \sum_{\alpha \in J} u_{F/H}(x_\alpha) \otimes_A i_\alpha$ in $F/H \otimes_A I$ maps to $0 = \sum_{\alpha \in J} u_{F/H_u}(x_\alpha) \otimes_A i_\alpha$ in $F/H_u \otimes_A I$, and we have proved that F/H_u is flat over A.

It remains to prove that every submodule H' of F such that the image of H' by the homomorphism $F \to F_0$ is H_0 and such that F/H' is flat over A is on the form H_u for exactly one map $u: H_0 \to F_0/H_0 \otimes_{A_0} I$. We construct the map

$$u: H_0 \to F_0/H_0 \otimes_{A_0} I$$

as follows:

For every $x_0 \in H'$ we choose an element $x' \in H'$ such that $u_{H'/IH'}(x') = x_0$ and we let $u(x_0 = u_{F/H}(x')$.

We have that $u_{F/H}(x')$ lies in the kernel of the homomorphism $F/H \to F_0/H_0$ because $H' \subseteq H + IF$ with $x' = x + y$ with $x \in I$ and $y \in IF$ and thus $u_{F/H}(x') = u_{F/H}(x) + u_{F/H}(y) = u_{F/H}(y) \in I(F/H)$. Moreover we have that $u_{F/H}(x')$ is independent of the choise of x' because if $u_{H'/IH'}(x') = u_{H'/IH'}(x'')$ for some $x'' \in H'$ then we have that $x' - x'' \in IH'$. However $IH' \subseteq IH + IIF = IH$ and $IH \subseteq IH'$ so that $u_{F/H}(x') = u_{F/H}(x' - x'') + u_{F/H}(x'') = u_{F/H}(x'')$. We have thus proved that u is well defined and has image in the kernel $I(F/H)$ of the map $F/H \to F_0/H_0$.

It is clear that $H' \subseteq H_u$ because if $x' \in H'$ we have that $x' = x + y$ with $x \in H$ and $y \in IF$ and we have that $u_{F/IF}(x') = u_{F/IF}(x)$. Hence we have that $u(u_{H/IH}(x)) = u(u_{H'/IH'}(x'))$. Moreover we have that $uu_{H'/IH'}(x') = u_{F/H}(x')$ by the definition of u. Hence we have that $u(u_{H/IH}(x')) = u_{F/H}(x') = u_{F/H}(x + y) = u_{F/H}(y)$ and thus $x + y \in H_u$.

The inclusion of H' in H_u gives a commutative diagram

$$
\begin{array}{ccccccc}
H' \otimes_A I & \longrightarrow & H' \otimes_A A & \longrightarrow & H' \otimes_A A_0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
H_u \otimes_A I & \longrightarrow & H_u \otimes_A A & \longrightarrow & H_u \otimes_A A_0 & \longrightarrow & 0
\end{array}
$$

where the right and left vertical maps are isomorphisms as we have seen above. Consequently the middle vertical map is a surjection. That is we have $H' = H_u$.

4. Base change.

(4.1) Setup. Given a noetherian ring A and a free A–module E of rank $r+1$. Let $R = \operatorname{Sym}_A(E)$ and let $\mathbf{P}(E) = \operatorname{Proj}(R))$. Moreover, given a noetherian scheme S and a morphism $f\colon X \to S$ which is separated of finite type. Let $\mathcal{F}$ be a quasi–coherent $\mathcal{O}_X$–module. For each point x of X we denote by $\kappa(x)$ the residue class of the local ring $\mathcal{O}_{X,x}$ at x modulo the maximal ideal.

(4.2) Remark. Let $g\colon T \to S$ be a morphism from a noetherian scheme T. We saw in (1.11.2) that there is a base change map

$$\mathcal{O}_T \otimes_{\mathcal{O}_S} \widetilde{H^i(X, \mathcal{F})} = g^* R^i f_* \mathcal{F} \to R^i f_{T*} \mathcal{F}_T.$$

and this map is an isomorphism if and only if the base change map

$$B \otimes_A H^i(X_{\operatorname{Spec} A}, \mathcal{F}_{\operatorname{Spec} A}) \to H^i(X_{\operatorname{Spec} B}, \mathcal{F}_{\operatorname{Spec} B})$$

of (1.9.3) is an isomorphism for all affine open subsets $\operatorname{Spec} A$ of S and $\operatorname{Spec} B$ of T such that $\operatorname{Spec} B$ maps to $\operatorname{Spec} A$. With the notation of Definition (1.7) we have the isomorphism $B \otimes_A \mathcal{F}_{\mathcal{U}} \to (\mathcal{F}_{\operatorname{Spec} B})_{\mathcal{V}}$ of B–modules of (1.7.1) and thus an isomorphism $H^i(B \otimes_A \mathcal{F}_{\mathcal{U}}) \to H^i(X_{\operatorname{Spec} B}, \mathcal{F}_{\operatorname{Spec} B})$ of B–modules. Hence the base change map is an isomorphism if and only if the base change map

$$B \otimes_A H^i(\mathcal{F}_{\mathcal{U}}) \to H^i(B \otimes_A \mathcal{F}_{\mathcal{U}})$$

is an isomorphism for all open affine subset $\operatorname{Spec} A$ of S and $\operatorname{Spec} B$ of T such that $\operatorname{Spec} B$ maps to $\operatorname{Spec} A$.

(4.3) Lemma. *With the notation of* (1.9) *we have that the base change map*

$$B \otimes_A H^i(F) \to H^i(B \otimes_A F)$$

of (1.9.3) *is an isomorphism if:*

(1) *The map $B \otimes_A B^{i+1}(F) \to B \otimes_A F^{i+1}$ is injective.*
(2) *The map $B \otimes_A Z^i(F) \to B \otimes_A F^i$ is injective.*

Proof. Assume that the conditions (1) and (2) hold. From the sequence (1.9.1) for the complexes F and $B \otimes_A F$ we obtain the following commutative diagram of B–modules:

$$\begin{tikzcd}
 & B \otimes_A Z^i(F) \arrow[r] \arrow[d] & B \otimes_A F^i \arrow[r] \arrow[d, equal] & B \otimes_A B^{i+1}(F) \arrow[d] \\
0 \arrow[r] & Z^i(B \otimes_A F) \arrow[r] & F^i(B \otimes_A F) \arrow[r] & B^{i+1}(B \otimes_A F).
\end{tikzcd}$$

with exact rows. Since the right vertical map is injective by assumption the left vertical map is surjective, and since $B \otimes_A Z^i(F) \to B \otimes_A F^i$ is injective by assumption, the left vertical map is an isomorphism.

From (1.9.2), for the modules F and $B \otimes_A F$, we obtain a commutative diagram of B–modules

$$\begin{tikzcd}
B \otimes_A B^i(F) \arrow[r] \arrow[d] & B \otimes_A Z^i(F) \arrow[r] \arrow[d] & B \otimes_A H^i(F) \arrow[r] \arrow[d] & 0 \\
B^i(B \otimes_A F) \arrow[r] & Z^i(B \otimes_A F) \arrow[r] & H^i(B \otimes_A F) \arrow[r] & 0.
\end{tikzcd}$$

with exact rows. We Noted in (1.10) that the left vertical map is surjective, and we just proved that the middle vertical map is an isomorphism. It follows that the right vertical map is an isomorphism.

(4.4) Theorem. (Flat base change) *Given a flat morphism $g: T \to S$ from a noetherian scheme T. Then the base change map*

$$g^* R^i f_* \mathcal{F} \to R^i f_{T*} \mathcal{F}_T$$

of Definition (1.11) *is an isomorphism for all i.*

Proof. The assertion is local on S and T. Hence we may assume that $S = \operatorname{Spec} A$ and $T = \operatorname{Spec} B$ for an A–algebra B. Then B is flat over A and consequently $B \otimes_A B^i(\mathcal{F}_\mathcal{U}) \to B \otimes_A (\mathcal{F}_\mathcal{U})^i$ and $B \otimes_A Z^i(\mathcal{F}_\mathcal{U}) \to B \otimes_A (\mathcal{F}_\mathcal{U})^i$ are injective for all i. It follows from Lemma (4.3) that the base change map $B \otimes_A H^i(\mathcal{F}_\mathcal{U}) \to H^i(B \otimes_A \mathcal{F}_\mathcal{U})$ is an isomorphism for all i. The Theorem therefore follows from Remark (4.2).

(4.5) Note. Given a field K and a morphism $\operatorname{Spec} K \to S$. Denote by s the image point. We have a field extension $\kappa(s) \to K$. It follows from Theorem (4.4) that we have an isomorphism

$$K \otimes_{\kappa(s)} H^i(X_{\operatorname{Spec} \kappa(s)}, \mathcal{F}_{\operatorname{Spec} \kappa(s)}) \to H^i(X_{\operatorname{Spec} K}, \mathcal{F}_{\operatorname{Spec} K})$$

of K–vectorspaces, for all i. In particular, if $g: T \to S$ is a morphism and t a point in T we obtain an isomorphism

$$\kappa(t) \otimes_{\kappa(g(t))} H^i(X_{\operatorname{Spec} \kappa(g(t))}, \mathcal{F}_{\operatorname{Spec} \kappa(g(t))}) \to H^i(X_{\operatorname{Spec} \kappa(t)}, \mathcal{F}_{\operatorname{Spec} \kappa(t)}). \quad (4.5.1)$$

(4.6) Proposition. *With the notation of Definition* (1.9), *assume that the A–modules $F^0, F^1, \dots$ of the complex F are flat and that $H^i(F)$ is a flat A–module for $i \geq p+1$. Then, for every A–algebra B, the base change map*

$$B \otimes_A H^i(F) \to H^i(B \otimes_A F) \tag{4.6.1}$$

is an isomorphism for $i \geq p$.

In particular, when $H^i(F) = 0$ for $i > 0$ then:

(1) *The base change map $B \otimes_A H^0(F) \to H^0(B \otimes_A F)$ is an isomorphism.*

(2) *We have that $H^i(B \otimes_A F) = 0$ for $i > 0$.*

Proof. Since $H^i(F)$ is flat for $i \geq p+1$, it follows from sequence (1.9.2) and Lemma (3.5) that $B \otimes_A B^i(F) \to B \otimes_A Z^i(F)$ is injective for $i \geq p+1$. It follows from Lemma (3.15) that $B^i(F)$ is flat for $i \geq p+1$. Hence it follows from the sequence (1.9.1) and Lemma (3.5) that $B \otimes_A Z^i(F) \to B \otimes_A F^i$ is injective for $i \geq p$. Conditions (1) and (2) of Lemma (4.3) are therefore satisfied. The Proposition is therefore a consequence of Lemma (4.3).

(4.7) Theorem. *Assume that $\mathcal{F}$ is flat over S and that $R^i f_* \mathcal{F} = 0$ for $i > 0$. Given a morphism $g\colon T \to S$ from a noetherian scheme T. Then:*

(1) *The $\mathcal{O}_T$–module $f_{T*}\mathcal{F}_T$ is flat.*

(2) *We have that $R^i f_{T*}\mathcal{F}_T = 0$ for $i > 0$.*

(3) *The base change map*

$$g^* f_* \mathcal{F} \to f_{T*}\mathcal{F}_T$$

is an isomorphism.

Proof. The assertions are local on S and T so we may assume that $S = \operatorname{Spec} A$ and that $T = \operatorname{Spec} B$ where B is an A–algebra.

With the notation of Definition (1.7) we have the isomorphism $B \otimes_A \mathcal{F}_{\mathcal{U}} \to (\mathcal{F}_{\operatorname{Spec} B})_{\mathcal{V}}$ of (1.7.1). When $\mathcal{F}$ is flat we noted in (3.11) that the complex $\mathcal{F}_{\mathcal{U}}$ consists of flat A–modules and since $H^i(\mathcal{F}_{\mathcal{U}}) = H^i(X, \mathcal{F}) = 0$ for $i > 0$ by assumption, it follows from Proposition (4.6) with $p = 0$ that $H^i(X_{\operatorname{Spec} B}, \mathcal{F}_{\operatorname{Spec} B}) = H^i((\mathcal{F}_{\operatorname{Spec} B})_{\mathcal{V}}) = H^i(B \otimes_A \mathcal{F}_{\mathcal{U}}) = 0$ for $i > 0$ and that the base change map

$$\begin{aligned} &B \otimes_A H^0(X, \mathcal{F}) = B \otimes_A H^0(\mathcal{F}_{\mathcal{U}}) \to \\ &\qquad H^0(B \otimes_A \mathcal{F}_{\mathcal{U}}) = H^0((\mathcal{F}_{\operatorname{Spec} B})_{\mathcal{V}}) = H^0(X_{\operatorname{Spec} B}, \mathcal{F}_{\operatorname{Spec} B}) \end{aligned}$$

is an isomorphism. We have proved assertions (2) and (3). Assertion (1) follows from (2) and Theorem (3.16)(1).

(4.8) Lemma. *Assume that A is local and let k be the residue field. With the notation of* (1.9) *assume that the A–modules $F^0, F^1, \dots$ of the complex F are flat and that $H^i(F)$ is a finitely generated A–module for all i. Moreover, assume that $H^i(k \otimes_A F) = 0$ for $i > 0$. Then we have that*

$$H^i(F) = 0$$

for $i > 0$.

Proof. We shall prove, by descending induction on p, that for $p > 0$ we have that $H^p(F) = 0$, that $Z^p(F)$ is flat, and that $k \otimes_A Z^p(F) \to Z^p(k \otimes_A F)$ is an isomorphism. These assertions hold for $p > r$. Assume that they hold for $p+1$. Then $B^{p+1}(F) = Z^{p+1}(F)$. By the assumption we have that $H^{p+1}(k \otimes_A F) = 0$ and thus $B^{p+1}(k \otimes_A F) = Z^{p+1}(k \otimes_A F)$. Since $B^{p+1}(F) = Z^{p+1}(F)$ is flat by the induction assumption it follows from the sequence (1.9.1) with $i = p$ and Lemma (3.18) that $Z^p(F)$ is flat.

From the sequence (1.9.1) for F and $k \otimes_A F$ we obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & k \otimes_A Z^p(F) \arrow[r] \arrow[d] & k \otimes_A F^p \arrow[r] \arrow[d, equal] & k \otimes_A Z^{p+1}(F) \arrow[d] \\
0 \arrow[r] & Z^p(k \otimes_A F) \arrow[r] & F^p(k \otimes_A F) \arrow[r] & Z^{p+1}(k \otimes_A F)
\end{tikzcd}$$

and it follows from Lemma (3.5) that the top row is exact. Hence the left vertical map is injective. Since the right vertical map is injective, by the induction assumption, we obtain that the left vertical map is surjective. The sequence (1.9.2) for $i = p$ applied to F and $k \otimes_A F$ gives a commutative diagram

$$\begin{tikzcd}
k \otimes_A B^p(F) \arrow[r] \arrow[d] & k \otimes_A Z^p(F) \arrow[r] \arrow[d] & k \otimes_A H^p(F) \arrow[r] \arrow[d] & 0 \\
B^p(k \otimes_A F) \arrow[r] & Z^p(k \otimes_A F) \arrow[r] & H^p(k \otimes_A F) \arrow[r] & 0
\end{tikzcd}$$

with exact rows. We have proved that the middle map is an isomorphism and noted in (1.10) that the left vertical map is surjective. Hence the right vertical map is an isomorphism. Since $H^p(F \otimes_A k) = 0$ for $p > 0$ it follows from Nakayama's Lemma that $H^p(F) = 0$ for $p > 0$.

(4.9) Theorem. *Assume that $\mathcal{F}$ is flat over S and that $R^i f_* \mathcal{F}$ is coherent for all i. Let s be a point of S be such that $H^i(X_{\operatorname{Spec} \kappa(s)}, \mathcal{F}_{\operatorname{Spec} \kappa(s)}) = 0$ for $i > 0$. Then $(R^i f_* \mathcal{F})_s = 0$ for $i > 0$.*

In particular, when $H^i(X_{\operatorname{Spec}\kappa(s)}, \mathcal{F}_{\operatorname{Spec}\kappa(s)}) = 0$ *for* $i > 0$ *and all* s *in* S*, we have that* $R^i f_* \mathcal{F} = 0$ *for* $i > 0$.

Proof. We can clearly assume that S is affine. Let $S = \operatorname{Spec} A$ and let P be the prime ideal in A corresponding to the point s. It follows from (1.7.4) that the assertion of the Theorem is equivalent to $(R^i f_* \mathcal{F})_s = A_P \otimes_A H^i(X, \mathcal{F}) = 0$ for $i > 0$.

Theorem (4.4) for the flat map $A \to A_P$ states that we have an isomorphism $A_P \otimes_A H^i(X, \mathcal{F}) \to H^i(X_{\operatorname{Spec} A_P}, \mathcal{F}_{\operatorname{Spec} A_P})$. Hence it suffices to prove the Theorem when $S = \operatorname{Spec} A_P$. That is, we can assume that A is local.

With the notation of Definition (1.7) with $B = \kappa(P)$ we have the isomorphism $\kappa(P) \otimes_A \mathcal{F}_{\mathcal{U}} \to (\mathcal{F}_{\operatorname{Spec}\kappa(P)})_{\mathcal{V}}$ of (1.7.1). Consequently it follows from the assumption that $H^i(\kappa(P) \otimes_A \mathcal{F}_{\mathcal{U}}) = H^i(X_{\operatorname{Spec}\kappa(P)}, \mathcal{F}_{\operatorname{Spec}\kappa(P)}) = 0$ for $i > 0$. When $\mathcal{F}$ is flat over S we Noted in (3.11) that the complex $\mathcal{F}_{\mathcal{U}}$ consists of flat modules and we have that $H^i(\mathcal{F}_{\mathcal{U}}) = H^i(X, \mathcal{F})$ is finitely generated for all i by assumption. It follows from Lemma (4.10) that $H^i(X, \mathcal{F}) = H^i(\mathcal{F}_{\mathcal{U}}) = 0$ for $i > 0$, as we wanted to prove.

(4.10) Proposition. *Assume that* $S = \operatorname{Spec} A$*, that* X *is a closed subscheme of* $\mathbf{P}(E)$*, and that* $\mathcal{F}$ *is coherent. Given a morphism* $g: T \to S$ *from a noetherian scheme* T*. Then there is an* m_0 *such that the base change map*

$$\mathcal{O}_T \otimes_{\mathcal{O}_{\operatorname{Spec} A}} H^0\widetilde{(X, \mathcal{F}}(m)) = g^* f_* \mathcal{F}(m) \to f_{T*} \mathcal{F}_T(m)$$

is an isomorphism for $m \geq m_0$.

Proof. The base change map is local in T. Hence it suffices to prove that the base change map

$$B \otimes_A H^0(X, \mathcal{F}(m)) \to H^0(X_{\operatorname{Spec} B}, \mathcal{F}_{\operatorname{Spec} B}(m))$$

is an isomorphism for m sufficiently big for every open affine subset $\operatorname{Spec} B$ of T.

With the notation of Setup (2.1) we have that $\mathcal{F} = \widetilde{M_{\mathcal{F}}}$ for a graded (R/I)–module $M_{\mathcal{F}}$, where I is an ideal in R defining X in $\mathbf{P}(E)$. Then we have that $\mathcal{O}_{\operatorname{Spec} B} \otimes_{\mathcal{O}_{\operatorname{Spec} A}} \mathcal{F} = B \widetilde{\otimes_A} M_{\mathcal{F}}$. We obtain a commutative diagram

$$\begin{tikzcd}
(M_{\mathcal{F}})_m \arrow[r] \arrow[d] & H^0(X, \mathcal{F}(m)) \arrow[d] \\
(B \otimes_A M_{\mathcal{F}})_m \arrow[r] & H^0(X_{\operatorname{Spec} B}, \mathcal{F}_{\operatorname{Spec} B}(m))
\end{tikzcd} \tag{4.8.1}$$

where the left vertical map sends $m \in (M_{\mathcal{F}})_m$ to $1 \otimes m \in (B \otimes_A M_{\mathcal{F}})_m$ and the right vertical map is the map $H^0(X, \mathcal{F}(m)) \to H^0(X_{\operatorname{Spec} B}, \mathcal{F}_{\operatorname{Spec} B}(m))$ of (1.7.3).

In diagram (4.8.1) we can extend the scalars of the modules in the top row from A to B. We obtain a commutative diagram

$$\begin{tikzcd} B \otimes_A (M_{\mathcal{F}})_m \arrow[r] \arrow[d, equal] & B \otimes_A H^0(X, \mathcal{F}(m)) \arrow[d] \\ (B \otimes_A M_{\mathcal{F}})_m \arrow[r] & H^0(X_{\operatorname{Spec} B}, \mathcal{F}_{\operatorname{Spec} B}(m)) \end{tikzcd} \quad . \tag{4.8.2}$$

where the right vertical map is the base change map of (1.11.1). It follows from Theorem (2.2)(1) that the horizontal maps of diagram (4.8.1) and thus of diagram (4.8.2) are isomorphisms for big m. Consequently the right vertical base change map of (4.8.2) is an isomorphism for big m.

(4.11) Lemma. *Assume that* $S = \operatorname{Spec} A$*, that* X *is a closed subscheme of* $\mathbf{P}(E)$*, and that* $\mathcal{F}$ *is coherent. There is an* m_0 *such that for all* $m \geq m_0$ *and for all points* $s \in \operatorname{Spec} A$ *the following two assertions hold:*

(1) *We have that* $H^i(X_{\operatorname{Spec} \kappa(s)}, \mathcal{F}_{\operatorname{Spec} \kappa(s)}(m)) = 0$ *for* $i > 0$.

(2) *The base change map*

$$\kappa(s) \otimes_A H^0(X, \mathcal{F}(m)) \to H^0(X_{\operatorname{Spec} \kappa(s)}, \mathcal{F}_{\operatorname{Spec} \kappa(s)}(m))$$

is an isomorphism.

Proof. It follows from Proposition (3.18) that we can find a finite number of locally closed reduced subschemes of $\operatorname{Spec} A$ that cover $\operatorname{Spec} A$ and such that $\mathcal{F}$ is flat over each of the subschemes. If necessary, covering each of these reduced subschemes with a finite number of open affine sets, we can cover $\operatorname{Spec} A$ with a finite number of locally closed reduced affine subschemes $S_j = \operatorname{Spec} B_j$ such that $\mathcal{F}_{S_j}$ is flat over S_j.

From Theorem (2.2)(2) it follows that we can find an m_1 such that we have $H^i(S_j, \mathcal{F}_{S_j}(m)) = 0$ for $m \geq m_1$ for all j and all $i > 0$. Hence it follows from Theorem (4.7) applied to the flat $\mathcal{O}_{S_j}$–module $\mathcal{F}_{S_j}$ that for $m \geq m_1$ and all j, and for all points $s \in S_j$, we have that $H^i(X_{\operatorname{Spec} \kappa(s)}, \mathcal{F}_{\operatorname{Spec} \kappa(s)}(m)) = 0$ for $i > 0$ and that the base change map

$$\kappa(s) \otimes_{B_j} H^0(X_{S_j}, \mathcal{F}_{S_j}(m)) \to H^0(X_{\operatorname{Spec} \kappa(s)}, \mathcal{F}_{\operatorname{Spec} \kappa(s)}(m)) \tag{4.11.1}$$

is an isomorphism. In particular we have proved assertion (1).

It follows from Proposition (4.10) that we can choose an m_2 such that the base change map

$$B_j \otimes_A H^0(X, \mathcal{F}(m)) \to H^0(S_j, \mathcal{F}_{S_j}(m)) \tag{4.11.2}$$

is an isomorphism for $m \geq m_2$, and for all j. Let $m \geq m_0 = \max(m_1, m_2)$ and let $s \in S$. Choose an S_j that contains s and let $S_j = \operatorname{Spec} B_j$. We obtain isomorphisms

$$\begin{aligned}\kappa(s) \otimes_A H^0(X, \mathcal{F}(m)) &= \kappa(s) \otimes_{B_j} (B_j \otimes_A H^0(X, \mathcal{F}(m)) \\ &\to \kappa(s) \otimes_{B_j} H^0(S_j, \mathcal{F}_{S_j}(m)) \to H^0(X_{\operatorname{Spec}\kappa(s)}, \mathcal{F}_{\operatorname{Spec}\kappa(s)}(m))\end{aligned}$$

where the left map is obtained from (4.11.2) and the right is given by (4.11.1). Clearly the composite map is the base change map for the point s of A and we have proved assertion (2).

We sum up the main results about projective spaces in this Section in the following result:

(4.12) Theorem. *Assume that $S = \operatorname{Spec} A$, that X is a closed subscheme of $\mathbf{P}(E)$, and that the $\mathcal{O}_X$–module $\mathcal{F}$ is coherent and flat over S. Then there is an m_0 such that for all $m \geq m_0$ we have that given morphisms $T \to \operatorname{Spec} A$ and $g\colon U \to T$ of noetherian schemes then:*

(1) *The $\mathcal{O}_T$–module $f_{T*}\mathcal{F}(m)$ is locally free.*
(2) *There is an equality $R^i f_{T*}\mathcal{F}(m) = 0$ for each $i > 0$.*
(3) *The base change map*

$$g^* f_{T*}\mathcal{F}_T(m) \to f_{U*}\mathcal{F}_U$$

is an isomorphism.

Proof. It follows from Lemma (4.11) that there is an m_0 such that for all $m \geq m_0$ and for all points s of S we have that $H^i(X_{\operatorname{Spec}\kappa(s)}, \mathcal{F}_{\operatorname{Spec}\kappa(s)}(m)) = 0$ for $i > 0$. Consequently it follows from (4.5.1) that for all $m \geq m_0$ and all points t of T we have that $H^i(X_{\operatorname{Spec}\kappa(t)}, \mathcal{F}_{\operatorname{Spec}\kappa(t)}(m)) = 0$ for $i > 0$. It follows from Theorem (2.7) that $R^i f_{T*}\mathcal{F}(m)$ is coherent for all i and m, and thus it follows from Theorem (4.9) that $R^i f_{T*}\mathcal{F}_T(m) = 0$ for $m \geq m_0$ and $i > 0$. Hence we have proved assertion (2). It follows from Theorem (4.7) that assertion (3) is a consequence of assertion (2).

Assertion (1) is a consequence of (2) and Theorem (3.16)(1).

5. Hilbert polynomials.

(5.1) Setup. Given a noetherian ring A and a free A–module E of rang $r+1$. We choose a basis $e_0, \dots, e_r$ of E. Denote by $R = \mathrm{Sym}_A(E)$ the symmetric algebra of E over A and write $\mathbf{P}(E) = \mathrm{Proj}(R)$.

Let X be a closed subscheme of $\mathbf{P}(E)$ with inclusion $\iota: X \to \mathbf{P}(E)$, and $\mathcal{F}$ a coherent $\mathcal{O}_X$–module.

(5.2) Definition. Denote by $\mathbf{Q}[t]$ the polynomial ring in the variable t over the rational numbers. For each positive integer d we define a polynomial $\binom{t}{d}$ in $\mathbf{Q}[t]$ by

$$\binom{t}{d} = \frac{t(t-1)(t-2)\cdots(t-d+1)}{d!} = t^d/d! + c_{d-1}t^{d-1} + \cdots + c_0$$

and we let $\binom{t}{0} = 1$.

(5.3) Note. For each positive integer e we define an operator Δ_e on all functions $f: \mathbf{Z} \to \mathbf{Z}$ by

$$\Delta_e f(m) = f(m+e) - f(m).$$

We let $\Delta = \Delta_1$. Then $\Delta\binom{t}{d} = \binom{t}{d-1}$.

For each non–negative integer d we have that $\binom{t}{d}$ defines a function $\mathbf{Z} \to \mathbf{Z}$ and we have that

$$\Delta_e\binom{t}{d} = \binom{t+e}{d} - \binom{t}{d} = e\frac{t^{d-1}}{(d-1)!} + b_{d-2}t^{d-2} + \cdots + b_0.$$

Thus the polynomials $\Delta_e\binom{t}{1}, \Delta_e\binom{t}{2}, \dots$ form a $\mathbf{Q}$–basis for $\mathbf{Q}[t]$. In particular every polynomial $Q \in \mathbf{Q}[t]$ of degree $d-1$ can be written in the form $\Delta_e P = Q$ for a polynomial $P(t)$ of degree d.

(5.4) Lemma. *Given a polynomial $P(t) \in \mathbf{Q}[t]$ of degree d.*

(1) *There is an m_0 such that $P(m) \in \mathbf{Z}$, for $m \geq m_0$, there exist integers $c_0, \dots, c_d$ such that*

$$P(t) = c_d\binom{t}{d} + c_{d-1}\binom{t}{d-1} + \cdots + c_0.$$

(2) *Given a function $f: \mathbf{Z} \to \mathbf{Z}$ and a polynomial $Q(t) \in \mathbf{Q}[t]$ of degree $d-1$ such that*

$$\Delta_e f(m) = f(m+e) - f(m) = Q(m),$$

for all m. Then there is a polynomial $P(t) \in \mathbf{Q}[t]$ of degree d such that

$$f(em) = P(em)$$

for all m. The polynomial $P(t)$ satisfies $\Delta_e P = Q$.

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Proof. Write $P(t) = c_d\binom{t}{d} + c_{d-1}\binom{t}{d-1} + \cdots + c_0$ with $c_0, \dots, c_d$ in $\mathbf{Q}$.

We prove assertion (1) by induction on d. The assertion holds trivially for $d = 0$. By the induction assumption the assertion holds for the polynomial $\Delta P(t) = c_d\binom{t}{d-1} + \cdots + c_1$ of degree $d-1$. We conclude that $c_1, \dots, c_d$ are integers. Then $P(m) - c_d\binom{m}{d} - \cdots - c_1\binom{m}{1} = c_0$ is an integer for $m \geq m_0$. We have proved the first assertion.

To prove the second assertion we use the first assertion to write $Q(t)$ in the form $Q(t) = b_{d-1}\binom{t}{d-1} + \cdots + b_0$ where $b_0, \dots, b_{d-1}$ are integers. We saw in Note (5.3) that there is a polynomial $P_1(t) = c_d\binom{t}{d} + \cdots + c_1\binom{t}{1}$ in $\mathbf{Q}[t]$ of degree d such that $\Delta_e P_1 = Q$. Then $\Delta_e(f - P_1) = 0$. Consequently we obtain that $(f - P_1)(em) = (f - P_1)(e(m-1)) = \cdots = (f - P_1)(0)$. Write $b_0 = (f - P_1)(0)$. Then $f(em) = (P_1 + b_0)(em)$ for all m and thus $f(em) = P(em)$ with $P = P_1 + b_0$. We have proved the first assertion of (2). The second assertion of the Lemma follows from the equality $\Delta_e(P) = \Delta_e P_1 = Q$.

(5.5) Theorem. *Assume that A is an artinian ring. Then*

$$\chi_{\mathcal{F}}(m) = \sum_{i=0}^{r} (-1)^i l_A\left(H^i(X, \mathcal{F}(m))\right)$$

is a polynomial in m of degree $\dim \operatorname{Supp} \mathcal{F}$*, and the coefficient of the term of highest degree is positive.*

Proof. To simplify the notation we observe that it follows from the equalities $H^i(X, \mathcal{F}(m)) = H^i(\mathbf{P}(E), \iota_*(\mathcal{F}(m))) = H^i(\mathbf{P}(E), (\iota_*\mathcal{F})(m))$ of Note (1.6) and Setup (2.1) that it suffices to prove the Theorem when $X = \mathbf{P}(E)$.

We shall prove the Theorem by induction on the dimension s of the support $\operatorname{Supp} \mathcal{F}$ of $\mathcal{F} = \widetilde{M}$, where $M = M_{\mathcal{F}}$ is the finitely generated R–module of Setup (2.1). When $s < 0$ we have that $\mathcal{F} = 0$ and the statement is true. Assume that $s \geq 0$. It follows from Lemma (2.6) that M has a finite filtration whose quotients are isomorphic to $(R/P)[d]$, where P is a prime ideal in R. The support of $\mathcal{F}$ is the union of the irreducible varieties $Z(P)$ in $\mathbf{P}(E)$. Since l_A and $\chi_{\mathcal{F}}$ are additive it suffices to prove that the Theorem holds when $\mathcal{F}$ is the sheaf associated to the R–module $L = (R/P)[d]$. We have that $\widetilde{L} = 0$ when P contains the ideal $(e_0, \dots, e_r)$. Hence we can assume that the ideal P does not contain $(e_0, \dots, e_r)$. Since we assumed that $s \geq 0$ there exists such an ideal P.

Let $P = P_0 \subset P_1 \subset \cdots \subset P_s$ be a maximal sequence of homogeneous prime ideals in R such that $(e_0, \dots, e_r)$ is not contained in P_s. Choose a homogeneous element $f \in P_1 \setminus P$ of degree d. (*Må gjøre $s = 0$. Ta $e_i \notin P = P_0$*) We obtain an exact sequence

$$0 \to L \xrightarrow{f} L[d] \to N \to 0. \tag{5.5.1}$$

The dimension of $\operatorname{Supp}\widetilde{N}$ is $s-1$ because f defines an isomorphism at the generic point P of $Z(P)$ and P_1 is contained in $\operatorname{Supp}\widetilde{N}$. From the long exact sequence of cohomology corresponding to the seqence $0 \to \mathcal{F} \xrightarrow{f} \mathcal{F}(d) \to \widetilde{N} \to 0$ associated to (5.5.1) we obtain that

$$\Delta_d\chi_{\mathcal{F}}(m) = \chi_{\mathcal{F}}(m+d) - \chi_{\mathcal{F}}(m) = \chi_{\widetilde{N}}(m).$$

It follows from the induction assumption that $\chi_{\widetilde{N}}(m)$ is a polynomial of degree $s-1$ whose coefficient of the term of degree $s-1$ is positive. We obtain from Lemma (5.4)(2) that there is a polynomial $P(t) \in \mathbf{Q}[t]$ of degree s whose coefficient of the term of degree s is positive and such that $\chi_{\mathcal{F}}(dm) = P(dm)$ for all m.

Since $(e_0, \dots, e_r)$ is not in P_1 we can choose an $e_i \notin P_1$. Then $e_i f \in P_1 \setminus P$. The same reasoning as above shows that $\chi_{\mathcal{F}}(m+d+1) - \chi_{\mathcal{F}}(m)$ is a polynomial in m. Consequently we have that

$$\Delta\chi_{\mathcal{F}}(m+d) = \chi_{\mathcal{F}}(m+d+1) - \chi_{\mathcal{F}}(m) + \chi_{\mathcal{F}}(m) - \chi_{\mathcal{F}}(m+d)$$

is a polynomial in m. It follows from Lemma (5.4)(2) with $e=1$ that there is a polynomial $P_1(t) \in \mathbf{Q}[t]$ such that $\chi_{\mathcal{F}}(m) = P_1(m)$. Then $P(md) = \chi_{\mathcal{F}}(md) = P_1(md)$ and thus $P(t) = P_1(t)$. Consequently $\chi_{\mathcal{F}}(m) = P(m)$ and we have proved that $\chi_{\mathcal{F}}$ is a polynomial of degree s whose term of degree s has positive coefficient.

(5.6) Corollary. *With the assumptions of the Theorem there is an m_0 such that $l_A((M_{\mathcal{F}})_m)$ is a polynomial in m for $m \geq m_0$, where $M_{\mathcal{F}}$ is the module of Setup* (2.1) *such that $\mathcal{F} = \widetilde{M_{\mathcal{F}}}$.*

Proof. The proof follows from the Theorem and Theorem (2.2)(1).

(5.7) Definition. The polynomial $\chi_{\mathcal{F}}$ of Theorem (5.5) is called the *Hilbert polynomial of $\mathcal{F}$* , and the polynomial of Corollary (5.6) that gives $l_A((M_{\mathcal{F}})_m)$ for big m is called the *Hilbert polynomial of the R/I–module M* . For any ring A we write

$$\chi_{\mathcal{F},s}(m) = \chi_{\mathcal{F},P}(m) = \sum_{i=0}^{r}(-1)^i \dim_{\kappa(s)} H^i(X_{\operatorname{Spec}\kappa(s)}, \mathcal{F}_{\operatorname{Spec}\kappa(s)}(m)).$$

for each point s of $\operatorname{Spec} A$ with corresponding prime ideal P.

(5.8) Note. Let K be a field and $\operatorname{Spec} K \to \operatorname{Spec} A$ a morphism. Denote by s the image point of the map. It follows from Note (4.5) that we have $\chi_{\mathcal{F},s} = \chi_{\mathcal{F}_{\operatorname{Spec} K},(o)}$. In particular, given a morphism $g\colon T \to \operatorname{Spec} A$, we obtain, for each point t of T that $\chi_{\mathcal{F},g(t)} = \chi_{\mathcal{F}_T,t}$.

Moreover, it follows from Proposition (4.10) and Theorem (2.2) that there is an m_0 depending on s such that $H^i(X_{\operatorname{Spec} K}, \mathcal{F}_{\operatorname{Spec} K}(m)) = 0$ for $i > 0$ and such that the base change map

$$K \otimes_A H^0(X, \mathcal{F}(m)) \to H^0(X_{\operatorname{Spec} K}, \mathcal{F}_{\operatorname{Spec} K}(m))$$

is an isomorphism for $m \geq m_0$. We obtain that

$$\chi_{\mathcal{F},s}(m) = \dim_K(K \otimes_A H^0(X, \mathcal{F}(m)))$$

for $m \geq m_0$ where m_0 depends on s.

(5.9) Lemma. *Given a local noetherian integral domain A and let k and K be the residue field, respectively the fraction field of A. Let F be a finitely generated A–module. If*

$$d = \dim_k(k \otimes_A F) = \dim_K(K \otimes_A F)$$

we have that F is a free A–module of rank d.

Proof. By assumption we have that $d = \dim_k(k \otimes_A F)$. It follows from Nakayama's Lemma that we have a surjective map $A^d \to F$ of A–modules. Let L be the kernel of this map. Since K is A–flat we obtain an exact sequence

$$0 \to K \otimes_A L \to K \otimes_A A^d \to K \otimes_A F \to 0.$$

of vectorspaces over K. Since $d = \dim_K(K \otimes_A F)$ by assumption the surjection $K \otimes_A A^d \to K \otimes_A F$ must be an isomorphism. Hence $K \otimes_A L = 0$. However the map $L \to K \otimes_A L$ which sends l to $1 \otimes l$ is injective because it is induced by the composite $L \to A^d \to K \otimes_A A^d$ of two injections. Hence $L = 0$, and F is isomorphic to A^d.

(5.10) Theorem. *Assume that* $\operatorname{Spec} A$ *is connected.*

(1) *If $\mathcal{F}$ is flat over* $\operatorname{Spec} A$ *then the polynomial $\chi_{\mathcal{F},s}$ is independent of $s \in$* $\operatorname{Spec} A$.
(2) *If A is integral and $\chi_{\mathcal{F},s}$ is independent of $s \in$* $\operatorname{Spec} A$*, then $\mathcal{F}$ is flat over* $\operatorname{Spec} A$.

Proof. Assume that $\mathcal{F}$ is flat over $\operatorname{Spec} A$. It follows from Theorem (2.2) that $H^i(X, \mathcal{F}(m)) = 0$ for $i > 0$ and for big m. Moreover it follows from Theorem (2.7) that $f_*\mathcal{F}(m)$ is coherent for all m. Consequently it follows from Theorem (3.16)(1) that $f_*\mathcal{F}(m)$ is locally free. Since $\operatorname{Spec} A$ is connected we have that $f_*\mathcal{F}(m)$ has constant rank $r(m)$ on $\operatorname{Spec} A$. It follows from the equality $H^0(\widetilde{X, \mathcal{F}(m)}) =$

$f_*\mathcal{F}(m)$ of (1.7.4) that $A_P \otimes_A H^0(X, \mathcal{F}(m))$ is a free A_P–module of rank $r(m)$ for all prime ideals P in A. Consequently we have that

$$r(m) = \dim_{\kappa(P)} \left(\kappa(P) \otimes_{A_P} A_P \otimes_A H^0(X, \mathcal{F}(m))\right)$$
$$= \dim_{\kappa(P)} \left(\kappa(P) \otimes_A H^0(X, \mathcal{F}(m))\right).$$

From Proposition (4.10) it follows that the base change map

$$\kappa(P) \otimes_A H^0(X, \mathcal{F}(m)) \to H^0(X_{\operatorname{Spec}\kappa(P)}, \mathcal{F}_{\operatorname{Spec}\kappa(P)}(m))$$

is an isomorphism for big m. Consequently we have that

$$r(m) = \dim_{\kappa(P)} \left(H^0(X_{\operatorname{Spec}\kappa(P)}, \mathcal{F}_{\operatorname{Spec}\kappa(P)}(m))\right) = \chi_{\mathcal{F},P}(m)$$

for big m. Hence $\chi_{\mathcal{F},P}$ is independent of P.

Conversely, assume that A is integral and that $\chi_{\mathcal{F},P}$ is independent of the prime ideal P of A. Denote by s the point corresponding to the prime ideal P. Let K be the fraction field of A. It follows from Proposition (4.10) applied to $\operatorname{Spec} A_P$ and the points s respectively (0) of $\operatorname{Spec} A_P$ that we, for big m, have isomorphisms

$$\kappa(P) \otimes_{A_P} H^0(X_{\operatorname{Spec} A_P}, \mathcal{F}_{\operatorname{Spec} A_P}(m)) \to H^0(X_{\operatorname{Spec}\kappa(P)}, \mathcal{F}_{\operatorname{Spec}\kappa(P)}(m)), \quad (5.9.1)$$

respectively

$$K \otimes_{A_P} H^0(X_{\operatorname{Spec} A_P}, \mathcal{F}_{\operatorname{Spec} A_P}(m)) \to H^0(X_{\operatorname{Spec} K}, \mathcal{F}_{\operatorname{Spec} K}(m)). \quad (5.9.2)$$

Since $\chi_{\mathcal{F},P}(m) = \chi_{\mathcal{F},(0)}(m)$ for all m, by assumption, and we have that both $H^0(X_{\operatorname{Spec}\kappa(P)}, \mathcal{F}_{\operatorname{Spec}\kappa(P)}(m))$ and $H^0(X_{\operatorname{Spec} K}, \mathcal{F}_{\operatorname{Spec} K}(m))$ are zero for big m by Theorem (2.2) we have that the right hand sides, and therefore the left hand sides, of (5.9.1) respectively (5.9.2) have the same dimension over $\kappa(P)$ respectively over K. It follows from Lemma (5.9) that $H^0(X_{\operatorname{Spec} A_P}, \mathcal{F}_{\operatorname{Spec} A_P}(m))$ is a free A_P–module. Since $\operatorname{Spec} A_P \to \operatorname{Spec} A$ is flat, it follows from Theorem (4.4) that we, for each m, have an isomorphism

$$A_P \otimes_A H^0(X, \mathcal{F}(m)) \to H^0(X_{\operatorname{Spec} A_P}, \mathcal{F}_{\operatorname{Spec} A_P}(m)).$$

Consequently we have that $A_P \otimes_A H^0(X, \mathcal{F}(m))$ is free for big m. Since the A–module $H^0(X, \mathcal{F}(m))$ is finitely generated by Theorem (2.7) we have that $f_*\mathcal{F}(m)$ is a locally free $\mathcal{O}_{\operatorname{Spec} A}$–module for big m. It follows from Theorem (3.16)(2) that $\mathcal{F}$ is flat over $\operatorname{Spec} A$.

(5.11) Proposition. *There is only a finite set of polynomials* $\{P_j\}_{j \in \mathcal{J}}$ *such that* $P_j(n) = \chi_{\mathcal{F},s}(n)$ *for some* $s \in \operatorname{Spec} A$.

Proof. It follows from Proposition (3.19) that we can find a finite number of locally closed reduced subschemes $S_1, \dots, S_m$ of $\operatorname{Spec} A$ that cover $\operatorname{Spec} A$ and such that $\mathcal{F}_{S_i}$ is flat over S_i. It follows from (5.10(1)) that $\chi_{\mathcal{F},s}$ is independent of $s \in S_i$ and we have proved the Proposition.

6. Castelnuovo–Mumford regularity.

(6.1) Setup. Assume that $k = A$ is a field and E a vector space of dimension $r+1$. We choose a basis $e_0, \dots, e_r$ of E. Denote by $R = \mathrm{Sym}_A(E)$ the symmetric algebra of E over A and write $\mathbf{P}(E) = \mathrm{Proj}(R)$.

Let $\iota: X \to \mathbf{P}(E)$ be a closed immersion of a scheme X into $\mathbf{P}(E)$ and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$–module. Given a closed immersion $j: H \to X$ we shall write $\mathcal{F}|H = j^*\mathcal{F}$.

(6.2) Definition. We say that $\mathcal{F}$ is *m–regular* if

$$H^i(X, \mathcal{F}(m-i)) = 0, \quad \text{for } i > 0.$$

→ **(6.3) Remark.** It follows from Theorem (2.2) there is an $m_0(\mathcal{F})$ such that $\mathcal{F}$ is m–regular for $m \geq m_0(\mathcal{F})$.

(6.4) Note. For every field extension $k \subseteq K$, we have:

(1) The $\mathcal{O}_X$–module $\mathcal{F}$ is m–regular if and only if $\mathcal{F}_{\mathrm{Spec}\,K}$ is m–regular.

→ (2) The map (2.4.2) for $k = A$

$$\beta_m(\mathrm{Spec}\,A) \colon H^0(X, \mathcal{F}(m)) \otimes_k H^0(X, \mathcal{O}_X(1)) \to H^0(X, \mathcal{F}(m+1)) \tag{6.4.1}$$

→ is surjective if and only if the map (2.4.1) for K

$$\beta_m(\mathrm{Spec}\,K) \colon H^0(X_{\mathrm{Spec}\,K}, \mathcal{F}_{\mathrm{Spec}\,K}(m)) \otimes_K H^0(X_{\mathrm{Spec}\,K}, \mathcal{O}_{X_K}(1)) \\ \to H^0(X_{\mathrm{Spec}\,K}, \mathcal{F}_{\mathrm{Spec}\,K}(m+1))$$

is surjective.

→ These assertions follow from Note (4.5).

(6.5) Lemma. *Assume that $k = A$ is an infinite field. Given a non–zero coherent sheaf $\mathcal{G}$ on $\mathbf{P}(E)$. For $h \in E$ we let $H = Z(h) = \mathbf{P}(E/Ah)$ be the corresponding hyperplane in $\mathbf{P}(E)$ and $j: H \to \mathbf{P}(E)$ the corresponding closed immersion. Then, for a general choice of h we have that the sequence*

$$0 \to \mathcal{G}(-1) \xrightarrow{h} \mathcal{G} \to j_*(\mathcal{G}|H) \to 0 \tag{6.5.1}$$

is exact, where the map $\mathcal{G}(-1) \xrightarrow{h} \mathcal{G}$ is obtained from multiplication by the element $h \in E$. Moreover we have that $\dim \mathrm{Supp}\, j_*(\mathcal{G}|H) < \dim \mathrm{Supp}\, \mathcal{G}$.

Proof. We check the exactness on the open subsets $U_i = D_+(e_i)$ of X. The Lemma assert that for a general linear form $h \in E$ we have that the map $M_{(e_i)} \xrightarrow{h/e_i} M_{(e_i)}$

is injective for $i = 0, \dots, r$, where $M = M_{\mathcal{G}}$ is the R–module of Setup (2.1) such that $\mathcal{G} = \widetilde{M}$. When $M_{(e_i)} = 0$ we can choose any h. Otherwise we must choose h such that h/e_i is not contained in any associated prime of $M_{(e_i)}$ in $A[e_0/e_i, \dots, e_r/e_i]$. For every associated prime P, the subspace E_P of E consisting of the elements h such that h/e_i is in P is a proper subspace, since $e_i/e_i = 1$ is not in E_P. Since $k = A$ is infinite E can not be the union of the vector spaces consisting subspaces E_P for the finite set of associated primes and all $i = 0, 1, \dots, r$. Any h outside of the union of these spaces will give a hyperplane satisfying the assertions of the Lemma.

(6.6) Note. Assume that $k = A$ is an infinite field. If follows from Lemma (6.5) that, for a general hyperplane $j\colon H \subseteq \mathbf{P}(E)$, we have an exact sequence

$$0 \to \mathcal{G}(-1) \xrightarrow{h} \mathcal{G} \to j_*(\mathcal{G}|H) \to 0.$$

Consequently we obtain a commutative diagram

$$\begin{tikzcd} H^0(\mathbf{P}(E), \mathcal{G}(m)) \otimes_k H^0(\mathbf{P}(E), \mathcal{O}_{\mathbf{P}(E)}(1)) \arrow[r, "\beta_m"] \arrow[d, "\rho_m \otimes \gamma"'] & H^0(\mathbf{P}(E), \mathcal{G}(m+1)) \arrow[d, "\rho_{m+1}"] \\ H^0(H, (\mathcal{G}|H)(m)) \otimes_k H^0(H, \mathcal{O}_H(1)) \arrow[r] & H^0(H, (\mathcal{G}|H)(m+1)). \end{tikzcd} \tag{6.6.1}$$

Here γ is surjective because $H^1(\mathbf{P}(E), \mathcal{O}_X(1)) = 0$.

(6.7) Definition. We say that $\mathcal{F}$ *is generated by global sections* if the map

$$f^* f_* \mathcal{F} \to \mathcal{F}$$

is surjective.

(6.8) Proposition. *Assume that $\mathcal{F}$ is m–regular. Then*

(1) *$\mathcal{F}$ is $(m+1)$–regular.*
(2) *The map*

$$H^0(X, \mathcal{F}(m)) \otimes_k H^0(X, \mathcal{O}_X(1)) \to H^0(X, \mathcal{F}(m+1))$$

is surjective.
(3) *$\mathcal{F}(m)$ is generated by global sections.*

Proof. To simplify the proof of the Proposition we observe that it follows from the equalities $H^i(X, \mathcal{F}(m)) = H^i(\mathbf{P}(E), \iota_*(\mathcal{F}(m))) = H^i(\mathbf{P}(E), (\iota_*\mathcal{F})(m))$ and the isomorphism $\iota^*\iota_*\mathcal{F} \to \mathcal{F}$ of Note (1.6) and Setup (2.1) that it suffices to prove the Theorem when $X = \mathbf{P}(E)$.

We observed in Note (2.4) that (3) is a consequence of (2). To prove the two first assertions it follows from Note (6.4) that we may assume that $k = A$ is infinite. We prove the Proposition by induction on the dimension r of $X = \mathbf{P}(E)$. The case $r = 0$ is clear. When $r > 0$ we choose a hyperplane $j\colon H \to \mathbf{P}(E)$ of $\mathbf{P}(E)$ as in Lemma (6.5). From the short exact sequence (6.5.1) tensored by $\mathcal{O}_{\mathbf{P}(E)}(m-i)$ we obtain the piece

$$\cdots \to H^i(\mathbf{P}(E), \mathcal{F}(m-i)) \to \\ H^i(\mathbf{P}(E), j_*(\mathcal{F}|H)(m-i)) \to H^{i+1}(\mathbf{P}(E), \mathcal{F}(m-i-1)) \to \cdots$$

of the corresponding long exact sequence. Since $\mathcal{F}$ is m–regular it follows that $\mathcal{F}|H$ is m–regular.

From the short exact sequence (6.5.1), with $\mathcal{F} = \mathcal{G}$ tensored by $\mathcal{O}_{\mathbf{P}(E)}(m+1)$ we obtain an exact sequence

$$H^i(\mathbf{P}(E), \mathcal{F}(m-i)) \to H^i(\mathcal{F}(m+1-i)) \to H^i(H, (\mathcal{F}|H)(m+1-i))$$

The left hand term is 0 by the m–regularity of $\mathcal{F}$ and the right hand term is 0 because $\mathcal{F}|H$ is $(m+1)$–regular by the induction assumption. Hence $H^i(\mathbf{P}(E), \mathcal{F}(m+1-i)) = 0$, and we have proved the first assertion of the Proposition.

To prove the second assertion of the Proposition we note that, by the induction assumption, we have that the bottom map of diagram (6.6.1) is surjective. Since $\mathcal{F}$ is m–regular we have that $\rho_m\colon H^0(\mathbf{P}(E), \mathcal{F}(m)) \to H^0(H, \mathcal{F}|H(m))$ is surjective, the cokernel being $H^1(\mathbf{P}(E), \mathcal{F}(m-1))$. Hence the map $\rho_m \otimes \gamma$ of diagram (6.6.1) is surjective. To prove that β_m is surjective it therefore suffices to check that $\operatorname{Ker}\rho_{m+1} \subseteq \operatorname{Im}\beta_m$. However, we have that $\operatorname{Ker}\rho_{m+1} = hH^0(\mathbf{P}(E), \mathcal{F}(m)) = \beta_m(H^0(\mathbf{P}(E), \mathcal{F}(m)) \otimes \langle h\rangle)$, where $h \in E$ is the linear form that defines H.

(6.9) Lemma. *Given a non–zero coherent $\mathcal{O}_{\mathbf{P}(E)}$–module $\mathcal{G}$. Let $j\colon H \subseteq \mathbf{P}(E)$ be a hyperplane such that the sequence*

$$0 \to \mathcal{G}(-1) \xrightarrow{h} \mathcal{G} \to j_*(\mathcal{G}|H) \to 0 \tag{6.9.1}$$

of (6.5.1) *is exact. Assume that $\mathcal{G}|H$ is m_1–regular. Then:*

(1) *We have that* $\dim_k H^1(\mathbf{P}(E), \mathcal{G}(m)) \leq \dim_k H^1(\mathbf{P}(E), \mathcal{G}(m-1))$*, for* $m \geq m_1$

(2) *If $m \geq m_1$ and $H^1(\mathbf{P}(E), \mathcal{G}(m-1)) \neq 0$, then*

$$\dim_k H^1(\mathbf{P}(E), \mathcal{G}(m)) < \dim_k H^1(\mathbf{P}(E), \mathcal{G}(m-1)).$$

In particular, if $\dim_k H^1(\mathbf{P}(E), \mathcal{G}(m-1)) = d+1$ *we have that* $H^1(\mathbf{P}(E), \mathcal{G}(m+d)) = 0$.

Proof. It follows from the long exact sequence associated to (6.9.1) tensored by $\mathcal{O}_{\mathbf{P}(E)}(m)$ that we have an exact sequence

$$H^0(\mathbf{P}(E), \mathcal{G}(m)) \xrightarrow{\rho_m} H^0(\mathbf{P}(E), j_*(\mathcal{G}|H)(m)) \to$$
$$H^1(\mathbf{P}(E), \mathcal{G}(m-1)) \to H^1(\mathbf{P}(E), \mathcal{G}(m)) \to 0 \quad (6.9.2)$$

for $m \geq m_1$. In (6.9.2) we have 0 to the right because $H^1(\mathbf{P}(E), j_*(\mathcal{G}|H)(m)) = H^1(H, (\mathcal{G}|H)(m)) = 0$, which follows from the assumption that $\mathcal{G}|H$ is m_1–regular and thus, by Proposition (6.8), is m–regular for all $m \geq m_1$. In particular we have that $\dim_k H^1(\mathbf{P}(E), \mathcal{G}(m)) \leq \dim_k H^1(\mathbf{P}(E), \mathcal{G}(m-1))$, which is the first assertion of the Lemma.

The second part of the Lemma asserts that when $m \geq m_1$ and $H^1(\mathbf{P}(E), \mathcal{G}(m-1)) \neq 0$, then ρ_m is not surjective. Assume, to the contrary, that $H^1(\mathbf{P}(E), \mathcal{G}(m-1)) \neq 0$ and that ρ_m is surjective. We shall prove by induction on n that ρ_n is surjective for $n \geq m$. Assume that ρ_n is surjective. Since $\mathcal{G}|H$ is m_1–regular it follows from Proposition (6.8)(2) that the bottom line of diagram (6.6.1) with $m = n$ is surjective. Since ρ_n surjective implies that the left vertical map of diagram (6.6.1) is surjective for $m = n$, we conclude that ρ_{n+1} is surjective. Since $\rho_m, \rho_{m+1}, \dots$ are surjective it follows from (6.9.2) that $H^1(\mathbf{P}(E), \mathcal{G}(m-1)) = H^1(\mathbf{P}(E), \mathcal{G}(m)) = \cdots = 0$, and it follows from Theorem (2.2) that $H^1(\mathbf{P}(E), \mathcal{G}(n)) = 0$ for big n and we obtain a contradiction to the assumption that $H^1(\mathbf{P}(E), \mathcal{G}(m-1)) \neq 0$. Hence, ρ_m is not surjective and we have proved the second part of the Lemma.

The last assertion follows from the inequalties $\dim_k H^1(\mathbf{P}(E), \mathcal{G}(m_1-1)) \geq \dim_k H^1(\mathbf{P}(E), \mathcal{G}(m_1)) \geq \cdots \geq \dim_k H^1(\mathbf{P}(E), \mathcal{G}(m_1+d))$, where we have that $\dim_k H^1(\mathbf{P}(E), \mathcal{G}(n-1)) > \dim_k H^1(\mathbf{P}(E), \mathcal{G}(n))$ if $H^1(\mathbf{P}(E), \mathcal{G}(n-1)) \neq 0$.

(6.10) Theorem. *Let* $P \in \mathbf{Q}[t]$ *be a polynomial. Then there is an integer* $m_0(P)$ *such the kernel of every surjection* $\mathcal{F} \to \mathcal{G}$ *to a coherent* $\mathcal{O}_X$*–module* $\mathcal{G}$ *with Hilbert polynomial* P *is* $m_0(P)$*–regular.*

Proof. It follows from Note (6.4) that we can assume that the field $k = A$ is infinite. We can also assume that $X = \mathbf{P}(E)$. Indeed the quotients $\mathcal{F} \to \mathcal{G}$ on X with kernel $\mathcal{K}$ give quotients $\iota_*\mathcal{F} \to \iota_*\mathcal{G}$ on $\mathbf{P}(E)$ with kernel $i_*\mathcal{K}$. and we have that $H^i X, \mathcal{K}(m)) = H^i(\mathbf{P}(E), \iota_*\mathcal{K}(m))$ by Note (1.6) and Setup (2.1).

We shall prove the Theorem by induction on the dimension r of $X = \mathbf{P}(E)$. The case $r = 0$ is clear. Assume that $r > 0$ and that the Theorem holds for $r-1$. Fix a quotient $\mathcal{F} \to \mathcal{G}$ with kernel $\mathcal{K}$. It follows from Lemma (6.5) that we can choose a hyperplane $j\colon H \subseteq \mathbf{P}(E)$ such that the sequences

$$0 \to \mathcal{G}(-1) \xrightarrow{h} \mathcal{G} \to j_*(\mathcal{G}|H) \to 0 \quad (6.10.1)$$

and

$$0 \to \mathcal{F}(-1) \xrightarrow{h} \mathcal{F} \to j_*(\mathcal{F}|H) \to 0$$

are exact. Hence we obtain a surjection $\mathcal{F}|H \to \mathcal{G}|H$ with kernel $\mathcal{K}|H$, and an exact sequence

$$0 \to \mathcal{K}(-1) \xrightarrow{h} \mathcal{K} \to j_*(\mathcal{K}|H) \to 0. \tag{6.10.2}$$

From Sequence (6.10.1) we obtain that

$$\chi_{\mathcal{G}}(m) - \chi_{\mathcal{G}}(m-1) = \chi_{\mathcal{G}|H}(m).$$

Hence the Hilbert polynomial Q of $\mathcal{G}|H$ is given by $P(m) - P(m-1) = Q(m)$, that depends on P only. It follows from the induction assumption that there is a number $m_0(Q) = m_1(P) \geq 0$ such that the kernel of all surjective maps $\mathcal{F}|H \to \mathcal{H}$, where $\mathcal{H}$ has Hilbert polynomial Q, have an $m_0(Q)$–regular kernel. In particular we have that $\mathcal{K}|H$ is $m_0(Q)$–regular. We choose $m_0(Q) > 0$ so big that $\mathcal{F}$ is m–regular for all $m \geq m_0(Q)$. This is possible, as noted in (6.3). From Sequence (6.10.2) tensored by $\mathcal{O}_{\mathbf{P}(E)}(m+1-i)$ and Note (1.6) we obtain the exact sequence

$$\begin{aligned} H^{i-1}(H, j_*(\mathcal{K}|H)(m+1-i)) &\to H^i(\mathbf{P}(E), \mathcal{K}(m-i)) \\ &\to H^i(\mathbf{P}(E), \mathcal{K}(m+1-i)) \to H^i(H, j_*(K|H)(m+1-i)). \end{aligned}$$

Since $\mathcal{K}|H$ is $m_0(Q)$–regular, and thus m–regular for $m \geq m_0(Q)$ by Proposition (6.8)(1), we have that the left and right hand terms are zero for $m \geq m_0(Q)$. Hence we obtain that $H^i(\mathbf{P}(E), \mathcal{K}(m-i)) = H^i(\mathbf{P}(E), \mathcal{K}(m+1-i))$, and thus $H^i(\mathbf{P}(E), \mathcal{K}(m-i)) = H^i(\mathbf{P}(E), \mathcal{K}(m-i+1)) = \cdots$ for $m \geq m_0(Q)$ and $i \geq 2$. It follows from Theorem (2.2) that $H^i(\mathbf{P}(E), \mathcal{K}(m-i)) = 0$ for $i \geq 2$. From the short exact sequence $0 \to \mathcal{K} \to \mathcal{F} \to \mathcal{G} \to 0$ tensored by $\mathcal{O}_{\mathbf{P}(E)}(m-i)$ we obtain the exact sequence

$$H^i(\mathbf{P}(E), \mathcal{F}(m-i)) \to H^i(\mathbf{P}(E), \mathcal{G}(m-i)) \to H^{i+1}(\mathbf{P}(E), \mathcal{K}(m-i)).$$

We have chosen $m_0(Q)$ so big that $\mathcal{F}$ is $m_0(Q)$–regular. Hence we have that $H^i(\mathbf{P}(E), \mathcal{K}(m-i)) = 0$ for $i \geq 2$ and $m \geq m_0(Q)$ we get that $H^i(\mathbf{P}(E), \mathcal{G}(m-i)) = 0$ for $i \geq 1$ and $m \geq m_0(Q)$. Consequently we have that $\mathcal{G}$ is $m_0(Q)$–regular. We obtain from Proposition (6.8)(1) that $H^i(\mathbf{P}(E), \mathcal{G}(m_0(Q)-1)) = 0$ for $i \geq 1$. Hence we have that $\dim_k H^0(\mathbf{P}(E), \mathcal{G}(m_0(Q)-1)) = \chi_{\mathcal{G}}(m_0(Q)-1) = P(m_0(Q)-1) = d_0(P)$, depends only on P. We have a surjection

$$H^0(\mathbf{P}(E), \mathcal{G}(m_0(Q)-1)) \to H^1(\mathbf{P}(E), \mathcal{K}(m_0(Q)-1))$$

because $H^1(\mathbf{P}(E), \mathcal{F}(m_0(Q)-1)) = 0$. Hence we have that

$$\dim_k H^1(\mathbf{P}(E), \mathcal{K}(m_0(Q)-1) \leq d_0(P).$$

→ From Lemma (6.9) it follows that

$$H^1(\mathbf{P}(E), \mathcal{K}(m_0(Q)+d_0(P)-1)) = 0.$$

Together with the equalities $H^i(\mathbf{P}(E), \mathcal{K}(m-i)) = 0$ for $i \geq 2$ and $m \geq m_0(Q)$ we see that if we choose

$$m_0(P) = m_0(Q) + d_0(P) = m_1(P) + d_0(P)$$

we have that $\mathcal{K}$ is $m_0(P)$–regular.

(6.11) Note. Let $\mathcal{K}$ be the kernel of a surjection $\mathcal{F} \to \mathcal{G}$ of coherent $\mathcal{O}_{\mathbf{P}(E)}$–
→ modules. We obtain that $\chi_{\mathcal{K}}(m) + \chi_{\mathcal{G}}(m) = \chi_{\mathcal{F}}(m)$. It follows from Note (6.3) that there is an integer $m_0(\mathcal{F})$ such that $\mathcal{F}$ is m–regular for all $m \geq m_0(\mathcal{F})$. From the exact sequence

$$H^{i-1}(\mathbf{P}(E), \mathcal{F}(m-i)) \to H^{i-1}(\mathbf{P}(E), \mathcal{G}(m-i)) \to \\ H^i(\mathbf{P}(E), \mathcal{K}(m-i)) \to H^i(\mathbf{P}(E), \mathcal{F}(m-i))$$

→ and Proposition (6.8)(1) we see that, for $m \geq m_0(\mathcal{F}) + r - 1$, we have that $\mathcal{K}$ is m regular if and only if $\mathcal{G}$ is $m-1$ regular and the map $H^0(\mathbf{P}(E), \mathcal{F}(m-i)) \to H^0(\mathbf{P}(E), \mathcal{G}(m-i))$ is surjective.

7. Fitting ideals.

(7.1) Setup. Given a ring A and a finitely generated A–module M. Fix a non–negative integer r. Choose generators $m_1, \dots, m_s$ for M and let

$$N = \{(a_1, \dots, a_s) \in A^n : a_1 m_1 + \cdots + a_s m_s = 0\}.$$

Moreover, choose generators $\{n_\alpha = (a_{\alpha,1}, \dots, a_{\alpha,s})\}_{\alpha \in \mathcal{I}}$ for the A–module N. We denote by I_r the ideal in A generated by the $(s-r)$–minors of the $(\#\mathcal{I} \times s)$–matrix $B = (a_{\alpha,1}, \dots, a_{\alpha,s})_{\alpha \in \mathcal{I}}$. When $(s-r) > \min(\#I, s)$ we let $I_r = (0)$ and when $(s-r) \leq 0$ we let $I_r = B$. We have that $0 = I_{-1} \subseteq I_0 \subseteq \cdots \subseteq I_s = B = I_{s+1} = \cdots$.

(7.2) Note. Given an element $n = (a_1, \dots, a_s)$ in N. Let J be the ideal in A generated by the $s-r$ minors of the $((\#\mathcal{I}+1) \times s)$–matrix C obtained from B by adding $(a_1, \dots, a_s)$ as the first row. Then $J = I_r$.

It is clear that $I_r \subseteq J$ because the matrix B is formed from the rows $2, 3, \dots$ of C.

To prove the opposite inclusion we only have to show that the $s-r$–minors containing the first row of C are contained in I_r. However, we have that $n = b_1 n_{\alpha_1} + \cdots b_s n_{\alpha_t}$, for some b_i in B, and α_i in $\mathcal{I}$. Hence, the first row of C is a sum of rows $\alpha_1+1, \cdots, \alpha_t+1$ multiplied with $b_1, \dots, b_t$ respectively. Hence the $(s-r)$–minors containing the first row can be expanded as a sum of the $(s-r)$–minors containing rows $\alpha_1+1, \dots, \alpha_t+1$ multiplied by $b_1, \dots, b_t$. We consequently have that $J \subseteq I_r$.

By (transfinite, if necessary) induction, we obtain that the ideal in A obtained from the $(s-r)$–minors of the matrix obtained by adding to B rows coming from any set of elements of N, is equal to I_r. In particular we obtain that the ideal I_r is independent of the choice of generators n_α of N. Indeed, if we chose another set of generators for N, we have that the ideal obtained from the union of the two sets of generators is equal to the ideal obtained from each set.

(7.3) Note. Let m be an element of M. Moreover, let

$$P = \{(a, a_1, \dots, a_s) \in A^{s+1} : am + a_1 m_1 + \cdots + a_s m_s = 0\}.$$

Then, if we write $m = -b_1 m_1 - \cdots - b_s m_s$, with b_i in A, we have that P contains the element $p = (1, b_1, \dots, b_s)$, and that the A–module P is generated by the element p and elements $\{p_\alpha = (0, a_{\alpha,1}, \dots, a_{\alpha,s})\}_{\alpha \in \mathcal{I}}$, where $n_\alpha = \{(a_{\alpha,1}, \dots, a_{\alpha,s})\}_{\alpha \in \mathcal{I}}$ are generators for N. Let J be the ideal in A generated by the $(s-r+1)$–minors of the $((\#\mathcal{I}+1) \times (s+1))$–matrix whose first row is the the element p and whose $(\alpha+1)$'st row consists of the coordinates of p_α. It is clear that we have an equality $J = I_r$
→ and it follows from Note (7.2) that J is independent of the choice of generators
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of P. We have shown that I_r is the ideal defined by the $(s+1-r)$–minors of the matrix obtained from $s+1$ generators $m_1, \dots, m_s, m$ of M and any set of generators of P. By induction on t we obtain that I_r is the ideal obtained from the $(s+t-r)$–minors of the $(\#\mathcal{I}+t, s+t)$–matrix obtained from $s+t$ elements $m_1, \dots, m_s, n_1, \dots, n_t$, and any set of generators of the A–module

$$\{(a_1, \dots, a_s, b_1, \dots, b_t) | a_1 m_1 + \cdots + a_s m_s + b_1 m_1 + \cdots + b_t m_t = 0\}.$$

In particular we have that the ideal I_r is independent of the choice of generators $m_1, \dots, m_s$ of M. Indeed, if we had another set of generators we have that the ideal obtained from the union of the two sets of generators is equal to the ideal obtained from each set.

(7.4) Definition. Let M be a finitely generated A–module and r a non–negative integer. We saw in Notes (7.2) and (7.3) that the ideal in A generated by the $(s-r)$–minors of the matrix obtained from a set of generators $m_1, \dots, m_s$ of M by taking as rows the set of generators for the A–module $N = \{(a_1, \dots, a_s) \in A^s | \sum_{i=1}^s a_i m_i = 0\}$ is independent of s, of the choice of generators of both of M, and of the corresponding N. Thus the ideal depends only on M and r. We denote the ideal by $F_r(M)$ and we call it *the r'th Fitting ideal of the A–module M* .

(7.5) Remark. We have inclusions $0 = F_{-1}(M) \subseteq F_0(M) \subseteq \cdots \subseteq F_{r-1}(M) \subseteq F_r(M)$. If M can be generated by s elements we have that $A = F_s(M) = F_{s+1}(M) = \cdots$.

(7.6) Note. Given generators $m_1, \dots, m_s$ for the A–module M. We obtain a surjection

$$A^s \to M$$

and it is clear that N of Setup (7.1) is the kernel to this map. The choice of generators $\{n_\alpha\}_{\alpha \in \mathcal{I}}$ for N gives an exact sequence

$$A^{\oplus \mathcal{I}} \to B^{\oplus s} \to M \to 0$$

of A–modules. It follows from Definition (7.4) that $F_r(M)$ is the ideal of A generated by the $(s-r)$–minors of the $((\#\mathcal{I}) \times s)$–matrix $A^{\oplus \mathcal{I}} \to A^{\oplus s}$.

(7.7) Lemma. *Let B be an A–algebra and let M be a finitely generated A–module. Then we have an equality*

$$F_r(M)B = F_r(B \otimes_A M)$$

of ideals in B.

→ *Proof.* It follows from Remark (7.5) that we have a presentation

$$A^{\oplus\mathcal{I}} \xrightarrow{\beta} A^{\otimes s} \to M \to 0$$

of M. We obtain a presentation

$$B \otimes_A A^{\oplus\mathcal{I}} = B^{\oplus\mathcal{I}} \xrightarrow{\beta\otimes\mathrm{id}_B} B \otimes_A A^{\oplus s} = B^s \to B \otimes_A M \to 0$$

→ of $B \otimes_A M$. It follows from Remark (7.6) that $F_r(M)$ and $F_r(B \otimes_A M)$ are generated by the $(s-r)$–minors of β respectively $\beta \otimes \mathrm{id}_B$. The Lemma follows since the images of the entires of β in B are the same as the entries of $\beta \otimes_A \mathrm{id}_B$.

(7.8) Proposition. *Given a noetherian ring A and a finitely generated A–module M. Let P be a prime ideal of A. Then r is the minimal number of generators for the A_P–module M_P if and only if $F_{r-1}(M) \subseteq P$ and $F_r(M) \not\subseteq P$.*

We have that M_P is free A_P–module of rank r if and only if $F_{r-1}(M)A_P = 0$ and $F_r(M) \not\subseteq P$.

Proof. When the minimal number of generators for M_P is r it follows from the
→ definition of Fitting ideals and Lemma (7.7) that $F_r(M)A_P = F_r(M_P) = A_P$. Thus we have that $F_r(M) \not\subseteq P$. It follows from Nakayamas Lemma that we have a presentation $A_P^t \xrightarrow{\beta} A_P^r \to M_P \to 0$ which induces an isomorphism $(A_P/PA_P)^r \to M_P/PM_P$. Hence all the elements of the matrix $\beta\colon A_P^t \to A_P^r$ are in PA_P. Since $F_{r-1}(M)A_P$ is generated by these elements it follows that $F_{r-1}(M)A_P \neq 0$. Multiplying, if necessary, with a unit in A_P we may assume that the matrix β is the image of a matrix with coefficients in A. Then $F_{r-1}(M) \subseteq P$.

When M_P is free we can choose $t = 0$ and thus obtain that $F_{r-1}(M) = 0$.

Conversely, assume that $F_r(M)A_P = A_P$ that $F_r(M)A_P \subseteq PA_P$. Choose a presentation $A_P^t \xrightarrow{\beta} A_P^s \to M_P$. If necessay, we may multiply the β with a unit in A_P such that the coefficients of β are images of elements in A. Since $F_r(M)A_P = A_P$ there is an $(s-r)$–minor of the matrix β which is invertible. Reordering, if necessary, the bases for A_P^s and A_P^t we can assume that this minor is the determinant of the matrix in the upper left corner of β.

Reordering the first $s-r$ rows and coluns, if necessary, and using row and column operations, we can make the upper left $(s-r)\times(s-r)$–matrix in the upper left corner the unit matrix. We can then use row and column operations on β to put β in a form where the $r \times (s-r)$–matrix in the lower left corner and the $(s-r)\times(t-s+r)$–matrix in the upper right corner are zero. Since we have assumed that $F_{r-1}(M)A_P \subseteq PA_P$ we have that the coordinates of the $r \times (t-s+r)$–matrix in the lower right corner are in PA_P. It follows that the surjection $(A_P/PA_P)^s \to M_P/PM_P$ induces an isomorphism between M_P/PM_P

and the vector subspace of $(A_P/PA_P)^s$ generated by the r last basis vectors. Hence the minimal number of generators for M is r.

When $F_{r-1}(M)A_P = 0$ we have that the $r \times (t-s+r)$–matrix in the lower left corner is zero and thus that $A_P^s \to M_P$ induces an isomorphism between M and the submodule of A_P^s generated by the last r basis vectors. Hence M_P is free of rank n.

(7.9) Definition. Let S be a scheme and $\mathcal{G}$ a coherent $\mathcal{O}_S$–module. It follows from Lemma (7.7) that the ideals $F_r(\mathcal{G}(\operatorname{Spec} A))$ for all open affine subschemes $\operatorname{Spec} A$ of S define a quasi–coherent ideal $F_r(\mathcal{G})$ of $\mathcal{O}_S$ such that $F_r(\mathcal{G})(\operatorname{Spec} A) = F_r(\mathcal{G}(\operatorname{Spec} A))$. We call this ideal the *r'th Fitting ideal of $\mathcal{G}$ in S* .

(7.10) Remark. Corresponding to the inclusion $0 = F_{r-1}(M) \subseteq F_0(M) \subseteq F_1(M) \subseteq \cdots \subseteq F_{r-1}(M) \subseteq F_r(M)$ of Remark (7.5) we obtain inclusions $0 = F_{-1}(\mathcal{G}) \subseteq F_0(\mathcal{G}) \subseteq F_1(\mathcal{G}) \subseteq \cdots \subseteq F_{r-1}(\mathcal{G}) \subseteq F_r(\mathcal{G})$.

(7.11) Proposition. *Let $g\colon T \to S$ be a morphism and $\mathcal{G}$ a coherent $\mathcal{O}_S$–module. We have that*

$$F_r(g^*\mathcal{G}) = g^*(F_r(\mathcal{G}))\mathcal{O}_T.$$

Proof. Let $\operatorname{Spec} A$ be an open subset of S and $\operatorname{Spec} B$ an open affine subset of T mapping to $\operatorname{Spec} A$ by g. Moreover, let $M = \mathcal{G}(\operatorname{Spec} A)$. By definition of the Fitting ideals of $\mathcal{G}$ we have that $F_r(g^*\mathcal{G}) = F_r(\widetilde{B \otimes_A M})$ and $g^*(F_r(\mathcal{G}))\mathcal{O}_{\operatorname{Spec} B} = \widetilde{F_r(M)B}$. Hence the Proposition follows from Lemma (7.7).

(7.12) Proposition. *Given a noetherian scheme S, a coherent $\mathcal{O}_S$–module $\mathcal{G}$ and a point s of S. Then r is the minimal number of generators for the $\mathcal{O}_{S,s}$–module $\mathcal{G}_s$ if and only if $s \in Z(F_{r-1}(\mathcal{G})) \setminus Z(F_r(\mathcal{G}))$.*

We have that $\mathcal{G}_s$ is a free $\mathcal{O}_{S,s}$–module of rank r if and only if $F_{r-1}(\mathcal{G})_s = 0$ and $F_r(\mathcal{G})_s = \mathcal{O}_{S,s}$.

Proof. Let $\operatorname{Spec} A$ be an open subset of S containing s and let P be the prime ideal in A corresponding to the point s. Moreover, let $M = \mathcal{G}(\operatorname{Spec}(A))$. Then $\mathcal{G}_s = M_P$ and $s \in Z(F_{r-1}(\mathcal{G}))$ if and only if $P \supseteq Z(F_{r-1}(M)$. The Proposition therefore follows from Proposition (7.8).

8. Flattening stratifications.

(8.1) Setup. Given a noetherian ring A and a free A–module E of rank $r+1$. Let S be a noetherian scheme and $f\colon X \to S$ a morphism from a scheme X. Moreover, let $\mathcal{F}$ be a coherent $\mathcal{O}_X$–module.

(8.2) Definition. A *flattening stratification* of $\mathcal{F}$ over S is a finite collection $\{S_i\}_{i\in\mathcal{I}}$ of disjoint locally closed subschemes of S such that S is the set theoretic union of the S_i, and such that, for each morphism $g\colon T \to S$, we have that $\mathcal{F}_T$ is flat over T if and only if $g^{-1}S_i$ is open and closed in T.

In other words, given a morphism $g\colon T \to S$ from a connected scheme T, then $\mathcal{F}_T$ is flat over T if and only if g factors via one of the S_i.

(8.3) Proposition. *Let $\mathcal{G}$ be a coherent $\mathcal{O}_S$–module. For each non–negative integer r there is a locally closed subscheme S_r of S such that a morphism $g\colon T \to S$ factors via S_r if and only if $g^*\mathcal{G}$ is locally free of rank r, and S_r is empty except for a finite number of integers. That is, the $\mathcal{O}_S$–module $\mathcal{G}$ has a flatttening stratification over S.*

Proof. We shall show that the locally closed subschemes

$$S_r = Z(F_{r-1}(\mathcal{G})) \setminus Z(F_r(\mathcal{G}))$$

→ of S form a flattening stratification for $\mathcal{G}$. It follows from Proposition (7.12) that $g^*\mathcal{G}$ is locally free of rank r if and only if $F_r(g^*\mathcal{G}) = \mathcal{O}_T$ and $F_{r-1}(g^*\mathcal{G}) =$
→ 0. However, it follows from Proposition (7.11) that $g^*F_r(\mathcal{G})\mathcal{O}_T = F_r(g^*\mathcal{G})$ and $g^*F_{r-1}(\mathcal{G})\mathcal{O}_T = F_{r-1}(g^*\mathcal{G})$. Hence $g^*\mathcal{G}$ is locally free if and only if we have that the map $g^*F_r(\mathcal{G}) \to \mathcal{O}_T$ is surjective and the map $g^*F_{r-1}(\mathcal{G}) \to \mathcal{O}_T$ is zero. The condition that the first map is surjective is equivalent to the condition that g factors via $S \setminus Z(F_r(\mathcal{G}))$, and the condition that the second is zero is equivalent to the condition that g factors via $Z(F_{r-1}(\mathcal{G}))$.

The rank of $\mathcal{G}$ is limited by the maximum of the dimensions $\dim_{\kappa(s)} \mathcal{G}_s \otimes_{\mathcal{O}_{S,s}} \kappa(s)$ for $s \in S$, and the dimension is upper semi–continuous and therefore limited since S is noetherian. Hence there is only a finite number of different schemes S_r.

(8.4) Lemma. *(alt nedenfor gjøres for fast m?) Assume that X is a closed subscheme of $\mathbf{P}(E)$ with structure map f. Given a morphism $g\colon T \to \operatorname{Spec} A$. Assume that m_0 is such that*

$$H^i(X_{\operatorname{Spec}\kappa(s)}, \mathcal{F}_{\operatorname{Spec}\kappa(s)}(m)) = 0$$

for $i > 0$ and $m \geq m_0$. Then $\mathcal{F}_T$ is flat over T if and only if $f_*\mathcal{F}(m)_T$ is locally free for $m \geq m_0$.
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When $\mathcal{F}_T$ is flat we have that the base change map $f_*\mathcal{F}_T(m) \to f_{T*}\mathcal{F}_T(m)$ is an isomorphism for $m \geq m_0$.

Proof. Assume that $\mathcal{F}_T$ is flat over T. It follows from Proposition (4.9) that $R^i f_{T*}\mathcal{F}_T(m) = 0$ for $i > 0$ and $m \geq m_0$. Consequently it follows from Theorem (4.7) that $f_*\mathcal{F}(m)_T = f_{T*}\mathcal{F}_T(m)$ for $m \geq m_0$. Moreover it follows from Theorem (3.16)(1) applied to $\mathcal{F}_T$ over T that $f_{T*}\mathcal{F}_T(m)$ is locally free for $m \geq m_0$. Hence $f_*\mathcal{F}(m)_T$ is locally free for $m \geq m_0$. We also proved the last assertion of the Lemma.

Conversely, assume that $f_*\mathcal{F}(m)_T$ is locally free for $m \geq m_0$. It follows from Proposition (4.10) that $f_*\mathcal{F}(m)_T = f_{T*}\mathcal{F}_T(m)$ for big m. Thus $f_{T*}\mathcal{F}_T(m)$ is locally free for big m. It follows from Theorem (3.16)(2) that $\mathcal{F}_T$ is flat over T.

(8.5) Theorem. *Assume that X is a closed subscheme of $\mathbf{P}(E)$. There is a flattening stratification $\{S_P\}_{P\in\mathbf{Q}[t]}$ of $\mathcal{F}$ over $\operatorname{Spec} A$ such that for every morphism $g\colon T \to \operatorname{Spec} A$ we have that g factors via S_P if and only if $\mathcal{F}_T$ is flat over T with Hilbert polynomial P.*

Proof. It follows from Lemma (4.11) that we can choose an m_0 such

$$H^i(X_{\operatorname{Spec}\kappa(s)}, \mathcal{F}_{\operatorname{Spec}\kappa(s)}(m)) = 0 \tag{8.5.1}$$

for $i > 0$ and

$$\kappa(s) \otimes_A H^0(X, \mathcal{F}(m)) \to H^0(X_{\operatorname{Spec}\kappa(s)}, \mathcal{F}_{\operatorname{Spec}\kappa(s)}(m))$$

is an isomorphism for $m \geq m_0$ and all points $s \in \operatorname{Spec} A$. For $m \geq m_0$ choose a stratification $\{S_i(m)\}_{i\in\mathcal{I}(m)}$ for $f_*\mathcal{F}(m) = H^0(X, \mathcal{F}(m))$ as in Proposition (8.3) such that $f_*\mathcal{F}(m)_{S_i(m)}$ is locally free of rank i. Since $S_i(m)$ is locally closed we have an equality on fibers $\kappa(s) \otimes_{\mathcal{O}_{S_i,s}} (f_*\mathcal{F}(m)_{S_i(m)})_s = \kappa(s) \otimes_{\mathcal{O}_{\operatorname{Spec} A,s}} f_*\mathcal{F}(m)_s$. The latter fiber is equal to $\kappa(s) \otimes_{A_P} H^0(X, \mathcal{F}(m))_P = \kappa(P) \otimes_A H^0(X, \mathcal{F}(m))$, where P is the prime ideal in A corresponding to the point s. We obtain that the rank of $f_*\mathcal{F}(m)_{S_i(m)}$ is equal to

$$\dim_{\kappa(s)} H^0(X_{\operatorname{Spec}\kappa(s)}, \mathcal{F}_{\operatorname{Spec}\kappa(s)}(m)) = \chi_{\mathcal{F},s}(m)$$

for $m \geq m_0$. Hence the underlying set of $S_i(m)$ is

$$\{s \in \operatorname{Spec} A\colon \chi_{\mathcal{F},s}(m) = i\}.$$

Denote by $\{T_j(n)\}_{j\in\mathcal{J}(n)}$ the stratification of Proposition (8.3) for the $\mathcal{O}_{\operatorname{Spec} A}$–module $\mathcal{N}_n = \oplus_{i=0}^n f_*\mathcal{F}(m_0+i)$. Since the sum $\mathcal{N}_n$ is locally free if and only if each summand is locally free we have, for a given $j \in \mathcal{J}(n)$, that $T_j(n)$ is the disjoint union of the sets

$$S_{i_0}(m_0) \cap \cdots \cap S_{i_n}(m_0 + n)$$

where $j = i_0 + \cdots + i_n$. We obtain that the underlying set of $T_j(n)$ is the disjoint union of the sets

$$\{s \in \operatorname{Spec} A \colon \chi_{\mathcal{F},s}(m_0 + h) = i_h, \quad \text{for } h = 0, \dots, n\}. \tag{8.5.2}$$

with $j = i_0 + \cdots + i_n$.

It follows from Theorem (5.5) that $\chi_{\mathcal{F},s}$ has degree at most r. When $n \geq r$ we therefore have that $\chi_{\mathcal{F},s}$ is defined by its values on $m_0, \dots, m_0 + r$. In particular the values $i_{r+1}, i_{r+2}, \dots$ are determined by $i_0, \dots, i_r$. It follows that the $\chi_{\mathcal{F},s}$ are the same for $s \in S(m_0) \cap \cdots \cap S(m_0 + r)$ and that for $n \geq r$ we have that the underlying set of $T_j(n)$ is the disjoint union of the sets

$$\{s \in \operatorname{Spec} A \colon \chi_{\mathcal{F},s}(m_0 + h) = i_h \quad \text{for } h = 0, \dots, r\}$$

where $j = \sum_{h=0}^{n} \chi_{\mathcal{F},s}(m_0 + h)$. We thus have a sequence $T_j(r) \supseteq T_j(r+1) \supseteq \cdots$ of locally closed subschemes of $\operatorname{Spec} A$ with the same underlying set. It follows that there is an $n_0 \geq r$ such that $T_j(n_0) = T_j(n_0 + 1) = \cdots$. Since there is only a finite number of j's, by the definition of a stratification we can choose n_0 such that the equality $T_j(n_0) = T_j(n_0 + 1) = \cdots$ holds for all indices j.

We have proved that a morphism $g \colon T \to \operatorname{Spec} A$ factors via $T_j(n_0)$ if and only if $f_*\mathcal{F}(m_0 + i)_{T_j(n_0)}$ is locally free of rank $P_j(m_0 + i)$ for $i = 0, 1, \dots$. It follows from (8.5.1) and Lemma (8.4) that g factors via $T_j(n_0)$ if and only if $\mathcal{F}_{T_j(n_0)}$ is flat over $T_j(n_0)$. It also follows that the rank of $f_{T_j(n_0)*}\mathcal{F}_{T_j(n_0)}(m)$ is $P_j(m)$ for big m. In particular the Hilbert polynomial of $\mathcal{F}_{T_j(n_0)}$ is P_j. We have proved that the finite collection of sets $\{T_j(n_0)\}_{j \in \mathcal{J}}$ gives the asserted flattening stratification for $\mathcal{F}$ over $\operatorname{Spec} A$.

(8.6) Note. It follows from the definition of a stratification that there is only a finite number of strata in Theorem (8.4). Moreover the strata are unique because if $\{S'_P\}$ is another stratum, then each S'_P must factor via S_P, and conversely. However, both are subschemes of $\operatorname{Spec} A$ and must therefore be equal.

9. Representation of functors.

(9.1) Setup. Given a scheme S and a contravariant functor F from schemes over S to sets. All schemes and morphisms will be taken over S. Given a scheme X over S, we denote by h_X the contravariant functor from schemes over S to sets which sends a scheme T to the set of S–homomorphisms $h_X(T) = \mathrm{Hom}_S(T, X)$ from T to X, and to a morphism $h: U \to T$ associates the map $h_X(h): h_X(T) \to h_X(U)$ given by $h_X(h)(g) = gh$, for all morphisms $g: T \to X$.

(9.2) Note. There is a natural bijection between elements in $F(X)$ and morphisms of functors $H: h_X \to F$.

Given an element $\xi \in F(X)$ we define a morphism

$$H_\xi: h_X \to F$$

by $H_\xi(T)(g) = F(g)(\xi)$ for all S–schemes T and all S–morphisms $g: T \to X$. In this way we clearly obtain a morphism of functors $h_X \to F$. We have that $H_\xi(X)(\mathrm{id}_X) = F(\mathrm{id}_X)(\xi) = \xi$.

Conversely, given a morphism of functors $H: h_X \to F$. We obtain an element $\xi_H = H(X)(\mathrm{id}_X)$ in $F(X)$ such that for all morphisms $g: T \to X$ we have that

$$H(T)(g) = H(T)h_X(g)(\mathrm{id}_X) = F(g)H(X)(\mathrm{id}_X) = F(g)(\xi_H).$$

In particular we have that $H = H_{\xi_H}$.

Hence we have that the map that sends ξ to H_ξ, and the map that sends H to ξ_H are inverses of each other.

(9.3) Definition. The functor F is *representable* , and is *represented by a scheme* X if there is an element $\xi \in F(X)$ such that the morphism $H_\xi: h_X \to F$ of Note
$\to$ (9.2) is an isomorphism. We call ξ the *universal element* .

(9.4) Note. It follows from the definition of a representable functor that the scheme X representing the functor F is determined up to isomorphisms.

(9.5) Definition. Given a contravariant functor G from schemes over S to sets. We say that G is a *subfunctor* of F if $G(T) \subseteq F(T)$ for all schemes T over S and we have that $G(g)(\eta) = F(g)(\eta)$ for all morphisms $g: U \to T$ over S and all $\eta \in G(T)$.

(9.6) Example. Given a scheme X over S and let $i: Y \to X$ be the immersion of a subscheme Y of X. Two different morphisms $g, h: T \to Y$ give different morphisms $ig, ih: T \to X$. Hence we have that h_Y is a subfunctor of h_X. We have that a morphism $g: T \to X$ lies in $h_Y(T)$ if and only if g factors via $i: Y \to X$.
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(9.7) Note. Given a subfunctor G of F and let $H_\xi: h_X \to F$ be the morphism of functors given by an element $\xi \in F(X)$. We obtain a subfunctor $h_X \times_F G$ of h_X which, for every scheme T over S is given by

$$(h_X \times_F G)(T) = h_X(T) \times_{F(T)} G(T) = \{g \in h_X(T) \colon H_\xi(T)(g) \in G(T)\}$$

and which to a morphism $h: U \to T$ associates the map

$$h_X(h) \times_{F(h)} G(h) \colon h_X(T) \times_{F(T)} G(T) \to h_X(U) \times_{F(U)} G(U).$$

(9.8) Definition. A subfunctor G of F is *locally closed* if there, for every morphism $H_\xi: h_X \to F$ of functors, is a subscheme $X_{G,\xi}$ of X such that $h_{X_{G,\xi}} = h_X \times_F G$, where $h_{X_{G,\xi}}$ is considered as a subfunctor of h_X via the immersion $i: X_{G,\xi} \to X$.

The subfunctor G is *open* or *closed* if the scheme $X_{G,\xi}$ is an open, respectively closed, subscheme of X.

→ **(9.9) Note.** It is immediate from Definition (9.8) that the subfunctor G of F is locally closed if and only if there, for every scheme X over S and every element $\xi \in F(X)$, is a subscheme $X_{G,\xi}$ of X such that a morphism $g: T \to X$ factors via $X_{G,\xi}$ if and only if $F(g)(\xi) \in G(T)$.

(9.10) Note. It follows from the Definition of a locally closed subfunctor that the associated scheme $X_{G,\xi}$ is unique.

(9.11) Note. Given a locally closed subfunctor G of F. Let $H_\xi: h_X \to F$ be the morphism associated to an element $\xi \in F(X)$, and let $i: X_{G,\xi} \to X$ be the corresponding subscheme of X. We have that i is the image of id_X by the map $h_X(X) \xrightarrow{h(i)} h_X(X_{G,\xi})$, and of $\mathrm{id}_{X_{G,\xi}}$ by the map $h_{X_{G,\xi}}(X_{G,\xi}) \to h_X(X_{G,\xi})$. It follows that $F(i)(\xi) = H_\xi(X_{G,\xi})(i)$, and that $F(i)(\xi) \in G(X_{G,\xi})$. We obtain that the morphism $h_{X_{G,\xi}} \to G$ induced by $H_\xi: h_X \to F$ is equal to $H_{F(i)(\xi)}$. In particular, if F is represented by (X, ξ), we have that G is represented by $(X_{G,\xi}, F(i)(\xi))$.

(9.12) Definition. A family $\{F_i\}_{i\in\mathcal{I}}$ of open subfunctors of F is *an open covering of* F if, for every scheme X over S and every element $\xi \in F(X)$, the open subschemes $X_{F_i,\xi_i}, X_{F_i,\xi}$ of X corresponding to F_i cover X.

(9.13) Definition. A functor F is a *Zariski sheaf* if, for every scheme T over S and every open covering $\{T_i\}_{i\in\mathcal{I}}$ of T the sequence

$$F(T) \to \prod_{i\in I} F(T_i) \underset{p_2}{\overset{p_1}{\rightrightarrows}} \prod_{i,j\in\mathcal{I}} F(T_i \cap T_j) \tag{9.13.1}$$

is exact. That is, the map p defined by the restrictions $F(T) \to F_i(T)$ is injective, and the image of p is the kernel

$$\{(f_i)_{i\in\mathcal{I}} \in \prod_{i\in\mathcal{I}} F(T_i) \colon p_1(f_i) = p_2(f_i) \text{ for all } i \in \mathcal{I}\}$$

of the projections p_1 and p_2 induced by the maps $F(T_i) \to F(T_i \cap T_j)$, respectively $F(T_j) \to F(T_i \cap T_j)$, for all $i, j \in \mathcal{I}$.

(9.14) Example. Given a scheme X over S and let Y be a subscheme of X with immersion $i \colon Y \to X$. We have that h_Y is a locally closed subfunctor and that $Y = X_{h_Y,i}$. Hence h_Y is an open or closed subfunctor of h_X if and only if Y is an open respectively closed subscheme of X.

Given an open covering $\{X_i\}_{i\in\mathcal{I}}$ of X. Then the subfunctors $\{h_{X_i}\}_{i\in\mathcal{I}}$ of h_X form an open covering of h_X. For an element $\xi \in h_X(Z)$ corresponding to a morphism $g \colon Z \to X$ we have that $Z_{h_{X_i},\xi} = g^{-1}(X_i)$

We have that the functor h_X is a Zariski sheaf. Given an open covering $\{T_i\}_{i\in\mathcal{I}}$ of the scheme T. The exactness of the sequence (9.13.1) for h_X means that a morphism $g \colon T \to X$ is determined by the restrictions $g|T_i \colon T_i \to X$ for $i \in \mathcal{I}$, and that morphisms $g_i \colon T_i \to X$ such that $g_i|T_i \cap T_j = g_j|T_j \cap T_i$ for all i, j in $\mathcal{I}$ uniquely determine a morphism $g \colon T \to X$ such that $g|T_i = g_i$.

(9.15) Theorem. *Given a functor F which is a Zariski sheaf and an open covering $\{F_i\}_{i\in\mathcal{I}}$ of F by representable functors F_i. Then F is representable.*

Proof. For all $i \in \mathcal{I}$ let the scheme X_i represent the functor F_i. By assumption we have an open cover $\{h_{X_i}\}_{i\in\mathcal{I}}$ of F. For every i and j in $\mathcal{I}$ we have a morphism $h_{X_i} \to F$ of functors, and an open subfunctor h_{X_j} of F. Hence there is an unique open subset $X_{i,j}$ of X_i which represents the functor $h_{X_i} \times_F h_{X_j} = h_{X_i} \cap h_{X_j}$. It follows from the definition of $X_{i,j}$ that, for all $i, j \in \mathcal{I}$, there is a canonical isomorphism $\rho_{i,j} \colon X_{i,j} \to X_{j,i}$, and this isomorphism sends $X_{i,j} \cap X_{i,k}$ isomorphically to $X_{j,i} \cap X_{j,k}$ for all indices k. Moreover we have that $(\rho_{j,k}|X_{j,k} \cap X_{j,i})(\rho_{i,j}|X_{i,j} \cap X_{i,k}) = \rho_{i,k}|X_{i,k} \cap X_{i,j}$, and that $\rho_{i,i} = \mathrm{id}_{X_i}$.

We can thus use the morphisms $\rho_{i,j}$ to glue the schemes $\{X_i\}_{i\in\mathcal{I}}$ into a scheme X with maps $\varphi_i \colon X_i \to X$ of X_i onto an open subset of X such that $\varphi_i|X_{i,j} = (\varphi_j|X_{j,i})\rho_{i,j}$.

Given a morphism $g \colon T \to X$. Let $T_i = g^{-1}(\varphi_i(X_i))$, and let $g_i \colon T_i \to \varphi(X_i)$ be the morphism induced by g. We obtain a unique morphism $\psi_i \colon T_i \to X_i$ such that $\varphi_i\psi_i = g_i$. Denote by σ_i the image of ψ_i by the inclusion $h_{X_i}(T_i) \to F(T_i)$. The element $(\sigma_i)_{i\in\mathcal{I}} \in \prod_{i\in\mathcal{I}} F(T_i)$ has the same image by p_1 and p_2 since $g_i|T_i \cap T_j = g_j|T_j \cap T_j$. Since F is a Zariski sheaf we obtain a unique element $\sigma \in F(T)$ that maps to $(\sigma_i)_{i\in\mathcal{I}}$ by $F(T) \to \prod_{i\in\mathcal{I}} F(T_i)$. We have thus constructed a map

$h_X(T) \to F(T)$. The map is injective since the map $\prod_{i\in\mathcal{I}} h_{X_i}(T_i) \to \prod_{i\in\mathcal{I}} F(T_i)$ is injective. It is clear that this construction is functorial in T. Hence we obtain a subfunctor $h_X \to F$ of F.

It remain to prove that $h_X(T) \to F(T)$ is surjective for all schemes T over S. Let $\sigma \in F(T)$. Since F is covered by the functors $h_{X_i} = F_i$ we can cover T by open subsets $T_i = T_{h_{X_i},\sigma}$ such that a homomorphism $h\colon U \to T$ factors via $T_i \to T$ if and only if the image of h by $F(T) \to F(U)$ lies in $h_{X_i}(U)$. In particular, when h is the inclusion $T_i \to T$, we obtain that the image σ_i of σ by the map $F(T) \to F(T_i)$ comes from a morphism $\psi_i\colon T_i \to X_i$. When h is the inclusion $T_i \cap T_j \to T$ or the inclusion $T_j \cap T_i \to T$ we obtain that the morphisms $\varphi_i\psi_i\colon T_i \to X$ and $\varphi_j\psi_j\colon T_j \to X$ are equal on $T_i \cap T_j = T_j \cap T_i$ because these restrictions maps are elements in $h_X(T_i \cap T_j) = h_X(T_j \cap T_i)$ that, by the injection $h_X(T_i \cap T_j) = h_X(T_j \cap T_i) \to F(T_i \cap T_j) = F(T_j \cap T_i)$, map to the image of σ. Hence the maps $\varphi_i\psi_i$ glue together to a morphism $g\colon T \to X$ such that $g|T_i = \varphi_i\psi_i$ for all $i \in \mathcal{I}$. We have that $g \in h_X(T)$ maps to $(\varphi_i\psi_i)_{i\in\mathcal{I}}$ in $\prod_{i\in\mathcal{I}} h_X(T_i)$ and $(\varphi_i\psi_i)_{i\in\mathcal{I}}$ maps to $(\sigma_i)_{i\in\mathcal{I}}$ in $\prod_{i\in\mathcal{I}} F(T_i)$. Since the map $\prod_{i\in\mathcal{I}} h_{X_i}(T_i) \to \prod_{i\in\mathcal{I}} F(T_i)$ is injective and since F and h_X are Zariski sheaves we have that g maps to σ by the map $h_X(T) \to F(T)$.

(9.16) Note. Given an open covering $\{S_i\}_{i\in\mathcal{I}}$ of S. For every morphism $g\colon T \to S$ we define

$$F_i(T) = \begin{cases} F(T) \text{ when } g \text{ factors via } S_i \\ \emptyset \text{ otherwise.} \end{cases}$$

It is clear that F_i is a subfunctor of F. We have that F_i is an open subfunctor. Indeed, given an S–scheme $f\colon X \to S$ and $\xi \in F(X)$. Let $X_{i,\xi} = f^{-1}(S_i)$. Then an S–morphism $h\colon T \to X$ factors via $X_{i,\xi}$ if and only if fh factors via S_i. Hence h factors via $X_{i,\xi}$ if and only if $F(h)(\xi)$ lies in $F_i(T)$, that is $X_{i,\xi} = X_{F_i,\xi}$. Since the X_i cover S we have that the $X_{i,\xi}$ cover X. Consequently we have that $\{F_i\}_{i\in\mathcal{I}}$ is an open covering of the functor F.

$\rightarrow$ In particular, it follows from Theorem (9.15) that, if F is a Zariski functor then F is representable if and only if all the F_i are representable.

(9.17) Definition. Given a morphism $f\colon X \to S$ of schemes, and an $\mathcal{O}_X$–module $\mathcal{F}$. Two surjections $\mathcal{F} \to \mathcal{G}$ and $\mathcal{F} \to \mathcal{G}'$ of $\mathcal{O}_X$–modules are *equivalent* if they have the same kernel.

Assume that $\mathcal{F}$ is quasi coherent.

Given a scheme T over S we let

$$\mathcal{Q}uot_{\mathcal{F}}(T) = \{\text{equivalence classes of } \mathcal{O}_T\text{–module surjections } \mathcal{F}_T \to \mathcal{G} \\ \text{to a quasi coherent } \mathcal{O}_{X_T}\text{–module } \mathcal{G} \text{ which is flat over } T\}.$$

For each morphism $h: U \to T$ we let

$$\mathcal{Q}uot_{\mathcal{F}}(h): \mathcal{Q}uot_{\mathcal{F}}(T) \to \mathcal{Q}uot_{\mathcal{F}}(U)$$

be the map sending a surjection $\alpha: \mathcal{F}_T \to \mathcal{G}$ to the surjection $h^*(\alpha): \mathcal{F}_U = h^*\mathcal{F}_T \to h^*\mathcal{G}$. It is clear that we obtain a contravariant functor $\mathcal{Q}uot_{\mathcal{F}}$ from schemes over S to sets, called the *quotient functor of* $\mathcal{F}$.

We call $\mathcal{Q}uot_{\mathcal{O}_X}$ the *Hilbert functor* and denote it by $\mathcal{H}ilb_{X/S}$. Given an S–scheme T we have that

$$\mathcal{H}ilb_{X/S}(T) = \{\text{closed subschemes of } T \times_S X \text{ such that the projection } Z \to T \text{ is flat}\}.$$

When $X = S$ We define, for each S–scheme T

$$\mathcal{G}rass_{\mathcal{F}}(T) = \{\text{equivalence classes of } \mathcal{O}_T\text{–module surjections } \mathcal{F}_T \to \mathcal{G} \text{ to a locally free} \mathcal{O}_T\text{–module of finite rank.} \mathcal{G}\}.$$

It is clear that $\mathcal{G}rass_{\mathcal{F}}$ is a subfunctor of $\mathcal{Q}uot_{\mathcal{F}}$.

For each non–negative integer r we let $\mathcal{G}rass^r_{\mathcal{F}}$ be the subfunctor

$$\mathcal{G}rass^r_{\mathcal{F}}(T) = \{\text{equivalence classes of surjections } \mathcal{F}_T \to \mathcal{G} \text{ to a locally free } \mathcal{O}_T\text{–module } \mathcal{G} \text{ of rank } r\}.$$

(9.18) Proposition. *Given a morphism $f: X \to S$ of schemes and a quasi–coherent $\mathcal{O}_X$–module $\mathcal{F}$. The functors $\mathcal{Q}uot_{\mathcal{F}}$, $\mathcal{G}rass_{\mathcal{F}}$ and $\mathcal{G}rass^r_{\mathcal{F}}$ are all Zariski sheaves.*

Proof. Let $g: T \to S$ be a morphism and let $\{T_i\}_{i\in\mathcal{I}}$ be an open covering of T. Consider the sequence

$$\mathcal{Q}uot_{\mathcal{F}}(T) \to \prod_{i\in\mathcal{I}} \mathcal{Q}uot_{\mathcal{F}}(T_i) \underset{p_2}{\overset{p_1}{\rightrightarrows}} \prod_{i,j\in\mathcal{I}} \mathcal{Q}uot(T_i \cap T_j).$$

Given $(f_i)_{i\in\mathcal{I}} \in \prod_{i\in\mathcal{I}} \mathcal{Q}uot_{\mathcal{F}}(T_i)$, where f_i is represented by surjections $\mathcal{F}_{T_i} \to \mathcal{G}_i$ on X_{T_i}. If $p_1(f_i) = p_2(f_i)$ we have that the restriction of $\mathcal{F}_{T_i} \to \mathcal{G}_i$ to $f_T^{-1}(T_i \cap T_j)$ is equivalent to the restriction of $\mathcal{F}_{T_j} \to \mathcal{G}_j$ to $f_T^{-1}(T_j \cap T_i)$, for all i and j. Consequently the kernels of the maps $\mathcal{F}_{T_i} \to \mathcal{G}_i$, for all i, define a submodule $\mathcal{K} \subseteq \mathcal{F}_T$, such that the restriction of the quotient $\mathcal{F}_T \to \mathcal{G}$ to $f_T^{-1}(T_i)$ is equivalent to $\mathcal{F}_{T_i} \to \mathcal{G}_i$ for all $i \in \mathcal{I}$. Hence $(f_i)_{i\in\mathcal{I}}$ is the image of the equivalence class of $\mathcal{F}_T \to \mathcal{G}$ in $\mathcal{Q}uot_{\mathcal{F}}(T)$. Clearly $\mathcal{F}_T \to \mathcal{G}$ is unique since it is determined by its restriction to T_i for all $i \in \mathcal{I}$.

The above proof shows that $\mathcal{G}rass_{\mathcal{F}}$ and $\mathcal{G}rass^r_{\mathcal{F}}$ also are Zariski sheaves.

(9.19) Note. We have a map of functors $\mathcal{Q}uot_{\mathcal{F}} \to h_S$ sending each element of $\mathcal{Q}uot_{\mathcal{F}}(U)$ to the structure morphism $U \to S$. Let $g: T \to S$ be the structure morphism of a scheme T over S. Given a scheme U over S, the elements in the product $\mathcal{Q}uot_{\mathcal{F}}(U) \times_{h_S(U)} h_T(U)$ consist of a quotient $\mathcal{F}_U \to \mathcal{G}$ and an S–morphism $h: U \to T$. Thus $\mathcal{F}_U \to \mathcal{G}$ is equal to $h^*_{X_T}\mathcal{F}_T \to \mathcal{G}$ which is an element in $\mathcal{Q}uot_{\mathcal{F}_T}(U)$. Clearly we obtain a morphism of functors

$$\mathcal{Q}uot_{\mathcal{F}} \times_{h_S} h_T \to \mathcal{Q}uot_{\mathcal{F}_T}.$$

The morphism is an isomorphism of functors with an inverse

$$\mathrm{Quot}_{\mathcal{F}_T} \to \mathrm{Quot}_{\mathcal{F}} \times_{h_S} h_T$$

which, given a morphism $h: U \to T$, sends the element $h^*_{X_T}\mathcal{F}_T \to \mathcal{G}$ in $\mathrm{Quot}_{\mathcal{F}_T}(U)$ to the element $(gh)^*_X\mathcal{F} = h^*_{X_T}\mathcal{F}_T \to \mathcal{G}$ in $\mathrm{Quot}_{\mathcal{F}}(U)$ and to h in $h_T(U)$.

In particular, if $c: S \to \mathbf{Z}$ is the canonical morphism and there is a scheme X_0 over $\operatorname{Spec}\mathbf{Z}$ and a quasi–coherent $\mathcal{O}_{X_0}$–module $\mathcal{F}_0$ such that $X = X_S$ and $\mathcal{F} = c^*_{X_0}\mathcal{F}_0$, we obtain that there is an isomorphism

$$\mathcal{Q}uot_{\mathcal{F}} = \mathcal{Q}uot^{r+1}_{\mathcal{F}_0} \times_{h_{\operatorname{Spec}\mathbf{Z}}} h_S.$$

Thus, when $\mathcal{F}$ is free, $S = \operatorname{Spec} A$ and $X = \mathbf{P}(\operatorname{Sym}_A(E))$ for a free A–module E of rank $r+1$ we have that

$$\mathcal{Q}uot_{\mathcal{F}} = \mathcal{Q}uot_{\mathcal{O}^{r+1}_{\operatorname{Spec}\mathbf{Z}}} \times_{h_{\operatorname{Spec}\mathbf{Z}}} h_S.$$

(9.20) Proposition. *The functor $\mathcal{G}rass^r_{\mathcal{F}}$ is representable.*

When $\mathcal{F}$ is locally free we have that the representing scheme has a natural open covering of the form $\mathbf{V}(\mathcal{E}^* \otimes_{\operatorname{Spec} A} \mathcal{G})$, where $\operatorname{Spec} A$ is an open subset of S over which $\mathcal{F}$ is free, and $\mathcal{E}$ is the free $\mathcal{O}_{\operatorname{Spec} A}$–submodule spanned by r basis elements of $\mathcal{F}|\operatorname{Spec} A$, and $\mathcal{G}$ the module spanned by the remaining basis elements.

Proof. We first reduce to the case when S is affine and $\mathcal{F}$ is a free $\mathcal{O}_S$–module.

Assume that we have a surjection $\mathcal{F}' \to \mathcal{F}$ of quasi–coherent $\mathcal{O}_X$–modules. For every scheme T over S we obtain an injection $\mathcal{G}rass^r_{\mathcal{F}}(T) \to \mathcal{G}rass^r_{\mathcal{F}'}(T)$ sending a quotient $\mathcal{F}_T \to \mathcal{G}$ to the quotient $\mathcal{F}'_T \to \mathcal{F}_T \to \mathcal{G}$. Clearly $\mathcal{G}rass^r_{\mathcal{F}}$ is a subfunctor of $\mathcal{G}rass^r_{\mathcal{F}'}$. It is a closed subfunctor. Indeed, given a morphism $g: T \to S$ and an element $\xi \in \mathcal{G}rass^r_{\mathcal{F}'}(T)$ represented by a quotient $\mathcal{F}'_T \to \mathcal{G}$. Denote by $\mathcal{H}$ the kernel of the map $\mathcal{F}'_T \to \mathcal{F}_T$ and by $\mathcal{I}$ the image of the map $\mathcal{H} \otimes_{\mathcal{O}_T} \mathcal{G}^* \to \mathcal{O}_T$ obtained from the composite map $\mathcal{H} \to \mathcal{F}'_T \to \mathcal{G}$. Let T_ξ be the closed subscheme of T defined by the ideal $\mathcal{I}$. Given a morphism $h: U \to T$ the quotient $\mathcal{F}'_U \to h^*\mathcal{G}$

belongs to $\mathcal{G}rass^r_{\mathcal{F}}(U)$ if and only if the composite map $h^*\mathcal{H}_U \to \mathcal{F}'_U \to g^*\mathcal{G}$ is zero, or equivalently when the map $h^*(\mathcal{H} \otimes_{O_T} \mathcal{G}^*) \to \mathcal{O}_U$ is zero. The image of the latter map is the image of $h^*\mathcal{I} \to \mathcal{O}_U$. Hence the quotient $\mathcal{F}'_U \to h^*\mathcal{G}$ belongs to $\mathcal{G}rass^r_{\mathcal{F}}(U)$ if and only if h factors via T_ξ.

It follows from Note (9.16) that in order to represent $\mathcal{G}rass^r_{\mathcal{F}}$ we can assume that S is an affine scheme $\operatorname{Spec} A$. Let $M = \mathcal{F}(\operatorname{Spec} A)$. Choose a surjection $F \to M$ from a free A–module F. Since we have proved that $\mathcal{G}rass^r_{\mathcal{F}}$ is a closed subfunctor of $\mathcal{G}rass^r_{\widetilde{F}}$ we have that it suffices to represent $\mathcal{G}rass^r_{\widetilde{F}}$. We may thus assume that $\mathcal{F}$ is a free $\mathcal{O}_{\operatorname{Spec} A}$–module.

When S is affine and $\mathcal{F}$ is a free $\mathcal{O}_S$–module, we shall cover the functor $\mathcal{G}rass^r_{\mathcal{F}}$ with open representable functors. Since we have prowed that $\mathcal{G}rass^r_{\mathcal{F}}$ is a Zariski functor it then follows from Proposition (9.15) that $\mathcal{G}rass^r_{\mathcal{F}}$ is representable.

Choose a free submodule $\mathcal{E}$ of $\mathcal{F}$ spanned by r basis vectors. For each scheme T over S we let

$$G_{\mathcal{E}}(T) = \{\mathcal{F}_T \to \mathcal{G} \text{ in } \mathcal{G}rass^r_{\mathcal{F}}(T) \text{ such that } \mathcal{E}_T \subseteq \mathcal{F}_T \to \mathcal{G} \text{ is surjective}\}.$$

Clearly we have that $G_{\mathcal{E}}$ is a subfunctor of $\mathcal{G}rass^r_{\mathcal{F}}$. It is an open subfunctor. Indeed, given a scheme $g: T \to S$ over S and an element $\xi \in \mathcal{G}rass^r_{\mathcal{F}}(T)$ corresponding to a quotient $\mathcal{F}_T \to \mathcal{G}$. The subset $T_{\mathcal{E},\xi}$ of T where the map $\mathcal{E}_T \to \mathcal{F} \to \mathcal{G}$ is surjective is open. Given a morphism $h: U \to T$. Then $\mathcal{E}_U \to h^*\mathcal{G}$ belongs to $G_{\mathcal{E}}(U)$ if and only if the map $\mathcal{E}_U \to \mathcal{F}_U \to h^*\mathcal{G}$ is surjective. Given a point $u \in U$. The determinant of the map $\mathcal{E}_{T,h(u)} \to \mathcal{G}_{h(u)}$ of free $\mathcal{O}_{T,h(u)}$–modules pulls back to the determinant of the map $\mathcal{E}_{U,u} = h^*(\mathcal{E}_{T,h(u)}) \to h^*(\mathcal{G}_{h(u)}) = (h^*\mathcal{G})_u$. We have that $T_{\mathcal{E},\xi}$ is exactly the open subscheme of T where the determinant of $\mathcal{E}_{T,h(u)} \to \mathcal{G}_{h(u)}$ is invertible, and thus where the determinant of $\mathcal{E}_{U,u} \to (h^*\mathcal{G})_u$ is invertible. The latter determinant is invertible if and only if $\mathcal{E}_U \to h^*\mathcal{G}$ is surjective at u. Hence we have that $\mathcal{E}_U \to \mathcal{F}_U \to h^*\mathcal{G}$ is surjective if and only if $h: U \to T$ factors via $T_{\mathcal{E},\xi}$.

The open subsets $T_{\mathcal{E},\xi}$ for varying $\mathcal{E}$ cover T because at every point t of T we can find a submodule $\mathcal{E}$ of $\mathcal{F}$ spanned by r basis vectors such that $\mathcal{E}_{T,t} \to \mathcal{F}_{T,t} \to \mathcal{G}_t$ is surjective, and thus that $\mathcal{E}_T \to \mathcal{F}_T \to \mathcal{G}$ is surjective in a neighbourhood of t. To obtain such a map we choose a map $\kappa(t) \otimes_{\mathcal{O}_{T,t}} \mathcal{F}_{T,t} \to \kappa(t) \otimes_{\mathcal{O}_{T,t}} \mathcal{G}_{T,t}$ that sends the vector space spanned by the images of vectors $e_1, \dots, e_r$ of a basis of $\mathcal{F}$ surjectively to $\kappa(t) \otimes_{\mathcal{O}_{T,t}} \mathcal{G}_{T,t}$. By Nakayamas Lemma this map can be lifted to a surjection $\mathcal{O}_{T_{\mathcal{E},\xi}} e_1 \oplus \cdots \oplus \mathcal{O}_{T_{\mathcal{E},\xi}} e_r \to \mathcal{G}$ in a neighbourhood $T_{\mathcal{E},\xi}$ of t in T.

It follows that it suffices to represent the functor $G_{\mathcal{E}}$. Write $\mathcal{F}$ as a direct sum $\mathcal{F} = \mathcal{E} \oplus \mathcal{F}'$ where $\mathcal{F}'$ is the sheaf spanned by the remaining basis vectors. A surjection $\mathcal{E}_T \to \mathcal{G}$ to a locally free sheaf of rank r is an isomorphism. Hence $\mathcal{G}$ must be free and surjections $\mathcal{E}_T \xrightarrow{i} \mathcal{F}_T \xrightarrow{\varphi} \mathcal{G}$ are in one to one correspondence with homomorphisms $\mathcal{F}_T \xrightarrow{\varphi} \mathcal{E}_T$ that are the identity on the component $\mathcal{E}_T$, via the homomorphism that send φ to $(\varphi i)^{-1}\varphi$. Hence surjections $\mathcal{E}_T \xrightarrow{i} \mathcal{F}_T \xrightarrow{\varphi} \mathcal{G}$ are the same as homomorphisms $\mathcal{F}'_T \to \mathcal{E}_T$. However, a homomorphism $\mathcal{F}'_T \to \mathcal{E}_T$ is the same as a homomorphism $\mathcal{E}^*_T \otimes_{\mathcal{O}_T} \mathcal{F}'_T \to \mathcal{O}_T$. It follows that $G_{\mathcal{E}}$ is representable, and represented by the affine scheme $\mathbf{V}(\mathcal{E}^*_T \otimes_{\mathcal{O}_T} \mathcal{F}'_T) = \operatorname{Spec}(\operatorname{Sym}_A(\mathcal{E}(\operatorname{Spec} A)^* \otimes_A \mathcal{F}'(\operatorname{Spec} A)))$.

We have proved that the functor $\mathcal{G}rass^r_{\mathcal{F}}$ is representable.

(9.21) Definition. The scheme that represents the functor $\mathcal{G}rass^r_{\mathcal{F}}$ is denoted by $\operatorname{Grass}^r(\mathcal{F})$ and called *the grassmannian of r–quotients of $\mathcal{F}$*. The universal element $\operatorname{id}_{\operatorname{Grass}^r(\mathcal{F})} \in \mathcal{G}rass^r_{\mathcal{F}}(\operatorname{Grass}^r(\mathcal{F}))$ corresponds to a universal quotient $\mathcal{F}_{\operatorname{Grass}^r(\mathcal{F})} \to \mathcal{Q}$ on $\operatorname{Grass}^r(\mathcal{E})$.

We write $\operatorname{Grass}^1(\mathcal{F}) = \mathbf{P}(\mathcal{F})$, and we call $\mathbf{P}(\mathcal{F})$ the *projective space* associated to $\mathcal{F}$. A scheme X over S is *projective* over S if there is a locally free $\mathcal{O}_S$–module $\mathcal{F}$ of finite rank such that X is a closed subscheme of $\mathbf{P}(\mathcal{F})$ and the structure morphism of X is induced by the structure morphism of $\mathbf{P}(\mathcal{F})$.

(9.22) Note. We have earlier used the projective r–dimensional space $\mathbf{P}(E)$ over $\operatorname{Spec} A$, where E is a free A–module spanned by vectors $e_0, \dots, e_r$. When $S = \operatorname{Spec} A$ and $\mathcal{F} = \widetilde{E}$ we have that this space is equal to the projective space $\mathbf{P}(\mathcal{F})$ defined in (9.21). Indeed, the latter is covered by affine schemes $\mathbf{V}(\mathcal{E}^*_i \otimes_{\mathcal{O}_{\operatorname{Spec} A}} \mathcal{G}) = \operatorname{Spec}(\operatorname{Sym}_{\mathcal{O}_{\operatorname{Spec} A}} \mathcal{G}_i)$, where $\mathcal{E}_i = \mathcal{O}_{\operatorname{Spec} A} e_i$ and $\mathcal{G}_i = \mathcal{O}_{\operatorname{Spec} A} e_0 \oplus \cdots \oplus \mathcal{O}_{\operatorname{Spec} A} e_{i-1} \oplus \mathcal{O}_{\operatorname{Spec} A} e_{i+1} \oplus \cdots \oplus \mathcal{O}_{\operatorname{Spec} A} e_r$, in exactly the same way as $\mathbf{P}(E)$ is covered by the affine schemes $\operatorname{Spec} A[\frac{x_0}{x_i}, \dots, \frac{x_r}{x_i}]$, where $A[x_0, \dots, x_r]$ is the polynomal ring in the variables $x_0, \dots, x_r$ over A. More precisely we have an isomorphism $A[x_0, \dots, x_r] \to \operatorname{Sym}_A(E)$ depending on the choise of basis $e_0, \dots, e_r$ and for each index i this gives an isomorphism $A[\frac{x_0}{x_i}, \dots, \frac{x_r}{x_i}] \cong \operatorname{Sym}_A(E^*_i \otimes_A G_i)$, where $E_i = Ae_i$ and $G_i = Ae_1 \oplus \cdots \oplus Ae_{i-1} \oplus e_{i+1} \oplus \cdots \oplus Ae_r$. This isomorphism sends $\frac{x_j}{x_i}$ to $e^*_i \otimes e_j$ for $j = 1, \dots, i-1, i+1, \dots, r$. Finally we have that $\operatorname{Spec}_{\mathcal{O}_{\operatorname{Spec} A}}(\mathcal{E}^*_i \otimes_{\mathcal{O}_{\operatorname{Spec} A}} \mathcal{G}_i) = \operatorname{Sym}_A\widetilde{(E^*_i \otimes_A G_i)} = \operatorname{Sym}_{\mathcal{O}_{\operatorname{Spec} A}}(\widetilde{E_i} \otimes_{\mathcal{O}_{\operatorname{Spec} A}} \widetilde{G_i})$, which gives the isomorphism $\mathbf{P}(E) \to \mathbf{P}(\mathcal{F})$ on the affine covering.

(9.23) Note. Let $U_i = \operatorname{Spec} A_i$ be an open affine covering of S. It follows from Note (9.16) that $\operatorname{Grass}^r(\mathcal{F})$ has an open covering of the schemes $\operatorname{Grass}^r(\mathcal{F}|U_i)$. In particular $\mathbf{P}(\mathcal{F})$ can be covered by projective spaces of the form $\mathbf{P}(E)$, where E is a free A–module and $\operatorname{Spec} A$ an open subset of S.

(9.24) Note. Assume that $\mathcal{F}$ is locally free of finite rank. The r–th exterior power $\wedge^r \mathcal{F}_{\operatorname{Grass}^r(\mathcal{F})} \to \wedge^r \mathcal{Q}$ gives rise to a morphism $\pi \colon \operatorname{Grass}^r(\mathcal{F}) \to \operatorname{Grass}^1(\wedge^r \mathcal{F}) = \mathbf{P}(\wedge^r \mathcal{F})$.

→ **(9.25) Proposition.** *The morphism* $\pi\colon \mathrm{Grass}^r(\mathcal{F}) \to \mathbf{P}(\wedge^r\mathcal{F})$ *of Note* (9.22) *is a closed embedding.*

Proof. It suffices to show that there is an open cover $P_\mathcal{E}$ of $\mathbf{P}(\wedge^r\mathcal{F})$ such that
→ $f^{-1}(P_\mathcal{E}) \to P_\mathcal{E}$ is a closed embedding. It follows from Note (9.16) that we may assume that S is affine given by $\operatorname{Spec} A$, and that $\mathcal{F}$ is a free $\mathcal{O}_S$–module. Let $\mathcal{E}$ be a free submodule of $\mathcal{F}$ spanned by r basis elements and let $P_\mathcal{E}$ be the open subscheme of $\mathbf{P}(\wedge^r\mathcal{F})$ where the map $\wedge^r\mathcal{E}_{\mathbf{P}(\wedge^r\mathcal{F})} \to \wedge^r\mathcal{F}_{\mathbf{P}(\wedge^r(\mathcal{F}))} \to \mathcal{O}_{\mathbf{P}(\wedge^r(\mathcal{F}))}(1)$ is surjective. Write $\mathcal{F} = \mathcal{E}\oplus\mathcal{G}$ and let $G_\mathcal{E}$ be the open subscheme of $\mathrm{Grass}^r(\mathcal{F})$ over which the composite map $\mathcal{E}_{\mathrm{Grass}^r(\mathcal{E})} \to \mathcal{F}_{\mathrm{Grass}^r(\mathcal{F})} \to \mathcal{Q}$ is surjective. We have that $\mathcal{E}_{\mathrm{Grass}^r(\mathcal{F})} \to \mathcal{Q}$ is surjective if and only if $\wedge^r\mathcal{E}_{\mathrm{Grass}^r(\mathcal{F})} \to \wedge^r\mathcal{Q}$ is surjective. Indeed, the second is the determinant of the first and both are surjective at the stalks where the determinant is invertible. Hence we have that $G_\mathcal{F} = \pi^{-1}(P_\mathcal{F})$.

We shall prove that the induced map $\pi_\mathcal{F}\colon \mathcal{G}_\mathcal{F} = \mathbf{V}(\mathcal{E}^*\otimes_{\mathcal{O}_S}\mathcal{G}) \to \mathbf{V}(\wedge^r\mathcal{E}^*\otimes_{\mathcal{O}_X}\mathcal{H}) = P_\mathcal{F}$ is a closed embedding. Write $\mathcal{H} = \wedge^{r-1}\mathcal{E}\otimes_{\mathcal{O}_S}\mathcal{G}\oplus\cdots\oplus\mathcal{E}\otimes_{\mathcal{O}_S}\wedge^{r-1}\mathcal{G}\oplus\wedge^r\mathcal{G}$. Then we have that $\wedge^r\mathcal{F} = \wedge^r\mathcal{E}\oplus\mathcal{H}$. For each i we have a canonical isomorphism $\wedge^{r-i}\mathcal{E}\otimes_{O_S}\wedge^r\mathcal{E}^* \to \wedge^i\mathcal{E}^*$. Hence we have that $\wedge^r\mathcal{E}^*\otimes_{O_S}\mathcal{H} = \mathcal{E}^*\otimes_{\mathcal{O}_S}\mathcal{G}\oplus\wedge^2\mathcal{E}^*\otimes_{O_S}\wedge^2\mathcal{G}\oplus\cdots\oplus\wedge^{r-1}\mathcal{E}^*\otimes_{O_S}\wedge^{r-1}\mathcal{G}\oplus\wedge^r\mathcal{E}^*\otimes_{O_S}\wedge^r\mathcal{G}$. The morphism $\pi_\mathcal{F}\colon\mathbf{V}(\mathcal{E}^*\otimes_{O_S}\mathcal{G}) \to \mathbf{V}(\wedge^r\mathcal{E}^*\otimes_{\mathcal{O}_S}\mathcal{H})$ is given on coordinate rings by an algebra homomorphism $\lambda\colon \mathrm{Sym}_{\mathcal{O}_S}(\mathcal{E}^*\otimes_{O_S}\mathcal{G}\oplus\cdots\oplus\wedge^r\mathcal{E}^*\otimes_{O_S}\wedge^r\mathcal{G}) \to \mathrm{Sym}_{O_S}(\mathcal{E}^*\otimes_{O_S}\mathcal{G})$. This map is determined on the linear part of the source, and given on the factors by the maps

$$\lambda_i\colon \wedge^i\mathcal{E}^*\otimes_{O_S}\wedge^i\mathcal{G} \to \mathrm{Sym}^i_{\mathcal{O}_S}(\mathcal{E}^*\otimes_{O_S}\mathcal{G})$$

defined by $\lambda_i(f_1^*\wedge\cdots\wedge f_i^*\otimes g_1\wedge\cdots\wedge g_i) = \sum_{\sigma\in\mathcal{S}_i}(-1)^{\operatorname{sign}\sigma}(f_1^*\otimes g_{\sigma(1)})\cdots(f_i^*\otimes g_{\sigma(i)})$, where $\mathcal{S}_i$ are the permutations of $\{1,\dots,i\}$. Since λ_1 is the identity we have that λ is surjective, and consequently that $\pi_\mathcal{F}$ is a closed imbedding.

(9.26) Definition. The morphism $\pi\colon \mathrm{Grass}^r(\mathcal{F}) \to (\wedge^r\mathcal{F})$ is called the *Plücker embedding* .

10. The Quotient functor.

(10.1) Setup. Given a scheme S and a locally free $\mathcal{O}_S$–module $\mathcal{E}$ of rank $r+1$. We assume that S is locally noetherian, that is, S can be covered by open affine subschemes $\operatorname{Spec} A$ such that A is a noetherian ring. Let $\mathbf{P}(\mathcal{E})$ be the r–dimensional projective space over S associated to $\mathcal{E}$ and let X be a closed subscheme of $\mathbf{P}(\mathcal{E})$ and $\iota\colon X \to \mathbf{P}(\mathcal{E})$ the corresponding closed immersion. Denote by $f\colon X \to S$ the structure morphism of X. Finally let $\mathcal{F}$ be a coherent $\mathcal{O}_X$–module.

(10.2) Note. Let $\mathcal{F}' \to \mathcal{F}$ be a surjection of coherent $\mathcal{O}_X$-modules. For every morphism $g\colon T \to \operatorname{Spec} A$ we get a map

$$\operatorname{Quot}_{\mathcal{F}}(T) \to \operatorname{Quot}_{\mathcal{F}'}(T)$$

sending the class of the quotient $\mathcal{F}_T \to \mathcal{G}$ to the composite $\mathcal{F}'_T \to \mathcal{F}_T \to \mathcal{G}$. It is clear that this map is injective and defines a map of functors $\operatorname{Quot}_{\mathcal{F}} \to \operatorname{Quot}_{\mathcal{F}'}$.

(10.3) Proposition. *Let $\mathcal{F}' \to \mathcal{F}$ be a surjection of coherent $\mathcal{O}_X$-modules. Then the injection of Note* (10.2) *makes $\operatorname{Quot}_{\mathcal{F}}$ a closed subfunctor of $\operatorname{Quot}_{\mathcal{F}'}$.*

Proof. It follows from Note (9.16) that $\mathcal{Q}uot_{\mathcal{F}'}$ and $\mathcal{Q}uot_{\mathcal{F}}$ are covered by open subfunctors $\mathcal{Q}uot_{\mathcal{F}'|f^{-1}(\operatorname{Spec} A)}$ and $\mathcal{Q}uot_{\mathcal{F}|f^{-1}(\operatorname{Spec} A)}$, where $\operatorname{Spec} A$ is an open affine subset of S. We can therefore assume that $S = \operatorname{Spec} A$.

We have to show that for every morphism $g\colon T \to \operatorname{Spec} A$ and every element $\mathcal{F}'_T \to \mathcal{G}$ in $\operatorname{Quot}_{\mathcal{F}'}(T)$ there is a closed subscheme T_0 of T such that a morphism $h\colon U \to T$ factors via T_0 if and only if $\mathcal{F}'_U \to h^*_{X_T}\mathcal{G}$ factors via $\mathcal{F}_U$. Such a scheme is clearly unique, if it exists. Hence we may assume that T is affine.

Let $\mathcal{K}$ be the kernel of $\mathcal{F}' \to \mathcal{F}$. It follows from Theorem (2.2)(2) and (3) that we can choose an m_0 such that $\mathcal{K}(m)$ and $\mathcal{G}(m)$ are generated by global sections, and such that $H^i(X_T, \mathcal{G}(m)) = 0$ for $i > 0$ and for $m \geq m_0$. Since $\mathcal{G}$ is flat over T it follows from Theorem (4.7) that $f_{U*}h^*_{X_T}\mathcal{G}(m)$ is locally free and the base change map

$$h^* f_{T*}\mathcal{G} = \mathcal{O}_U \otimes_{\mathcal{O}_T} \widetilde{H^0(X_T, \mathcal{G}(m))} \to f_{U*}h^*_{X_T}\mathcal{G}(m) \tag{10.3.1}$$

is an isomorphism for $m \geq m_0$.

Since $\mathcal{K}(m_0)$ is generated by global sections we can, as we saw in Note (2.3) choose a surjection $\mathcal{O}^n_{X_T} \to \mathcal{K}(m_0)$. We have that the the map $\mathcal{F}'_U \to h^*_{X_T}\mathcal{G}$ factors via $\mathcal{F}_U$ if and only if the composite map $\mathcal{O}^n_{X_U} \to \mathcal{K}_U(m_0) \to \mathcal{F}'_U(m_0) \to h^*_{X_T}\mathcal{G}(m_0)$ is zero. By adjunction there is a bijection between $\mathcal{O}_{X_U}$–module homomorphisms $\mathcal{O}^n_{X_U} = f^*_U\mathcal{O}^n_U \to h^*_{X_T}\mathcal{G}(m_0)$ and $\mathcal{O}_U$–module homomorphisms $\mathcal{O}^n_U \to f_{U*}h^*_{X_T}\mathcal{G}(m_0)$. Consequently we have that $\mathcal{O}^n_{X_U} \to h^*_{X_T}\mathcal{G}(m_0)$ is zero if and only if $\mathcal{O}^n_U \to f_{U*}h^*_{X_T}\mathcal{G}(m_0)$ is zero. The latter map is the composite map $\alpha\colon \mathcal{O}_U^n \to f_{U*}f_U^*\mathcal{O}_U^n = f_{U*}h_{X_T}^*\mathcal{O}_{X_T}^n \to f_{U*}h_{X_T}^*\mathcal{G}(m_0)$ obtained from the map $\mathcal{O}_{X_T}^n \to \mathcal{K}(m_0) \to \mathcal{G}(m_0)$. By the base change map (10.3.1) the map α is the same as the map $\mathcal{O}_U^n = h^*\mathcal{O}_T^n \to h^*f_{X_T*}f_T^*\mathcal{O}_T^n = h^*f_{T*}\mathcal{O}_{X_T}^n \to h^*f_{T*}\mathcal{G}(m_0)$. We have proved that the map $\mathcal{O}_U^n \to f_{U*}h_{X_T}^*\mathcal{G}(m_0)$ is zero if and only if the pull back by h of the map $\mathcal{O}_T^n \to f_{T*}f_T^*\mathcal{O}_T^n = f_{T*}\mathcal{O}_{X_T}^n \to f_{T*}\mathcal{G}(m_0)$ is zero.

Since $f_{T*}\mathcal{G}(m_0)$ is locally free we can therefore define T_0 on each component of T to be the $(\mathrm{rk}(f_{X_T*}\mathcal{G}(m_0))-1)$'st Fitting ideal of the cokernel of $\mathcal{O}_T^n \to f_{X_T*}\mathcal{G}(m_0)$.

(10.4) Definition. Given a morphism $g\colon T \to S$ and an element $\mathcal{F}_T \to \mathcal{G}$ in $\mathcal{Q}uot_{\mathcal{F}}(T)$ Let $t \in T$. For each open affine neighbourhood $\operatorname{Spec} A$ of t such that $\mathcal{E}|\operatorname{Spec} A$ is free we have defined in (5.7) the Hilbert polynomial $\chi_{\mathcal{G}|f^{-1}(\operatorname{Spec} A),t}$ of $\mathcal{G}|f^{-1}(\operatorname{Spec} A)$ at t. Clearly we obtain the same Hilbert polynomial indepently of which connected neighbourhood of t we choose. We can therefore define the *Hilbert polynomial* $\chi_{\mathcal{G},t}$ of $\mathcal{G}$ as $\chi_{\mathcal{G}|f^{-1}(\operatorname{Spec} A),t}$ for any connected neighbourhood $\operatorname{Spec} A$ of t.

For $P \in \mathbf{Z}[t]$ we let

$$\mathcal{Q}uot_{\mathcal{F}}^P(T) = \{\mathcal{F}_T \to \mathcal{G} \text{ in } \mathcal{Q}uot_{\mathcal{F}}(T)\colon \chi_{\mathcal{G},t} = P \text{ for all } t \in T\}.$$

It follows from Note (5.8) that $\mathcal{Q}uot_{\mathcal{F}}^P$ is a subfunctor of $\mathcal{Q}uot_{\mathcal{F}}$.

(10.5) Note. We have that $\mathcal{Q}uot_{\mathcal{F}}^P$ is an open subfunctor of $\mathcal{Q}uot_{\mathcal{F}}$. To prove this we must show that for every morphism $T \to S$ and every element $\mathcal{F}_T \to \mathcal{G}$ in $\mathcal{Q}uot_{\mathcal{F}}$ there is an open subset T_P of T such that a morphism $h\colon U \to T$ factors via T_P if and only if $h^*\mathcal{G}$ has Hilbert polynomial P. However, it follows from (5.10) that $\mathcal{G}$ has constant Hilbert polynomial on every connected component of T. Consequently $\mathcal{G}$ has Hilbert polynomial P on an open, possibly empty, subscheme T_P of T. It follows from Note (5.8) that T_P is the open set we are looking for.

(10.6) Note. Given an integer n. For every morphism $g\colon T \to \operatorname{Spec} A$ we have a map

$$\mathcal{Q}uot_{\mathcal{F}}(T) \to \mathcal{Q}uot_{\mathcal{F}(n)}(T)$$

which sends the class of $\mathcal{F}_T \to \mathcal{G}$ to the class of $\mathcal{F}_T(n) \to \mathcal{G}(n)$. It is clear that this gives an isomorphism of functors

$$\mathcal{Q}uot_{\mathcal{F}} \to \mathcal{Q}uot_{\mathcal{F}(n)}.$$

We have that $\chi_{\mathcal{G}}(m+n) = \chi_{\mathcal{G}(n)}(m)$. Consequently we obtain an isomorphism of functors

$$\mathcal{Q}uot_{\mathcal{F}}^P \to \mathcal{Q}uot_{\mathcal{F}(n)}^Q$$

where P and Q are elements in $\mathbf{Q}[t]$ related by $P(m+n) = Q(m)$ for all m.

(10.7) Note. Given a closed immersion $\varepsilon\colon Y \to Z$ of schemes, and let $\mathcal{G}$ be an $\mathcal{O}_Y$–module. We have an isomorphism of rings $(\varepsilon_*\mathcal{O}_Y)_{\varepsilon(y)} \to \mathcal{O}_{Y,y}$ for all points y of Y. Via this isomorphism we have an isomorphism $(\varepsilon_*\mathcal{G})_{\varepsilon(y)} \to \mathcal{G}_y$ of $\mathcal{O}_{Y,y}$–modules, and we have that $(\varepsilon_*\mathcal{G})_z = 0$ when $z \in Z \setminus \varepsilon(Y)$. In particular we have an isomorphism $\varepsilon^*\varepsilon_*\mathcal{G} \to \mathcal{G}$ of $\mathcal{O}_Y$–modules. Moreover we have that given an quotient $\mathcal{G} \to \mathcal{H}$ of $\mathcal{O}_Y$–modules, then we obtain a quotient $\varepsilon_*\mathcal{G} \to \varepsilon_*\mathcal{H}$ of $\mathcal{O}_Z$–modules, via the homomorphism $\mathcal{O}_Z \to \varepsilon_*\mathcal{O}_Y$, and $\varepsilon_*\mathcal{H}_{\varepsilon(y)} \to \mathcal{H}_{\varepsilon(y)}$ is an isomorphism of $\mathcal{O}_{Y,y}$–modules, and that $\mathcal{H}_z = 0$ for $z \in Z \setminus \varepsilon(Y)$.

Given a quotient $\varepsilon_*\mathcal{G} \to \mathcal{K}$ of $\mathcal{O}_Z$–modules. We obtain a quotient $\varepsilon^*\varepsilon_*\mathcal{G} = \mathcal{G} \to \varepsilon^*\mathcal{K}$ of $\mathcal{O}_Y$–modules.

It is clear that we in this way obtain a bijection between $\mathcal{O}_Y$–module quotients of $\mathcal{G}$ and $\mathcal{O}_Z$–module quotients of $\varepsilon_*\mathcal{G}$. Since the fibers of modules corresponding to each other by this bijection are either isomorphic or zero we have that the bijection takes quotients that are flat over a morphism $Z \to T$ into quotients that are flat over the restriction $Y \to T$, and conversely. In particular we see that we have an isomorphism of functors $\mathcal{Q}uot_{\mathcal{F}} \to \mathcal{Q}uot_{i_*\mathcal{F}}$ from the closed immersion $\iota\colon X \to \mathbf{P}(\mathcal{E})$.

(10.8) Theorem. *For each $P \in \mathbf{Q}[t]$ we have that the functor $\mathcal{Q}uot_{\mathcal{F}}^P$ is representable by a quasi projective scheme.*

Proof. It follows from Note (9.16) that $\mathcal{Q}uot_{\mathcal{F}}^P$ can be covered by open subfunctors $\mathcal{Q}uot_{\mathcal{F}|f^{-1}(\operatorname{Spec} A)}$ where $\operatorname{Spec} A$ is an open affine subset of S. Since $\mathcal{Q}uot_{\mathcal{F}}$ and thus $\mathcal{Q}uot_{\mathcal{F}}^P$ are Zariski functors it follows from Proposition (9.18) that we may assume that $S = \operatorname{Spec} A$. It follows from Note (10.6) and Theorem (2.2) that we may assume that $\mathcal{F}$ is generated by global sections. We can then, as we saw in Note (2.3), find a surjection $\mathcal{O}_X^n \to \mathcal{F}$. Consequently, it follows from Proposition (10.3) that we may assume that $\mathcal{F}$ is a free $\mathcal{O}_X$–module of finite rank. Finally it follows from Note (10.7) that we may assume that $X = \mathbf{P}(E)$, where E is a free A–module of finite rank. Then we have that $\mathcal{F}$ is flat over S.

Let T be a scheme over $\operatorname{Spec} A$. For every exact sequence

$$0 \to \mathcal{K} \to \mathcal{F}_T \to \mathcal{G} \to 0$$

of $\mathcal{O}_{X_T}$-modules, with $\mathcal{G}$ flat over T, it follows from Lemma (3.5) that the sequence

$$0 \to \mathcal{K}_{\operatorname{Spec} \kappa(t)} \to \mathcal{F}_{\operatorname{Spec} \kappa(t)} \to \mathcal{G}_{\operatorname{Spec} \kappa(t)} \to 0$$

is exact, for all $t \in T$. Since $\mathcal{F}$ is free, by assumption, *(er dette nødvendig her?)* it follows from Theorem (6.10) that there is an m_0 such that, for all schemes T over $\operatorname{Spec} A$, for all quotients $\mathcal{F}_T \to \mathcal{G}$ in $\mathcal{Q}uot_{\mathcal{F}}^P(T)$, and for all points $t \in T$, we have that the kernel *($\xi_{K.t} = \chi_{\mathcal{F},t} - \xi_{\mathcal{G},t}$ så kjernene har samme Hilbert polynom.)* $\mathcal{K}_{\operatorname{Spec}\kappa(t)}$ of $\mathcal{F}_{\operatorname{Spec}\kappa(t)} \to \mathcal{G}_{\operatorname{Spec}\kappa(t)}$ is an m_0-regular $\mathcal{O}_{X_{\operatorname{Spec}\kappa(t)}}$-module. Hence it follows from Proposition (6.8) that $\mathcal{K}_{\operatorname{Spec}\kappa(t)}$ is m–regular for $m \geq m_0$. It follows from Note (6.11) that $\mathcal{G}_{\operatorname{Spec}\kappa(t)}$ is also m-regular for $m \geq m_0$. Hence we have that

$$H^i(X_{\operatorname{Spec}\kappa(t)}, \mathcal{K}_{\operatorname{Spec}\kappa(t)}(m)) = H^i(X_{\operatorname{Spec}\kappa(t)}, \mathcal{G}_{\operatorname{Spec}\kappa(t)}(m)) = 0,$$

for $i > 0$ and $m \geq m_0$.

Since $\mathcal{G}$ and thus $\mathcal{K}$ are flat over T it follows from Theorem (4.9) that

$$R^i f_{T*}\mathcal{K}(m) = 0 = R^i f_{T*}\mathcal{G}(m) \quad \text{for } i > 0 \text{ and } m \geq m_0,$$

and thus it follows from Theorem (3.19)(1) that $f_{T*}\mathcal{G}(m)$ is locally free of rank $\dim_{\kappa(t)} H^0(X_{\operatorname{Spec}\kappa(t)}, \mathcal{G}_{\operatorname{Spec}\kappa(t)}(m)) = \chi_{\mathcal{G},t}(m) = P(m)$ for $m \geq m_0$. It also follows that we have an exact sequence

$$0 \to f_{T*}\mathcal{K}(m_0) \to f_{T*}\mathcal{F}_T(m_0) \to f_{T*}\mathcal{G}(m_0) \to 0 \tag{10.8.1}$$

of $\mathcal{O}_T$-modules.

Since $\mathcal{F}$ is assumed to be free and X to be $\mathbf{P}(E)$ it follows from Setup (2.1) that $H^0(X, \mathcal{F}(m))$ is a free A-module and that $H^i(X, \mathcal{F}(m)) = 0$ for $i > 0$ and for $m \geq 0$. Hence it follows from Theorem (4.7) that we have an isomorphism

$$\mathcal{O}_T \otimes_{\mathcal{O}_{\operatorname{Spec} A}} H^0(\widetilde{X, \mathcal{F}}(m)) \to f_{T*}\mathcal{F}_T(m)$$

for $m \geq 0$. Let $V = H^0(X, \mathcal{F}(m_0))$ and $\mathcal{V} = \widetilde{V}$. Then we obtain an exact sequence

$$0 \to f_{T*}\mathcal{K}(m_0) \to \mathcal{V}_T \to f_{T*}\mathcal{G}(m_o) \to 0$$

of $\mathcal{O}_T$–modules.

We have thus obtained a map

$$\mathcal{Q}uot^P_{\mathcal{F}}(T) \to \mathcal{G}rass^{P(m_0)}(\mathcal{V})(T)$$

which sends the quotient $\mathcal{F}_T \to \mathcal{G}$ to the quotient $\mathcal{V}_T \to f_{T*}\mathcal{F}_T(m_0)$. These maps, for all S-schemes T define a morphism of functors

$$\mathcal{Q}uot^P_{\mathcal{F}} \to \mathcal{G}rass^{P(m_0)}(\mathcal{V}).$$

Indeed, given a morphism $h\colon U \to T$ we obtain a commutative diagram

$$\begin{array}{ccccccc}
\mathcal{V}_U = h^*\mathcal{V}_T & \longrightarrow & h^* f_{T*}\mathcal{F}(m_0) & \longrightarrow & h^* f_{T*}\mathcal{G}(m_0) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathcal{V}_U & \longrightarrow & f_{U*}h^*_{X_T}\mathcal{F}(m_0) & \longrightarrow & f_{U*}h^*_{X_T}\mathcal{G}(m_0) & \longrightarrow & 0
\end{array}$$

(dette kan skrives mye bedre) where the upper row is the composite map $\mathcal{V}_T \to f_{T*}\mathcal{F}_T(m_0) \to f_{T*}\mathcal{G}(m_0)$ pulled back to T, the lower row is the surjection on U that we obtain by the above construction when we start with $\mathcal{F}_T \to \mathcal{G}$ pulled back to X_U by h_{X_T}, and the vertical maps are the base change maps. Since the higher cohomology of $\mathcal{F}(m_0)$ and $\mathcal{G}(m_0)$ vanishes it follows from Proposition (4.7) that the base change maps are isomorphisms. It follows that the construction which to $\mathcal{F}_T \to G$ in $\mathcal{Q}uot_{\mathcal{F}}(T)$ associates the surjection $\mathcal{V}_T \to f_{T*}\mathcal{G}(m_0)$ is functorial, as we wanted to prove.

The morphism $\mathcal{Q}uot^P_{\mathcal{F}} \to \mathcal{G}rass^{P(m_0)}(\mathcal{V})$ is injective. Indeed, given a morphism $g\colon T \to \operatorname{Spec} A$ and an element $\mathcal{F}_T \to \mathcal{G}$ in $\mathcal{Q}uot^P_{\mathcal{F}}$. The image of the quotient in $\mathcal{G}rass^{P(m_0)}(\mathcal{V})$ is equivalent to $f_{T*}\mathcal{F}_T(m_0) \to f_{T*}\mathcal{G}(m_0)$. As we have seen *(spørsmålet er om vi bare har vist dette på fibre. Vi må da vise at det følger fra resultatet på fibre ved basisbytte)* it follows from (6.8) that $\mathcal{K}(m_0)$ is generated by global sections. Hence the top map in the commutative diagram

$$\begin{array}{ccc}
f^*_T f_{T*}\mathcal{K}(m_0) & \longrightarrow & \mathcal{K}(m_0) \\
\downarrow & & \downarrow \\
f^*_T f_{T*}\mathcal{F}(m_0) & \longrightarrow & \mathcal{F}(m_0)
\end{array}$$

is surjective. It follows that $\mathcal{G}(m_0)$ is the cokernel of the composed map

$$f^*_T f_{T*}\mathcal{K}(m_0) \to f^*_T f_{T*}\mathcal{F}_T(m_0) \to \mathcal{F}_T(m_0) \tag{10.8.2}$$

Since the map (10.8.2) has kernel $f_{T*}\mathcal{K}(m_0)$ by (10.8.1) we can consequently restore the quotient $\mathcal{F}_T \to G$ from $f_{T*}\mathcal{F}_T(m_0) \to f_{T*}\mathcal{G}(m_0)$.

Finally we shall prove that the functor $\mathcal{Q}uot^P_{\mathcal{F}}$ is a locally closed subfunctor of $\mathcal{G}rass^{P(m_0)}_{\mathcal{V}}$. Since $\mathcal{G}rass^{P(m_0)}_{\mathcal{V}}$ is represented by a scheme $G = \operatorname{Grass}^{P(m_0)}(\mathcal{V})$, with universal quotient

$$\xi\colon \mathcal{F}V_G \cong f_{G*}\mathcal{F}_G(m_0) \to \mathcal{Q}$$

of $\mathcal{O}_G$-modules we must show that there is a locally closed subscheme G_ξ of G such that a morphism $h\colon T \to G$ factors via G_ξ if and only if $h^* f_{G*}\mathcal{F}_G(m_0) \to h^*\mathcal{Q}$ is the image of a quotient $\mathcal{F}_T \to \mathcal{G}$ in $\mathcal{Q}uot^P_{\mathcal{F}}(T)$. That is, we have that $h^* f_{G*}\mathcal{F}_G(m_0) \to h^*\mathcal{Q}$, or $f_{T*}\mathcal{F}_T(m_0) \to h^*\mathcal{Q}$, is equivalent to $f_{T*}\mathcal{F}_T(m_0) \to f_{T*}\mathcal{G}(m_0)$. Let $p\colon G \to \operatorname{Spec} A$ be the structure morphism and let $\mathcal{R}$ be the kernel of the map $f_{G*}\mathcal{F}_G(m_0) = f_{G*}p_X^*\mathcal{F}(m_0) \to \mathcal{Q}$ *(dette er gjort for komplisert)* corresponding to the canonical morphism $\mathcal{V}_G = p^* f_*\mathcal{F}(m_0) \to \mathcal{Q}$ via the isomorphism $p^* f_*\mathcal{F} \to f_{G*}p_X^*\mathcal{F}$. On X_G we obtain an exact sequence

$$0 \to f_G^*\mathcal{R} \to f_G^* f_{G*}\mathcal{F}_G(m_0) \to f_G^*\mathcal{Q} \to 0$$

Let $\mathcal{H}$ be the $\mathcal{O}_{X_G}$-module such that $\mathcal{H}(m_0)$ is the cokernel of the map

$$f_G^*\mathcal{R} \to f_G^* f_{G*}\mathcal{F}_G(m_0) \to \mathcal{F}_G(m_0).$$

Moreover, let G_P be the locally closed subscheme of G which is the part of the flattening stratification of $\mathcal{H}$ that corresponds to P. The scheme exists by Theorem
$\to$ (8.5) and is unique by Note (8.6). We shall show that $G_\xi = G_P$.

Assume first that $f_{T*}\mathcal{F}_T(m_0) \to h^*\mathcal{Q}$ is the image of $\mathcal{Q}uot^P_{\mathcal{F}}(T)$. That is, the quotients $f_{T*}\mathcal{F}_T(m_0) \to h^*\mathcal{Q}$ and $f_{T*}\mathcal{F}_T(m_0) \to f_{T*}\mathcal{G}(m_0)$ are equivalent for some quotient $\mathcal{F}_T \to \mathcal{G}$ in $\mathcal{Q}uot^P_{\mathcal{F}}(T)$. Then the kernel $\mathcal{R}_T = h^*\mathcal{R}$ of $f_{T*}\mathcal{F}_T(m_0) \to h^*\mathcal{Q}$ is equal to the kernel $f_{T*}\mathcal{K}(m_0)$ of $f_{T*}\mathcal{F}_T(m_0) \to f_{T*}\mathcal{G}(m_0)$. We obtain a commutative and exact diagram *(tvilsom notasjon bruk h)*

$$\begin{tikzcd}
f_T^*\mathcal{R}_T = f_T^* f_{T*}\mathcal{K}(m_0) \arrow[r] \arrow[d] & f_T^* f_{T*}\mathcal{F}_T(m_0) \arrow[r] \arrow[d] & f_T^* f_{T*}\mathcal{G}(m_0) \arrow[r] & 0 \\
\mathcal{K}(m_0) \arrow[r] & \mathcal{F}_T(m_0) & &
\end{tikzcd}$$

where the left vertical map is surjective, as we have seen above. It follows that $\mathcal{H}_T(m_0) = \mathcal{G}(m_0)$, such that $\mathcal{H}_T = \mathcal{G}$. We have that $\mathcal{G}$ is flat over $\operatorname{Spec} A$ with Hilbert polynomial $\chi_{\mathcal{G},t} = P$ for all $t \in T$. The same therefore holds for $\mathcal{H}_T$. From the definition of G_P as the flattening stratification of $\mathcal{H}$ corresponding to P it follows that $h\colon T \to G$ factors via G_P.

Conversely, assume that $h\colon T \to G$ factors via G_P. We have that $\mathcal{H}_T(m_0)$ is the cokernel of the map $f_T^*\mathcal{R}_T \to f_T^* f_{T*}\mathcal{F}_T(m_0) \to \mathcal{F}_T(m_0)$ of $\mathcal{O}_{X_T}$-modules. Let $\mathcal{L}$ be the $\mathcal{O}_T$-module such that $\mathcal{L}(m_0)$ is the kernel of $\mathcal{F}_T(m_0) \to \mathcal{H}_T(m_0)$. We obtain an exact commutative diagram of $\mathcal{O}_{X_T}$-modules

$$\begin{tikzcd}
0 \arrow[r] & f_T^*\mathcal{R}_T \arrow[r] \arrow[d] & f_T^* f_{T*}\mathcal{F}_T(m_0) \arrow[r] \arrow[d] & f_T^*\mathcal{Q}_T \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{L}(m_0) \arrow[r] & \mathcal{F}_T(m_0) \arrow[r] & \mathcal{H}_T(m_0) \arrow[r] & 0.
\end{tikzcd} \tag{10.8.2}$$

By the definition of $\mathcal{H}$ the left vertical map is surjective. Since h factors through G_P we have that $\mathcal{H}_T$ is flat over T and that $\chi_{\mathcal{H}_{T,t}} = P$ for all $t \in T$. Hence we have that $\mathcal{F}_T \to \mathcal{H}_T$ is in $\mathcal{Q}uot^P_{\mathcal{F}}(T)$. It follows from Theorem (6.10) that $\mathcal{L}_{\operatorname{Spec}\kappa(t)}$ is m_0-regular for all points t of T. Hence it follows from Note (6.11) that $\mathcal{H}_{\operatorname{Spec}\kappa(t)}$ is m_0-regular, and from Theorem (4.7) it follows that $f_{T*}\mathcal{H}_T(m_0)$ is locally free of rank $P(m_0)$. From diagram (10.8.2) we obtain, using f_{T*} an exact diagram of $\mathcal{O}_T$-modules

$$\begin{array}{ccccccccc}
0 & \longrightarrow & h^*\mathcal{R} & \longrightarrow & f_{T*}\mathcal{F}_T(m_0) & \longrightarrow & h^*\mathcal{Q} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & f_{T*}\mathcal{L}(m_0) & \longrightarrow & f_{T*}\mathcal{F}_T(m_0) & \longrightarrow & f_{T*}\mathcal{H}_T(m_0) & \longrightarrow & 0.
\end{array}$$

Since the middle vertical map is the identity the right vertical map is surjective. Both $\mathcal{Q}_T$ and $f_{T*}\mathcal{H}_T(m_0)$ are locally free modules of rank $P(m_0)$. Consequently the right vertical map is an isomorphism. We conclude that the map $f_{T*}\mathcal{F}_T(m_0) \to \mathcal{Q}_T = h^*\mathcal{Q}$ is equivalent to $f_{T*}\mathcal{F}_T(m_0) \to f_{T*}\mathcal{H}_T(m_0)$, which comes from the quotient $\mathcal{F}_T \to \mathcal{H}_T$ in $\mathcal{Q}uot^P_{\mathcal{F}}(T)$.

We have proved that $\mathcal{Q}uot^P_{\mathcal{F}}$ is a locally closed subfunctor of $\mathcal{G}rass^{P(m_0)}(V)$, and consequently it is represented by the subscheme G_P of G.

(10.9) The differential structure.

Let Y be an S-scheme and let y be a point on Y. We let s be the image of y by the structure map $Y \to S$ and let $Y_s = Y \times_S \operatorname{Spec}\kappa(s)$ be the fiber of $Y \to S$ over s. Let $Y_y = Y_s \times_{\operatorname{Spec}(\kappa(s))} \operatorname{Spec}(\kappa(y)) = Y \times_S \operatorname{Spec}(\kappa(y))$ be the extension of $Y_s \to \operatorname{Spec}(\kappa(s))$ to $\operatorname{Spec}(\kappa(y))$ by the augmentation map $\operatorname{Spec}(\kappa(y)) \to \operatorname{Spec}(\kappa(s))$. The point $y \in Y$ induces a point $\operatorname{Spec}(\kappa(y)) \to Y_s$ and a section $\operatorname{Spec}(\kappa(y)) \to Y_y$ of the structure map $Y_y \to \operatorname{Spec}(\kappa(y))$.

We shall determine the tangent space $\mathcal{T}(Y_y)_y$ of Y_y at the point y.

Let $B = (\mathcal{O}_{Y_s})_y$. The structure map $Y_s \to \operatorname{Spec}(\kappa(s))$ gives B the structure of an $\kappa(s)$-algebra, and we have an $\kappa(s)$-algebra homomorphism $B \to \kappa(y)$ corresponding to the point y of Y_s. We have a multiplication map

$$\psi : B \otimes_{\kappa(s)} \kappa(y) \to \kappa(y)$$

that is a $\kappa(y)$-homomorphism. Let $\mathfrak{m} = (m_{Y_s})_y$ be the kernel of ψ. We have that $(B \otimes_{\kappa(s)} \kappa(y))_{\mathfrak{m}} = (\mathcal{O}_{Y_y})_y$. In particular we have that $\mathfrak{m}/\mathfrak{m}^2 = (\mathfrak{m}_{Y_y})_y/(\mathfrak{M}_{Y_y})^2_y$ as $\kappa(y)$-modules. We have an isomorphism of $\kappa(y)$-modules

$$\mathfrak{m}/\mathfrak{m}^2 \to \Omega^1_{B\otimes^{\kappa}_{\kappa(s)}(y)/\kappa(y)} \otimes_{B\otimes_{\kappa(s)}\kappa(y)} \kappa(y)$$

that sends the class in $\mathfrak{m}/\mathfrak{m}^2$ of $g \otimes_{\kappa(s)} 1 \in \mathfrak{m}$ to $d_{B\otimes_{\kappa(s)}\kappa(y)/\kappa(y)}(g) \otimes_{B\otimes_{\kappa(s)}\kappa(y)} \kappa(y)$. In order to define an inverse we consider the homomorphism of $\kappa(y)$-modules

$$B \otimes_{\kappa(s)} \kappa(y) \to \mathfrak{m}/\mathfrak{m}^2$$

that maps $g \otimes_{\kappa(s)} h$ to $g \otimes_{\kappa(s)} h - \iota\psi(g \otimes_{\kappa(s)} h)$, where $\iota : \kappa(y) \to B \otimes_{\kappa(s)} \kappa(y)$ is defined by $\iota(h) = 1 \otimes_{\kappa(s)} h$ for all $h \in \kappa(y)$. The formula

$$\begin{aligned}
gg' \otimes_{\kappa(s)} hh' - \iota\psi(gg' \otimes_{\kappa(s)} hh')& \\
&= (g \otimes_{\kappa(s)} h)(g' \otimes_{\kappa(s)} h' - \iota\psi(g' \otimes_{\kappa(s)} h')) \\
&+ (g' \otimes_{\kappa(s)} h')(g \otimes_{\kappa(s)} h - \iota\psi(g \otimes_{\kappa(s)} h)) \\
&- (g \otimes_{\kappa(s)} h - \iota\psi(g \otimes_{\kappa(s)} h))(g' \otimes_{\kappa(s)} h' - \iota\psi(g' \otimes_{\kappa(s)} h'))
\end{aligned}$$

shows that D is a $\kappa(y)$-derivation. This gives a $B \otimes_{\kappa(s)} \kappa(y)$-linear homomorphism

$$\Omega^1_{B\otimes_{\kappa(s)}\kappa(y)/\kappa(y)} \to \mathfrak{m}/\mathfrak{m}^2.$$

→ We obtain the inverse of the map (?) by extension of the variables by ψ.

Note that $\Omega^1_{B\otimes_{\kappa(s)}\kappa(y)/\kappa(y)} = \Omega^1_{B/\kappa(y)} \otimes_B (B \otimes_{\kappa(s)} \kappa(y))$. Consequently we have that $\Omega^1_{B\otimes_{\kappa(s)}\kappa(y)/\kappa(y)} \otimes_{B\otimes_{\kappa(s)}\kappa(y)} \kappa(y) = \Omega^1_{B/\kappa(y)} \otimes_B \kappa(y)$. Hence we have an isomorphism $\mathfrak{m}/\mathfrak{m}^2 \otimes_{B\otimes_{\kappa(s)}\kappa(y)} \kappa(y) \to \Omega^1_{B/\kappa(y)} \otimes_B \kappa(y)$.

By standard equivalences we get bijections $\mathrm{Hom}_{\kappa(y)-\mathrm{alg}}(B \otimes_{\kappa(s)} \kappa(y), \kappa(y)[\varepsilon]) = \mathrm{Der}_{\kappa(y)}(B \otimes_{\kappa(s)} \kappa(y), \kappa(y)) = \mathrm{Hom}_{B\otimes_{\kappa(s)}\kappa(y)}(\Omega^1_{B\otimes_{\kappa(s)}\kappa(y)/\kappa(y)}, \kappa(y))$, and as we have seen all these sets are in bijection with the sets $\mathrm{Hom}_{B\otimes_{\kappa(s)}\kappa(y)}(\mathfrak{m}/\mathfrak{m}^2, \kappa(y)) = (\mathcal{T}_{Y_y})_y$.

We have shown that there is a bijection between the tangent space to Y_y at y and all $k(y)$-algebra homomorphism $B \otimes_{\kappa(s)} \kappa(y) \to \kappa(y)[\varepsilon]$, or equivalently with all morphisms $\mathrm{Spec}(\kappa(y)[\varepsilon]) \to Y$ that gives the point $\mathrm{Spec}(\kappa(y)) \to Y$ when composed with the augmentation morphism $\mathrm{Spec}(\kappa(y)) \to \mathrm{Spec}(\kappa(y)[\varepsilon])$.

Let $X \to S$ be a scheme and let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume that the functor $\mathrm{Quot}_{\mathcal{F}}$ is representable, and represented by the scheme Q. Let $y \in Q$ be a point. The morphism $\mathrm{Spec}(\kappa(y)) \to Y$ be a point of Q. This point corresponds to a quotient $\mathcal{F}_{\mathrm{Spec}(\kappa(y))} \to \mathcal{G}$ of $\mathcal{O}_{Q_y}$-modules on the fiber $Q_y = Q \times_S \mathrm{Spec}(\kappa(y))$ to $Q \to S$ over y.

A morphism $\mathrm{Spec}(\kappa(y)[\varepsilon]) \to Q$ coresponds to a quotient $\mathcal{F}_{\mathrm{Spec}(\kappa(y)[\varepsilon])} \to \mathcal{G}_\varepsilon$ of $\mathcal{O}_{Q\times_S\mathrm{Spec}(\kappa(y)[\varepsilon])}$-modules such that $\mathcal{G}_\varepsilon$ is flat over $\mathrm{Spec}(\kappa(y)[\varepsilon])$. That the morphism $\mathrm{Spec}(\kappa(y)[\varepsilon]) \to Q$ composed with the augmentation $\mathrm{Spec}(\kappa(y)) \to \mathrm{Spec}(\kappa(y)[\varepsilon])$ gives the point y means that the restriction of $\mathcal{F}_{\mathrm{Spec}(\kappa(y)[\varepsilon])} \to \mathcal{G}_\varepsilon$ is

$\mathcal{F}_{\mathrm{Spec}(\kappa(y))} \to \mathcal{G}$ by the extension $Q_y \to Q \times_S \mathrm{Spec}(\kappa(y)[\varepsilon]))$ of the augmentation $\mathrm{Spec}(\kappa(y)) \to \mathrm{Spec}(\kappa(y)[\varepsilon]))$.

→ It follows from Lemma (3.21) glabalized that the tangent space to $Q_y = Q \times_S \mathrm{Spec}(\kappa(y))$ at y is bijective to

$$\mathrm{Hom}_{\mathcal{O}_{Q_y}}(\mathcal{H}, \mathcal{G})$$

where $\mathcal{H}$ is the kernel of $\mathcal{F}_{\mathrm{Spec}(\kappa(s))} \to \mathcal{G}$.

1. Results that are to be included.

(1.7) Definition. Let

$$F\colon 0 \to F^0 \xrightarrow{d^0} F^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{r-1}} F^r \to 0$$

be a complex of A-modules. We write $Z^i = Z^i(F) = \operatorname{Ker} d^i$, and $B^i = B^i(F) = \operatorname{Im} d^{i-1}$. Moreover we let $H^i(F) = Z^i(F)/B^i(F)$. There are exact sequences

$$0 \to Z^i(F) \to F^i \to B^{i+1}(F) \to 0 \tag{1.7.1}$$

and

$$0 \to B^i(F) \to Z^i(F) \to H^i(F) \to 0. \tag{1.7.2}$$

for $i - 0, \dots, r$

Given an A-algebra B. We obtain a complex

$$F \otimes_A B\colon 0 \to F^0 \otimes_A B \xrightarrow{d^0 \otimes \mathrm{id}_B} F^1 \otimes_A B \xrightarrow{d^1 \otimes \mathrm{id}_B} \cdots \xrightarrow{d^{r-1} \otimes \mathrm{id}_B} F^r \otimes_A B \to 0.$$

Consider $F \otimes_A B$ as an A-module. Then we obtain an A-linear map

$$F \to F \otimes_A B \tag{1.7.3}$$

of complexes, which sends m to $m \otimes 1$.

(1.8) Lemma. *Given an A-algebra B.*

→ (1) *The map* (1.7.3) *induces a natural map*

$$H^i(F) \otimes_A B \to H^i(F \otimes_A B)$$

of B-modules.

(2) *Assume that the map $B^j(F) \otimes_A B \to F^j \otimes_A B$ is injective for $j = i, i+1$, and that the map $Z^i(F) \otimes_A B \to F^i \otimes_A B$ is injective. Then the map of assertion* (1) *is an isomorphism.*

→ *Proof.* The map (1.7.3) induces a map $H^i(F) \to H^i(F \otimes_A B)$ of A-modules. We extend it to the B-module map of assertion (1).

Assume that the assertions of (2) hold. We have that $(F \otimes_A B)^j = F^j \otimes_A B$. In particular we obtain that the map $F^j \to (F \otimes_A B)^j$ induces a surjective map $B^j \otimes_A B \to B^j(F \otimes_A B)$ and since $B^j \otimes_A B \to F^j \otimes_A B$ is injective for $j = i, i+1$ by assumption we obtain that $B^j \otimes_A B \to B^j(F \otimes_A B)$ is an isomorphism for $j = i, i+1$.

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→ From (1.7.1), for F and $F \otimes_A B$, we obtain the following commutative diagram of B-modules:

$$\begin{array}{ccccccccc}
 & & Z^i(F) \otimes_A B & \longrightarrow & F^i \otimes_A B & \longrightarrow & B^{i+1}(F) \otimes_A B & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & Z^i(F \otimes_A B) & \longrightarrow & F^i \otimes_A B & \longrightarrow & B^{i+1}(F \otimes_A B) & \longrightarrow & 0.
\end{array}$$

with exact rows. We have seen that the left and middle vertical maps are isomorphisms. Hence the right vertical map is an isomorphism.

→ From (1.7.2), for the modules F and $F \otimes_A B$, we obtain a commutative diagram of B-modules

$$\begin{array}{ccccccccc}
 & & B^i(F) \otimes_A B & \longrightarrow & Z^i(F) \otimes_A B & \longrightarrow & H^i(F) \otimes_A B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B^i(F \otimes_A B) & \longrightarrow & Z^i(F \otimes_A B) & \longrightarrow & H^i(F \otimes_A B) & \longrightarrow & 0.
\end{array}$$

with exact rows. When the conditions of part (2) are satisfied we have seen that the two left vertical maps of the last diagram are isomorphisms. Hence the right vertical map is also an isomorphism.

(1.9) Lemma. *Let*

$$F\colon 0 \to F^0 \xrightarrow{d^0} F^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{r-1}} F^r \to 0$$

be a sequence of flat A-modules. Assume that $H^i(F)$ is a flat A-module for $i \geq p$. Then B^i is flat for $i \geq p$ and Z^i is flat for $i \geq p-1$. Moreover, for every A-algebra B the base change map

$$H^i(F) \otimes_A B \to H^i(F \otimes_A B) \qquad ((3.9.1))$$

→ *of Lemma* (1.8) *is an isomorphism for $i \geq p$.*

In particular, when $H^i(F) = 0$ for $i > 0$. we have, for every A algebra B, that:

(1) *$H^i(F \otimes_A B) = 0$ for $i > 0$.*
(2) *$H^0(F \otimes_A B)$ is a flat B-module.*
(3) *The base change map*

$$H^i(F) \otimes_A B \to H^i(F \otimes_A B)$$

is an isomorphism for all i.

Proof. We prove the first assertion of the Lemma by descending induction on p. The Lemma holds for $p > r$. Assume that it holds for $p+1$ and assume that $H^p(F)$ is flat. By the induction assumption we have that B^i is flat for $i > p$ and Z^i is flat for $i \geq p$. From the sequence (1.7.2) with $i = p$ and Lemma (1.3(2)) we conclude that B^p is flat. Similarly, from the sequence (1.7.1) with $i = p-1$ and Lemma (1.3(2)) we conclude that Z^{p-1} is flat over A.

To prove that the base change map is an isomorphism we note that, since Z^i and $H^i(F)$ are flat for $i = p, p+1$, it follows from sequence (1.7.2) and Lemma (3.3(1)) that $B^i \otimes_A B \to Z^i \otimes_A B$ is injective for $i = p, p+1$. Moreover, since B^{p+1} is flat, it follows from sequence (1.7.1) and Lemma (3.3(1)) that $Z^p \otimes_A B \to F^p \otimes_A B$ is injective. The two conditions of Lemma (1.8(2)) with $i = p$ are therefore satisfied and consequently formula (1.6.1) holds for $i = p$.

The second assertion of the Lemma follows from the first for $p = 1$. Indeed, when $p = 1$ it follows that $H^0(F) = Z^0$ is flat. Consequently B^0 is also flat.