Topological spaces

1. Notation and Zorns Lemma.

(1.1) Notation. Let !!I be an index set and !!$\{E_\alpha\}_{\alpha\in I}$ a collection of sets E_α indexed by I. We denote by !!$\prod_{\alpha\in I} E_\alpha$ the *cartesian product* of the sets E_α. The elements of $\prod_{\alpha\in I} E_\alpha$ we denote by !!$(g_\alpha)_{\alpha\in I}$ with $g_\alpha \in E_\alpha$ for all $\alpha \in I$. When $I = \{1, 2, \ldots, n\}$ we write $E_1 \times E_2 \times \cdots \times E_n$ for the product.

We can interpret the elements of $\prod_{\alpha\in I} E_\alpha$ as applications !!$\varphi : I \to \cup_{\alpha\in I} E_\alpha$ from I to the union of the sets E_α such that $\varphi(\alpha) \in E_\alpha$ for all $\alpha \in I$. With this interpretation the relation between the functions $\varphi : I \to \cup_{\alpha\in I} E_\alpha$ and the elements $(g_\alpha)_{\alpha\in I} \in \prod_{\alpha\in I} E_\alpha$ is given by $\varphi(\alpha) = g_\alpha$ for all $\alpha \in I$. When all the sets E_α are equal to the same set E we write !!E^I for the cartesian product. That is E^I consists of all maps $I \to E$.

When !!$\{F_\alpha\}_{\alpha\in I}$ is another collection of sets and we have maps !!$u_\alpha : E_\alpha \to F_\alpha$ for all $\alpha \in I$, we obtain a map of sets !!$\prod_{\alpha\in I} u_\alpha : \prod_{\alpha\in I} E_\alpha \to \prod_{\alpha\in I} F_\alpha$ defined by $(\prod_{\alpha\in I} u_\alpha)(g_\alpha)_{\alpha\in I} = (u_\alpha(g_\alpha))_{\alpha\in I}$.

Let $\{E_\alpha\}_{\alpha\in I}$ be a collection of sets and let J be a subset of I. The map $\prod_{\alpha\in I} E_\alpha \to \prod_{\beta\in J} E_\beta$ that sends $(x_\alpha)_{\alpha\in I}$ to $(x_\beta)_{\beta\in J}$ we call a *projection*.

(1.2) Definition. A *partially ordered set* is a set E with a relation !!$\leq$ such that, for all !!α, β, γ in E, satisfies the conditions:

(1) $\alpha \leq \alpha$.
(2) If $\alpha \leq \beta$ and $\beta \leq \alpha$ then $\alpha = \beta$.
(3) If $\alpha \leq \beta$ and $\beta \leq \gamma$ then $\alpha \leq \gamma$.

The partially ordered set E is *upper filtrating* or *upper directed* if there, for all α, β in E, is a γ in E such that $\alpha \leq \gamma$ and $\beta \leq \gamma$.

Let E be a non-empty partially ordered set. A *chain* !!F in E is a subset of E such that for α, β in F we have that either $\alpha \leq \beta$ or $\beta \leq \alpha$.

(1.3) Zorns Lemma. Let E be a non-empty partially ordered set. If all chains F in E have an upper bound in E, that is, there is an element α in E such that $\beta \leq \alpha$ for all β in F, then E has at least one maximal element. In other words, there is a γ in E such that no element !!δ of E different from γ satisfies $\gamma \leq \delta$.

(1.4) Definition. A partially ordered set E satisfies the *maximum condition* if every non-empty subset F has a *maximal* element, that is, there is an element $\beta \in F$

such that $\alpha \leq \beta$ for all $\alpha \in F$. It satisfies the *minimum condition* if every non-empty subset F has a minimal element, that is, there is an element $\beta \in F$ such that $\beta \leq \alpha$ for all $\alpha \in F$.

(1.5) Lemma. *Let E be a partially ordered set. The following conditions are equivalent:*

n (1) *Every increasing chain !!$\alpha_1 \leq \alpha_2 \leq \dots$ is stationary, that is, there is a positive integer n such that $\alpha_n = \alpha_{n+1} = \cdots$.*
(2) *The set E satisfies the maximum condition.*

Proof. (1) Assume that E satisfies the maximum condition, and let $\alpha_1 \leq \alpha_2 \leq \dots$ be a chain in E. If α_m is a maximal element for the subset $\{\alpha_n\}_{n\in\mathbf{N}}$ we have that $\alpha_m = \alpha_{m+1} = \cdots$.

(2) Assume that every sequence is stationary, and let F be a subset of E. By induction on n we can clearly find a sequence $\alpha_1 < \alpha_2 < \cdots < \alpha_{n-1} < \alpha_n < \dots$, which is not stationary.

(1.6) Exercises.

n **1.** Let E be a set and let !!$\mathcal{E} = \{E_\alpha\}_{\alpha\in I}$ be a collection of subsets of E. We write $E_\alpha \leq E_\beta$ when E_α is contained in E_β. Show that $\mathcal{E}$ with the relation $\leq$ is a partially ordered set.

n **2.** Let E be a set. The set !$\mathbf{Z}^E$! consists of all functions $\varphi : E \to \mathbf{Z}$ from E to the
n integers $\mathbf{Z}$. We denote by !$\mathcal{I}$! the family consisting of all subsets !$\mathfrak{a}$! of $\mathbf{Z}^E$ different from $\mathbf{Z}^E$ satisfying the two conditions:

(i) If $\varphi \in \mathfrak{a}$ and $\chi \in \mathbf{Z}^E$ then $\varphi\chi$ is in $\mathfrak{a}$.
(ii) If φ, χ are in $\mathfrak{a}$ then $\varphi + \chi$ is in $\mathfrak{a}$.

(1) Show that the sets $\alpha(F) = \{\varphi \in \mathbf{Z}^E : \varphi(x) = 0 \text{ for all } x \in F\}$ for all non-empty subsets F of E satisfy conditions (i) and (ii).
(2) Show that when we order $\mathcal{I}$ partially by inclusion of the sets $\mathfrak{a}$, then $\mathcal{I}$ will contain maximal elements.

3. Let $\mathcal{H}$ be the collection of subsets of the set $\mathbf{Z}^{\mathbf{N}}$ of functions $\varphi : \mathbf{N} \to \mathbf{Z}$ from the
n natural numbers !$\mathbf{N}$! to the integers consisting of sets H such that if φ and χ are in H then $\varphi + \chi$ is also in H.

n (1) Show that for all integers n and prime numbers p the set !!$H_{n,p} = \{(m_i)_{i\in\mathbf{N}} : p \text{ divides } m_n\}$ is a maximal subset of $\mathcal{H}$.
(2) Show that there are other maximal subsets of $\mathfrak{B}$ than those of the form $H_{n,p}$.
n (3) Let !$\mathbf{Z}^{(\mathbf{N})}$! be the subset of $\mathbf{Z}^{\mathbf{N}}$ consisting of functions with *finite support*, that is $\varphi(n) = 0$ except for a finite number of natural numbers n. Is $\mathbf{Z}^{(\mathbf{N})}$ a maximal subset of $\mathbf{Z}^{\mathbf{N}}$?

n **4.** Let I be a partially ordered upper directed set. An *inductive system* of sets !! $\{E_\alpha, \rho^\alpha_\beta\}_{\alpha,\beta\in I,\alpha\leq\beta}$ consists of a collection of sets $\{E_\alpha\}_{\alpha\in I}$, and maps $\rho^\alpha_\beta : E_\alpha \to E_\beta$ for all pairs of elements α, β in I with $\alpha \leq \beta$.

Let $\{E_\alpha, \rho^\alpha_\beta\}_{\alpha,\beta\in I,\alpha\leq\beta}$ be an inductive system indexed by the partially ordered
n upper directed set I. Denote by !!$E = \cup_{\alpha\in I}(E_\alpha \times \{\alpha\})$ the disjoint union of the sets
n E_α for $\alpha \in I$. We define on E a relation !!$\sim$ by $(x_\alpha, \alpha) \sim (x_\beta, \beta)$ if there is a $\gamma \in I$ satisfying $\alpha \leq \gamma$ and $\beta \leq \gamma$ such that $\rho^\alpha_\gamma(x_\alpha) = \rho^\beta_\gamma(x_\beta)$.

(1) Show that $\sim$ is an *equivalence relation* on E.
(2) Let $\varinjlim_{\alpha\in I} E_\alpha = E/\sim$ be the residue classes of E modulo the equivalence relation $\sim$, and let $u_\alpha : E_\alpha \to \varinjlim_{\alpha\in I} E_\alpha$ be the map that sends an element $x_\alpha \in E_\alpha$ to the equivalence class of (x_α, α). Show that $u_\alpha = u_\beta\rho^\alpha_\beta$ when $\alpha \leq \beta$. We call the module $\varinjlim_{\alpha\in I} E_\alpha$ together with the maps u_α the *direct limit* of the inductive system. It is often convenient to call $\varinjlim_{\alpha\in I} E_\alpha$ simply the *direct limit* of the system.
(3) Show that for every element x in $\varinjlim_{\alpha\in I} E_\alpha$ there is an index $\alpha \in I$ and an $x_\alpha \in E_\alpha$ such that $u_\alpha(x_\alpha) = x$.
(4) Show that if $x_\alpha \in E_\alpha$ and $x_\beta \in E_\beta$ are such that $u_\alpha(x_\alpha) = u_\beta(x_\beta)$ then there is an index $\gamma \in I$ satisfying $\alpha \leq \gamma$ and $\beta \leq \gamma$ such that $\rho^\alpha_\gamma(x_\alpha) = \rho^\beta_\gamma(x_\beta)$.
(5) Show that if x and y are elements in E there is an index $\alpha \in I$ and x_α, y_α in E_α such that $u_\alpha(x_\alpha) = x$ and $u_\alpha(y_\alpha) = y$.
(6) Show that the inductive limit has the following *universal property*:
For all $\alpha \in I$ let $v_\alpha : E_\alpha \to F$ be maps into a set F such that $v_\alpha = v_\beta\rho^\alpha_\beta$ for all α, β in I with $\alpha \leq \beta$. Then there is a unique map $u : \varinjlim_{\alpha\in I} E_\alpha \to F$ such that $v_\alpha = uu_\alpha$ for all $\alpha \in I$.
(7) Show that the universal property characterizes the direct limit up to isomorphisms. That is, if $w_\alpha : E_\alpha \to G$ are maps for $\alpha \in I$ with the universal property that for all $v_\alpha : E_\alpha \to F$ such that $v_\alpha = w_\beta\rho^\alpha_\beta$ there is a unique map $w : G \to F$ satisfyting $v_\alpha = vu_\alpha$, then there are unique maps $\varinjlim_{\alpha\in I} E_\alpha \to G$ and $G \to \varinjlim_{\alpha} E_\alpha$ that are inverses.
n (8) Let !!$\{F_\alpha, \sigma^\alpha_\beta\}_{\alpha,\beta\in I,\alpha\leq\beta}$ be another inductive system of sets. Assume that we
n for every $\alpha \in I$ has a map !!$u_\alpha : E_\alpha \to F_\alpha$ such that $u_\beta\rho^\alpha_\beta = \sigma^\alpha_\beta u_\alpha$ for all α, β in I with $\alpha \leq \beta$. We call the collection $\{u_\alpha\}_{\alpha\in I}$ a *map of inductive systems.* We call the collection $\{u_\alpha\}_{\alpha\in I}$ a *map of inductive systems.*
Show that there is a unique map $\varinjlim_{\alpha\in I} u_\alpha : \varinjlim_{\alpha\in I} E_\alpha \to \varinjlim_{\alpha\in I} F_\alpha$ such that $\varinjlim_{\alpha\in I} u_\alpha\rho_\alpha = \sigma_\alpha u_\alpha$ for all $\alpha \in I$, where $\sigma_\alpha : F_\alpha \to \varinjlim_{\alpha\in I} F_\alpha$ is the canonical map for the inductive system $\{F_\alpha, \sigma^\alpha_\beta\}_{\alpha,\beta\in I,\alpha\leq\beta}$.

2. Categories.

(2.1) Definition. A *category* !!$\boldsymbol{K}$ consists of a collection of *objects* !!$\mathrm{Obj}(\boldsymbol{K})$ and, for every pair of objects !!A, B a set !!$\mathrm{Mor}(A, B)$ of *morphisms* with the property that for !!$u \in \mathrm{Mor}(A, B)$ and !!$v \in \mathrm{Mor}(B, C)$ there is a *composition* vu in $\mathrm{Mor}(A, C)$ such that:

(1) For every object A in $\mathrm{Obj}(\boldsymbol{K})$ there is an element !!id_A in $\mathrm{Mor}(A, A)$ such that $u = u\,\mathrm{id}_A$ and $u = \mathrm{id}_B\,u$.

(2) If !!$w \in \mathrm{Mor}(C, D)$ then $w(vu) = (wv)u$ in $\mathrm{Mor}(A, D)$.

Often we simply say that $\mathrm{Obj}(\boldsymbol{K})$ is a category and we write !!$u : A \to B$ instead of $u \in \mathrm{Mor}(A, B)$. A morphism $u : A \to B$ is an *isomorphism* if there is a morphism $v : B \to A$ such that $vu = \mathrm{id}_A$ and $uv = \mathrm{id}_B$.

(2.2) Example. Let $\mathrm{Obj}(\boldsymbol{K})$ be the collection of *all* sets, and for every pair of sets A, B we let !!$\mathrm{Mor}(A, B)$ be all maps from A to B. Then $\mathrm{Obj}(\boldsymbol{K})$ with these morphisms form a category called the *category of sets.*

(2.3) Example. Let E be a partially ordered set under a relation $\leq$, and let $\mathrm{Obj}(\boldsymbol{K})$ consist of the elements of E. For two elements α, β in E we let $\mathrm{Mor}(\alpha, \beta)$ consist of all relations $\alpha \leq \beta$. That is, a morphism $\alpha \to \beta$ is a relation $\alpha \leq \beta$. Then $\mathrm{Obj}(\boldsymbol{K})$ with the sets $\mathrm{Mor}(\alpha, \beta)$ is a category.

(2.4) Definition. Let $\boldsymbol{K}$ and !!$\boldsymbol{L}$ be categories. A *covariant*, respectively *contravariant, functor* !!$\mathcal{F}$ from $\boldsymbol{K}$ to $\boldsymbol{L}$ is a map that associates to every object A in $\mathrm{Obj}(\boldsymbol{K})$ an object $\mathcal{F}(A)$ of $\mathrm{Obj}(\boldsymbol{L})$ and that to each morphism $\varphi : A \to B$ in $\mathrm{Mor}(A, B)$ associates a morphism $\mathcal{F}(\varphi) : \mathcal{F}(A) \to \mathcal{F}(B)$, respectively $\mathcal{F}(\varphi) : \mathcal{F}(B) \to \mathcal{F}(A)$, such that:

(1) $\mathcal{F}(\mathrm{id}_B) = \mathrm{id}_{\mathcal{F}(B)}$.

(2) If $\chi : B \to C$ is another morphism then $\mathcal{F}(\chi\varphi) = \mathcal{F}(\chi)\mathcal{F}(\varphi)$, respectively $\mathcal{F}(\chi\varphi) = \mathcal{F}(\varphi)\mathcal{F}(\chi)$.

We usually simply say that $\mathcal{F}$ is *functorial in* A.

A *natural transformation* $u : \mathcal{F} \to \mathcal{G}$ of two functors from the category $\boldsymbol{K}$ to the category $\boldsymbol{L}$ is a map $u(A) : \mathcal{F}(A) \to \mathcal{G}(A)$ for each $A \in \mathrm{Obj}(\boldsymbol{K})$ such that for each morphism $\varphi \in \mathrm{Mor}(A, B)$ in the category $\boldsymbol{K}$ we have that $u(B)\mathcal{F}(\varphi) = \mathcal{G}(\varphi)u(A)$. We say that the natural transformation is an *isomorphism* if it has an inverse.

Two categories $\boldsymbol{K}$ and $\boldsymbol{L}$ are *equivalent* if there are functors $\mathcal{F}$ and $\mathcal{G}$ from $\boldsymbol{K}$ to $\boldsymbol{L}$, respectively from $\boldsymbol{L}$ to $\boldsymbol{K}$, such that $\mathcal{GF}$ is isomorphic to the identity $\mathrm{id}_{\boldsymbol{K}}$ and $\mathcal{FG}$ is isomorphic to $\mathrm{id}_{\boldsymbol{L}}$.

(2.5) Remark. When $\mathcal{F}$ is a functor from the category $\boldsymbol{K}$ to the category $\boldsymbol{L}$, and $\mathcal{G}$ is a functor from $\boldsymbol{L}$ to the category $\boldsymbol{M}$ we have that the composite $\mathcal{GF}$ is a functor from $\boldsymbol{K}$ to $\boldsymbol{M}$. Moreover, the identity map on objects and morphisms in the category $\boldsymbol{K}$ is a functor $\mathrm{id}_{\boldsymbol{K}}$ from $\boldsymbol{K}$ to itself. Hence categories with natural transformations form a category.

(2.6) Products. Let $\boldsymbol{K}$ be a category and let $\{A_\alpha\}_{\alpha\in I}$ be a collection of objects in
n Obj($\boldsymbol{K}$). The *product* of the objects A_α is an object !!$\prod_{\alpha\in I} A_\alpha$ in Obj($\boldsymbol{K}$) together
with a morphism $u_\alpha : \prod_{\alpha\in I} A_\alpha \to A_\alpha$ for each $\alpha \in I$ such that for every object B of $\boldsymbol{K}$ and every collection of morphisms $v_\alpha : B \to A_\alpha$ for all $\alpha \in I$ there is a unique morphism $v : B \to \prod_{\alpha\in I} A_\alpha$ such that $v_\alpha = u_\alpha v$ for all $\alpha \in I$.

(2.7) Example. Let I be an index set. In the category of sets we have that the product $\prod_{\alpha\in I} E_\alpha$ of the sets $\{E_\alpha\}_{\alpha\in I}$ is the product in the *categorical sense*, that is, the product in the category of sets.

(2.8) Coproducts. Let $\boldsymbol{K}$ be a category and $\{A_\alpha\}_{\alpha\in I}$ be a collection of objects
n in Obj($\boldsymbol{K}$). The *coproduct* of the objects is an object !!$\coprod_{\alpha\in I} A_\alpha$ in Obj($\boldsymbol{K}$) together
with a morphism

$$u_\alpha : A_\alpha \to \coprod_{\alpha\in I} A_\alpha$$

for each $\alpha \in I$ such that for every object B of $\boldsymbol{K}$ and every collection of morphisms $v_\alpha : A_\alpha \to B$ for all $\alpha \in I$ there is a unique morphism $v : \coprod_{\alpha\in I} A_\alpha \to B$ such that $v_\alpha = v u_\alpha$ for all $\alpha \in I$.

(2.9) Exercises.

1. Let $\{E_\alpha\}_{\alpha\in I}$ be a collection of sets. Show that the disjoint union $E = \cup_{\alpha\in I}(E_\alpha \times \{\alpha\})$ together with the maps $u_\alpha : E_\alpha \to E$ defined by $u_\alpha(x_\alpha) = (x_\alpha, \alpha)$ for all $\alpha \in I$ is the coproduct in the category of sets.

2. Let E be a set and let Obj($\boldsymbol{K}$) consist of all subsets of E. For two subsets A and B of E we let $\mathrm{Mor}(A, B)$ consist of all maps $A \to B$. Show that Obj($\boldsymbol{K}$) with the maps $\mathrm{Mor}(A, B)$ is a category.

3. Let E be a set and let Obj($\boldsymbol{K}$) consist of all subsets of E. For two subsets A and B of E we let $\mathrm{Mor}(A, B)$ be the inclusion map if $A \subseteq B$ and otherwise be empty. Show that Obj($\boldsymbol{K}$) with the maps $\mathrm{Mor}(A, B)$ is a category.

→ **4.** Let $\boldsymbol{K}$ be the category of Exercise (3) and let $\boldsymbol{L}$ be the category of Exercise (2). Show that the map $\mathcal{F} : \mathrm{Obj}(\boldsymbol{K}) \to \mathrm{Obj}(\boldsymbol{L})$ defined by $\mathcal{F}(A) = A$ for all subsets A of E, and by $\mathcal{F}(u) = u$ for all maps $u : A \to B$ in $\boldsymbol{K}$ is a covariant functor.

3. Topological spaces.

n **(3.1) Definition.** A *topological space* is a set !!X together with a collection !!$\{U_\alpha\}_{\alpha \in I}$
of subsets U_α of X such that

(1) The set X and the empty set $\emptyset$ are in the collection $\{U_\alpha\}_{\alpha \in I}$.
n (2) For every subset !!J of I the union $\cup_{\beta \in J} U_\beta$ of the sets in $\{U_\beta\}_{\beta \in J}$ is in the collection $\{U_\alpha\}_{\alpha \in I}$.
(3) For every *finite* subset J of I the intersection $\cap_{\beta \in J} U_\beta$ of the sets in $\{U_\beta\}_{\beta \in J}$ is in the collection $\{U_\alpha\}_{\alpha \in I}$.

n The sets U_α are called the *open* sets of X and the complement !!$X \setminus U_\alpha$ of the open
sets are called *closed*. We say that the collection of sets $\{U_\alpha\}_{\alpha \in I}$ is a *topology* on X. Often we simply say that X is a *topological space*.

n Let !!x be a point of X. A subset !!Y of X that contains x is a *neighbourhood* of
x if there exists an open subset U of X such that $x \in U \subseteq Y$. A collection $\{U_\beta\}_{\beta \in J}$ of open sets in X is called an *open covering* of X if the union of the sets is X, that is $X = \cup_{\beta \in J} U_\beta$.

(3.2) Example. Let X be a set. The set X with the collection $\{\emptyset, X\}$ consisting of the empty set and X itself is a topological space. This topology is called the *trivial topology* on X.

(3.3) Example. Let X be a set. The set X with the collection of all subsets of X is a topological space. This topology is called the *discrete topology*.

(3.4) Example. Let X be a set. The set X with the collection of sets consisting of
n $\emptyset$ and all the subsets U of X whose complement !!$X \setminus U$ is a finite set is a topological
space. We call this topology the *finite complement topology*.

(3.5) Remark. Let X be a topological space with open sets $\{U_\alpha\}_{\alpha \in I}$. For every subset Y of X we have that the collection of sets $\{U_\alpha \cap Y\}_{\alpha \in I}$ are the open subsets of a topology on Y. We call this topology on Y the topology *induced* by the topology on X, and we say that Y is a *subspace* of X.

(3.6) Definition. Let X be a topological space and let x be a point of X. A
n collection of sets !!$\mathfrak{B} = \{U_\beta\}_{\beta \in J}$ consisting of open neighbourhoods U_β of x is a *basis*
for the neighbourhoods of x if there, for every open neighbourhood U of x, is an open set U_β belonging to $\mathfrak{B}$ such that $x \in U_\beta \subseteq U$.

A collection of subsets $\mathfrak{B} = \{U_\gamma\}_{\gamma \in K}$ of X consisting of open sets U_γ of X is a
n *basis for the topology* if the members !!$\mathfrak{B}_x = \{U \in \mathfrak{B} : x \in U\}$ containing x is a basis
for the neighbourhoods of x for every point $x \in X$.

(3.7) Example. The collection of all open sets is a basis for the topology on X.

n **(3.8) Definition.** For every subset !!Y of X we denote by !!$\overline{Y}$ the intersection of
all the closed sets that contain Y. Equivalently $\overline{Y}$ is the set consisting of all points x in X such that every open neighbourhood of x contains at least one point of Y. We call $\overline{Y}$ the *closure* of the set Y.

n **(3.9) Definition.** Let X and Y be topological spaces. A map !!$\psi : X \to Y$ is called *continuous* if the inverse image $\psi^{-1}(V)$ of every open subset V of Y is open in X.

The map is an *isomorphism*, or *homeomorphism*, if there is a continuous map
n !!$\omega : Y \to X$ which is *inverse* to ψ. That is $\omega\psi = \mathrm{id}_X$ and $\psi\omega = \mathrm{id}_Y$.

(3.10) Example. Let X be a topological space and Y a subset considered as a topological space with the induced topology. Then the inclusion map $Y \to X$ is continuous.

(3.11) Example. The set theoretic inverse of a bijecive continuous map $\psi : X \to Y$ is not necessarily bijective. For example the identity map $\mathrm{id}_X : X' \to X''$ from the topological space X' with X as underlying set and the discrete topology to the topological space X'' with X as underlying set and trivial topology is always continous. However, the inverse, which is also id_X is not continous if X has more than one point.

(3.12) Remark. For every topological space X the map id_X is continuous. When $\psi : X \to Y$ and $\omega : Y \to Z$ are continuous maps of topological spaces we have that $\omega\psi : X \to Z$ is continuous. In other words the topological spaces with continuous maps form a category, called the *category of topological spaces.*

(3.13) Remark. Let X be a topological space and let $\mathrm{Obj}(\boldsymbol{K})$ be the collection of open sets of X. For each pair of open sets U, V in X we let $\mathrm{Hom}(U, V)$ consist of the inclusion map of U in V if U is contained in V, and otherwise let $\mathrm{Hom}(U, V)$ be empty. Then $\mathrm{Obj}(\boldsymbol{K})$ with these morphisms form a category.

(3.14) Exercises.

1. Let X be a set and let $X = U_0 \supset U_1 \supset U_2 \supset \cdots$ be a sequence of subsets.

(1) Show that the sets $\emptyset$ and $\{U_n\}_{n\in\mathbf{N}}$ are the open sets of a topology of X.
(2) Show that if $\cap_{n\in\mathbf{N}} U_n \neq 0$ the set $\cap_{n\in\mathbf{N}} U_n$ is not open in X.

2. Let X be a set and let x_0 be an elements of X.

(1) Show that X with the collection of all subsets of X that contain x_0 is a topological space.
(2) Show that X has a basis for the topology with open sets consisting of 1 or 2 elements.
(3) Find the closed points of X.

3. Let $Y = \{y, X\}$ be the disjoint union of a point y and the underlying set X of a topological space with open sets $\{U_\alpha\}_{\alpha\in I}$. Show that Y with the family of sets $\{y, U_\alpha\}_{\alpha\in I}$ is a topological space.

4. Let X and Y be topological spaces and $\psi : X \to Y$ a map.

(1) Show that when X has the discrete topology then ψ is continuous.
(2) Show that when Y has the trivial topology then ψ is continuous.

→ **5.** Give another example than (?) of a continuous bijective homomorphism $\psi : X \to Y$ of topological spaces which is not an isomorphism.

6. Let X and Y be topological spaces and $\mathfrak{B}$ a basis for the topology on Y. Show that a map $\psi : X \to Y$ is continuous if and only if the inverse image of every open set belonging to $\mathfrak{B}$ is open in X.

7. Let X be a set and let $\mathfrak{B} = \{U_\alpha\}_{\alpha \in I}$ be a family of subsets U_α with the property that for every pair of sets U_α, U_β in the family $\mathfrak{B}$ and every point $x \in U_\alpha \cap U_\beta$ there is a U_γ in $\mathfrak{B}$ such that $x \in U_\gamma \subseteq U_\alpha \cap U_\beta$. Let $\mathcal{U}$ be the family of all subsets U of X such that for every point $x \in U$ there is a U_α in $\mathfrak{B}$ such that $x \in U_\alpha \subseteq U$.

(1) Show that X with the family of sets $\mathcal{U}$ is a topological spaces.
→ (2) Show that the sets of $\mathfrak{B}$ form a basis for the topological space of part (1).

n **8.** Let X and Y be topological spaces and let !!$\mathcal{V}$ be the collection of subsets of the cartesian product $X \times Y$ of the form $U \times V$, where U is open in the X and V is open in Y.

n (1) Show that $X \times Y$ with the sets !!$\mathcal{U}$ which consists of all the unions of the sets in $\mathcal{V}$ form a topological space. We call this topology the *product topology* on $X \times Y$.
n (2) Show that the *projection* !!$\pi : X \times Y \to X$ defined by $\pi(x, y) = x$ is continuous when $X \times Y$ has the product topology.
(3) Assume that X and Y have the finite complement topology. Show that in most cases the finite complement topology on $X \times Y$ is different from the product topology.

n **9.** Let $X = \mathbf{Z}$. An arithmetic progression consists of numbers of the form !!$V_{p,q} = \{pn + q : n \in \mathbf{Z}\}$ where p and q are integers, and $p \neq 0$.

(1) Show that for every integer m we have that $V_{p,q} = V_{p,mp+q}$.
(2) Let p', p'', q', q'' be natural numbers. Show that for every number n in $V_{p',q'} \cap V_{p'',q''}$ there are natural numbers p, q such that $n \in V_{p,q} \subseteq V_{p',q'} \cap V_{p'',q''}$.
→ (3) Show that the collection of all subsets of $\mathbf{Z}$ that are arithmetic progressions satisfy the conditions of Exercise (6), and consequently is the basis for a topology on X.
→ (4) Show that all the arithmetic progressions $V_{p,q}$ are closed in the topology of part (3).
(5) Let Y be the union of all the sets $V_{p,0}$ where p is a prime number. Show that $X \setminus Y = \{-1, 1\}$ and that $\{-1, 1\}$ is not open in X.
→ (6) Use part (4) and (5) to prove that there exists infinitely many prime numbers.

10. Let X be a set with a *metric*, that is, for each pair of points x, y of X there is a real number $d(x, y)$ such that for all elements x, y, z of X we have:

(1) $d(x, y) \geq 0$.
(2) $d(x, y) = 0$ if and only if $x = y$.
(3) $d(x, y) = d(y, x)$.
(4) $d(x, z) \leq d(x, y) + d(y, z)$.

Let $\mathcal{U}$ consist of all sets U with the property that for every point x of U there is a real number ε_x such that the set $\{y \in X : d(x, y) < \varepsilon_x\}$ is contained in U.

(1) Show that X with the family $\mathcal{U}$ is a topological space.

(2) Show that for each point x of X the sets $U_{x,n} = \{y \in X : d(y, x) < 1/n\}$ for all natural numbers n form a basis for the neighbourhoods of x.

4. Irreducible sets.

(4.1) Definition. A topological space X is *irreducible* if X is non-empty, and if any two non-empty open subsets of X intersect. Equivalently X is irreducible if $X \neq \emptyset$ and X is *not* the union of two closed subsets different from X. A subset Y of X is irreducible if it is an irreducible topological space with the induced topology.

(4.2) Proposition. *Let X be a topological space.*

(1) *A subset Y of X is irreducible if and only if the closure $\overline{Y}$ is irreducible.*
(2) *Every irreducible subset Y of X is contained in a maximal irreducible subset.*
(3) *The maxmial irreducible subsets of X are closed, and they cover X.*

Proof. (i) The first claim follows easily from the observation that every open subset that intersects $\overline{Y}$ also intersects Y.

n (ii) Let Y be an irreducible subset of X, and let !!$\mathcal{I}$ be the family consisting of
n all irreducible subsets of X that contain Y. For every chain !!$\mathcal{J} = \{Z_\alpha\}_{\alpha \in I}$ in $\mathcal{I}$ we have that $Z = \cup_{\alpha \in I} Z_\alpha$ is irreducible. This is because, when U and V are open sets that intersect Z there are α and β in $\mathcal{I}$ such that $U \cap Z_\alpha$ and $V \cap Z_\beta$ are non-empty. Since $\mathcal{J}$ is a chain we have that either the sets $U \cap Z_\alpha$ and $V \cap Z_\alpha$, or the sets $U \cap Z_\beta$ and $V \cap Z_\beta$, are non-empty. In particular $(U \cap Z) \cap (V \cap Z)$ is non-empty. Since all chains have maximal elements it follows from Zorns Lemma that $\mathcal{I}$ has maximal elements.

→ (iii) The third claim is an immediate consequence of assertions (1) and (2).

(4.3) Definition. The maximal irreducible subsets of X are called the *irreducible components* of X.

(4.4) Example. The irreducible components of the topological space with the trivial topology is X itself.

(4.5) Example. The irreducible components of the topological space X with the discrete topology are the points of X.

(4.6) Example. The topological space X with the finite complement topology is irreducible exactly when X consists of infinitely many points, or consists of one point.

(4.7) Example. Let x be a point of the topological space X. Then the closure
n !!$\overline{\{x\}}$ is irreducible.

(4.8) Definition. Let X be an irreducible topological space. If there is a point x in X such that $X = \overline{\{x\}}$ we call x a *generic point* of X.

(4.9) Definition. A topological space X is *compact* if every open covering $\{U_\alpha\}_{\alpha \in I}$ has a finite subcover, that is, there is a *finite* subset J of I such that $X = \cup_{\beta \in J} U_\beta$.

(4.10) Example. The topological space X with the trivial topology is compact.

(4.11) Example. The topological space X with the discrete topology is compact if and only if the set X is finite.

(4.12) Example. The topological space X with the finite complement topology is compact.

(4.13) Definition. The *combinatorial dimension*, or simply the *dimension*, of a topological space X is the supremum of the length n of all chains

$$X_0 \subset X_1 \subset \cdots \subset X_n$$

of irreducible closed subsets X_i of X. We denote the dimension of X by $\dim(X)$.

Let Y be a closed irreducible subset of X. The *combinatorial codimension*, or simply the *codimension*, of Y in X is the supremum of the length n of all chains

$$Y = X_0 \subset X_1 \subset \cdots \subset X_n$$

of irreducible closed subsets X_i of X. We denote the codimension of Y in X by $\operatorname{codim}(Y, X)$.

(4.14) Example. The topological space X with the trivial topology has dimension 0.

(4.15) Example. The topological space with the discrete topology has dimension 0.

(4.16) Example. Let $X = \{x_0, x_1\}$ be the topological space consisting of two points and with open sets $\{\emptyset, X, \{x_0\}\}$. Then X has dimension 1.

(4.17) Remark. Let X be a topological space and $\{X_\alpha\}_{\alpha \in I}$ its irreducible components. Then $\dim(X) = \sup_{\alpha \in I} \dim(X_\alpha)$.

(4.18) Remark. For every subset Y of X with the induced topology we have that $\dim(Y) \leq \dim(X)$. This is because when Z is closed and irreducible in Y, then the closure $\overline{Z}$ of Z in X is irreducible by Proposition (4.2), and since Z is closed in Y we obtain that $\overline{Z} \cap Y = Z$.

(4.19) Remark. A topological space X is *noetherian* if the open subsets of X satisfy the maximum condition. That is, every chain of open subsets of X has a maximal element. Equivalently the space X is noetherian if the closed subsets of X satisfy the minimum condition. That is, every chain of closed subsets have a minimal element. A space is *locally noetherian* if every point $x \in X$ has a neighbourhood that is noetherian.

(4.20) Example. The topological space X with the trivial topology is noetherian.

(4.21) Example. The topological space X with the discrete topology is noetherian exactly when the space consists of a finite number of points.

(4.22) Example. A topological space with the finite complement topology is noetherian.

(4.23) Remark. Let X be a noetherian topological space. Then every subspace Y of X is noetherian. This is because a chain $\{Z_\alpha\}_{\alpha\in I}$ of closed subsets in Y gives a chain $\{\overline{Z}_\alpha\}_{\alpha\in I}$ of closed subsets in X, where $\overline{Z}_\alpha$ is the closure of Z_α in X. We have that $\overline{Z}_\alpha \cap Y = Z_\alpha$ and consequently that when $Z_\alpha \subset Z_\beta$ then $\overline{Z}_\alpha \subset \overline{Z}_\beta$.

(4.24) Remark. A noetherian topological space X is compact. This is because if $\{U_\alpha\}_{\alpha\in I}$ is an open covering of X without a finite subcovering we can find, by induction on n, a sequence of indices $\alpha_1, \alpha_2, \dots, \alpha_n, \dots$ in I such that $U_{\alpha_1} \subset U_{\alpha_1} \cup U_{\alpha_2} \subset U_{\alpha_1} \cup U_{\alpha_2} \cup U_{\alpha_3} \subset \cdots$. Hence X is not noetherian.

Conversely, if every open subset of X is compact, then X is noetherian. This is because if X is not noetherian then we can find an infinite sequence of open subsets $U_1 \subset U_2 \subset \cdots$ of X. Then the union $\cup_{n=1}^\infty U_n$ is an open subset of X with a covering $\{U_n\}_{n\in\mathbf{N}}$ that does not have a finite subcovering.

(4.25) Proposition. *A noetherian topological space X has only a finite number of distinct irreducible components $X_1, X_2, \dots, X_n$. Moreover we have that X is not contained in $\cup_{i\neq j} X_j$ for $i = 1, 2, \dots, n$.*

Proof. Let $\mathcal{I}$ be the collection of all closed subsets of the topological space X for which the Lemma does not hold. Assume that $\mathcal{I}$ is not empty. Since X is noetherian the collection $\mathcal{I}$ then has a minimal element Y. Then Y can not be irreducible, so Y is the union $Y = Y' \cup Y''$ of two closed subsets Y', Y'' different from Y. By the minimality of Y the sets Y' and Y'' both have a finite number of irreducible components. Consequently Y can be written as a union of a finite number of closed irreducible subsets. It follows from Proposition (4.2) that Y has only a finite number of irreducible components. This contradicts the assumption that $\mathcal{I}$ is not empty. Hence $\mathcal{I}$ is empty and the Proposition holds.

If i is such that $X_i \subseteq \cup_{i\neq j} X_j$ we have that X_i is covered by the closed subsets $X_i \cap X_j$ for $i \neq j$. Since X_i is irreducible it follows that X_i must be contained in one of the X_j, which contradicts the maximality of X_i.

(4.26) Exercises.

1. Find the generic points of the topological space X with the trivial topology.

Let X with a distinguished element x_0 be the topological space with open subsets consisting of all subsets that contain x_0.

(1) Find the irreducible subsets of X.
(2) Find the generic point of all the irreducible subsets.

2. A topological space X is called a *Kolmogorov space* if there for every pair x, y of distinct points of X is an open set which contains one of the points, but not the other. Show that when X is a Kolmogorov space which is irreducible and has a generic point, then there is only one generic point.

3. A topological space is called a *Hausdorff space* if there for every pair of distinct points x, y of X are two open disjoint subsets of X such that one contains x and the other contains y. Determine the irreducible components of a Hausdorff space.

4. Let X be an irreducible topological space, and $f : X \to Y$ a continuous map to a topological space Y.

(1) Show that the the image $f(X)$ of X is an irreducible subset of Y.
(2) Show that if x is a generic point of X, then $f(x)$ is a generic point of $f(X)$.

5. Let X be an irreducible topological space. Show that all open subsets are irreducible.

6. Let $X = \mathbf{N}$ be the natural numbers and let $\mathcal{U}$ be the collection of sets consisting of X, $\emptyset$ and the subsets $\{0, 1, \ldots, n\}$ for all $n \in \mathbf{N}$.

(1) Show that X with the collection of sets $\mathcal{U}$ is a topological space.
$\rightarrow$ (2) Show that the topological space of part (1) is irreducible.
$\rightarrow$ (3) Show that the topological space of part (1) has exactly one generic point.
(4) What is the dimension of X?

Rings

1. Groups.

n **(1.1) Definition.** An *abelian*, or *commutative*, *group* !!G is a set with an addition $+$ that to every pair of elements x, y of G associates an element $x + y$ of G satisfying the following properties:

(1) There is an element 0 in G such that $0 + x = x + 0 = x$ for all x in G.
(2) For every x in G there is an element y in G such that $x + y = y + x = 0$.
(3) For all elements x, y, z of G we have that $(x + y) + z = x + (y + z)$.
(4) For every pair of elements x, y of G we have $x + y = y + x$.

→ **(1.2) Remark.** The element 0 of part (1) is unique because if $0' + x = x + 0' = x$ for all x in G then $0' = 0' + 0 = 0$. We call 0 the *zero element* of the group G.

→ The element y in part (2) is also unique for if $x+y' = y'+x = 0$ then $y' = y'+0 = y' + (x + y) = (y' + x) + y = 0 + y = y$. We call the element $y = y'$ the *inverse* of x and write it $-x$.

n **(1.3) Example.** The integers !!$\mathbf{Z}$, the rational numbers !!$\mathbf{Q}$, the real numbers !!$\mathbf{R}$,
n and the complex numbers !!$\mathbf{C}$ are abelian groups under addition. On the other hand
n the natural numbers !!$\mathbf{N}$ is not a group since all elements do not have an inverse.

n The non-zero rational, real and complex numbers !!$\mathbf{Q}^* = \mathbf{Q} \setminus \{0\}$, !!$\mathbf{R}^* = \mathbf{R} \setminus \{0\}$
n respectively !!$\mathbf{C}^* = \mathbf{C} \setminus \{0\}$ are groups under multiplication. On the other hand the non-zero integers do not form a group under multiplication because all elements do not have an inverse under multiplication.

n **(1.4) Definition.** A *subgroup* !!H of G is a subset of G such that for every pair x, y of elements in H we have that $x + y \in H$ and $-x \in H$.

(1.5) Remark. A *subgroup* H of G is a group under the addition induced by the addition on G.

(1.6) Example. Each group in the sequence $\mathbf{Z} \subseteq \mathbf{Q} \subseteq \mathbf{R} \subseteq \mathbf{C}$ is a subgroup of the following group. The same is true for the sequence $\mathbf{Q}^* \subseteq \mathbf{R}^* \subseteq \mathbf{C}^*$.

n **(1.7) Definition.** A map !!$u : G \to H$ from a group G to a group H is a *group*
n *homomorphism*, or a *homomorphism of groups*, if for all pairs !!x, y of elements of G we have

$$u(x + y) = u(x) + u(y).$$

A homomorphism of groups is an *isomorphism* if it has an inverse, or equivalently if it is bijective. The inverse is then automatically a homomorphism.

(1.8) Example. Let $u : G \to H$ be a homomorphism of groups. The set $\{x \in G : u(x) = 0\}$ of all elements that are mapped to zero in H is a subgroup of G. Moreover the image $\{u(x) : x \in G\}$ of the elements of G by u is a subgroup of H. The homomorphism u is injective if and only if $\{0\} = \{x : u(x) = 0\}$ because $u(x) = u(y)$ if and only if $u(x - y) = 0$.

(1.9) Remark. Let G be a group. Then id_G is a group homomorphism. Moreover, if $u : F \to G$ and $v : G \to H$ are group homomomorphisms then $vu : F \to H$ is a group homomorphism. In other words the groups with group homomorphisms form a category. We call this category the *category of groups*.

(1.10) Residue class groups. Let H be a subgroup of an abelian group G. When
n x, y are elements of G such that $x - y \in H$ we write !!$x \equiv y \pmod H$. It is clear that
the relation $\equiv \pmod H$ defines an equivalence relation on G, and we say that x is
n *equivalent* to y *modulo* H. The collection of equivalence classes we denote by !!G/H.
There is a unique way of defining an addition on G/H such that G/H becomes a group
n and the canonical map !!$u_{G/H} : G \to G/H$ becomes a homomorphism of groups.
The addition on G/H is given by $u_{G/H}(x) + u_{G/H}(y) = u_{G/H}(x + y)$ for all pairs of elements x, y of G. It is clear that the definition of multiplication is independent of the choice of representatives x and y for the classes $u_{G/H}(x)$, respectively $u_{G/H}(y)$. We call the group G/H the *residue class group* of G with respect to H.

The canonical homomorphism $u_{G/H} : G \to G/H$ is surjective and $H = \{x \in G : u_{G/H}(x) = 0\}$.

(1.11) Definition. Let $u : G \to H$ be a homomorphism of groups. We call
n the sub-group !!$\mathrm{Ker}(u) = \{x \in G : u(x) = 0\}$ the *kernel* of u, and the sub-group
n !!$\mathrm{Im}(u) = \{u(x) : x \in G\}$ of H the *image* of u. The group $H/\mathrm{Im}(u)$ is called the
cokernel of u.

(1.12) Operations on groups. Let $\{G_\alpha\}_{\alpha \in I}$ be a family of abelian groups G_α. The cartesian product $\prod_{\alpha \in I} G_\alpha$ becomes an abelian group under *pointwise addition.* That is, for elements $(x_\alpha)_{\alpha \in I}$ and $(y_\alpha)_{\alpha \in I}$ we define the sum by $(x_\alpha)_{\alpha \in I} + (y_\alpha)_{\alpha \in I} = (x_\alpha + y_\alpha)_{\alpha \in I}$.

n We denote by !!$\oplus_{\alpha \in I} G_\alpha$ the subset of $\prod_{\alpha \in I} G_\alpha$ consisting of the elements $(x_\alpha)_{\alpha \in I}$
with *finite support*, that is the elements $(x_\alpha)_{\alpha \in I}$ such that $x_\alpha = 0$ except for finitely many α. It is clear that the addition on $\prod_{\alpha \in I} G_\alpha$ induces an addition on $\oplus_{\alpha \in I} G_\alpha$, and that $\oplus_{\alpha \in I} G_\alpha$ with the induced addition becomes a subgroup of $\prod_{\alpha \in I} G_\alpha$. We call this subgroup the *direct sum* of the groups G_α for $\alpha \in I$. When $I = \{1, 2, \dots, n\}$ we write $\oplus_{\alpha \in I} G_\alpha = G_1 \oplus G_2 \oplus \cdots \oplus G_n$.

When all the groups G_α are isomorphic to the same group G we denote the direct
n sum by !!$G^{(I)}$. Then $G^{(I)}$ is the subgroup of G^I consisting of functions $\varphi : I \to G$
such that $\varphi(\alpha) = 0$ except for a finite number of $\alpha \in I$.

n Let $(x_\alpha)_{\alpha\in I}$ be an element of $G^{(I)}$. We shall write !!$\sum_{\alpha\in I} x_\alpha$ for the sum $\sum_{\beta\in J} x_\beta$ for every finite subset J of I such that $x_\alpha = 0$ when $\alpha \in I \setminus J$.

Let $\{G_\alpha\}_{\alpha\in I}$ be a family of subgroups of a group G. It is clear that the intersection $\cap_{\alpha\in I} G_\alpha$ of the groups G_α is a subgroup of G. Then the intersection of the subgroups of G that contain the groups G_α for $\alpha \in I$ is a group that we denote by $\sum_{\alpha\in I} G_\alpha$. It is the smallest group that contains all the subgroups G_α, and it is clear that $\sum_{\alpha\in I} G_\alpha$ consists of all the elements of the form $\sum_{\beta\in J} x_\beta$ for all finite subsets J of I and elements $x_\beta \in G_\beta$ for $\beta \in J$. That is, the group $\sum_{\alpha\in I} G_\alpha$ consists of all elements of the form $\sum_{\alpha\in I} x_\alpha$ with $x_\alpha \in G_\alpha$ for all $\alpha \in I$ and where $x_\alpha = 0$ except for a finite number of indices $\alpha \in I$.

(1.13) Remark. The product $\prod_{\alpha\in I} G_\alpha$ and sum $\oplus_{\alpha\in I} G_\alpha$ of a collection of groups $\{G_\alpha\}_{\alpha\in I}$ are the product, respectively coproduct in the *categorical sense*, that is, the product and co-product in the category of groups.

(1.14) Exercises.

1. Show that all the subgroups of the integers $\mathbf{Z}$ are of the form $m\mathbf{Z} = \{mn : n \in \mathbf{Z}\}$ for some integer m.

n **2.** Let !$\zeta_n = \cos(2\pi/n) + i\sin(2\pi/n)$! where i is the complex number $\sqrt{-1}$.

n (1) Show that !!$\mu_n = \{\zeta_n^i : i \in \mathbf{Z}\}$ is an abelian group under multiplication of complex numbers.

(2) Show that the abelian groups μ_n and $\mathbf{Z}/n\mathbf{Z}$ are isomorphic groups.

(3) Are all abelian groups with n elements isomorphic?

3. Let G and H be abelian groups, and $\mathrm{Hom}(G, H)$ the set of all group homomorphism from G to H. We define an addition on the set $\mathrm{Hom}(G, H)$ *pointwise*, that is the sum $u + v$ of two group homomorphism $u : G \to H$ and $v : G \to H$ is defined by $(u + v)(x) = u(x) + v(x)$. Show that $\mathrm{Hom}(G, H)$ is an abelian group under this addition.

4. Let $\{G_\alpha, \rho_\beta^\alpha\}_{\alpha,\beta\in I,\alpha\le\beta}$ be an inductive system of groups such that the maps ρ_β^α are group homomorphisms.

(1) Show that then $\varinjlim_{\alpha\in I} G_\alpha$ has a unique structure of group such that the canonical maps ρ_α are group homomorphisms for all $\alpha \in I$.

(2) Let $\{H_\alpha, \sigma_\beta^\alpha\}_{\alpha,\beta\in I,\alpha\le\beta}$ be another inductive system, and let $u_\alpha : G_\alpha \to H_\alpha$, for $\alpha \in I$ be a map of inductive systems.

Show that the resulting map $\varinjlim_{\alpha\in I} u_\alpha : \varinjlim_{\alpha\in I} G_\alpha \to \varinjlim_{\alpha\in I} H_\alpha$ is a group homomorphism.

5. Let n be a natural number. Find all group homomorphisms $\mathbf{Z}/n\mathbf{Z} \to \mathbf{Z}$.

2. Rings.

(2.1) Definition. A *commutative ring with unity*, which we simply call a *ring* below,
n is an abelian group !!A under an addition $+$, that also has a multiplication which
associates to each pair of elements f, g in A an element fg in A that satisfies the
n following properties for all elements !!f, g, h of A:

(1) There is a *unit* element $1 = 1_A$ in A such that $1f = f1 = f$.
(2) $f(gh) = (fg)h$.
(3) $f(g + h) = fg + fh$ and $(f + g)h = fh + gh$.
(4) $fg = gf$

(2.2) Example. The integers $\mathbf{Z}$, the rational numbers $\mathbf{Q}$, the real numbers $\mathbf{R}$, and the complex numbers $\mathbf{C}$ are all rings.

n **(2.3) Example.** Let n be a positive integer and let !!$\mathbf{Z}/n\mathbf{Z}$ be the residue group of
the integers modulo the subgroup $n\mathbf{Z}$. We use the canonical homomorphism $u_{\mathbf{Z}/n\mathbf{Z}} : \mathbf{Z} \to \mathbf{Z}/n\mathbf{Z}$ to give a multiplication on $\mathbf{Z}/n\mathbf{Z}$ by $u_{\mathbf{Z}/n\mathbf{Z}}(p)u_{\mathbf{Z}/n\mathbf{Z}}(q) = u_{\mathbf{Z}/n\mathbf{Z}}(pq)$ for all integers p and q. This multiplication makes the group $\mathbf{Z}/n\mathbf{Z}$ into a ring.

(2.4) Example. Let E be a set and A a ring. We define addition and multiplication on the set A^E of all maps from E to A *pointwise*, that is for all maps $\varphi : E \to A$ and $\chi : E \to A$ we define the sum by $(\varphi + \chi)(x) = \varphi(x) + \chi(x)$ and the product by $(\varphi\chi)(x) = \varphi(x)\chi(x)$ for all $x \in E$. With the pointwise addition and multiplication A^E becomes a ring.

(2.5) Polynomials in one variable. Let A be a ring. A *formal expression* of the
n form !!$f(t) = f_0 + f_1 t + \cdots + f_m t^m$, where m is a natural number and where the
elements $f_0, f_1, \ldots, f_m$ are in A, we call a polynomial. The set of all polynomials we denote by $A[t]$. When $f_m \neq 0$ we call m the *degree* of the polynomial $f(t)$, and we let $f_i = 0$ for $i > m$. Two polynomials $f_0 + f_1 t + \cdots + f_m t^m$ and $g_0 + g_1 t + \cdots + g_n t^n$ are equal when they are identical, that is when $f_i = g_i$ for $i = 0, 1, \ldots$.

We define addition of the polynomials $(f_0 + f_1 t + \cdots + f_m t^m)$ and $(g_0 + g_1 t + \cdots + g_n t^n)$ in $A[t]$ by

$$(f_0+f_1t+\cdots+f_mt^m)+(g_0+g_1t+\cdots+g_nt^n) = (f_0+g_0)+(f_1+g_1)t+\cdots+(f_p+g_p)t^p,$$

where $p = \max(m, n)$, and we define multiplication of the polynomials by

$$(f_0+f_1t+\cdots+f_mt^m)(g_0+g_1t+\cdots+g_nt^n) = f_0g_0+(f_0g_1+f_1g_0)t+\cdots+f_mg_nt^{m+n}.$$

With this addition and multiplication $A[t]$ becomes a ring which we call the *polynomial ring in the variable t* over A, or the *ring of polynomials in t with coefficients in A*.

Instead of introducing polynomials by the somewhat vague notion of *formal expressions* we can be more precise and define the polynomial ring as the set $A^{(\mathbf{N})}$

with *pointwise addition* and *convolution product.* That is, the sum of two functions $\varphi : \mathbf{N} \to A$ and $\chi : \mathbf{N} \to A$ is the function $\varphi + \chi : \mathbf{N} \to A$ defined by $(\varphi + \chi)(n) = \varphi(n) + \chi(n)$ for all !!n in $\mathbf{N}$, and their product !!$\varphi\chi : \mathbf{N} \to A$ is defined by $(\varphi\chi)(n) = \sum_{p+q=n} \varphi(p)\chi(q)$ for all n in $\mathbf{N}$. It is clear that $A^{(\mathbf{N})}$ with this addition and product is a ring. Let $t \in A^{(\mathbf{N})}$ be the function defined by $t(1) = 1$, and $t(n) = 0$ for $n \neq 1$. For each natural number n we have that $t^n(n) = 1$, and $t^n(m) = 0$ for $m \neq n$, where t^n is the convolution product of t with itself n times. It follows that every function $\varphi : \mathbf{N} \to A$ with finite support can be written uniquely as $\varphi = \sum_{n=0}^{\infty} \varphi(n)t^n$. Hence we have a bijection between $A^{(\mathbf{N})}$ and the polynomial ring $A[t]$ which maps $\sum_{n=0}^{\infty} \varphi(n)t^n$ considered as a function $\mathbf{N} \to A$, to the same element $\sum_{n=0}^{\infty} \varphi(n)t^n$ considered as a formal expression $A[t]$. Easy calculations show that this bijection is an isomorphism of rings.

(2.6) Power series rings. Let A be a ring. We define the sum of two elements in $A^{\mathbf{N}}$ pointwise. That is, the sum $\varphi + \chi$ of two elements $\varphi : \mathbf{N} \to A$ and $\chi : \mathbf{N} \to A$ is defined by $(\varphi + \chi)(n) = \varphi(n) + \chi(n)$ for all $n \in \mathbf{N}$. It is clear that under this addition $A^{\mathbf{N}}$ becomes an abelian group.

Moreover, we define a *convolution product*, by $\varphi\chi(n) = \sum_{p+q=n} \varphi(p)\chi(q)$. It is easily checked that the group $A^{\mathbf{N}}$ with this product becomes a ring. We denote this ring by !!$A[[t]]$. The ring $A[[t]]$ we call the *power series ring* in the variable t over A, or the *ring of power series* in the variable t with coefficients in A. We call the elements of $A[[t]]$ *power series* in the variable t.

Every element $\varphi : \mathbf{N} \to A$ in $A[[t]]$ is determined by the family $(\varphi(0), \varphi(1), \dots)$ of its values. In analogy with Example (2.5) we write

$$(f_0, f_1, f_2, \dots) = f_0 + f_1 t + f_2 t^2 + \cdots ,$$

where the expression $f_0 + f_1 t + f_2 t^2 + \cdots$ is just a formal way of writing the function $\varphi : \mathbf{N} \to A$ given by $\varphi(n) = f_n$. With this notation, addition and multiplication of power series take the form

$$(f_0 + f_1 t + f_2 t^2 + \cdots) + (g_0 + g_1 t + g_2 t^2 + \cdots) = (f_0 + g_0) + (f_1 + g_1)t + (f_2 + g_2)t^2 + \cdots$$

and

$$\begin{aligned}(f_0 + f_1 t + f_2 t^2 + \cdots)&(g_0 + g_1 t + g_2 t^2 + \cdots) \\ &= f_0 g_0 + (f_1 g_0 + f_0 g_1)t + (f_2 g_0 + f_1 g_1 + f_0 g_2)t^2 + \cdots\end{aligned}$$

analogously to the expressions in Example (2.5).

Let $t = t^1 : \mathbf{N} \to A$ be the map given by $t(1) = 1$ and $t(n) = 0$ when $n \neq 1$. Then $t^n(n) = 1$ and $t^n(m) = 0$ when $m \neq n$. Addition and multiplication on $A^{\mathbf{N}}$ clearly induces an addition and multiplication on the subset $A^{(\mathbf{N})}$. With this addition and multiplication the subgroup $A^{(\mathbf{N})}$ of $A^{\mathbf{N}}$ becomes the polynomial ring $A[t]$ as defined in Example (2.5). The elements of $A[t]$ are exactly the elements in $A[[t]]$ of the form $f_0 + f_1 t + f_2 t^2 + \cdots + f_m t^m$ for some elements $f_0, f_1, \dots, f_m$ in A. This explains the use of formal expansion $f_0 + f_1 t + f_2 t^2 + \cdots$ for power series.

(2.7) Polynomials in several variables. Let A be a ring. We defined in Example (?) the polynomial ring $B = A[u]$ in the variable u over A. Similarly we can define the ring $B[v] = A[u][v]$ in the variable v over B. Changing the notation we obtain in the same way a ring $A[v][u]$. It is clear that there is a bijection between $A[u][v]$ and $A[v][u]$ mapping u to v and v to u, and that this bijection preserves addition and multiplication. We therefore can identify the two rings and write $A[u, v] = A[u][v] = A[v][u]$. By induction on n we can thus define the polynomial ring $A[t_1, t_2, \dots, t_n]$ in the variables $t_1, t_2, \dots, t_n$ over A for every natural number n.

(2.8) Polynomials in an arbitrary set of variables. We can use the inductive procedure of Section (2.7) to define polynomial rings in any finite set of variables. In analogy with Section (2.5) we prefer however to define polynomial rings directly in terms of functions.

Let I be an index set. We define an additon on $\mathbf{N}^{(I)}$ pointwise, that is, the sum $\mu + \nu$ of two functions !!$\mu : I \to \mathbf{N}$ and !!$\nu : I \to \mathbf{N}$ is defined by $(\mu + \nu)(\alpha) = \mu(\alpha) + \nu(\alpha)$ for all $\alpha \in I$. We consider I as a subset of $\mathbf{N}^{(I)}$ identifying $\alpha \in I$ with the function that maps α to 1 and all other elements in I to 0, and we write 0 for the element of $\mathbf{N}^{(I)}$ that maps all $\alpha \in I$ to 0.

Let $A[t_\alpha]_{\alpha \in I}$ be the set of maps $\mathbf{N}^{(I)} \to A$ with finite support. We define addition on $A[t_\alpha]_{\alpha \in I}$ *pointwise* and multiplication by *convolution*. That is the sum and product $f + g$, respectively fg, of two functions !!$f : \mathbf{N}^{(I)} \to A$ and !!$g : \mathbf{N}^{(I)} \to A$ with finite support is given by:

$$(f+g)(\mu) = f(\mu) + g(\mu), \quad \text{and} \quad (fg)(\mu) = \sum_{\mu = \nu + \pi} f(\nu) g(\pi)$$

for all $\mu \in \mathbf{N}^{(I)}$. It is clear that $A[t_\alpha]_{\alpha \in I}$ becomes a ring with this addition and multiplication. The unit element 1 is defined by $1(\mu) = 1$ if $\mu = 0$ and $1(\mu) = 0$ if $\mu \neq 0$. We call this ring the *polynomial ring in the variables* t_α *over* A, or the *ring of polynomials in the variables* t_α with coefficients in A.

For every $\alpha \in I$ we have a map $t_\alpha : \mathbf{N}^{(I)} \to A$ defined by $t_\alpha(\alpha) = 1$ and $t_\alpha(\mu) = 0$ for $\mu \neq \alpha$. We have for every integer n_α that $t_\alpha^{n_\alpha}(\beta) = 0$ when $\beta \neq n_\alpha \alpha$ and $t_\alpha^{n_\alpha}(n_\alpha \alpha) = 1$, where $t_\alpha^{n_\alpha}$ is the product of t_α with itself n_α times in the convolution product. For every $\mu \in \mathbf{N}^{(I)}$ we let !!$t^\mu = \prod_{\alpha \in I} t_\alpha^{\mu(\alpha)} = \prod_{\alpha \in J} t^{\mu(\alpha)}$ for every finite subset J of I such that $\mu(\alpha) = 0$ for $\alpha \in I \setminus J$. Then $t^\mu(\nu)$ is equal to 1 when $\nu = \sum_{\alpha \in I} \mu(\alpha)\alpha$ and otherwise is equal to 0. Consequently we have that every element $f : \mathbf{N}^{(I)} \to A$ in $A[t_\alpha]_{\alpha \in I}$ with finite support can be written uniquely in the form $f = \sum_{\mu \in \mathbf{N}^{(I)}} f(\mu) t^\mu$.

(2.9) Remark. When $I = \{1, 2, \dots, n\}$ we have that Example (2.7) and Example (2.8) give the same ring $A[t_\alpha]_{\alpha \in I} = A[t_1, t_2, \dots, t_n]$.

(2.10) Definition. A *subring* B of a ring A is a subgroup of A that contains the unit 1_A of A, with the property that for every pair f, g of elements in B the product fg is in B.

(2.11) Remark. A *subring* B of A is a ring under the addition and multiplication induced by the addition and multiplication of A, and with the same unit as that of A.

(2.12) Example. Each ring in the sequence $\mathbf{Z} \subseteq \mathbf{Q} \subseteq \mathbf{R} \subseteq \mathbf{C}$ is a subring of the following ring.

(2.13) Example. The polynomial ring $A[t]$ in the variable t over the ring A is a subring of the power series ring $A[[t]]$ in the variable t over A when we identify a polynomial $f_0+f_1t+\cdots f_nt^n$ in $A[t]$ with the power series $f_0+f_1t+\cdots f_nt^n+0t^{n+1}+0t^{n+2}+\cdots$.

n **(2.14) Definition.** A *ring homomorphism*, or a *homomorphism of rings*, !!$\varphi : A \to B$ from a ring A to a ring B is a map such that, for all elements f, g in A, the following properties hold:

(1) $\varphi(f+g) = \varphi(f) + \varphi(g)$.
(2) $\varphi(fg) = \varphi(f)\varphi(g)$.
(3) $\varphi(1) = 1$.

A homomorphism φ is an *isomorphism* if it has an inverse. Equivalently a homomorphism is an isomorphism if it is bijective. The inverse is then automatically a ring homomorphism.

n **(2.15) Remark.** Let $\varphi : A \to B$ and !!$\chi : B \to C$ be homomorphism of rings. Then id_A and $\chi\varphi : A \to C$ are ring homomorphisms. In other words, the rings and their homomorphisms form a category. We call this category the *category of rings*.

→ **(2.16) Example.** The homomorphism $u_{\mathbf{Z}/n\mathbf{Z}} : \mathbf{Z} \to \mathbf{Z}/n\mathbf{Z}$ defined in Example (2.3) is a ring homomorphism by definition.

(2.17) Example. Let A be a ring and g an element in A. The map $A[t] \to A$ which to a polynomial $f(t) = f_0 + f_1t + \cdots + f_nt^n$ associates the element $f(g) = f_0 + f_1g + \cdots + f_ng^n$ is a ring homomorphism.

(2.18) Definition. An element f in a ring A is a *zero divisor* if $f \neq 0$ and there is an element $g \neq 0$ in A such that $fg = 0$. A ring where all elements are non-zero divisors is called an *integral domain*.

A *nilpotent* element in A is an element f such that $f^n = 0$ for some natural number n. We call the ring *reduced* when it has no non-zero nilpotent elements.

We call an element f in A a *unit*, or an *invertible element*, if there is an element g in A such that $fg = 1$. The element g is unique, for if g' is another element such that $fg' = 1$ then $g' = g'(fg) = (g'f)g = g$. We call the element $g = g'$ the *inverse*
n of the element f and denote it by !!f^{-1}.

If $1 \neq 0$ in A and all non-zero elements in A are units we call A a *field*.

(2.19) Example. Let p and q be prime numbers. In the ring $\mathbf{Z}/(p^n)$ all the elements in the ideal $(p)/(p^n)$ are nilpotent. All other elements are invertible. This is because, for any integer m, we can find integers r and s such that $rm + sp = 1$. We have that $\mathbf{Z}/(p)$ is a field.

In the ring $\mathbf{Z}/(pq)$ all non zero elements in the ideals $(p)/(pq)$ and $(q)/(pq)$ are zero divisors. The ring is reduced. All the elements that are not contained in the ideal $(p)/(pq)$ or $(q)/(pq)$ are invertible. This is because, for any integer m, we can find integers r and s such that $rm + spq = 1$.

(2.20) Example. Let $f(t) = f_0 + f_1 t + \cdots + f_n t^n$ be a polynomial in the variable t with coefficients in the ring A. When $f(t)$ is a zero-divisor in $A[t]$ there is an element $h \in A$ such that $hf(t) = 0$. In order to prove this we let $g(t) = g_0 + g_1 t + \cdots + g_p t^p$ be a non-zero polynomial of minimal degree in t with coefficients in A such that $f(t)g(t) = 0$. We first prove by descending induction on q that $f_q g(t) = 0$ for $q = 0, 1, \dots, n$. Assume that we have shown that $f_{q+1} g(t) = f_{q+2} g(t) = \cdots = f_n g(t) = 0$ for some integer q satisfying $0 \leq q \leq n$. Then we have that $f(t)g(t) = (f_0 + f_1 t + \cdots + f_q t^q)g(t) = 0$. However, then we have that $f_q g_p = 0$. In particular we have that $f_q g(t)$ is of degree at most $p - 1$ and $f_q g(t) f(t) = 0$. It follows from the minimality of the degree of $g(t)$ that $f_q g(t) = 0$. We have thus proved that $f_q g(t) = 0$ for $q = 0, 1, \dots, n$. Consequently we have that $f_q g_p = 0$ for $q = 0, 1, \dots, n$, and consequently that $g_p f(t) = 0$.

In particular we have that if $f(t)$ has one coefficient that is not a zero-divisor in A then $f(t)$ is not a zero divisor in $A[t]$.

We can generalize the above Example to several variables. Let $A[t_1, t_2, \dots, t_n]$ be the ring of polynomials in the independent variables $t_1, t_2, \dots, t_n$ with coefficients in A. When $f(t_1, t_2, \dots, t_n)$ is a zero-divisor in $A[t_1, t_2, \dots, t_n]$ there is an element h in A such that $hf(t_1, t_2, \dots, t_n) = 0$. In fact, write $f(t_1, t_2, \dots, t_n) = \sum_{\mu \in \mathcal{I}} f_\mu t_1^{\mu(1)} t_2^{\mu(2)} \cdots t_n^{\mu(n)}$ with f_μ in A, where $\mathcal{I}$ is a finite subset of $\mathbf{N}^n$. We choose an integer m which is strictly larger than all the coordinates $\mu(1), \mu(2), \dots, \mu(n)$ of μ, for all μ in $\mathcal{I}$. Then all the numbers $\mu(1) + \mu(2)m + \cdots + \mu(n)m^{n-1}$ for μ in $\mathcal{I}$ are different. We have that

$$f(t, t^m, \dots, t^{m^{n-1}}) = \sum_{\mu \in \mathcal{I}} f_\mu t^{\mu(1) + \mu(2)m + \cdots + \mu(n)m^{n-1}}$$

and we can use the first part of the example to conclude that there is an element h in A such that $hf_\mu = 0$ for all $\mu \in \mathcal{I}$. Hence we have that $hf(t_1, t_2, \dots, t_n) = \sum_{\mu \in \mathcal{I}} h f_\mu t_1^{\mu(1)} t_2^{\mu(2)} \cdots t_n^{\mu(n)} = 0$ as asserted.

(2.21) Operations on rings. Let $\{A_\alpha\}_{\alpha \in I}$ be a collection of rings. As we saw in
→ (?) the product $\prod_{\alpha \in I} A_\alpha$ is in a natural way an abelian group. We define a product on $\prod_{\alpha \in I} A_\alpha$ by $(f_\alpha)_{\alpha \in I} (g_\alpha)_{\alpha \in I} = (f_\alpha g_\alpha)_{\alpha \in I}$. With this multiplication the product $\prod_{\alpha \in I} A_\alpha$ becomes a ring.

(2.22) Remark. The product $\prod_{\alpha \in I} A_\alpha$ of a collection of rings $\{A_\alpha\}_{\alpha \in I}$ is a product in the *categorical sense*. In other words $\prod_{\alpha \in I} A_\alpha$ is the product in the category of rings.

(2.23) Notation. Let A be a ring and let $\{f_\alpha\}_{\alpha \in I}$ be a collection of elements in
n A. For every element $\mu \in \mathbf{N}^{(I)}$ we write !!$f^\mu = \prod_{\alpha \in I} f_\alpha^{\mu(\alpha)} = \prod_{\alpha \in J} f_\alpha^{\mu(\alpha)}$ for every finite subset J of I such that $\mu(\alpha) = 0$ for all $\alpha \in I \setminus J$.

(2.24) Exercises.

1. Let A be a ring. Determine all ring homomorphisms $\mathbf{Z} \to A$.

2. Determine the units in the ring $\mathbf{Z}$ of integers.

3. Let A be a ring and let $\mathcal{P}(A^A)$ be the subset of A^A of *polynomial maps*, that is maps $\varphi : A \to A$ with the property that there exists a natural number n and elements $f_0, f_1, \dots, f_n$ in A such that $\varphi(g) = f_0 + f_1 g + \cdots + f_n g^n$ for all g in A.

(1) Show that the ring structure of A^A with pointwise addition and multiplication induces a ring structure on $\mathcal{P}(A^A)$.

(2) Show that there is a natural surjective ring homomorphism $A[x] \to \mathcal{P}(A^A)$ sending x to the identity map $A \to A$.

$\to$ (3) Give an example where the map of part (2) is not an isomorphism.

4. Show that there are no ring homomorphisms $\mathbf{C} \to \mathbf{R}$ from the complex to the real numbers.

5. Show that the rational numbers $\mathbf{Q}$, the real numbers $\mathbf{R}$, and the complex numbers $\mathbf{C}$ are fields.

n **6.** Let !K! be a field.

(1) Determine all the units in the polyomial ring $K[t]$ in the variable t over K.

(2) Show that every homomorphism $K \to A$ of rings is injective.

7. Let $\{A_\alpha\}_{\alpha \in I}$ be a family of rings. Moreover, let $\oplus_{\alpha \in I} A_\alpha$ be the group which is the direct sum of the rings A_α considered as groups.

(1) Show that the multiplication on $\prod_{\alpha \in I} A_\alpha$ induces a multiplication on the sum $\oplus_{\alpha \in I} A_\alpha$.

(2) Show that $\oplus_{\alpha \in I} A_\alpha$ with addition and multiplication induced from $\prod_{\alpha \in I} A_\alpha$ is not a ring.

8. Let $\{A_\alpha, \rho_\beta^\alpha\}_{\alpha,\beta \in I, \alpha \le \beta}$ be an inductive system of rings such that the maps ρ_β^α are ring homomorphisms for all $\alpha \le \beta$.

(1) Show that the group $\varinjlim_{\alpha \in I} A_\alpha$ has a unique product that makes the group $\varinjlim_{\alpha \in I} A_\alpha$ into a ring, in such a way that the canonical maps $\rho_\alpha : A_\alpha \to \varinjlim_{\alpha \in I} A_\alpha$ are ring homomorphisms.

(2) Let $\{B_\alpha, \sigma_\beta^\alpha\}_{\alpha,\beta \in I, \alpha \le \beta}$ be another inductive system of rings, and let $\varphi_\alpha : A_\alpha \to B_\alpha$ for $\alpha \in I$ be a map of inductive systems, where each φ_α is a ring homomorphism.

Show that the resulting map $\varinjlim_{\alpha \in I} \varphi_\alpha : \varinjlim_{\alpha \in I} A_\alpha \to \varinjlim_{\alpha \in I} B_\alpha$ is a homomorphism of rings.

9. Let G be an abelian group. We define an addition on the set $\mathbf{Z} \times G$ by $(m, x) + (n, y) = (m + n, x + y)$, and a multiplication by $(m, x)(n, y) = (mn, nx + my)$.

(1) Show that $\mathbf{Z} \times G$ with this addition and multiplication is a ring.
(2) Find all the zero divisors in $\mathbf{Z} \times G$.
(3) Find all the nilpotent elements of $\mathbf{Z} \times G$.

10. Let p be prime number and let $\mathbf{Z}_{(p)}$ be all rational numbers of the form m/n
n such that p does not divide n. Show that !$\mathbf{Z}_{(p)}$! is a ring.

11. Let A be a ring and let f, g be elements of A.

(1) Assume that f is a unit and that g is nilpotent. Show that the element $f + g$ is a unit.
(2) Assume that f and g are nilpotent. Show that the element $f + g$ is nilpotent.

3. Algebras.

(3.1) Definition. Let A be a ring. An A-*algebra* is a ring B with a fixed ring homomorphism $\varphi : A \to B$. For simplicity we often say that $\varphi : A \to B$ is an A-algebra. In cases when it is unnecessary to refer explicitely to φ, it is more convenient to say that B is an A-*algebra*. When φ is injective we often identify A with its image.

An A-algebra $\psi : A \to C$ is a *subalgebra* an A-algebra $\varphi : A \to B$ if C is a subring of B and $\psi(f) = \varphi(f)$ for all f in A.

Let $\varphi : A \to B$ and $\chi : A \to C$ be two A-algebras. An A-*algebra homomorphism*, or a *homomorphism* of A-algebras, is a ring homomorphism $\psi : B \to C$ such that $\chi = \psi\varphi$.

(3.2) Example. All rings A are $\mathbf{Z}$-algebras under the unique homomorphism $\mathbf{Z} \to A$ that maps n to the sum $n1_A$ of the unit of A with itself n times. A ring homomorphism is the same as a $\mathbf{Z}$-algebra homomorphism.

(3.3) Example. The polynomial ring $A[t_\alpha]_{\alpha\in I}$ in the variables t_α over A, for α in an index set I, is canonically an A-algebra under the ring homomorphism $\varphi : A \to A[t_\alpha]_{\alpha\in I}$ that maps $f \in A$ to the *constant polynomial* f. That is, to the polynomial $\mathbf{N}^{(I)} \to A$ mapping 0 to f and the other $\mu \in \mathbf{N}^{(I)}$ to 0. The homomorphism φ is clearly injective and we identify A with its image.

(3.4) Example. The power series in t with coefficients in A is canonically an A-algebra under the ring homomorphism $\varphi : A \to A[[t]]$ that maps $f \in A$ to the *constant power series*. That is, to the power series that maps 0 to f and all the other natural numbers to 0. The homomorphism φ is clearly injective. We identify A with its image.

→ With the ideantification of Example (2.?) we have that $A[t]$ is a subalgebra of $A[[t]]$.

(3.5) Notation. Let $\varphi : A \to B$ be an A-algebra. When the reference to φ is clear we write $fg = \varphi(f)g$ in B when $f \in A$ and $g \in B$.

(3.6) Proposition. *Let $\varphi : A \to B$ be an A-algebra, and let $\{h_\alpha\}_{\alpha\in I}$ be a collection of elements of B. Then there is a unique homomorphism*

$$A[t_\alpha]_{\alpha\in I} \to B$$

of A-algebras that maps t_α to h_α for all $\alpha \in I$.

Proof. Since every element f in $A[t_\alpha]_{\alpha\in I}$ can be written in a unique way in the form $f(t) = \sum_{\mu\in\mathbf{N}^{(I)}} f(\mu)t^\mu$, where only a finite number of the $f(\mu)$ are different from zero and where $t^\mu = \prod_{\alpha\in I} t_\alpha^{\mu(\alpha)}$, it is clear that an A-algebra homomorphism $\varphi : A[t_\alpha]_{\alpha\in I} \to B$ is uniquely determined by the equations $\varphi(t_\alpha) = h_\alpha$ for all α in I. Moreover it follows that we can define a map $\varphi : A[t_\alpha]_{\alpha\in I} \to B$ of sets by

$\varphi(f) = \sum_{\mu \in \mathbf{N}^{(I)}} f(\mu)h^\mu$, where $h^\mu = \prod_{\alpha \in I} h_\alpha^{\mu(\alpha)}$. This map satisfies the relation $\varphi(t^\mu) = h^\mu$ for all $\mu \in \mathbf{N}^{(I)}$, and consequently we have that $\varphi(t_\alpha) = h_\alpha$ for all $\alpha \in I$.

It remains to prove that φ is a homomorphism of A-algebras. To this end let $f(t) = \sum_{\mu \in \mathbf{N}^{(I)}} f(\mu)t^\mu$ and $g(t) = \sum_{\mu \in \mathbf{N}^{(I)}} g(\mu)t^\mu$ be elements of $A[t_\alpha]_{\alpha \in I}$, and let e be an element of A. The following three sets of equalities express that φ is a homomorphism of A-algebras:

$$\begin{aligned}\varphi(f+g) &= \varphi\Big(\sum_{\mu \in \mathbf{N}^{(I)}} f(\mu)t^\mu + \sum_{\mu \in \mathbf{N}^{(I)}} g(\mu)t^\mu\Big) = \varphi\Big(\sum_{\mu \in \mathbf{N}^{(I)}} (f(\mu)+g(\mu))t^\mu\Big) \\ &= \sum_{\mu \in \mathbf{N}^{(I)}} (f(\mu)+g(\mu))h^\mu = \sum_{\mu \in \mathbf{N}^{(I)}} f(\mu)h^\mu + \sum_{\mu \in \mathbf{N}^{(I)}} g(\mu)h^\mu = \varphi(f) + \varphi(g),\end{aligned}$$

$$\begin{aligned}\varphi(ef) &= \varphi\Big(e\sum_{\mu \in \mathbf{N}^{(I)}} f(\mu)t^\mu\Big) = \varphi\Big(\sum_{\mu \in \mathbf{N}^{(I)}} ef(\mu)t^\mu\Big) \\ &= \sum_{\mu \in \mathbf{N}^{(I)}} ef(\mu)h^\mu = e\sum_{\mu \in \mathbf{N}^{(I)}} f(\mu)h^\mu = e\varphi(f),\end{aligned}$$

$$\begin{aligned}\varphi(fg) &= \varphi\Big(\sum_{\mu=\nu+\pi} f(\nu)g(\pi)t^\mu\Big) \\ &= \sum_{\mu=\nu+\pi} f(\nu)g(\pi)h^\mu = \sum_{\nu \in \mathbf{N}^{(I)}} f(\nu)h^\nu \sum_{\pi \in \mathbf{N}^{(I)}} g(\pi)h^\pi = \varphi(f)\varphi(g).\end{aligned}$$

(3.7) Remark. Let B be an A-algebra via the homomorphism $\varphi : A \to B$, and let $\{g_\alpha\}_{\alpha \in I}$ be a collection of elements in B. It follows from Proposition (3.6) that we have a unique homomorphism $\varphi : A[t_\alpha]_{\alpha \in I} \to B$ defined by $\varphi(t_\alpha) = g_\alpha$. Since $A[t_\alpha]_{\alpha \in I}$ is an A-algebra it follows that the image of φ is an A-algebra. The image consists of all elements in B of the form $\sum_{\mu \in \mathbf{N}^{(I)}} f_\mu g^\mu$ with $f_\mu \in A$ and where only a finite number of the f_μ are non-zero.

(3.8) Definition. Let B be an A-algebra via the homomorphism $\varphi : A \to B$, and let $\{g_\alpha\}_{\alpha \in I}$ be a collection of elements in B. We denote by $A[g_\alpha]_{\alpha \in I}$ the A-algebra in B consisting of the elements of the form $\sum_{\mu \in \mathbf{N}^{(I)}} f_\mu g^\mu = \sum_{\mu \in \mathbf{N}^{(I)}} \varphi(f_\mu)g^\mu$ with $f_\mu \in A$, and where only a finite number of the f_μ are different from 0. The algebra $A[g_\alpha]_{\alpha \in I}$ is called the A-algebra *generated by the elements* $\{g_\alpha\}_{\alpha \in I}$. When $B = A[g_\alpha]_{\alpha \in I}$ we say that B is *generated by the elements* $\{g_\alpha\}_{\alpha \in I}$, and the elements g_α are called the *generators* of B as an A-algebra. We say that B is a *finitely generated* A-algebra, or that the homomorphism φ is of *finite type*, if $B = A[g_1, g_2, \dots, g_n]$ for some elements $g_1, g_2, \dots, g_n$ of B.

(3.9) Example. Let A be a ring and $A[t_\alpha]_{\alpha \in I}$ the polynomial ring in the variables $\{t_\alpha\}_{\alpha \in I}$ over A. Then $A[t_\alpha]_{\alpha \in I}$ is an A-algebra which is generated by the variables t_α for $\alpha \in I$. It is finitely generated if exactly when the set I is finite.

(3.10) Exercises.

1. Let B and C be A-algebras and $\varphi : B \to C$ a surjective map of A-algebras. Show that when B is of finite type then C is of finite type.

2. Let B be a finitely generated A-algebra and C a finitely generated B-algebra. Show that C is a finitely generated A-algebra.

3. Show that an A-algebra B is of finite type if and only if there is a surjective map $A[t_1, t_2, \ldots, t_n] \to B$ of A-algebras.

4. Show that $\mathbf{Q}$ is not of finite type as a $\mathbf{Z}$-algebra.

n **5.** Let $A[u, v]$ be a polynomial ring in the variables $!u, v!$ over the ring A.

(1) Show that all the elements of $A[u, v]$ of the form $\sum_{m \leq n} f_{mn} u^m u^n$ with $f_{mn} \in A$ and where only a finite number of of the f_{mn} are non-zero, form an A-algebra.

$\rightarrow$ (2) Is the A-algebra of part (1) of finite type?

4. Ideals.

n **(4.1) Definition.** An *ideal* !$\mathfrak{a}$! in a ring A is a subgroup of A such that for all elements f in A and g in $\mathfrak{a}$ we have that $fg \in \mathfrak{a}$.

(4.2) Notation. Let $\{f_\alpha\}_{\alpha\in I}$ be a family of elements f_α in the ring A. We denote
n by !!$(f_\alpha)_{\alpha\in I} = \sum_{\alpha\in I} Af_\alpha$ the subset of A consisting of all sums

$$\sum_{\beta\in J} g_\beta f_\beta$$

for all *finite* subsets J of I with $g_\beta \in A$. It is clear that $(f_\alpha)_{\alpha\in I} = \sum_{\alpha\in I} Af_\alpha$ is an ideal in A, and that it is the smallest ideal of A that contains the elements f_α for all $\alpha \in I$.

When there is only one element f in the family we have that $(f) = Af$, the set of all elements gf with g in A.

(4.3) Definition. We call $(f_\alpha)_{\alpha\in I} = \sum_{\alpha\in I} Af_\alpha$ the *ideal generated by the elements* f_α *for* α *in* I. The ideals of the form Af are called *principal ideals.*

(4.4) Example. The kernel $\mathrm{Ker}(\varphi) = \{f \in A : \varphi(f) = 0\}$ of a ring homomorphism $\varphi : A \to B$ is an ideal in A.

(4.5) Residue class rings. Let $\mathfrak{a}$ be an ideal in the ring A. The residue class
~~n~~ group $A/\mathfrak{a}$ of (?) has a unique multiplication that makes !!$A/\mathfrak{a}$ into a ring in such a
n way that the canonical homomorphism!!

$$\varphi_{A/\mathfrak{a}} : A \to A/\mathfrak{a}$$

becomes a homomorphism of rings. This multiplication is defined by the equalities $\varphi_{A/\mathfrak{a}}(f)\varphi_{A/\mathfrak{a}}(g) = \varphi_{A/\mathfrak{a}}(fg)$ for all elements f, g in A.

(4.6) Lemma. *Let* $\mathfrak{a}$ *be an ideal in the ring* A *and let* $\varphi_{A/\mathfrak{a}} : A \to A/\mathfrak{a}$ *be the*
n *canonical homomorphism. The image* $\varphi_{A/\mathfrak{a}}(\mathfrak{b})$ *of an ideal* !!$\mathfrak{b}$ *that contains* $\mathfrak{a}$ *is an ideal in* $A/\mathfrak{a}$. *Moreover the correspondence that maps* $\mathfrak{b}$ *to* $\varphi_{A/\mathfrak{a}}(\mathfrak{b})$ *gives a bijection between ideals in* A *that contain* $\mathfrak{a}$ *and the ideals in* $A/\mathfrak{a}$.

Proof. It is clear that $\varphi_{A/\mathfrak{a}}(\mathfrak{b})$ is an ideal in $A/\mathfrak{a}$ and that $\varphi_{A/\mathfrak{a}}^{-1}(\varphi_{A/\mathfrak{a}}(\mathfrak{b})) \supseteq \mathfrak{b}$. Moreover it is clear that the inverse image by $\varphi_{A/\mathfrak{a}}$ of an ideal in $A/\mathfrak{a}$ is an ideal in A that contains $\mathfrak{a}$.

To prove that the correspondence of the Lemma is a bijection it therefore suffices to prove the inclusion $\varphi_{A/\mathfrak{a}}^{-1}(\varphi_{A/\mathfrak{a}}(\mathfrak{b})) \subseteq \mathfrak{b}$. Let $f \in \varphi_{A/\mathfrak{a}}^{-1}(\varphi_{A/\mathfrak{a}}(\mathfrak{b}))$. Then we have that $\varphi_{A/\mathfrak{a}}(f) = \varphi_{A/\mathfrak{a}}(g)$ for some $g \in \mathfrak{b}$. Consequently we have that $f - g$ is in the kernel $\mathfrak{a}$ of $\varphi_{A/\mathfrak{a}}$, that is $f = g + h$ for some $h \in \mathfrak{a}$. Since both g and h are in $\mathfrak{b}$ we have that $f \in \mathfrak{b}$ as we wanted to prove.

(4.7) Operations on ideals. Let $\{\mathfrak{a}_\alpha\}_{\alpha\in I}$ be a family of ideals in the ring A. The intersection $\cap_{\alpha\in I}\mathfrak{a}_\alpha$ of the ideals $\mathfrak{a}_\alpha$ is an ideal in A. The smallest subgroup $\sum_{\alpha\in I}\mathfrak{a}_\alpha$ of A that contains all the groups $\mathfrak{a}_\alpha$ is clearly an ideal of A. It is the smallest ideal containing all the ideals $\mathfrak{a}_\alpha$, and it consists of the elements of the form $\sum_{\beta\in J} f_\beta$ with $f_\beta \in \mathfrak{a}_\beta$ for all *finite* subsets J of I.

Let $\mathfrak{a}_1, \mathfrak{a}_2, \dots, \mathfrak{a}_n$ be ideals in A. We denote by $\mathfrak{a}_1\mathfrak{a}_2\cdots\mathfrak{a}_n$ the smallest ideal containing all products of the form $f_1 f_2\cdots f_n$ with $f_i \in \mathfrak{a}_i$ for $i = 1, 2, \dots, n$. The ideal $\mathfrak{a}_1\mathfrak{a}_2\cdots\mathfrak{a}_n$ we call the *product* of the ideals $\mathfrak{a}_1, \mathfrak{a}_2, \dots, \mathfrak{a}_n$. It is clear that the product consists of all finite sums of elements of the form $f_1 f_2\cdots f_n$ with $f_i \in \mathfrak{a}_i$ for $i = 1, 2, \dots, n$.

(4.8) Remark. Let $f_1, f_2, \dots, f_m$ be nilpotent elements in the ring A, and let $\mathfrak{a}$ be the ideal generated by these elements. Then there is an integer n such that $\mathfrak{a}^n = 0$. This is because the elements of $\mathfrak{a}^n$ are sums of product of n elements on the form $g_1 f_1 + g_2 f_2 + \cdots + g_m f_m$ with $g_i \in A$. However, the product of n such elements is a sum of elements of the form $g f_1^{n_1} f_2^{n_2}\cdots f_m^{n_m}$ with $g \in A$ and $n_1 + n_2 + \cdots + n_m = n$. When we choose an integer p such that $f_i^p = 0$ for $i = 1, 2, \dots, m$ we have that $f_1^{n_1} f_2^{n_2}\cdots f_m^{n_m} = 0$ when $n = n_1 + n_2 + \cdots + n_m$ and $n \ge mp$. Consequently we obtain that $\mathfrak{a}^n = 0$ when $n \ge mp$.

n **(4.9) Definition.** An ideal !!$\mathfrak{p}$ in the ring A is a *prime ideal* if it is different from A, and if f, g are elements in A such that if $fg \in \mathfrak{p}$, then $f \in \mathfrak{p}$ or $g \in \mathfrak{p}$. Equivalently we have that an ideal $\mathfrak{p}$ is prime if for each pair of elements f, g in A that are not in $\mathfrak{p}$ we have that fg is not in $\mathfrak{p}$.

n An ideal !!$\mathfrak{m}$ of A is *maximal* if it is different from A and it is not contained in any ideal in A different from A and $\mathfrak{m}$. A ring with only one maximal ideal is called a *local ring*. We shall denote by $\mathfrak{m} = \mathfrak{m}_A$ the maximal ideal in a local ring A. A homomorphism $\varphi : A \to B$ of local rings is called *local* if it maps the maximal ideal in A to the maximal ideal in B, or equivalently if we have $\varphi^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$.

(4.10) Example. The prime ideals in $\mathbf{Z}$ are the ideals $(p) = p\mathbf{Z}$ generated by the prime numbers p, and the ideal (0). We have that the maximal ideals are those generated by the prime numbers.

(4.11) Example. Let p be a prime number and let $\mathbf{Z}_{(p)}$ be the rational numbers of the form m/n where p does not divide n. Then the ideal $\mathfrak{p} = p\mathbf{Z}_{(p)}$ generated by p is a prime ideal which is maximal, and (0) is a prime ideal. There are no other maximal ideals in $\mathbf{Z}_{(p)}$ because if $\mathfrak{q}$ is an ideal that is not contained in $\mathfrak{p}$ it must contain an element m/n such that p divides neither n nor m. However, then m/n is invertible in A and therefore generates the ideal A. Thus $\mathfrak{q} = A$. Hence $\mathbf{Z}_{(p)}$ is a local ring.

(4.12) Example. Let K be a field. The ideals of the form $(t_1, t_2, \dots, t_m) = \sum_{i=1}^m K[t_1, t_2, \dots, t_n]t_i$ of the polynomial ring $K[t_1, t_2, \dots, t_n]$ are prime ideals, and $(t_1, t_2, \dots, t_n)$ is maximal.

(4.13) Proposition. *Let $\mathfrak{a}$ be an ideal in A.*

(1) *The ideal $\mathfrak{a}$ is prime if and only if the ring $A/\mathfrak{a}$ is an integral domain.*
(2) *The ideal $\mathfrak{a}$ is maximal if and only if the ring $A/\mathfrak{a}$ is a field.*

Proof. (1) Let $\varphi_{A/\mathfrak{a}} : A \to A/\mathfrak{a}$ be the canonical homomorphism. Then f is not in
→ $\mathfrak{a}$ if and only if $\varphi_{A/\mathfrak{a}}(f)$ is not zero in $A/\mathfrak{a}$. Hence assertion (1) follows from the equality $\varphi_{A/\mathfrak{a}}(f)\varphi_{A/\mathfrak{a}}(g) = \varphi_{A/\mathfrak{a}}(fg)$ valid for all pairs of elements f, g of A.

(2) We have that $\mathfrak{a}$ is maximal if and only if $Af + \mathfrak{a} = A$ for all elements $f \notin \mathfrak{a}$. However the equality $Af + \mathfrak{a} = A$ is equivalent to the existence of an element g of A such that $gf + h = 1$ for some element $h \in \mathfrak{a}$, that is the existence of an element g such that $\varphi_{A/\mathfrak{a}}(g)\varphi_{A/\mathfrak{a}}(f) = 1$. Hence $\mathfrak{a}$ is maximal if and only if $\varphi_{A/\mathfrak{a}}(f)$ is a unit in $A/\mathfrak{a}$ for all $f \in A \setminus \mathfrak{a}$.

n **(4.14) Definition.** A non-empty subset !!S of a ring A is called *multiplicatively closed* if it contains 1 and for every pair s, t of elements in S the product st lies in S.

(4.15) Example. Let A be a ring and let $\mathfrak{p}$ be a prime ideal. Then $A \setminus \mathfrak{p}$ is a multiplicatively closed subset of A. Let $f \in A$. Then $\{1, f, f^2, \dots\}$ is a multiplicatively closed subset of A. We also have that the collection of all elements of A different from 0 that are not zero divisors is a multiplicatively closed subset of A.

(4.16) Lemma. *Let $\mathfrak{a}$ be an ideal in a ring A and let S be a multiplicatively closed subset of A that does not intersect $\mathfrak{a}$. Then there is a prime ideal $\mathfrak{p}$ in A that contains $\mathfrak{a}$ and that does not intersect S.*

Moreover, every ideal in the ring A which is different from A is contained in a maximal ideal.

n *Proof.* Let !!$\mathcal{I}$ be the set of all ideals in A that contain $\mathfrak{a}$ and that do not intersect S. We order the elements of $\mathcal{I}$ by inclusion. For every chain $\{\mathfrak{a}_\alpha\}_{\alpha\in I}$ of elements in $\mathcal{I}$ the union $\mathfrak{a} = \cup_{\alpha\in I}\mathfrak{a}_\alpha$ of the ideals $\mathfrak{a}_\alpha$ clearly is an ideal in A which is different from A and does not intersect S. Hence every chain in $\mathcal{I}$ has a maximal element. It follows from Zorns Lemma that the set $\mathcal{I}$ contains a maximal element $\mathfrak{p}$. When $S = \{1\}$ we have that $\mathfrak{p}$ is a maximal ideal in A.

It remains to prove that $\mathfrak{p}$ is a prime ideal. Let f and f' be elements in A that are not in $\mathfrak{p}$. We must show that ff' is not in $\mathfrak{p}$. Since $\mathfrak{p}$ is maximal in $\mathcal{I}$ the ideals $fA + \mathfrak{p}$ and $f'A + \mathfrak{p}$ of A both intersect S. Hence there are elements g, g' in A, h, h' in $\mathfrak{p}$, and s, s' in S such that $fg + h = s$ and $f'g' + h' = s'$. Since $ss' \in S$ the product $(fg + h)(f'g' + h')$ is not in $\mathfrak{p}$. Since h and h' both are in $\mathfrak{p}$, we therefore have that ff' can not be in $\mathfrak{p}$, as we wanted to prove.

(4.17) Remark. Let A be a ring and $\mathfrak{m}$ an ideal in A. Then A is a local ring with maximal ideal $\mathfrak{m}$ if and only if all elements in $A \setminus \mathfrak{m}$ are invertible in A.

It is clear that if all the elements of $A \setminus \mathfrak{m}$ are invertible then $\mathfrak{m}$ is maximal and A can not have other maximal ideals than $\mathfrak{m}$. Conversely if A is a local ring with maximal ideal $\mathfrak{m}$ and $f \in A\setminus\mathfrak{m}$, then f must be invertible. In fact, if f is not invertible
→ we can by Lemma (?) find a maximal ideal containing f, and thus different from $\mathfrak{m}$.

(4.18) Proposition. *Let $\mathfrak{a}$ be an ideal in the ring A. The intersection of all prime ideals that contain $\mathfrak{a}$ is equal to the set of nilpotent elements $\{f \in A : f^{n_f} \in \mathfrak{a}$ for some natural number $n_f\}$ of A.*

In particular the nilpotent elements of A form an ideal of A.

Proof. If $f^n \in \mathfrak{a}$ we have have that f is contained in all the prime ideals that contain $\mathfrak{a}$ and hence in the intersection of all these ideals.

It remains to prove that, if f is an element in A such that f^n is not in $\mathfrak{a}$ for any positive integer n, then f is not in some prime ideal containing $\mathfrak{a}$. However, when f^n is not in $\mathfrak{a}$ for any positive integer n the ideal $\mathfrak{a}$ does not intersect the multiplicatively closed subset $\{1, f, f^2, \dots\}$ of A. It follows from Lemma (4.2) that we can find a prime ideal $\mathfrak{p}$ of A containing $\mathfrak{a}$ that does not intersect S, as we wanted to prove.

n **(4.19) Definition.** Let A be a ring and let E be a subset of A. We write !!$\mathfrak{r}(E) = \mathfrak{r}_A(E) = \{f \in A : f^{n_f} \in E$ for some natural number $n_f\}$. Let $\mathfrak{a}$ be an ideal in A. The ideal $\mathfrak{r}(\mathfrak{a})$ we call the *radical* of $\mathfrak{a}$, and the ideal $\mathfrak{r}(0) = \mathfrak{r}_A(0)$ we call the *radical* of A.

(4.20) Example. When $\mathfrak{p}$ is a prime ideal of the ring A we have that $\mathfrak{r}(\mathfrak{p}^n) = \mathfrak{p}$ for all positive integers n. It is clear that $\mathfrak{p} \subseteq \mathfrak{r}(\mathfrak{p}^n)$. Conversely, if $f \notin \mathfrak{p}$ we have that $f^m \notin \mathfrak{p}$ for all natural numbers m. In particular $f^m \notin \mathfrak{p}^n$ for all natural numbers m. That is, we have that $f \notin \mathfrak{r}(\mathfrak{p}^n)$.

(4.21) Remark. Let $\mathfrak{a}$ be an ideal in the ring A and let $\varphi_{A/\mathfrak{a}} : A \to A/\mathfrak{a}$ be the canonical homomorphism. Then $\mathfrak{r}_A(\mathfrak{a}) = \varphi_{A/\mathfrak{a}}^{-1}(\mathfrak{r}_{A/\mathfrak{a}}(0))$. This is because for $f \in A$ we have that $f^n \in \mathfrak{a}$ if and only if $\varphi_{A/\mathfrak{a}}(f)^n = 0$.

(4.22) Proposition. *Let A be a ring and let $\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_n$ be prime ideals in A.*

(1) *If $\mathfrak{a}$ is an ideal of A such that $\mathfrak{a} \subseteq \cup_{i=1}^n \mathfrak{p}_n$, then $\mathfrak{a} \subseteq \mathfrak{p}_i$ for some i.*
(2) *If $\mathfrak{a}_1, \mathfrak{a}_2, \dots, \mathfrak{a}_n$ are ideals in A and $\mathfrak{p}$ is a prime ideal of A such that $\cap_{i=1}^n \mathfrak{a}_i \subseteq \mathfrak{p}$, then $\mathfrak{a}_i \subseteq \mathfrak{p}$ for some i.*

Proof. (1) We show assertion (1) by induction on n. For $n = 1$ the assertion is clear. Assume that assertion (1) holds for $n - 1$. If $\mathfrak{a} \subseteq \cup_{j \in J} \mathfrak{p}_i$ for some subset J of $\{1, 2, \dots, n\}$ with less than n elements assertion (1) follows by the induction hypothesis. We prove assertion (1) by showing that if $\mathfrak{a}$ is is not contained in the union of the ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_{i-1}, \mathfrak{p}_{i+1}, \dots, \mathfrak{p}_n$ for any i, then $\mathfrak{a}$ can not be contained in $\cup_{i=1}^n \mathfrak{p}_i$. In fact we can then find, for each i, an element $f_i \in \mathfrak{a} \cap \mathfrak{p}_i$ such that $f_i \notin \mathfrak{p}_j$ when $i \neq j$. Then

$$f_1 \cdots f_{i-1} f_{i+1} \cdots f_n = \begin{cases} \notin \mathfrak{p}_i & \\ \in \mathfrak{p}_j & j \neq i \end{cases}$$

Hence exactly one term in the sum $f = \sum_{i=1}^n f_1 \cdots f_{i-1} f_{i+1} \cdots f_n$ is not in $\mathfrak{p}_j$, and thus $f \notin \mathfrak{p}_j$ for $j = 1, 2, \dots, n$. Since $f \in \mathfrak{a}$ we can not have that $\mathfrak{a} \subseteq \cup_{i=1}^n \mathfrak{p}_i$.

(2) If $\mathfrak{a}_i \not\subseteq \mathfrak{p}$ for all i we can find an $f_i \in \mathfrak{a}_i \backslash \mathfrak{p}$ for $i = 1, 2, \dots, n$. Then we have that $f_1 f_2 \cdots f_n \in \mathfrak{a}_1 \mathfrak{a}_2 \cdots \mathfrak{a}_n \subseteq \cap_{i=1}^n \mathfrak{a}_i$. On the other hand we have that $f_1 f_2 \cdots f_n \notin \mathfrak{p}$, which contradicts the assumption that $\cap_{i=1}^n \mathfrak{a}_i \subseteq \mathfrak{p}$.

(4.23) Exercises.

1. Let K be a field and let $K[t]$ be the polynomial ring in the variable t over K.

(1) Find all non-zero divisors in the residue class ring $k[t]/(t^2)$.
(2) Find all the units in the residue class ring $k[t]/(t^2)$.

2. Let n be a positive integer.

(1) Determine for which integers n the ring $\mathbf{Z}/n\mathbf{Z}$ is an integral domain.
(2) Determine for which integers n the ring $\mathbf{Z}/n\mathbf{Z}$ is a field.

3. Let $\mathbf{Z}[t]$ be the polynomial ring in the variable t over $\mathbf{Z}$.

(1) Show that $(2, t^2 + 1)$ is a maximal ideal.
(2) Is the ideal $(t + 3, t^2 + 2)$ a maximal ideal?

4. Let K be a field, and let $K[u, v]$ be the polynomial ring in the independent variables u, v over K. Show that $K[u, v]/(v^2 - u^3)$ is an integral domain.

5. Let K be a field and let I be an infinite set.

(1) For each $\beta \in I$ we let $\mathfrak{a}_\beta = \{(f_\alpha)_{\alpha \in I} : f_\beta = 0\}$. Show that the ideals $\mathfrak{a}_\beta$ are maximal ideals in K^I.
(2) Show that there are other maximal ideals than the ideals $\mathfrak{a}_\beta$ for $\beta \in I$.

6. Let p be a prime number and let $\mathbf{Z}_{(p)}$ be the ring of rational numbers of the form m/n such that p does not divide n. Let $\mathfrak{m}$ be the maximal ideal $p\mathbf{Z}_{(p)}$ of $\mathbf{Z}_{(p)}$. Show that the residue ring $\mathbf{Z}_{(p)}/\mathfrak{m}$ is canonically isomorphic to $\mathbf{Z}/(p)$.

7. Let $A[t]$ be the polynomial ring in the variable t with coefficient in A and let $\mathfrak{p}$ be a prime ideal in A. Show that the set $\mathfrak{p}A[t]$ of all polynomials with coefficietns in $\mathfrak{p}$ form a prime ideal in $A[t]$.

8. Let $K[u, v]$ be the polynomial ring in the two variables u and v over the field K. Is the union $(u) \cup (v)$ of the two ideals (u) and (v) of $K[u, v]$ an ideal?

9. Show that if A is a ring such that $1 \neq 0$ then A has minimal prime ideals.

10. Let K be a field an let $K[t]$ be the ring of polynomials in the variable t over K. Moreover, let $A = K[t]/(t^2(t-1)^3)$.

(1) Determine the radical of A.
(2) Determine the prime ideals of A whose intersection is the radical of A.

11. Let K be a field and let $K[u, v]$ be the ring of polynomials in the variables u, v with coefficients in K. Moreover, let $A = K[u, v]/(u^2, uv)$.

(1) Find the radical of A.
(2) Determine the prime ideals of A whose intersection is the radical of A.

12. Let A be a ring and let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals in A.

(1) Show that if $\mathfrak{a} \subseteq \mathfrak{b}$ then $\mathfrak{r}(\mathfrak{a}) \subseteq \mathfrak{r}(\mathfrak{b})$.
(2) Show that $\mathfrak{r}(\mathfrak{r}(\mathfrak{a})) = \mathfrak{r}(\mathfrak{a})$.

13. Let A be a ring. Show that the following assertions are equivalent:

(1) Every element in A is either a unit or is nilpotent.
(2) The ring A has exactly one prime ideal.
(3) The ring $A/\mathfrak{r}(0)$ is a field.

14. Let $D[u, v, w]$ be the polynomial ring in the variables u, v, w over a field K. Moreover, let $\mathfrak{a} = (u, v)$ and $\mathfrak{b} = (v, w)$. Is the set $\{fg : f \in \mathfrak{a}, g \in \mathfrak{b}\}$ an ideal?

15. Let A be a local ring. Show that if f is a non-zero element in A that is *nilpotent*, that is $f^2 = f$, then f is a unit in A.

5. The Zariski topology.

(5.1) Notation. Let A be a ring. We denote by !!Spec(A) the set consisting of all prime ideals in A. For every subset E of A we denote by !!$V(E)$ the subset of Spec(A) consisting of all prime ideals that contain E.

Let f be an element of A. We write !!$D(f) = \operatorname{Spec}(A) \setminus V(f)$ for the set of prime ideals not containing f.

It is often useful to distiguish between the prime ideals of A and the elements of the set Spec(A). We therefore denote by !!$\mathfrak{j}_x$ the prime ideal corresponding to the element x of Spec(A).

(5.2) Remark. Let E and F be subsets of a ring A.

(1) If $E \subseteq F$ then $V(F) \subseteq V(E)$.

(2) Let $\mathfrak{a}$ be the ideal generated by the elements of E. Then $V(\mathfrak{a}) = V(E)$.

The inclusion $V(\mathfrak{a}) \subseteq V(E)$ follows from the first Remark. To prove the opposite inclusion we take a prime ideal $\mathfrak{p}$ that does not contain $\mathfrak{a}$. Then it can not contain all elements of E. Consequently $\mathfrak{p}$ does not contain E, and $V(E) \subseteq V(\mathfrak{a})$ as we claimed.

(5.3) Proposition. *Let A be a ring.*

(1) *We have that $V(0) = \operatorname{Spec}(A)$, and $V(1) = \emptyset$.*

(2) *For every collection $\{\mathfrak{a}_\alpha\}_{\alpha \in I}$ of ideals $\mathfrak{a}_\alpha$ in A we have that*

$$\bigcap_{\alpha \in I} V(\mathfrak{a}_\alpha) = V\Big(\sum_{\alpha \in I} \mathfrak{a}_\alpha\Big).$$

(3) *Let $\mathfrak{a}_1, \mathfrak{a}_2, \dots, \mathfrak{a}_n$ be ideals in A. Then*

$$V(\mathfrak{a}_1) \cup V(\mathfrak{a}_2) \cup \cdots \cup V(\mathfrak{a}_n) = V(\mathfrak{a}_1 \mathfrak{a}_2 \cdots \mathfrak{a}_n).$$

Proof. (1), (2) The assertions (1) and (2) are easily checked.

(3) If we can prove the third assertion for $n = 2$, we can prove it for any n by induction on n. To prove the assertion for $n = 2$ we observe that the inclusion $V(\mathfrak{a}_1) \cup V(\mathfrak{a}_2) \subseteq V(\mathfrak{a}_1\mathfrak{a}_2)$ is obvious. We shall prove the opposite inclusion. Let $\mathfrak{p}$ be a prime ideal that contains neither $\mathfrak{a}_1$ nor $\mathfrak{a}_2$. Then we can find elements $f_1 \in \mathfrak{a}_1 \setminus \mathfrak{p}$ and $f_2 \in \mathfrak{a}_2 \setminus \mathfrak{p}$. Since the ideal $\mathfrak{p}$ is prime it follows that $f_1 f_2$ is not in $\mathfrak{p}$. Hence $\mathfrak{p}$ does not contain the ideal $\mathfrak{a}_1\mathfrak{a}_2$ and we have proved that $V(\mathfrak{a}_1\mathfrak{a}_2) \subseteq V(\mathfrak{a}_1) \cup V(\mathfrak{a}_2)$.

(5.4) The Zariski topology. It follows from Proposition (5.3) that the collection of subsets $X \setminus V(\mathfrak{a})$ of Spec(A) for all ideals $\mathfrak{a}$ in A makes Spec(A) into a topological space with open sets $X \setminus V(\mathfrak{a})$. The closed sets of the topology are the sets $V(\mathfrak{a})$ for all ideals $\mathfrak{a}$ in A, or equivalently the sets $V(E)$ for all subsets E of A.

The sets of the form $D(f)$ for $f \in A$ are open, and the collection of open sets $\{D(f)\}_{f \in A}$ is a basis for the topology of Spec(A). In fact we take an open subset $U = X \setminus V(\mathfrak{a})$ in X and $x \in U$. That is $\mathfrak{a} \not\subseteq \mathfrak{j}_x$. Then there is an element $f \in \mathfrak{a} \setminus \mathfrak{j}_x$. Hence $x \in D(f)$ and $D(f) \subseteq X \setminus V(\mathfrak{a}) = U$.

(5.5) Example. When A is a field the topological space $\mathrm{Spec}(A)$ consists of the point corresponding to the ideal (0) of A.

(5.6) Example. We have that $\mathrm{Spec}(\mathbf{Z})$ consists of the closed points corresponding to prime numbers of $\mathbf{Z}$, and the generic point corresponding to the ideal (0).

(5.7) Example. Let p be a prime number and $\mathbf{Z}_{(p)}$ the ring of rational numbers of the form m/n where p does not divide n. Then $\mathrm{Spec}(\mathbf{Z}_{(p)})$ consists of the closed point corresponding to the maximal ideal $p\mathbf{Z}_{(p)}$, and the generic point corresponding to the ideal (0) of $\mathbf{Z}_{(p)}$.

(5.8) Remark. Under the correspondence that maps a point $x \in \mathrm{Spec}(A)$ to the prime ideal $\mathfrak{j}_x$ in A, the closed points correspond to the maximal ideals. In fact the points in the closure $\overline{\{x\}}$ of x in $\mathrm{Spec}(A)$ correspond to the prime ideals of A containing $\mathfrak{j}_x$.

(5.9) Remark. Let $\mathfrak{a}$ be an ideal in the ring A. Then the bijection between ideals in the residue ring $A/\mathfrak{a}$ and the ideals of A containing $\mathfrak{a}$, via the canoni-
$\rightarrow$ cal homomorphism $\varphi_{A/\mathfrak{a}} : A \to A/\mathfrak{a}$ descibed in Section (?), gives a bijection between $\mathrm{Spec}(A/\mathfrak{a})$ and $V(\mathfrak{a})$. We give $V(\mathfrak{a})$ the topology induced from the topology on $\mathrm{Spec}(A)$. Then the bijection is an isomorphism of topological spaces because $D(\varphi_{A/\mathfrak{a}}(f))$ in $\mathrm{Spec}(A/\mathfrak{a})$ corresponds to $D(f)$ in $\mathrm{Spec}(A)$ for all $f \in A$.

(5.10) Maps. Let $\varphi : A \to B$ be a homomorphism of rings. For every prime ideal $\mathfrak{q}$ in B we have that $\varphi^{-1}(\mathfrak{q})$ is a prime ideal in A. Consequently we obtain a map of
n topological spaces $!!^a\varphi : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$.

(5.11) Proposition. *Let $\varphi : A \to B$ be a homomorphism of rings. For each ideal $\mathfrak{a}$ of A we have that ${}^a\varphi^{-1}(V(\mathfrak{a})) = V(\varphi(\mathfrak{a}))$.*

In particular ${}^a\varphi^{-1}(D(f)) = D(\varphi(f))$ for all $f \in A$, and ${}^a\varphi$ is a continuous map of topological spaces.

Proof. For $x \in \mathrm{Spec}\, B$ we have that ${}^a\varphi(x) \in V(\mathfrak{a})$ if and only if $\varphi^{-1}(\mathfrak{j}_x) \supseteq \mathfrak{a}$, that is, if and only if $\mathfrak{j}_x \supseteq \varphi(\mathfrak{a})$. However $\mathfrak{j}_x \supseteq \varphi(\mathfrak{a})$ if and only if $x \in V(\varphi(\mathfrak{a}))$. Hence ${}^a\varphi(x) \in V(\mathfrak{a})$ if and only if $x \in V(\varphi(\mathfrak{a}))$ and we have proved the first assertion. The last assertions follow from the first since $D(f) = \mathrm{Spec}(A) \setminus V(f)$.

(5.12) Remark. Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals in the ring A. We have that $V(\mathfrak{a}) = V(\mathfrak{r}(\mathfrak{a}))$ and $V(\mathfrak{a}) \subseteq V(\mathfrak{b})$ if and only if $\mathfrak{b} \subseteq \mathfrak{r}(\mathfrak{a})$. In fact $\mathfrak{r}(\mathfrak{a})$ is the intersection of all prime ideals of A containing $\mathfrak{a}$. We have, in particular that $V(\mathfrak{a}) = V(\mathfrak{b})$ if and only if $\mathfrak{r}(\mathfrak{a}) = \mathfrak{r}(\mathfrak{b})$.

(5.13) Proposition. *Let A be a ring and $\mathfrak{a}$ an ideal in A. The closed subset $V(\mathfrak{a})$ of the topological space* $\mathrm{Spec}(A)$ *is irreducible if and only if the radical $\mathfrak{r}(\mathfrak{a})$ of A is a prime ideal.*

In particular the correspondence that maps a prime ideal $\mathfrak{p}$ to the irreducible closed subset $V(\mathfrak{p})$ of $\mathrm{Spec}(A)$ *is a bijection between the prime ideals of A and the irreducible closed subsets of* $\mathrm{Spec}(A)$.

Proof. It follows from the isomorphism of topological spaces between $\mathrm{Spec}(A/\mathfrak{a})$ and $V(\mathfrak{a})$ of Remark (5.9), and from Remark (4.21) that it suffices to consider the particular case when $\mathfrak{a} = 0$.

The topological space $\mathrm{Spec}(A)$ is irreducible if and only if, for each pair of elements f, g in A we have that $D(f) \neq \emptyset$ and $D(g) \neq \emptyset$ implies that $D(fg) = D(f) \cap D(g) \neq \emptyset$. It follows from Proposition (4.18) that $D(h) = \emptyset$ for an element $h \in A$ if and only if h is nilpotent, that is, if and only if $h \in \mathfrak{r}(0)$. Hence $\mathrm{Spec}(A)$ is irreducible if and only if $f \notin \mathfrak{r}(0)$ and $g \notin \mathfrak{r}(0)$ implies that $fg \notin \mathfrak{r}(0)$. That is, $\mathrm{Spec}(A)$ is irreducible if and only if $\mathfrak{r}(0)$ is a prime ideal.

(5.14) Proposition. *Let $\varphi : A \to B$ be a homomorphism of rings. For every ideal $\mathfrak{b}$ in B we have that $\overline{{}^a\varphi(V(\mathfrak{b}))} = V(\varphi^{-1}\mathfrak{b})$.*

Proof. It is clear that ${}^a\varphi(V(\mathfrak{b})) \subseteq V(\varphi^{-1}\mathfrak{b})$, and consequently that we have an inclusion $\overline{{}^a\varphi(V(\mathfrak{b}))} \subseteq V(\varphi^{-1}\mathfrak{b})$. To prove the opposite inclusion we take a point $x \notin \overline{{}^a\varphi(V(\mathfrak{b}))}$. Then there is a neighbourhood $D(f)$ of x for some $f \in A$ that does not intersect ${}^a\varphi(V(\mathfrak{b}))$, that is $f \notin \mathfrak{j}_x$ and $f \in \varphi^{-1}(\mathfrak{j}_y)$ for all $y \in V(\mathfrak{b})$. Hence $\varphi(f) \in \mathfrak{j}_y$ for all $y \in V(\mathfrak{b})$, that is, the element $\varphi(f)$ is in $\mathfrak{r}(\mathfrak{b})$. However, then $\varphi(f^n) \in \mathfrak{b}$ for some positive integer n, and thus $f^n \in \varphi^{-1}(\mathfrak{b})$. Since $f^n \notin \mathfrak{j}_x$ we have that $\mathfrak{j}_x \not\supseteq \varphi^{-1}(\mathfrak{b})$, that is $x \notin V(\varphi^{-1}(\mathfrak{b}))$. We obtain the inclusion $V(\varphi^{-1}(\mathfrak{b})) \subseteq \overline{{}^a\varphi(V(\mathfrak{b}))}$ that we wanted to prove.

(5.15) Definition. Let A be a ring. The *dimension* of A is the dimension of the topological space $\mathrm{Spec}(A)$. That is, the dimension of A is the supremum of the length n of the chains $\mathfrak{p}_0 \supset \mathfrak{p}_1 \supset \cdots \supset \mathfrak{p}_n$ of prime ideals $\mathfrak{p}_i$ in A. We denote the dimension of A by $\dim(A)$. For every prime ideal $\mathfrak{p}$ of A the *height* of $\mathfrak{p}$ in A is the codimension of the irreducible subset $V(\mathfrak{a})$ of $\mathrm{Spec}(A)$. That is, the height of $\mathfrak{p}$ is the supremum of the lengths n of the chains $\mathfrak{p} = \mathfrak{p}_0 \supset \mathfrak{p}_1 \supset \cdots \supset \mathfrak{p}_n$ of prime ideals $\mathfrak{p}_i$ of A. We denote the height of $\mathfrak{p}$ by $\mathrm{ht}(\mathfrak{p}) = \mathrm{ht}_A(\mathfrak{p})$. When there exists arbitrary long chains we say that A has *infinite dimension.*

(5.16) Example. We have that $\dim(\mathbf{Z}) = 1$.

(5.17) Example. Let p be a prime number and $\mathbf{Z}_{(p)}$ the ring of rational numbers of the form m/n such that p does not divide n. Then $\dim(\mathbf{Z}_{(p)}) = 1$.

(5.18) Example. When A is a field $\dim(A) = 0$.

(5.19) Example. Let $K[t_1, t_2, \ldots]$ be the polynomial ring in the infinitely many variables $t_1, t_2, \ldots$ over K. Then the dimension of $K[t_1, t_2, \ldots]$ is infinite because there is an infinite chain $(t_1) \subset (t_1, t_2) \subset \cdots$ of prime ideals $(t_1, t_2, \ldots, t_n) = \sum_{i=1}^n K[t_1, t_2, \ldots]t_i$.

(5.20) Theorem. *Let $\{f_\alpha\}_{\alpha \in I}$ be a family of elements in A, and let $f \in A$.*

(1) *We have that $D(f) \subseteq \cup_{\alpha \in I} D(f_\alpha)$ if and only if $f \in \mathfrak{r}(\sum_{\alpha \in I} A f_\alpha)$.*

(2) (Partition of unity) *When $D(f) \subseteq \cup_{\alpha \in I} D(f_\alpha)$ there is a finite subset J of I with the property that for every family !!$\{n_\beta\}_{\beta \in J}$ of positive integers n_β*

there is a positive integer n and a family $\{g_\beta\}_{\beta\in J}$ of elements g_β of A such that

$$f^n = \sum_{\beta\in J} f_\beta^{n_\beta} g_\beta.$$

(3) *The open subset $D(f)$ of* $\operatorname{Spec}(A)$ *is compact.*

Proof. (1) We let $\mathfrak{a} = \sum_{\alpha\in I} Af_\alpha$ be the ideal generated by the elements f_α for all $\alpha \in I$. We have that $D(f) \subseteq \cup_{\alpha\in I} D(f_\alpha)$ if and only if every prime ideal $\mathfrak{p}$ that does not contain f does not contain f_α for some $\alpha \in I$. That is, if and only if every prime ideal that contains the elements f_α for all $\alpha \in I$ also contains f. Hence $D(f) \subseteq \cup_{\alpha\in I} D(f_\alpha)$ if and only if $f \in \mathfrak{r}(\mathfrak{a})$.

(2) When $D(f) \subseteq \cup_{\alpha\in I} D(f_\alpha)$ we have that $f \in \mathfrak{r}(\mathfrak{a})$. Consequently there is a positive integer n and a finite subset J of I such that $f^n = \sum_{\beta\in J} f_\beta h_\beta$ with $h_\beta \in A$ for all β in a finite set J. Hence $f \in \mathfrak{r}((f_\beta)_{\beta\in J})$ and it follows from assertion (1) that $D(f) \subseteq \cup_{\beta\in J} D(f_\beta)$. However $D(f_\beta) = D(f_\beta^{n_\beta})$ for all positive integers n_β. Consequently $D(f) \subseteq \cup_{\beta\in J} D(f_\beta^{n_\beta})$, and using assertion (1) once more we obtain the inclusion $f \in \mathfrak{r}((f_\beta^{n_\beta})_{\beta\in J})$ which is equivalent to the equality of assertion (2).

(3) When $D(f)$ is covered by a family $\{U_\alpha\}_{\alpha\in I}$ of open sets, we can cover each U_α with open sets of the form $D(g)$ for some $g \in A$. To show that $D(f)$ is compact it therefore suffices to prove that when $D(f) \subseteq \cup_{\alpha\in I} D(f_\alpha)$ there is a finite subset J of I such that $D(f) \subseteq \cup_{\beta\in J} D(f_\beta)$. However, this we verified in the proof of assertion (2).

(5.21) Exercises.

1. Show that the space $\operatorname{Spec}(A)$ is Kolmogorov.

2. Find all the closed sets subset of the topological space $\operatorname{Spec}(\mathbf{Z})$.

3. Let $\mathbf{Z}_{(p)}$ be the ring of all rational numbers of the form m/n such that p does not divide n. Let x_0 and x_1 be the points of $\operatorname{Spec}(\mathbf{Z}_{(p)})$ corresponding to the prime ideals (0), respectively $p\mathbf{Z}_{(p)}$, of $\mathbf{Z}_{(p)}$, and let x be the point of $\operatorname{Spec}(\mathbf{Q})$.

(1) Find all the open and closed subsets of the topological space $\operatorname{Spec}(\mathbf{Z}_{(p)})$.

(2) Let ${}^a\varphi : \operatorname{Spec}(\mathbf{Q}) \to \operatorname{Spec}(\mathbf{Z}_{(p)})$ be the continuous map corresponding to the inclusion $\mathbf{Z}_{(p)} \subseteq \mathbf{Q}$. Describe the image of x by ${}^a\varphi$.

(3) Let $\psi : \operatorname{Spec}(\mathbf{Q}) \to \operatorname{Spec}(\mathbf{Z}_{(p)})$ be the map defined by $\psi(x) = x_1$. Show that ψ is continuous, but does not come from a ring homomorphism $\mathbf{Z}_{(p)} \to \mathbf{Q}$.

4. Let G be a group. What is the dimension of the ring $\mathbf{Z} \times G$ with additon defined by $(m, x) + (n, y) = (m + n, x + y)$, and multiplication defined by $(m, x)(n, y) = (mn, my + nx)$?

5. Let K be a field and let $A = K[t_1, t_2, \ldots, t_n]$ be the polynomial ring in the variables $t_1, t_2, \ldots, t_n$ over K. Denote by $\mathfrak{a}$ the ideal generated by the elements t_it_j for $i, j = 1, 2, \ldots, n$. Determine the dimension of the ring $A/\mathfrak{a}$.

6. Let K be a field and let $A = K^n$ be the product of the field K with itself n times.

(1) Describe $\mathrm{Spec}(A)$.
(2) What is the dimension of $\mathrm{Spec}(A)$?

7. Let $A = \mathbf{Z}^n$ be the cartesian product of the ring $\mathbf{Z}$ with itself n times.

(1) Describe $\mathrm{Spec}(A)$.
(2) What is the dimension of $\mathrm{Spec}(A)$?

8. A topological space X is *connected* if there is no non-empty subset of X different from X. Let A be a ring and let $X = \mathrm{Spec}(A)$. Show that the following assertions are equivalent:

(1) The space $X = \mathrm{Spec}(A)$ is not connected.
(2) There are elements f and g in A such that $fg = 0$, $f^2 = f$, $g^2 = g$, and $f + g = 1$.
(3) The ring A is isomorphic to a direct product $B \times C$ of two rings B and C.

9. Let A be a ring and let $f \in A$. Show that the open subset $D(f)$ of $\mathrm{Spec}(A)$ is empty if and only if f is nilpotent in A.

Sheaves

1. Sheaves.

(1.1) Presheaves and sheaves. Let X be a topological space and $\mathfrak{B}$ a basis for
n the topology. A *presheaf* !!$\mathcal{F}$ on $\mathfrak{B}$ consists of a set $\mathcal{F}(U)$ for each subset U of X belonging to $\mathfrak{B}$, and for every inclusion $U \subseteq V$ of subsets of X belonging to $\mathfrak{B}$ a map
n !!

$$\rho_U^V = (\rho_{\mathcal{F}})_U^V : \mathcal{F}(V) \to \mathcal{F}(U)$$

such that for all inclusions $U \subseteq V \subseteq W$ of sets belonging to $\mathfrak{B}$ we have:

(1) $\rho_U^U = \mathrm{id}_{\mathcal{F}(U)}$.
(2) $\rho_U^W = \rho_U^V \rho_V^W$.

We call the elements in $\mathcal{F}(U)$ *sections* of $\mathcal{F}$ over U. The maps ρ_U^V we call *restriction maps*, and for a section $s \in \mathcal{F}(V)$ over V we call $\rho_U^V(s)$ the *restriction of s to U*.

The presheaf $\mathcal{F}$ is called a *sheaf* on $\mathfrak{B}$ if we for all collections $\{U_\alpha\}_{\alpha \in I}$ of subsets U_α of X belonging to $\mathfrak{B}$ with union $U = \cup_{\alpha \in I} U_\alpha$ belonging to $\mathfrak{B}$ have:

(F1) For every pair of sections s, t in $\mathcal{F}(U)$ such that

$$\rho_{U_\alpha}^U(s) = \rho_{U_\alpha}^U(t)$$

for all $\alpha \in I$, we have that $s = t$.

(F2) For every collection $\{s_\alpha\}_{\alpha \in I}$ of sections $s_\alpha \in \mathcal{F}(U_\alpha)$ that satisfy the condition

$$\rho_V^{U_\alpha}(s_\alpha) = \rho_V^{U_\beta}(s_\beta)$$

for all α, β in I, and all V belonging to $\mathfrak{B}$ with $V \subseteq U_\alpha \cap U_\beta$, there is a section s in $\mathcal{F}(U)$ restricting to U_α for all $\alpha \in I$, that is,

$$\rho_{U_\alpha}^U(s) = s_\alpha$$

for all $\alpha \in I$.

A presheaf, or sheaf, that is defined on all the open subsets of X is called a *presheaf* respectively a *sheaf* on X.

→ **(1.2) Remark.** If follows from property (F1) that the section s of property (F2)
→ i unique. Moreover from the equality $\emptyset = \emptyset \cup \emptyset$ is follows from property (F2) for sheaves that when not all the $\mathcal{F}(U)$ with U belonging to $\mathfrak{B}$ are empty we have that $\mathcal{F}(\emptyset)$ consists of exactly one element.

(1.3) Remark. A presheaf is the same as a contravariant functor from the category of open sets of X, with inclusions as maps, to the category of sets.

(1.4) Example. Let X be a topological space and E a set. Let $\mathcal{F}(U) = E$ for all non-empty open subsets U of X, and let $\rho_U^V = \mathrm{id}_E$ for all inclusions $U \subseteq V$ of non-empty open subsets of X. Then $\mathcal{F}$ is a presheaf that we call the *constant presheaf* with fiber E. This presheaf is not necessarily a sheaf. If there are two disjoint non-empty subsets U and V of X and there are two different elements s and t in E, there can be no section in $\mathcal{F}(U \cup V) = E$ which maps to s and t by the restriction $\rho_U^{U\cup V}$ respectively $\rho_V^{U\cup V}$.

(1.5) Example. Let X and Y be be topological spaces. For every open subset U of X we let $\mathcal{F}(U)$ be all continuous maps $U \to Y$, and for each inclusion $U \subseteq V$ of open subsets of X we let ρ_U^V be the map that takes a continuous map $\varphi : V \to Y$ to its restriction $\varphi|U : U \to Y$ to U. Then $\mathcal{F}$ with the maps ρ_U^V is a sheaf on X.

(1.6) Example. Let X be a topological space and let $\{E_x\}_{x\in X}$ be a collection of sets. For every open subset U of X we let $\mathcal{F}(U) = \prod_{x\in U} E_x$, and for every inclusion $U \subseteq V$ of open subsets of X we let $\rho_U^V : \prod_{x\in V} E_x \to \prod_{x\in U} E_x$ be the projection. Then $\mathcal{F}$ with the maps ρ_U^V is a sheaf on X.

(1.7) Example. Let $\mathcal{F}$ be a presheaf on X and let W be an open subset of X. We define a presheaf $\mathcal{F}|W$ on W by $(\mathcal{F}|W)(U) = \mathcal{F}(U)$ for all open subsets U of X contained in W, and take $(\rho_{\mathcal{F}|W})_U^V = (\rho_{\mathcal{F}})_U^V$ for all inclusions $U \subseteq V$ of open subsets of X contained in W.

When $\mathcal{F}$ is a sheaf on X we have that $\mathcal{F}|W$ is a sheaf on W. We call $\mathcal{F}|W$ the *restriction* of $\mathcal{F}$ to W.

(1.8) Example. Let X be a topological space and let Y be a closed subset. We give Y the topology induced by the topology on X. Let $\mathcal{G}$ be a presheaf on Y. We define a presheaf $\mathcal{F}$ on X by $\mathcal{F}(U) = \mathcal{G}(U \cap Y)$ for every open subset U of X, and $(\rho_{\mathcal{F}})_U^V = (\rho_{\mathcal{G}})_{U\cap Y}^{V\cap Y}$ for all inclusions $U \subseteq V$ of open subsets of X.

When $\mathcal{G}$ is a sheaf we have that $\mathcal{F}$ is a sheaf. We call $\mathcal{F}$ the *extension* of $\mathcal{G}$ to X.

(1.9) Definition. Let X be a topological space and $\mathcal{F}$ and $\mathcal{G}$ presheaves defined on
n a basis $\mathfrak{B}$ of the topology. A *homomorphism* !!$u : \mathcal{F} \to \mathcal{G}$ of presheaves is a map !!

$$u_U : \mathcal{F}(U) \to \mathcal{G}(U)$$

for each subset U of X belonging to $\mathfrak{B}$, such that if $U \subseteq V$ is an inclusion of subsets of X belonging to $\mathfrak{B}$ then the diagram

$$\begin{tikzcd}
\mathcal{F}(V) \arrow[r, "u_V"] \arrow[d, "(\rho_{\mathcal{F}})_U^V"'] & \mathcal{G}(V) \arrow[d, "(\rho_{\mathcal{G}})_U^V"] \\
\mathcal{F}(U) \arrow[r, "u_U"'] & \mathcal{G}(U)
\end{tikzcd}$$

commutes.

A *homomorphism* of sheaves, or a *homomorphism* from a presheaf to a sheaf on $\mathfrak{B}$, is a homomorphism of presheaves, when we consider the sheaves as presheaves. When $\mathfrak{B}$ consists of all open sets in X we say that u is a *homomorphism* of presheaves on X. All homomorphisms $u : \mathcal{F} \to \mathcal{G}$ of sheaves, or presheaves, on X we denote by $\operatorname{Hom}(\mathcal{F}, \mathcal{G})$.

An *isomorphism* of sheaves is a homomorphism $u : \mathcal{F} \to \mathcal{G}$ that has an inverse. Equivalently the maps u_U for all U in $\mathfrak{B}$ are isomorphisms. The inverses of the maps u_U then define a homomorphism of sheaves $\mathcal{G} \to \mathcal{F}$.

(1.10) Remark. Let $u : \mathcal{F} \to \mathcal{G}$ and $v : \mathcal{G} \to \mathcal{H}$ be homomorphisms of presheaves on $\mathfrak{B}$. Then $\mathrm{id}_{\mathcal{F}}$ and $vu : \mathcal{F} \to \mathcal{H}$ are homomorphisms of presheaves on $\mathfrak{B}$. In other words the presheaves on $\mathfrak{B}$ together with the homomorphisms of presheaves on $\mathfrak{B}$ form a category. Hence the sheaves defined on $\mathfrak{B}$ also form a category.

(1.11) Stalks. Let X be a topological space and $\mathfrak{B}$ a basis for the topology on X. Moreover, let $\mathcal{F}$ be a presheaf on $\mathfrak{B}$. For every point x of X we define the *stalk* !!$\mathcal{F}_x$ of $\mathcal{F}$ at x by:

Take !!$\mathcal{R}_x$ to be the family of all pairs (U, t) where U is a neighbourhood of x belonging to $\mathfrak{B}$, and $t \in \mathcal{F}(U)$. We define a relation !!$\sim$ on $\mathcal{R}_x$ by $(U, s) \sim (V, t)$ if there is an open neighbourhood W of x belonging to $\mathfrak{B}$ contained in $U \cap V$ such that $\rho^U_W(s) = \rho^V_W(t)$. It is clear that $\sim$ is an equivalence relation on $\mathcal{R}_x$. We define $\mathcal{F}_x$ as the equivalence classes !!$\mathcal{R}_x/\sim$ of $\mathcal{R}_x$ modulo the relation $\sim$.

For every neighbourhood U of x belonging to $\mathfrak{B}$ we have a canonical map!!

$$\rho^U_x = (\rho_{\mathcal{F}})^U_x : \mathcal{F}(U) \to \mathcal{F}_x$$

that takes a section s of $\mathcal{F}(U)$ to the class of (U, s) in $\mathcal{F}_x$. We write !!$\rho^U_x(s) = s_x$ and call s_x the *germ* of the section s at x.

For every inclusion $U \subseteq V$ of open neighbourhoods of x belonging to $\mathfrak{B}$ we have

$$\rho^V_x = \rho^U_x \rho^V_U.$$

When $\mathcal{F}$ is a sheaf on $\mathfrak{B}$ we let $\mathcal{F}_x$ be the *stalk* of $\mathcal{F}$ when $\mathcal{F}$ is considered as a presheaf.

When $u : \mathcal{F} \to \mathcal{G}$ is a homomorphism of presheaves on $\mathfrak{B}$ we obtain for each point x in X a map of stalks!!

$$u_x : \mathcal{F}_x \to \mathcal{G}_x$$

by mapping the class s_x of a pair (U, s) where U is a neighbourhood of x belonging to $\mathfrak{B}$ and $s \in \mathcal{F}(U)$ to the class of $(U, u_U(s))$ in $\mathcal{G}_x$. It is clear that the map is independent of the choice of representative (U, s) of the class s_x, and that for all U belonging to $\mathfrak{B}$ and for all $x \in U$ we have that

$$u_x(\rho_{\mathcal{F}})^U_x = (\rho_{\mathcal{G}})^U_x u_U.$$

(1.12) Example. Let X be a topological space and let E be a set. When $\mathcal{F}$ is the presheaf defined in Exercise (?) we have that $\mathcal{F}_x = E$ for all $x \in X$.

(1.13) Example. Let X be a topological space and let W be an open subset of X. Moreover let $\mathcal{F}$ be a sheaf on X and $\mathcal{F}|W$ the restriction of $\mathcal{F}$ to W defined in Example (?). Then $(\mathcal{F}|W)_x = \mathcal{F}_x$ for all $x \in W$.

(1.14) Example. Let X be a topological space and let Y be a closed subset of X. Moreover let $\mathcal{G}$ be a presheaf on Y and $\mathcal{F}$ the extension of $\mathcal{G}$ to X defined in Excercise (?). Then $\mathcal{F}_x = \mathcal{G}(\emptyset)$ for all $x \in X \setminus Y$ and $\mathcal{F}_x = \mathcal{G}_x$ for all $x \in Y$.

(1.15) Characterization of sheaves. Let X be a topological space and let $\mathfrak{B}$ be a basis for the topology. Moreover let $\mathcal{F}$ be a presheaf on $\mathfrak{B}$. For every open subset U belonging to $\mathfrak{B}$ we let

$$\pi_U : \mathcal{F}(U) \to \prod_{x \in U} \mathcal{F}_x$$

be the map that takes s to $(s_x)_{x \in U}$. The following assertions are equivalent:

(1) The presheaf $\mathcal{F}$ is a sheaf.
(2) We have
  (i) The maps π_U are injective for all open subsets U belonging to $\mathfrak{B}$.
  (ii) The image of $\mathcal{F}(U)$ by π_U consists exactly of the elements $(s_x)_{x \in U}$ of $\prod_{x \in U} \mathcal{F}_x$ with the property that for each $x \in U$ there is a neighbourhood U_x of x belonging to $\mathfrak{B}$ contained in U, and a section $s(x) \in \mathcal{F}(U_x)$ such that $s_y = s(x)_y$ for all $y \in U_x$.

(2) $\Rightarrow$ (1) With the notation of property (F1) for sheaves we have that $s_x = t_x$ for all $x \in U_\alpha$, and consequently $s_x = t_x$ for all $x \in U$. Since π_U is injective by property (i) we have that $s = t$. Hence the property (F1) for sheaves holds.

We shall prove that property (F2) of sheaves holds. With the notation of property (F2) for sheaves we have that $(s_\alpha)_x = (s_\beta)_x$ for all x contained in an open set V belonging to $\mathfrak{B}$ and contained in $U_\alpha \cap U_\beta$. Hence $(s_\alpha)_x = (s_\beta)_x$ for all $x \in U_\alpha \cap U_\beta$. We can therefore define an element $(s_x)_{x \in U} \in \prod_{x \in U} \mathcal{F}_x$ by $s_x = (s_\alpha)_x$ for any α such that $x \in U_\alpha$. Then $(s_x)_{x \in U}$ is in $\pi_U(\mathcal{F}(U))$ by property (ii) because for each $x \in U$ we can take $U_x = U_\alpha$ and take $s(x) = s_\alpha \in \mathcal{F}(U_x)$ for any α such that $x \in U_\alpha$. Then $s_y = (s_\alpha)_y = s(x)_y$ for all $y \in U_x = U_\alpha$. Finally $\rho^U_{U_\alpha}(s) = s_\alpha$ since the projection of $s = (s_x)_{x \in U}$ by $\prod_{x \in U} \mathcal{F}_x \to \prod_{x \in U_\alpha} \mathcal{F}_x$ is $(s_x)_{x \in U_\alpha} = ((s_\alpha)_x)_{x \in U_\alpha} = s_\alpha$. Hence we have proved that property (F2) for sheaves holds.

(1) $\Rightarrow$ (2) Let $\mathcal{F}$ be a sheaf. We first prove that property (i) of assertion (2) holds. If s and t are sections of $\mathcal{F}(U)$ such that $s_x = t_x$ for all $x \in U$, then, for each $x \in U$, there exists a neighbourhood U_x of x contained in U such that $\rho^U_{U_x}(s) = \rho^U_{U_x}(t)$. Since the sets U_x for all $x \in U$ cover U it follows from property (F1) for sheaves that $s = t$. Hence property (i) holds.

The images by π_U of the sections of $\mathcal{F}(U)$ satisfy property (ii) of assertion (2) since for $s \in \mathcal{F}(U)$ we can take $U_x = U$ and $s(x) = s$ for all $x \in U$.

Conversely let $(s_x)_{x \in U} \in \prod_{x \in U} \mathcal{F}_x$ satisfy the condition of (ii), We shall show that $(s_x)_{x \in U}$ is in the image of π_U. By property (F1) of sheaves we then have

$\rho^{U_x}_{U_x\cap U_{x'}}(s(x)) = \rho^{U_{x'}}_{U_x\cap U_{x'}}(s(x'))$ for all x, x' in U. Hence $\rho^{U_x}_V(s(x)) = \rho^{U_{x'}}_V(s(x'))$ for all neighbourhoods V belonging to $\mathfrak{B}$ and contained in $U_x \cap U_{x'}$ and it follows from property (F2) for sheaves that there is an $s \in \mathcal{F}(U)$ such that $\rho^U_{U_x}(s) = s(x)$ for all $x \in U$. It follows that $\rho^U_x(s) = \rho^{U_x}_x \rho^U_{U_x}(s) = \rho^{U_x}_x(s(x)) = s_x$ for all $x \in U$. Hence $\pi_U(s) = (s_x)_{x\in U}$ as we wanted to prove.

(1.16) Remark. When $\mathcal{F}$ is a sheaf on $\mathfrak{B}$ we shall we shall always identify $\mathcal{F}(U)$ with its image by the map $\pi_U : \mathcal{F}(U) \to \prod_{x\in U} \mathcal{F}_x$, for all open subsets U of X .

(1.17) Remark. Let $u : \mathcal{F} \to \mathcal{G}$ be a map of sheaves defined on $\mathfrak{B}$. For every open set U belonging to $\mathfrak{B}$ we obtain a map $\prod_{x\in U} u_x : \prod_{x\in U} \mathcal{F}_x \to \prod_{x\in U} \mathcal{G}_x$. It is clear that this map induces the map $u_U : \mathcal{F}(U) \to \mathcal{G}(U)$. In particular the map $u : \mathcal{F} \to \mathcal{G}$ is determined by the maps $u_x : \mathcal{F}_x \to \mathcal{G}_x$ for all $x \in U$.

(1.18) Remark. Let X be a topological space with a basis $\mathfrak{B}$ of the topology, and let $\mathcal{F}$ be a presheaf on X. We denote by $\mathcal{F}|\mathfrak{B}$ the *restriction* of $\mathcal{F}$ to $\mathfrak{B}$. That is for each inclusion $U \subseteq V$ of open sets belonging to $\mathfrak{B}$, we let $(\mathcal{F}|\mathfrak{B})(U) = \mathcal{F}(U)$ and $(\rho_{\mathcal{F}|\mathfrak{B}})^V_U = (\rho_{\mathcal{F}})^V_U$. For all $x \in X$ we have a canonical bijection $i : (\mathcal{F}|\mathfrak{B})_x \to \mathcal{F}_x$ which maps the class in $(\mathcal{F}|\mathfrak{B})_x$ of a pair (U, s) with U belonging to $\mathfrak{B}$ to the class in $\mathcal{F}_x$ of the same pair. To prove that the map i is a bijection we define its inverse. Let s_x be an element in $\mathcal{F}_x$ which is the class of a pair (V, s) where V is a neighbourhood of x and $s \in \mathcal{F}(V)$. Since $\mathfrak{B}$ is a basis for the topology there is a neighbourhood U of x belonging to $\mathfrak{B}$ with $U \subseteq V$. We map s_x to the class of the pair $(U, (\rho_{\mathcal{F}})^V_U(s))$ in $(\mathcal{F}|\mathfrak{B})_x$. It is clear that the map is independent of the choice of the pair (V, s) and of U, and that the map is the inverse of i.

Let $\mathcal{G}$ be another presheaf on X and let $u : \mathcal{F} \to \mathcal{G}$ be a homomorphism of presheaves. We let $u|\mathfrak{B} : \mathcal{F}|\mathfrak{B} \to \mathcal{G}|\mathfrak{B}$ denote the homomorphism of presheaves defined by $(u|\mathfrak{B})_U = u_U$ for all U belonging to $\mathfrak{B}$. Clearly $(u|\mathfrak{B})_x = u_x$ for all $x \in X$.

(1.19) Exercises.

1. Let !!E, F and G be three sets. A sequence !!

$$E \xrightarrow{u} F \underset{w}{\overset{v}{\rightrightarrows}} G \to 0$$

is *exact* if u is injective, if $vu = wu$, and if for every element $y \in F$ that satisfies the equation $v(y) = w(y)$ there is an element $x \in E$ such that $u(x) = y$.

Let X be a topological space and let $\mathcal{F}$ a presheaf on X. For every open set U of X and covering $\{U_\alpha\}_{\alpha\in I}$ of U:

(1) Show that the restrictions $\rho^U_{U_\alpha}$ define a natural map !!$\mathcal{F}(U) \xrightarrow{u} \prod_{\alpha\in I} \mathcal{F}(U_\alpha)$.

(2) Show that the restictions $\rho^{U_\alpha}_{U_\alpha\cap U_\beta}$ and $\rho^{U_\alpha}_{U_\beta\cap U_\alpha}$ for all α, β define two natural maps !!$\prod_{\alpha\in I} \mathcal{F}(U_\alpha) \underset{w}{\overset{v}{\rightrightarrows}} \prod_{\alpha,\beta\in I} \mathcal{F}(U_\alpha \cap U_\beta)$.

(3) Show that $\mathcal{F}$ is a sheaf if and only if the sequence of sets

$$\mathcal{F}(u) \xrightarrow{u} \prod_{\alpha\in I} \mathcal{F}(U_\alpha) \underset{w}{\overset{v}{\rightrightarrows}} \prod_{\alpha,\beta\in I} \mathcal{F}(U_\alpha \cap U_\beta)$$

is exact.

2. Let X be a topological space and E a set. We let $\mathcal{F}(\emptyset) = \{\emptyset\}$ and for all other open subsets U of X we let $\mathcal{F}(U) = E^U$ be all maps $U \to E$, and for all inclusions $U \subseteq V$ of open subsets of X we let ρ_U^V be the map that takes a function $\varphi : V \to E$ to its restriction $\varphi|U : U \to E$.

(1) Show that $\mathcal{F}$ is a sheaf on X.
(2) Show that there is a natural inclusion $E \subseteq \mathcal{F}_x$ for all $x \in X$
(3) Fix a point $x \in X$ and let Y be the intersection of all open sets in X that contain x. Show that there is a canonical map $\mathcal{F}_x \to E^Y$.
$\to$ (4) Is the map of part (3) always an isomorphism?

3. Let X be a topological space. Define $\mathcal{F}(U)$ on all open subsets U of X by $\mathcal{F}(X) = \mathbf{Z}$ and $\mathcal{F}(U) = \{0\}$ for all U different from X.

(1) Show that $\mathcal{F}$ with the only possible group homomorphisms $\rho_U^V : \mathcal{F}(V) \to \mathcal{F}(U)$ is a presheaf on X.
(2) Is $\mathcal{F}$ a sheaf on X?

4. Let X be a topological space. Define $\mathcal{F}(U)$ for all open subsets U of X by $\mathcal{F}(\emptyset) = \{0\}$, $\mathcal{F}(X) = \{0\}$, and $\mathcal{F}(U) = \mathbf{Z}$ for all the other open sets U of X. For each inclusion $U \subseteq V$ of open subsets of X let $\rho_U^V : \mathcal{F}(V) \to \mathcal{F}(U)$ be the identity map when $U \subseteq V$ are different from $\emptyset$ and X and otherwise the only possible group homomorphism.

(1) Show that $\mathcal{F}$ with these maps is a presheaf on X.
(2) Is $\mathcal{F}$ a sheaf on X?

5. Let X be a topological space and let $x_1, x_2, \ldots, x_n$ be closed points. Moreover let $G_1, G_2, \ldots, G_n$ be groups. For each open subset U of X we let $\mathcal{F}(U) = G_{j_1} \times G_{j_2} \times \cdots \times G_{j_m}$ when U contains the points $x_{j_1}, x_{j_2}, \ldots, x_{j_m}$ and no other of the points $x_1, x_2, \ldots, x_n$. When $U \subseteq V$ is an inclusion of open sets we let $\rho_U^V : \mathcal{F}(V) = G_{i_1} \times G_{i_2} \times \cdots \times G_{i_l} \to G_{j_1} \times G_{j_2} \times \cdots \times G_{j_m} = \mathcal{F}(U)$ be the natural projecton.

(1) Show that $\mathcal{F}$ is a presheaf on X.
(2) Describe the stalk $\mathcal{F}_x$ of $\mathcal{F}$ for all $x \in X$.
(3) Is $\mathcal{F}$ a sheaf on X?

6. Let X be a set and let $X = U_0 \supset U_1 \supset U_2 \supset \cdots$ be a sequence of subsets U_n strictly contained in each other. Give X the topology consisting of the open sets $\emptyset$ and $\{U_n\}_{n\in\mathbf{N}}$. Let $\{E_n\}_{n\in\mathbf{N}}$ be a collection of sets and $\{\rho_n\}_{n\in\mathbf{N}}$ a collection of maps $\rho_n : E_n \to E_{n+1}$. For all $n \in \mathbf{N}$ we write $\mathcal{F}(U_n) = E_n$, and for all m, n in $\mathbf{N}$ with $n \leq m$ we let $\rho_m^n : \mathcal{F}(U_n) \to \mathcal{F}(U_m)$ be the identity on E_n when $m = n$ and $\rho_m^{m-1} \cdots \rho_{n+2}^{n+1} \rho_{n+1}^n$ when $n < m$.

(1) Show that $\mathcal{F}$ with the restriction maps ρ_m^n for all $n \leq m$ is a presheaf.

→ (2) Show that the presheaf $\mathcal{F}$ of part (1) is a sheaf when $\mathcal{F}(\emptyset) = \{\emptyset\}$.
(3) Find the stalks of $\mathcal{F}$ over the points of $\cap_{n\in\mathbf{N}} U_n$, when $E_n = E$ and $\rho_n = \mathrm{id}_E$ for all $n \in \mathbf{N}$.

7. let X be a topological space with a basis $\mathfrak{B}$ for the topology. For every open subset U of X we consider U as a topological space with the topology induced by that of X, and we let $\mathfrak{B}_U$ be the basis for U consisting of open sets V belonging to $\mathfrak{B}$ that are contained in U.

For every presheaf $\mathcal{F}$ defined on $\mathfrak{B}$ we let $\mathcal{F}|U$ be the presheaf on $\mathfrak{B}_U$ defined by $(\mathcal{F}|U)(V) = \mathcal{F}(V)$ for all V belonging to $\mathfrak{B}_U$ and $(\rho_{\mathcal{F}|U})_V^W = (\rho_{\mathcal{F}})_V^W$ for all inclusions $V \subseteq W$ of open sets belonging to $\mathfrak{B}_U$.

Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves defined on $\mathfrak{B}$. For every open subset U belonging to $\mathfrak{B}$ we write $\mathcal{H}om(\mathcal{F},\mathcal{G})(U) = \mathrm{Hom}(\mathcal{F}|U, \mathcal{G}|U)$ for the set of all presheaf homomorphisms from $\mathcal{F}|U$ to $\mathcal{G}|U$.

(1) Show that for all inclusions $U \subseteq V$ of open sets belonging to $\mathfrak{B}$ we have a canonical map

$$\rho_U^V : \mathrm{Hom}(\mathcal{F}|V, \mathcal{G}|V) \to \mathrm{Hom}(\mathcal{F}|U, \mathcal{G}|U)$$

that maps a homomorphism $u : \mathcal{F}|V \to \mathcal{G}|V$ to the restriction $u|U : \mathcal{F}|U \to \mathcal{G}|U$ to U.
(2) Show that $\mathcal{H}om(\mathcal{F},\mathcal{G})$ with the restriction maps $\rho_U^V : \mathcal{H}om(\mathcal{F},\mathcal{G})(V) \to \mathcal{H}om(\mathcal{F},\mathcal{G})(U)$ for all inclusions $U \subseteq V$ of open subsets belonging to $\mathfrak{B}$ is a presheaf on $\mathfrak{B}$.
(3) Show that for all $x \in X$ we have a canonical map of stalks

$$\mathcal{H}om(\mathcal{F},\mathcal{G})_x \to \mathrm{Mor}(\mathcal{F}_x, \mathcal{G}_x)$$

that maps the class of a pair (U, u), where $u : \mathcal{F}|U \to \mathcal{G}|U$ is a homomorphism of presheaves, to the map $u_x : \mathcal{F}_x \to \mathcal{G}_x$, and where $\mathrm{Mor}(\mathcal{F}_x, \mathcal{G}_x)$ is the collection of maps from $\mathcal{F}_x$ to $\mathcal{G}_x$.
(4) Show that when $\mathcal{F}$ and $\mathcal{G}$ are sheaves on $\mathfrak{B}$ then $\mathcal{H}om(\mathcal{F},\mathcal{G})$ is a sheaf on $\mathfrak{B}$.
(5) Let $X = \{x_0, x_1\}$ be the topological space with open sets $\{\emptyset, \{x_0\}, X\}$. Moreover let $\mathcal{F}$ be the sheaf defined by $\mathcal{F}(X) = (0) = \mathcal{F}(\emptyset)$ and $\mathcal{F}(\{x_0\}) = \mathbf{Z}$, and let $\mathcal{G}$ be the sheaf defined by $\mathcal{G}(X) = \mathbf{Z}$ and $\mathcal{G}(\{x_0\}) = (0) = \mathcal{G}(\emptyset)$, both with the only possible restriction maps. finally let $\mathcal{H}$ be the sheaf defined by $\mathcal{H}(\emptyset) = 0$, $\mathcal{H}(X) = \mathbf{Z} = \mathcal{H}(\{x_0\})$, and with $(\rho_{\mathcal{H}})_{\{x_0\}}^X = \mathrm{id}_{\mathbf{Z}}$.
  (a) Show that the map

$$\mathcal{H}om(\mathcal{F},\mathcal{F})_{x_1} \to \mathrm{Mor}(\mathcal{F}_{x_1}, \mathcal{F}_{x_1})$$

  is not injective.
  (b) Show that the map

$$\mathcal{H}om(\mathcal{G},\mathcal{H})_{x_1} \to \mathrm{Mor}(\mathcal{G}_{x_1}, \mathcal{H}_{x_1})$$

  is not surjective.

8. Let X be a topological space with open sets $\{U_{n+1}\}_{n\in\mathbf{N}}$ and $\emptyset$ where $X = U_1 \supset U_2 \supset \cdots$. Define a sheaf $\mathcal{F}$ on X by $\mathcal{F}(U_i) = \{n \in \mathbf{N} : n < i\}$, and with restriction maps $(\rho_{\mathcal{F}})^{U_i}_{U_j} = \mathrm{id}_{\mathbf{N}}\,|\mathcal{F}(U_i)$ for $i \leq j$. Then we have that $\mathcal{F}_x = \mathbf{N}$ for all $x \in \cap_{n=1}^{\infty} U_n$. Let $\mathcal{G}$ be the simple sheaf with stalk $\mathbf{N}$. Moreover let $\mathcal{H}$ be the sheaf defined by $\mathcal{H}(U_i) = \{x \in \mathbf{Q} : x < (1/i)\}$ and with $(\rho_{\mathcal{H}})^{U_i}_{U_j} = x$ when $x \in U_i$ and $(\rho_{\mathcal{H}})^{U_i}_{U_j} = 0$ when $x \notin U_i$ for $i \leq j$.

(1) Show that for all $v \in \mathrm{Hom}(\mathcal{F}|U_i, \mathcal{G}|U_i)$ we have that $v_x(n) = 0$ when $n \geq i$, and $x \in \cup_{n=1}^{\infty} U_n$.

(2) Show that when $x \in \cap_{n=1}^{\infty} U_n$ there is no element in $\mathcal{H}om(\mathcal{F}, \mathcal{G})_x$ that maps to $\mathrm{id}_{\mathbf{Z}} : \mathcal{F}_x \to \mathcal{G}_x$ by the map

$$\mathcal{H}om(\mathcal{F}, \mathcal{G})_x \to \mathrm{Mor}(\mathcal{F}_x, \mathcal{G}_x)$$

→ in Excercise (?).

(3) Show that for all $x \in \cap_{n=1}^{\infty} U_n$ we have that $\mathcal{H}_x = 0$.

(4) Show that we have homomorphisms of sheaves $u : \mathcal{H} \to \mathcal{G}$ defined by $u_{U_n} : \mathcal{H}(U_n) \to \mathcal{G}(U_n)$ that sends 0 to 0 and all other elements in $\mathcal{H}(U_n)$ to 1.

(5) Show that we have a homomorphism of sheaves $v : \mathcal{H} \to \mathcal{G}$ that sends all elements in $\mathcal{H}(U_n)$ to 0.

(6) Show that u and v induce different elements in $\mathcal{H}om(\mathcal{H}, \mathcal{G})_x$.

(7) Show that the map

$$\mathcal{H}om(\mathcal{H}, \mathcal{G})_x \to \mathrm{Mor}(\mathcal{H}_x, \mathcal{G}_x)$$

→ in Excercise (?) is not injective when $x \in \cap_{n=1}^{\infty} U_n$.

9. Let $X = \{x_0, x_1\}$ be the topological space with open sets $\emptyset, X, \{x0\}$. Moreover let $\mathcal{F}$ be the sheaf on X defined by $\mathcal{F}(\emptyset) = \{0\}, \mathcal{F}(X) = \mathbf{Z}$ and $\mathcal{F}(\{x_0\}) = \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$, and with the restrictions maps beeing zero except for $\rho^X_{\{x_0\}}$ which sends n to $(u_{\mathbf{Z}/2\mathbf{Z}}(n), 0)$.

(1) Show that $\mathcal{H}om(\mathcal{F}, \mathcal{F})_{x_1} = \mathrm{Hom}(\mathcal{F}, \mathcal{F})$.

(2) Show that $\mathrm{Mor}(\mathcal{F}_{x_1}, \mathcal{F}_{x_1}) = \mathrm{Mor}(\mathbf{Z}, \mathbf{Z})$.

(3) Show that there does not exist a homomorphism of sheaves $u : \mathcal{F} \to \mathcal{F}$ such that $u_X(0) = 0$ and $u_X(2) = 1$.

(4) Show that the map

$$\mathcal{H}om(\mathcal{F}, \mathcal{F})_{x_1} \to \mathrm{Mor}(\mathcal{F}_{x_1}, \mathcal{F}_{x_1})$$

→ defined in Excercise (?) is not surjective.

(5) Show that there is a homomorphism of sheaves $v : \mathcal{F} \to \mathcal{F}$ such that $v_X = \mathrm{id}_{\mathcal{F}(X)}$ and $v_{\{x_0\}}(m, n) = (m, 0)$ for all m, n in $\mathbf{Z}/2\mathbf{Z}$.

(6) Show that the map v is different from the identity map.

(7) Show that the map

$$\mathcal{H}om(\mathcal{F}, \mathcal{F})_{x_1} \to \mathrm{Mor}(\mathcal{F}_{x_1}, \mathcal{F}_{x_1})$$

→ defined in Excercise (?) is not injective.

10. Let X be a topological space with basis $\mathfrak{B}$ for the topology. Moreover let $\{\mathcal{F}_\alpha\}_{\alpha\in I}$ be a collection of presheaves on $\mathfrak{B}$. For every open subset U of $\mathfrak{B}$ we let $\mathcal{F}(U) = \prod_{\alpha\in I}\mathcal{F}_\alpha(U)$, and for every inclusion $U\subseteq V$ of open subsets belonging to $\mathfrak{B}$ we let $(\rho_{\mathcal{F}})_U^V = \prod_{\alpha\in I}(\rho_{\mathcal{F}_\alpha})_U^V$.

(1) Show that $\mathcal{F}$ with the restrictions $(\rho_{\mathcal{F}})_U^V$ is a presheaf on $\mathfrak{B}$.

(2) For every $\alpha\in I$ and every open subset U of $\mathfrak{B}$ we have a projection map $(p_\alpha)_U : \prod_{\alpha\in I}\mathcal{F}_\alpha(U)\to\mathcal{F}_\alpha(U)$. Show that the maps $(p_\alpha)_U$ for all U belonging to $\mathfrak{B}$ define a homomorphism
$$p_\alpha : \prod_{\alpha\in I}\mathcal{F}_\alpha\to\mathcal{F}_\alpha$$
of presheaves.

(3) Show that $\mathcal{F}$ with the projections p_α is a product of the presheaves $\mathcal{F}_\alpha$ in the category of presheaves.

(4) Show that when all the presheaves $\mathcal{F}_\alpha$ are sheaves then $\mathcal{F}$ is a sheaf. We denote this sheaf by $\prod_{\alpha\in I}\mathcal{F}_\alpha$ and call it the *product* of the sheaves $\mathcal{F}_\alpha$.

(5) Show that the sheaf $\prod_{\alpha\in I}\mathcal{F}_\alpha$ together with the projections p_α is a product of the sheaves $\mathcal{F}_\alpha$ in the category of sheaves.

(6) For every $x\in X$ and for every $\alpha\in I$ we have a map $(p_\alpha)_x : (\prod_{\alpha\in I}\mathcal{F}_\alpha)_x\to(\mathcal{F}_\alpha)_x$. Show that these maps, for all $\alpha\in I$, give a map
$$(\prod_{\alpha\in I}\mathcal{F}_\alpha)_x\to\prod_{\alpha\in I}(\mathcal{F}_\alpha)_x.$$

(7) Assume that X has a sequence $X = U_0\supset U_1\supset U_2\supset\cdots$ of subsets U_n strictly contained in each other. Then the set X is a topological space with the collection $\{\emptyset,\{U_n\}_{n\in\mathbf{N}}\}$ as open sets. For every $p\in\mathbf{N}$ we let $F_{p,n}=\mathbf{Z}$ for $n\le p$ and $F_{p,n}=(0)$ for $n>p$. Moreover we let $(\rho_p)_n^m : F_{p,m}\to F_{p,n}$ be $\mathrm{id}_{\mathbf{Z}}$ for all m,n,p in $\mathbf{N}$ such that $m\le n\le p$ and otherwise zero. We denote by $\mathcal{F}_p$ the sheaf on X such that $\mathcal{F}_p(U_n)=F_{p,n}$ for all $n\in\mathbf{N}$ and with restricions $(\rho_{\mathcal{F}_p})_{U_n}^{U_m}$ for all m,n in $\mathbf{N}$ with $m\le n$.

Let $\mathcal{F}$ be the simple sheaf with stalk $\mathbf{Z}$.

(a) For all $p\in\mathbf{N}$ we have a map $s_p:\mathcal{F}\to\mathcal{F}_p$ of sheaves given by $(u_p)_{U_n}$: $\mathcal{F}(U_n)\to\mathcal{F}_p(U_n)$ which is the identity on $\mathbf{Z}$ when $n\le p$ and otherwise zero. Show that this defines a map of sheaves
$$\mathcal{F}\to\prod_{p\in\mathbf{N}}\mathcal{F}_p$$
such that $\mathcal{F}(U_n)\to(\prod_{p\in\mathbf{N}}\mathcal{F}_p)(U_n)$ is injective for all $n\in\mathbf{N}$.

(b) Show that the map
$$\mathcal{F}_x\to(\prod_{p\in\mathbf{N}}\mathcal{F}_p)_x$$

is injective for all $x \in X$.

(c) Show that for all points $x \in \cap_{n\in\mathbf{N}} U_n$ we have that $(\mathcal{F}_p)_x = 0$, and that $\prod_{p\in\mathbf{N}} (\mathcal{F}_p)_x = (0)$.

(d) Show that the map

$$(\prod_{p\in\mathbf{N}} \mathcal{F}_p)_x \to \prod_{p\in\mathbf{N}} (\mathcal{F}_p)_x$$

is not injective.

(8) Let $G_{p,n} = \mathbf{Z}$ when $n > p$ and $G_{p,n} = (0)$ when $n \leq p$. Moreover let $(\sigma_p)_n^M : G_{p,m} \to G_{p,n}$ be the identity for all m, n, p in $\mathbf{N}$ such that $p < m \leq n$, and otherwise zero. We denote by $\mathcal{G}_p$ the sheaf defined by $\mathcal{G}_p(U_n) = G_{p,n}$ and with restriction maps $(\rho_{\mathcal{G}_p})_{U_n}^{U_m} = (\sigma_p)_n^m$ for all m, n in $\mathbf{N}$ with $m \leq n$.

(a) Show that for all $x \in \cap_{n\in\mathbf{N}} U_n$ we have that $(\mathcal{G}_p)_x = \mathbf{Z}$, and that $(\sigma_p)_n^m = \mathrm{id}_{\mathbf{Z}}$ when $p < m \leq n$.

(b) Show that all sections $(s_p)_{p\in\mathbf{N}}$ in $(\prod_{p\in\mathbf{N}} \mathcal{G}_p)(U_n)$ satisfy $s_p = 0$ for $n \leq p$.

(c) Show that the element $(x_p)_{p\in\mathbf{N}} \in \prod_{n\in\mathbf{N}} (\mathcal{G}_n)_x$ with $x_p = 1$ for all p can not be in the image of

$$(\prod_{p\in\mathbf{N}} \mathcal{G}_p)_x \to \prod_{p\in\mathbf{N}} (\mathcal{G}_p)_x.$$

11. Let X be a topological space with open sets $\emptyset$ and $\{U_n\}_{n\in\mathbf{N}}$ where $X = U_0 \supset U_1 \supset \cdots$. Moreover let $\mathcal{F}$ be the sheaf defined by $\mathcal{F}(U_n) = \mathbf{N}^{U_n}$ and with restriction maps $\rho_{U_m}^{U_n}(s) = s|U_m$ for all $s \in \mathcal{F}(U_n)$ and $m \geq n$. Let $\mathcal{F}_n = \mathcal{F}$ for $n = 0, 1, 2, \dots$ and let $s_n \in \mathcal{F}(X)$ be the function defined by $s_n(x) = 1$ when $x \in U_n$ and $s_n(x) = 0$ when $x \notin U_n$. Moreover let $s \in \mathcal{F}(X)$ be defined by $s(x) = 1$ for all $x \in X$.

(1) Show that for all $x \in \cap_{n=0}^{\infty} U_n$ we have that $\rho_x^X(s_n) = \rho_x^X(s)$ in $\mathcal{F}_x$.

(2) Show that the elements $(s_n)_{n\in\mathbf{N}}$ and $(s)_{n\in\mathbf{N}}$ in $\prod_{n=0}^{\infty} \mathcal{F}_n(X)$ do not have the same class in $(\prod_{n=0}^{\infty} \mathcal{F}_n)_x$.

(3) Show that the map

$$(\prod_{n=0}^{\infty} \mathcal{F}_n)_x \to \prod_{n=0}^{\infty} (\mathcal{F}_n)_x$$

→ of Exercise (?) is not injective.

2. Direct and inverse images.

(2.1) The direct image. Let X and Y be two topological spaces and $\psi : X \to Y$ a continuous map. Moreover let $\mathcal{F}$ be a presheaf on X. We define a presheaf !!$\psi_*(\mathcal{F})$ on Y as follows:

For each open subset V of Y we let $\psi_*(\mathcal{F})(V) = \mathcal{F}(\psi^{-1}(V))$, and for each inclusion $V \subseteq W$ of open sets of Y we let

$$\rho_V^W = (\rho_{\psi_*(\mathcal{F})})_V^W = (\rho_{\mathcal{F}})_{\psi^{-1}(V)}^{\psi^{-1}(W)} : \mathcal{F}(\psi^{-1}(W)) \to \mathcal{F}(\psi^{-1}(V)).$$

It is clear that $\psi_*(\mathcal{F})$ is a presheaf on Y, and that when $\mathcal{F}$ is a sheaf on X then $\psi_*(\mathcal{F})$ is a sheaf on Y. We call $\psi_*(\mathcal{F})$ the *direct image* of $\mathcal{F}$ by ψ.

It is clear that when $u : \mathcal{F} \to \mathcal{G}$ is a homomorphism of presheaves on X we obtain a homomorphism $\psi_*(u) : \psi_*(\mathcal{F}) \to \psi_*(\mathcal{G})$ of presheaves on Y defined for each open subset V in Y by $\psi_*(u)_V = u_{\psi^{-1}(V)}$.

We have $\psi_*(\mathrm{id}_{\mathcal{F}}) = \mathrm{id}_{\psi_*(\mathcal{F})}$, and when $v : \mathcal{G} \to \mathcal{H}$ is another homomorphism of presheaves on X then $\psi_*(vu) = \psi_*(v)\psi_*(u)$.!! In other words, we have that ψ_* is a functor from presheaves, respectively sheaves, on X to presheaves, respectively sheaves, on Y.

For all x in X we have a canonical map of stalks:!!

$$\psi_x = (\psi_{\mathcal{F}})_x : \psi_*(\mathcal{F})_{\psi(x)} \to \mathcal{F}_x \tag{2.1.1}$$

that takes the class of the pair (V, t), where V is an open neighbourhood of $\psi(x)$ and $t \in \psi_*(\mathcal{F})(V) = \mathcal{F}(\psi^{-1}(V))$, to the class in $\mathcal{F}_x$ of the pair $(\psi^{-1}(V), t)$. It is clear that the map $(\psi_{\mathcal{F}})_x$ is independent of the choice of the representative (V, t) of the class $t_{\psi(x)}$. Let V be an open subset of Y and U an open subset of $\psi^{-1}(V)$. We clearly have that

$$(\rho_{\mathcal{F}})_x^U (\rho_{\mathcal{F}})_U^{\psi^{-1}(V)} = (\psi_{\mathcal{F}})_x (\rho_{\psi_*\mathcal{F}})_{\psi(x)}^V.$$

When $u : \mathcal{F} \to \mathcal{G}$ is a homomorphism of presheaves we have that $u_x(\psi_{\mathcal{F}})_x = (\psi_{\mathcal{G}})_x(\psi_*(u))_{\psi(x)}$.

When $\omega : Y \to Z$ is a continuous map of topological spaces then $(\omega\psi)_*(\mathcal{F}) = \omega_*(\psi_*(\mathcal{F}))$, and $(\mathrm{id}_X)_*(\mathcal{F}) = \mathcal{F}$.

(2.2) The inverse image. Let $\psi : X \to Y$ be a continuous map of topological spaces X and Y, and let $\mathfrak{B}$ be a basis for the topology on Y. For each presheaf $\mathcal{G}$ defined on $\mathfrak{B}$ we define a sheaf !!$\psi^*(\mathcal{G})$ on X as follows:

For every open subset U of X we let $\psi^*(\mathcal{G})(U)$ be the subset of the product $\prod_{x \in U} \mathcal{G}_{\psi(x)}$ that consists of the collections !!$(t_{\psi(x)})_{x \in U}$ with the property:

For every point $x \in U$ there exists a neighbourhood !!$V_{\psi(x)}$ of $\psi(x)$ belonging to $\mathfrak{B}$, a section !!$t(x) \in \mathcal{G}(V_{\psi(x)})$, and an open neighbourhood !!U_x of x contained in $U \cap \psi^{-1}(V_{\psi(x)})$ such that for all $y \in U_x$ we have $t_{\psi(y)} = t(x)_{\psi(y)} = (\rho_{\mathcal{G}})_{\psi(y)}^{V_{\psi(x)}}(t(x))$.

It is clear that for every inclusion $U \subseteq V$ of open sets on X the projection $\prod_{x \in V} \mathcal{G}_{\psi(x)} \to \prod_{x \in U} \mathcal{G}_{\psi(x)}$ induces a map

$$\rho_U^V = (\rho_{\psi^*(\mathcal{G})})_U^V : \psi^*(\mathcal{G})(V) \to \psi^*(\mathcal{G})(U),$$

and that $\psi^*(\mathcal{G})$, with these maps, is a presheaf on X.

We have that $\psi^*(\mathcal{G})$ is a sheaf on X.

To see that $\psi^*(\mathcal{G})$ is a sheaf we let U be an open subset of X, and we let $\{U_\alpha\}_{\alpha \in I}$ be an open covering of U. A section $t = (s_{\psi(x)})_{x \in U}$ in $\psi^*(\mathcal{G})(U) \subseteq \prod_{x \in U} \mathcal{G}_{\psi(x)}$ is completely determined by the values $s_{\psi(x)}$ for all $x \in U$. Since $\rho_{U_\alpha}^U$ is given by the projection $\prod_{x \in U} \mathcal{G}_{\psi(x)} \to \prod_{x \in U_\alpha} \mathcal{G}_{\psi(x)}$ two sections of $\psi^*(\mathcal{G})(U)$ that have the same restriction to $\psi^*(\mathcal{G})(U_\alpha)$ for all $\alpha \in I$ must be equal. Thus property (F1) for sheaves is satisfied by the presheaf $\psi^*(\mathcal{G})$.

Let $s_\alpha \in \psi^*(\mathcal{G})(U_\alpha)$ be a section on U_α for all $\alpha \in I$ such that s_α and s_β have the same restriction to $U_\alpha \cap U_\beta$ for all α, β in I. For each $\alpha \in I$ we have that $s_\alpha = ((t_\alpha)_{\psi(x)})_{x \in U_\alpha} \in \psi^*(\mathcal{G})(U_\alpha)$ for some $(t_\alpha)_{\psi(x)} \in \mathcal{G}_{\psi(x)}$, and since the restriction of s_α and s_β to $U_\alpha \cap U_\beta$ are equal we have that $(t_\alpha)_{\psi(x)} = (t_\beta)_{\psi(x)}$ for all α, β in I and all $x \in U_\alpha \cap U_\beta$. Hence we can define $t_{\psi(x)} \in \mathcal{G}_{\psi(x)}$ by $t_{\psi(x)} = (t_\alpha)_{\psi(x)}$ for any $\alpha \in I$ such that $x \in U_\alpha$. We thus obtain an element $s = (t_{\psi(x)})_{x \in U} \in \prod_{x \in U} \mathcal{G}_{\psi(x)}$. Since $(t_{\psi(x)})_{x \in U_\alpha} = ((t_\alpha)_{\psi(x)})_{x \in U_\alpha} \in \psi^*(\mathcal{G})(U_\alpha)$ for all $\alpha \in I$ it follows from the definition of $\psi^*(\mathcal{G})(U)$ that $s = (t_{\psi(x)})_{x \in U} \in \psi^*(\mathcal{G})(U)$. It also follows that $\rho_{U_\alpha}^U(s) = s_\alpha$. Hence we have proved that property (F2) for sheaves is satisfied for the presheaf $\psi^*(\mathcal{G})$. We have thus proved that $\psi^*(\mathcal{G})$ is a sheaf.

It follows from the definition of $\psi^*(\mathcal{G})$ that for all subsets W of Y that belong to $\mathfrak{B}$ there is a map

$$\mathcal{G}(W) \to \psi^*(\mathcal{G})(\psi^{-1}(W)) = \psi_*(\psi^*(\mathcal{G}))(W)$$

that takes a section t in $\mathcal{G}(W)$ to the section $(t_{\psi(x)})_{x \in \psi^{-1}(W)}$ in $\psi^*(\mathcal{G})(\psi^{-1}(W))$. These maps define a homomorphism of presheaves on $\mathfrak{B}$: !!

$$\rho_{\mathcal{G}} : \mathcal{G} \to \psi_*(\psi^*(\mathcal{G})). \tag{2.2.1}$$

When $u : \mathcal{F} \to \mathcal{G}$ is a homomorphism of presheaves on $\mathfrak{B}$ the maps $\prod_{x \in U} u_{\psi(x)} : \prod_{x \in U} \mathcal{F}_{\psi(x)} \to \prod_{x \in U} \mathcal{G}_{\psi(x)}$ for each open set U in X induce a homomorphism $\psi^*(u) : \psi^*(\mathcal{F}) \to \psi^*(\mathcal{G})$ of presheaves on X. We have that $\psi^*(\mathrm{id}_{\mathcal{G}}) = \mathrm{id}_{\psi^*(\mathcal{G})}$, and when $v : \mathcal{G} \to \mathcal{H}$ is another homomorphism of presheaves on $\mathfrak{B}$ we have that $\psi^*(vu) = \psi^*(v)\psi^*(u)$. In other words ψ^* is a functor from presheaves on $\mathfrak{B}$ to sheaves on X.

Let $\omega : Y \to Z$ be a continuous map of topological spaces, and $\mathcal{H}$ a presheaf defined on a basis of Z. Then we have that $(\omega\psi)^*(\mathcal{H}) = \psi^*(\omega^*(\mathcal{H}))$.

For every point x in X we obtain from the homomorphism (2.2.1) and the homomorphism (2.1.1) a map !!

$$(\iota_{\mathcal{G}})_x = (\psi_{\psi^*(\mathcal{G})})_x (\rho_{\mathcal{G}})_{\psi(x)} : \mathcal{G}_{\psi(x)} \to \psi^*(\mathcal{G})_x$$

that takes the class of a pair (W, t) where W is a neighbourhood of $\psi(x)$ belonging to $\mathfrak{B}$ and $t \in \mathcal{G}(W)$ to the class of $(\psi^{-1}(W), (t_{\psi(x)})_{x \in \psi^{-1}(W)})$ in $\psi^*(\mathcal{G})_x$.

(2.3) Proposition. *The map* $(\iota_{\mathcal{G}})_x = (\psi_{\psi^*(\mathcal{G})})_x(\rho_{\mathcal{G}})_{\psi(x)} : \mathcal{G}_{\psi(x)} \to \psi^*(\mathcal{G})_x$ *is a bijection.*

Proof. We construct the inverse of the map $(\iota_{\mathcal{G}})_x$. Let s_x be an element in $\psi^*(\mathcal{G})_x$. Then s_x is the class of a pair $(U_x, (t(x)_{\psi(y)})_{y \in U_x})$, where $t(x) \in \mathcal{G}(V_{\psi(x)})$ for a neighbourhood $V_{\psi(x)}$ of $\psi(x)$ belonging to $\mathfrak{B}$, and where U_x is a neighbourhood of x contained in $U \cap \psi^{-1}(V_{\psi(x)})$. We map s_x to $t(x)_{\psi(x)}$. It is clear that this map is independent of the representative $(U_x, (t(x)_{\psi(y)})_{y \in U_x})$, and of $V_{\psi(x)}$.

(2.4) Remark. We have two descriptions of the inverse image $\psi^*(\mathcal{G})$ of $\mathcal{G}$ by ψ. Firstly it follows from the definition of $\psi^*(\mathcal{G})$ that for all U belonging to $\mathfrak{B}$ we have that $\psi^*(\mathcal{G})(U) \subseteq \prod_{x \in U} \mathcal{G}_{\psi(x)}$. Secondly since $\psi^*(\mathcal{G})$ is a sheaf it follows from Remark (?) that $\psi^*(\mathcal{G})(U) \subseteq \prod_{x \in U} \psi^*(\mathcal{G})_x$. From the first description we can write every section $s \in \psi^*(\mathcal{G})(U)$ on the form $s = (t_{\psi(x)})_{x \in U}$ with $t_{\psi(x)} \in \mathcal{G}_{\psi(x)}$, and from the second we have that $s = (s_x)_{x \in U}$ where $s_x = (\rho_{\psi^*\mathcal{G}})^U_x(s) \in (\psi^*\mathcal{G})_x$. The two descriptions are linked by the formula

$$(\iota_{\mathcal{G}})_x(t_{\psi(x)}) = s_x$$

for all $x \in U$.

When $v : \mathcal{G} \to \mathcal{H}$ is a homomorphism of presheaves on $\mathfrak{B}$ we have that

$$\psi^*(v)_x(\iota_{\mathcal{G}})_x = (\iota_{\mathcal{H}})_x v_{\psi(x)}.$$

(2.5) Definition. Let $\mathcal{G}$ be a presheaf defined on a basis $\mathfrak{B}$ of the topological space Y. The sheaf $(\mathrm{id}_Y)^*(\mathcal{G})$ on Y is called the *sheaf associated to the presheaf* $\mathcal{G}$. It comes with the canonical homomorphism

$$\rho_{\mathcal{G}} : \mathcal{G} \to (\mathrm{id}_Y)^*(\mathcal{G}) = (\mathrm{id}_Y)_*(\mathrm{id}_Y)^*(\mathcal{G}).$$

(2.6) Remark. Let $\mathcal{G}$ be a presheaf defined on a basis $\mathfrak{B}$ of the topological space Y. It follows from Proposition (?) that the map $(i_{\mathcal{G}})_y : \mathcal{G}_y \to (\mathrm{id}_Y)^*(\mathcal{G})_y$ is the identity map on $\mathcal{G}_y$.

For every open subset V of Y we have that $(\mathrm{id}_Y)^*(\mathcal{G})(V) \subseteq \prod_{y \in V} \mathcal{G}_y$. It follows from the definition of the associated sheaf $(\mathrm{id}_Y)^*(\mathcal{G})$ and the characterization (?) of sheaves that $\mathcal{G}$ is a sheaf on $\mathfrak{B}$ if and only if the homomorphism $(\rho_{\mathcal{G}})_V : \mathcal{G}(V) \to (\mathrm{id}_Y)^*(\mathcal{G})(V)$ is an isomorphism for all open subsets V of X belonging to $\mathfrak{B}$.

When $\mathcal{G}$ is a sheaf on $\mathfrak{B}$ we have that $\mathcal{G}(V)$ and $(\mathrm{id}_Y)^*(\mathcal{G})(V)$ is the same subset of $\prod_{y \in V} \mathcal{G}_y$ for all open sets V belonging to $\mathfrak{B}$.

(2.7) Example. Let X be a topological space and let E be a set. The associated sheaf to the constant presheaf with fiber E has fiber E at all poins. We call the associated sheaf the *simple sheaf* with fiber E.

(2.8) Proposition. *Let $\psi : X \to Y$ be a continuous map of topological spaces and let $\mathfrak{B}$ be a basis for Y.*

(1) *When $\mathcal{G}$ is a presheaf defined on Y, and $\mathcal{G}|\mathfrak{B}$ its restriction to $\mathfrak{B}$, we have that $\psi^*(\mathcal{G}|\mathfrak{B}) = \psi^*(\mathcal{G})$.*

(2) *When $u : \mathcal{G} \to \mathcal{H}$ is a homomorphism between presheaves on Y we have that $\psi^*(u) = \psi^*(u|\mathfrak{B})$.*

Proof. (1), (2) We have that $\psi^*(\mathcal{H})(U)$ is determined, as a subset of $\prod_{x\in U}\mathcal{H}_{\psi(x)}$, by conditions on arbitrarily small neighbourhoods U_x of x contained in $\psi^{-1}(V_{\psi(x)})$
→ for an arbitrarily small neighbourhood $V_{\psi(x)}$ of $\psi(x)$. Hence part (1) and (2) follow immediately from the definitions of $\psi^*(\mathcal{G}|\mathfrak{B}) < \psi^*(\mathcal{G}), \psi^*(u)$ and $\psi^*(u|\mathfrak{B})$.

→ **(2.9) Adjunction.** For every presheaf $\mathcal{G}$ on $\mathfrak{B}$ we saw in (?) that we have a homomorphism of presheaves

$$\rho_{\mathcal{G}} : \mathcal{G} \to \psi_*(\psi^*(\mathcal{G})).$$

n On the other hand we have, for every sheaf $\mathcal{F}$ on X, a homomorphism of sheaves:!!

$$\sigma_{\mathcal{F}} : \psi^*(\psi_*(\mathcal{F})) \to \mathcal{F}.$$

In order to define $\sigma_{\mathcal{F}}$ we take an open neighbourhood U of X and a section $s \in \psi^*(\psi_*(\mathcal{F}))(U)$. For every point $x \in U$ there is an open neighbourhood $V_{\psi(x)}$ of $\psi(x)$, a section $t(x) \in \psi_*(\mathcal{F})(V_{\psi(x)}) = \mathcal{F}(\psi^{-1}(V_{\psi(x)}))$, and an open neighbourhood U_x of x contained in $U \cap \psi^{-1}(V_{\psi(x)})$ such that $s_y = (\iota_{\psi_*(\mathcal{F})})_y(t(x)_{\psi(y)})$ for all $y \in U_x$, as we ob-
→ served in Remark (2.2.3). Let $s(x) = (\rho_{\mathcal{F}})_{U_x}^{\psi^{-1}(V_{\psi(x)})}(t(x)) \in \mathcal{F}(U_x)$. Since $\mathcal{F}$ is a sheaf
→ it follows from Remark (1.?) and Remark (?) that the section $s(x)$ of $\mathcal{F}(U_x)$ is determined uniquely by $s(x)_y = (\rho_{\mathcal{F}})_y^{U_x}(\rho_{\mathcal{F}})_{U_x}^{\psi^{-1}(V_{\psi(x)})}(t(x)) = (\psi_{\mathcal{F}})_y(\rho_{\psi_*(\mathcal{F})})_{\psi(y)}^{V_{\psi(x)}}(t(x)) = (\psi_{\mathcal{F}})_y(t(x)_{\psi(y)})$, and therefore by $s_y = (\iota_{\psi_*(\mathcal{F})})_y(t(x)_{\psi(y)})$ for all $y \in U_x$. Hence the sections $s(x)$ for all $x \in U$ define a section $\sigma_{\mathcal{F}}(s) \in \mathcal{F}(U)$. It is clear that the definition of $\sigma_{\mathcal{F}}(s)$ is independent of the choices of $V_{\psi(x)}$, $t(x)$ and U_x for $x \in U$.

Let $\mathcal{F}$ be a sheaf on X and $\mathcal{G}$ a presheaf on $\mathfrak{B}$. For every homomorphism of
n presheaves !!$u : \mathcal{G} \to \psi_*(\mathcal{F})$ on $\mathfrak{B}$ we obtain a homomorphism !!

$$u^\sharp = \sigma_{\mathcal{F}}\psi^*(u) : \psi^*(\mathcal{G}) \to \mathcal{F}$$

of presheaves on $\mathfrak{B}$. In order to describe $u^\sharp$ explicitely let $s \in \psi^*(\mathcal{G})(U)$ be a section of $\psi^*(\mathcal{G})$ over an open neighbourhood U of X. For every point $x \in U$ there is an open neighbourhood $V_{\psi(x)}$ of $\psi(x)$ belonging to $\mathfrak{B}$, a section $t(x) \in \mathcal{G}(V_{\psi(x)})$, and a neighbourhood U_x of x contained in $U \cap \psi^{-1}(V_{\psi(x)})$ such that $s_y = (\iota_{\mathcal{G}})_y(t(x)_{\psi(y)})$ for all $y \in U_x$. Then $u_U^\sharp(s) \in \mathcal{F}(U)$ is determined as a subset of $\prod_{x\in U}\mathcal{F}_x$ by $u_U^\sharp(s)_x = (\psi_{\mathcal{F}})_x u_{\psi(x)}(t(x)_{\psi(x)})$ for all $x \in U$.

For every homomorphism !!$v : \psi^*(\mathcal{G}) \to \mathcal{F}$ of sheaves on X we obtain a homomorphism !!

$$v^\flat = \psi_*(v)\rho_{\mathcal{G}} : \mathcal{G} \to \psi_*(\mathcal{F})$$

of presheaves on $\mathfrak{B}$. In order to describe $v^\flat$ explicitely we let $t \in \mathcal{G}(V)$ be a section over a subset V of Y belonging to $\mathfrak{B}$. Let s be the section of $\psi^*(\mathcal{G})(\psi^{-1}(V))$ determined by $s_x = (\iota_{\mathcal{G}})_x t_{\psi(x)}$ for all $x \in \psi^{-1}(V)$. Then we have that $v^\flat_V(t) \in \psi_*(\mathcal{F})(V) = \mathcal{F}(\psi^{-1}(V))$ is determined as a subset of $\prod_{x \in \psi^{-1}(V)} \mathcal{F}_x$ by the equalities $v^\flat(t)_x = v_x(s_x)$ for all $x \in \psi^{-1}(V)$.

It follows from the explicit expression for $u^\sharp$ and $v^\flat$ that $(u^\sharp)^\flat = u$ and $(v^\flat)^\sharp = v$. We have thus shown:

The map

$$\mathrm{Hom}_X(\psi^*(\mathcal{G}), \mathcal{F}) \to \mathrm{Hom}_Y(\mathcal{G}, \psi_*(\mathcal{F})) \tag{2.9.1}$$

that takes $v : \psi^*(\mathcal{G}) \to \mathcal{F}$ *to* $v^\flat : \mathcal{G} \to \psi_*(\mathcal{F})$ *is a bijection. The inverse map takes* $u : \mathcal{G} \to \psi_*(\mathcal{F})$ *to* $u^\sharp : \psi^*(\mathcal{G}) \to \mathcal{F}$.

In particular we obtain, for every presheaf $\mathcal{G}$ *defined on a basis* $\mathfrak{B}$ *of* Y *and for every homomorphism* $w : \mathcal{G} \to \mathcal{H}$ *to a sheaf* $\mathcal{H}$ *on* Y*, a unique homomorphism* $w^\sharp : (\mathrm{id}_Y)^*(\mathcal{G}) \to \mathcal{H}$ *of sheaves such that* $w = w^\sharp \rho_{\mathcal{G}}$.

(2.10) The image of homomorphisms of sheaves. Let X be a topological space with a basis $\mathfrak{B}$ for the topology. Moreover let $u : \mathcal{F} \to \mathcal{G}$ be a homomorphism of presheaves on $\mathfrak{B}$. For every open set U belonging to $\mathfrak{B}$ we let !!$\mathcal{H}(U) = \mathrm{Im}(u_U)$. It is clear that for every inclusion $U \subseteq V$ of open sets belonging to $\mathfrak{B}$ the restriction map $(\rho_{\mathcal{G}})^V_U : \mathcal{G}(V) \to \mathcal{G}(U)$ induces a restriction map $(\rho_{\mathcal{H}})^V_U : \mathcal{H}(V) \to \mathcal{H}(U)$, and that $\mathcal{H}$ with these restriction maps becomes a presheaf on $\mathfrak{B}$. Moreover it is clear that the surjections $\mathcal{F}(U) \to \mathcal{H}(U)$ and the inclusions $\mathcal{H}(U) \to \mathcal{G}(U)$ induce homomorphisms!!

$$v : \mathcal{F} \to \mathcal{H}$$

respectively!!

$$i : \mathcal{H} \to \mathcal{G}$$

of presheaves on $\mathfrak{B}$.

We write!!

$$w = \rho_{\mathcal{H}} v : \mathcal{F} \to (id_X)^*(\mathcal{H}).$$

When $\mathcal{G}$ is a sheaf on $\mathfrak{B}$ it follows from Remark (?) that the homomorphism i induces a homomorphism!!

$$j = i^\sharp : (\mathrm{id}_X)^*(\mathcal{H}) \to \mathcal{G}$$

of sheaves on $\mathfrak{B}$ such that $i = j\rho_{\mathcal{H}}$. We then have that $u = iv = j\rho_{\mathcal{H}} v = jw$.

(2.11) Example. Even when the homomorphism $u : \mathcal{F} \to \mathcal{G}$ is a homomorphism of sheaves the presheaf $\mathcal{H}$ of Section (2.?) does not have to be a sheaf.

Let $X = \{x_0, x_1, x_2\}$ be the topological space with open sets $\emptyset, X, U_0 = \{x_0\}, U_1 = \{x_0, x_1\}, U_2 = \{x_0, x_2\}$. The constant presheaf $\mathcal{F}$ on X with fiber $\mathbf{Z}$ is in this case a sheaf, and thus equal to the simple sheaf with fiber $\mathbf{Z}$.

Let $\mathcal{G}$ be the sheaf defined by $\mathcal{G}(\emptyset) = \{0\}$, $\mathcal{G}(X) = \mathbf{Z} \oplus \mathbf{Z}$, and $\mathcal{G}(U_i) = \mathbf{Z}$ for $i = 0, 1, 2$, and with restrictions $(\rho_{\mathcal{G}})_{U_0}^{U_i} = \mathrm{id}_{\mathbf{Z}}$, and $(\rho_{\mathcal{G}})_{U_i}^{X}$ the projection on the i'the factor, for $i = 1, 2$.

It is clear that the map $u : \mathcal{F} \to \mathcal{G}$ given by $u_X : \mathbf{Z} \to \mathbf{Z} \oplus \mathbf{Z}$ with $u_X(n) = (n, n)$, by $u_\emptyset = \mathrm{id}_{\{0\}}$, and where $u_{U_i} = \mathrm{id}_{\mathbf{Z}}$ for $i = 0, 1, 2$, is a homomorphism of sheaves. We have that $\mathcal{H}(\emptyset) = \{0\}$, $\mathcal{H}(X)$ is isomorphic to $\mathbf{Z}$, $\mathcal{H}(U_0) = \{0\}$, and $\mathcal{H}(U_i) = \mathbf{Z}$ for $i = 1, 2$. Then $\mathcal{H}$ is not a sheaf because sections of $\mathcal{H}(U_1) = \mathbf{Z}$ and $\mathcal{H}(U_2) = \mathbf{Z}$ that are represented by different integers can not come from a section of $\mathcal{H}(X)$.

(2.12) Lemma. *Let X be a topological space with a basis $\mathfrak{B}$ of the topology, and let $u : \mathcal{F} \to \mathcal{G}$ be a homomorphism of presheaves defined on $\mathfrak{B}$.*

(1) *If $u_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for all open sets U belonging to $\mathfrak{B}$ we have that the map $u_x : \mathcal{F}_x \to \mathcal{G}_x$ is injective for all $x \in X$.*
(2) *If $u_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is surjective for all open sets U belonging to $\mathfrak{B}$ we have that the map $u_x : \mathcal{F}_x \to \mathcal{G}_x$ is surjective.*
(3) *If $\mathcal{F}$ and $\mathcal{G}$ are sheaves on $\mathfrak{B}$ and the map $u_x : \mathcal{F}_x \to \mathcal{G}_x$ is injective for all $x \in X$, then we have that $u_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for all open sets U belonging to $\mathfrak{B}$. In particular, the homomorphism u induces an isomorphism $\mathcal{F} \xrightarrow{\sim} \mathcal{H}$.*
(4) *If $\mathcal{F}$ and $\mathcal{G}$ are sheaves on $\mathfrak{B}$ and the map $u_x : \mathcal{F}_x \to \mathcal{G}_x$ is surjective for all x in X, then we have that $j_U : (\mathrm{id}_X)^*(\mathcal{H})(U) \to \mathcal{G}(U)$ is an isomorphism for all open sets U belonging to $\mathfrak{B}$. That is, the homomorphism u induces an isomorphism $(\mathrm{id}_X)^*(\mathcal{H}) \xrightarrow{\sim} \mathcal{G}$.*

In particular we have that when $\mathcal{F}$ and $\mathcal{G}$ are sheaves, then $u : \mathcal{F} \to \mathcal{G}$ is an isomorphism if and only if the induced map on stalks $u_x : \mathcal{F}_x \to \mathcal{G}_x$ is an isomorphism for all points $x \in X$.

Proof. (1) Let s_x and t_x be elements in $\mathcal{F}_x$ such that $u_x(s_x) = u_x(t_x)$. We can find a neighbourhood V of x belonging to $\mathfrak{B}$ and sections s, t in $\mathcal{F}(V)$ such that $(\rho_{\mathcal{F}})_x^V(s) = s_x$ and $(\rho_{\mathcal{F}})_x^V(t) = t_x$. Then $(V, u_V(s))$ and $(V, u_V(t))$ have the same class in $\mathcal{G}_x$. Consequently there is a neighbourhood U of x belonging to $\mathfrak{B}$ and contained in V such that $(U, (\rho_{\mathcal{G}})_U^V(u_V(s))) = (U, (\rho_{\mathcal{G}})_U^V(u_V(s)))$. That is $(U, u_U((\rho_{\mathcal{F}})_U^V(s))) = (U, u_U((\rho_{\mathcal{F}})_U^V(t)))$. Since u_U is injective by assumption we have that $(\rho_{\mathcal{F}})_U^V(s) = (\rho_{\mathcal{F}})_U^V$, and consequently that $s_x = t_x$. Hence we have proved that u_x is injective.

(2) Let $t_x \in \mathcal{G}_x$. Then there is an open neighbourhood U of x and $t \in \mathcal{G}(U)$ such that $(\rho_{\mathcal{G}})_x^V(t) = t_x$. Since u_U is surjective we can find a section $s \in \mathcal{F}(U)$ such that $u_U(s) = t$. Then $u_x(s_x) = t_x$ and we have proved that u_x is surjective.

(3) Since the maps u_x for $x \in X$ are injective, we have an injective map $\prod_{x \in U} u_x :$

$\prod_{x \in U} \mathcal{F}_x \to \prod_{x \in U} \mathcal{G}_x$. It follows from Remark (?) that it suffices to prove that if $(s_x)_{x \in U}$ lies in $\mathcal{F}(U)$ then $(u_x(s_x))_{x \in U}$ lies in $\mathcal{G}(U)$. However it follows from the characterization on sheaves that if $(s_x)_{x \in U} \in \mathcal{F}(U)$ then there is, for every $x \in X$, a neighbourhood U_x of x belonging to $\mathfrak{B}$ and a section $s(x) \in \mathcal{F}(U_x)$ such that $s_y = s(x)_y$ for all $y \in U_x$. Let $t(x) = u_{U_x}(s(x))$. Then $t(x) \in \mathcal{G}(U_x)$ and for all $y \in U_x$ we have that $t(x)_y = u_y(s(x)_y) = u_y(s_y)$. From the characterization (?) of $\mathcal{G}(U)$ as a subset of $\prod_{x \in U} \mathcal{G}_x$ it follows that $(u_x(s_x))_{x \in U} \in \mathcal{G}(U)$.

(4) Since $\mathcal{H}(U) \subseteq \mathcal{G}(U)$ for all U belonging to $\mathfrak{B}$ it follows from part (?) that $i_x : \mathcal{H}_x \to \mathcal{G}_x$ is injective for all $x \in X$. It follows from the definition of the homomorphism j that the isomorphism $\prod_{x \in U} i_x : \prod_{x \in U} \mathcal{H}_x \to \prod_{x \in U} \mathcal{G}_x$ induces the map $j_U : (\mathrm{id}_X)^*(\mathcal{H})(U) \to \mathcal{G}(U)$ for every open set U belonging to $\mathfrak{B}$. Hence j_U is injective.

In order to show that j_U is also surjective we choose a section $(t_x)_{x \in U}$ in $\mathcal{G}(U)$ with $t_x \in \mathcal{G}_x$. We shall show that $(t_x)_{x \in U}$ is contained in $(\mathrm{id}_X)^*(\mathcal{H})(U)$ when $(\mathrm{id}_X)^*(\mathcal{H})(U)$ is considered as a subset of $\prod_{x \in U} \mathcal{H}_x \subseteq \prod_{x \in U} \mathcal{G}_x$. It follows from the characterization of sheaves (?) that we, for each $x \in U$, can find an open neighbourhood U_x of x belonging to $\mathfrak{B}$ and a section $t(x) \in \mathcal{G}(U_x)$ such that $t_y = t(x)_y$ for all $y \in U_x$. We have that $\mathcal{H}(U_x) \subseteq \mathcal{G}(U_x)$, and since $i_x : \mathcal{H}_x \to \mathcal{G}_x$ is surjective we can find a neighbourhood V_x of x belonging to $\mathfrak{B}$ and a section $s(x) \in \mathcal{H}(V_x)$ such that t_x is the class of $(V_x, s(x))$ when we consider $s(x)$ as a section of $\mathcal{G}(V_x)$. Since t_x is also the class of $(U_x, t(x))$ we can find a neighbourhood W_x of x belonging to $\mathfrak{B}$, and contained in $U_x \cap V_x$ such that $(\rho_\mathcal{H})^{V_x}_{W_x}(s(x)) = (\rho_\mathcal{G})^{U_x}_{W_x}(t(x))$ in $\mathcal{H}(W_x) \subseteq \mathcal{G}(W_x)$. Consequently we have found a neighbourhood W_x of x belonging to $\mathfrak{B}$ and a section $r(x) = (\rho_\mathcal{H})^{V_x}_{W_x}(s(x))$ in $\mathcal{H}(W_x)$ such that $t_y = t(x)_y = (\rho_\mathcal{G})^{W_x}_y (\rho_\mathcal{G})^{U_x}_{W_x}(t(x)) = (\rho_\mathcal{H})^{W_x}_y (\rho_\mathcal{H})^{V_x}_{W_x}(s(x)) = (\rho_\mathcal{H})^{W_x}_y(r(x)) = r(x)_y$ for all $y \in W_x$. Consequently we have that $(t_x)_{x \in U}$ lies in $(\mathrm{id}_X)^*(\mathcal{H})(U)$ and we have shown that j_U is surjective.

(2.13) Definition. Let X be a topological space with a basis $\mathfrak{B}$ for the topology. Moreover let $u : \mathcal{F} \to \mathcal{G}$ be a homomorphism of sheaves on $\mathfrak{B}$. The sheaf $(\mathrm{id}_X)^*(\mathcal{H})$ is called the *image* of u and is denoted by !!Im(u) or by !!$u(\mathcal{F})$. A sheaf is a *subsheaf* of $\mathcal{G}$ if it is of the form $\mathrm{Im}(u)$ for some homomorphism $u : \mathcal{F} \to \mathcal{G}$ of sheaves.

We identify $\mathrm{Im}(u)_x$ with a subset of $\mathcal{G}_x$ via j_x and $\mathrm{Im}(u)(U)$ with a subset of $\mathcal{G}(U)$ via the inclusion $\prod_{x \in U} j_x : \prod_{x \in U} \mathrm{Im}(u)_x \to \prod_{x \in U} \mathcal{G}_x$.

The homomorphism u is *injective* if $w : \mathcal{F} \to \mathrm{Im}(u)$ is an isomorphism, and it is *surjective* if $j : \mathrm{Im}(u) \to \mathcal{G}$ is an isomorphism. When u is injective we sometimes write !!Im$(u) = u(\mathcal{F})$.

(2.14) Remark. Let $u : \mathcal{F} \to \mathcal{G}$ be a homomorphism of presheaves on $\mathfrak{B}$. It follows from Lemma (?) that for all $x \in X$ the map $v_x : \mathcal{F}_x \to \mathcal{H}_x$ is surjective and $i_x : \mathcal{H}_x \to \mathcal{G}_x$ is injective. Because of the inclusions $\mathcal{H}(U) \subseteq \mathcal{G}(U)$ for all U belonging to $\mathfrak{B}$, it is natural to identify $\mathcal{H}_x$ with a subset of $\mathcal{G}_x$ via the homomorphism i_x.

When $\mathcal{F}$ and $\mathcal{G}$ are sheaves it follows from Remark (?) that

$$w_x : \mathcal{F}_x \to \mathrm{Im}(u)_x$$

is surjective, and that

$$j_x : \mathrm{Im}(u)_x \to \mathcal{G}_x$$

is injective.

(2.15) Remark. It follows from Lemma (?) that a homomorphism $u : \mathcal{F} \to \mathcal{G}$ of sheaves on $\mathfrak{B}$ is injective if and only if $u_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for all open sets U belonging to $\mathfrak{B}$, or equivalently, if and only if $u_x : \mathcal{F}_x \to \mathcal{G}_x$ is injective for all $x \in X$. Moreover it follows from Lemma (?) that the homomorphism u is surjective if and only if $j_U : \mathrm{Im}(u)(U) \to \mathcal{G}(U)$ is surjective for all open sets U belonging to $\mathfrak{B}$, or equivalently, if and only if $u_x : \mathcal{F}_x \to \mathcal{G}_x$ is surjective for all $x \in X$.

In particular it follows from Lemma (?) that the homomorphism $w : \mathcal{F} \to \mathrm{Im}(u)$ is surjective, and from Lemma (?) that $j : \mathrm{Im}(u) \to \mathcal{G}$ is injective. Since $u = jw$ and w_x is surjective we have that

$$\mathrm{Im}(u_x) = \mathrm{Im}(j_x) = \mathrm{Im}(u)_x$$

as subsets of $\mathcal{G}_x$, for all $x \in X$.

(2.16) Exercises.

1. Let X be a topological space and let Y be a closed subset of X. Denote by $\iota : Y \to X$ the inclusion map. For every sheaf $\mathcal{G}$ on Y, describe the stalks of $\iota_*(\mathcal{G})$ at all points of X.

2. Let X be the topological space with two points x and Y with open sets $\{\emptyset, X, Y\}$. Moreover, let $\iota : Y \to X$ be the inclusion map. For every sheaf $\mathcal{G}$ on Y describe the sheaf $\iota_*(\mathcal{G})$ and its stalks.

3. Let X be a topological space with the discret topology. For each open subset U of X that contains at least two points let $\mathcal{F}(U) = \mathbf{Z}$ and let $\mathcal{F}(U) = 0$ otherwise. When $U \subseteq V$ is an inclusion of open sets in X we let $\rho_U^V = \mathrm{id}_{\mathbf{Z}}$ if U contains at least two points and otherwise be 0.

(1) Show that $\mathcal{F}$ is a presheaf on X.
(2) Describe the associated sheaf $(\mathrm{id}_X)^*(\mathcal{F})$ of $\mathcal{F}$.

4. Let Y be a topological space and $X = \{y\}$ the topological space that consist of a closed point y of Y. Moreover let $\psi : X \to Y$ be the inclusion map and let $\mathcal{G}$ be a sheaf on Y.

(1) Describe the sheaf $\psi^*(\mathcal{G})$.
(2) Describe the map $\rho_{\mathcal{G}} : \mathcal{G} \to \psi_*(\psi^*(\mathcal{G}))$.

5. Let X be a topological space and let $\psi : X \to Y$ be the map into a topological space Y consisting of a single point. Moreover let $\mathcal{F}$ be a sheaf on Y.

(1) Describe the sheaf $\psi_*(\mathcal{F})$.
(2) Describe the map $\sigma_{\mathcal{F}} : \psi^*(\psi_*(\mathcal{F})) \to \mathcal{F}$.

6. Let $\psi : X \to Y$ be a continuous map of two topological spaces X and Y. Moreover let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on X, respectively Y.

(1) Which map $\psi^*(\psi_*(\mathcal{F})) \to \mathcal{F}$ corresponds to the identity map $\psi_*(\mathcal{F}) \to \psi_*(\mathcal{F})$ by *adjunction*?
(2) Which map $\mathcal{G} \to \psi_*(\psi^*(\mathcal{G}))$ corresponds to the identity map $\psi^*(\mathcal{G}) \to \psi^*(\mathcal{G})$ by *adjunction*.

7. Let X be a topological space with the discrete topology, and let $\{G_x\}_{x \in X}$ be a collection of commutative groups.

(1) Let $\mathcal{F}(X) = (0)$ and $\mathcal{F}(U) = \prod_{x \in U} G_x$ for all non empty open subsets U of X different from X. Moreover let ρ_U^X be the zero map, and let ρ_U^V be the projection when $U \subseteq V$ and $U \neq \emptyset$ and $V \neq X$. Show that $\mathcal{F}$ is a presheaf and describe the associated sheaf.
(2) Let $\mathcal{F}(X) = \prod_{x \in X} G_x$ and let $\mathcal{F}(U) = (0)$ for all open subsets U of X different from X. For all inclusions $U \subseteq V$ of open sets in X different from X we let ρ_U^V be the zero map and we let $\rho_X^X = \mathrm{id}_X$. Show that $\mathcal{F}(X)$ is a presheaf and describe the associated sheaf.

8. Let X be a topological space. Moreover let $\mathcal{F}(X) = \mathbf{Z}$ and let $\mathcal{F}(U) = (0)$ for all other open subsets of X.

(1) Show that $\mathcal{F}$ with the restriction maps that are zero is a presheaf.
(2) Describe the associated sheaf.

9. Let X be a topological space and let $\{G_x\}_{x \in X}$ be a collection of commutative groups. Moreover let $\mathcal{F}(X) = (0)$ and let $\mathcal{F}(U) = \prod_{x \in U} G_x$ for all open subsets of X different from X. Define the restriction maps $\rho_U^V : \mathcal{F}(V) \to \mathcal{F}(U)$ to be the projections for all inclusions $U \subseteq V$ of open subsets of X with $V \neq X$, and otherwise to be the zero map.

(1) Show that $\mathcal{F}$ with the restriction maps ρ_U^V is a presheaf.
(2) Describe the associated sheaf.

10. Let X and Y be topological spaces where Y has the discrete topology. Moreover let $\mathcal{F}$ be the constant presheaf with fiber Y, and let $\mathcal{G}$ be the sheaf where $\mathcal{G}(U)$ consists of all continous maps $U \to Y$ for all open subsets U of X and the restrictions $(\rho_{\mathcal{G}})_U^V : \mathcal{G}(V) \to \mathcal{G}(U)$ sends $\varphi : V \to Y$ to its restriction $\varphi|U : U \to Y$ for all inclusions $U \subseteq V$ of open subsets of X.

Show that $\mathcal{G}$ is the sheaf associated to the presheaf $\mathcal{F}$.

11. Let $\mathcal{F}$ be a preshaf on the topological space X. Morover let $Y = \cup_{x \in X}(\mathcal{F}_x, x)$ be the disjoint union of the fibers $\mathcal{F}_x$ of $\mathcal{F}$ for all $x \in X$. We have a map $\varphi : Y \to X$ defined by mapping the pair (s_x, x) with $s_x \in \mathcal{F}_x$ to x. For every open subset U in X, and every section $s \in \mathcal{F}(U)$ we have a map $s_U : U \to Y$ that maps x to the pair (s_x, x). This map satisfies the equation $\pi s_U = \mathrm{id}_U$. The maps $s : U \to Y$ such that $\pi s = \mathrm{id}_U$ are called *sections* of π over U.

(1) Let $\{V_\alpha\}_{\alpha \in J}$ be the collection of all subsets of Y such that $s_U^{-1}(V_\alpha)$ is open

in U for all open subsets U of X and all sections $s \in \mathcal{F}(U)$. Show that Y is a topological space with the collection $\{V_\alpha\}_{\alpha \in J}$ as open sets.

(2) Show that the map π is continous for this topology on Y.

(3) For every open subset U of X we let $\mathcal{G}(U)$ be the collection of all continous sections of π over U, that is, the continous maps $\varphi : U \to Y$ such that $\pi\varphi = \mathrm{id}_U$, and let $(\rho_\mathcal{G})^V_U$ be the restriction of functions on V to functions on U. Show that $\mathcal{G}$ is a sheaf.

(4) Show that $\mathcal{G}$ is the associated sheaf of $\mathcal{F}$.

12. Let X be a topological space and let G be an abelian group. Fix a point $x \in X$. Let $\mathcal{G}(U) = G$ if $x \in U$ and let $\mathcal{G}(U) = (0)$ otherwise. Moreover define the restriction maps to be $(\rho_\mathcal{G})^V_U = \mathrm{id}_G$ if $x \in U$, and otherwise to be zero.

(1) Show that $\mathcal{G}$ is a presheaf on X.

(2) Describe the fiber of $\mathcal{G}$ at each point in the closure $\overline{\{x\}}$ of x in Y.

(3) Show that if $\iota : \overline{\{x\}} \to X$ is the inclusion map and $\mathcal{F}$ is the simple sheaf on $\overline{\{x\}}$ with fiber G, then $\mathcal{G} = \iota_*(\mathcal{F})$.

13. Let X be a topological space with the discrete topology and let $\{G_x\}_{x \in X}$ be a collection of commutative groups. For each open subset U of X we let $\mathcal{F}(U) = \prod_{x \in U} G_x$, and for every inclusion of open sets $U \subseteq V$ of X we let $\rho^V_U : \mathcal{F}(V) \to \mathcal{F}(U)$ be the projection. Moreover, for every point $x \in X$ and every open set U of X we let $i^x_U : G_x \to \mathcal{F}(U)$ be the map that sends $z \in G_x$ to $(z_y)_{y \in U} \in \mathcal{F}(U)$ with $z_x = z$ and $z_y = 0$ when $x \neq y$. Fix a point $x' \in X$. For each open set U of X we define a map $u_U : \mathcal{F}(U) \to \mathcal{F}(U)$ by $u_U = i^{x'}_U \rho^U_x$ when $x' \in U$ and $u_U = 0$ otherwise.

(1) Show that $\mathcal{F}$ with the restrictions ρ^V_U is a sheaf on X.

(2) Find the fiber $\mathcal{F}_x$ of $\mathcal{F}$ at x.

(3) Show that every sheaf $\mathcal{G}$ on X with fiber $\mathcal{G}_x = G_x$ is equal to $\mathcal{F}$.

(4) Show that $u : \mathcal{F} \to \mathcal{F}$ is a homomorphism of sheaves.

(5) Let $\mathcal{K}(U) = \mathrm{Ker}(u_U)$ for all open subsets U of X. Show that $\mathcal{K}$ with the restriction maps $(\rho_\mathcal{K})^V_U$ induced by the maps ρ^V_U is a sheaf.

(6) Let $\mathcal{G}(U) = \mathrm{Im}(u_U)$ for all open subsets U of X. Show that $\mathcal{G}$ with the restriction maps $(\rho_\mathcal{G})^V_U$ induced by the maps ρ^V_U is a presheaf.

(7) Let $\mathcal{H}(U) = \mathrm{Coker}(u_U)$ for all open subsets U of X. Show that $\mathcal{H}$ with the restriction maps $(\rho_\mathcal{H})^V_U$ induced by the maps $(\rho_\mathcal{G})^V_U$ is a presheaf.

(8) Is the presheaf $\mathcal{G}$ always a sheaf?

(9) Is the presheaf $\mathcal{H}$ always a sheaf?

3. Sheaves of groups and rings.

(3.1) Definition. Let X be a topological space and let $\mathfrak{B}$ be a basis for the topology. Moreover let $!\mathcal{F}!$ and $!\mathcal{A}!$ be presheaves defined on $\mathfrak{B}$. We say that $\mathcal{F}$ *takes values in groups*, or is a *presheaf of groups*, on $\mathfrak{B}$ if for every subset U of X belonging to $\mathfrak{B}$ we have that $\mathcal{F}(U)$ is a group, and for every inclusion $U \subseteq V$ of subsets of X belonging to $\mathfrak{B}$ the map $(\rho_\mathcal{F})^V_U : \mathcal{F}(V) \to \mathcal{F}(U)$ is a group homomorphism. Similarly we say that $\mathcal{A}$ *takes values in rings*, or is a *presheaf of rings* on $\mathfrak{B}$, if $\mathcal{A}(U)$ is a ring, and $(\rho_\mathcal{A})^V_U$ is a homomorphism of rings for all inclusions $U \subseteq V$ of sets belonging to $\mathfrak{B}$.

When $\mathcal{F}$ and $\mathcal{A}$ are sheaves on $\mathfrak{B}$ we say that $\mathcal{F}$ is a *sheaf of groups*, respectively that $\mathcal{A}$ is a *sheaf of rings*, on $\mathfrak{B}$ if they are presheaves of groups, respectively rings, when considered as presheaves on $\mathfrak{B}$.

(3.2) Remark. It follows from Remark (?) that for a sheaf of groups $\mathcal{F}$ we have that $\mathcal{F}(\emptyset) = (0)$.

(3.3) Stalks. Let $\mathcal{F}$ and $\mathcal{A}$ be presheaves of groups, respectively rings, defined on a basis $\mathfrak{B}$ of the topological space X. For every point x in X the stalk $\mathcal{F}_x$ has a natural structure as a group in such a way that the map $(\rho_\mathcal{F})^U_x : \mathcal{F}(U) \to \mathcal{F}_x$ is a group homomorphism, and $\mathcal{A}_x$ has a natural structure as a ring in such a way that the map $(\rho_\mathcal{A})^U_x$ is a ring homomorphism for all neighbourhoods U of x belonging to $\mathfrak{B}$. In order to define the addition on $\mathcal{F}_x$, and the multiplication on $\mathcal{A}_x$ we let s_x and t_x in $\mathcal{F}_x$ be the classes of pairs (V, s) and (W, t) where V and W belong to $\mathfrak{B}$, and $s \in \mathcal{F}(V)$ and $t \in \mathcal{F}(W)$. Then there is a neighbourhood U of x belonging to $\mathfrak{B}$ contained in $V \cap W$. We define the sum $s_x + t_x$ of s_x and t_x as the class in $\mathcal{F}_x$ of the pair $(U, (\rho_\mathcal{F})^V_U(s) + (\rho_\mathcal{F})^W_U(t))$. It is clear that the definition is independent of the choice of the representatives (V, s) and (W, t) of the classes s_x and t_x and of U. Moreover it is clear that $\mathcal{F}_x$ with the addition becomes a group in such a way that $(\rho_\mathcal{F})^U_x$ is a homomorphism of groups. When $s \in \mathcal{A}(V)$ and $t \in \mathcal{A}(W)$ we define the product $s_x t_x$ of s_x and t_x as the class of the pair $(U, (\rho_\mathcal{A})^V_U(s)(\rho_\mathcal{A})^W_U(t))$ in $\mathcal{A}_x$. It is clear that the definition is independent of the choice of the representatives (V, s), (W, t) of s_x and t_x, and of U. Moreover it is clear that $\mathcal{A}_x$ with the given addition and multiplication becomes a ring in such a way that $(\rho_\mathcal{A})^U_x$ is a homomorphism of rings.

(3.4) Definition. Let $\mathcal{F}$ and $!\mathcal{G}!$ be presheaves of groups on a basis $\mathfrak{B}$ of the topological space X. A homomorphism $u : \mathcal{F} \to \mathcal{G}$ of presheaves is a *homomorphism of presheaves of groups* on $\mathfrak{B}$ if for every subset U of X belonging to $\mathfrak{B}$ we have that $u_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is a group homomorphism. Let $\mathcal{A}$ and $!\mathcal{B}!$ be presheaves of rings defined on $\mathfrak{B}$. We have that a homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ of presheaves is a *homomorphism of presheaves of rings* on $\mathfrak{B}$ if for every subset U of X belonging to $\mathfrak{B}$ we have that $\varphi_U : \mathcal{A}(U) \to \mathcal{B}(U)$ is a homomorphism of rings.

When $\mathcal{A}$, $\mathcal{B}$, $\mathcal{F}$ and $\mathcal{G}$ are sheaves of rings respectively groups we say that the homomorphisms are homomorphisms of sheaves of groups respectively rings. A sheaf of rings $\mathcal{B}$ together with a homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ we call an $\mathcal{A}$-*algebra*.

(3.5) Remark. We easily see that if u and φ are homomorphisms of groups, respectively rings, then the map of stalks $u_x : \mathcal{F}_x \to \mathcal{G}_x$, respectively $\varphi_x : \mathcal{A}_x \to \mathcal{B}_x$, are homomorphisms of groups, respectively rings for all $x \in X$.

(3.6) The direct image. Let $\psi : X \to Y$ be a continuous map of topological spaces and $\mathcal{F}$ a presheaf of groups on X. The direct image $\psi_*(\mathcal{F})$ is then a presheaf of groups on Y. In fact, for every inclusion $V \subseteq W$ of open subsets of Y we have that $\psi_*(\mathcal{F})(V) = \mathcal{F}(\psi^{-1}(V))$ is a group, and $(\rho_{\mathcal{F}})_{\psi^{-1}(V)}^{\psi^{-1}(W)}$ is a group homomorphism. Hence $(\rho_{\psi_*(\mathcal{F})})_V^W = (\rho_{\mathcal{F}})_{\psi^{-1}(V)}^{\psi^{-1}(W)}$ is a group homomorphism. When $\mathcal{A}$ is a presheaf of rings we have that the direct image $\psi_*(\mathcal{A})$ is a presheaf of rings. In fact $\psi_*(\mathcal{A})(V) = \mathcal{A}(\psi^{-1}(V))$ is a ring and $(\rho_{\psi_*(\mathcal{A})})_V^W = (\rho_{\mathcal{A}})_{\psi^{-1}(V)}^{\psi^{-1}(W)}$ is a homomorphism of rings for all open subsets V, and all inclusions $V \subseteq W$ of open subsets of Y.

For every point x of X the map $(\psi_{\mathcal{F}})_x : \psi_*(\mathcal{F})_{\psi(x)} \to \mathcal{F}_x$ is a homomorphism of groups, and the map $(\psi_{\mathcal{A}})_x : \psi_*(\mathcal{A})_{\psi(x)} \to \mathcal{A}_x$ is a homomorphism of rings.

When $u : \mathcal{F} \to \mathcal{G}$ is a homomorphism of presheaves of groups on X we have that $\psi_*(u) : \psi_*(\mathcal{F}) \to \psi_*(\mathcal{G})$ is a homomorphism of presheaves of groups. In fact, for every open subset U of V, the map $\psi_*(u)_U$ comes from a homomorphism of groups $u_{\psi^{-1}(U)} : \mathcal{F}(\psi^{-1}(U)) \to \mathcal{G}(\psi^{-1}(U))$.

When $\varphi : \mathcal{A} \to \mathcal{B}$ is a homomorphism of presheaves of rings we have correspondingly that $\psi_*(\varphi) : \psi_*(\mathcal{A}) \to \psi_*(\mathcal{B})$ is a homomorphism of presheaves of rings.

(3.7) The inverse image. Let $\psi : X \to Y$ be a continuous map of topological spaces, and let $\mathfrak{B}$ be a basis for Y. Moreover let $\mathcal{G}$ be a presheaf of groups on $\mathfrak{B}$ and $\mathcal{B}$ a presheaf of rings on $\mathfrak{B}$. The inverse image $\psi^*(\mathcal{G})$ of $\mathcal{G}$ is a sheaf of groups on X and $\psi^*(\mathcal{B})$ is a sheaf of rings on X. In fact, we shall verify that for every open subset U of X the group structure on $\prod_{x \in U} \mathcal{G}_{\psi(x)}$ induces a group structure on $\psi^*(\mathcal{G})(U)$. Let $(s_{\psi(x)})_{x \in U}$ and $(t_{\psi(x)})_{x \in U}$ be elements of $\prod_{x \in U} \mathcal{G}_{\psi(x)}$ that belong to the subset $\psi^*(\mathcal{G})(U)$. Then there is an open nieghbourhood $V_{\psi(x)}$ of $\psi(x)$ belonging to $\mathfrak{B}$ and $s(x), t(x)$ in $\mathcal{G}(V_{\psi(x)})$ such that for all y in a neighbourhood of x contained in $U \cap \psi^{-1}(V_{\psi(x)})$ we have that $s_{\psi(x)} = s(x)_y$ and $t_{\psi(y)} = t(x)_y$. We have that $s(x) + t(x) \in \mathcal{G}(V_{\psi(x)})$ and $(s(x) + t(x))_y = s(x)_y + t(x)_y = s_{\psi(y)} + t_{\psi(y)}$. Hence $(s_{\psi(x)})_{x \in U} + (t_{\psi(x)})_{x \in U} = (s_{\psi(x)} + t_{\psi(x)})_{x \in U} = (s(x)_x + t(x)_x) = (s(x) + t(x))_x$. Hence we have that $(s_{\psi(x)})_{x \in U} + (t_{\psi(x)})_{x \in U}$ is contained in $\psi^*(\mathcal{G})(U)$ as we wanted to verify.

Moreover, for every inclusion $U \subseteq V$ of open subsets of X we have that the projection $\prod_{x \in V} \mathcal{G}_{\psi(x)} \to \prod_{x \in U} \mathcal{G}_{\psi(x)}$ is a group homomorphism that induces a group homomorphism $(\rho_{\psi^*(\mathcal{G})})_U^V : \psi^*(\mathcal{G})(V) \to \psi^*(\mathcal{G})(U)$. Similarly the ring structure on $\prod_{x \in U} \mathcal{B}_{\psi(x)}$ induces a ring structure on $\psi^*(\mathcal{B})(U)$, and the projection $\prod_{x \in V} \mathcal{B}_{\psi(x)} \to \prod_{x \in U} \mathcal{B}_{\psi(x)}$ is a ring homomorphism inducing a ring homomorphism $(\rho_{\psi^*(\mathcal{B})})_U^V : \psi^*(\mathcal{B})(V) \to \psi^*(\mathcal{B})(U)$, for every inclusion $U \subseteq V$ of sets belonging to $\mathfrak{B}$. For every $x \in X$ we have that the map $(\iota_{\mathcal{G}})_x : \mathcal{G}_{\psi(x)} \to \psi^*(\mathcal{G})_x$ is a group homomorphism, and the map $(\iota_{\mathcal{B}})_x : \mathcal{B}_{\psi(x)} \to \psi^*(\mathcal{B})_x$ is a ring homomorphism.

(3.8) Adjunction. Let $\mathcal{F}$ and $\mathcal{A}$ be sheaves of groups, respectively rings, on X, and let $\mathcal{G}$ and $\mathcal{B}$ be presheaves of groups, respectively rings, on Y. The *adjunction maps* $\rho_{\mathcal{G}} : \mathcal{G} \to \psi_*(\psi^*(\mathcal{G}))$ and $\sigma_{\mathcal{F}} : \psi^*(\psi_*(\mathcal{F})) \to \mathcal{F}$ are both homomorhism of presheaves of groups, and $\rho_{\mathcal{B}}$ and $\sigma_{\mathcal{A}}$ are homomorphisms of presheaves of rings. Consequently the adjunction $\mathrm{Hom}_X(\psi^*(\mathcal{G}), \mathcal{F}) \to \mathrm{Hom}_Y(\mathcal{G}, \psi_*(\mathcal{F}))$ of (3.9.1) and $\mathrm{Hom}_X(\psi^*(\mathcal{B}), \mathcal{A}) \to \mathrm{Hom}_Y(\mathcal{B}, \psi_*(\mathcal{A}))$ induce bijections between the subset consisting of homomorphisms of presheaves of groups, respectively of presheaves of rings.

(3.9) Definition. A *ringed space* is a pair consisting of a topological space X and a sheaf of rings $\mathcal{A}$. We shall often denote a ringed space by $!(X, \mathcal{O}_X)!$ where $\mathcal{O}_X$ is the sheaf of rings on the topological space X. The stalk of $\mathcal{O}_X$ at a point x of X we denote by $\mathcal{O}_{X,x}$. A homomorphism $(\psi, \theta) : (X, \mathcal{A}) \to (Y, \mathcal{B})$ of ringed spaces consists of a continuous map $\psi : X \to Y$ of topological spaces and a homomorphism $\theta : \mathcal{B} \to \psi_*(\mathcal{A})$ of sheaves of rings. We say that $(X, \mathcal{A})$ is a *local ringed space* if $\mathcal{A}_x$ is a local ring for all $x \in X$. A *local homomorphism of local ringed spaces* (ψ, θ) : $(X, \mathcal{A}) \to (Y, \mathcal{B})$ is a homomorphism of ringed spaces such that $(\theta)^\sharp_x : \psi^*(\mathcal{B})_x \to \mathcal{A}_x$ maps the maximal ideal in $\psi^*(\mathcal{B})_x = \mathcal{B}_{\psi(x)}$ to the maximal ideal in $\mathcal{A}_x$ for all $x \in X$.

(3.10) Remark. The ringed spaces with morphism form a category, as does the locally ringed spaces with local homomorphisms.

(3.11) Exercises.

1. Let X be a topological space and G an abelian group. For every non-empty open subset U of X we let $\mathcal{F}(U) = G^U$ be all maps $U \to G$. Let $\mathcal{F}(\emptyset) = \{0\}$. For every inclusion $U \subseteq V$ of open subsets of X we define $\rho^V_U : \mathcal{F}(V) \to \mathcal{F}(U)$ to be the map that takes a section $s : V \to G$ to its restriction $s|U : U \to G$. Show that $\mathcal{F}$ with the maps ρ^V_U has a natural structure as a sheaf of groups.

2. Let X be a topological space and A a ring. For every non-empty open subset U of X we let $\mathcal{A}(U) = A^U$ be all maps $U \to A$. For every inclusion $U \subseteq V$ of open subsets of X we define $\rho^V_U : \mathcal{A}(V) \to \mathcal{A}(U)$ to be the map that takes a section $s : V \to A$ to its restriction $s|U : U \to A$. Let $\mathcal{A}(\emptyset) = \{\emptyset\}$. Show that $\mathcal{A}$ with the maps ρ^V_U has a natural structure as a sheaf of rings.

3. Let $X = \{x_0, x_1\}$ have the topology with open sets $\emptyset, X, \{x_0\}$. We let $\mathcal{A}(x) = \mathbf{Z}_{(p)}$, $\mathcal{A}(x_0) = \mathbf{Q}$, and $\mathcal{F}(\emptyset) = \{0\}$. Moreover, we let $\rho^X_{x_0} : \mathbf{Z}_{(p)} \to \mathbf{Q}$ be the inclusion map.

(1) Show that $\mathcal{A}$ is a sheaf and that the pair $(X, \mathcal{A})$ is a local ringed space.
(2) Let $Y = \{y_0\}$ and let $\mathcal{B}$ be the sheaf $\mathcal{B}(Y) = \mathbf{Q}$ on Y. Moreover let $\psi : Y \to X$ be the map that takes y_0 to x_0. Show that there is a unique homomorphism of sheaves of algebras $\mathcal{A} \to \psi_*(\mathcal{B})$ which is the identity on $\mathbf{Q}$ over $\{x_0\}$, and the inclusion $\mathbf{Z}_{(p)} \to \mathbf{Q}$ on X.
(3) Show that the map of part (2) is a local map of local ringed spaces.
(4) Let $\psi : Y \to X$ be the map that takes y_0 to x_1. Show that there is a map of sheaves of rings $\mathcal{A} \to \psi_*(\mathcal{B})$ which is the inclusion $\mathbf{Z}_{(p)} \to \mathbf{Q}$ on X and the zero map $\mathbf{Q} \to \{0\}$ on $\{x_0\}$. Show that this is not a local homomorphism of local ringed spaces.

4. Let X be a topological space and let $\mathcal{F}$ be a sheaf of groups on X. Show that for every open subset U of X, and every section $s \in \mathcal{F}(U)$ we have that the set consisting of $x \in U$ such that $s_x = 0$ is open in X.

Modules

1. Modules.

n **(1.1) Definition.** Let A be a ring. An A-*module* is an abelian group !!M and an
n *operation* of A on M which to !!$f \in A$ and !!$x \in M$ gives a product $fx \in M$ such that for all f, g in A and x, y in M we have:

(1) $1x = x$.
(2) $(f + g)x = fx + gx$.
(3) $f(x + y) = fx + fy$.
(4) $f(gx) = (fg)x$.

(1.2) Remark. An operation of A on M, or a product of the elements of A with the elements of M, is the same as a map $A \times M \to M$.

(1.3) Example. Every abelian group G is a $\mathbf{Z}$-module under the multiplication that to a positive integer n and an element $x \in G$ associates the sum $nx = x + x + \cdots + x$ of x with itself n times, and $(-n)x = -nx = -x - x - \cdots - x$ is the sum of $-x$ with itself n times. We let $0x = 0$.

(1.4) Example. Let A be a ring. Multiplication on A makes A into an A-module.

n **(1.5) Definition.** Let A be a ring and M an A-module. A *submodule* !!L of M is a subgroup of M such that for all f in A and $x \in L$ we have $fx \in L$.

We say that a submodule L of M is *properly contained* in M if L is different from M, and that L is a *proper* submodule of M if it is non-zero and properly contained in M.

(1.6) Remark. Let M be an A-module and L a submodule. The A-module structure on M *induces* an A-module structure on L.

(1.7) Example. Consider the ring A as a module over itself. A subgroup $\mathfrak{a}$ of A is an ideal if and only if $\mathfrak{a}$ is a submodule of A.

n **(1.8) Definition.** Let A be a ring and M and N two modules. A map !!$u : M \to N$ is A-*linear* or an A-*module homomorphism* if, for all $f \in A$ and x, y in M, we have:

(1) $u(fx) = fu(x)$.
(2) $u(x + y) = u(x) + u(y)$.

An A-module homomorphism is an *isomorphism* if it has an inverse, or equivalently if it is bijective. The set theoretic inverse is then automatically a homomorphism of A-modules. We denote the A-module homomorphisms from M to N by $\mathrm{Hom}_A(M, N)$.

(1.9) Remark. We know that $\mathrm{Hom}_A(M,N)$ is a group under the addition that takes $u : M \to N$ and $v : M \to N$ to the homomorphism $(u+v) : M \to N$ defined by $(u+v)(x) = u(x)+v(x)$ for all $x \in M$. The ring A operates on $\mathrm{Hom}_A(M,N)$ by the product that takes $f \in A$ and u to fu defined by $(fu)(x) = f(u(x))$ for all $x \in M$. It is clear that the group $\mathrm{Hom}_A(M,N)$ becomes an A-module under this operation.

n **(1.10) Remark.** Let $u : M \to N$ and !!$v : N \to P$ be A-module homomorphisms. Then id_M and $vu : M \to P$ are A-module homomorphisms. In other words, the A-modules with A-linear homomorphisms form a category. We call this category the *category of* A-modules.

(1.11) Residue modules. Let M be an A-module and L a submodule. The A-module structure on M induces a unique A-module structure on the residue group
n M/L such that the canonical homomorphism !!$u_{M/L} : M \to M/L$ is an A-module homomorphism. The multiplication of an element $f \in A$ with the residue class $u_{M/L}(x)$ of an element $x \in M$ is defined by $fu_{M/L}(x) = u_{M/L}(fx)$. It is clear that the definition is independent of the choice of representative x of the class $u_{M/L}(x)$.

(1.12) Example. Let A be a ring and $u : M \to N$ a homomorphism of A-modules. The *kernel* $\mathrm{Ker}(u) = \{x \in M : u(x) = 0\}$ of u is a submodule, and the *image* $\mathrm{Im}(u) = \{u(x) : x \in M\}$ is a submodule of N. The cokernel $N/\mathrm{Im}(u)$ of N is an A-module under the multiplication defined by $fu_{N/\mathrm{Im}(u)}(y) = u_{N/\mathrm{Im}(u)}(fy)$ for all $f \in A$ and $y \in N$.

(1.13) Lemma. *Let A be a ring, and let $u : M \to N$ be a homomorphism of A-modules. Moreover let L be a submodule of M.*

(1) *The homomorphism u factors via the canonical map $u_{M/L} : M \to M/L$ and an A-linear homomorphism $v : M/L \to N$ if and only if $u(L) = 0$. When v exists it is unique.*
(2) *If v exists, then it is injective if and only if $L = \mathrm{Ker}(u)$.*

Proof. (1) Assume that v exists. Then, for each $x \in L$, we have $u(x) = vu_{M/L}(x) = v(0) = 0$. Conversely, if $u(L) = 0$ we can define the homomorphism $v : M/L \to N$ by $v(u_{M/L}(x)) = u(x)$. The homomorphism v is independent of the choice of representative x of the class of $u_{M/L}(x)$. In fact if $u_{M/L}(x) = u_{M/L}(y)$ we have that $x-y \in L$ and consequently that $u(x) = u(x-y+y) = u(x-y)+u(y) = 0+u(y) = u(y)$. Finally, since $u_{M/L}$ is surjective, we have that v is uniquely determined by the relation $u(x) = v(u_{M/L}(x))$.

(2) Since $u(x) = v(u_{M/L}(x))$ we have that $\mathrm{Ker}(v) = 0$ if and only if $u(x) = 0$ is equivalent to $u_{M/L}(x) = 0$. However we have that $u_{M/L}(x) = 0$ if and only if $x \in L$.

(1.14) Lemma. *Let N be an A-module and L, M two submodules. Then there is an isomorphism $L/(L \cap M) \to (L+M)/M$ of A-modules which maps the class of an element $x \in L$ in $L/(L \cap M)$ to the class of x in $(L+M)/M$.*

Proof. It is clear that the map $L/L \cap M \to (L+M)/M$ is well defined and gives an A-module homomorphism. The homomorphism is injective because the class of x is mapped to zero in $(L+M)/M$ exactly when $x \in M$ and thus $x \in (L \cap M)$. It is surjective because every class in $(L+M)/M$ is represented by an element $x \in L$.

(1.15) Operations on modules. Let A be a ring and !!$\{M_\alpha\}_{\alpha \in I}$ a collection of A modules M_α. The *direct product* !!$\prod_{\alpha \in I} M_\alpha$ of the groups M_α becomes an A-module when we define the *product* of $f \in A$ and $(x_\alpha)_{\alpha \in I} \in \prod_{\alpha \in I} M_\alpha$ by $f(x_\alpha)_{\alpha \in I} = (fx_\alpha)_{\alpha \in I}$. We call the A-module $\prod_{\alpha \in I} M_\alpha$ the *direct product* of the modules M_α. We have a *canonical projection* $\prod_{\alpha \in I} M_\alpha \to M_\beta$ for all $\beta \in I$. We have that $p_\beta((x_\alpha)_{\alpha \in I}) = x_\beta$.

The direct sum !!$\oplus_{\alpha \in I} M_\alpha$ of the groups M_α for $\alpha \in I$ becomes an A-submodule of the product $\prod_{\alpha \in I} M_\alpha$. We call the A-module $\oplus_{\alpha \in I} M_\alpha$ the *direct sum* of the A-modules M_α. The direct sum comes with a canonical homomorphism $h_\beta \to \oplus_{\alpha \in I} M_\alpha$ to factor β for all $\beta \in I$. We have that $h_\beta(x) = (x_\alpha)_{\alpha \in I}$ with $x_\beta = x$ and $x_\alpha = 0$ when $\alpha \neq \beta$.

More generally, if !!$\{A_\alpha\}_{\alpha \in I}$ is a collection of rings A_α and $\{M_\alpha\}_{\alpha \in I}$ is a collection of A_α-modules M_α we have that $\prod_{\alpha \in I} M_\alpha$ becomes an $(\prod_{\alpha \in I} A_\alpha)$-module when we define the product of $(f_\alpha)_{\alpha \in I} \in \prod_{\alpha \in I} A_\alpha$ and $(x_\alpha)_{\alpha \in I} \in \prod_{\alpha \in I} M_\alpha$ by $(f_\alpha)_{\alpha \in I}(x_\alpha)_{\alpha \in I} = (f_\alpha x_\alpha)_{\alpha \in I}$. The *direct sum* $\oplus_{\alpha \in I} M_\alpha$ becomes a sub $(\prod_{\alpha \in I} A_\alpha)$-module of $\prod_{\alpha \in I} M_\alpha$.

When all the modules M_α are submodules of the same A-module M the sum $\sum_{\alpha \in I} M_\alpha$ of the groups M_α is an A-submodule of M. It is the smallest submodule of M containing the submodules M_α for $\alpha \in I$. We have that $\sum_{\alpha \in I} M_\alpha$ consists of all sums $\sum_{\beta \in J} x_\beta$ for all finite subsets J of I and all $x_\beta \in M_\beta$ for $\beta \in J$.

When $I = \{1, 2, \dots, n\}$ we write !!$M_1 \times M_2 \times \cdots \times M_n$ for the direct product and !!$M_1 \oplus M_2 \oplus \cdots \oplus M_n$ for the direct sum of the modules $M_1, M_2, \dots, M_n$.

(1.16) Remark. The direct product $\prod_{\alpha \in I} M_\alpha$ with the canonical projections $p_\beta : \prod_{\alpha \in I} M_\alpha \to M_\beta$ and the direct sum $\oplus_{\alpha \in I} M_\alpha$ with the canonical homomorphism $h_\beta : M_\beta \to \oplus_{\alpha \in I} M_\alpha$ are the product, respectively coproduct, in the category of A-modules. That is the direct product and direct sum of a collection of modules is a product, respectively a coproduct, in the *categorical sense.*

(1.17) Definition. Let M be an A-module and let $\{x_\alpha\}_{\alpha \in I}$ be a family of elements in M. The sum $\sum_{\alpha \in I} Ax_\alpha$ of the submodules Ax_α of M we call the *submodule of M generated by the elements x_α*. When $x = \sum_{\beta \in J} f_\beta x_\beta$ with $f_\beta \in A$ and where J is a finite subset of I we say that x is a *linear combination* of the elements x_α.

We say that the *elements $\{x_\alpha\}_{\alpha \in I}$ generate M* when $M = \sum_{\alpha \in I} Ax_\alpha$. A module that is generated by a finite number of elements is called *finitely generated.*

The elements $\{x_\alpha\}_{\alpha \in I}$ are *linearly independent* if a relation $\sum_{\beta \in J} f_\beta x_\beta = 0$ with $f_\beta \in A$ and where J is a finite subset of I implies that $f_\beta = 0$ for all $\beta \in J$. A relation $\sum_{\beta \in J} f_\beta x_\beta = 0$ is called a *linear relation* between the elements x_α.

The module M is *free with basis* $\{x_\alpha\}_{\alpha \in I}$ if the set $\{x_\alpha\}_{\alpha\in I}$ generates M and consists of linearly independent elements.

(1.18) Remark. The elements $\{x_\alpha\}_{\alpha\in I}$ in the A-module M are linearly independent if and only if each element x of the module $\sum_{\alpha\in I} Ax_\alpha$ can be expressed *uniquely* on the form $x = \sum_{\alpha\in I} f_\alpha x_\alpha$ with $f_\alpha \in A$ where at most a finite number of the f_α are different from 0. In fact if every such expression is unique, we have that $\sum_{\beta\in J} f_\beta x_\beta = 0 = \sum_{\beta\in J} 0x_\beta$ for a finite subset J of I implies that $f_\beta = 0$ for all $\beta \in J$. Conversely if the elements x_α are linearly independent two expressions $\sum_{\alpha\in I} f_\alpha x_\alpha = x = \sum_{\alpha\in I} g_\alpha x_\alpha$ for x with f_α and g_α in A where at most a finite number of the f_α and g_α are different from zero imply that $\sum_{\alpha\in I}(f_\alpha - g_\alpha)x_\alpha = 0$ and consequently that $f_\alpha = g_\alpha$ for all $\alpha \in I$.

(1.19) Example. Let A be a ring and I a set. For each $\alpha \in I$ we denote by $e(\alpha)$ the element $(e_\beta)_{\beta\in I}$ in $A^{(I)}$ with $e_\alpha = 1$ and $e_\beta = 0$ when $\alpha \neq \beta$. Then $A^{(I)}$ is a free A-module with basis $\{e(\alpha)\}_{\alpha\in I}$. In fact the elements $\{e(\alpha)\}_{\alpha\in I}$ generate $A^{(I)}$ because the coordinates of an element $(f_\alpha)_{\alpha\in I}$ in $A^{(I)}$ satisfy $f_\alpha = 0$ except for at most a finite number of $\alpha \in I$, and consequently $(f_\alpha)_{\alpha\in I} = \sum_{\alpha\in I} f_\alpha e(\alpha)$. Moreover the elements $e(\alpha)$ for $\alpha \in I$ are linearly independent because a relation $\sum_{\alpha\in I} f_\alpha e(\alpha) = 0$ with $f_\alpha \in A$ and at most a finite number of the f_α different from zero implies that $(f_\alpha)_{\alpha\in I} = \sum_{\alpha\in I} f_\alpha e(\alpha) \in A^{(I)}$ is 0. That is $f_\alpha = 0$ for $\alpha \in I$. We call the basis $\{e(\alpha)\}_{\alpha\in I}$ the *canonical basis* of $A^{(I)}$.

(1.20) Proposition. *Let M be a free A-module with basis $\{x_\alpha\}_{\alpha\in I}$. Moreover let N be an A-module and let $\{y_\alpha\}_{\alpha\in I}$ be elements in N. There is a unique A-module homomorphism $u : M \to N$ such that $u(x_\alpha) = y_\alpha$ for all $\alpha \in I$.*

Proof. Since the elements $\{x_\alpha\}_{\alpha\in I}$ generate M every element $x \in M$ can be written as $x = \sum_{\beta\in J} f_\beta x_\beta$ where J is a finite subset of I and $f_\beta \in A$. If u exists we have that $u(x) = \sum_{\beta\in J} f_\beta u(x_\beta)$ and the conditions $u(x_\alpha) = y_\alpha$ for $\alpha \in I$ determine u uniquely.

Every element x in M can be written uniquely as $x = \sum_{\alpha\in I} f_\alpha x_\alpha$ where $f_\alpha \in A$ and with at most a finite number of the f_α different from zero. Consequently we can define a map $u : M \to N$ by $u(x) = \sum_{\alpha\in I} f_\alpha y_\alpha$. We have that u is A-linear for if $x' = \sum_{\alpha\in I} f'_\alpha x_\alpha$ with the $f'_\alpha \in A$ and at most a finite number of the f'_α different from zero we obtain that $u(x + x') = u(\sum_{\alpha\in I}(f_\alpha + f'_\alpha)x_\alpha) = \sum_{\alpha\in I}(f_\alpha + f'_\alpha)y_\alpha = \sum_{\alpha\in I} f_\alpha y_\alpha + \sum_{\alpha\in I} f'_\alpha y_\alpha = u(x) + u(x')$. Moreover for $f \in A$ we have that $u(fx) = u(\sum_{\alpha\in I} ff_\alpha x_\alpha) = \sum_{\alpha\in I} ff_\alpha y_\alpha = f\sum_{\alpha\in I} f_\alpha y_\alpha = fu(x)$.

(1.21) Definition. When A is a field we call an A-module an *A-vector space*, and the elements of M we call *vectors*. We also say that the vector space is *defined over* K. A submodule of M is called a *subspace*.

(1.22) Theorem. *Let $M \neq 0$ be a vector space over a field K. Moreover let $\{x_\alpha\}_{\alpha\in J}$ be generators for M such that $\{x_\alpha\}_{\alpha\in H}$ with $H \subseteq J$ are linearly independent vectors.*

Then there is a collection I of indices with $H \subseteq I \subseteq J$ such that $\{x_\alpha\}_{\alpha\in I}$ is a basis for M.

In particular every vector space has a basis.

Proof. Let $\mathcal{I}$ be the collection of subsets I' of J containing H such that the vectors $\{x_\alpha\}_{\alpha\in I'}$ are linearly independent. We have that $\mathcal{I}$ is not empty since it contains
n H. Every chain in $\mathcal{I}$ has a maximal element. In fact let !!$\{I'_\beta\}_{\beta\in G}$ be a chain of subsets I'_β in $\mathcal{I}$. Then the collection of elements $\cup_{\beta\in G}\{x_\alpha\}_{\alpha\in I'_\beta}$ consists of linearly independent elements since linear relations between the elements $\{x_\alpha\}_{\alpha\in J}$ involve only a finite number of these elements. It follows from Zorns Lemma that $\mathcal{I}$ contains a maximal subset I.

Let L be the subspace generated by the linearly independent vectors $\{x_\alpha\}_{\alpha\in I}$. We shall prove that $L = M$. Assume to the contrary that $L \subset M$. Since the elements $\{x_\alpha\}_{\alpha\in J}$ generate M there must then be an index $\beta \in J \setminus I$ such $x_\beta \in M \setminus L$. We shall prove that then the vectors $\{x_\beta\} \cup \{x_\alpha\}_{\alpha\in I}$ are linearly independent. This is impossible since I is maximal in $\mathcal{I}$, and thus contradicts the assumption that $L \subset M$. To prove that the vectors $\{x_\beta\} \cup \{x_\alpha\}_{\alpha\in I}$ are linearly independent we observe that a relation $fx_\beta + \sum_{\alpha\in I} f_\alpha x_\alpha = 0$ with f, f_α in A where at most a finite number of the f_α are different from zero, implies that $f \neq 0$ since the x_α for $\alpha \in I$ are linearly independent. Hence we have the equality $x_\beta = -\sum_{\alpha\in I}(f_\alpha/f)x_\alpha$, which is impossible since $x_\beta \notin L$. This contradicts the assumption that $L \subset M$. We have proved that $M = L$ and thus that the first assertion of the Theorem holds.

The second part follows since we can take $\{x_\alpha\}_{\alpha\in J}$ to be the collection of all vectors in M, and H to be empty.

(1.23) Proposition. *Let $M \neq 0$ be a vector space defined over a field K. Moreover let $\{x_\alpha\}_{\alpha\in I}$ be a basis for M, and let $\{y_\gamma\}_{\gamma\in J}$ be a collection of linearly independent*
n *vectors. Then there is an injective map of sets !!$\iota : J \to I$ such that $\{x_\alpha\}_{\alpha\in I\setminus\iota(J)} \cup \{y_\gamma\}_{\gamma\in J}$ is a basis for M.*

In particular, when M is a finitely generated over K, the least number n of generators of M is equal to the largest number of linearly independent elements of M, and n is equal to the number of elements of any basis of M.

n *Proof.* Let !!$\mathcal{L}$ be the collection of all pairs (L, ι) consisting of a subset L of J and an injective map of sets $\iota : L \to I$, and where $\{x_\alpha\}_{\alpha\in I\setminus\iota(L)} \cup \{y_\gamma\}_{\gamma\in L}$ is a basis for M. Then $\mathcal{L}$ is not empty because it contains the empty set. We order the elements in $\mathcal{L}$ by $(L', \iota') \leq (L'', \iota'')$ if $L' \subseteq L''$ and $\iota''|L' = \iota'$. Since every element in M can be expressed as a linear combination of a finite number of basis elements we have that every chain in $\mathcal{L}$ has a maximal elements. It follows from Zorns Lemma that $\mathcal{L}$ has a maximal elements (L, ι). To prove the first part of the Proposition it suffices to show that $L = J$. Assume to the contrary that $L \subset J$. Choose an element $\delta \in J \setminus L$. Then we have that $y_\delta = \sum_{\alpha\in I\setminus\iota(L)} f_\alpha x_\alpha + \sum_{\gamma\in L} g_\gamma y_\gamma$ where f_α and g_γ are in A and at most a finite number for the f_α and g_γ are different from zero. Since the elements $\{y_\gamma\}_{\gamma\in J}$ are linearly independent we have that there there is a $\beta \in I \setminus \iota(L)$ such that

$f_\beta \neq 0$. Then $x_\beta = f_\beta^{-1}(y_\delta - \sum_{\alpha \in I \setminus \{\iota(L) \cup \{\beta\}\}} f_\alpha x_\alpha - \sum_{\gamma \in L} g_\gamma x_\gamma)$. It follows that $\{y_\gamma\}_{\gamma \in L \cup \{\delta\}} \cup \{x_\alpha\}_{\alpha \in I \setminus \{\iota(L) \cup \{\beta\}\}}$ is a basis for M. Hence the pair $(L \cup \{\delta\}, \iota_\delta)$ with $\iota_\delta(\delta) = \beta$ and $\iota_\delta | L = \iota$ is in $\mathcal{L}$ and it is strictly greater than the pair (L, ι). This is impossible since (L, ι) is maximal in $\mathcal{L}$, and therefore contradicts the assumption that $L \subset J$. Hence $L = J$ and we have proved the first part of the Proposition.

Let $x_1, x_2, \dots, x_n$ be a set of generators of M with the least number n of elements. It follows from Proposition (?) that we can find a subset of $\{x_1, x_2, \dots, x_n\}$ which is a basis for M. Since the elements of a basis generate M and n is the minimal number of elements in a system of generators we have that $x_1, x_2, \dots, x_n$ is a basis. Hence it follows from the first part of the Proposition that any set of linearly independent elements has at most n elements. In particular any basis for M has at most n elements. However, every basis for M generates M and thus has at least n elements by the minimality of n. Hence each basis has exactly n elements. We have proved the last part of the Proposition.

(1.24) Proposition. *Let $M \neq 0$ be a finitely generated free A-module. Then all the bases of M have the same number of elements as the least number of generators for M.*

Proof. Let $\{x_\alpha\}_{\alpha \in I}$ be a basis for the A-module M. Choose a maximal ideal $\mathfrak{m}$ of A. The classes z_α of x_α in the $A/\mathfrak{m}$-vector space $N = M/\mathfrak{m}M$ for all $\alpha \in I$ is a basis of N. In fact a relation between the elements z_α is the same as a relation $\sum_{\alpha \in I} f_\alpha x_\alpha \in \mathfrak{m}$ with $f_\alpha \in A$ and at most a finite number of the f_α different from zero, and where $f_\alpha \in A \setminus \mathfrak{m}$ whenever $f_\alpha \neq 0$. That is, we have $\sum_{\alpha \in I} f_\alpha x_\alpha = \sum_{\alpha \in I} g_\alpha x_\alpha$ with all the $g_\alpha \in \mathfrak{m}$ and at most a finite number of the g_α different from zero. Since the elements $\{x_\alpha\}_{\alpha \in I}$ is a basis for M we must have that $f_\alpha = g_\alpha$ for $\alpha \in I$. But this is impossible unless $f_\alpha = 0$ for all α because $f_\alpha \notin \mathfrak{m}$ when $f_\alpha \neq 0$, and $g_\alpha \in \mathfrak{m}$.

Let $y_1, y_2, \dots, y_n$ be a set of generators of M with the least number of elements. Then the classes of $y_1, y_2, \dots, y_n$ in N generate N. Since the $\{z_\alpha\}_{\alpha \in I}$ is a basis for N it follows from Proposition (?) that the set I is finite and has at most n elements. Since the elements $\{x_\alpha\}_{\alpha \in I}$ for $\alpha \in I$ is a basis for M by assumption and in particular generate M we have that I has at least n elements. Hence I contains exactly n elements, and we have proved the Proposition.

(1.25) Definition. Let $M \neq 0$ be a finitely generated free A-module. The common number of elements of the bases for M is called the *rank* of M and denoted $\mathrm{rk}(M) = \mathrm{rk}_A(M)$. When K is a field and N is a finitely generated vector space over K the rank of N is called the *dimension* of N and written $\dim_K(N)$. We let $\mathrm{rk}_A(0) = 0$ and $\dim_K(0) = 0$.

(1.26) Remark. Let L be a subspace of a finitely generated vector space M over a field K. Then L is finitely generated and if L is a proper subspace then $\dim_K(L) < \dim_K(M)$. This is because every basis of L can be extended to a basis of M.

(1.27) Theorem. (Nakayamas Lemma) *Let A be a ring and let $\mathfrak{a}$ be an ideal of A. Moreover let M be a be finitely generated A-module. The following assertions hold:*

(1) *When $M = \mathfrak{a}M$ there is an $f \in \mathfrak{a}$ such that $(1+f)M = 0$.*
(2) *When $M = \mathfrak{a}M$ and $\mathfrak{a}$ is contained in all the maximal ideals of A we have that $M = 0$.*
(3) *When L is a submodule of M such that $M = L + \mathfrak{a}M$, and $\mathfrak{a}$ is contained in all the maximal ideals of A, then we have that $M = L$.*

Proof. (1) We show assertion (1) by induction on the least number of generators for the module M. When M has one generator the assertion is clear. Assume that assertion (1) holds for all modules that can be generated by $n-1$ elements, and assume that M has n generators $x_1, x_2, \dots, x_n$. The residue module M/Ax_n can be generated by $n-1$ elements and $\mathfrak{a}(M/Ax_n) = \mathfrak{a}M + Ax_n/Ax_n = M + Ax_n/Ax_n = M/Ax_n$. Hence it follows from the induction assumption that there is an elements $g \in \mathfrak{a}$ such that $(1+g)(M/Ax_n) = 0$, that is, such that $(1+g)M \subseteq Ax_n$. It follows that there are elements $g_1, g_2, \dots, g_{n-1}$ in A such that $(1+g)x_i = g_i x_n$ for $i = 1, 2, \dots, n-1$. Since $\mathfrak{a}M = M$ we can find elements $h_1, h_2, \dots, h_n$ in $\mathfrak{a}$ such that $x_n = h_1x_1 + h_2x_2 + \cdots + h_nx_n$. We obtain that $(1+g)(1-h_n)x_n = (1+g)h_1x_1 + (1+g)h_2x_2 + \cdots + (1+g)h_{n-1}x_{n-1} = g_1h_1x_n + g_2h_2x_n + \cdots + g_{n-1}h_{n-1}x_n$. Let $f_n = (1-g)(1-h_n) - g_1h_1 - g_2h_2 - \cdots - g_{n-1}h_{n-1} - 1$. Then we have that $f_n \in \mathfrak{a}$ and $(1+f_n)x_n = 0$. Similarly we can find elements $f_1, f_2, \dots, f_{n-1}$ in $\mathfrak{a}$ such that $(1+f_i)x_i = 0$ for $i = 1, 2, \dots, n-1$. Let $f = (1+f_1)(1+f_2)\cdots(1+f_n) - 1$. Then $f \in \mathfrak{a}$ and $(1+f)M = 0$, and we have proved assertion (1).

(2) It follows from assertion (1) that there is an element $f \in \mathfrak{a}$ such that $(1+g)M = 0$. When $\mathfrak{a}$ is contained in all maximal ideals of A the element $1+f$ is a unit in A. In fact if $1+f$ is not a unit there is a maximal ideal $\mathfrak{m}$ in A containing $1+f$. However this is impossible since $f \in M$ such that $1+f \in \mathfrak{m}$ implies that $1 \in \mathfrak{m}$. Since $1+f$ is a unit there is an element $g \in A$ such that $g(1+f) = 1$, and we obtain that $M = g(1+f)M = 0$, as we wanted to prove.

(3) We have that $\mathfrak{a}(M/L) = (\mathfrak{a}M + L)/L = M/L$. It follows from assertion (2) that $M/L = 0$, that is, we have $M = L$.

(1.28) Lemma. *Let A be a ring and M an A-module. Moreover let $\sum_{j=1}^n f_{ij}x_j = 0$ be equations in M for $i = 1, 2, \dots, n$, with $f_{ij} \in A$ and $x_j \in M$ for $i, j = 1, 2, \dots, n$. Then there are equations $\sum_{j=m}^n g_{ij}x_j = 0$ in M for $i = m, m+1, \dots, n$, where the elements g_{ij} are sums of elements of the form $\pm\prod_{i,j=1}^n f_{ij}^{n_{ij}}$ with $\sum_{i,j=1}^n n_{ij} = 2^{m-1}$. Moreover it is only the coefficients g_{ii} for $i = 1, 2, \dots, n$ where the sums contain products of the form $\pm\prod_{i=1}^n f_{ii}^{n_{ii}}$, and in g_{ii} there is exactly one such term $f_{11}^{2^{m-2}} f_{22}^{2^{m-3}} \cdots f_{(m-1)(m-1)} f_{ii}$.*

Proof. We prove the Lemma by induction on m. The Lemma clearly holds for $m = 1$. Assume that it holds for $m-1$. Then we have equations $\sum_{j=m-1}^n h_{ij}x_j = 0$ for $i = m-1, m, \dots, n$ where the coefficients h_{ij} are as described in the Lemma. Multiply both sides of the equation $\sum_{j=m-1}^{n} h_{(m-1)j}x_j = 0$ with $h_{p(m-1)}$, and both sides of the equation $\sum_{j=m-1}^{n} h_{pj}x_j = 0$ with $h_{(m-1)(m-1)}$. Subtract the first equation from the second for $p = m, m+1, \dots, n$. We obtain the equations $\sum_{j=m}^{n}(h_{(m-1)(m-1)}h_{pj} - h_{p(m-1)}h_{(m-1)j})x_j = 0$ for $p = m, m+1, \dots, n$. It is clear that the coefficients $g_{ij} = h_{(m-1)(m-1)}h_{ij} - h_{i(m-1)}h_{(m-1)j}$ are of the form described in the Lemma.

(1.29) Theorem. *Let A be a ring and let $\mathfrak{a}$ be an ideal. Moreover let M be a finitely generated A-module and $u : M \to M$ an A-module homomorphism such that $u(M) \subseteq \mathfrak{a}M$. Then there are elements $f_1, f_2, \dots, f_n$ in $\mathfrak{a}$ such that the endomorphism*

$$u^n + f_1u^{n-1} + \cdots + f_n : M \to M$$

defined by $(u^n + f_1u^{n-1} + \cdots + f_n)(x) = u^n(x) + f_1u^{n-1}(x) + \cdots + f_nx$, for all $x \in M$, is equal to zero.

Proof. We have that the polynomial ring $A[t]$ in the variable t over A operates on M by $(g_0t^n + g_1t^{n-1} + \cdots + g_m)(x) = g_0u^n(x) + g_1u^{n-1}(x) + \cdots + g_0x$, for all $g_0, g_1, \dots, g_n$ in A and x in M. It is clear that under this action M becomes an $A[t]$-module. The Proposition states that we can find elements $f_1, f_2, \dots, f_n$ in $\mathfrak{a}$ such that $(t^n + f_1t^{n-1} + \cdots + f_n)(x) = 0$ for all $x \in M$.

Let $x_1, x_2, \dots, x_m$ be generators for the A-module M. Then we have that $tx_i = \sum_{j=1}^{m} f_{ij}x_j$ with $f_{ij} \in \mathfrak{a}$ for $i, j = 1, 2, \dots, m$, and thus we have equations $\sum_{j=1}^{m}(t\delta_{ij} - f_{ij})x_j = 0$ for $i = 1, 2, \dots, m$, where $\delta_{ii} = 1$ and $\delta_{ij} = 0$ when $i \neq j$. We apply Lemma (?) to these equations with $A[t]$ instead of A and with m instead of n. We obtain an equation $g_{mm}x_m = 0$ where g_{mm} is the sum of terms $\pm\prod_{i,j=1}^{m}(t\delta_{ij} - f_{ij})^{m_{ij}}$ with $\sum_{i,j=1^m} m_{ij} = 2^{m-1}$. From the description of the coefficient g_{mm} in The Lemma we see that all the terms are polynomials in t with all their coefficents in $\mathfrak{a}$ except the term $(t - f_{11})^{2^{m-2}}(t - f_{22})^{2^{m-3}} \cdots (t - f_{(m-1)(m-1)})(t - f_{mm})$. We obtain that g_{mm} is a polynomial of the form $g_{mm}(t) = t^{2^{m-1}} + h_{m1}t^{2^{m-1}-1} \cdots + h_{m2^{m-1}}$ with $h_{ij} \in \mathfrak{a}$, and whose product with x_m is 0.

Renumbering the generators $x_1, x_2, \dots, x_m$ we see that for each i we have a polynomial $g_{ii}(t) = t^{2^{m-1}} + h_{i1}t^{2^{m-2}-1} + \cdots + h_{i2^{m-1}}$ with coefficients $h_{ij} \in \mathfrak{a}$ whose product with x_i is 0. Then $g_{11}g_{22} \cdots g_{mm} = t^n + f_1t^{n-1} + \cdots + f_n$ is a polynomial with coefficients in $\mathfrak{a}$ whose product with all the elements of M is 0.

(1.30) Definition. Let $\{M_n\}_{n \in \mathbf{Z}}$ be a *sequence* of A modules M_n, and let $u_n : M_n \to M_{n+1}$ for all $n \in \mathbf{Z}$ be A-module homomorphisms. We say that!!

$$\cdots \to M_{n-1} \xrightarrow{u_{n-1}} M_n \xrightarrow{u_n} M_{n+1} \to \cdots$$

is a *complex* of A-modules if $\mathrm{Im}(u_n) \subseteq \mathrm{Ker}(u_{n+1})$ for all $n \in \mathbf{Z}$. The complex is *exact* if $\mathrm{Im}(u_n) = \mathrm{Ker}(u_{n+1})$ for all $n \in \mathbf{Z}$. If there are integers p or q such that $M_n = 0$ for $n > p$ or $n < q$ we often write the terms!! $\cdots \xrightarrow{u_{p-2}} M_{p-1} \xrightarrow{u_{p-1}} M_p \to 0$, or !!$0 \to M_q \xrightarrow{u_q} M_{q+1} \xrightarrow{u_{q+1}} \cdots$ only.

n An exact complex of A-modules!!

$$0 \to M \to N \to P \to 0$$

is said to be *short exact.*

(1.31) Example. Let $u : M \to N$ be an A-module homomorphism. We obtain two short exact sequences:

$$0 \to \mathrm{Ker}(u) \to M \to \mathrm{Im}(u) \to 0$$

and

$$0 \to \mathrm{Im}(u) \to N \to \mathrm{Coker}(u) \to 0.$$

(1.32) Remark. Let $v : M' \to M$ and $w : N \to N'$ be homomorphisms of A-modules. We obtain an A-module homomorphism

$$\mathrm{Hom}_A(v, w) : \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(M', N')$$

that maps $u : M \to N$ to $wuv : M' \to N'$. The correspondence that maps an A-module M to the A-module $\mathrm{Hom}_A(M, N)$ for fixed N is clearly a *contravariant* functor from A-modules to A-modules. Similarly the functor the correspondence that maps an A-module N to $\mathrm{Hom}_A(M, N)$ for fixed M is a *covariant* functor from A-modules to A-modules.

(1.33) Lemma. *We have*

(1) *Let*

$$L \xrightarrow{u} M \xrightarrow{v} N \to 0 \tag{1.33.1}$$

be a complex of A-modules. The complex is exact if and only if the complex

$$0 \to \mathrm{Hom}_A(N, P) \xrightarrow{\mathrm{Hom}_A(v, \mathrm{id}_P)} \mathrm{Hom}_A(M, P) \xrightarrow{\mathrm{Hom}_A(u, \mathrm{id}_P)} \mathrm{Hom}_A(L, P) \tag{1.33.2}$$

is exact for all A-modules P.

(2) *Let*

$$0 \to L \xrightarrow{u} M \xrightarrow{v} N \tag{1.33.3}$$

be a complex of A-modules. The complex is exact if and only if the complex

$$\mathrm{Hom}_A(P, L) \xrightarrow{\mathrm{Hom}_A(\mathrm{id}_P, u)} \mathrm{Hom}_A(P, M) \xrightarrow{\mathrm{Hom}_A(\mathrm{id}_P, v)} \mathrm{Hom}_A(P, N) \to 0 \tag{1.33.4}$$

is exact for all A-modules P.

Proof. It is clear that if (1.33.1) and (1.33.3) are exact complexes then (1.33.2), respectively (1.33.4), are exact complexes.

To prove the converse implications we assume that (1.33.2) is exact for all P. First we let $P = N/\operatorname{Im}(v)$. The canonical homomorphism $u_{N/\operatorname{Im}(v)} : N \to N/\operatorname{Im}(v)$ maps to zero in $\operatorname{Hom}_A(M, N/\operatorname{Im}(v))$. Since $\operatorname{Hom}_A(\mathrm{id}_P, u)$ is injective by assumption we obtain that $u_{N/\operatorname{Im}(v)} = 0$, that is $N/\operatorname{Im}(v) = 0$. Hence $N = \operatorname{Im}(v)$, and v is surjective.

Secondly let $P = M/\operatorname{Im}(u)$. The canonical homomorphism $u_{M/\operatorname{Im}(u)} : M \to M/\operatorname{Im}(u)$ maps to zero in $\operatorname{Hom}_A(L, M/\operatorname{Im}(u))$. Since the sequence (1.33.2) is exact by assumption there is an A-modules homomorphism $w : N \to M/\operatorname{Im}(u)$ such that $u_{M/\operatorname{Im}(u)} = wv$. In particular the kernel $\operatorname{Ker}(v)$ of v is contained in the kernel $\operatorname{Im}(u)$ of $u_{M/\operatorname{Im}(u)}$. However $\operatorname{Im}(u) \subseteq \operatorname{Ker}(v)$ since (1.33.1) is a complex. Hence $\operatorname{Im}(u) = \operatorname{Ker}(v)$, and we have proved that (1.33.1) is exact.

Similar reasoning gives that if (1.33.4) is exact for all A-modules P then (1.33.3) is exact.

(1.34) Proposition. *Let*

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r, "u"] \arrow[d, "w'"'] & M \arrow[r, "v"] \arrow[d, "w"'] & M'' \arrow[r] \arrow[d, "w''"'] & 0 \\ 0 \arrow[r] & N' \arrow[r, "u'"'] & N \arrow[r, "v'"'] & N'' \arrow[r] & 0 \end{tikzcd} \tag{1.34.1}$$

be a commutative diagram of A-modules with exact horizontal sequences. Then there is a natural exact sequence of A-modules!!

$$0 \to \operatorname{Ker}(w') \xrightarrow{u^0} \operatorname{Ker}(w) \xrightarrow{v^0} \operatorname{Ker}(w'') \xrightarrow{d}$$
$$\operatorname{Coker}(w') \xrightarrow{(u')^1} \operatorname{Coker}(w) \xrightarrow{(v')^1} \operatorname{Coker}(w'') \to 0$$

where u^0, v^0 are induced by the restrictions of u and v respectively, and $(u')^1, (v')^1$ are induced by u' respectively v'.

Proof. We first define the homomorphism d. Let $x'' \in \operatorname{Ker} w''$. Choose an $x \in M$ such that $v(x) = x''$. Then we have that $v'w(x) = w''v(x) = w''(x'') = 0$. Since the bottom horizontal sequence is exact there is a unique element $y' \in N'$ such that $w(x) = u'(y')$. We take $d(x'')$ to be the class of y' in $N'/\operatorname{Im}(w')$. The definition of d is independent of the choice of x. In fact, since the top horizontal sequence of diagram (1.34.1) is exact, we have that $v(x_1) = x''$ and $w(x_1) = u'(y_1')$ with $x_1 \in M$ and $y_1' \in N'$. In particular $v(x - x_1) = v(x) - v(x_1) = x'' - x'' = 0$, and since the top horizontal sequence is exact we have that $x - x_1 = u(x')$ for some $x' \in M'$. Then $w(x) - w(x_1) = w(x - x_1) = wu(x') = u'w'(x')$, and thus

$u'(y') = w(x) = w(x_1) + u'w'(x') = u'(y_1') + u'w'(x') = u'(y_1' + w'(x'))$. Since u' is injective we have $y' = y_1' + w'(x')$. Thus y and y_1 belong to the same class in $N'/\operatorname{Im}(w')$.

Simple calculations show that the sequence is exact.

(1.35) Exercises.

1. Let A be a ring and let M, M' and M'' be A-modules. We have defined what it means that a sequence of sets $M' \to M \rightrightarrows M''$ is exact and what it means that a sequence of A-modules $0 \to M' \to M \to M'' \to 0$ is exact. Give the connection between the two notions of exactness.

2. Show that the polynomial ring $A[t]$ in the variable t with coefficients in the ring A is a free A-module with basis $\{1, t, t^2, \dots\}$.

3. Let M be an A-module. Define the sum and product of the elements in the cartesian product $A \times M$ by $(f, x) + (g, y) = (f + g, x + y)$, and $(f, x)(g, y) = (fg, gx + fy)$. Show that $A \times M$ with this sum and product is a ring which is an A-algebra under the map $A \to A \times M$ that sends $f \in A$ to $(f, 0)$ in $A \times M$. We denote this A-algebra by $A[M]$.

4. Let $A \neq 0$ be a ring and denote by A^n the direct sum n times of the A-module A with itself. Assume that $u : M \to A^n$ be a surjection of A-modules. Show that there is a submodule L of M such that M is isomorphic to the A-module $L \oplus \operatorname{Ker} u$.

5. Let $\{M_\alpha, \rho^\alpha_\beta\}_{\alpha,\beta\in I,\alpha\leq\beta}$ be an inductive system of A modules.

(1) Show that the group $\varinjlim_{\alpha\in I} M_\alpha$ has a unique structure of an A-module such that the canonical homomorphisms $\varphi_\beta : M_\beta \to \varinjlim_{\alpha\in I} M_\alpha$ of groups are all A-module homomorphisms.

(2) Let $\{N_\alpha\}_{\alpha\in I}$ be another family of A-modules, and let $\{u_\alpha\}_{\alpha\in I}$ be a map of the inductive systems. Show that the resulting map $\varinjlim_{\alpha\in I} u_\alpha : \varinjlim_{\alpha\in I} M_\alpha \to \varinjlim_{\alpha\in I} N_\alpha$ is an A-module homomorphism.

6. Let $A \neq 0$ be a ring and let A^n be the direct sum of the A-module A with itself n times. Let $u : A^m \to A^n$ be an A-linear map.

(1) Show that for every ideal $\mathfrak{a}$ of A the map u induces a canonical A-linear map $u_{\mathfrak{a}} : (A/\mathfrak{a})^m \to (A/\mathfrak{a})^n$ that sends $(u_{A/\mathfrak{a}}(f_1), u_{A/\mathfrak{a}}(f_2), \dots, u_{A/\mathfrak{a}}(f_m))$ to $u_{A/\mathfrak{a}}(u_1(x)), u_{A/\mathfrak{a}}(u_2(x)), \dots, u_{A/\mathfrak{a}}(u_n(x)))$ for all $x = (f_1, f + 2, \dots, f_m)$ in A^m, where we have written $u(x) = (u_1(x), u_2(x), \dots, u_n(x))$.

(2) Show that if $u : A^m \to A^n$ is surjective then $m \geq n$.

(3) Show that if u is surjective and injective then $m = n$.

7. Let A be a ring and let $M \neq 0$ be a free A-module of finite rank. Moreover let $\{x_\alpha\}_{\alpha\in I}$ be a collection of elements of M.

(1) Is it true that when the elements $\{x_\alpha\}_{\alpha\in I}$ generate M then we can find a subset J of I such that the elements $\{x_\beta\}_{\beta\in J}$ form a basis for M?

(2) Is it true that when the elements $\{x_\alpha\}_{\alpha \in I}$ are linearly independent then there is a set K containing I and elements $\{x_\gamma\}_{\gamma \in K}$ that form a basis for M?

8. Let I be an infinite set, and let A be a ring. For each $\alpha \in I$ we let $e(\alpha) = (e_\beta)_{\beta \in I} \in A^I$ be the element given by $e_\alpha = 1$ and $e_\beta = 0$ when $\alpha \neq \beta$.

(1) Are the elements $\{e(\alpha)\}_{\alpha \in I}$ linearly independent in A^I?
(2) Are the elements $\{e(\alpha)\}_{\alpha \in I}$ generators for the A-module A^I?

9. Let $\cdots \to M_{n-1} \xrightarrow{u_{n-1}} M_n \xrightarrow{u_n} M_{n+1} \to \cdots$ be a complex of A-modules M_n.

(1) Show that we obtain complexes $0 \to \mathrm{Ker}(u_n) \to M_n \to \mathrm{Coker}(u_{n-1}) \to 0$ for $n \in \mathbf{Z}$.
(2) Show that the complexes of (1) are exact for all $n \in \mathbf{Z}$ if and only if the complex $\cdots \to M_{n-1} \xrightarrow{u_{n-1}} M_n \xrightarrow{u_n} M_{n+1} \to \cdots$ is exact.

10. Show that when K is a field and when

$$0 \to M_0 \to M_1 \to \cdots \to M_n \to 0$$

is an exact sequence of finitely generated K-vector spaces, then $\sum_{i=0}^n \dim_K(M_i) = 0$.

2. Tensor products.

(2.1) Definition. Let A be a ring and let M, N and P be A-modules. A map
n !!$b : M \times N \to P$ from the direct product of M and N to P is called A-*bilinear* if we for all elements x, x' in M, y, y' in N and f in A have:

(1) $b(fx, y) = fb(x, y) = b(x, fy)$.
(2) $b(x + x', y) = b(x, y) + b(x', y)$.
(3) $b(x, y + y') = b(x, y) + b(x, y')$.

→ **(2.2) Construction of tensor products.** We saw in (?) that the the A-module $A^{(M\times N)}$ is free and has a canonical basis consisting of the family $\{e(x, y)\}_{(x,y)\in M\times N}$ where $e(x, y) = (e_{(x',y')})_{(x',y')\in M\times N}$ with $e_{(x,y)} = 1$ and $e_{(x'y')} = 0$ if $(x, y) \neq (x', y')$. Let L be the submodule of $A^{(M\times N)}$ generated by the elements

$$\begin{aligned} &e(x + x', y) - e(x, y) - e(x', y),\\ &e(x, y + y') - e(x, y) - e(x, y'),\\ &e(fx, y) - fe(x, y),\\ &e(x, fy) - fe(x, y), \end{aligned} \tag{2.2.1}$$

n for all elements x, x' in M, y, y' in N and f in A. Moreover let !!

$$M \otimes_A N = A^{(M\times N)}/L,$$

and let $u_{M\otimes_A N} : A^{(M\times N)} \to M \otimes_A N$ be the canonical homomorphism. We denote
n the residue class of $e(x, y)$ in $M \otimes_A N$ by !!$x \otimes_A y$. It follows from the definition of the A-module $M \otimes_A N$ that $M \otimes_A N$ is generated as an A-module by the elements of the form $x \otimes_A y$ for all $x \in M$ and $y \in N$, and that we have relations

$$(x + x') \otimes_A y = x \otimes_A y + x' \otimes_A y, \qquad x \otimes_A (y + y') = x \otimes_A y + x \otimes_A y',$$
$$fx \otimes_A y = f(x \otimes_A y) = x \otimes_A fy. \tag{2.2.2}$$

n Finally we let !!

$$b_{M\otimes_A N} : M \times N \to M \otimes_A N$$

be the homomorphism defined by $b_{M\otimes_A N}(x, y) = x \otimes_A y$.

→ **(2.3) Remark.** It follows from the equalities (2.2.2) that if M and N are generated as A-modules by the elements $\{x_\alpha\}_{\alpha\in I}$ respectively by $\{y_\beta\}_{\beta\in J}$ then the A-module $M \otimes_A N$ is generated by the elements $\{x_\alpha \otimes_A y_\beta\}_{(\alpha,\beta)\in I\times J}$.

(2.4) Proposition. *Let M, N and P be A-modules. The canonical homomorphism $b_{M\otimes_A N} : M \times N \to M \otimes_A N$ is A-bilinear and it has the following universal property:*

If $b : M \times N \to P$ is an A-bilinear map, then there is a unique A-linear homomorphism $u : M \otimes_A N \to P$ such that $b = ub_{M\otimes_A N}$.

Proof. Since $b_{M\otimes_A N}(x,y) = x\otimes_A y$ for all $(x,y)\in M\times N$, the bilinearity of the map $b_{M\otimes_A N}$ follows from the relations (2.2.1).

The homomorphism u is unique when it exists, because the A-module $M\otimes_A N$ is generated by the elements $x\otimes_A y$ for all $(x,y)\in M\times N$, and $u(x\otimes_A y) = u(b_{M\otimes_A N}(x,y)) = b(x,y)$.

In order to show the existence of u we observe first that it follows from Proposition (?) that there is an A-linear homomorphism $v : A^{(M\times N)} \to P$ defined by $v(e(x,y)) = b(x,y)$ for all $(x,y)\in M\times N$ where $\{e(x,y)\}_{(x,y)\in M\times N}$ is the canonical basis for $A^{(M\times N)}$. Since the map b is A-bilinear it follows that v is zero on the submodule L. Hence it follows from Proposition (?) that v factors via an A-linear homomorphism $u_{M\otimes_A N} : A^{(M\times N)} \to M\otimes_A N$ and an A-linear homomorphism $u : M\otimes_A N\to P$. We have that $b(x,y) = v(e(x,y)) = u(u_{M\otimes_A N}(e(x,y))) = u(x\otimes_A y) = u(b_{M\otimes_A N}(x,y))$. Hence we have that $b = ub_{M\otimes_A N}$.

(2.5) Remark. The universal property determines $b_{M\otimes_A N} : M\times N\to M\otimes_A N$ uniquely, up to an A-module isomorphisms. In fact if $c : M\times N\to T$ is a bilinear map of A-modules such that for each A-bilinear map $b : M\times N\to P$ there is a unique homomorphism $u : T\to P$ with $b = uc$, then the universal properties for $b_{M\otimes_A N}$ and c define unique A-linear homomorphisms $v : M\otimes_A N\to T$ and $w : T\to M\otimes_A N$ such that $b = vb_{M\otimes_A N}$ and $b_{M\otimes_A N} = wb$. Hence $b = vwb$ and $b_{M\otimes_A N} = wvb_{M\otimes_A N}$, and again by uniqueness v and w must be inverses.

(2.6) Definition. The module $M\otimes_A N$ is called the *tensor product* of M and N.

(2.7) Example. When p and q are different prime numbers we have that $\mathbf{Z}/p\mathbf{Z}\otimes_{\mathbf{Z}} \mathbf{Z}/q\mathbf{Z} = 0$. In fact, it follows from the Euclidian algorithm that we can find an integer n such that $nq\equiv 1 \pmod p$. Hence, when we denote by $\overline{n}$ the class of an integer n in $\mathbf{Z}/p\mathbf{Z}$ and $\mathbf{Z}/q\mathbf{Z}$, we have that $1\otimes_{\mathbf{Z}} 1 = \overline{nq}\otimes_{\mathbf{Z}} 1 = \overline{n}\otimes_{\mathbf{Z}}\overline{q} = 0$ in $\mathbf{Z}/p\mathbf{Z}\otimes_{\mathbf{Z}}\mathbf{Z}/q\mathbf{Z}$.

On the other hand we have that $\mathbf{Z}/p\mathbf{Z}\otimes_{\mathbf{Z}}\mathbf{Z}/p\mathbf{Z}$ is isomorphic to $\mathbf{Z}/p\mathbf{Z}$, because for all integers m and n, we have that $\overline{m}\otimes_{\mathbf{Z}}\overline{n} = 1\otimes_{\mathbf{Z}}\overline{mn}$. Hence the homomorphism $\mathbf{Z}/p\mathbf{Z}\otimes_{\mathbf{Z}}\mathbf{Z}/p\mathbf{Z}\to\mathbf{Z}/p\mathbf{Z}$ that maps $\overline{m}\otimes_{\mathbf{Z}}\overline{n}$ to $\overline{mn}$ is an isomorphism with inverse mapping $\overline{m}$ to $1\otimes_{\mathbf{Z}}\overline{m}$.

(2.8) Multilinear maps. Let A be a ring and let $M_1, M_2, \dots, M_n$ be A-modules. A map !!

$$m : M_1\times M_2\times\cdots\times M_n\to P$$

is A-multilinear if for all $f\in A$ and x_i, x_i' in M_i for $i = 1,2,\dots,n$ we have that

$$\begin{aligned} m(x_1,\dots,x_i+x_i',\dots,x_n) &= m(x_1,\dots,x_i,\dots,x_n) + m(x_1,\dots,x_i',\dots,x_n)\\ m(x_1,\dots,fx_i,\dots,x_n) &= fm(x_1,\dots,x_i,\dots,x_n)\end{aligned}$$

for $i = 1,2,\dots,n$. An analogous construction to that giving the tensor product of two modules will give the *tensor product* !!$M_1\otimes_A M_2\otimes_A\cdots\otimes_A M_n$ of the modules $M_1, M_2, \dots, M_n$ and an A-multilinear homomorphism !!$m_{M_1\otimes_A M_2\otimes_A\cdots\otimes_A M_n}$:

$M_1 \times M_2 \times \cdots \times M_n \to M_1 \otimes_A M_2 \otimes_A \cdots \otimes_A M_n$ with the universal property that if $m : M_1 \times M_2 \times \cdots \times M_n \to P$ is an A-multilinear map then there is a unique A-linear homomorphism $u : M_1 \otimes_A M_2 \otimes_A \cdots \otimes_A M_n \to P$ such that $m = u m_{M_1 \otimes_A M_2 \otimes_A \cdots \otimes_A M_n}$. The universal property again determines $M_1 \otimes_A M_2 \otimes_A \cdots \otimes_A M_n$ up to an A-module isomorphism. We denote the image of the elements $(x_1, x_2, \dots, x_n)$ by the homomor-
n phism $m_{M_1 \otimes_A M_2 \otimes_A \cdots \otimes_A M_n}$ by !!$x_1 \otimes_A x_2 \otimes_A \cdots \otimes_A x_n$.

(2.9) Maps of tensor products. Let $u : M \to N$ and $u' : M' \to N'$ be homomorphisms of A-modules. We obtain a map $b : M \times M' \to N \otimes_A N'$ defined by $b(x, x') = u(x) \otimes_A u(x')$ for all $x \in M$ and $x' \in M'$. It is clear that the map b is
n A-bilinear. Consequently we obtain an A-linear homomorphism !!

$$u \otimes_A u' : M \otimes_A M' \to N \otimes_A N'.$$

When $v : N \to P$ and $v' : N' \to P'$ are A-linear homomorphisms we clearly have that:

$$(v \otimes_A v')(u \otimes_A u') = vu \otimes_A v'u'.$$

(2.10) Remark. For fixed N we have that the correspondence that sends an A-module M to $M \otimes_A N$ is a *covariant* functor from the category of A-modules to the category of A-modules. Similarly for fixed M the correspondence that sends the A-module N to $M \otimes_A N$ is a *covariant* functor between the same categories.

(2.11) Lemma. *Let M, N and P be A-modules.*

1. *We have an isomorphism of A-modules $M \otimes_A A \to M$ that is uniquely determined by mapping $x \otimes_A f$ to fx for all $f \in A$ and $x \in M$.*
2. *We have an isomorphism of A-modules $(M \otimes_A N) \otimes_A P \to M \otimes_A N \otimes_A P$ that is uniquely determined by mapping $(x \otimes_A y) \otimes_A z$ to $x \otimes_A y \otimes_A z$.*

Proof. (1) The map $M \times A \to M$ that takes (x, f) to fx is A-bilinear. Consequently there is an A-linear homomorphism $M \otimes_A A \to M$ that maps $x \otimes_A f$ to fx. We also have an A-linear homomorphism $M \to M \otimes_A A$ that maps x to $x \otimes_A 1$. It is clear that the two homomorphisms are inverses of each other.

(2) For every element z in P we have a map $M \times N \to M \otimes_A N \otimes_A P$ that takes (x, y) to $x \otimes_A y \otimes_A z$. This map is clearly A-bilinear. Consequently there is an A-linear homomorphism $u_z : M \otimes_A N \to M \otimes_A N \otimes_A P$ that maps w to $u_z(w) = w \otimes_A z$. In particular it maps $x \otimes_A y$ to $x \otimes_A y \otimes_A z$. The map $(M \otimes_A N) \times P \to M \otimes_A N \otimes_A P$ that takes (w, z) to $u_z(w)$ is clearly A-bilinear. Consequently we have an A-linear homomorphism $(M \otimes_A N) \otimes_A P \to M \otimes_A N \otimes_A P$ that maps $(x \otimes_A y) \otimes_A z$ to $x \otimes_A y \otimes_A z$. We also have a map $M \times N \times P \to (M \otimes_A N) \otimes_A P$ that takes (x, y, z) to $(x \otimes_A y) \otimes_A z$. This map is clearly A-multilinear, and thus defines an A-linear homomorphism $M \otimes_A N \otimes_A P \to (M \otimes_A N) \otimes_A P$. The two homomorphisms between $(M \otimes_A N) \otimes_A P$ and $M \otimes_A N \otimes_A P$ are clearly inverses of each other.

(2.12) Proposition. *Let $\{M_\alpha\}_{\alpha\in I}$ be a collection of A-modules, and let N be an A-module. Then there is an isomorphism of A-modules*

$$u : (\oplus_{\alpha\in I} M_\alpha) \otimes_A N \xrightarrow{\sim} \oplus_{\alpha\in I}\, (M_\alpha \otimes_A N)$$

that is uniquely determined by $u(x_\alpha \otimes_A y) = x_\alpha \otimes_A y$ for all $x_\alpha \in M_\alpha$ and $y \in N$.

Proof. It is clear that u is unique if it exists.

We have a map $(\oplus_{\alpha\in I} M_\alpha) \times N \to \oplus_{\alpha\in I}(M_\alpha \otimes_A N)$ that takes $(\sum_{\alpha\in I} x_\alpha, y)$ to $\sum_{\alpha\in I} x_\alpha \otimes_A y$ for all $\sum_{\alpha\in I} x_\alpha \in \oplus_{\alpha\in I} M_\alpha$ and $y \in N$. This map is clearly A-bilinear and thus defines an A-linear homomorphism $u : (\oplus_{\alpha\in I}) M_\alpha \otimes_A N \to \oplus_{\alpha\in I}(M_\alpha \otimes_A N)$.

To show that u is an isomorphism we shall construct the inverse homomorphism. For each $\alpha \in I$ we have a map $M_\alpha \times N \to (\oplus_{\alpha\in I} M_\alpha) \otimes_A N$ which takes (x_α, y) to $x_\alpha \otimes_A y$ for all $x_\alpha \in M_\alpha$ and $y \in N$. The map is clearly A-bilinear. Consequently we obtain an A-linear homomorphism $M_\alpha \otimes_A N \to (\oplus_{\alpha\in I} M_\alpha) \otimes_A N$ for all $\alpha \in I$. From the categorical definition of the sum of modules we have an A-linear homomorphism $\oplus_{\alpha\in I}(M_\alpha \otimes_A N) \to (\oplus_{\alpha\in I} M_\alpha) \otimes_A N$. It is clear that the latter homomorphism is the inverse of u.

(2.13) Corollary. *Let M be a free A-module with basis $\{x_\alpha\}_{\alpha\in I}$, and let N be an A-module. Then every element in $M \otimes_A N$ can be written uniquely on the form $\sum_{\alpha\in I} x_\alpha \otimes_A y_\alpha$ with $y_\alpha \in N$, and with $y_\alpha = 0$ except for at most a finite number of the $\alpha \in I$.*

Proof. Since M is free it is isomorphic to the direct sum $\oplus_{\alpha\in I} Ax_\alpha$ via the homomorphism that maps $\sum_{\alpha\in I} f_\alpha x_\alpha$ in M to $\sum_{\alpha\in I} f_\alpha x_\alpha$ in $\oplus_{\alpha\in I} Ax_\alpha$. It follows from the Proposition that $M \otimes_A N$ is isomorphic to $\oplus_{\alpha\in I}(Ax_\alpha \otimes_A N)$. We have that the map $A \to Ax_\alpha$ that sends f to fx_α is an isomorphism. Hence it follows from Lemma (?) that $Ax_\alpha \otimes_A N$ is isomorphic to N. Hence $M \otimes_A N$ is isomorphic to $\oplus_{\alpha\in I} N$ and we have proved the Corollary.

(2.14) Proposition. *Let*

$$M' \xrightarrow{u'} M \xrightarrow{u''} M'' \to 0$$

be an exact sequence of A-modules. For every A-module N the sequence

$$M' \otimes_A N \xrightarrow{u'\otimes_A \mathrm{id}_N} M \otimes_A N \xrightarrow{u''\otimes_A \mathrm{id}_N} M'' \otimes_A N \to 0$$

is exact.

Proof. Since u'' is surjective and $M'' \otimes_A N$ is generated by the elements $x'' \otimes_A z$ with $x'' \in M''$ and $z \in N$ we have that $u'' \otimes_A \mathrm{id}_N$ is surjective. Moreover, since $u''u' = 0$, it is clear that $\mathrm{Im}(u' \otimes_A \mathrm{id}_N) \subseteq \mathrm{Ker}(u'' \otimes_A \mathrm{id}_N)$.

It remains to prove that $\mathrm{Im}(u' \otimes_A \mathrm{id}_N) = \mathrm{Ker}(u'' \otimes_A \mathrm{id}_N)$. Let $L = \mathrm{Im}(u' \otimes_A \mathrm{id}_N)$. It follows from Proposition (?) that we have an A-linear homomorphism

$u : (M \otimes_A N)/L \to M'' \otimes_A N$ that maps the class of $x \otimes_A z$ to $u''(x) \otimes_A z$, and that we have an equality $L = \mathrm{Ker}(u'' \otimes_A \mathrm{id}_N)$ if and only if the homomorphism u is injective. To prove injectivity of u it suffices to prove that there is a homomorphism $v : M'' \otimes_A N \to (M \otimes_A N)/L$ such that vu is the identity on $(M \otimes_A N)/L$. In order to prove the existence of v we let $b : M'' \times N \to (M \otimes_A N)/L$ be the map that takes $(u''(x), z)$ to the class of $x \otimes_A z$ for all $x \in M$ and $z \in N$. The map b is well defined if $u''(x) = u''(y)$ we have that $x - y$ is in the kernel of u'', and thus $x - y = u'(x')$ for some $x' \in M'$. Then we have that $x \otimes_A z = y \otimes_A z + u'(x') \otimes_A z = y \otimes_A z + (u' \otimes_A \mathrm{id}_N)(x' \otimes_A z)$, and consequently $x \otimes_A z$ and $y \otimes_A z$ have the same class in $(M \otimes_A N)/L$. It is clear that b is A-bilinear. Hence we obtain an A-linear homomorphism $v : M'' \otimes_A N \to (M \otimes_A N)/L$. Since the A-module $M \otimes_A N$ is generated by elements of the form $x \otimes_A z$, it suffices, in order to prove that vu is the identity homomorphism, to check that vu is the identity on the classes of the elements $x \otimes_A z$. However the class of $x \otimes_A z$ is mapped by u to $u''(x) \otimes_A z$ and $v(u''(x) \otimes_A z) = x \otimes_A z$.

(2.15) Remark. We express the conclusion of the Proposition by saying that the tensor product is *exact to the right* or *right exact*. It is not *left exact* because the homomorphism $\mathbf{Z} \xrightarrow{2} \mathbf{Z}$ given by multiplication by 2 is an injection of $\mathbf{Z}$-modules. It
→ follows from Proposition (?) that $\mathbf{Z} \otimes_{\mathbf{Z}} \mathbf{Z}/2\mathbf{Z} = \mathbf{Z}/2\mathbf{Z}$. However the homomorphism $\mathbf{Z} \otimes_{\mathbf{Z}} (\mathbf{Z}/2\mathbf{Z}) \xrightarrow{2_{\mathbf{Z}} \otimes_{\mathbf{Z}} \mathrm{id}_{\mathbf{Z}/2\mathbf{Z}}} \mathbf{Z} \otimes_{\mathbf{Z}} (\mathbf{Z}/2\mathbf{Z})$ is zero because $2_{\mathbf{Z}} \otimes_{\mathbf{Z}} \mathrm{id}_{\mathbf{Z}/2\mathbf{Z}} = \mathrm{id}_{\mathbf{Z}} \otimes_{\mathbf{Z}} 2_{\mathbf{Z}/2\mathbf{Z}} = 0$.

(2.16) Tensor products of algebras. Let A be a ring and let $\varphi : A \to B$ and $\chi : A \to C$ be A-algebras. Moreover let N be a B-module and P a C-module. We consider N and P as A-modules the product $gy = \varphi(g)y$ and $gz = \chi(g)z$ for all $g \in A$, $y \in N$ and $z \in P$. We have a map

$$B \times C \times N \times P \to N \otimes_A P$$

which takes (g, h, y, z) to $gy \otimes_A hz$. It is clear that this map is A-multilinear. Consequently we obtain an A-linear homomorphism

$$B \otimes_A C \otimes_A N \otimes_A P \to N \otimes_A P.$$

→ Using Lemma (?) repeatedly we obtain an A-module isomorphism $(B \otimes_A C) \otimes_A (N \otimes_A P) \to B \otimes_A C \otimes_A N \otimes_A P$ which maps $(g \otimes_A h) \otimes_A (y \otimes_A z)$ to $g \otimes_A h \otimes_A y \otimes_A z$. Consequently we have an A-module homomorphism $(B \otimes_A C) \otimes_A (N \otimes_A P) \to N \otimes_A P$ that gives an A-bilinear map

$$(B \otimes_A C) \times (N \otimes_A P) \to N \otimes_A P$$

that takes $((g \otimes_A h), (y \otimes_A z))$ to $gy \otimes_A hz$. We have thus defined an operation of the tensor product $B \otimes_A C$ on the group $N \otimes_A P$ such that the product of $\sum_{i=1}^m g_i \otimes_A h_i$

and $\sum_{i=1}^n y_i \otimes_A z_i$, with $g_i \in B$, $h_i \in C$, $y_j \in N$ and $z_j \in P$ for $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$, is given by

$$\left(\sum_{i=1}^m g_i \otimes_A h_i\right)\left(\sum_{i=1}^n y_i \otimes_A z_i\right) = \sum_{i=1}^m \sum_{j=1}^n g_i y_j \otimes_A h_i z_j. \tag{2.16.1}$$

It follows from Formula (2.16.1) that when $B = N$ and $C = P$ the group $B \otimes_A C$ with this product is a ring with unit $1 \otimes_A 1$, and that $N \otimes_A P$ becomes a $B \otimes_A C$-module. It is easy to see that the homomorphism

$$\psi : A \to B \otimes_A C \tag{2.16.2}$$

defined by $\psi(f) = \varphi(f) \otimes_A 1 = 1 \otimes_A \chi(f)$ is a ring homomorphism which gives $B \otimes_A C$ a natural structure as an A-algebra.

(2.17) Restriction and extension of scalars. Let A be a ring and let $\varphi : A \to B$ be an A-algebra. Moreover let M be an A-module and N a B-module.

We obtain an operation of A on the group N by defining the product of $f \in A$ and $y \in N$ by $fy = \varphi(f)y$. It is easy to check that N with this operation by A becomes an A-module. We say that N is an A-module via the A-algebra *structure* on B and denote this A-module by !!$N_{[\varphi]}$. The A-module $N_{[\varphi]}$ we call the A-module obtained from the B-module N by *restriction of scalars* to A.

The tensor product $M \otimes_A B$ has a natural structure as an A-module when the product of $f \in A$ with $x \otimes_A g \in M \otimes_A B$ is given by $f(x \otimes_A g) = fx \otimes_A g = x \otimes_A fg$. As we saw in Section (2.16) the group $M \otimes_A B$ has a natural structure as a $(A \otimes_A B$-module, and consequently by Lemma (?) a structure as a B-module. The product of g with $\sum_{i=1}^n x_i \otimes_A h_i$ is defined by

$$g\left(\sum_{i-1}^n x_i \otimes_A h_i\right) = u_g\left(\sum_{i=1}^n x_i \otimes_A h_i\right) = \sum_{i=1}^n x_i \otimes_A gh_i.$$

It is easy to verify that $M \otimes_A B$ with this product becomes a B-module. We say that $M \otimes_A B$ is the B-module obtained from the A-module M by *extension of scalars* to B.

Let $v : M \otimes_A B \to N$ be a homomorphism of B-modules. The composite of v with the homomorphism $M \to M \otimes_A B$ that maps $x \in M$ to $x \otimes_A 1$ gives a map of A-modules $M \to M_{[\varphi]}$. Hence we have defined a map

$$\operatorname{Hom}_B(M \otimes_A B, N) \to \operatorname{Hom}_A(M, N_{[\varphi]}). \tag{2.17.1}$$

This map is a bijection. To construct an inverse we let $u : M \to N$ be a homomorphism of groups such that $u(fx) = \varphi(f)u(x)$ for all $f \in A$ and $x \in M$. That is, u is an A-module homomorphism $u : M \to N_{[\varphi]}$. Then the A-bilinear map $M \times B \to N$ that takes (x, g) to $\varphi(g)u(x) = gu(x)$ for all $g \in B$ and $x \in M$ gives an A-linear homomorphism $M \otimes_A B \to N$ that maps $x \otimes_A g$ to $gu(x)$. If follows from the definition of the B-module structure on $M \otimes_A B$ that this is a B-module homomorphism, and it is clear that it gives an inverse to the map (2.17.1).

(2.18) Definition. Let $\varphi : A \to B$ be an A-algebra, and let M be an A-module, and N a B-module. A homomorphism of groups $u : M \to N$ is *compatible with the algebra structure* φ, or is a *φ-module homomorphism* if for all $f \in A$ and $x \in M$ we have $u(fx) = \varphi(f)u(x)$. Equivalently we have that u is compatible with φ if $u : M \to N_{[\varphi]}$ is a homomorphism of A-modules.

(2.19) Remark. It is clear that the map (2.17.1)

$$\operatorname{Hom}_A(M, N_{[\varphi]}) \xrightarrow{\sim} \operatorname{Hom}_B(M \otimes_A B, N)$$

is a φ-module isomorphism.

(2.20) Remark. Let $\varphi : A \to B$ be an A-algebra. Moreover let M be an A-module and N a B-modules. We have an φ-module isomorphism

$$M \otimes_A N_{[\varphi]} \xrightarrow{\sim} (M \otimes_A B) \otimes_B N \tag{2.20.1}$$

which is uniquely determined by mapping $x \otimes_A y$ to $(x \otimes_A 1) \otimes_B y$ for all $x \in M$ and $y \in N$. The inverse homomorphism of (2.20.1) maps $(x \otimes_A g) \otimes_B y$ to $x \otimes_A gy$ for all $g \in B$, $x \in M$ and $y \in N$. In fact we have an A-bilinear map $M \times N_{[\varphi]} \to (M \otimes_A B) \otimes_B N_{[\varphi]}$ which takes (x, y) to $(x \otimes_A 1) \otimes_B y$. This map gives an A-linear homomorphism $M \otimes_A N_{[\varphi]} \to (M \otimes_A B) \otimes_B N$.

To define the inverse homomorphism we consider the B-bilinear map $(M \otimes_A B) \times N \to M \otimes_A N$ that maps $(\sum_{i=1}^n x_i \otimes_A g_i, y)$ to $\sum_{i=1}^n x_i \otimes_A g_i y$ for all $x_i \in M$, $g_i \in B$ and $y \in N$. We obtain a B-linear homomorphism $(M \otimes_A B) \otimes_B N \to M \otimes_A N$. The latter homomorphism defines, by restriction of scalars, the inverse of the homomorphism (2.20.1).

It is clear that the map (2.20.1) is compatible with φ.

(2.21) Remark. Let M be a free A-module with a basis $\{x_\alpha\}_{\alpha \in I}$ and let $\varphi : A \to B$ be a homomorphism of rings. Then $M \otimes_A B$ is a free B-module with basis $\{x_\alpha \otimes_A 1\}_{\alpha \in I}$. This follows immediately from Corollary (?).

(2.22) Example. Let $\varphi : A \to B$ be a homomorphism of rings, and let $A[t]$ and $B[t]$ be the polynomials rings in the variable t with coefficients in A, respectively B. Then there is an isomorphism $A[t] \otimes_A B \to B[t]$ uniquely determined by mapping $f(t) \otimes_A g$ to $f(t)g$ for all $f(t) \in A[t]$ and $g \in B$. The existence and the uniqueness is clear. That the homomorphism is an isomorphism follows from Corollary (?).

(2.23) Lemma. *Let $\varphi : A \to B$ be an A-algebra. Moreover let N and N' be B-modules. Then we have a canonical φ-module homomorphism $N_{[\varphi]} \otimes_A N'_{[\varphi]} \to N \otimes_B N'$ which maps $y \otimes_A y'$ to $y \otimes_B y'$.*

In particular, if M and M' are A-modules and $u : M \to N$ and $u' : M' \to N'$ are φ-module homomorphisms we have a natural φ-module homomorphism

$$u \otimes_\varphi u' : M \otimes_A M' \to N \otimes_B N'$$

that maps $x \otimes_A x'$ to $u(x) \otimes_B u'(x')$. When $v : N \to P$ and $v' : N' \to P'$ are χ-module homomorphisms for a B-algebra $\chi : B \to C$ we have that $(vu) \otimes_{\chi\varphi} (v'u') = (v \otimes_\chi v')(u \otimes_\varphi u')$

Proof. We have a map $N_{[\varphi]} \times N'_{[\varphi]} \to N \otimes_B N'$ that takes (y, y') to $y \otimes_B y'$. This map is A-bilinear because, for all $f \in A$, we have that $(fy, y') = (\varphi(f)y, y')$ maps to the element $\varphi(f)y \otimes_B y' = y \otimes_B \varphi(f)y'$ which is also the image of $(y, \varphi(f)y') = (y, fy')$. The remaining properties for A-bilinearity are clearly satisfied. Hence we obtain an A-module homomorphism $N_{[\varphi]} \otimes_A N'_{[\varphi]} \to N \otimes_B N'$. It is clear from the explicit descrition of the map that it is a φ-module homomorphism.

For the last part we take the composite of the homomorphism $N_{[\varphi]} \otimes_A N'_{[\varphi]} \to N \otimes_B N'$ with the natural homomorphism $M \otimes_A M' \to N_{[\varphi]} \otimes_A N'_{[\varphi]}$ of A-modules coming from $u \otimes_A u'$. The composite homomorphism $u \otimes_\varphi u' : M \otimes_A M' \to N \otimes_B N'$ maps $x \otimes_A x'$ to $u(x) \otimes_B u(x')$, and is therefore a φ-module homomorphism.

(2.24) Exercises.

1. Show that there is a canonical isomorphism $M \otimes_A N \to N \otimes_A M$ of A-modules.

2. Let A be a ring and let M be an A-module. Show that for all ideals $\mathfrak{a}$ of A there is a canonical isomorphism between $M \otimes_A (A/\mathfrak{a})$ and $M/\mathfrak{a}M$.

3. Let $\varphi : A \to B$ be an A-algebra, and let $A[t_\alpha]_{\alpha \in I}$ and $B[t_\alpha]_{\alpha \in I}$ be the polynomial rings in the variables t_α over A respectively B. Show that $A[t_\alpha]_{\alpha \in I} \otimes_A B$ is canonically isomorphic to $B[t_\alpha]_{\alpha \in I}$.

4. Let $A[u]$ be the polynomial ring in the variable u over the ring $A \neq 0$, and let $A[[t]]$ be the power series ring in the variable t over A.

(1) Show that there is a homomorphism of rings $A[u] \otimes_A A[[t]] \to A[u][[t]]$, uniquely determined by mapping $f \otimes_A g$ to fg for all $f \in A[u]$ and $g \in A[[t]]$.
(2) Show that the homomorphism in (1) is injective.
(3) Show that the homomorphism in (1) is not surjective.

5. Let A be a ring and let $\mathfrak{a}$ be an ideal in A. Moreover let M be an A-module.

(1) Show that the $A/\mathfrak{a}$-module $M \otimes_A A/\mathfrak{a}$ is canonically isomorphic to the $A/\mathfrak{a}$ module $M/\mathfrak{a}M$.
(2) Show that the ring $(A/\mathfrak{a}) \otimes_A (A/\mathfrak{a})$ is canonically isomorphic to $A/\mathfrak{a}$.
(3) Show that the let $A/\mathfrak{a}$-module $(A/\mathfrak{a}) \otimes_A \mathfrak{a}$ is canonically isomorphic to $\mathfrak{a}/\mathfrak{a}^2$.

6. Let $\{M_\alpha\}_{\alpha \in I}$ be a family of A-modules, and N an A-module. Show that $\oplus_{\alpha \in I}(M_\alpha \otimes_A N)$ is isomorphic to $(\oplus_{\alpha \in I} M_\alpha) \otimes_A N$.

7. Is the tensor product $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C}$ of the complex numbers over the rational numbers a local ring?

3. Localization.

(3.1) Construction. Let S be a multiplicatively closed subset of the ring A. For
n every A-module M we define a relation !!$\sim$ on the cartesian product $M \times S$ by $(x, s) \sim (y, t)$ if there is an element $r \in S$ such that $r(tx - sy) = 0$ in M. It is clear that the relation $\sim$ is *reflexive*, that is $x \sim x$, and *symmetric*, that is $x \sim y$ implies that $y \sim x$. It is *transitive* because if $(x, s) \sim (x', s')$ and $(x', s') \sim (x'', s'')$ there are elements t, t' in S such that $t(s'x - sx') = 0$ and $t'(s''x' - s'x'') = 0$. Then we have that $tt's'(s''x - sx'') = tt's's''x - tt's'sx'' = t's''tsx' - tst's''x' = 0$, and consequently that $(x, s) \sim (x'', s'')$.

n Let !!$S^{-1}M = M \times S/\sim$ be the residue classes of $M \times S$ modulo the equivalence relation $\sim$. The class of the element (x, s) we denote by x/s. There is a canonical
n map!!

$$i_M^S : M \to S^{-1}M$$

defined by $i_M^S(x) = x/1$.

On the set $S^{-1}M$ there is a unique addition such that $S^{-1}M$ becomes a group and such that the canonical map i_M^S is a group homomorphism. The sum of two elements x/s and y/t in $S^{-1}M$ is defined by $x/s + y/t = (tx + sy)/st$. We have that the addition is independent of the choice of representative (x, s) for the class x/s because if $x/s = x'/s'$ there is an element $r \in S$ such that $r(s'x - sx') = 0$. Consequently we have that $r(s't(tx + sy) - st(tx' + s'y)) = t^2rs'x + rs'tsy - rst^2x' - rsts'y = 0$, and thus $(tx + sy)/st = (tx' + s'y)/s't$. Symmetrically the addition is independent of the choice of representative (y, t) of the class y/t. It is easily checked that $S^{-1}M$ with this addition becomes an abelian group with $0 = 0/1$.

We define the product of an element $f/s \in S^{-1}A$ with an element $x/t \in S^{-1}M$ by $(f/s)(x/t) = (fx)/(st)$. A simple calculation shows that the multiplication is independent of the choice of representatives (f, s) and (x, t) of the classes f/s, respectively x/t. In particular we obtain a multiplication on $S^{-1}A$ and it is easily seen that this multiplication toghether with the group structure on $S^{-1}A$ makes $S^{-1}A$ into a ring. Moreover, with this ring structure the above operation of $S^{-1}A$ on $S^{-1}M$ makes $S^{-1}M$ into a $(S^{-1}A)$-module.

n The canonical map!!

$$i_A^S : A \to S^{-1}A$$

that maps an element f in A to $f/1$ is a ring homomorphism.

(3.2) Definition. We call $S^{-1}M$ the *localization* of M by the *multiplicative set S*.

(3.3) Proposition. *Let A be a ring and S a multiplicatively closed subset. Moreover, let M be an A-module and N an $S^{-1}A$-module. For every homomorphism*

$$u : M \to N_{[i_M^S]}$$

of A-modules there is a unique $S^{-1}A$-module homomorphism

$$v : S^{-1}M \to N_{[i_M^S]}$$

such that $u = vi_M^S$.

The canonical map $i_A^S : A \to S^{-1}A$ has the universal property:

For every homomorphism of rings $\varphi : A \to B$ where $\varphi(s)$ is invertible in B for every element $s \in S$, there is a unique ring homomorphism $\chi : S^{-1}A \to B$ such that $\varphi = \chi i_A^S$.

Proof. If v exists we have, for all $x \in M$ and $s \in S$, that $v(x/s) = v((1/s)i_M^S(x)) = (1/s)v(i_M^S(x)) = (1/s)u(x)$. Hence v is uniquely determined if it exists.

To show that v exists we let $v(x/s) = (1/s)u(x)$ for all $x \in M$ and $s \in S$. This definition is independent of the choise of representative (x, s) for the class x/s because if $x/s = y/t$ with $y \in M$ and $t \in S$ there is an $r \in S$ such that $u(r(tx - sy)) = ru(tx - sy) = 0$. Hence we have that $r(t(u(x) - su(y)) = 0$ in N and consequently that $u(x)/s = u(y)/t$ in the $S^{-1}A$-module N. It is clear that v is an $S^{-1}A$-module homomorphism and that $u = vi_M^S$.

Finally when $\varphi : A \to B$ is a ring homomorphism such that $\varphi(s)$ is invertible in B for all $s \in S$ we have that B is an $S^{-1}A$-module by the multiplication $(f/s)g = \varphi(f)\varphi(s)^{-1}g$ for all $f/s \in S^{-1}A$ and $g \in B$. It is easily checked that the definition is independent of the representative (f, s) of the element f/s and that B becomes an $S^{-1}A$-module. Hence it follows from the first part of the Proposition that we have a map $\chi : S^{-1}A \to B$ of $S^{-1}A$-modules, and it is clear that χ is a ring homomorphism.

(3.4) Remark. The universal property characterizes $i_A^S : A \to S^{-1}A$ up to an isomorphism of rings. In fact let $\psi : A \to T$ be a homomorphism of rings with the same universal property as i_A^S. That is, for each homomorphism of rings $\varphi : A \to B$ with $\varphi(s)$ invertible in B for all $s \in S$ there is a unique homomorphism $\tau : T \to B$ such that $\varphi = \tau\psi$. Then the universal properties give unique ring homomorphisms $\omega : S^{-1}A \to T$ and $\tau : T \to S^{-1}A$ such that $\omega i_A^S = \psi$ and $\tau\psi = i_A^S$. Hence we have that $i_A^S = \tau\omega i_A^S$ and $\psi = \omega\tau\psi$. By unicity, we obtain that τ and ω are inverse maps.

(3.5) Example. Let $S = \mathbf{Z} \setminus \{0\}$. Then $S^{-1}\mathbf{Z}$ are the rational numbers $\mathbf{Q}$.

(3.6) Example. Let A be a ring and S a multiplicatively closed subset of A containing 0. Then $S^{-1}A = 0$.

(3.7) Example. Let $A = \mathbf{Z}/6\mathbf{Z}$ and let $S = \{1, 2, 2^2, \dots\}$. Then we have that $S^{-1}A = \mathbf{Z}/3\mathbf{Z}$. The map $i_A^S : \mathbf{Z}/6\mathbf{Z} \to \mathbf{Z}/3\mathbf{Z}$ coincides with the canonical residue map of $\mathbf{Z}/6\mathbf{Z}$ modulo the ideal $3\mathbf{Z}/6\mathbf{Z}$. In particular we have that i_A^S is not injective.

(3.8) Remark. Let A be a ring and let S be a multiplicatively closed subset that consist of non-zero divisors different from 0. Then the map $i_A^S : A \to S^{-1}A$ is injective. We often identify A with its image by i_A^S. When A is an integral domain and $S = A \setminus \{0\}$ we have that $S^{-1}A$ is a field.

(3.9) Definition. The *total quotient ring*, or *total fraction ring* of a ring A is the localization $S^{-1}A$ of A in the multiplicative set S consisting of all non-zero divisors different from 0. When A is an integral domain we call the field $S^{-1}A$ the *quotient field*, or *field of fractions* of A.

(3.10) Notation. Let f be an element of A. The set $S = \{1, f, f^2, \dots, \}$ is a
n multiplicatively closed subset of A. We write !!$S^{-1}M = M_f$. Let $\mathfrak{p}$ be a prime ideal
n of A. Then the set $T = A \setminus \mathfrak{p}$ is a multiplicatively closed subset of A. We write !! $T^{-1}M = M_{\mathfrak{p}}$. The $A_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$ is called the *localization* of M at $\mathfrak{p}$. Moreover we write $i_M^S = i_M^f$ and $i_M^T = i_M^{\mathfrak{p}}$.

(3.11) Proposition. *Let A be a ring and S a multiplicatively closed subset. For*
n *every prime ideal $\mathfrak{p}$ in A that does not intersect S we have that !!$\mathfrak{p}S^{-1}A = \{f/s \in S^{-1}A : f \in \mathfrak{p}\}$ is a prime ideal in $S^{-1}A$. The correspondence that maps $\mathfrak{p}$ to $\mathfrak{p}S^{-1}A$ is a bijection between the prime ideals in A that do not intersect S and the prime ideals of $S^{-1}A$. The inverse correspondence associates to a prime ideal $\mathfrak{q}$ in $S^{-1}A$ the ideal $(i_A^S)^{-1}(\mathfrak{q})$ in A.*

Proof. Let $\mathfrak{q}$ be a prime ideal in $S^{-1}A$. It is clear that $(i_A^S)^{-1}(\mathfrak{q})$ is a prime ideal in A that does not intersect S.

Let $\mathfrak{p}$ be a prime ideal in A that does not intersect S. If $(f/s)(g/t) \in \mathfrak{p}S^{-1}A$ there is an $r \in S$ such that $rfg \in \mathfrak{p}$. Since $r \notin \mathfrak{p}$ we have that f or g are in $\mathfrak{p}$, and thus that f/s or g/t is in $\mathfrak{p}S^{-1}A$. Moreover we have that $(i_A^S)^{-1}(\mathfrak{p}S^{-1}A) = \mathfrak{p}$ since, if $i_A^S(f) = g/t$ with $g \notin \mathfrak{p}$, then there is an $r \notin \mathfrak{p}$ such that $r(tf - g) = 0$ in A. We obtain that $rtf \notin \mathfrak{p}$, and thus that $f \notin \mathfrak{p}$.

It remains to prove that if $\mathfrak{p} = (i_A^S)^{-1}(\mathfrak{q})$ then $\mathfrak{p}S^{-1}A = \mathfrak{q}$. However it is clear that $\mathfrak{p}S^{-1}A \subseteq \mathfrak{q}$. Conversely if $f/s \in \mathfrak{q}$ we must have that $f \in \mathfrak{p}$.

(3.12) Corollary. *Let $\mathfrak{p}$ be a prime ideal in the ring A. Then the localization $A_{\mathfrak{p}}$ of A at $\mathfrak{p}$ is a local ring with maximal ideal $\mathfrak{p}A_{\mathfrak{p}}$.*

Proof. In this case $S = A \setminus \mathfrak{p}$ so $\mathfrak{p}$ is maximal among the ideals in A that do not intersect S.

(3.13) Remark. Let $\mathfrak{b}$ be an ideal in $S^{-1}A$ and let $\mathfrak{a} = (i_A^S)^{-1}(\mathfrak{b})$. Then, $\mathfrak{b} = \mathfrak{a}S^{-1}A = \{f/s \in S^{-1}A : f \in \mathfrak{a}, s \in S\}$. It is clear that $\mathfrak{a}S^{-1}A \subseteq \mathfrak{b}$. Conversely, when $f/s \in \mathfrak{b}$ we have that $f/1 \in \mathfrak{b}$ and consequently $f \in \mathfrak{a}$. Hence $f/s = (f/1)(1/s) \in \mathfrak{a}S^{-1}A$.

(3.14) Proposition. *There is a canonical isomorphism of $S^{-1}A$-modules*

$$M \otimes_A S^{-1}A \to S^{-1}M \tag{3.14.1}$$

that is uniquely determined by mapping $x \otimes_A (f/s)$ to $(fx)/s$ for all $f \in A$, $s \in S$ and $x \in M$.

→ *Proof.* It follows from the explicit description of the map (3.14.1) that it is a map of $S^{-1}A$-modules if it exists.

To prove the existence we consider the map $M \times S^{-1}A \to S^{-1}M$ that maps $(x, f/s)$ to $(fx)/s$. It is clear that this map is A-bilinear. Consequently we obtain an A-linear map $M \otimes_A S^{-1}A \to S^{-1}M$ that maps $x \otimes_A f/s$ to $(fx)/s$. It is clear that this map is an $(S^{-1}A)$-homomorphism.

In order to show that the map is an isomorphism we construct an inverse $S^{-1}M \to M \otimes_A S^{-1}A$ by mapping x/s to $x \otimes_A 1/s$. The latter map is independent of the choice of representative (x, s) of the class of x/s. In fact if $x/s = y/t$ there is an $r \in S$ such that $r(tx - sy) = 0$ in A. We obtain that $x \otimes_A (1/s) = x \otimes_A ((rt)/(rst)) = rtx \otimes_A (1/(rst)) = rsy \otimes_A (1/(rst)) = y \otimes_A ((rs)/(rst)) = y \otimes_A (1/t)$.

It is clear that the two maps are inverses of each other.

(3.15) Homomorphisms. Let S be a multiplicatively closed subset of A, and let $u : M \to N$ be a homomorphism of A-modules. There is a canonical map of
n $S^{-1}A$-modules:!!

$$S^{-1}u : S^{-1}M \to S^{-1}N$$

that maps x/s to $u(x)/s$ for all $s \in S$ and $x \in M$. The map is independent of the choice of representative (x, s) of the class x/s because if $x/s = y/t$ there is an $r \in S$ such that $r(tx - sy) = 0$, and thus $u(x)/s = u(y)/t$. It follows from the explicit form of $S^{-1}u$ that it is an $S^{-1}A$-module homomorphism.

(3.16) Remark. When $v : N \to P$ is a homomorphism of A-modules we have that $S^{-1}(vu) = S^{-1}vS^{-1}u$, and $S^{-1}\,\mathrm{id}_M = \mathrm{id}_{S^{-1}M}$. In other words, the correspondence that maps an A-module M to the $(S^{-1}A)$-module $S^{-1}M$ is a *covariant* functor from A-modules to $(S^{-1}A)$-modules.

(3.17) Notation. Let f be an element of A and let $S = \{1, f, f^2, \ldots, \}$. Moreover let $\mathfrak{p}$ be a prime ideal of A, and let $T = A \setminus \mathfrak{p}$. For every homomorphism $u : M \to N$
n we write !!$u_f = S^{-1}u$ and !!$u_{\mathfrak{p}} = T^{-1}u$. Moreover we write the canonical maps
n !!$i_A^S = i_A^f$ and !!$i_A^T = i_A^{\mathfrak{p}}$.

(3.18) Proposition. *Let A be a ring and S a multiplicatively closed subset. Moreover let*

$$0 \to M' \xrightarrow{u} M \xrightarrow{v} M'' \to 0$$

be an exact sequence of A-modules. Then the sequence

$$0 \to S^{-1}M' \xrightarrow{S^{-1}u} S^{-1}M \xrightarrow{S^{-1}v} S^{-1}M'' \to 0$$

is an exact sequence of $S^{-1}A$-modules.

Proof. We first show that $S^{-1}u$ is injective. If $x' \in M'$ and $s \in S$, and $S^{-1}u(x'/s) = u(x')/s = 0$ there is a $t \in S$ such that $u(tx') = tu(x') = 0$. Since u is injective we have that $tx' = 0$ and consequently $x'/s = 0$.

It is clear that we have an inclusion $\mathrm{Im}(S^{-1}u) \subseteq \mathrm{Ker}(S^{-1}v)$. We will show that the opposite inclusion holds. Let $x/s \in \mathrm{Ker}(S^{-1}v)$, that is $v(x)/s = 0$. Then there is a $t \in S$ such that $v(tx) = tv(x) = 0$. Since $\mathrm{Im}(u) = \mathrm{Ker}(v)$ there is an $x' \in M'$ such that $u(x') = tx$. Consequently $S^{-1}u(x'/(st)) = u(x')/(st) = (tx)/(st) = x/s$, and thus $x/s \in \mathrm{Im}(S^{-1}u)$.

Finally it is obvious that $S^{-1}v$ is surjective.

(3.19) Remark. We paraphrase Proposition (3.17) by saying that the functor S^{-1} is *exact*.

(3.20) Proposition. *Let A be a ring and let S be a multiplicatively closed subset. Moreover let $\{M_\alpha\}_{\alpha \in I}$ be a collection of A-modules. Then there is a canonical isomorphism of A-modules*

$$\oplus_{\alpha \in I} S^{-1}M_\alpha \xrightarrow{\sim} S^{-1}(\oplus_{\alpha \in I} M_\alpha) \tag{3.20.1}$$

such that the composite of the map (3.20.1) *with the canonical map $v_\beta : S^{-1}M_\beta \to \oplus_{\alpha \in I} S^{-1}M_\alpha$ to factor β is the localization $S^{-1}u_\beta : S^{-1}M_\beta \to S^{-1}(\oplus_{\alpha \in I} M_\alpha)$ of the canonical map $u_\beta : M_\beta \to \oplus_{\alpha \in I} M_\alpha$ to factor β.*

Proof. The canonical map $u_\beta : M_\beta \to \oplus_{\alpha \in I} M_\alpha$ gives a map $S^{-1}u_\beta : S^{-1}M_\beta \to S^{-1}(\oplus_{\alpha \in I} M_\alpha)$ and by the universal property of direct products we obtain the map $\oplus_{\alpha \in I}(S^{-1}M_\alpha) \to S^{-1}(\oplus_{\alpha \in I} M_\alpha)$ of (3.20.1).

To show that the map is an isomorphism we construct the inverse. The canonical maps $M_\alpha \to S^{-1}M_\alpha$ for $\alpha \in I$ define a homomorphism $\oplus_{\alpha \in I} M_\alpha \to \oplus_{\alpha \in I} S^{-1}M_\alpha$. Consequently it follows from Proposition (3.3) that we have a canonical homomorphism $S^{-1}(\oplus_{\alpha \in I} M_\alpha) \to \oplus_{\alpha \in I} S^{-1}M_\alpha$ and it is clear that this map is the inverse of the map (3.20.1).

(3.21) Proposition. *Let A be a ring and let M be an A-module. The following conditions are equivalent:*

(1) $M = \{0\}$.
(2) $M_{\mathfrak{p}} = \{0\}$ *for all prime ideals $\mathfrak{p}$ of A.*
(3) $M_{\mathfrak{m}} = \{0\}$ *for all maximal ideals $\mathfrak{m}$ of A.*

Proof. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are clear.

(3) $\Rightarrow$ (1) Let $x \in M$ and let $\mathfrak{a}_x = \{f \in A : fx = 0\}$. It is clear that $\mathfrak{a}_x$ is an ideal in A. We shall show that $\mathfrak{a}_x = A$, and hence in particular that $1x = x = 0$. Assume to the contrary that $\mathfrak{a}_x \not\subseteq A$. Then there is a maximal ideal $\mathfrak{m}$ of A that contains $\mathfrak{a}_x$. Since $M_{\mathfrak{m}} = 0$ we can find an element $s \in A \setminus \mathfrak{m}$ such that $sx = 0$. Then $s \in \mathfrak{a}_x$ which is impossible since $\mathfrak{a}_x \subseteq \mathfrak{m}$. This contradicts the assumption that $\mathfrak{a}_x \not\subseteq A$. Hence we have proved that $\mathfrak{a}_x = A$ for all $x \in M$ and consequently that $M = 0$.

(3.22) Proposition. *Let A be a ring and $u : M \to N$ an A-linear homomorphism. The following conditions are equivalent:*

(1) *u is injective, respectively surjective.*

(2) $u_{\mathfrak{p}}$ *is injective, respectively surjective, for all prime ideal* $\mathfrak{p}$ *of* A.
(3) $u_{\mathfrak{m}}$ *is injective, respectively surjective, for all maximal ideals* $\mathfrak{m}$ *of* A.

Proof. We prove the equivalence of the conditions for injective maps:

(1) $\Rightarrow$ (2) It follows from Proposition (?) that condition (2) follows from conditon (1).

(2) $\Rightarrow$ (3) This implication is clear.

(3) $\Rightarrow$ (1) Let $L = \operatorname{Ker}(u)$. When $u_{\mathfrak{m}}$ is injective it follows from Proposition (?) that $L_{\mathfrak{m}} = 0$ for all maximal primes $\mathfrak{m}$ of A. Hence it follows from Proposition (?) that $L = 0$ and thus that u is injective.

Similar arguments show the equivalence of the assertions for surjective maps.

(3.23) Corollary. *Let* $f \neq 0$ *be an element of* A. *We have:*

(1) *If* f *is not a zero divisor in* A *then* $f/1$ *is not a zero divisor in the localization* $A_{\mathfrak{p}}$ *of* A *in* $\mathfrak{p}$ *for all prime ideals* $\mathfrak{p}$ *of* A.
(2) *If* $f/1$ *is not a zero divisor in* $A_{\mathfrak{m}}$ *for all maximal ideals* $\mathfrak{m}$ *of* A *then* f *is not a zero divisor in* A.

Proof. We have that f is not a zero divisor in A if and only if the multiplication map $f_A : A \to A$ is injective, and $f/1$ is not a zero divisor in $A_{\mathfrak{p}}$ if and only if the multiplication map $(f/1)_{A_{\mathfrak{p}}} : A_{\mathfrak{p}} \to A_{\mathfrak{p}}$ is not injective. Hence the Corollary follows from the Proposition.

(3.24) Exercises.

1. Let K be a field and let $K[u, v]$ be the polynomial ring in the variables u, v with coefficients in K. Moreover let $A = K[u, v]/(uv)$.

(1) Show that the ideal $\mathfrak{p} = (u)/(uv)$ is a prime ideal in A.
(2) Describe the localization $A_{\mathfrak{p}}$.

2. Let M and N be A-modules and let S be a multiplicatively closed subset of A. Show that the $S^{-1}A$-modules $S^{-1}(M \otimes_A N)$ and $S^{-1}M \otimes_{S^{-1}A} S^{-1}N$ are canonically isomorphic.

3. Let f be a nilpotent element in A, and M an A-module. Determine M_f.

4. For every $f \in A$ and every prime ideal $\mathfrak{p}$ of A we let $f(\mathfrak{p})$ be the image of f by the composite map $A \xrightarrow{i_A^{\mathfrak{p}}} A_{\mathfrak{p}} \xrightarrow{\varphi_{A_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}}} A_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}$. Show that $f(\mathfrak{p}) = 0$ for all prime ideals $\mathfrak{p}$ if and only if f is contained in the radical $\mathfrak{r}(A)$ of A.

5. Let $\varphi : A \to B$ be a homomorphism of rings, and let S be a multiplicatively closed subset in A.

(1) Show that $T = \varphi(S)$ is a multiplicatively closed subset of B.
(2) Show that there is a canonical isomorphism between the $S^{-1}A$-modules $T^{-1}B$ and $S^{-1}B = B \otimes_A S^{-1}A$.

6. Let A be a ring and $\mathfrak{p}$ a prime ideal.

(1) Show that if the local ring $A_{\mathfrak{p}}$ has no nilpotent elements different from zero for all prime ideals $\mathfrak{p}$ in A then A has no nilpotent elements different from zero.

(2) Is it true that if A has no nilpotent elements different from zero then $A_{\mathfrak{p}}$ has no nilpotent elements different from zero for all prime ideals $\mathfrak{p}$ of A?

7. Let M be a finitely generated A module, and let S be a multiplicatively closed subset of M. Show that $S^{-1}M = 0$ if and only if there is an element $s \in S$ such that $sM = 0$.

n **8.** Let !!$\mathcal{P}$ be the set of all prime number in $\mathbf{Z}$

(1) Show that the map $\mathbf{Z} \to \prod_{p \in \mathcal{P}} \mathbf{Z}/p\mathbf{Z}$ that sends an integer n to $(n, n, \dots)$ is injective.

(2) Show that for all injective maps $u : G \to H$ of groups the map $u \otimes_{\mathbf{Z}} \mathrm{id}_{\mathbf{Q}} : G \otimes_{\mathbf{Z}} \mathbf{Q} \to H \otimes_{\mathrm{id}_{\mathbf{Z}}} \mathbf{Q}$ is injective.

(3) Show that $(\prod_{p \in \mathcal{P}} \mathbf{Z}/p\mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{Q}$ is not zero

(4) Show that $(\prod_{p \in \mathcal{P}} \mathbf{Z}/p\mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{Q}$ is not isomorphic to $\prod_{p \in \mathcal{P}} (\mathbf{Z}/p\mathbf{Z} \otimes_{\mathbf{Z}} \mathbf{Q})$.

9. Let A be a ring and S a multiplicatively closed subset. Moreover let M be an A-module. Describe the kernel of the A-module homomorphism $M \to M \otimes_A S^{-1}A$ that maps $x \in M$ to $x \otimes_A 1$.

10. Let $A \neq 0$ be a ring and $u : A^m \to A^n$ an A-linear map. Moreover let $\mathfrak{p}$ be a minimal prime ideal in A.

(1) Let $f_1, f_2, \dots, f_m$ be elements in $\mathfrak{p}$. Show that the ideal $\mathfrak{b}$ in $A_{\mathfrak{p}}$ generated by the elements $f_1/1, f_2/1, \dots, f_n/1$ is *nilpotent*, that is, we have $\mathfrak{b}^m = (0)$ for some positive integer m.

(2) Let p be the integer such that $\mathfrak{b}^p \neq (0)$ and $\mathfrak{b}^{p+1} = (0)$ in $A_{\mathfrak{p}}$. Show that for all elements $f \in \mathfrak{b}^p$ we have that $fs \neq 0$ for all $s \in A \setminus \mathfrak{p}$, and that $f_i f t = 0$ for some $t \in A \setminus \mathfrak{p}$ for $i = 1, 2, \dots, m$.

(3) Show that if u is injective then the map

$$u_{\mathfrak{p}} : (A/\mathfrak{p})^m \to (A/\mathfrak{p})^n$$

is injective, where the $A/\mathfrak{p}$-module homomorphism $u_{\mathfrak{p}}$ is defined by

$$u_{\mathfrak{p}}((u_{A/\mathfrak{p}}(f_1), u_{A/\mathfrak{p}}(f_2), \dots, u_{A/\mathfrak{p}}(f_n)) \\ = (u_{A/\mathfrak{p}}(u_1(x)), u_{A/\mathfrak{p}}(u_2(x)), \dots, u_{A/\mathfrak{p}}(u_n(x)))$$

for all $x = (f_1, f_2, \dots, f_m)$ in A^m and where $u(x) = (u_1(x), u_2(x), \dots, u_n(x))$ in A^n.

(4) Show that when u is injective then $m \leq n$.

4. Annihilators, associated ideals and primary modules.

(4.1) Notation. Let M be an A-module, and let E and F be subsets of M. We
n write!!

$$(E : F) = (E : F)_A = \{f \in A : fx \in E \text{ for all } x \in F\}.$$

n When E or F consists of one point x, respectively y, we write!! $(x : F) = (\{x\}, F)$,
respectively $(E : y) = (E : \{y\})$. Moreover we write $\mathfrak{r}((E : F)) = \mathfrak{r}(E : F) = \mathfrak{r}_A(E : F)$.

n For every element $f \in A$ we let !!$f_M : M \to M$ be the A-module homomorphism
defined by $f_M(x) = fx$ for all $x \in M$.

(4.2) Remark. It is clear that when E is a submodule of M we have that $(E : F)$ is an ideal in A.

(4.3) Definition. Let M be an A-module. For each element $x \in M$ we write
n !!$\mathrm{Ann}(x) = \mathrm{Ann}_A(x) = (0 : x)$, and call the ideal $\mathrm{Ann}(x)$ the *annihilator* of x. The
n *annihilator* of M is the ideal !!$\mathrm{Ann}(M) = \mathrm{Ann}_A(M) = (0 : M)$.

(4.4) Example. Let $\mathfrak{a}$ be an ideal in A. Consider $A/\mathfrak{a}$ as an A-module via the canonical map $\varphi_{A/\mathfrak{a}}$. Then $\mathrm{Ann}_A(A/\mathfrak{a}) = \mathfrak{a}$.

(4.5) Remark. Let M be an A-module. For all $x \in M$ we have an isomorphism $A/\mathrm{Ann}(x) \to Ax$ of A-modules that send the class in $A/\mathrm{Ann}(x)$ of $f \in A$ to fx. In particular we have an injection $A/\mathrm{Ann}(x) \to M$.

(4.6) Example. Let K be a field and let $A = K[u, v]$ be the polynomial ring in two variables u, v over K. Moreover, let $M = K[u, v]/(u^2, uv)$. Then $\mathrm{Ann}_A(u) = (u, v)$ and $\mathrm{Ann}_A(v) = (u)$.

(4.7) Definition. Let M be an A-module and $f \in A$. The homomorphism f_M is *nilpotent* if there is a positive integer n_f such that $f_M^{n_f} = 0$, that is, if $f \in \mathfrak{r}(\mathrm{Ann}(M))$. We say that f_M is *locally nilpotent* if there for every element $x \in M$ is a positive integer n_x such that $f_M^{n_x}(x) = f^{n_x}x = 0$, that is, $f \in \cap_{x\in M}\mathfrak{r}(\mathrm{Ann}(x))$.

An element $f \in A$ is *M-regular*, or *regular for M*, if the map $f_M : M \to M$ is injective. When $M = A$ we simply say that f is a *regular element* of A. A sequence of elements $f_1, f_2, \dots, f_n$ of A is *M-regular* if f_i is $(M/(f_1, f_2, \dots, f_{i-1})M)$-regular for $i = 1, 2, \dots, n$.

(4.8) Remark. When M is finitely generated we have that f_M is nilpotent if and only if it is locally nilpotent.

(4.9) Remark. The A-regular elements of A are the non-zero divisors of A different from 0.

(4.10) Definition. Let M be an A-module. A prime ideal $\mathfrak{p}$ in the ring A is *associated to M* if $\mathfrak{p} = \mathrm{Ann}(x)$ is the annihilator of an element x in M. The *support*
n !!$\mathrm{Supp}(M)$ of the module M is the set of prime ideals $\mathfrak{p}$ in A such that $M_\mathfrak{p} \neq 0$.

(4.11) Example. Let K be a field and let $A = K[u,v]$ be the polynomial ring in the variables u, v over K. Moreover let $M = K[u,v]/(u^2, uv)$. We have that the ideals (u,v) and (u) are associated to the A-module M. They are the only ideals associated to M since (u,v) and (u) are the only prime ideals in A that contain the annihilator $\mathrm{Ann}(M) = (u^2, uv)$.

(4.12) Example. Let A be a ring and let $\mathfrak{a} \subseteq A$ be an ideal. Then the support of the A-module $A/\mathfrak{a}$ is the set $V(\mathfrak{a})$ of prime ideals of A containing $\mathfrak{a}$.

(4.13) Remark. Let A be a ring and let $\mathfrak{p}$ be a prime ideal of A. For each elements $x \in M$ we have that $(Ax)_{\mathfrak{p}} \neq 0$ if and only if $\mathrm{Ann}(x) \subseteq \mathfrak{p}$. It follows that the associated ideals of the A-module M are contained in the support of M. In fact if $\mathfrak{p} = \mathrm{Ann}(x)$ is associated to M then $(Ax)_{\mathfrak{p}} \neq 0$, and consequently it follows from Proposition (3.18) that $M_{\mathfrak{p}} \neq 0$.

(4.14) Remark. When M is a finitely generated A-module we have that the support $\mathrm{Supp}(M)$ of M is a closed subset of $\mathrm{Spec}(A)$. In fact choose generators $x_1, x_2, \dots, x_n$ of M. If $\mathfrak{p} \notin \mathrm{Supp}(M)$ we have $M_{\mathfrak{p}} = 0$. Consequently there are elements $s_1, s_2, \dots, s_n$ in $A \setminus \mathfrak{p}$ such that $s_i x_i = 0$ for $i = 1, 2, \dots, n$. Let $f = s_1 s_2 \cdots s_n$. Then $f \notin \mathfrak{p}$ and $f x_i = 0$ for $i = 1, 2, \dots, n$. Hence, for each prime ideal $\mathfrak{q}$ not containing f, we have that $M_{\mathfrak{q}} = 0$. That is, the open set $D(f)$ is a neighbourhood of $\mathfrak{p}$ such that for $\mathfrak{q} \in D(f)$ we have that $M_{\mathfrak{q}} = 0$. Hence the complement of $\mathrm{Supp}(M)$ is open in $\mathrm{Spec}(A)$.

(4.15) Example. When M is not finitely generated it is not always true that $\mathrm{Supp}(M)$ is closed. For example, let $\mathcal{P}$ be an infinite set of prime numbers of $\mathbf{Z}$ such that there are infinitely many prime numbers of $\mathbf{Z}$ that are not contained in $\mathcal{P}$. Then the support of the $\mathbf{Z}$-module $\oplus_{p \in \mathcal{P}} \mathbf{Z}/p\mathbf{Z}$ is equal to $\mathcal{P}$, which is neither closed, nor open, in $\mathrm{Spec}(\mathbf{Z})$.

(4.16) Definition. Let M be an A-module and let L be a submodule of M. The *radical* of L is the ideal $\mathfrak{r}_M(L) = \mathfrak{r}_A(L : M)$.

(4.17) Remark. When $\mathfrak{a}$ is an ideal of A, the radical of $\mathfrak{a}$ as a module coincide with the radical of the ideal $\mathfrak{a}$ in the ring A.

(4.18) Remark. We have that $\mathfrak{r}_M(L) = \mathfrak{r}_{M/L}(0)$.

(4.19) Definition. Let M be an A-module. A submodule L of M is *primary* if the map $f_{M/L} : M/L \to M/L$ is either injective, or nilpotent for all elements $f \in A$. An ideal $\mathfrak{a}$ in A is *primary* if it is primary considered as an A-module.

(4.20) Remark. Clearly an ideal $\mathfrak{a}$ of A is primary if and only if $fg \in \mathfrak{a}$ and $g \notin \mathfrak{a}$ implies that $f^n \in \mathfrak{a}$ for some integer n.

(4.21) Lemma. *If L is a primary submodule of an A-module M we have that the radical $\mathfrak{r}_M(L)$ of L is a prime ideal in A.*

Proof. When $fg \in \mathfrak{r}_M(L)$ and $f \notin \mathfrak{r}_M(L)$ we have that $f^n_{M/L} g^n_{M/L} = (fg)^n_{M/L} = 0$ for some positive integer n. Since L is primary and $f \notin \mathfrak{r}_M(L)$ we have that $f^n_{M/L} : M/L \to M/L$ is injective. Consequently we have that $g^n_{M/L} = 0$, or equivalently that $g \in \mathfrak{r}_M(L)$.

(4.22) Definition. Let M be an A-module and L a primary submodule of M. The prime ideal $\mathfrak{r}_M(L)$ is called the *prime ideal belonging to* the submodule L. We say that L is $\mathfrak{r}_M(L)$-*primary.*

(4.23) Example. Let A be a ring. An ideal $\mathfrak{a}$ whose radical $\mathfrak{r}_A(\mathfrak{a})$ is maximal is primary. This is because the image of $\mathfrak{r}(\mathfrak{a})$ by the canonical map $\varphi_{A/\mathfrak{a}} : A \to A/\mathfrak{a}$ is the radical $\mathfrak{r}_{A/\mathfrak{a}}(0)$ of $A/\mathfrak{a}$. When $\mathfrak{r}(\mathfrak{a})$ is maximal the same is true for $\mathfrak{r}_{A/\mathfrak{a}}(0)$. The ring $A/\mathfrak{a}$ is local and the only prime ideal is $\mathfrak{r}_{A/\mathfrak{a}}(0)$ because $\mathfrak{r}_{A/\mathfrak{a}}(0)$ is contained in every prime ideal of $A/\mathfrak{a}$. For every element $f \in A$ we consequently have that $f_{A/\mathfrak{a}} : A/\mathfrak{a} \to A/\mathfrak{a}$ is an isomorphism when $f \notin \mathfrak{r}(\mathfrak{a})$ and is nilpotent when $f \in \mathfrak{r}(\mathfrak{a})$.

In particular we have that every power of a maximal ideal in A i primary.

(4.24) Example. It is not true that powers of prime ideals necessarily are primary. → In Example (?) the image of the prime ideal (u) of $K[u,v]$ by the canonical map $\varphi_{K[u,v]/(u^2,uv)} : K[u,v] \to K[u,v]/(u^2,uv)$ is a prime ideal $\mathfrak{q}$ such that $\mathfrak{q}^2 = 0$. However the ideal (0) in $K[u,v]/(u^2,uv)$ is not primary. In fact the map $v_{K[u,v]/(u^2,uv)} : K[u,v]/(u^2,uv) \to K[u,v]/(u^2,uv)$ is not injective since the class of u maps to zero, and it is not nilpotent since $v^n \notin (u^2,uv)$ for all positive integers n.

(4.25) Proposition. *Let M be an A-module M, and let L be a submodule of M.*

(1) *Every prime ideal associated to the module L is associated to the module M.*
(2) *The associated primes ideals of the module M are associated to either the module L or to the module M/L.*

Proof. (1) The first assertion is clear.

(2) Let $\mathfrak{p}$ be associated to M. Then $\mathfrak{p} = \mathrm{Ann}(x)$ for some $x \in M$. If $Ax \cap L = 0$ we have that $\mathfrak{p}$ is associated to M/L. If $Ax \cap L \neq 0$ we choose a non zero element $y = fx$ with $f \in A$. Then $\mathfrak{p} = \mathrm{Ann}(y)$ because, on the one hand $\mathfrak{p} \subseteq \mathrm{Ann}(y)$, and on the other hand $g \in \mathrm{Ann}(y)$ for some $g \in A$ implies that $gf \in \mathfrak{p}$. Since $f \notin \mathfrak{p}$ we have that $g \in \mathfrak{p}$.

(4.26) Exercises.

1. Let A be a ring and S a multiplicatively closed subset. What is the relation between the associated prime ideals of an A-modules M and those of the $S^{-1}A$-module $S^{-1}M$?

2. Let L and M be submodules of an A-module P.

(1) Show that $(L : M) = \mathrm{Ann}((L+M)/L)$.
(2) Show that $\mathrm{Ann}(L+M) = \mathrm{Ann}(L) \cap \mathrm{Ann}(M)$.

3. Let N be an A-module, and let L and M be submodules. Prove the following four assertions:

(1) $\mathfrak{r}_A(\mathfrak{r}_N(M)) = \mathfrak{r}_N(M)$.
(2) $\mathfrak{r}_N(L \cap M) = \mathfrak{r}_N(L) \cap \mathfrak{r}_N(M)$.
(3) $\mathfrak{r}_N(M) = A$ if and only if $M = L$.
(4) $\mathfrak{r}_N(L + M) = \mathfrak{r}_A(\mathfrak{r}_N(L) + \mathfrak{r}_N(M))$.

4. Let S be a multiplicatively closed subset of the ring A. Moreover, let P be an A-module and M and N submodules. Show that $S^{-1}(M : N) = (S^{-1}M : S^{-1}N)$.

5. Let A be a ring and M a finitely generated A-module. Show that we have $\mathrm{Supp}(M) = V(\mathrm{Ann}(M))$ in $\mathrm{Spec}(A)$.

6. Let A be a ring and let S be a multiplicatively closed subset. Moreover let $\mathfrak{q}$ be a primary ideal and let $\mathfrak{p}$ be the prime ideal belonging to $\mathfrak{q}$.

(1) Show that the ideal $\mathfrak{q}$ instersects S if and only if the ideal $\mathfrak{p}$ instersects S.
(2) Show that when the ideal $\mathfrak{q}$ does not intersect S then $S^{-1}\mathfrak{q}$ is a primary ideal in $S^{-1}A$ and that the prime ideal $S^{-1}\mathfrak{p}$ belongs to $S^{-1}\mathfrak{q}$.

7. Show that the ideal $(4, t)$ in the polynomial ring $\mathbf{Z}[t]$ in the variable t over the integers is primary. Find the prime ideal that belongs to $(4, t)$.

8. Let $A[t]$ be the polynomial ring in the variable t over the ring A. For each ideal $\mathfrak{a}$ in A we write $\mathfrak{a}[t]$ for the subset of $A[t]$ consisting of all polynomials $f_0 + f_1 t = \cdots + f_n t^n$ with coefficients $f_0, f_1, \ldots, f_n$ in A.

(1) Show that $\mathfrak{a}[t]$ is the smallest ideal in $A[t]$ that contains the ideal $\mathfrak{a}$.
(2) Show that if $\mathfrak{p}$ is a prime ideal in A then $\mathfrak{p}[t]$ is a prime ideal in $A[t]$.
(3) Show that if $\mathfrak{a}$ is a $\mathfrak{p}$-primary ideal in A, then $\mathfrak{a}[t]$ is $\mathfrak{p}[t]$-primary ideal in $A[t]$.

5. Differentials.

(5.1) Definition. Let A be a ring and let B be an A-algebra. Moreover, let N be a B-module that we consider as an A-module via restriction of scalars. An A-*derivation* from B to N is an A-linear homomorphism $D : B \to N$ such that for all g, h in B we have!!

$$D(gh) = gD(h) + hD(g). \tag{5.1.1}$$

The collection of all A-derivations from B to N we denote by $\mathrm{Der}_A(B, N)$.

(5.2) Remark. We have that $D(1) = 0$ because $D(1) = D(1 \cdot 1) = 1D(1) + 1D(1) = 2D(1)$. Consequently, when $\varphi : A \to B$ is the algebra structure, we have for all $f \in A$ that $D(\varphi(f)) = D(\varphi(f)1) = \varphi(f)D(1) = 0$ for all $f \in A$.

For every element $g \in B$ and every natural number n we have that $D(g^n) = ng^{n-1}D(g)$. This is easily shown by induction on n. When $g \in B$ is an invertible element we have that $D(g^{-1}) = -g^{-2}D(g)$. In fact, when we take the derivative of both sides of the equality $gg^{-1} = 1$ we obtain that $gD(g^{-1}) + g^{-1}D(g) = 0$.

(5.3) Remark. We have that $\mathrm{Der}_A(B, N)$ is a B-module. The sum $D + E$ of two A-derivations $D : B \to N$ and $E : B \to N$ is defined by $(D + E)(h) = D(h) + E(h)$ for all $h \in B$, and the product gD of D with an element $g \in B$ is defined by $(gD)(h) = gD(h)$. It is easy to check that $D + E$ and gD are derivations and that the sum and product make $\mathrm{Der}_A(B, N)$ into a B-module.

(5.4) Remark. When the A-algebra B is generated by elements $\{g_\alpha\}_{\alpha \in I}$ we have that an A-derivation $D : B \to N$ to a B-module N is determined by the values $D(g_\alpha)$ for all $\alpha \in I$. This follows by repeated application of the derivation rule (5.1.1) to expressions of the form $\prod_{\beta \in J} g_\beta^{n_\beta}$ for a finite subset J of I, and with $g_\beta \in B$ and $n_\beta \in \mathbf{N}$.

(5.5) Example. Let $A[t_\alpha]_{\alpha \in I}$ be the polynomial ring in the variables t_α over the ring A. For each $\alpha \in I$ there is a unique A-derivation $D_\alpha : A[t_\alpha]_{\alpha \in I} \to A[t_\alpha]_{\alpha \in I}$ determined by $D_\alpha(t_\alpha) = 1$ and $D_\alpha(t_\beta) = 0$ when $\alpha \neq \beta$. For every finite subset J of I, and $n_\beta \in \mathbf{N}$ the derivation D_α maps the monomial $\prod_{\beta \in J} t_\beta^{n_\beta}$ to 0 if $\alpha \notin J$ and to $n_\alpha t_\alpha^{n_\alpha - 1} \prod_{\beta \in J \setminus \{\alpha\}} t_\beta^{n_\beta}$ when $\alpha \in J$. It is clear that for every derivation $D : A[t_\alpha]_{\alpha \in I} \to N$ to an $(A[t_\alpha]_{\alpha \in I})$-module N we have that $D(f(t)) = \sum_{\alpha \in I} D_\alpha(f)D(t_\alpha)$ for all $f(t) \in A[t_\alpha]_{\alpha \in I}$. Note that $D_\alpha(f) = 0$ for all but a finite number of α because each polynomial $f(t)$ is expressed in a finite number of variables t_α. Moreover, for every ideal $\mathfrak{b}$ of $A[t_\alpha]_{\alpha \in I}$ an A-derivation $D : A[t_\alpha]_{\alpha \in I} \to N$ factors via an A-derivation $A[t_\alpha]_{\alpha \in I}/\mathfrak{b} \to N$ if and only if $\sum_{\alpha \in I} D_\alpha(f)D(t_\alpha) = 0$ for all $f \in \mathfrak{b}$.

(5.6) Example. Let $\varphi : A \to B$ be an A-algebra and $\chi : B \to C$ a homomorphism of A-algebras. The homomorphism χ gives C a structure as a B-algebra. Moreover let P be a C-module that we consider as a B-module by restriction of scalars. We denote by $C[P]$ the C-algebra $C \times P$ with addition defined by $(\chi(g), x) + (\chi(h), y) =$

$(\chi(g)+\chi(h), x+y)$ and product defined by $(\chi(g), x)(\chi(h), y) = (\chi(g)\chi(h), gy+hx)$ for all g, h in B and x, y in P. Write $g+x = (g, x)$. We consider $C[P]$ as an C-algebra via the map $C \to C[P]$ that sends h to $(h, 0)$ and identify C with its image by this map.

Let $\psi : B \to C[P]$ and $D : B \to P$ be maps that are related by $\psi(h) = \chi(h)+D(h)$ for all $h \in B$. It is clear that ψ is an A-linear homomorphism if and only if D is an A-linear homomorphism. When ψ and D are both A-linear homomorphism we have that ψ is an A-algebra homomorphism if and only if D is an A-derivation because $\psi(gh)-\psi(g)\psi(h) = \chi(gh)+D(gh)-(\chi(g)+D(g))(\chi(h)+D(h)) = D(gh)-gD(h)-hD(g)$ for all g, h in B. In this way we obtain a bijection

$$\mathrm{Hom}_{A\text{-alg}}(B, C[P]) \to \mathrm{Der}_A(B, P).$$

(5.7) Example. Let $\varphi : A \to B$ be an A-algebra and let S and T be multiplicatively closed subsets of A respectively B such that $\varphi(S) \subseteq T$. Then every A-derivation $D : B \to N$ from B to a B-module N defines a unique $(S^{-1}A)$-derivation $T^{-1}D : T^{-1}B \to T^{-1}N$ such that $D(g)/1 = T^{-1}D(g/1)$ for all $g \in B$.

It is clear that $T^{-1}D$ is unique if it exists because for all elements $g \in B$ and $t \in T$ we have that $T^{-1}D(g/t) = T^{-1}D((t/1)^{-1}(g/1)) = -(g/1)(t/1)^{-2}T^{-1}D(t/1) + (t/1)^{-1}T^{-1}D(g/1) = -(g/t^2)(D(t)/1) + (1/t)(D(g)/1)$ in $T^{-1}N$.

We define $T^{-1}D$ by $T^{-1}D(g/t) = D(g)/t - (gD(t))/t^2$ for all $g \in B$ and $t \in T$. The definition is independent of the representation g/t because if $g/t = g'/t'$ in $T^{-1}B$ with $t' \in T$ and $g' \in B$ there is a $t'' \in T$ such that we have an equality $t''(gt'-tg') = 0$ in B. Derivation of both sides of the equality gives $D(t'')(gt'-tg') + t''(gD(t') + t'D(g) - g'D(t) - tD(g')) = 0$. Multiplication of both sides of the latter equality with t'' and division by tt' in $T^{-1}N$ gives the equality $0 = (t'')^2((gD(t')/(tt') + D(g)/t - (g'D(t))/(tt') - D(g')/t') = (t'')^2((g'D(t'))/(t')^2 + D(g)/t - (gD(t))/t^2 - D(g')/t')$. Hence we obtain that $D(g)/t - (gD(t))/t^2 = D(g')/t' - (g'D(t'))/(t')^2$ in $T^{-1}N$, and thus $T^{-1}D(g/t) = T^{-1}D(g'/t')$. It is easy to check that $T^{-1}D$ is an $(S^{-1}A)$-derivation.

(5.8) Functoriality. Let B be an A-algebra and N a B-module. For every homomorphism $v : N \to N'$ of B-modules we obtain a homomorphism of B-modules

$$v_0 : \mathrm{Der}_A(B, N) \to \mathrm{Der}_A(B, N')$$

that maps the A-derivation $D : B \to N$ to the A-derivation $vD : B \to N'$. It is clear that the correspondence that maps a B-module N to $\mathrm{Der}_A(B, N)$ with fixed A and B is a functor from B-modules to B-modules.

Let $\chi : B' \to B$ be a homorphism of A-algebras. We consider N as a B'-module by restriction of scalars via χ. Then we obtain a χ-module homomorphism

$$\chi^0 : \mathrm{Der}_A(B, N) \to \mathrm{Der}_A(B', N)$$

that maps the A-derivation $D : B \to N$ to the A-derivation $D\chi : B' \to N$.

Let $\varphi : A' \to A$ be a homomorphism of rings. We consider B as an A'-algebra, and N as an A'-module, by restriction of scalars. Then we obtain a homomorphism of B-modules

$$\varphi^0 : \mathrm{Der}_A(B, N) \to \mathrm{Der}_{A'}(B, N)$$

that maps the A-derivation $D : B \to N$ to itself considered as an A'-derivation by restriction of scalars.

(5.9) Theorem. *Let $\varphi : A \to B$ be an A-algebra and let $\chi : B \to C$ be a B-algebra. Moreover let P be a C-module. We obtain an exact sequence of B-modules*

$$0 \to \mathrm{Der}_B(C, P) \xrightarrow{\varphi^0} \mathrm{Der}_A(C, P) \xrightarrow{\chi^0} \mathrm{Der}_A(B, P),$$

where we consider the C-module $\mathrm{Der}_A(C, P)$ and $\mathrm{Der}_B(C, P)$ as B-modules by restriction of scalars.

Proof. It is clear that φ^0 is injective. Moreover we have that $\mathrm{Im}(\varphi^0) \subseteq \mathrm{Ker}(\chi^0)$ because, when $D : C \to P$ is a B-derivation, we have that $(D\chi)(g) = D(\chi(g)) = 0$ → for all $g \in B$ by Remark (?).

We shall show that $\mathrm{Im}(\varphi^0) = \mathrm{Ker}(\chi^0)$. Let $D : C \to P$ in $\mathrm{Ker}(\chi^0)$ be an A derivation, that is, a derivation such that $D\chi : B \to P$ is zero. For all $g \in B$ and $h \in C$ we have that $D(gh) = D(\chi(g)h) = hD(\chi(g)) + \chi(g)D(h) = \chi(g)D(h)$. Consequently $D : C \to P$ is a B-derivation. Hence the image of the A-derivation D by φ^0 is itself considered as a B-derivation.

(5.10) Remark. Let A be a ring and let B and C be A-algebras. Moreover let $\chi : B \to C$ be a surjection of A-algebras with kernel $\mathfrak{b}$. Every A-derivation $D : B \to P$ into a C-module P induces a B-linear map $v = D|\mathfrak{b} : \mathfrak{b} \to P$. In fact, for all $g \in B$ and $h \in \mathfrak{b}$ we have that $v(gh) = D(gh) = gD(h) + hD(g) = \chi(g)D(h) + \chi(h)D(g) = \chi(g)D(h) = D(gh) = v(gh)$. We also see that when $g \in \mathfrak{b}$ we have that $v(gh) = 0$. Hence v is zero on the ideal $\mathfrak{b}^2$ and induces a B-module homomorphism $\mathfrak{b}/\mathfrak{b}^2 \to P_{'\xi]}$. Since $C = B/\mathfrak{b}$ and P are C-modules we obtain a C-module homomorphism $w : \mathfrak{b}/\mathfrak{b}^2 \to P$. It is clear the correspondence that sends D to w gives a B-module homomorphism

$$u : \mathrm{Der}_A(B, P) \to \mathrm{Hom}_C(\mathfrak{b}/\mathfrak{b}^2, P)$$

where $\mathrm{Hom}_C(\mathfrak{b}/\mathfrak{b}^2, P)$ is a B-module via χ.

(5.11) Theorem. *Let $\chi : B \to C$ be a surjection of A-algebras, and let $\mathfrak{b}$ be the kernel of χ. For every C-module P there is an exact sequence of B-modules*

$$0 \to \mathrm{Der}_A(C, P) \xrightarrow{\chi^0} \mathrm{Der}_A(B, P) \xrightarrow{u} \mathrm{Hom}_C(\mathfrak{b}/\mathfrak{b}^2, P),$$

→ *where u is the map of Remark* (5.10).

Proof. We have that χ^0 is injective since χ is surjective. Moreover $\mathrm{Im}(\chi^0) \subseteq \mathrm{Ker}(u)$ because, if $D : C \to P$ is an A-derivation, then the map $(D\chi)|\mathfrak{b} : \mathfrak{b} \to P$ is zero, since for all $h \in \mathfrak{b}$ we have $(D\chi)(h) = D(\chi(h)) = D(0) = 0$.

It remains to prove that $\mathrm{Im}(\chi^0) = \mathrm{Ker}(u)$. Let $D : B \to P$ be an A-derivation such
→ that $u(D) = 0$. Then $D(h) = 0$ for all $h \in \mathfrak{b}$. Hence it follows from Lemma (1.13) that the A-linear homomorphism $D : B \to P$ factors via $\chi : B \to C$, and an A-linear homomorphism $E : C \to P$. It is easy to verify that since D is an A-derivation the A-linear homomorphism E is also an A-derivation. Clearly we have that $\chi^0(E) = D$, and we have proved the Theorem.

(5.12) Kähler differentials. Let B be an A-algebra. We consider the A-algebra $B \otimes_A B$ as a B-module via multiplication by B in the left factor of $B \otimes_A B$. That is, for all f, g, h in B we let $f(g \otimes_A h) = fg \otimes_A h$.

The multiplication $B \otimes_A B \to B$ that maps $g \otimes_A h$ to gh is a ring homomorphism, and a B-module homomorphism. We denote the kernel of the multiplication map by $\mathfrak{I} = \mathfrak{I}_{B/A}$. Then we have an exact sequence of B-modules

$$0 \to \mathfrak{I}_{B/A} \to B \otimes_A B \to B \to 0.$$

The B-module $\mathfrak{I}_{B/A}$ is generated by the elements $\{1 \otimes_A g - g \otimes_A 1\}_{g \in B}$. In fact if $\sum_{i=1}^n g_i \otimes_A h_i$ with g_i and h_i in B is in $\mathfrak{I}_{B/A}$, that is $\sum_{i=1}^n g_i h_i = 0$, we have that $\sum_{i=1}^n g_i \otimes_A h_i = \sum_{i=1}^n g_i \otimes_A h_i - \sum_{i=1}^n g_i h_i \otimes_A 1 = \sum_{i=1}^n g_i(1 \otimes_A h_i - h_i \otimes_A 1)$. We
n write!!

$$\Omega^1_{B/A} = \mathfrak{I}_{B/A}/\mathfrak{I}^2_{B/A}.$$

The B-module $\Omega^1_{B/A}$ is generated by the classes of the elements $\{1 \otimes_A g - g \otimes_A 1\}_{g \in B}$.
n Let !!

$$d_{B/A} : B \to \Omega^1_{B/A}$$

be the map that takes an element $g \in B$ to the class $d_{B/A}(g)$ of $1 \otimes_A g - g \otimes_A 1$ in $\mathfrak{I}_{B/A}/\mathfrak{I}^2_{B/A}$. The map $d_{B/A}$ is A-linear. In fact it is clearly a group homomorphism, and for $f \in A$ and $g \in B$ we have that $d_{B/A}(fg)$ is the class in $\Omega^1_{B/A}$ of the element $1 \otimes_A fg - fg \otimes_A 1 = f \otimes_A g - fg \otimes_A 1 = f(1 \otimes_A g - g \otimes_A 1)$, and thus $d_{B/A}(fg) = f d_{B/A}(g)$. Moreover we have that $d_{B/A}$ is an A-derivation. In fact for g, h in B we have that $d_{B/A}(gh)$ is the class in $\mathfrak{I}_{B/A}/\mathfrak{I}^2_{B/A}$ of

$$\begin{aligned} 1 \otimes_A gh - gh \otimes_A 1 &= g(1 \otimes_A h - h \otimes_A 1) \\ &\quad + h(1 \otimes_A g - g \otimes_A 1) + (1 \otimes_A g - g \otimes_A 1)(1 \otimes_A h - h \otimes_A 1). \end{aligned}$$

(5.13) Definition. We call the B-module $\Omega^1_{B/A}$ the *Kähler differentials* of the A-algebra B, and we call the A-derivation $d_{B/A} : B \to \Omega^1_{B/A}$ the *exterior derivation.* For $f \in B$ we call $d_{B/A}(f)$ the *differential* of the element f.

(5.14) Remark. We have that the B-module $\Omega^1_{B/A}$ is generated by the elements $\{d_{B/A}(g)\}_{g\in B}$. When the A-algebra B is generated by the elements $\{g_\alpha\}_{\alpha\in I}$ we have that the B-module $\Omega^1_{B/A}$ is generated by the elements $\{d_{B/A}(g_\alpha)\}_{\alpha\in I}$. In fact, this follows from Remark (5.4) since $d_{B/A}$ is a derivation.

(5.15) Proposition. *Let B be an A-algebra. The exterior derivation $d_{B/A}: B \to \Omega^1_{B/A}$ has the following universal property:*

For every A-derivation $D: B \to N$ to a B-module N there is a unique B-linear homomorphism $v: \Omega^1_{B/A} \to N$ such that $D = vd_{B/A}$.

The map that sends D to v is an isomorphism of B-modules

$$\mathrm{Der}_A(B, N) \xrightarrow{\sim} \mathrm{Hom}_B(\Omega^1_{B/A}, N).$$

Proof. Since the B-module Ω^1_{BA} is generated by the elements $\{d_{B/A}(g)\}_{g\in B}$ we have that if v exists then it is uniquely determined by $v(d_{B/A}(g)) = D(g)$ for all $g \in B$.

To show that v exists we observe that we have an A-bilinear map $B \times B \to N$ that sends a pair (g, h) to $gD(h)$. The A-bilinear map gives an A-linear homomorphism $w: B \otimes_A B \to N$ determined by $w(g \otimes_A h) = gD(h)$ for all g, h in B. We see that w is also B-linear. The homomorphism w is zero on $\mathfrak{I}^2_{B/A}$ because for f, g, h in B we have

$$\begin{aligned} w(h(1\otimes_A f - f\otimes_A 1)(1\otimes_A g - g\otimes_A 1)) &= w(h\otimes_A fg - gh\otimes_A f - hf\otimes_A g + hfg\otimes_A 1) \\ = hD(fg) - hgD(f) - hfD(g) + hfgD(1) &= h(D(fg) - fD(g) - gD(f)) = 0. \end{aligned}$$

Consequently w induces a B-linear homomorphism $v: \mathfrak{I}_{B/A}/\mathfrak{I}^2_{B/A} \to N$ such that $D = vd_{B/A}$. Hence we have proved the first part of the Proposition.

It is clear that the map $\mathrm{Der}_A(B, N) \to \mathrm{Hom}_B(\Omega^1_{B/A}, N)$ that sends D to v is a B-module homomorphism, and the uniqueness of v implies that it is injective. The map is surjective because if $w: \Omega^1_{B/A} \to N$ is a B-module homomorphism we obtain that $wd_{B/A}: B \to N$ is an A-derivation that maps to w.

(5.16) Remark. The map $d_{B/A}: B \to \Omega^1_{B/A}$ is uniquely determined in the sense that if $d: B \to \Omega$ is another A-derivation that has the same universal property, that is, for every A-derivation $D: B \to N$ there is a unique B-module map $v: \Omega \to N$ such that $D = vd$, then there is a B-module isomorphism $w: \Omega^1_{B/A} \to \Omega$ such that $d = wd_{B/A}$. In fact w is obtained by the universality of $d_{B/A}$ and the universality of d defines a unique B-module homomorphism $v: \Omega \to \Omega^1_{B/A}$ such that $vd = d_{B/A}$. We have that $d = wvd$ and $d_{B/A} = vwd_{B/A}$ and by the universality of Ω and $\Omega^1_{B/A}$ we have that $wv = \mathrm{id}_\Omega$ and $vw = \mathrm{id}_{\Omega^1_{B/A}}$, and thus that v and w are inverse maps.

(5.17) Example. Let $B = A[t_\alpha]_{\alpha\in I}$ be the polynomial ring in the variables t_α over the ring A. It follows from Example (5.5) that $\Omega^1_{B/A}$ is the free B-module with basis $\{d_{B/A}(t_\alpha)\}_{\alpha\in I}$.

Let $\mathfrak{b}$ be an ideal of B, and let $C = B/\mathfrak{b}$. It follows from Example (5.5) that $\Omega^1_{C/A}$ is equal to the C-module which is the free C-module $\Omega^1_{B/A} \otimes_B C$ modulo the C-module generated by the elements $\sum_{\alpha \in I} D_\alpha(f) d_{B/A}(t_\alpha)$ for all $f \in B$.

(5.18) Functoriality. Let $\chi : B \to C$ be a homomorphism of A-algebras. We obtain a C-linear homomorphism

$$\chi_{C/B/A} : \Omega^1_{B/A} \otimes_B C \to \Omega^1_{C/A} \tag{5.18.1}$$

that is uniquely determined by $\chi_{C/B/A}(d_{B/A}(g) \otimes_B h) = hd_{C/A}(\chi(g))$ for all g in B and h in C. In fact the A-derivation $d_{C/A} : C \to \Omega^1_{C/A}$ gives an A-derivation $d_{C/A}\chi : B \to \Omega^1_{C/A}$, and consequently an A-derivation $\psi : \Omega^1_{B/A} \to \Omega^1_{C/A}$ such that $\chi = \psi d_{B/A}$. We obtain the map (5.18.1) by extension of scalars.

Let $\varphi : A \to B$ be a ring homomorphism and let C be an B-algebra that we consider as an A-algebra by restriction of scalars. We have a C-linear homomorphism

$$\varphi_{C/B/A} : \Omega^1_{C/A} \to \Omega^1_{C/B}$$

that is uniquely determined by $\varphi_{C/B/A}(d_{C/A}(h)) = d_{C/B}(h)$ for all $h \in C$. In fact the B-derivation $d_{C/B} : C \to \Omega_{C/B}$ is also an A-derivation via φ and by the universal property gives a C-linear map $\varphi_{C/B/A} : \Omega^1_{C/A} \to \Omega^1_{C/B}$ such that $\varphi_{C/B/A} d_{C/A} = d_{C/B}$.

(5.19) Lemma. *Let B be an A-algebra and let Ω be a B-module. Moreover let $d : B \to \Omega$ be an A-derivation that satisfies the conditions:*

(1) *There is a B-linear homomorphism $v : \Omega \to \Omega^1_{B/A}$ such that $d_{B/A} = vd$.*
(2) *The B-module Ω is generated by the elements $\{d(g)\}_{g \in B}$.*

Then v is an isomorphism.

Proof. Since d is a B-derivation there is a unique B-linear homomorphism $w : \Omega^1_{B/A} \to \Omega$ such that $d = wd_{B/A}$. Since the B-module Ω is generated by the elements $\{d(g)\}_{g \in B}$ it follows that w is surjective. We have that $vwd_{B/A} = vd = d_{B/A}$. It follows from the universality of $\Omega^1_{B/A}$ that $vw = \mathrm{id}_{\Omega^1_{B/A}}$. In particular we have that w is injective. Consequently w is an isomorphism and the same is therefore true for v.

(5.20) Proposition. *Let B be an A-algebra and let C be a B-algebra. We have an isomorphism of $(B \otimes_A C)$-modules*

$$\Omega^1_{B/A} \otimes_A C \xrightarrow{\sim} \Omega^1_{B \otimes_A C/C} \tag{5.20.1}$$

that is uniquely determined by mapping the element $d_{B/A}(g) \otimes_A h$ to the element $d_{B \otimes_A C/C}(g \otimes_A h) = hd_{B \otimes_A C/C}(g \otimes_A 1)$ for all $g \in B$ and $h \in C$.

Proof. The A-algebra structure $\psi : A \to C$ on C gives a $(B \otimes_A C)$-linear homomorphism $\psi_{B\otimes_A C/C/A} : \Omega^1_{B\otimes_A C/A} \to \Omega^1_{B\otimes_A C/C}$. We obtain a $(B \otimes_A C)$-linear homomorphism $\Omega^1_{B/A} \otimes_B (B \otimes_A C) \to \Omega^1_{B\otimes_A C/C}$. It follows from Lemma (2.11) that we have a canonical $(B \otimes_A C)$-module isomorphism $\Omega^1_{B/A} \otimes_B (B \otimes_A C) \xrightarrow{\sim} \Omega^1_{B/A} \otimes_A C$. We consequently have constructed the $(B \otimes_A C)$-linear homomorphism $v : \Omega^1_{B/A} \otimes_A C \to \Omega^1_{B\otimes_A C/C}$ of the Proposition. We have that the A-derivation $d_{B/A} : B \to \Omega^1_{B/A}$ gives a C-derivation $d = d_{B/A} \otimes_A \mathrm{id}_C : B \otimes_A C \to \Omega^1_{B/A} \otimes_A C$. It is clear that the $(B \otimes_A C)$-module $\Omega^1_{B/A} \otimes_A C$ is generated by the elements $\{d(g \otimes_A 1)\}_{g\in B}$, and that $d_{B\otimes_A C/C} = vd_{B\otimes_A C/C}$. It follows from Lemma (5.19) that the homomorphism (5.20.1) is an isomorphism.

(5.21) Proposition. *Let B be an A-algebra via the ring homomorphism $\varphi : A \to B$. Moreover let S and T be multiplicatively closed subsets of A, respectively B, such that $\varphi(S) \subseteq T$. Then there is an isomorphism of $T^{-1}B$-modules*

$$\Omega^1_{B/A} \otimes_B T^{-1}B \xrightarrow{\sim} \Omega^1_{T^{-1}B/S^{-1}A} \tag{5.21.1}$$

that is uniquely determined by mapping $d_{B/A}(g) \otimes_A (h/t)$ to $(h/t)d_{T^{-1}B/S^{-1}A}(g/1)$ for all g, h in B and $t \in T$.

Proof. Functoriality of the differentials gives homomorphisms of $(T^{-1}B)$-modules $\Omega^1_{B/A} \otimes_B T^{-1}B \to \Omega^1_{T^{-1}B/A}$ and $\Omega^1_{T^{-1}B/A} \to \Omega^1_{T^{-1}B/S^{-1}A}$. We consequently have constructed the $T^{-1}B$-module homomorphism $v : \Omega^1_{B/A} \otimes_B T^{-1}B \to \Omega^1_{T^{-1}B/S^{-1}A}$ of the Proposition. It follows from Example (5.7) that the A-derivation $d_{B/A} : B \to \Omega^1_{B/A}$ gives an $S^{-1}A$-derivation $T^{-1}d_{B/A} : T^{-1}B \to T^{-1}\Omega_{B/A} = \Omega_{B/A} \otimes_B T^{-1}B$ defined by $T^{-1}d_{B/A}(g/t) = -(g/t^2)(d_{B/A}(t)/1) + (d_{B/A}(g)/t)$ for all $g \in B$ and $t \in T$. The elements $\{d_{B/A}(g)\}_{g\in B}$ clearly generate the $T^{-1}B$-module $\Omega_{B/A} \otimes_B T^{-1}B$, and $d_{T^{-1}B/S^{-1}A} = vT^{-1}d_{B/A}$. It follows from Lemma (5.19) that the homomorphism (5.21.1) is an isomorphism.

(5.22) Theorem. *Let $\varphi : A \to B$ be an A-algebra and let $\chi : B \to C$ be a B-algebra. We consider C as an A-algebra by restriction of scalars. There is an exact sequence of C-modules*

$$\Omega^1_{B/A} \otimes_B C \xrightarrow{\chi_{C/B/A}} \Omega^1_{C/A} \xrightarrow{\varphi_{C/B/A}} \Omega^1_{C/B} \to 0. \tag{5.22.1}$$

Proof. It is clear that that (5.22.1) is a complex. Let P be a C-module that we consider as a B-module by restriction of scalars. We obtain a complex of C-modules

$$0 \to \mathrm{Hom}_C(\Omega^1_{C/B}, P) \xrightarrow{\mathrm{Hom}_C(\varphi_{C/B/A}, \mathrm{id}_P)} \mathrm{Hom}_C(\Omega_{C/A}, P) \xrightarrow{\mathrm{Hom}_C(\chi_{C/B/A}, \mathrm{id}_P)} \mathrm{Hom}_C(\Omega^1_{B/A} \otimes_B C, P). \tag{5.22.2}$$

If follows from Remark (2.19) that we have a canonical χ-module homomorphism $\operatorname{Hom}_C(\Omega^1_{B/A} \otimes_B C, P) \xrightarrow{\sim} \operatorname{Hom}_B(\Omega^1_{B/A}, P)$. Hence it follows from Proposition (5.15) that the complex is the same as the complex

$$0 \to \operatorname{Der}_B(C, P) \xrightarrow{\varphi^0} \operatorname{Der}_A(C, P) \xrightarrow{\chi^0} \operatorname{Der}_A(B, P) \tag{5.22.3}$$

which is eact by Theorem (5.9). Since the sequence (5.22.3) is exact for all C-modules P it follows from Lemma (1.33) that the sequence (5.22.1) is exact.

(5.23) Remark. Let $\chi : B \to C$ be a surjective map of A-algebras with kernel $\mathfrak{b}$. In Remark (5.10) we saw that to every A-derivation $D : B \to P$ we can associate a canonical C-module homomorphism $w : \mathfrak{b}/\mathfrak{b}^2 \to P$. In particular we obtain from the A-derivation $D : B \to \Omega^1_{B/A} \otimes_B C$ which maps $g \in B$ to $d_{B/A}(g) \otimes_B 1$ a C-module homomorphism

$$\delta_{C/B/A} : \mathfrak{b}/\mathfrak{b}^2 \to \Omega^1_{B/A} \otimes_B C$$

that is uniquely determined by mapping the class in $\mathfrak{b}/\mathfrak{b}^2$ of $g \in B$ to $d_{B/A}(g) \otimes_B 1$ for all $g \in B$.

(5.24) Theorem. *Let $\chi : B \to C$ be a surjection of A-algebras and let $\mathfrak{b} = \operatorname{Ker}(\chi)$. Then there is an exact sequence of C-modules*

$$\mathfrak{b}/\mathfrak{b}^2 \xrightarrow{\delta_{C/B/A}} \Omega^1_{B/A} \otimes_B C \xrightarrow{\chi_{C/B/A}} \Omega_{C/A} \to 0. \tag{5.24.1}$$

Proof. It is clear that (5.24.1) is a complex of C-modules. Let P be a C-module. The complex (5.24.1) gives a complex of C-modules

$$\begin{aligned} 0 \to \operatorname{Hom}_C(\Omega^1_{C/A}, P) &\xrightarrow{\operatorname{Hom}_C(v_\chi, \mathrm{id}_P)} \operatorname{Hom}_C(\Omega^1_{B/A} \otimes_B C, P) \\ &\xrightarrow{\operatorname{Hom}_C(u, \mathrm{id}_P)} \operatorname{Hom}_C(\mathfrak{b}/\mathfrak{b}^2, P). \end{aligned} \tag{5.24.2}$$

It follows from Remark (2.19) that we have a canonical isomorphism of χ-modules $\operatorname{Hom}_C(\Omega^1_{B/A} \otimes_B C, P) \xrightarrow{\sim} \operatorname{Hom}_B(\Omega^1_{B/A}, P)$. Hence it follows from Proposition (5.15) that the complex (5.24.2) is the same as the complex

$$0 \to \operatorname{Der}_A(C, P) \xrightarrow{\chi^0} \operatorname{Der}_A(B, P) \xrightarrow{u} \operatorname{Hom}_C(\mathfrak{b}/\mathfrak{b}^2, P) \tag{5.24.3}$$

which is exact by Theorem (5.11). Since the complex (5.24.3) is exact for all C-modules P is follows from Lemma (1.33) that the complex (5.24.3) is exact.

(5.25) Example. Let $\varphi : A \to B$ be an A-algebra and $\chi : B \to C$ be an A-algebra homomorphism. We let $\psi : B \otimes_A C \to C$ be the C-algebra homomorphism determined by $\psi(g \otimes_A h) = \chi(g)h$ for all $g \in B$ and $h \in C$. Denote the kernel of ψ by $\mathfrak{b}$. We have that the homomorphism of C-modules

$$\delta_{C/B\otimes_A C/C} : \mathfrak{b}/\mathfrak{b}^2 \to \Omega^1_{B\otimes_A C/C} \otimes_{B\otimes_A C} C.$$

We shall prove that $\delta_{C/B\otimes_A C/C}$ is an isomorphism. It follows from Proposition (5.20) that we have a canonical isomorphism $\Omega^1_{B/A} \otimes_A C \xrightarrow{\sim} \Omega^1_{B\otimes_A C/C}$. The homomorphism of C-modules $\Omega_{B\otimes_A C/C} \to \Omega_{B\otimes_A C/C} \otimes_{B\otimes_A C} C$ that sends $d_{B\otimes_A C/C}(g \otimes_A h)$ to $d_{B\otimes_A C/C}(g \otimes_A h) \otimes_{B\otimes_A C/C} 1$ for all $g \in B$ and $h \in C$ is clearly an isomorphism. Hence the homomorphism $\delta_{C/B/A}$ is the same as a homomorphism of C-modules

$$\mathfrak{b}/\mathfrak{b}^2 \to \Omega^1_{B/A} \otimes_B C \tag{5.25.1}$$

that sends the class in $\mathfrak{b}/\mathfrak{b}^2$ of $g \otimes_A h \in B \otimes_A C$ to $d_{B/A}(g) \otimes_B h$ for all $g \in B$ and $h \in C$. In order to show that the homomorphism (5.25.1) is an isomorphism we construct an inverse. Let $D : B \to \mathfrak{b}/\mathfrak{b}^2$ be the map that sends $g \in B$ to the class in $\mathfrak{b}/\mathfrak{b}^2$ of the elements $g \otimes_A 1 - 1 \otimes_A \chi(g)$ in $B \otimes_A C$. It is clear that the map D is A-linear. It is a derivation because

$$\begin{aligned} gh \otimes_A 1 - 1 \otimes_A \chi(gh) &= (g \otimes_A 1)(h \otimes_A 1 - 1 \otimes_A \chi(h)) \\ &+ (h \otimes_A 1)(g \otimes_A 1 - 1 \otimes_A \chi(g)) + (g \otimes_A 1 - 1 \otimes_A \chi(g))(h \otimes_A 1 - g \otimes_A \chi(g)). \end{aligned}$$

From the A-derivation D we obtain a B-linear homomorphism $\Omega^1_{B/A} \to \mathfrak{b}/\mathfrak{b}^2$, and by extension of scalars we obtain a C-module homomorphism $\Omega_{B/A} \otimes_B C \to \mathfrak{b}/\mathfrak{b}^2$ that is clearly the inverse of the homomorphism (5.25.1).

The most important application of the Example is when B is a local ring that is an algebra over a field K and when $C = B/\mathfrak{m}_B = \boldsymbol{\kappa}$. We obtain an isomorphism of $\boldsymbol{\kappa}$-vector spaces

$$\mathfrak{m}/\mathfrak{m}^2 \to \Omega_{B\otimes_K \boldsymbol{\kappa}/\boldsymbol{\kappa}} \otimes_{B\otimes_K \boldsymbol{\kappa}} \boldsymbol{\kappa}$$

where $\mathfrak{m}$ is the kernel of the multiplication map $B \otimes_K \boldsymbol{\kappa} \to \boldsymbol{\kappa}$.

(5.26) Exercises.

1. Let $K[u,v]$ be a polynomial ring in the two variables u and v over the field K, and let $B = K[u,v]_{(u,v)}/(v - u^2)$ and $C = K[u,v]_{(u,v)}/(v^2 - u^3)$, where $K[u,v]_{(u,v)}$ is the polynomial ring $K[u,v]$ localized in the prime ideal (u,v).

(1) Give generators and relations that determine the B-module $\Omega^1_{B/K}$.
(2) Give generators and relations that determine the C-module $\Omega^1_{C/K}$.
(3) Is $\Omega^1_{B/A}$ a free B-module?
(4) Is $\Omega^1_{C/A}$ a free C-module?
(5) What is the dimension of the $(B/\mathfrak{m}_B)$-vector space $\Omega^1_{B/K}/\mathfrak{m}_B\Omega^1_{B/K}$?
(6) What is the dimension of the $(C/\mathfrak{m}_C)$-vector space $\Omega^1_{C/K}/\mathfrak{m}_C\Omega^1_{C/K}$?

2. Let A be a ring and let $B = A[t_1, t_2, \dots, t_n]$ be the polynomial ring in the n variables $t_1, t_2, \dots, t_n$ over A.

(1) Show that $\Omega^1_{B/A}$ is a free A-module of rank 1 generated by the elements $dt_1, dt_2, \dots, dt_n$.

(2) Show that the exterior derivative $d_{B/A}(f)$ of an element $f \in B$ can be written uniquely in the form $d_{B/A}(f) = (\partial f/\partial t_1)dt_1 + (\partial f/\partial t_2)dt_2 + \cdots + (\partial f/\partial t_n)dt_n$ for uniquely determined elements $\partial f/\partial t_1, \partial f/\partial t_2, \dots, \partial f/\partial t_n$ in B.

(3) Show that the map $\partial/\partial t_i : B \to B$ that sends an element $f \in B$ to $\partial f/\partial t_i$ is an A-derivation.

3. Let $K[t_1, t_2, \dots, t_n]$ be the polynomial ring in the variables $t_1, t_2, \dots, t_n$ over the field K. Moreover let $f \in K[t_1, t_2, \dots, t_n]$ be a polynomial without constant term and with linear term $a_1t_1 + a_2t_2 + \cdots a_nt_n$ with $a_i \in K$. Let $B = K[t_1, t_2, \dots, t_n]/(f)$ be the residue ring of $K[t_1, t_2, \dots, t_n]$ with respect to the ideal generated by f.

Determine the dimension of the K-vector space $\Omega^1_{B/K} \otimes_B B/\mathfrak{m}$ where $\mathfrak{m}$ is the maximal ideal of B generated by the residue classes in B of the variables $t_1, t_2, \dots, t_n$.

4. Let A be a ring and let B and C be two A-algebras.

(1) Show that the map $\varphi : B \to B \otimes_A C$ induces a map $u : \Omega^1_{B/A} \otimes_A B \to \Omega^1_{B \otimes_A C/A}$.

(2) Show that the map u together with the corresponding map for the algebra C defines an isomorphism of $B \otimes_A C$-modules

$$(\Omega^1_{B/A} \otimes_A B) \oplus (\Omega^1_{C/A} \otimes_A C) \to \Omega^1_{B \otimes_A C/A}.$$

5. Let B be an A-algebra, and let $\mathfrak{I}_{B/A}$ be the kernel of the multiplication map $B \otimes_A B \to B$ that maps $f \otimes_A g$ to fg for all f, g in B. Moreover let $d : B \to \mathfrak{I}_{B/A}/\mathfrak{I}^2_{B/A}$ be the A-module homomorphism defined by $d(f) = 1 \otimes_A f - f \otimes_A 1$.

(1) Show that the homomorphism d is an A-derivation.

(2) Show that there is an exact sequence of B-modules

$$0 \to \mathfrak{I}_{B/A}/\mathfrak{I}^2_{B/A} \xrightarrow{u} \Omega^1_{B \otimes_A B/A} \otimes_{B \otimes_A B} B \xrightarrow{v} \Omega^1_{B/A} \to 0.$$

(3) Show that there are maps $s : \Omega^1_{B \otimes_A B/A} \otimes_{B \otimes_A B} B \to \mathfrak{I}_{B/A}/\mathfrak{b}^2$ and $t : \Omega^1_{B/A} \to \Omega^1_{B \otimes_A B/A} \otimes_{B \otimes_A B} B$ such that su and vt are the identiy maps.

6. Let A be a ring and B an A-algebra. For all elements D and E in $\mathrm{Der}_A(B, B)$ we let $[D, E] = DE - ED$. Show that $\mathrm{Der}_A(B, B)$ with the bracket $[,]$ is a *Lie algebra*. That is, show that

(1) The $[,] : \mathrm{Der}_A(B, B) \times \mathrm{Der}_A(B, B) \to \mathrm{Der}_A(B, B)$ defines a product on $\mathrm{Der}_A(B, B)$ which together with the A-module structure on $\mathrm{Der}_A(B, B)$ satisfies all the properties of an A-algebra except commutativity and associativity.

(2) For all $D \in \mathrm{Der}_A(B, B)$ we have that $[D, D] = 0$.
(3) For all D, E, F in $\mathrm{Der}_A(B, B)$ the *Jacobi identity* holds, that is $[[D, E], F] + [[E, F], D] + [[F, D], E] = 0$.

7. Let A be a ring and B an A-algebra. For all elements D and E in $\mathrm{Der}_A(B, B)$:

(1) Show that $[D, E] = DE - ED$ is an A-derivation.
(2) Show that for all g, h in B we have

$$[gD, hE] = gh[D, E] + gD(h)E - hD(g)D.$$

8. Let B be an A-algebra and let $D \in \mathrm{Der}_A(B, B)$.

(1) Show that $C = \{g \in B : D(g) = 0\}$ is an A algebra in such a way that the map $C \to B$ defining C as a subset of B is an A-algebra homomorphism.
(2) Let p be a prime number such that $D^p = 0$, and such that $pg = 0$ for all elements $g \in B$. Moreover let $f \in B$ be such that $D(f) = 1$. Show that B is a C-algebra generated by the elements $1, f, \ldots, f^{p-1}$.
(3) Show that B is free module over C with basis $1, f, \ldots, f^{p-1}$.

Affine schemes

1. Sheaves of modules.

n **(1.1) Notation.** We shall follow usual notation and write!! $\Gamma(U, \mathcal{F})$ instead of $\mathcal{F}(U)$ for the sections of a presheaf $\mathcal{F}$ over an open subset U.

(1.2) Definition. Let X be a topological space with basis $\mathfrak{B}$, and let $\mathcal{F}$ and $\mathcal{A}$ be presheaves of groups, respectively rings, on $\mathfrak{B}$. We say that the presheaf $\mathcal{F}$ is an $\mathcal{A}$-*module* on $\mathfrak{B}$ if for every U in $\mathfrak{B}$ we have that $\Gamma(U, \mathcal{F})$ is a $\Gamma(U, \mathcal{A})$-module, and for every inclusion $U \subseteq V$ of open sets belonging to $\mathfrak{B}$ the map $(\rho_{\mathcal{F}})^V_U : \Gamma(V, \mathcal{F}) \to \Gamma(U, \mathcal{F})$ is a map of $(\rho_{\mathcal{A}})^V_U : \Gamma(V, \mathcal{A}) \to \Gamma(U, \mathcal{A})$-modules. That is, for every pair of sections s in $\Gamma(V, \mathcal{F})$ and t in $\Gamma(V, \mathcal{A})$ we have that

$$(\rho_{\mathcal{F}})^V_U(ts) = (\rho_{\mathcal{A}})^V_U(t)(\rho_{\mathcal{F}})^V_U(s).$$

When $\mathcal{A}$ and $\mathcal{F}$ are both sheaves we simply say that $\mathcal{F}$ is an $\mathcal{A}$-module.

(1.3) Remark. Let X be a topological space and $\mathfrak{B}$ a basis for the topology. Moreover let $\mathcal{A}$ be a presheaf of rings on $\mathfrak{B}$ and let $\mathcal{F}$ be a presheaf of $\mathcal{A}$-modules on $\mathfrak{B}$. For every point $x \in X$ the group $\mathcal{F}_x$ becomes an $\mathcal{A}_x$-module. The product of the class s_x of a pair (U, s) with $s \in \Gamma(U, \mathcal{F})$, and the class t_x of a pair (V, t) with $t \in \Gamma(V, \mathcal{A})$ is given by the class $t_x s_x$ of $(U \cap V, \rho^U_{U \cap V}(t)\rho^V_{U \cap V}(s))$. It is clear that the definition of the product is independent of the choices (U, s) and (V, t) of representatives of the classes t_x and s_x.

For every open set U belonging to $\mathfrak{B}$ and every point $x \in U$ we have that the map $(\rho_{\mathcal{F}})^U_x : \Gamma(U, \mathcal{F}) \to \mathcal{F}_x$ is a homomorphism of $(\rho_{\mathcal{A}})^U_x : \Gamma(U, \mathcal{A}) \to \mathcal{A}_x$-modules.

(1.4) Definition. Let X be a topological space with basis $\mathfrak{B}$ for the topology and let $\mathcal{A}$ be a presheaf of rings defined on $\mathfrak{B}$. Moreover let $\mathcal{F}$ and $\mathcal{G}$ be presheaves defined on $\mathfrak{B}$ that are $\mathcal{A}$-modules. A *homomorphism* $u : \mathcal{F} \to \mathcal{G}$ of presheaves of $\mathcal{A}$-modules is a homomorphism of presheaves of groups such that for all U belonging to $\mathfrak{B}$ the map $u_U : \Gamma(U, \mathcal{F}) \to \Gamma(U, \mathcal{G})$ is a homomorphism of $\Gamma(U, \mathcal{A})$-modules. When $\mathcal{A}$ is a sheaf and $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{A}$-modules we say that u is a *homomorphism of* $\mathcal{A}$-modules.
n The set of $\mathcal{A}$-module homomorphisms we denote by $\mathrm{Hom}_{\mathcal{A}}(\mathcal{F}, \mathcal{G})$.!!

(1.5) Remark. Let $\mathcal{A}$ be a presheaf of rings on a basis $\mathfrak{B}$ of a topological space X and let $u : \mathcal{F} \to \mathcal{G}$ be a homomorphism of presheaves of $\mathcal{A}$-modules on $\mathfrak{B}$. Then the map $u_x : \mathcal{F}_x \to \mathcal{G}_x$ is an $\mathcal{A}_x$-module homomorphism.

(1.6) The direct image. Let $\psi : X \to Y$ be a continous map of topological spaces. Moreover let $\mathcal{A}$ be a presheaf of rings on X and let $\mathcal{F}$ be a presheaf of $\mathcal{A}$-modules. By the definiton of the direct images $\psi_*(\mathcal{A})$ and $\psi_*(\mathcal{F})$ we have for every open subset V of Y that $\Gamma(V, \psi_*(\mathcal{A})) = \Gamma(\psi^{-1}(V), \mathcal{A})$ and $\Gamma(V, \psi_*(\mathcal{F})) = \Gamma(\psi^{-1}(V), \mathcal{F})$. Hence $\Gamma(V, \psi_*(\mathcal{F}))$ is a $\Gamma(V, \psi_*(\mathcal{A}))$-module. It is clear that these modules, for all open subsets V of Y, make $\psi_*(\mathcal{F})$ into a presheaf of $\psi_*(\mathcal{A})$-modules. When $\mathcal{A}$ is a sheaf and $\mathcal{F}$ is an $\mathcal{A}$-module we have that $\psi_*(\mathcal{A})$ is a sheaf and that $\psi_*(\mathcal{F})$ is a $\psi_*(\mathcal{A})$-module.

Let $u : \mathcal{F} \to \mathcal{G}$ be an $\mathcal{A}$-homomorphism of presheaves on X. We have that $\psi_*(u) : \psi_*(\mathcal{F}) \to \psi_*(\mathcal{G})$ is a homomorphism of presheaves of $\mathcal{A}$-modules.

(1.7) The inverse image. Let $\psi : X \to Y$ be a continous map of topological spaces and let $\mathfrak{B}$ be a basis for the topology of Y. Moreover let $\mathcal{B}$ be a presheaf of rings on $\mathfrak{B}$ and let $\mathcal{G}$ be a presheaf of $\mathcal{B}$-modules on $\mathfrak{B}$. For all $y \in Y$ the stalk $\mathcal{G}_y$ of $\mathcal{G}$ at y is a $\mathcal{B}_y$-module. Consequently the product $\prod_{x\in U} \mathcal{G}_{\psi(x)}$ is a $\prod_{x\in U} \mathcal{B}_{\psi(x)}$-module for all open subsets U of X. By the definition of the inverse image in Section (?) we have that $\Gamma(U, \psi^*(\mathcal{B}))$ is a subring of $\prod_{x\in U} \mathcal{B}_{\psi(x)}$ and that $\Gamma(U, \psi^*(\mathcal{G}))$ is a subgroup of $\prod_{x\in U} \mathcal{G}_{\psi(x)}$. We shall show that the $\prod_{x\in U} \mathcal{B}_{\psi(x)}$-module structure on $\prod_{x\in U} \mathcal{G}_{\psi(x)}$ induces a $\Gamma(U, \psi^*(\mathcal{B}))$-module structure on $\Gamma(U, \psi^*(\mathcal{G}))$. Let $(s_{\psi(x)})_{x\in U} \in \Gamma(U, \psi^*(\mathcal{B}))$ and $(t_{\psi(x)})_{x\in U} \in \Gamma(U, \psi^*(\mathcal{G}))$. For all $x \in U$ there is a neighbourhood $V_{\psi(x)}$ of $\psi(x)$ belonging to $\mathfrak{B}$ and sections $s(x) \in \Gamma(V_{\psi(x)}, \mathcal{B})$ and $t(x) \in \Gamma(V_{\psi(x)}, \mathcal{G})$ such that for all y in a neighbourhood U_x of x contained in $U \cap \psi^{-1}(V_{\psi(x)})$ we have that $s_{\psi(y)} = s(x)_y$ and $t_{\psi(y)} = t(x)_y$. We have that $s(x)t(x) \in \Gamma(V_{\psi(x)}, \mathcal{G})$ and $(s(x)t(x))_y = s(x)_y t(x)_y = s_{\psi(y)} t_{\psi(y)}$. Consequently $(s_{\psi(x)})_{x\in U}(t_{\psi(x)})_{x\in U} = (s_{\psi(x)}t_{\psi(x)})_{x\in U}$ is in $\Gamma(U, \psi^*(\mathcal{G})$. and the $\prod_{x\in U} \mathcal{B}_{\psi(x)}$-module structure on $\prod_{x\in U} \mathcal{G}_{\psi(x)}$ induces a $\Gamma(U, \psi^*(\mathcal{B}))$-modules structure on $\Gamma(U, \psi^*(\mathcal{G}))$. We easily see that $\psi^*(\mathcal{G})$ becomes a $\psi^*(\mathcal{B})$-module. In particular the associated sheaf $\mathrm{id}_Y^*(\mathcal{G})$ of $\mathcal{G}$ becomes a module over the associated sheaf $\mathrm{id}_Y^*(\mathcal{B})$ of $\mathcal{B}$.

When $u : \mathcal{G} \to \mathcal{H}$ is a homomorphism of presheaves of $\mathcal{B}$-modules on the basis $\mathfrak{B}$ we have that $\psi^*(u) : \psi^*(\mathcal{G}) \to \psi^*(\mathcal{H})$ is a homomorphism of $\psi^*(\mathcal{B})$-modules.

(1.8) The tensor product. Let X be a topological space with basis $\mathfrak{B}$ for the topology. Moreover let $\mathcal{A}$ be a presheaf of rings on $\mathfrak{B}$ and let $\mathcal{F}$ and $\mathcal{G}$ be presheaves of $\mathcal{A}$-modules. For every open subset U of X belonging to $\mathfrak{B}$ we have that $\Gamma(U, \mathcal{F})$ and $\Gamma(U, \mathcal{G})$ are $\Gamma(U, \mathcal{A})$-modules. Let $\Gamma(U, \mathcal{H}) = \Gamma(U, \mathcal{F}) \otimes_{\Gamma(U,\mathcal{A})} \Gamma(U, \mathcal{G})$. When $U \subseteq V$ is an inclusion of open sets belonging to $\mathfrak{B}$ it follows from Lemma (?) that we have a map

$$\Gamma(V, \mathcal{F}) \otimes_{\Gamma(V,\mathcal{A})} \Gamma(V, \mathcal{G}) \xrightarrow{(\rho_{\mathcal{F}})_U^V \otimes_{(\rho_{\mathcal{A}})_U^V} (\rho_{\mathcal{G}})_U^V} \Gamma(U, \mathcal{F}) \otimes_{\Gamma(U,\mathcal{A})} \Gamma(U, \mathcal{G}).$$

We thus obtain a presheaf $\mathcal{H}$ of $\mathcal{A}$-modules on $\mathfrak{B}$. The associated sheaf of $\mathcal{H}$ we denote by !!$\mathcal{F} \otimes_{\mathcal{A}} \mathcal{G}$ and call the *tensor product* of $\mathcal{F}$ and $\mathcal{G}$ over $\mathcal{A}$. It follows from Section (1.?) that $\mathcal{F} \otimes_{\mathcal{A}} \mathcal{G}$ is a module over the associated sheaf $\mathrm{id}_X^*(\mathcal{A})$ of $\mathcal{A}$.

(1.9) Lemma. *For all points $x \in X$ there is a canonical isomorphism of A_x-modules*

$$(\mathcal{F} \otimes_{\mathcal{A}} \mathcal{G})_x \to \mathcal{F}_x \otimes_{\mathcal{A}_x} \mathcal{G}_x.$$

Proof. For each open set U belonging to $\mathfrak{B}$ we have maps $(\rho_{\mathcal{F}})^U_x : \Gamma(U, \mathcal{F}) \to \mathcal{F}_x$, $(\rho_{\mathcal{G}})^U_x : \Gamma(U, \mathcal{G}) \to \mathcal{G}_x$ and $(\rho_{\mathcal{A}})^U_x : \Gamma(U, \mathcal{A}) \to \mathcal{A}_x$. Moreover we have a map $\mathcal{H}(U) \to \mathcal{F}_x \otimes_{\mathcal{A}_x} \mathcal{G}_x$ that takes the class of $s \otimes_{\Gamma(U,\mathcal{A})} t$ with $s \in \Gamma(U, \mathcal{F})$ and $t \in \Gamma(U, \mathcal{G})$ to $s_x \otimes_{\mathcal{A}_x} t_x$. Hence we obtain a map $\mathcal{H}_x \to \mathcal{F}_x \otimes_{\mathcal{A}_x} \mathcal{G}_x$ that sends the class of $(U, s \otimes_{\Gamma(U,\mathcal{A})} t)$ with $s \in \Gamma(U, \mathcal{F})$ and $t \in \Gamma(U, \mathcal{G})$ to $s_x \otimes_{\mathcal{A}_x} t_x$. It is clear that the map is independent of the choice of representative $(U, s \otimes_{\Gamma(U,\mathcal{A})} t)$ of the class in $\mathcal{H}_x$. The inverse map sends $s_x \otimes_{\mathcal{A}_x} t_x$ to the class of $(U \cap V, (\rho_{\mathcal{F}})^U_{U\cap V}(s) \otimes_{\Gamma(U\cap V,\mathcal{A})} (\rho_{\mathcal{G}})^V_{U\cap V}(t))$ when (U, s) represents s_x and (V, t) represents t_x. The inverse is clearly independent of the choice of representatives (U, s) and (V, t) of the classes s_x, respectively t_x.

(1.10) Example. We shall show that the presheaf $\mathcal{H}$ used in the definition of the
→ tensor product in Section (1.?) is not necessarily a sheaf. Let $X = \{x_0, x_1, x_2\}$ be the topological spaces with open sets $\emptyset$, X, $U_0 = \{x_0\}$, $U_1 = \{x_0, x_1\}$, and $U_2 = \{x_0, x_2\}$. We define a presheaf of rings by $\Gamma(\emptyset, \mathcal{A}) = \{0\}$, $\Gamma(X, \mathcal{A}) = \mathbf{Z}$, $\Gamma(U_1, \mathcal{A}) = \mathbf{Z}$, $\Gamma(U_2, \mathcal{A}) = \mathbf{Z}$, $\Gamma(U_0, \mathcal{A}) = \mathbf{Z}$ with $(\rho_{\mathcal{A}})^V_U = \mathrm{id}_{\mathbf{Z}}$ when $U \neq \emptyset$. Moreover we define a presheaf of groups by $\Gamma(\emptyset, \mathcal{F}) = (0)$, $\Gamma(X, \mathcal{F}) = \mathbf{Z}$, $\Gamma(U_1, \mathcal{F}) = \mathbf{Z}/2\mathbf{Z}$, $\Gamma(U_2, \mathcal{F}) = \mathbf{Z}$, $\Gamma(x_0, \mathcal{F}) = \mathbf{Z}/2\mathbf{Z}$ where $(\rho_{\mathcal{F}})^X_{U_1}$ is the residue map and the remaining restriction maps are the identity. Finally let $\Gamma(\emptyset, \mathcal{G}) = \{0\}$, $\Gamma(X, \mathcal{G}) = \mathbf{Z}$, $\Gamma(U_1, \mathcal{G}) = \mathbf{Z}$, $\Gamma(U_2, \mathcal{G}) = \mathbf{Z}$, $\Gamma(U_0, \mathcal{G}) = \mathbf{Q}$ with restriction maps being the natural inclusions. Then $\mathcal{A}$ with the given restriction maps is a sheaf of rings, and $\mathcal{F}$ and $\mathcal{G}$ with the given restriction maps are $\mathcal{A}$-modules.

We let $\Gamma(U, \mathcal{H}) = \Gamma(U, \mathcal{F}) \otimes_{\Gamma(U,\mathcal{A})} \Gamma(U, \mathcal{G})$ for all open subsets U of X. Then $\Gamma(X, \mathcal{H}) = \mathbf{Z}$, $\Gamma(U_1, \mathcal{H}) = \mathbf{Z}/2\mathbf{Z}$, $\Gamma(U_2, \mathcal{H}) = \mathbf{Z}$ and $\Gamma(U_0, \mathcal{H}) = \{0\}$ and $(\rho_{\mathcal{H}})^X_{U_1}$ is the residue map and the remaining projection maps are the identity except those of the form $(\rho_{\mathcal{H}})^U_{U_0}$ that are zero. Hence $\mathcal{H}$ is not a presheaf. The associated sheaf $\mathcal{F} \otimes_{\mathcal{A}} \mathcal{G}$ has the same sections as $\mathcal{H}$ over all open subsets of X except over X where $\Gamma(X, \mathcal{F} \otimes_{\mathcal{A}} \mathcal{G}) = \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}$.

(1.11) Restriction and extension of scalars. Let X be a topological space with a basis $\mathfrak{B}$ for the topology and let $\varphi : \mathcal{A} \to \mathcal{B}$ be a homomorphism of presheaves of rings on $\mathfrak{B}$. Moreover let $\mathcal{F}$ be a presheaf of $\mathcal{A}$-modules and $\mathcal{G}$ a presheaf of $\mathcal{B}$-modules on $\mathfrak{B}$.

For every open subset U of $\mathfrak{B}$ we obtain a homomorphism of rings $\varphi_U : \Gamma(U, \mathcal{A}) \to \Gamma(U, \mathcal{B})$, and $\Gamma(U, \mathcal{G})$ is a $\Gamma(U, \mathcal{B})$-module. By restriction of scalars we have that $\Gamma(U, \mathcal{G})$ becomes a $\Gamma(U, \mathcal{A})$-module and the map $(\rho_{\mathcal{F}})^V_U : \Gamma(V, \mathcal{G}) \to \Gamma(U, \mathcal{G})$ is a $(\rho_{\mathcal{A}})^V_U$-module homomorphism for each inclusion $U \subseteq V$ of open subsets of $\mathfrak{B}$. Hence $\mathcal{G}$ becomes a presheaf of $\mathcal{A}$-modules on $\mathfrak{B}$. We say that $\mathcal{G}$ becomes an $\mathcal{A}$-module by
n *restriction of scalars*, and we denote $\mathcal{G}$ considered as an $\mathcal{A}$-module by $!!\mathcal{G}_{[\varphi]}$.

For every open subset U of $\mathfrak{B}$ we obtain, by extension of scalars, a $\Gamma(U,\mathcal{B})$-module $\Gamma(U,\mathcal{G})\otimes_{\Gamma(U,\mathcal{A})}\Gamma(U,\mathcal{B})$. The homomorphism $(\rho_{\mathcal{F}})^V_U\otimes_{(\rho_{\mathcal{A}})^V_U}(\rho_{\mathfrak{B}})^V_U$ is a $(\rho_{\mathcal{B}})^V_U$-homomorphism. It follows that the presheaf $\mathcal{H}$ defined in Section (1.?) becomes a presheaf of $\mathcal{B}$-modules. We say that $\mathcal{H}$ becomes a $\mathcal{B}$-module by *extension of scalars.* When $\mathcal{B}$ is a sheaf it follows from Section (1.?) that $\mathcal{F}\otimes_{\mathcal{A}}\mathcal{B}$ is a $\mathcal{B}$-module. We say that $\mathcal{F}\otimes_{\mathcal{A}}\mathcal{B}$ becomes a $\mathcal{B}$-module by *extension of scalars.*

(1.12) Direct images on ringed spaces. Let $(X,\mathcal{A})$ and $(Y,\mathcal{B})$ be two ringed spaces, and let $\Psi=(\psi,\theta):(X,\mathcal{A})\to(Y,\mathcal{B})$ be a morphism of ringed spaces, that is, the map $\psi:X\to Y$ is a continuous map of topological spaces and $\theta:\mathcal{B}\to\psi_*\mathcal{A}$ is a homomorphism of sheaves of rings on Y. Moreover let $\mathcal{F}$ and $\mathcal{G}$ be an $\mathcal{A}$-module respectively a $\mathcal{B}$-module.

We have that the direct image $\psi_*(\mathcal{F})$ is a $\psi_*(\mathcal{A})$-module. From the homomorphism of sheaves of rings $\theta:\mathcal{B}\to\psi_*(\mathcal{A})$ on Y we obtain, by restriction of scalars, that $\psi_*(\mathcal{F})$ becomes a $\mathcal{B}$-module. We denote this $\mathcal{B}$-module by !!$\Psi_*(\mathcal{F})$ and call it the *direct image* of $\mathcal{F}$ by Ψ.

When $u:\mathcal{F}\to\mathcal{G}$ is a homomorphism of $\mathcal{A}$-modules on X we have that the homomorphism $\psi_*(u):\psi_*(\mathcal{F})\to\psi_*(\mathcal{G})$ of sheaves of groups is a homomorphism of sheaves of $\psi_*(\mathcal{A})$-modules. Hence, by restriction of scalars $\psi_*(u)$ becomes a homomorphism of sheaves of $\mathcal{B}$-modules that we denote by !!$\Psi_*(u):\Psi_*(\mathcal{F})\to\Psi_*(\mathcal{G})$.

We have that $\Psi_*(\mathrm{id}_{\mathcal{F}})=\mathrm{id}_{\Psi_*(\mathcal{F})}$, and when $v:\mathcal{G}\to\mathcal{H}$ is a homomorphism of $\mathcal{B}$-modules we have that $\Psi_*(vu)=\Psi_*(v)\Psi_*(u)$. In other words Ψ_* is a functor from $\mathcal{A}$-modules on X to $\mathcal{B}$-modules on Y.

(1.13) Inverse images on ringed spaces. Let $(X,\mathcal{A})$ and $(Y,\mathcal{B})$ be two ringed spaces, and let $\Psi=(\psi,\theta):(X,\mathcal{A})\to(Y,\mathcal{B})$ be a morphism of ringed spaces, that is, the map $\psi:X\to Y$ is a continuous map of topological spaces and $\theta:\mathcal{B}\to\psi_*\mathcal{A}$ is a homomorphism of sheaves of rings on Y. Moreover let $\mathcal{F}$ and $\mathcal{G}$ be an $\mathcal{A}$-module respectively a $\mathcal{B}$-module.

We have that $\psi^*(\mathcal{G})$ is a $\psi^*(\mathcal{B})$-module. From the adjoint homomorphism $\theta^\sharp:\psi^*(\mathcal{B})\to\mathcal{A}$ of the homomorphism $\theta:\mathcal{B}\to\psi_*\mathcal{A}$ we obtain, by extension of scalars, that $\psi^*(\mathcal{G})\otimes_{\psi^*(\mathcal{B})}\mathcal{A}$ is an $\mathcal{A}$-module. We denote this $\mathcal{A}$-module by !!$\Psi^*(\mathcal{G})$ and call it the *inverse image* of $\mathcal{G}$ by the map Ψ.

When $v:\mathcal{G}\to\mathcal{H}$ is a homomorphism of $\mathcal{B}$-modules on the basis $\mathfrak{B}$ of the topology it is clear that the map $\psi^*(u):\psi^*(\mathcal{G})\to\psi^*(\mathcal{H})$ induces a homomorphism

$$\psi^*(\mathcal{G})\otimes_{\psi^*(\mathcal{B})}\mathcal{A}\xrightarrow{\psi^*(u)\otimes_{\psi^*(\mathcal{B})}\mathrm{id}_{\mathcal{A}}}\psi^*(\mathcal{H})\otimes_{\psi^*(\mathcal{B})}\mathcal{A},$$

that is, a homomorphism of $\mathcal{A}$-modules !!

$$\Psi^*(u):\Psi^*(\mathcal{G})\to\Psi^*(\mathcal{H}).$$

It is clear that $\Psi^*(\mathrm{id}_{\mathcal{G}})=\mathrm{id}_{\Psi^*(\mathcal{G})}$, and if $wu:\mathcal{F}\to\mathcal{G}$ is a homomorphism of $\mathcal{B}$-modules we have that $\Psi^*(vu)=\Psi^*(v)\Psi^*(u)$. In other words, we have that Ψ^* is a covariant functor from $\mathcal{B}$-modules on $\mathfrak{B}$ to $\mathcal{A}$-modules on X.

For every point $x \in X$ we have, by Lemma (1.9) applied to the $\Psi^*(\mathcal{B})$-modules $\Psi^*(\mathcal{G})$ and $\mathcal{A}$, an isomorphism $\Psi^*(\mathcal{G}) = (\psi^*(\mathcal{G}) \otimes_{\psi^*(\mathcal{B})} \mathcal{A})_x \xrightarrow{\sim} \psi^*(\mathcal{G})_x \otimes_{\psi^*(\mathcal{B})_x} \mathcal{A}_x$. Moreover it follows from Lemma (Modules 2.23) that there is a homomorphism of $\mathcal{A}_x$-modules $(\rho_{\mathcal{G}})_x \otimes_{(\rho_{\mathcal{B}})_x} (\mathrm{id}_{\mathcal{A}})_x : \mathcal{G}_{\psi(x)} \otimes_{\mathcal{B}_{\psi(x)}} \mathcal{A}_x \to \psi^*(\mathcal{G})_x \otimes_{\psi^*(\mathcal{B})_x} \mathcal{A}_x$, and from (Sheaves 2.3) that the latter homomorphism is an isomorphism. Consequently we have a canonical isomorphism of $\mathcal{A}_x$-modules

$$\mathcal{G}_{\psi(x)} \otimes_{\mathcal{B}_{\psi(x)}} \mathcal{A}_x \xrightarrow{\sim} \Psi^*(\mathcal{G})_x = (\psi^*(\mathcal{G}) \otimes_{\psi^*(\mathcal{B})} \mathcal{A})_x. \tag{1.13.1}$$

(1.14) Remark. Let $\Psi = (\psi, \theta) : (X, \mathcal{A}) \to (Y, \mathcal{B})$ be a homomorphism of ringed spaces. Then $\Psi^*(\mathcal{B}) = \mathcal{A}$. In fact we have that $\Gamma(U, \psi^*(\mathcal{B}))$ is canonically isomorphic to $\Gamma(U, \psi^*(\mathcal{B})) \otimes_{\Gamma(U, \psi^*(\mathcal{B}))} \Gamma(U, \mathcal{A}) = \Gamma(U, \mathcal{A})$ for all open subsets of X belonging to $\mathfrak{B}$.

(1.15) Adjunction. Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be two ringed spaces and let $\Psi = (\psi, \theta) : (X, \mathcal{A}) \to (Y, \mathcal{B})$ be a homomorphism of ringed spaces. Moreover let $\mathcal{F}$ be an $\mathcal{A}$-module and $\mathcal{G}$ be a $\mathcal{B}$-module. The adjunction maps $\rho_{\mathcal{B}} : \mathcal{B} \to \psi_*(\psi^*(\mathcal{B}))$ and $\sigma_{\mathcal{A}} : \psi^*(\psi_*(\mathcal{A})) \to \mathcal{A}$ are homomorphisms of rings. Clearly the adjunction map $\rho_{\mathcal{G}} : \mathcal{G} \to \psi_*(\psi^*(\mathcal{G}))$ of (Sheaves 3.8) is a homomorphism of $\rho_{\mathcal{B}}$-modules and the adjunction map $\sigma_{\mathcal{F}} : \psi^*(\psi_*(\mathcal{F})) \to \mathcal{F}$ of (Sheaves 3.8) is a homomorphism of $\sigma_{\mathcal{A}}$-modules. Hence we obtain an adjunction map $\rho_{\mathcal{G}} : \mathcal{G} \to \Psi_*(\Psi^*(\mathcal{G}))$ of $\mathcal{A}$-modules and an adjunction map $\sigma_{\mathcal{F}} : \Psi^*(\Psi_*(\mathcal{F})) \to \mathcal{F}$ of $\mathcal{B}$-modules. We obtain, by extension and restriction of scalars, an adjunction map

$$\mathrm{Hom}_{\mathcal{A}}(\Psi^*(\mathcal{G}), \mathcal{F}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{B}}(\mathcal{G}, \Psi_*(\mathcal{F}))$$

which is a bijection between $\mathcal{A}$-modules homomorphisms $\Psi^*(\mathcal{G}) \to \mathcal{F}$, and $\mathcal{B}$-module homomorphisms $\mathcal{G} \to \Psi_*(\mathcal{F})$.

(1.16) Kernels and cokernels of homomorphisms of modules. Let X be a topological space with a basis $\mathfrak{B}$ for the topology. Moreover let $\mathcal{A}$ be a presheaf of rings on $\mathfrak{B}$ and let $u : \mathcal{F} \to \mathcal{G}$ be a homomorphism of presheaves of $\mathcal{A}$-modules on $\mathfrak{B}$. Let $\mathcal{H}$ be the presheaf of Section (Sheaves 2.10) defined by $\Gamma(U, \mathcal{H}) = \mathrm{Im}(u_U)$ and where $(\rho_{\mathcal{H}})_U^V$ is induced by $(\rho_{\mathcal{G}})_U^V$ for all inclusions $U \subseteq V$ of open subsets belonging to $\mathfrak{B}$. Clearly $\mathcal{H}$ is a presheaf of $\mathcal{A}$-modules. For every open subset U belonging to $\mathfrak{B}$ we obtain commutative diagrams of $\mathcal{A}$-modules with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ker}(u_V) & \longrightarrow & \Gamma(V, \mathcal{F}) & \longrightarrow & \Gamma(V, \mathcal{H}) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \sigma_U^V} & & \Big\downarrow{\scriptstyle (\rho_{\mathcal{F}})_U^V} & & \Big\downarrow{\scriptstyle (\rho_{\mathcal{H}})_U^V} & & \\
0 & \longrightarrow & \mathrm{Ker}(u_U) & \longrightarrow & \Gamma(U, \mathcal{F}) & \longrightarrow & \Gamma(U, \mathcal{H}) & \longrightarrow & 0,
\end{array}$$

and

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Gamma(V, \mathcal{H}) & \longrightarrow & \Gamma(V, \mathcal{G}) & \longrightarrow & \mathrm{Coker}(u_V) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle (\rho_{\mathcal{H}})_U^V} & & \Big\downarrow{\scriptstyle (\rho_{\mathcal{G}})_U^V} & & \Big\downarrow{\scriptstyle (\tau)_U^V} & & \\
0 & \longrightarrow & \Gamma(U, \mathcal{H}) & \longrightarrow & \Gamma(U, \mathcal{G}) & \longrightarrow & \mathrm{Coker}(u_U) & \longrightarrow & 0,
\end{array}$$

where σ_U^V is induced by $(\rho_{\mathcal{G}})_U^V$ and where τ_U^V is induced by $(\rho_{\mathcal{G}})_U^V$. Let $\Gamma(U, \mathcal{F}') = \mathrm{Ker}(u_U)$ and let $\Gamma(U, \mathcal{G}') = \mathrm{Coker}(u_U)$ for all open subsets U belonging to $\mathfrak{B}$. For all inclusions $U \subseteq V$ of open sets belonging to $\mathfrak{B}$ we let $(\rho_{\mathcal{F}'})_U^V = \sigma_U^V$ and $(\rho_{\mathcal{G}'})_U^V = \tau_U^V$. It is clear that $\mathcal{F}'$ and $\mathcal{G}'$, with the restriction maps $(\rho_{\mathcal{F}'})_U^V$ respectively $(\rho_{\mathcal{G}'})_U^V$, are presheaves of $\mathcal{A}$-modules on $\mathfrak{B}$.

Assume that $\mathcal{A}$, $\mathcal{F}$ and $\mathcal{G}$ are sheaves on $\mathfrak{B}$. It is easy to check that $\mathcal{F}'$ is an $\mathcal{A}$-module on $\mathfrak{B}$, and that the inclusions $\Gamma(U, \mathcal{F}') \subseteq \Gamma(U, \mathcal{F})$ for all open sets belonging to $\mathfrak{B}$ make $\mathcal{F}'$ into a subsheaf of $\mathcal{F}$. The surjections $\Gamma(U, \mathcal{G}) \to \Gamma(U, \mathcal{G}')$ for all open sets U belonging to $\mathfrak{B}$ define a homomorphism $\mathcal{G} \to \mathcal{G}'$ of presheaves of $\mathcal{A}$-modules, and it follows from Lemma (Sheaves 2.12) that the resulting map on stalks $\mathcal{G}_x \to \mathcal{G}'_x$ is surjective for all $x \in X$. The composite of the map $\mathcal{G} \to \mathcal{G}'$ with $\rho_{\mathcal{G}'} : \mathcal{G}' \to (id_X)^*(\mathcal{G}')$ gives a homomorphism $\mathcal{G} \to (\mathrm{id}_X)^*(\mathcal{G}')_x$ of $\mathcal{A}$-modules, and it follows from Remark (?) that the the resulting map on stalks $\mathcal{G}_x \to (\mathrm{id}_X)^*(\mathcal{G}')_x$ is surjective. Hence it follows from Proposition (Sheaves 2.3) that the map $\mathcal{G} \to (\mathrm{id}_X)^*(\mathcal{G}')$ is surjective. We have that the subsheaf $\mathrm{Im}(u) = (\mathrm{id}_X)^*(\mathcal{H})$ of $\mathcal{G}$ is an $\mathcal{A}$-module, and that the homomorphism $j : \mathrm{Im}(u) \to \mathcal{G}$ is a homomorphism of $\mathcal{A}$-modules.

(1.17) Example. Even when $\mathcal{A}$, $\mathcal{F}$ and $\mathcal{G}$ are sheaves the presheaf $\mathcal{G}'$ of Section (1.16) is not necessarily a sheaf.

In fact in Example (Sheaves 2.11) the sheaf $\mathcal{F}$ is a sheaf of rings. Let $\mathcal{F} = \mathcal{A}$. It is clear that the sheaf $\mathcal{G}$ of Example (Sheaves 2.11) is an $\mathcal{A}$-module, and that $u : \mathcal{F} \to \mathcal{G}$ is a homomorphism of $\mathcal{A}$-modules. We have that $\Gamma(\emptyset, \mathcal{G}') = \{0\}$, $\Gamma(X, \mathcal{G}')$ is isomorphic to $\mathbf{Z}$, and $\Gamma(U_i, \mathcal{G}') = \{0\}$ for $i = 0, 1, 2$. Hence $\mathcal{G}'$ is not a sheaf.

We note that the presheaf $\mathcal{H}$ of Example (Sheaves 2.11) is an $\mathcal{A}$-module. Hence $\mathcal{H}$ is not necessarily a sheaf even when u is a homomorphism of $\mathcal{A}$-modules.

(1.18) Definition. Let X be a topological space with a basis $\mathfrak{B}$ for the topology. Moreover let $\mathcal{A}$ be a sheaf of rings on $\mathfrak{B}$ and let $u : \mathcal{F} \to \mathcal{G}$ be a homomorphism of $\mathcal{A}$-modules. The subsheaf $\mathcal{F}'$ of $\mathcal{F}$ defined in Remark (1.16) we call the *kernel* of u, and we denote it by !!Ker(u). We call the $\mathcal{A}$-module $(\mathrm{id}_X)^*(\mathcal{G}')$ the *cokernel* of u and we denote it by !!Coker(u). When u is injective we often write $\mathrm{Coker}(u) = \mathcal{G}/\mathcal{F}$. An $\mathcal{A}$-*submodule* of $\mathcal{G}$ is a sheaf of the form $\mathrm{Im}(u)$ for some $\mathcal{A}$-module homomorphism $u : \mathcal{F} \to \mathcal{G}$. A submodule of the sheaf of rings $\mathcal{A}$ is called an *ideal* of $\mathcal{A}$.

A sequence of $\mathcal{A}$-modules

$$\cdots \longrightarrow \mathcal{F}_{n-1} \xrightarrow{u_{n-1}} \mathcal{F}_n \xrightarrow{u_n} \mathcal{F}_{n+1} \longrightarrow \cdots$$

we call a *complex* when $\mathrm{Im}(u_{n-1}) \subseteq \mathrm{Ker}(u_n)$ for all n, and we say that the complex is *exact* if $\mathrm{Im}(u_{n-1}) = \mathrm{Ker}(u_n)$ for all n.

(1.19) Remark. For all points $x \in X$ we clearly have an injection $\mathrm{Ker}(u)_x \subseteq \mathrm{Ker}(u_x)$ of $\mathcal{A}_x$-modules contained in $\mathcal{F}_x$. This injection is an equality of sets

$$\mathrm{Ker}(u)_x = \mathrm{Ker}(u_x).$$

In fact, every element in $\operatorname{Ker}(u_x)$ is represented by a pair (U, s) where U is open in $\mathfrak{B}$, and $s \in \Gamma(U, \mathcal{F})$ is such that $u_U(s) = 0$ and (U, s) represents an element in $\operatorname{Ker}(u)_x$.

For all points $x \in X$ we have a cononical ismorphism of $\mathcal{A}_x$-modules

$$\operatorname{Coker}(u_x) \xrightarrow{\sim} \operatorname{Coker}(u)_x \tag{1.19.1}$$

between quotient modules of $\mathcal{G}_x$. In fact, it is clear that we have a surjective $\mathcal{A}_x$-module homomorphism $\operatorname{Coker}(u_x) \to \operatorname{Coker}(u)_x$. This homomorphism is an isomorphism because every element r in $\operatorname{Coker}(u_x)$ is represented by a pair (V, t) where V is open in $\mathfrak{B}$ and $t \in \mathcal{G}(V)$. If r is mapped to zero by (1.19.1) there is an open neighbourhoood U of x belonging to $\mathfrak{B}$ and contained in V such that $(\rho_{\mathcal{G}})_U^V(t) = u_U(s)$ for a section $s \in \Gamma(U, \mathcal{F})$. The pairs (V, t) and $(U, u_U(s))$ represent the same element r in $\operatorname{Coker}(u_x)$. Since $(U, u_U(s))$ represents the class of $u_x(s_x)$ in $\operatorname{Coker}(u_x)$, and this class is zero, we have that $r = 0$, and hence that (1.19.1) is injetive.

(1.20) Proposition. *A complex*

$$\mathcal{F}' \xrightarrow{u'} \mathcal{F} \xrightarrow{u''} \mathcal{F}'' \tag{1.20.1}$$

of $\mathcal{A}$-modules is exact if and only if the resulting complex of stalks of $\mathcal{A}_x$-modules

$$\mathcal{F}'_x \xrightarrow{u'_x} \mathcal{F}_x \xrightarrow{u''_x} \mathcal{F}''_x \tag{1.20.2}$$

is exact for all $x \in X$.

In particular we obtain two exact complexes of $\mathcal{A}$-modules

$$0 \to \operatorname{Ker}(u) \to \mathcal{F} \to \operatorname{Im}(u) \to 0$$

and

$$0 \to \operatorname{Im}(u) \to \mathcal{G} \to \operatorname{Coker}(u) \to 0.$$

The homomorphism u is injective if and only if $\operatorname{Ker}(u) = 0$ *and surjective if and only if* $\operatorname{Coker}(u) = 0$.

Proof. When the sequence (1.20.1) is exact we have that $\operatorname{Im}(u')_x = \operatorname{Ker}(u'')_x$ for all points $x \in X$. Hence it follows from Remark (Sheaves 2.15) and Remark (1.19) that $\operatorname{Im}(u'_x) = \operatorname{Ker}(u''_x)$ and thus that the sequence (1.20.2) is exact.

Conversely, if the sequence (1.20.2) is exact for all points $x \in X$, it follows from Remark (Sheaves 2.5) and Remark (1.19) that $\operatorname{Im}(u')_x = \operatorname{Ker}(u'')_x$ for all points $x \in X$. Hence it follows from Lemma (Sheaves 2.12) that the inclusion $\operatorname{Im}(u') \subseteq \operatorname{Ker}(u'')$ is an equality. That is, the sequence (1.20.1) is exact.

The last part of the Proposition follows from the two exact sequences of Example (Modules 1.31) associated to the $\mathcal{A}_x$-module map $u_x : \mathcal{F}_x \to \mathcal{G}_x$, and from the equalities $\operatorname{Ker}(u)_x = \operatorname{Ker}(u_x)$, $\operatorname{Coker}(u)_x = \operatorname{Coker}(u_x)$, and $\operatorname{Im}(u)_x = \operatorname{Im}(u_x)$ of Remark (Sheaves 2.15) and Remark (1.19).

(1.21) Proposition. *Let $\Psi = (\psi, \theta) : (X, \mathcal{A}) \to (Y, \mathcal{B})$ be a homomorphism of ringed spaces and let*

$$0 \to \mathcal{F}' \xrightarrow{u'} \mathcal{F} \xrightarrow{u''} \mathcal{F}'' \qquad (1.21.1)$$

be an exact sequence of $\mathcal{A}$-modules. Then we have an exact sequence of $\mathcal{B}$-modules

$$0 \to \Psi_*(\mathcal{F}') \xrightarrow{\Psi_*(u')} \Psi_*(\mathcal{F}) \xrightarrow{\Psi_*(u'')} \Psi_*(\mathcal{F}'') \qquad (1.21.2)$$

Proof. The map $\mathcal{F}' \to \mathrm{Im}(u')$ induced by u' is an isomorphism, and the map $\mathrm{Im}(u')$ is injective. Hence the map $u'_U : \Gamma(U, \mathcal{F}') \to \Gamma(U, \mathcal{F})$ is injective and has image $\mathrm{Im}(u')_U$. Hence, by assumption $\mathrm{Im}(u') = \mathrm{Ker}(u'')$ and by the definition of $\mathrm{Ker}(u'')$ we have that $\mathrm{Ker}(u'')_U = \mathrm{Ker}(u''_U)$. Hence $Im(u')_U = \mathrm{Ker}(u'')_U = \mathrm{Ker}(u''_U)$. Hence we have an exact sequence

$$0 \to \Gamma(U, \mathcal{F}') \xrightarrow{u'_U} \Gamma(U, \mathcal{F}) \xrightarrow{u''_U} \Gamma(U, \mathcal{F}'')$$

of $\Gamma(U, \mathcal{A})$-modules for all open subsets U of X. Hence we have that the sequence

$$0 \to \Gamma(V, \Psi_*(\mathcal{F}')) \xrightarrow{\Psi_*(u')_V} \Gamma(V, \Psi_*(\mathcal{F})) \xrightarrow{\Psi_*(u'')_V} \Gamma(V, \Psi_*(\mathcal{F}))$$

of $\Gamma(V, \Psi_*(\mathcal{A}))$-modules is exact for all open subsets V of Y. It follows that $\Psi_*(u')$ induces an isomorphism $\mathcal{F}' \to \mathrm{Im}(\Psi_*(u'))$. Moreover, since $\Gamma(V, \mathrm{Ker}(\Psi_*(u'')) = \mathrm{Ker}(\Psi_*(u''))_V \, \mathrm{Ker}(\Psi_*(u'')_V)$, we obtain that $\mathrm{Im}(\Psi_*(u')) = \mathrm{Ker}(\Psi_*(u''))$. Hence we have that the sequence (1.21.2) is exact.

(1.22) Proposition. *Let $\Psi = (\psi, \theta) : (X, \mathcal{A}) \to (Y, \mathcal{B})$ be a homomorphism of ringed spaces and let*

$$0 \to \mathcal{G}' \xrightarrow{v'} \mathcal{G} \xrightarrow{v''} \mathcal{G}'' \to 0 \qquad (1.22.1)$$

be an exact sequence of $\mathcal{B}$-modules. Then

$$0 \to \psi^*(\mathcal{G}') \xrightarrow{\psi^*(v')} \psi^*(\mathcal{G}) \xrightarrow{\psi^*(v'')} \psi^*(\mathcal{G}'') \to 0 \qquad (1.22.2)$$

is an exact sequence of $\psi^*(\mathcal{B})$-modules. In particular

$$\Psi^*(\mathcal{G}') \xrightarrow{\Psi^*(v')} \Psi^*(\mathcal{G}) \xrightarrow{\Psi^*(v'')} \Psi^*(\mathcal{G}'') \to 0 \qquad (1.22.3)$$

is an exact sequence of $\mathcal{A}$-modules.

Proof. It follows from Proposition (1.20) that the sequence (1.22.1) is exact if and only if the sequence

$$0 \to \mathcal{G}'_y \xrightarrow{v'_y} \mathcal{G}_y \xrightarrow{v''_y} \mathcal{G}''_y \to 0 \qquad (1.22.4)$$

is an exact sequence of $\mathcal{B}_y$-modules for all $y \in Y$. From Proposition (Sheaves 2.2) it follows that when the sequence (1.22.4) is exact for all $y \in Y$ the sequence of $\Psi^*(\mathcal{B})_x$-modules

$$0 \to \psi^*(\mathcal{G}')_x \xrightarrow{\psi^*(v')_x} \psi^*(\mathcal{G})_x \xrightarrow{\psi^*(v'')_x} \psi^*(\mathcal{G}'')_x \to 0$$

is exact for all $x \in X$. It follows from Proposition (1.20) that then (1.22.2) is exact.

From the exatness of the sequence

$$\psi^*(\mathcal{G}')_x \xrightarrow{\psi^*(v')_x} \psi^*(\mathcal{G})_x \xrightarrow{\psi^*(v'')_x} \psi^*(\mathcal{G}'')_x \to 0$$

and from Lemma (1.9) we obtain an exact sequence

$$\psi^*(\mathcal{G}')_x \otimes_{\psi^*(\mathcal{B})_x} \mathcal{A}_x \to \psi^*(\mathcal{G})_x \otimes_{\psi^*(\mathcal{B})_x} \mathcal{A}_x \to \psi^*(\mathcal{G}'')_x \otimes_{\psi^*(\mathcal{B})_x} \mathcal{A}_x \to 0$$

of $\mathcal{A}_x$-modules for all points $x \in X$. That is, we have an exact sequence

$$\Psi^*(\mathcal{G}')_x \xrightarrow{\Psi^*(v')_x} \Psi^*(\mathcal{G})_x \xrightarrow{\Psi^*(v'')_x} \Psi^*(\mathcal{G}'')_x \to 0$$

of $\mathcal{A}_x$-modules for all points $x \in X$. It follows from Proposition (1.20) that the sequence (1.22.3) is exact.

(1.23) Direct sums of modules of sheaves. Let X be a topological space with a basis $\mathfrak{B}$ for the topology. Moreover let $\mathcal{A}$ be a presheaf of rings on $\mathfrak{B}$ and let $\{\mathcal{F}_\alpha\}_{\alpha \in I}$ be a collection of presheaves of $\mathcal{A}$-modules. For every open set U belonging to $\mathfrak{B}$ we let $\Gamma(U, \mathcal{F}) = \oplus_{\alpha \in I} \Gamma(U, \mathcal{F}_\alpha)$ be the direct sum of the $\Gamma(U, \mathcal{A})$-modules $\Gamma(U, \mathcal{F}_\alpha)$ for $\alpha \in I$. Moreover, for all inclusions $U \subseteq V$ of open sets belonging to $\mathfrak{B}$, we let $(\rho_{\mathcal{F}})_U^V : \Gamma(V, \mathcal{F}) \to \Gamma(U, \mathcal{F})$ be the map induced by the maps $(\rho_{\mathcal{F}_\alpha})_U^V : \Gamma(V, \mathcal{F}_\alpha) \to \Gamma(U, \mathcal{F}_\alpha)$ for all $\alpha \in I$. It is clear that $\mathcal{F}$ with the restriction maps $(\rho_{\mathcal{F}})_U^V$ is a presheaf of $\mathcal{A}$-modules on $\mathfrak{B}$.

For all $\beta \in I$, and for every open set U belonging to $\mathfrak{B}$ there is a canonical map $\Gamma(U, \mathcal{F}_\beta) \to \oplus_{\alpha \in I} \Gamma(U, \mathcal{F}_\alpha)$ of $\Gamma(U, \mathcal{A})$-modules. It is clear that these maps, for fixed β, and for all open sets U belonging to $\mathfrak{B}$, define a canonical map of presheaves of $\mathcal{A}$-modules

$$i_\beta : \mathcal{F}_\beta \to \mathcal{F}.$$

For every point $x \in X$ we have a map $(i_\beta)_x : (\mathcal{F}_\beta)_x \to \mathcal{F}_x$ of $\mathcal{A}_x$-modules, and consequently a map of $\mathcal{A}_x$-modules

$$\oplus_{\alpha \in I} (\mathcal{F}_\alpha)_x \xrightarrow{\sim} \mathcal{F}_x. \tag{1.23.1}$$

The map (1.23.1) is an isomorphism of $\mathcal{A}_x$-modules. In fact we shall construct the inverse to the map (1.23.1). Each element in $\mathcal{F}_x$ is represented by a pair $(U, \oplus_{\alpha \in J} s_\alpha)$ where U is an open set belonging to $\mathfrak{B}$ and $\oplus_{\alpha \in J} s_\alpha \in \Gamma(U, \mathcal{F}) = \oplus_{\alpha \in I} \Gamma(U, \mathcal{F}_\alpha)$, with $s_\alpha \in \Gamma(U, \mathcal{F}_\alpha)$, and where J is a finite subset of I. We map the class of the pair $(U, \oplus_{\alpha \in J} s_\alpha)$ in $\mathcal{F}_x$ to $\oplus_{\alpha \in J} (s_\alpha)_x \in \oplus_{\alpha \in I} (\mathcal{F}_\alpha)_x$. It is clear that the resulting element in $\oplus_{\alpha \in I} (\mathcal{F}_\alpha)_x$ is independent of choise of the representative $(U, \oplus_{\alpha \in J} s_\alpha)$ of the element in $\mathcal{F}_x$, and that we obtain a map $\mathcal{F}_x \to \oplus_{\alpha \in I} (\mathcal{F}_\alpha)_x$ which is the inverse of the map (1.23.1).

(1.24) Example. Even when all the sheaves $\mathcal{F}_\alpha$ are $\mathcal{A}$-modules the presheaf $\mathcal{F}$ of Section (?) is not necessarily a sheaf. Let X be the topological space which consists of the set $\mathbf{N}$ of natural numbers with the discrete topology. We let $\mathcal{A}$ be the simple sheaf of rings on X with fiber $\mathbf{Z}$. Moreover, for every $n \in \mathbf{N}$ we let $\mathcal{F}_n$ be the sheaf on X defined by $\Gamma(U, \mathcal{F}_n) = \mathbf{Z}$ when $n \in U$ and $\Gamma(U, \mathcal{F}_n) = \{0\}$ otherwise, and where the restriction map $(\rho_{\mathcal{F}_n})_U^V = \mathrm{id}_\mathbf{Z}$ when $n \in U$ and the zero map otherwise. Clearly $\mathcal{F}_n$ is an $\mathcal{A}$-module via the multiplication of $\Gamma(U, \mathcal{A})$ on $\Gamma(U, \mathcal{F}_n)$ defined by $(f_i)_{i \in U} x = f_n x$.

The presheaf $\mathcal{F}$ associated to the collection $\{\mathcal{F}_n\}_{n \in \mathbf{N}}$ in Remark (?) is not a sheaf. In fact let $s_n = 1 \in \Gamma(\{n\}, \mathcal{F}_n)$ for every member $\{n\}$ in the open covering $\{\{n\}\}_{n \in \mathbf{N}}$ of X. Then the restriction of s_m and s_n to the intersecton $\{m\} \cap \{n\} = \emptyset$ is 0 for all pairs of integers m, n. However there is no section in $\Gamma(X, \mathcal{F}) = \oplus_{n \in \mathbf{N}} \Gamma(X, \mathcal{F}_n)$ that restricts to s_n on the open set $\{n\}$ for all $n \in \mathbf{N}$.

(1.25) Definition. Let X be a topological space with a basis $\mathfrak{B}$ for the topology. Moreover let $\mathcal{A}$ be a sheaf of rings on $\mathfrak{B}$ and let $\{\mathcal{F}_\alpha\}_{\alpha \in I}$ be a collection of $\mathcal{A}$-modules. The *direct sum* of the $\mathcal{A}$-modules $\mathcal{F}_\alpha$ is the associated sheaf $(\mathrm{id}_X)^*(\mathcal{F})$ of the presheaf $\mathcal{F}$ defined in Section (1.23) from the collection $\{\mathcal{F}_\alpha\}_{\alpha \in I}$. We denote the direct sum by !!$\oplus_{\alpha \in I} \mathcal{F}_\alpha$, and the canonical $\mathcal{A}$-module homomorphism we obtain by composing $i_\beta : \mathcal{F}_\beta \to \mathcal{F}$ with $\rho_\mathcal{F} : \mathcal{F} \to (\mathrm{id}_X)^*(\oplus_{\alpha \in I} \mathcal{F}_\alpha)$ we denote by !!

$$h_\beta : \mathcal{F}_\beta \to \oplus_{\alpha \in I} \mathcal{F}_\alpha.$$

When $\mathcal{F}_\alpha = \mathcal{G}$ for all $\alpha \in I$ we write !!$\oplus_{\alpha \in I} \mathcal{F}_\alpha = \mathcal{G}^{(I)}$, and when I is finite and consists of n elements we write $\mathcal{G}^{(I)} = \mathcal{G}^p$.

(1.26) Remark. It is clear that the $\mathcal{A}$-module $\oplus_{\alpha \in I} \mathcal{F}_\alpha$ together with the canonical maps $h_\alpha : \mathcal{F}_\alpha \to \oplus_{\alpha \in I} \mathcal{F}_\alpha$ is the direct sum of the collection of $\mathcal{A}$-modules $\{\mathcal{F}_\alpha\}_{\alpha \in I}$ in the category of $\mathcal{A}$-modules. Moreover it follows from Proposition (Sheaves 2.3) and the isomorphism (1.23.1) that we have a canonical isomorphism of $\mathcal{A}_x$-modules

$$\oplus_{\alpha \in I} (\mathcal{F}_\alpha)_x \xrightarrow{\sim} (\oplus_{\alpha \in I} \mathcal{F}_\alpha)_x$$

for all $x \in X$.

(1.27) Lemma. *Let $(X, \mathcal{A})$ be a ringed space and let $\{\mathcal{F}_\alpha\}_{\alpha \in I}$ be a collection of $\mathcal{A}$-modules. Moreover let $h_\beta : \mathcal{F}_\beta \to \oplus_{\alpha \in I} \mathcal{F}_\alpha$ be the canonical homomorphism to factor β. For every $\mathcal{A}$-modules $\mathcal{G}$ there is a canonical $\mathcal{A}$-module isomorphism*

$$\oplus_{\alpha \in I} (\mathcal{F}_\alpha \otimes_\mathcal{A} \mathcal{G}) \xrightarrow{\sim} (\oplus_{\alpha \in I} \mathcal{F}_\alpha) \otimes_\mathcal{A} \mathcal{G} \tag{1.27.1}$$

that composed with the homomorphism $h_\beta \otimes_\mathcal{A} \mathrm{id}_\mathcal{G} : \mathcal{F}_\beta \otimes_\mathcal{A} \mathcal{G} \to (\oplus_{\alpha \in I} \mathcal{F}_\alpha) \otimes_\mathcal{A} \mathcal{G}$ is the canonical homomorphism $\mathcal{F}_\beta \otimes_\mathcal{A} \mathcal{G} \to \oplus_{\alpha \in I} (\mathcal{F}_\alpha \otimes_\mathcal{A} \mathcal{G})$ to factor β for all $\beta \in I$.

Proof. From the universal property of direct sums of $\mathcal{A}$-modules the maps $h_\beta \otimes_\mathcal{A} \mathrm{id}_\mathcal{G} : \mathcal{F}_\beta \otimes_\mathcal{A} \mathcal{G} \to (\oplus_{\alpha \in I} \mathcal{F}_\alpha) \otimes_\mathcal{A} \mathcal{G}$ for $\beta \in I$ define a canonical homomorphism (1.27.1) with the properties described in the Lemma. We must show that the homomorphism (1.27.1) is an isomorphism. It follows from Proposition (1.20) that it suffices to prove that the induced map on stalks at $x \in X$ is an isomorphism for all $x \in X$. We saw in Remark (1.26) that up to isomorphisms the map of stalks at a point $x \in X$ is given by the homomorphism $\oplus_{\alpha\in I}(\mathcal{F}_\alpha \otimes_{\mathcal{A}} \mathcal{G})_x \to ((\oplus_{\alpha\in I}\mathcal{F}_\alpha) \otimes_{\mathcal{A}} \mathcal{G})_x$ obtained from the homomorphisms $(h_\beta \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{G}})_x : (\mathcal{F}_\beta \otimes_{\mathcal{A}} \mathcal{G})_x \to ((\oplus_{\alpha\in I}\mathcal{F}_\alpha) \otimes_{\mathcal{A}} \mathcal{G})_x$ for all $\beta \in I$. Moreover it follows from Lemma (1.9) that the latter homomorphism up to isomorphisms is given by the map $\oplus_{\alpha\in I}((\mathcal{F}_\alpha)_x \otimes_{\mathcal{A}_x} \mathcal{G}_x) \to (\oplus_{\alpha\in I}\mathcal{F}_\alpha)_x \otimes_{\mathcal{A}_x} \mathcal{G}_x$ obtained from the homomorphisms $(h_\beta)_x \otimes_{\mathcal{A}_x} \mathrm{id}_{\mathcal{G}_x} : (\mathcal{F}_\beta)_x \otimes_{\mathcal{A}_x} \mathcal{G}_x \to (\oplus_{\alpha\in I}\mathcal{F}_\alpha)_x \otimes_{\mathcal{A}_x} \mathcal{G}_x$. It follows from Remark (1.26) applied to the collection of modules $\{\mathcal{F}_\alpha \otimes_{\mathcal{A}} \mathcal{G}\}_{\alpha\in I}$ that the latter map is an isomorphism.

(1.28) Proposition. *Let $\Psi = (\psi, \theta) : (X, \mathcal{A}) \to (Y, \mathcal{B})$ be a homomorphism of ringed spaces and let $\{\mathcal{G}_\alpha\}_{\alpha\in I}$ be a collection of $\mathcal{B}$-modules. Then there is a canonical isomorphism of $\mathcal{A}$-modules*

$$\oplus_{\alpha\in I}\Psi^*(\mathcal{G}_\alpha) \xrightarrow{\sim} \Psi^*(\oplus_{\alpha\in I}\mathcal{G}_\alpha) \tag{1.28.1}$$

such that the composite of the homomorphism (1.28.1) *with the canonical homomorphism $\Psi^*(\mathcal{G}_\beta) \to \oplus_{\alpha\in I}\Psi^*(\mathcal{G}_\alpha)$ is the map $\Psi^*(h_\beta) : \Psi^*(\mathcal{G}_\beta) \to \Psi^*(\oplus_{\alpha\in I}\mathcal{G}_\alpha)$, where h_β is defined in* Section (1.25).

Proof. For every $\alpha \in I$ the canonical homomorphism $h_\beta : \mathcal{G}_\beta \to \oplus_{\alpha\in I}\mathcal{G}_\alpha$ gives a canonical homomorphism $\Psi^*(h_\beta) : \Psi^*(\mathcal{G}_\beta) \to \Psi^*(\oplus_{\alpha\in I}\mathcal{G}_\alpha)$. Consequently we obtain a canonical homomorphism $\oplus_{\alpha\in I}\Psi^*(\mathcal{G}_\alpha) \to \Psi^*(\oplus_{\alpha\in I}\mathcal{G}_\alpha)$ that composed with the canonical homomorphism of $\mathcal{A}$-modules $\Psi^*(\mathcal{G}_\beta) \to \oplus_{\alpha\in I}\Psi^*(\mathcal{G}_\alpha)$ is $\Psi^*(h_\beta)$. We have thus constructed the homomorphism (1.28.1).

It remains to prove that the homomorphism (1.28.1) is an isomorphism. It follows from Remark (1.26) and the isomorphism (1.13) that we have canonical isomorphisms $(\oplus_{\alpha\in I}\Psi^*(\mathcal{G}_\alpha))_x \xrightarrow{\sim} \oplus_{\alpha\in I}\,(\Psi^*(\mathcal{G})_\alpha)_x \xrightarrow{\sim} \oplus_{\alpha\in I}\,(\psi^*(\mathcal{G}_\alpha)_x \otimes_{\psi^*(\mathcal{B})_x} \mathcal{A}_x)$ of $\mathcal{A}_x$-modules for each point $x \in X$ i, and from Proposition (Sheaves 2.3) we obtain a canonical homomorphism of $\mathcal{A}_x$-modules $\oplus_{\alpha\in I}(\Psi^*(\mathcal{G}_\alpha)_x \otimes_{\Psi^*(\mathcal{B})_x} \mathcal{A}_x) \xrightarrow{\sim} \oplus_{\alpha\in I}\,(\mathcal{G}_\alpha)_{\psi(x)\otimes_{\mathcal{B}_{\psi(x)}}}\mathcal{A}_x)$. From the isomorphism (1.13.1) and Remark (1.26) we obtain isomorphisms $\Psi^*(\oplus_{\alpha\in I}\mathcal{G}_\alpha)_x \xrightarrow{\sim} (\oplus_{\alpha\in I}\mathcal{G}_\alpha)_{\psi(x)} \otimes_{\mathcal{B}_{\psi(x)}} \mathcal{A}_x \xrightarrow{\sim} \oplus_{\alpha\in I}\,(\mathcal{G}_\alpha)_{\psi(x)} \otimes_{\mathcal{B}_{\psi(x)}} \mathcal{A}_x$. It is easy to check that via these isomorphisms the homomorphism (1.28.1) induces the identity on $\oplus_{\alpha\in I}(\mathcal{G}_\alpha)_{\psi(x)} \otimes_{\mathcal{B}_{\psi(x)}} \mathcal{A}_x$. It consequently follows from Proposition (1.20) that (1.28.1) is an isomorphism.

(1.29) Definition. Let X be a topological space and $\mathcal{A}$ a sheaf of rings on X. The *support* !!Supp($\mathcal{F}$) of a $\mathcal{A}$-module $\mathcal{F}$ is the subset of X consisting of points x where $\mathcal{F}_x \neq 0$.

(1.30) Exercises.
1. Let A be a ring and let M be an A-module. Moreover let X be a topological space. For all open non-empty subsets U of X we let $\Gamma(U, \mathcal{A}) = A^U$ and $\Gamma(U, \mathcal{F}) = M^U$,

and let $\Gamma(\emptyset, \mathcal{A}) = \{0\}$, and $\Gamma(\emptyset, \mathcal{F}) = \{0\}$. For every inclusion $U \subseteq V$ of open subsets of X we define $(\rho_{\mathcal{A}})_U^V : \Gamma(V, \mathcal{A}) \to \Gamma(U, \mathcal{A})$ and $(\rho_{\mathcal{F}})_U^V : \Gamma(V, \mathcal{F}) \to \Gamma(U, \mathcal{F})$ as the restrictions of functions on V to functions on U. Show that the sheaf $\mathcal{F}$ with the restriction maps $(\rho_{\mathcal{F}})_U^V$ is a module over the sheaf $\mathcal{A}$ with the restriction maps $(\rho_{\mathcal{A}})_U^V$.

2. Let X be a topological space and let $\{A_x\}_{x\in X}$ and $\{M_x\}_{x\in X}$ be collections of rings A_x respectively A_x-modules M_x. Moreover let $\Gamma(U, \mathcal{A}) = \prod_{x\in U} A_x$ and $\Gamma(U, \mathcal{F}) = \prod_{x\in U} M_x$ for all open subsets U of X. Then $\Gamma(U, \mathcal{F})$ has a natural structure as a $\Gamma(U, \mathcal{A})$-module for all open subsets U of X. Let $(\rho_{\mathcal{A}})_U^V : \Gamma(V, \mathcal{A}) \to \Gamma(U, \mathcal{A})$ and $(\rho_{\mathcal{F}})_U^V : \Gamma(V, \mathcal{F}) \to \Gamma(U, \mathcal{F})$ be the projections.

(1) Show that $\mathcal{A}$ with the given restrictions maps is a sheaf of rings.
(2) Show that $\mathcal{F}$ with the given restriction maps is an $\mathcal{A}$-module.

3. Let X, with a fixed point x_0, be the topological space where the non-empty open subsets are all subsets of X that contain the point x_0. Let $\{A_x\}_{x\in X}$ and $\{M_x\}_{x\in X}$ be collection of rings A_x respectively A_x-modules M_x. Assume that we for all points $x \in X$ have a ring homomorphism $\varphi_x : A_x \to A_{x_0}$ such that $\varphi_{x_0} = \mathrm{id}_{A_{x_0}}$ and a φ_x-module homomorphism $u_x : M_x \to M_{x_0}$ such that $u_{x_0} = \mathrm{id}_{M_{x_0}}$. For all open subsets U of X let $\Gamma(U, \mathcal{A})$ be the subset of $\prod_{x\in U} A_x$ consisting of elements $(f_x)_{x\in U}$ such that $\varphi_x(f_x) = f_{x_0}$ for all $x \in U$, and let $\Gamma(U, \mathcal{F})$ be the subset of $\prod_{x\in U} M_x$ consisting of elements $(z_x)_{x\in U}$ such that $u_x(z_x) = z_{x_0}$ for all $x \in U$.

(1) Show that for every inclusion $U \subseteq V$ of open subsets of X the projections $\prod_{x\in V} A_x \to \prod_{x\in U} A_x$ and $\prod_{x\in V} M_x \to \prod_{x\in U} M_x$ induce restriction maps $(\rho_{\mathcal{A}})_U^V : \Gamma(V, \mathcal{A}) \to \Gamma(U, \mathcal{A})$, respectively $(\rho_{\mathcal{F}})_U^V : \Gamma(V, \mathcal{F}) \to \Gamma(U, \mathcal{F})$.
(2) Show that $\mathcal{A}$ with the restriction maps $(\rho_{\mathcal{A}})_U^V$ is a sheaf of rings on X and that $\mathcal{F}$ with the restriction maps $(\rho_{\mathcal{F}})_U^V$ is an $\mathcal{A}$-module.
(3) Describe the $\mathcal{A}_x$-module $\mathcal{F}_x$ for all points $x \in X$.
(4) Show that all sheaves of rings $\mathcal{B}$, and $\mathcal{B}$-modules $\mathcal{G}$ on X, are obtained from collections $\{B_x\}_{x\in X}$ and $\{N_x\}_{x\in X}$ of rings B_x and B_x-modules N_x in the same way as $\mathcal{A}$ and $\mathcal{F}$ are obtained from the collections $\{A_x\}_{x\in X}$ and $\{M_x\}_{x\in X}$.

4. Let X be a topological space and let $\mathcal{A}$ be a sheaf of rings on X. Moreover let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{A}$-modules.

(1) Show that the $\mathcal{A}$-modules $\mathcal{F} \otimes_{\mathcal{A}} \mathcal{A}$ and $\mathcal{F}$ are canonically isomorphic.
(2) Show that the $\mathcal{A}$-modules $\mathcal{F} \otimes_{\mathcal{A}} \mathcal{G}$ and $\mathcal{G} \otimes_{\mathcal{A}} \mathcal{F}$ are canonically isomorphic.

5. Let $\mathcal{A}$ and $\mathcal{B}$ be sheaves of rings on a toplogical spaces X. Moreover let $\mathcal{F}$ be an $\mathcal{A}$-modules and let $\mathcal{G}$ be a $\mathcal{B}$-modules. Show that there is a canonical homomorphism of groups

$$\mathrm{Hom}_{\mathcal{A}}(\mathcal{F}, \mathcal{G}_{[\varphi]}) \to \mathrm{Hom}_{\mathcal{B}}(\mathcal{F} \otimes_{\mathcal{A}} \mathcal{B}, \mathcal{G}).$$

6. Let $X = \{x, y\}$ be the topological space with two point x and y and the discret topology. Define sheaves of groups $\mathcal{F}$ and $\mathcal{G}$ by $\Gamma(X, \mathcal{F}) = \mathbf{Z}/6\mathbf{Z}$, $\Gamma(\{x\}, \mathcal{F}) = \mathbf{Z}/2\mathbf{Z}$, $\Gamma(\{y\}, \mathcal{F}) = \mathbf{Z}/3\mathbf{Z}$ with the natural residue maps as restrictions, and $\Gamma(X, \mathcal{G}) =$

$\mathbf{Z}/6\mathbf{Z}$, $\Gamma(\{x\},\mathcal{G}) = \mathbf{Z}/3\mathbf{Z}$, $\Gamma(\{y\},\mathcal{G}) = \mathbf{Z}/2\mathbf{Z}$ with the natural residue maps as restrictions. Give $\mathcal{F}$ and $\mathcal{G}$ a structure as modules over a ring $\mathcal{A}$ in such a way that the presheaf $\mathcal{H}$ of Section (?) is different from $\mathcal{F} \otimes_{\mathcal{A}} \mathcal{G}$.

7. Let X be a topological space with a basis $\mathfrak{B}$ for the topology. For every open subset U of X we consider U as a topological space with the topology induced by that of X, and we let $\mathfrak{B}_U$ be the basis for U consisting of open sets V belonging to $\mathfrak{B}$ that are contained in U.

For every presheaf $\mathcal{F}$ defined on $\mathfrak{B}$ we let $\mathcal{F}|U$ be the presheaf on $\mathfrak{B}_U$ defined by $\Gamma(V,\mathcal{F}|U) = \Gamma(V,\mathcal{F})$ for all V belonging to $\mathfrak{B}_U$ and $(\rho_{\mathcal{F}|U})_V^W = (\rho_{\mathcal{F}})_V^W$ for all inclusions $V \subseteq W$ of open sets belonging to $\mathfrak{B}_U$.

Let $\mathcal{A}$ be a presheaf of rings defined on $\mathfrak{B}$ and let $\mathcal{F}$ and $\mathcal{G}$ be presheaves of $\mathcal{A}$-modules defined on $\mathfrak{B}$. We write $\mathcal{H}om_{\mathcal{A}}(\mathcal{F},\mathcal{G})(U) = \mathrm{Hom}_{\mathcal{A}|U}(\mathcal{F}|U,\mathcal{G}|U)$ for the group of all homomorphisms of presheaves of $\mathcal{A}|U$-modules from $\mathcal{F}|U$ to $\mathcal{G}|U$, for every open subset U belonging to $\mathfrak{B}$.

(1) Show that for all inclusions $U \subseteq V$ of open sets belonging to $\mathfrak{B}$ we have a canonical map
$$\rho_U^V : \mathrm{Hom}_{\mathcal{A}|V}(\mathcal{F}|V,\mathcal{G}|V) \to \mathrm{Hom}_{\mathcal{A}|U}(\mathcal{F}|U,\mathcal{G}|U)$$
that maps a homomorphism $u : \mathcal{F}|V \to \mathcal{G}|V$ to the restriction $u|U : \mathcal{F}|U \to \mathcal{G}|U$ to U.

(2) Show that $\mathcal{H}om_{\mathcal{A}}(\mathcal{F},\mathcal{G})$ with the restriction maps $\rho_U^V : \mathcal{H}om_{\mathcal{A}}(\mathcal{F},\mathcal{G})(V) \to \mathcal{H}om_{\mathcal{A}}(\mathcal{F},\mathcal{G})(U)$ for all inclusions $U \subseteq V$ of open subsets belonging to $\mathfrak{B}$ is a presheaf of $\mathcal{A}$-modules on $\mathfrak{B}$.

(3) Show that there is a canonical homomorphism
$$\mathcal{H}om_{\mathcal{A}}(\mathcal{A},\mathcal{G}) \to \mathcal{G}.$$

(4) Show that for all $x \in X$ we have a canonical homomorphism of stalks
$$\mathcal{H}om_{\mathcal{A}}(\mathcal{F},\mathcal{G})_x \to \mathrm{Hom}_{\mathcal{A}_x}(\mathcal{F}_x,\mathcal{G}_x)$$
that maps the class of a pair (U,u), where $u : \mathcal{F}|U \to \mathcal{G}|U$ is a homomorphism of presheaves, to the map $u_x : \mathcal{F}_x \to \mathcal{G}_x$.

(5) Show that when $\mathcal{A}$, $\mathcal{F}$ and $\mathcal{G}$ are sheaves on $\mathfrak{B}$ then $\mathcal{H}om_{\mathcal{A}}(\mathcal{F},\mathcal{G})$ is an $\mathcal{A}$-module on $\mathfrak{B}$.

(6) Let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{A}$-modules. Assume that there is a neighbourhood U of x and an exact sequence
$$\mathcal{A}^m|U \to \mathcal{A}^n|U \to \mathcal{F}|U \to 0$$
of $\mathcal{A}$-modules. Show that then the homomorphism
$$\mathcal{H}om_{\mathcal{A}}(\mathcal{F},\mathcal{G})_x \to \mathrm{Hom}_{\mathcal{A}_x}(\mathcal{F}_x,\mathcal{G}_x)$$

is an isomorphism.

(7) Let $0 \to \mathcal{G}' \to \mathcal{G} \to \mathcal{G}''$ be an exact sequence of $\mathcal{A}$-modules, and let $\mathcal{F}$ be an $\mathcal{A}$-module. Show that the sequence of $\mathcal{A}$-modules

$$0 \to \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G}') \to \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G}'')$$

is exact.

(8) Let $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be an exact sequence of $\mathcal{A}$-modules, and let $\mathcal{G}$ be an $\mathcal{A}$-module. Show that the sequence of $\mathcal{A}$-modules

$$0 \to \mathcal{H}om_{\mathcal{A}}(\mathcal{F}'', \mathcal{G}) \to \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om_{\mathcal{A}}(\mathcal{F}', \mathcal{G})$$

is exact.

(9) Let $X = \{x_0, x_1\}$ be the topological space with open sets $\{\emptyset, \{x_0\}, X\}$, and let $\mathcal{A}$ be the simple sheaf with stalks $\mathbf{Z}$. Moreover let $\mathcal{F}$ be the sheaf defined by $\Gamma(X, \mathcal{F}) = \{0\} = \Gamma(\emptyset, \mathcal{F})$ and $\Gamma(\{x_0\}, \mathcal{F}) = \mathbf{Z}$, and let $\mathcal{G}$ be the sheaf defined by $\Gamma(X, \mathcal{G}) = \mathbf{Z}$ and $\Gamma(\{x_0\}, \mathcal{G}) = (0) = \Gamma(\emptyset, \mathcal{G})$, both with the only possible restriction maps. Finally let $\mathcal{H}$ be the sheaf defined by $\mathcal{H}(\emptyset) = \{0\}$ and $\mathcal{H}(X) = \mathbf{Z} = \mathcal{H}(\{x_0\})$ and with $(\rho_{\mathcal{H}})^X_{x_0} = \mathrm{id}_{\mathbf{Z}}$.

(a) Show that $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{A}$-modules.

(b) Show that the map

$$\mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{F})_{x_1} \to \mathrm{Hom}_{\mathcal{A}_{x_1}}(\mathcal{F}_{x_1}, \mathcal{F}_{x_1})$$

is not injective.

(c) Show that the map

$$\mathcal{H}om_{\mathcal{A}}(\mathcal{G}, \mathcal{H})_{x_1} \to \mathrm{Hom}_{\mathcal{A}_{x_1}}(\mathcal{G}_{x_1}, \mathcal{H}_{x_1})$$

is not surjective.

8. Let X be a topological space with basis $\mathfrak{B}$ for the topology. Moreover let $\mathcal{A}$ be a presheaf of rings on $\mathfrak{B}$ and let $\{\mathcal{F}_\alpha\}_{\alpha \in I}$ be a collection of presheaves of $\mathcal{A}$-modules on $\mathfrak{B}$. For every open subset U of $\mathfrak{B}$ we let $\Gamma(U, \mathcal{F}) = \prod_{\alpha \in I} \Gamma(U, \mathcal{F}_\alpha)$, and for every inclusion $U \subseteq V$ of open subsets belonging to $\mathfrak{B}$ we let $(\rho_{\mathcal{F}})^V_U = \prod_{\alpha \in I} (\rho_{\mathcal{F}_\alpha})^V_U$.

(1) Show that $\mathcal{F}$ with the restriction maps $(\rho_{\mathcal{F}})^V_U$ is a presheaf on $\mathcal{A}$-modules on $\mathfrak{B}$.

(2) For every $\alpha \in I$ and every open subset U of $\mathfrak{B}$ we have a projection map $u_U : \prod_{\alpha \in I} \Gamma(U, \mathcal{F}_\alpha) \to \Gamma(U, \mathcal{F}_\alpha)$. Show that the maps u_U for all U belonging to $\mathfrak{B}$ defines a homomorphism of presheaves of $\mathcal{A}$-modules

$$p_\alpha : \prod_{\alpha \in I} \mathcal{F}_\alpha \to \mathcal{F}_\alpha.$$

(3) Show that $\mathcal{F}$ with the projections p_α is a product of the presheaves $\mathcal{F}_\alpha$ in the category of presheaves of $\mathcal{A}$-modules.
(4) Show that when $\mathcal{A}$ is a sheaf all the presheaves $\mathcal{F}_\alpha$ are $\mathcal{A}$-modules then $\mathcal{F}$ is an $\mathcal{A}$-module. We denote this $\mathcal{A}$-module by $\prod_{\alpha \in I} \mathcal{F}_\alpha$ and call it the *product* of the sheaves $\mathcal{F}_\alpha$.
(5) Show that the $\mathcal{A}$-module $\prod_{\alpha \in I} \mathcal{F}_\alpha$ together with the projections p_α is a product of the $\mathcal{A}$-modules $\mathcal{F}_\alpha$ in the category of $\mathcal{A}$-modules.
(6) For every $x \in X$ and for every $\alpha \in I$ we have a map $(p_\alpha)_x : (\prod_{\alpha \in I} \mathcal{F}_\alpha)_x \to (\mathcal{F}_\alpha)_x$. Show that these maps, for all $\alpha \in I$, give a map of $\mathcal{A}_x$-modules

$$(\prod_{\alpha \in I} \mathcal{F}_\alpha)_x \to (\mathcal{F}_\alpha)_x.$$

9. Let X be a topological space and let $\mathcal{A}$ be a sheaf of rings on X. Moreover let $\{\mathcal{F}_\alpha\}_{\alpha \in I}$ be a collection of $\mathcal{A}$-modules.

(1) Show that $\mathrm{Supp}(\oplus_{\alpha \in I} \mathcal{F}_\alpha) = \cup_{\alpha \in I} \mathrm{Supp}(\mathcal{F}_\alpha)$.
(2) Is it true that $\mathrm{Supp}(\prod_{\alpha \in I} \mathcal{F}_\alpha) = \cup_{\alpha \in I} \mathrm{Supp}(\mathcal{F}_\alpha)$?

2. Quasi-coherent modules.

(2.1) Sections. Let $(X, \mathcal{O}_X)$ be a ringed space and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. An $\mathcal{O}_X$-module homomorphism $u : \mathcal{O}_X \to \mathcal{F}$ gives a section $s = u_X(1)$ in $\Gamma(X, \mathcal{F})$. Conversely a section $s \in \Gamma(X, \mathcal{F})$ defines an $\mathcal{O}_X$-module homomorphism $u : \mathcal{O}_X \to \mathcal{F}$ determined on each open subsets U of X, and every section $t \in \Gamma(U, \mathcal{O}_X)$, by $u_U(t) = t(\rho_{\mathcal{F}})^X_U(s)$. We call u the *homomorphism induced* by the section s.

In this way we obtain a bijection between the sections of $\Gamma(X, \mathcal{F})$ and the $\mathcal{O}_X$-module homomorphisms $\mathcal{O}_X \to \mathcal{F}$. Let I be a collection of indices. We obtain a bijection between $\mathcal{O}_X$-module homomorphisms $u : \mathcal{O}_X^{(I)} \to \mathcal{F}$ and families of sections $(s_\alpha)_{\alpha \in I}$ of $\Gamma(X, \mathcal{F})$. Under this bijection the homomorphism $u : \mathcal{O}_X^{(I)} \to \mathcal{F}$ corresponds to the family $(uh_\alpha)_{\alpha \in I}$ where $uh_\alpha : \mathcal{O}_X \to \mathcal{F}$ is the composite map of u with the canonical homomorphisms $h_\alpha : \mathcal{O}_X \to \mathcal{O}_X^{(I)}$ to factor α for all $\alpha \in I$.

(2.2) Definition. Let $(X, \mathcal{O}_X)$ be a ringed space and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. We say that a collection of section $(s_\alpha)_{\alpha \in I}$ of $\Gamma(X, \mathcal{F})$ *generate* $\mathcal{F}$ if the resulting homomorphism $\mathcal{O}_X^{(I)} \to \mathcal{F}$ is surjective. The sheaf $\mathcal{F}$ is *generated by global section* over X if it is generated by a collection of sections of $\Gamma(X, \mathcal{F})$. That is, there is a surjection $\mathcal{O}_X^{(I)} \to \mathcal{F}$ for some family I of indices.

→ **(2.3) Remark.** It follows from Lemma (?) and Lemma (?) that $\mathcal{F}$ is generated by the collection of sections $(s_\alpha)_{\alpha \in I}$ with $s_\alpha \in \Gamma(X, \mathcal{F})$ if and only if the $\mathcal{O}_x$-module $\mathcal{F}_x$ is generated by the elements $(s_\alpha)_x$ of the collection $((s_\alpha)_x)_{\alpha \in I}$ for all $x \in X$.

(2.4) Example. Not all modules are generated by their global sections.

Let $X = \{x_0, x_1\}$ be the topological space with open sets $\{\emptyset, X, \{x_0\}\}$. Moreover let $\mathcal{O}_X$ be the simple sheaf with fibers $\mathbf{Z}$, and let $\mathcal{F}$ be the submodule of $\mathcal{O}_X$ defined by $\Gamma(\emptyset, \mathcal{F}) = \{0\} = \Gamma(X, \mathcal{F})$, and $\Gamma(\{x_0\}, \mathcal{F}) = \mathbf{Z}$ with $(\rho_{\mathcal{F}})^V_U = 0$ for all $U \subseteq V$. Since $\mathcal{F}(X) = \{0\}$ and $\mathcal{F}_{x_0} = \mathbf{Z}$ the $\mathcal{O}_X$-module $\mathcal{F}$ can not be generated by global sections.

(2.5) Definition. Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-module $\mathcal{F}$ is *quasi-coherent* if there, for all $x \in X$, is an open neighborhood U of x such that $\mathcal{F}|U$ is the cokernel of a homomorphism $\mathcal{O}_X^{(I)}|U \to \mathcal{O}_X^{(J)}|U$ of $\mathcal{O}_X$-modules for some collections of indices I and J.

An $\mathcal{O}_X$-algebra $\mathcal{A}$ is *quasi-coherent* if it is quasi-coherent as an $\mathcal{O}_X$-module.

(2.6) Example. We have that $\mathcal{O}_X$ is quasi-coherent, and every direct sum of quasi-coherent $\mathcal{O}_X$-modules is quasi-coherent.

(2.7) Example. Even residue modules of $\mathcal{O}_X$ are not necessarily quasi-coherent.

→ Let $\mathcal{O}_X$ and $\mathcal{F}$ be as in Example (2.4). We have that the sections of $\mathcal{O}_X/\mathcal{F}$ are given by $\Gamma(\emptyset, \mathcal{O}_X/\mathcal{F}) = \{0\} = \Gamma(\{x_0\}, \mathcal{O}_X/\mathcal{F})$ and $\Gamma(X, \mathcal{O}_X/\mathcal{F}) = \mathbf{Z}$. The sheaf $\mathcal{O}_X/\mathcal{F}$ is not coherent because if $\mathcal{O}_X/\mathcal{F}$ were the cokernel of $u : \mathcal{O}_X^{(I)} \to \mathcal{O}_X^{(J)}$ it follows from Theorem (1.20) that we would have an exact sequence $\mathcal{O}_{X,x_i}^{(I)} \xrightarrow{u_{x_i}} \mathcal{O}_{X,x_i}^{(J)} \to \mathcal{F}_{x_i} \to 0$ for i=1,2. For $i = 0$ we obtain the exact sequence $\mathbf{Z}^{(I)} \xrightarrow{u_{x_0}} \mathbf{Z}^{(J)} \to \mathbf{Z} \to 0$, and for $i = 1$ we obtain the exact sequence $\mathbf{Z}^{(I)} \xrightarrow{u_{x_1}} \mathbf{Z}^{(J)} \to 0$. This is however impossible since $(\rho_{\mathcal{O}_X})_{x_0}^X = \mathrm{id}_{\mathbf{Z}}$ and thus $u_{x_0} = u_{x_1}$.

(2.8) Proposition. *Let $\Psi = (\psi, \theta) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a homomorphism of ringed spaces and let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_Y$-module. Then the $\mathcal{O}_X$-module $\Psi^*(\mathcal{G})$ is quasi-coherent.*

Proof. Let x be a point of X and let $y = \psi(x)$. Since the sheaf $\mathcal{F}$ is quasi-coherent there is an open neighbourhood V of y and an exact sequence of $(\mathcal{O}_Y|V)$-modules $\mathcal{O}_Y^{(I)}|V \to \mathcal{O}_Y^{(J)}|V \to \mathcal{F}|V \to 0$. Let $U = \psi^{-1}(V)$. Then U is a neighbourhood of x and it follows from Proposition (1.22) that there is an exact sequence of $(\Psi^*(\mathcal{O}_Y)|U)$-modules $\Psi^*(\mathcal{O}_Y^{(I)})|U \to \Psi^*(\mathcal{O}_Y^{(J)})|U \to \Psi^*(\mathcal{G})|U \to 0$. It follows from Proposition (1.28) that we obtain an exact sequence $\Psi^*(\mathcal{O}_Y)^{(I)}|U \to \Psi^*(\mathcal{O}_Y)^{(J)}|U \to \Psi^*(\mathcal{G})|U$. Since $\Psi^*(\mathcal{O}_Y) = \mathcal{O}_X$ we have that $\Psi^*(\mathcal{G})$ is quasi coherent.

(2.9) Definition. Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-modules $\mathcal{F}$ is of *finite type* it there for every $x \in X$ is an open neighbourhood U of x such that $\mathcal{F}|U$ is generated by a finite collection of sections of $\Gamma(U, \mathcal{F})$. That is, we have a surjection $(\mathcal{O}_X|U)^n \to \mathcal{F}|U$of $\mathcal{O}_X$-modules.

(2.10) Example. The sheaf $\mathcal{O}_X$ is of finite type. Every quotient module of a sheaf of finite type is of finite type. A finite direct sum of modules of finite type is of finite type, and it follows from Lemma (1.9) and Lemma (Sheaves 2.12) that the tensor product of two modules of finite type is of finite type.

(2.11) Remark. When $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite type the support $\mathrm{Supp}(\mathcal{F})$ is a closed subset of X. In fact if $x \notin \mathrm{Supp}(\mathcal{F})$ we have that $\mathcal{F}_x = 0$. Since $\mathcal{F}$ is of finite type we can find an open neighbourhood V of x and sections $s_1, s_2, \dots, s_n$ in $\Gamma(V, \mathcal{F})$ that generate the $\mathcal{O}_{X,y}$-module $\mathcal{F}_y$ for all $y \in U$. Since $(s_i)_x = 0$ in $\mathcal{F}_x$ there is a neighbourhood U_i of x contained in V such that $(s_i)_y = 0$ for $y \in U_i$. Let $U = \cap_{i=1}^n U_i$. Then U is an open neighbourhood of x and $\mathcal{F}_y = 0$ for all $y \in U$, and thus the complement of $\mathrm{Supp}(\mathcal{F})$ is open in X.

(2.12) Definition. Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-modules $\mathcal{F}$ is *coherent* if it satisfies the following two conditions:

(1) It is of finite type.
(2) For every open subset U of X the kernel of each homomorphism $\mathcal{O}_X^n|U \to \mathcal{F}|U$ of $\mathcal{O}_X$-modules is of finite type.

An $\mathcal{O}_X$-algebra $\mathcal{A}$ is *coherent* if it is coherent as an $\mathcal{O}_X$-module.

(2.13) Example. A coherent module is of finite type, and it is quasi-coherent.

(2.14) Remark. A submodule $\mathcal{F}$ of finite type of a coherent sheaf $\mathcal{G}$ is coherent. In fact, every homomorphism $\mathcal{O}_X^n|U \to \mathcal{F}$ gives a map $\mathcal{O}_X^n|U \to \mathcal{G}$ with the same kernel. Since $\mathcal{G}$ is coherent the kernel is of finite type.

(2.15) Theorem. *Let $(X, \mathcal{O}_X)$ be a ringed space and let*

$$0 \to \mathcal{F} \xrightarrow{u} \mathcal{G} \xrightarrow{v} \mathcal{H} \to 0$$

be an exact sequence of $\mathcal{O}_X$-modules. When two of the three modules $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ are coherent, then the third module is coherent.

Proof. (1) Assume that the modules $\mathcal{G}$ and $\mathcal{H}$ are coherent. Let $w : \mathcal{O}_X^n|U \to \mathcal{G}$ be a surjective $\mathcal{O}_X$-modules homomorphism. We obtain a commutative diagram of $(\mathcal{O}_X|U)$-modules with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}((v|U)w) \arrow[r] \arrow[d] & \mathcal{O}_X^n|U \arrow[r, "(v|U)w"] \arrow[d, "w"'] & \mathcal{H}|U \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{F}|U \arrow[r, "u|U"'] & \mathcal{G}|U \arrow[r, "v|U"'] & \mathcal{H}|U \arrow[r] & 0,
\end{tikzcd}$$

where the left vertical homomorphism is induced by w. The left vertical homomorphism is surjective, and $\mathrm{Ker}((v|U)w)$ is of finite type since $\mathcal{H}$ is coherent. Hence $\mathcal{F}$ is of finite type. Since $\mathcal{G}$ is quasi-coherent it follows from Remark (2.14) that $\mathcal{F}$ is coherent.

(2) Assume that $\mathcal{F}$ and $\mathcal{G}$ are coherent modules. Since $\mathcal{G}$ is of finite type we have that $\mathcal{H}$ is of finite type. Hence it remains to prove that the second condition of Definition (2.12) is fulfilled for $\mathcal{H}$.

Let $w : \mathcal{O}_X^n|U' \to \mathcal{H}$ be a homomorphism of $(\mathcal{O}_X|U)$-modules, and let $t_1, t_2, \dots, t_n$ in $\Gamma(U', \mathcal{H})$ be sections that define w. Since $\mathcal{F}$ is of finite type we can find an open neighbourhood V of x contained in U' and a surjective homomorphism $r : \mathcal{O}_X^m|V \to \mathcal{F}$. The map $v_x : \mathcal{G}_x \to \mathcal{H}_x$ is surjective. Hence we can, for each point $x \in X$, find an open neighbourhood W of x contained in V and sections $s_1, s_2, \dots, s_n$ in $\Gamma(W, \mathcal{G})$ such that $v_x((s_i)_x) = (t_i)_x$ for $i = 1, 2, \dots, n$. Then the pairs $(W, v_W(s_i))$ and $(W, (\rho_{\mathcal{H}})_W^{U'}(t_i))$ define the same class in $\mathcal{H}_x$. We can therefore find an open neighbourhood U of x contained in W such that $v_U((\rho_{\mathcal{G}})_U^W(s_i)) = (\rho_{\mathcal{H}})_U^{U'}(t_i)$ for $i = 1, 2, \dots, n$. The sections $s_1, s_2, \dots, s_n$ define a homomorphism $s : \mathcal{O}_X^n|U \to \mathcal{G}$ such that $(w|U) = (v|U)(s|U)$. Together with $(u|U)(r|U)$ we the homomorphism s defines a homomorphism $q = ((u|U)(r|U) + s) : \mathcal{O}_X^m|U \oplus \mathcal{O}_X^n|U \to \mathcal{G}|U$ such that we obtain a comutative of $\mathcal{O}_X|U$-modules with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_X^m|U \arrow[r, "h|U"] \arrow[d, "r|U"'] & \mathcal{O}_X^m|U \oplus \mathcal{O}_X^n|U \arrow[r, "p"] \arrow[d, "q"'] & \mathcal{O}_X^n|U \arrow[r] \arrow[d, "w|U"'] & 0 \\
0 \arrow[r] & \mathcal{F}|U \arrow[r, "u|U"] & \mathcal{G}|U \arrow[r, "v|U"] & \mathcal{H}|U \arrow[r] & 0,
\end{tikzcd} \tag{2.15.1}$$

where h is the canonical homomorphism to the first factor and p is the projection to the second factor. Since $\mathcal{G}$ is coherent the kernel of the homomorphism q is of finite type. Since $r|U$ is surjective this kernel maps by p onto the kernel of the homomorphism $w|U$. Hence the kernel of $w|U$ is finitely generated. We have proved that the module $\mathcal{H}$ satisfies the second condition of Definition (2.12), and consequently that $\mathcal{H}$ is a coherent $\mathcal{O}_X$-module.

(3) Assume that the modules $\mathcal{F}$ and $\mathcal{H}$ are coherent. As in the case when $\mathcal{F}$ and $\mathcal{G}$ were coherent we can construct a commutative diagram (2.15.1). Since $\mathcal{H}$ is assumed to be coherent, and thus of finite type, we can choose the homomorphism w in the diagram to be surjective. Then the homomorphism q is surjective hence $\mathcal{G}$ is of finite type.

It remains to prove that the second condition of Definition (2.12) holds for the module $\mathcal{G}$. Let $w : \mathcal{O}_X^n|U \to \mathcal{G}|U$ be a homomorphism of $\mathcal{O}_X|U$-modules. We obtain a homomorphism $(v|U)w : \mathcal{O}_X^n|U \to \mathcal{H}|U$. Since $\mathcal{H}$ is coherent there is a homomorphism $s : \mathcal{O}_X^m|U \to \mathcal{O}_X^n|U$ that maps onto the kernel of $(v|U)w$. Since $(v|U)ws = 0$ and $\mathrm{Ker}(v) = \mathrm{Im}(u)$ we have that the image of $ws : \mathcal{O}_X^m|U \to \mathcal{G}$ lies in $\mathrm{Im}(u|U)$. Moreover since $\mathcal{F}$ is coherent and is isomorphic to $\mathrm{Im}(u|U)$ we have that $\mathrm{Im}(u|U)$ is coherent. Hence $\mathrm{Ker}(ws)$ is of finite type. We have that $\mathrm{Ker}(ws)$ is mapped by $s : \mathcal{O}_X^m|U \to \mathcal{O}_X^n|U$ onto $\mathrm{Ker}(w)$. Since $\mathrm{Ker}(ws)$ is of finite type we have that $\mathrm{Ker}(w)$ is of finite type, and consequently $\mathcal{G}$ satisfies the second condition of (2.12).

(2.16) Corollary. *Let $u : \mathcal{F} \to \mathcal{G}$ be a homomorphism of coherent $\mathcal{O}_X$-modules. Then we have that* $\mathrm{Ker}(u)$, $\mathrm{Im}(u)$, *and* $\mathrm{Coker}(u)$ *are coherent $\mathcal{O}_X$-modules.*

Proof. Since u induces a surjection $\mathcal{F} \to \mathrm{Im}(u)$ and $\mathcal{F}$ is coherent we have that $\mathrm{Im}(u)$ is of finite type. However, since $\mathcal{G}$ is a coherent module, it follows from Remark (2.14) that $\mathrm{Im}(u)$ is coherent. Hence it follows from the exact sequences of Proposition (1.20) that $\mathrm{Ker}(u)$ and $\mathrm{Coker}(u)$ also are coherent modules.

(2.17) Remark. Let $(X, \mathcal{O}_X)$ be a ringed space and let $\mathcal{F}$ and $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules. We have an exact sequence of $\mathcal{O}_X$-modules $0 \to \mathcal{F} \xrightarrow{h} \mathcal{F} \oplus \mathcal{G} \xrightarrow{p} \mathcal{G} \to 0$ where h is the canonical map to the first factor, and p is the projection to the second factor. It follows from Theorem (2.15) that $\mathcal{F} \oplus \mathcal{G}$ is coherent.

Assume that $\mathcal{F}$ and $\mathcal{G}$ are submodules of a coherent $\mathcal{O}_X$-modules $\mathcal{H}$. The inclusion maps $\mathcal{F} \to \mathcal{H}$ and $\mathcal{G} \to \mathcal{H}$ define a canonical O_X-modules homomorphism $u : \mathcal{F} \oplus \mathcal{G} \to \mathcal{H}$. The image of u is called the *sum* of the submodules $\mathcal{F}$ and $\mathcal{G}$ and is denoted by !!$\mathcal{F} + \mathcal{G}$. Since $\mathcal{F} \oplus \mathcal{G}$ is of finite type we have that $\mathcal{F} + \mathcal{G}$ is of finite type. It consequently follows from Remark (2.14) that $\mathcal{F} + \mathcal{G}$ is coherent.

The residue maps $\mathcal{H} \to \mathcal{H}/\mathcal{F}$ and $\mathcal{H} \to \mathcal{H}/\mathcal{G}$ define a homomorphism of $\mathcal{O}_X$-modules $v : \mathcal{H} \to \mathcal{H}/\mathcal{F} \oplus \mathcal{H}/\mathcal{G}$. The kernel of the map v is called the *intersection* of the submodules $\mathcal{F}$ and $\mathcal{G}$ and is denoted by !!$\mathcal{F} \cap \mathcal{G}$. It follows follows from Theorem (2.15) that $\mathcal{H}/\mathcal{F}$ and $\mathcal{H}/\mathcal{G}$ are coherent. Hence the image of $\mathcal{H}$ by v is coherent by Corollay (2.16). It follows from Theorem (2.15) that the intersection $\mathcal{F} \cap \mathcal{G}$ is coherent.

In particular we see that when $\mathcal{H}$ is coherent then the intersection of two submodules of finite type is coherent.

(2.18) Example. Let $(X, \mathcal{O}_X)$ be a ringed space. Then $\mathcal{O}_X$ is not necessarily coherent, even when X consist of one point. It follows from Remark (2.17) that to give an example where $\mathcal{O}_X$ is not coherent it suffices to find a ring A and two finitely generated ideals $\mathfrak{a}$ and $\mathfrak{b}$ that have an intersection that is not finitely generated. Then we can take $X = \{x\}$ and define $\mathcal{O}_X$ by $\Gamma(X, \mathcal{O}_X) = A$.

I order to find such a ring A we consider the polynomial ring $K[u, v, t_1, t_2, \ldots]$ in the independent variables $u, v, t_1, t_2, \ldots$ over a ring K. Let $\mathfrak{c}$ be the ideal generated by the elements uv, $ut_i - vt_i$, and $t_i t_j$ for $i, j = 1, 2, \ldots$. Let $A = K[u, v, t_1, t_2, \ldots]/\mathfrak{c}$ and let x,y, and f_i be the residue classes of u, v, respectively t_i for $i = 1, 2, \ldots$. We then have relations $xy = 0 = f_i f_j$ and $xf_i = yf_i$ in A for $i, j = 1, 2, \ldots$. From the relations it follows that every element in A can be written uniquely in the form $f(x) + g(y) + \sum_{i=1}^{m} h_i(x) f_i$ where $f(u)$ and $h(u)$ are polynomials in $K[u]$, and $g(v)$ is a polynomial in $K[v]$ such that $g(0) = 0$.

We have that $x(f(x) + g(y) + \sum_{i=1}^{m} h_i(x) f_i) = xf(x) + \sum_{i=1}^{m} h_i(x) x f_i$ and that $y(f(x) + g(y) + \sum_{i=1}^{m} h_i(x) f_i) = yf(0) + yg(y) + \sum_{i=1}^{m} h_i(x) x f_i$. From these expressions we see that $(x) \cap (y) = (xf_1, xf_2, \ldots)$. The ideal $(xf_1, xf_2, \ldots)$ is not finitely generated. In fact if $\sum_{j=1}^{m_i} h_{ij}(x) x f_j$ for $i = 1, 2, \ldots, n$ were generators and m is an integer strictly greater than $m_1, m_2, \ldots, m_n$ then xf_i^m can not be in the ideal $(g_1, g_2, \ldots, g_n)$ generated by $g_1, g_2, \ldots, g_n$ since $f_i f_j = 0$.

(2.19) Exercises.

1. Let $(X, \mathcal{O}_X)$ be a ringed space. Show that when $\mathcal{F}$ and $\mathcal{G}$ are coherent $\mathcal{O}_X$-modules then $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is a coherent $\mathcal{O}_X$-module.

2. Let $(X, \mathcal{O}_X)$ be a ringed space. Show that when $\mathcal{F}$ and $\mathcal{G}$ are coherent $\mathcal{O}_X$-modules then the $\mathcal{O}_X$-module $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ defined in Excercise (?) is coherent.

3. Let $(X, \mathcal{O}_X)$ be a ringed space. Moreover let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module and let $\mathcal{I}$ be a coherent ideal in $\mathcal{O}_X$.

(1) Show that there is a canonical homomorphism $\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{F}$ of $\mathcal{O}_X$-modules.
(2) Show that the image $\mathcal{I}\mathcal{F}$ of this homomorphism is coherent.

4. Let $(X, \mathcal{O}_X)$ be a ringed space and let $\mathcal{F}$ be an $\mathcal{O}_X$-module.

(1) Show that here is a canonical homomorphism of $\mathcal{O}_X$-modules

$$\mathcal{O}_X \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$$

that maps a section $s \in \Gamma(U, \mathcal{O}_X)$ to the canonical multiplication by s in $\mathrm{Hom}(\mathcal{F}|U, \mathcal{F}|U)$ for all open subsets U of X. The kernel of this homomorphism is called the *annihilator* of $\mathcal{F}$.

(2) Show that when $\mathcal{O}_X$ and $\mathcal{F}$ are coherent $\mathcal{O}_X$-modules then the annihilator is coherent.

5. Let $(X, \mathcal{O}_X)$ be a ringed space with $\mathcal{O}_X$-coherent, and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Moreover let $x \in X$ and let M be an $\mathcal{O}_{X,x}$-submodule of the stalk $\mathcal{F}_x$. Show that there is an open neighbourhood U of x and a coherent $(\mathcal{O}_X|U)$-submodule $\mathcal{G}$ of $\mathcal{F}|U$ such that $\mathcal{G}_x = M$.

6. Let $X = \{x_0, x_1\}$ be the topological space with open sets $\emptyset$, $\{x_0\}$ and X. Moreover let $\mathcal{O}_X$ be the simple sheaf on X with fibers $\mathbf{Z}$, and let $\mathcal{F}$ be the $\mathcal{O}_X$-module defined by $\Gamma(\emptyset, \mathcal{F}) = 0 = \Gamma(X, \mathcal{F})$ and $\Gamma(\{x_0\}, \mathcal{F}) = \mathbf{Z}$ and with the restrictions being the only possible maps.

(1) Is the sheaf $\mathcal{O}_X$ quasi-coherent?
(2) Is the sheaf $\mathcal{O}_X$ of finite type?
(3) Is the sheaf $\mathcal{O}_X$ coherent?
(4) Is the sheaf $\mathcal{F}$ quasi-coherent?
(5) Is the sheaf $\mathcal{F}$ of finite type?
(6) Is the sheaf $\mathcal{F}$ coherent?

3. Modules and affine schemes.

(3.1) Change of multiplicative subsets. Let $\varphi : B \to A$ be a homomorphism of rings and let T and S be multiplicative closed subsets of B, respectively A, such that $\varphi(T) \subseteq S$. It follows from the universal property of localization that the composite map $B \to S^{-1}A$ of $\varphi : B \to A$ with $i_A^S : A \to S^{-1}A$, factors via the canonical map $i_B^T : B \to T^{-1}B$ and a unique algebra homomorphism

$$\varphi^{S,T} : T^{-1}B \to S^{-1}A.$$

Let N be an B-module and M a A-module, and let $u : N \to M$ be a φ-module homomorphism. We obtain a unique $\varphi^{S,T}$-module homomorphism

$$u^{S,T} : T^{-1}N \to S^{-1}M$$

such that $i_M^S u = u^{S,X} i_N^T$. We have that $u^{S,T}(x/s) = u(x)/\varphi(s)$. It is clear that the definition of $u^{S,T}$ is independent of the choise of representative (x, s) of the class x/s and it follows from the explicit form of the map that it is a $\varphi^{S,T}$-module homomorphism.

For every commutative diagram of A-modules

$$\begin{tikzcd} M \arrow[r, "u"] \arrow[d, "v"'] & N \arrow[d, "w"] \\ M' \arrow[r, "u'"] & N' \end{tikzcd}$$

we have a commutative diagram

$$\begin{tikzcd} T^{-1}M \arrow[r, "u^{S,T}"] \arrow[d, "T^{-1}v"'] & S^{-1}N \arrow[d, "S^{-1}w"'] \\ T^{-1}M' \arrow[r, "(u')^{S,T}"'] & S^{-1}N'. \end{tikzcd}$$

(3.2) Proposition. *Let S and T be multiplicatively closed subsets of the ring A. Assume that there for every element $s \in S$ are elements $t \in T$ and $f \in A$ such that $t = sf$. For every A-module M the following assertion shold:*

n (1) *There is a canonical homomorphism of groups!!*

$$\rho^{T,S} = (\rho_M)^{T,S} : S^{-1}M \to T^{-1}M.$$

(2) *When $t = sf$ with $s \in S$ and $t \in T$ we have that $(\rho_M)^{T,S}(x/s) = (fx)/t$.*

(3) *When s is in S, x, y are in M, and t, t' are in T and $t'(tx - sy) = 0$ in M we have that $(\rho_M)^{T,S}(x/s) = y/t$ in $T^{-1}M$.*
(4) *We have that $(\rho_A)^{T,S} : S^{-1}A \to T^{-1}A$ is a homomorphism of rings, and $(\rho_M)^{T,S}$ is a $(\rho_A)^{T,S}$-module homomorphism.*
(5) *We have that $(\rho_M)^{S,S} = \mathrm{id}_{S^{-1}M}$, and when R is a multiplicatively closed subset of A with the property that for every $r \in R$ there is an $s \in S$ and a $g \in A$ such that $s = rg$ then*

$$(\rho_M)^{R,S} = (\rho_M)^{R,T}(\rho_M)^{T,S}.$$

Proof. (1) Let $s \in S$ and $x \in M$. By assumption there is a $t \in T$ and an $f \in A$ such that $t = sf$. We let $(\rho_M)^{T,S}(x/s) = (fx)/t$. The definition is independent of the representation $t = sf$ because if $t' = sf'$ then we have that $t'fx - tf'x = (sf'f - sff')x = 0$, and hence that $(fx)/t = (f'x)/t'$. We have proved assertion (1).

(2) Assertion (2) follows from the definition of $(\rho_M)^{T,S}$

(3) To prove assertion (3) we assume that $t'(tx - sy) = 0$. By assumption there is a $t'' \in T$ and an $f \in A$ such that $t'' = sf$. Then $(\rho_M)^{T,S}(x/s) = (fx)/t''$. We must prove that $(fx)/t'' = y/t$ in $T^{-1}M$. However $t'(tfx - t''y) = ft'sy - t'sfy = 0$. That is, we have $(fx)/t'' = y/t$ in $T^{-1}M$ as we wanted to prove.

(4), (5) The remaining properties are easy to check from the explicit description of $(\rho_A)^{T,S}$ and $(\rho_M)^{T,S}$.

(3.3) Corollary. *Let f, g be elements in the ring A such that $D(f) \supseteq D(g)$ in*
n *$X = \mathrm{Spec}(A)$. For every A-module M there is a canonical homomorphism!!*

$$\rho_{g,f} = (\rho_M)_{g,f} : M_f \to M_g$$

such that $(\rho_A)_{g,f}$ is a ring homomorphism, and $(\rho_M)_{g,f}$ is a $(\rho_A)_{g,f}$-module homomorphism. when $g^m = hf$ we have that $\rho_{g,f}(x/f^n) = h^n x/g^{mn}$, and when $g^m(g^n x - f^p y) = 0$ then $\rho_{g,f}(x/f^p) = y/g^n$.

We have that $\rho_{f,f} = \mathrm{id}_{M_f}$, and for every $h \in A$ such that $D(f) \supseteq D(g) \supseteq D(h)$ we have that

$$\rho_{h,f} = \rho_{h,g}\rho_{g,f}.$$

→ *Proof.* If $D(f) \supseteq D(g)$ it follows from Proposition (?) that $g \in \mathfrak{r}(f)$. In other words there is a positive integer n and an $h \in A$ such that $g^n = fh$. This means that the multiplicatively closed subsets $S = \{1, f, f^2, \dots\}$ and $T = \{1, g, g^2, \dots\}$ of A satisfy the condition of the Proposition. Hence the Corollary follows from the Proposition.

(3.4) Corollary. *Let $\mathfrak{p}$ be a prime ideal in A, and let $f \in A \setminus \mathfrak{p}$. For every A-module*
n *M there is a canonical homomorphism!!*

$$\rho_{\mathfrak{p}}^f = (\rho_M)_{\mathfrak{p}}^f : M_f \to M_{\mathfrak{p}}$$

such that if $g \in A$ and $D(f) \supseteq D(g)$ we have that

$$\rho_{\mathfrak{p}}^{f} = \rho_{\mathfrak{p}}^{g} \rho_{g,f}.$$

Proof. Since $f \notin \mathfrak{p}$ we have that $S = \{1, f, f^2, \dots\} \subseteq A \setminus \mathfrak{p}$. The Corollary therefore follows from the Proposition with $S = \{1, f, f^2, \dots, \}$ and $T = A \setminus \mathfrak{p}$.

(3.5) Theorem. *Let A be a ring and M an A-module. Moreover let f be an element of A and $\{f_\alpha\}_{\alpha \in I}$ a family of elements of A such that $D(f) = \cup_{\alpha \in I} D(f_\alpha)$ in* $\operatorname{Spec}(A)$.

(1) *Let $x/f^n \in M_f$ be such that $\rho_{f_\alpha, f}(x/f^n) = 0$ in M_{f_α} for all $\alpha \in I$. Then $x/f^n = 0$ in M_f.*

(2) *Let $x_\alpha / f^{n_\alpha} \in M_{f_\alpha}$ be elements such that for every point $x \in D(f_\alpha) \cap D(f_\beta)$ and every neighbourhood $D(f_\gamma)$ of x contained in $D(f_\alpha) \cap D(f_\beta)$ we have that $\rho_{f_\gamma, f_\alpha}(x_\alpha / f^{n_\alpha}) = \rho_{f_\gamma, f_\beta}(x_\beta / f^{n_\beta})$. Then there is an element $x/f^n \in M_f$ such that $\rho_{f_\alpha, f}(x/f^n) = x_\alpha / f^{n_\alpha}$ for all $\alpha \in I$.*

Proof. (1) Since $D(f) \supseteq D(f_\alpha)$ it follows from Proposition (?) that there is a positive integer m_α and an element $g_\alpha \in A$ such that $f_\alpha^{m_\alpha} = g_\alpha f$. We have that $\rho_{f_\alpha, f}(x/f^n) = (g_\alpha^n x)/f_\alpha^{m_\alpha n} = 0$ in M_{f_α}. Consequently there is a positive integer q_α such that $f_\alpha^{q_\alpha} g_\alpha^n x = 0$ in M. We multiply the latter equation with f^n and obtain that $f_\alpha^{p_\alpha} x = 0$ for some positive integer p_α. It follows from Theorem (?) that there is a finite subset J of I such that $D(f) = \cup_{\beta \in J} D(f_\beta)$. Choose a positive integer p such that $p \geq p_\beta$ for all $\beta \in J$. Then $f_\beta^p x = 0$ for all $\beta \in J$. It follows from Theorem (?) that there is a positive integer m and elements $h_\beta \in A$ for $\beta \in J$ such that $f^m = \sum_{\beta \in J} h_\beta f_\beta^p$. Then $f^m x = 0$, and consequently $x/f^n = 0$ in M_f as we wanted to prove.

(2) We just observed that $D(f) = \cup_{\beta \in J} D(f_\beta)$ for a finite subset J of I. In order to prove assertion (2) it suffices to prove that there is an element $x/f^n \in M_f$ such that $\rho_{f_\beta, f}(x/f^n) = x_\beta / f_\beta^{n_\beta}$ for all $\beta \in J$. This is because for every $\alpha \in I$ we have that $D(f_\alpha) \cap D(f_\beta)$ can be covered by opens sets of the form $D(f_\gamma)$, such that $\rho_{f_\gamma, f_\alpha}(x_\alpha / f_\alpha^{n_\alpha}) = \rho_{f_\gamma, f_\beta}(x_\beta / f_\beta^{n_\beta}) = \rho_{f_\gamma, f_\beta} \rho_{f_\beta, f}(x/f^n) = \rho_{f_\gamma, f}(x/f^n) = \rho_{f_\gamma, f_\alpha} \rho_{f_\alpha, f}(x/f^n)$. Hence it follows from part (1) that $x_\alpha / f^{n_\alpha} = \rho_{f_\alpha, f}(x/f^n)$.

Moreover, when α, β are in J we obtain for every $D(f_\gamma) \subseteq D(f_\alpha) \cap D(f_\beta) = D(f_\alpha f_\beta)$ equalities $\rho_{f_\gamma, f_\alpha f_\beta} \rho_{f_\alpha f_\beta, f_\alpha}(x_\alpha / f_\alpha^{n_\alpha}) = \rho_{f_\gamma f_\alpha}(x_\alpha / f_\alpha^{n_\alpha}) = \rho_{f_\gamma f_\beta}(x_\beta / f_\beta^{n_\beta}) = \rho_{f_\gamma, f_\alpha f_\beta} \rho_{f_\alpha f_\beta, f_\beta}(x_\beta / f_\beta^{n_\beta})$. Hence it follows from part (1) that $\rho_{f_\alpha f_\beta, f_\alpha}(x_\alpha / f_\alpha^{n_\alpha}) = \rho_{f_\alpha f_\beta, f_\beta}(x_\beta / f_\beta^{n_\beta})$ in $M_{f_\alpha f_\beta}$ for all α, β in J, and we can find a positive integer m such that $(f_\alpha f_\beta)^m (f_\beta^{n_\beta} x_\alpha - f_\alpha^{n_\alpha} x_\beta) = 0$ in M for all α, β in J. It follows from Theorem (?) that there is a positive integer n and elements $g_\beta \in A$ for $\beta \in J$ such that $f^n = \sum_{\beta \in J} g_\beta f_\beta^{m + n_\beta}$. Let $x = \sum_{\beta \in J} g_\beta f_\beta^m x_\beta$. Then we have, for all α, β in J, the equalities $f_\alpha^{m + n_\alpha} x = \sum_{\beta \in J} g_\beta f_\alpha^{m + n_\alpha} f_\beta^m x_\beta = \sum_{\beta \in J} g_\beta f_\beta^{m + n_\beta} f_\alpha^m x_\alpha = f^n f_\alpha^m x_\alpha$. It follows from Proposition (?) that $\rho_{f_\alpha, f}(x/f^n) = x_\alpha / f_\alpha^{n_\alpha}$ in M_{f_α} as we wanted to prove.

(3.6) Sheaves associated to modules and rings. Let A be a ring and M an A-module, and let $X = \operatorname{Spec}(A)$. To every open subset $D(f)$ in X we associate the
→ localized module M_f. It follows from Proposition (?) that whenever $D(f) \supseteq D(g)$
there is a canonical map $\rho_{g,f} : M_f \to M_g$ and that we in this way we obtain a presheaf on the basis $\{D(f)\}_{f \in A}$ with restrictions $\rho_{g,f} : M_f \to M_g$ when $D(f) \supseteq D(g)$. It
→ follows from Theorem (?) that this presheaf is in fact a sheaf on $\{D(f)\}_{f \in A}$. We
~~n~~ denote the associated sheaf on X by !!$\widetilde{M}$. Hence it follows from Remark (?) that
$\Gamma(D(f), \widetilde{M}) = M_f$, and in particular that $\Gamma(X, \widetilde{M}) = \Gamma(D(1), \widetilde{M}) = M$.

It is clear that $\widetilde{M}$ is a sheaf of groups and that $\widetilde{A}$ is a sheaf of rings, and it follows
→ from Proposition (?) that $\widetilde{M}$ is an $\widetilde{A}$-module.

n **(3.7) Notation.** For all points $x \in X = \operatorname{Spec}(A)$ we let !!$M_x = M_{\mathfrak{j}_x}$ and we let
n !!$\rho_x^f : M_f \to M_x$ be the canonical map.

(3.8) Lemma. *There is a canonical isomorphism of groups $M_x \to \widetilde{M}_x$ such that the map $\rho_x^{D(f)} : \widetilde{M}(D(f)) \to \widetilde{M}_x$ corresponds to the localization map $\rho_x^f : M_f \to M_x$.*

n *In particular we obtain a homomorphism !!$\varphi_x : A_x \to \widetilde{A}_x$. The latter homomorphism is an isomorphism of rings, and the isomorphism $M_x \to \widetilde{M}_x$ is an φ_x-module homomorphism.*

Proof. Let $\mathfrak{j}_x = \mathfrak{p}$. We have that $\widetilde{M}_x$ consist of equivalence classes $(D(f), y/f^n)$ with $y/f^n \in M_f$ and $f \notin \mathfrak{p}$. Moreover $(D(f), y/f^n) \sim (D(g), z/g^m)$ if there is an $h \notin \mathfrak{p}$ such that $\rho_{h,f}(y/f^n) = \rho_{h,g}(z/g^m)$ in M_h, that is when $h^p(g^m y - f^n z) = 0$ in M for some positive integer p. Let $M_x \to \widetilde{M}_x$ be the map that maps y/s with $s \notin \mathfrak{p}$ to the class of $(D(s), y/s)$. The definition is independent of the representative (y, s) of the class of y/s because if $y/s = z/t$ then there is an $r \notin \mathfrak{p}$ such that $r(ty - sz) = 0$. Consequently $(D(s), y/s)$ and $(D(t), z/t)$ are both in the same class as $(D(rst), (rty)/(rst)) = (D(rst), (rsz)/(rst))$. The map is surjective because the class represented by $(D(f), y/f^n)$ is the image of the element $y/f^n \in M_x$. It is also injective because if y/s in M_x is mapped to zero, then the pair $(D(s), y/s)$ is equivalent to the class of $(D(t), 0)$ for some $t \notin \mathfrak{p}$. Hence $t^n y = 0$ for some positive integer n, and thus $y/s = 0$ in M_x.

(3.9) Remark. Let $u : M \to N$ be a homomorphism of A-modules. It follows from
→ Proposition (?) that the maps $u_f : M_f \to N_f$ for all $f \in A$ induce a homomorphism
$\widetilde{u} : \widetilde{M} \to \widetilde{N}$ of $\widetilde{A}$-modules. This homomorphism is uniquely determined by $\widetilde{u}(D(f)) =$
→ u_f for all $f \in A$. It follows from Proposition (?) that $\widetilde{\mathrm{id}_M} = \mathrm{id}_{\widetilde{M}}$ and that when
$v : N \to P$ is a map of A-modules we have that $\widetilde{vu} = \widetilde{v}\widetilde{u}$. In other words the correspondence that associates the $\widetilde{A}$-module $\widetilde{M}$ to the A-module M is a functor from A-modules to $\widetilde{A}$-modules.

(3.10) Proposition. *The map $\operatorname{Hom}_A(M, N) \to \operatorname{Hom}_{\widetilde{A}}(\widetilde{M}, \widetilde{N})$ that sends u to $\widetilde{u}$, is a bijection.*

n *Proof.* We shall show that the map !!$\Gamma : \mathrm{Hom}_{\widetilde{A}}(\widetilde{M}, \widetilde{N}) \to \mathrm{Hom}_A(M, N)$ wich sends $v : \widetilde{M} \to \widetilde{N}$ to $v_{D(1)} : M \to N$ is an inverse of the map of the Proposition. Let $u : M \to N$ be a homomorphism of A-modules. It follows from the definition of $\widetilde{u}$ that $\Gamma(\widetilde{u}) = \widetilde{u}_{D(1)} = u$. Hence it remains to prove that $v = \widetilde{\Gamma(v)}$ for all $\widetilde{A}$-module homomorphisms $v : \widetilde{M} \to \widetilde{N}$. For all $f \in A$ we have a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{v_{D(1)} = \Gamma(v)} & N \\ {\scriptstyle (\rho_{\widetilde{M}})_{f,1}}\big\downarrow & & \big\downarrow{\scriptstyle (\rho_{\widetilde{N}})_{f,1}} \\ M_f & \xrightarrow{v_{D(f)}} & N_f . \end{array}$$

By the definition of $\rho_{\widetilde{M}}$ we have that $(\rho_{\widetilde{M}})_{f,1} = i_M^f$ and $(\rho_{\widetilde{N}})_{f,1} = i_N^f$. Hence it
→ follows from (3.1) that $v_{D(f)} = \Gamma(v)_f$, and consequently that $v = \widetilde{\Gamma(v)}$.

(3.11) Proposition. *Let A be a ring and let*

$$M \xrightarrow{u} N \xrightarrow{v} P$$

be an exact sequence of A-modules. Then

$$\widetilde{M} \xrightarrow{\widetilde{u}} \widetilde{N} \xrightarrow{\widetilde{v}} \widetilde{P} \tag{3.11.1}$$

is an exact sequence of $\widetilde{A}$-modules

→ *Proof.* For all points $x \in \mathrm{Spec}(A)$ it follows from Lemma (3.8) that the sequence
→ (3.11.1) gives rise to an exact sequence $M_x \to N_x \to P_x$ of A_x-modules. It follows
→ from Lemma (3.8) that $\widetilde{M}_x \to \widetilde{N}_x \to \widetilde{P}_x$ is exact. Hence it follows from Theorem
→ (1.20) that the sequence (3.11.1) is exact.

(3.12) Proposition. *Let A be a ring and let $u : M \to N$ be a homomorphism of A-modules.*

(1) *The $\widetilde{A}$-modules $\widetilde{\mathrm{Ker}(u)}$, $\widetilde{\mathrm{Im}(u)}$, and $\widetilde{\mathrm{Coker}(u)}$ associated to the A-modules* $\mathrm{Ker}(u)$, $\mathrm{Im}(u)$, *respectively* $\mathrm{Coker}(u)$ *are the $\widetilde{A}$-modules* $\mathrm{Ker}(\widetilde{u})$, $\mathrm{Im}(\widetilde{u})$, *respectively* $\mathrm{Coker}(\widetilde{u})$.

   In particular we have that u is injective, surjective, or bijective, if and only if $\widetilde{u}$ is injective, surjective, respectively bijective.

(2) *Let $\{M_\alpha\}_{\alpha \in I}$ be a collection of A-modules. Then there is a canonical isomorphism of $\widetilde{A}$-modules*

$$\oplus_{\alpha \in I} \widetilde{M_\alpha} \xrightarrow{\sim} \widetilde{\oplus_{\alpha \in I} M_\alpha} \tag{3.12.1}$$

that composed with the canonical homomorphism $\widetilde{M_\beta} \to \oplus_{\alpha \in I} \widetilde{M_\alpha}$ is the homomorphism $\widetilde{h_\beta} : \widetilde{M_\beta} \to \widetilde{M}$ where $h_\beta : M_\beta \to \oplus_{\alpha \in I} M_\alpha = M$ is the canonical homomorphism to factor β.

Proof. (1) It follows from Proposition (3.11), applied to the sequences

$$0 \to \mathrm{Ker}(u) \to M \to \mathrm{Im}(u) \to 0$$

$$0 \to \mathrm{Im}(u) \to N \to \mathrm{Coker}(u) \to 0$$

obtained in Remark (?) from the factorization $M \to \mathrm{Im}(u) \to N$ of u, that we have exact sequences

$$0 \to \widetilde{\mathrm{Ker}(u)} \to \widetilde{M} \to \widetilde{\mathrm{Im}(u)} \to 0 \tag{3.12.2}$$

$$0 \to \widetilde{\mathrm{Im}(u)} \to \widetilde{N} \to \widetilde{\mathrm{Coker}(u)} \to 0 \tag{3.12.3}$$

obtained from the factorization $\widetilde{M} \to \widetilde{\mathrm{Im}(u)} \to \widetilde{N}$ of $\widetilde{u}$. From the sequences (3.12.2) and (3.12.3) it follows that $\widetilde{\mathrm{Ker}(u)} = \mathrm{Ker}(\widetilde{u})$, $\widetilde{\mathrm{Im}(u)} = \mathrm{Im}(\widetilde{u})$, and $\widetilde{\mathrm{Coker}(u)} = \mathrm{Coker}(\widetilde{u})$.

The last part of assertion (1) is clear.

(2) The canonical map $h_\beta : M_\beta \to \sum_{\alpha\in I} M_\alpha$ gives a canonical map $\widetilde{h_\beta} : \widetilde{M_\beta} \to \widetilde{\sum_{\alpha\in I} M_\alpha}$ of $\widetilde{A}$-modules. Consequently we obtain a homomorphism $\oplus_{\alpha\in I}\widetilde{M_\alpha} \to \widetilde{\oplus_{\alpha\in I} M_\alpha}$. We have thus constructed the map (3.12.1).

It remains to prove that the map (3.12.1) is an isomorphism. For every point $x \in \mathrm{Spec}(A)$ it follows from Remark (1.26) and from Lemma (3.8) that we have canonical isomorphisms $(\oplus_{\alpha\in I}\widetilde{M_\alpha})_x \xrightarrow{\sim} \oplus_{\alpha\in I}\,(\widetilde{M_\alpha})_x \xrightarrow{\sim} \oplus_{\alpha\in I}\,(M_\alpha)_x$, and also the canonical isomorphims $(\widetilde{\oplus_{\alpha\in I} M_\alpha})_x \xrightarrow{\sim} (\oplus_{\alpha\in I} M_\alpha)_x \xrightarrow{\sim} \oplus_{\alpha\in I}\,(M_\alpha)_x$. It is clear from Lemma (3.8) and Proposition (Modules 3.20) that via these isomorphisms the homomorphism (3.12.1) gives the identity map on $\oplus_{\alpha\in I}(M_\alpha)_x$. Hence the map $(\oplus_{\alpha\in I}\widetilde{M_\alpha})_x \to \oplus_{\alpha\in I}\widetilde{(M_\alpha)_x}$ is an isomorphism for all $x \in X$. It follows from Theorem (1.20) that the map (3.12.1) is an isomorphism.

(3.13) Proposition. *Let A be a ring and let M and N be A-modules. The $\widetilde{A}$-module associated to the A-module $M \otimes_A N$ is canonically isomorphic to $\widetilde{M} \otimes_{\widetilde{A}} \widetilde{N}$.*

Proof. We have that the sheaf $\widetilde{M} \otimes_{\widetilde{A}} \widetilde{N}$ is the sheaf associated to the presheaf $\mathcal{F}$ whose sections over the open subset U of X is $\Gamma(U,\mathcal{F}) = \Gamma(U,\widetilde{M}) \otimes_{\Gamma(U,\widetilde{A})} \Gamma(U,\widetilde{N})$. It follows from (?) that $\Gamma(D(f),\mathcal{F}) = M_f \otimes_{A_f} N_f$. However it follows from Proposition (3.6) and Proposition (?) that we have canonical isomorphisms $M_f \otimes_{A_f} N_f \xrightarrow{\sim} M \otimes_A A_f \otimes_{A_f} N_f \xrightarrow{\sim} M \otimes_A N_f \xrightarrow{\sim} M \otimes_A N \otimes_A A_f \xrightarrow{\sim} (M \otimes_A N)_f = \Gamma(D(f), \widetilde{M \otimes_A N})$. We consequently have a canonical isomorphism

$$\Gamma(D(f),\mathcal{F}) \xrightarrow{\sim} \Gamma(D(f), \widetilde{M \otimes_A N}).$$

It is clear that these isomorphisms for $f \in A$ are compatible with the restriction maps $(\rho_{\mathcal{F}})^{D(f)}_{D(g)}$ and $(\rho_{\widetilde{M\otimes_A N}})^{D(f)}_{D(g)}$ for all f, g in A such that $D(f) \supseteq D(g)$. Consequently we have a canonical isomorphism $\mathcal{F} \xrightarrow{\sim} \widetilde{M \otimes_A N}$.

(3.14) Criteria for quasi-coherence. Let A be a ring and let $X = \operatorname{Spec}(A)$. Moreover let U be an open subset U of X, and let $\mathcal{F}$ be an $\widetilde{A}|U$-module. For every element $f \in A = \Gamma(X, \widetilde{A})$ we have the two conditions:

$d(f, U)$:

(1) For every section $s \in \Gamma(D(f) \cap U, \mathcal{F})$ there is an integer $n \geq 0$ such that $f^n s$ can be *extended* to a section of $\Gamma(U, \mathcal{F})$. That is, the element $f^n s$ is in the image of $(\rho_{\mathcal{F}})^U_{D(f)\cap U}$.

(2) For every section $t \in \Gamma(U, \mathcal{F})$ with *restriction* $(\rho_{\mathcal{F}})^U_{D(f)\cap U}(t)$ to $D(f) \cap U$ equal to zero there is an integer $n \geq 0$ such that $f^n t = 0$ in $\Gamma(U, \mathcal{F})$.

(3.15) Lemma. *Let A be a ring and let $g_1, g_2, \dots, g_p$ be elements in A. Moreover let $X = \operatorname{Spec}(A)$, let $U = \cup_{i=1}^p D(g_i)$ be the union of the open subsets $D(g_i)$ of X, and let $\mathcal{F}$ be an $\widetilde{A}|U$-module.*

Assume that the conditons $d(f, D(g_i))$ and $d(f, D(g_i g_j))$ are fulfilled for all $f \in A$ such that $D(f) \subseteq D(g_i)$ respectively $D(f) \subseteq D(g_i g_j) = D(g_i) \cap D(g_j)$ for all $i, j = 1, 2, \dots, p$. Then the conditions $d(f, U)$ hold for all $f \in A$.

Proof. We first show that the second condition of $d(f, U)$ holds. Let $t \in \Gamma(U, \mathcal{F})$ have restriction equal to zero on $D(f) \cap U$. Since $D(fg_i) \subseteq D(g_i)$ and $d(fg_i, D(g_i))$ holds by assumption, we can find integers $n_i \geq 0$ such that $(fg_i)^{n_i} t = 0$ in $\Gamma(D(g_i), \mathcal{F})$ for $i = 1, 2, \dots, p$. We have that the image of g_i in $A_{g_i} = \Gamma(D(g_i), \widetilde{A})$ is invertible. Consequently $f^{n_i} t = 0$ in $\Gamma(D(g_i), \mathcal{F})$. Let n be an integer greater than or equal to $n_1, n_2, \dots, n_p$. We then have that $f^n t = 0$ in $\Gamma(D(g_i), \mathcal{F})$ for $i = 1, 2, \dots, p$. Since $\mathcal{F}$ is a sheaf we have that $f^n t = 0$ in $\Gamma(U, \mathcal{F})$. Hence the second condition of $d(f, U)$ holds.

We next show that the first condition of $d(f, U)$ holds. Let $s \in \Gamma(D(f) \cap U, \mathcal{F})$. Since $D(fg_i) \subseteq D(g_i)$ and $d(fg_i, D(g_i))$ holds by assumption we can find integers $n_i \geq 0$, and sections $s'_i \in \Gamma(D(g_i), \mathcal{F})$ with restriction to $D(f) \cap D(g_i)$ equal to $(fg_i)^{n_i} s$. The image of g_i in $A_{g_i} = \Gamma(D(g_i), \widetilde{A})$ is invertible. Hence we can find a section s_i in $\Gamma(D(g_i), \mathcal{F})$ such that $s'_i = g_i^{n_i} s_i$. Then the restriction of s_i to $D(f) \cap D(g_i)$ is $f^{n_i} s$.

Let n be an integer which is greater than or equal to all the $n_1, n_2, \dots, n_p$. Then the restriction of $f^{n-n_i} s_i = f^{n-n_j} s_j$ to $D(f) \cap D(g_i) \cap D(g_j) = D(fg_i g_j)$ is zero. Since the second condition of $d(f, D(g_i g_j))$ holds when $D(fg_i g_j) \subseteq D(g_i g_j)$ by assumption, we obtain integers $n_{ij} \geq 0$ such that $(fg_i g_j)^{n_{ij}}(f^{n-n_i} s_i - f^{n-n_j} s_j)$ is zero in $\Gamma(D(g_i g_j, \mathcal{F})$. We have that the image of $g_i g_j$ in $A_{g_i g_j} = \Gamma(D(g_i g_j), \widetilde{A})$ is invertible. Hence, when m is an integer which is greater or equal to n_{ij} for $i, j = 1, 2, \dots, p$, we have that $f^m(f^{n-n_i} s_i - f^{n-n_j} s_j)$ is zero in $\Gamma(D(g_i g_j), \mathcal{F}) = \Gamma(D(g_i) \cap D(g_j), \mathcal{F})$ for $i, j = 1, 2, \dots, p$. Since $\mathcal{F}$ is a sheaf we can find a section s' in $\Gamma(U, \mathcal{F})$ with restriction $f^{m+n-n_i} s_i$ to $D(g_i)$ for $i = 1, 2, \dots, p$. The restriction of $f^{m+n-n_i} s_i$ to $D(f) \cap D(g_i)$ is $f^{m+n-n_i} f^{n_i} s = f^{m+n} s$. Consequently the restriction of s' to $D(f) \cap U$ is equal to $f^{m+n} s$. Hence the first property of $d(f, U)$ holds.

(3.16) Proposition. *Let A be a ring and let $X = \operatorname{Spec}(A)$. Moreover let U be a* *compact open subset of X and let $\mathcal{F}$ be an $(\mathcal{O}_X|U)$-module. We have equivalently:*

(1) *The sheaf $\mathcal{F}$ is quasi-coherent.*

(2) *There is an open covering $U = \cup_{i=1}^{p} D(f_i)$ of U by open sets $D(f_i)$ with $f_i \in A$, and A_{f_i}-modules M_i such that $\widetilde{M}_i|D(f_i)$ is isomorphic to $\mathcal{F}|D(f_i)$ for $i = 1, 2, \dots, p$.*

Proof. (1)⇒(2) When $\mathcal{F}$ is quasi-coherent every point $x \in U$ has a neighbourhood $D(f)$ such that $\mathcal{F}|D(f)$ is isomorphic to the cokernel of a map $\widetilde{A}_f^{(I)} \to \widetilde{A}_f^{(J)}$. It follows from Proposition (?) that such a homomorphism is associated to a homomorphism $u : A_f^{(I)} \to A_f^{(J)}$ of A-modules. Hence it follows from Proposition (3.12) that $\mathcal{F}|D(f)$ is associated to the cokernel of u. Since U is compact we can cover it with a finite number of neighbourhoods of the form $D(f)$. Consequently condition (1) implies that condition (2) holds.

(2)⇒(1) Every A-module M is the quotient of a map $A^{(I)} \to A^{(J)}$ for some set of indices I and J. It follows from Proposition (3.11) that the resulting sequence $\mathcal{O}_X^{(I)} \to \mathcal{O}_X^{(J)} \to \widetilde{M} \to 0$ is exact. Hence every sheaf associated to a module is quasi-coherent. Condition (1) consequently follows from condition (2).

(3.17) Theorem. *Let A be a ring and let $X = \operatorname{Spec}(A)$. Moreover let U be a compact subset of X, and let $\mathcal{F}$ be an $(\mathcal{O}_X|U)$-module. We have equivalently:*

(1) *There is an A-module M such that $\widetilde{M}|U$ is isomorphic to $\mathcal{F}$.*

(2) *The sheaf $\mathcal{F}$ is quasi-coherent.*

(3) *The conditions $d(f, U)$ of Section (?) holds for all $f \in A$ such that $D(f) \subseteq U$.*

Proof. (1)⇒(2) Since the open sets $D(f)$ form a basis for the topology of X and U is compact it follows that the second condition of Proposition (3.16) is fulfilled. Hence assertion (2) holds.

(2)⇒(3). We shall show that assertion (3) follows from the second condition of Proposition (3.16), and thus from assertion (2). It follows from Lemma (3.15) that we may assume that $U = D(g)$ and that $\mathcal{F}$ is associated to an A_g-module N. In order to show that the second condition of Proposition (3.16) implies assertion (3) thus we may replace X by U and A by A_g. Hence we assume that $U = \operatorname{Spec}(A)$ and that $\mathcal{F}$ is associated to the A-module N.

Let $s \in \Gamma(D(f), \widetilde{N}) = N_f$. Then s can be written in the form z/f^n with $z \in N$ and $f \in A$. Then $f^n s$ is the restriction to $D(f)$ of the section z in $\Gamma(U, \widetilde{N}) = N$ and we have shown that the first condition of $d(f, U)$ holds.

Let $t \in \Gamma(U, \widetilde{N}) = N$ be a section that has restriction to $D(f)$ equal to zero. That is, the image of t by the canonical map $i_N^f : N \to N_f = \Gamma(D(f), \widetilde{N})$ is zero. Then $f^n t = 0$ in N for some integer $n \geq 0$. Hence we have proved that the second condition of $d(f, U)$ holds.

(3)⇒(1) Let $g \in A$ be such that $D(g) \subseteq U$. We note first that the conditions $d(f, D(g))$ are fulfilled for the sheaf $\mathcal{F}|D(g)$ and all $f \in A$ such that $D(f) \subseteq D(g)$. It is clear that the first condition of $d(f, U)$ for all $f \in A$ such that $D(f) \subseteq U$ implies that the first condition of $d(f, D(g))$ holds for all $f \in A$ such that $D(f) \subseteq U$. Moreover when $t \in \Gamma(D(g), \mathcal{F})$ is a section with restriction zero to $D(f)$ it follows from the first condition of $d(f, U)$ that there is a section $s \in \Gamma(U, \mathcal{F})$ and an integer $m \geq 0$ such that the restriction of s to $D(g)$ is $g^m t$. Then it follows from the second condition of $d(f, U)$ that there is an integer $n \geq 0$ such that $f^n g^m t = 0$ in $\Gamma(D(g), \mathcal{F})$. Since the image of g in A_g is invertible we have that $f^n t = 0$ in $\Gamma(D(g), \mathcal{F})$. Hence we have proved that the second condition of $d(f, D(g))$ is fulfilled for all $f \in A$ such that $D(f) \subseteq U$. We have shown that the conditions $d(f, D(g))$ are fulfilled for all $f \in A$ such that $D(f) \subseteq D(g)$. Hence it follows from Lemma (Modules ?) that $d(f, U)$ holds for all $f \in A$.

Let $M = \Gamma(U, \mathcal{F})$ and let $j : U \to X$ be the canonical inclusion. Moreover let $f \in A$. The image of f in A_f is invertible. Hence it follows from Proposition (?) that the restriction map $(\rho_{\mathcal{F}})^U_{D(f) \cap U} : M = \Gamma(D(f), \mathcal{F}) \to \Gamma(D(f), j_*(\mathcal{F})) = \Gamma(D(f) \cap U, \mathcal{F})$ factorizes via the canonical map $i^f_M : M \to M_f$ and a unique A_f-module homomorphism

$$u_{D(f)} : M_f = \Gamma(D(f), \widetilde{M}) \to \Gamma(D(f), j_*(\mathcal{F})) = \Gamma(D(f) \cap U, \mathcal{F}).$$

It is clear that the maps $u_{D(f)}$, for all $f \in A$, define a homomorphism of $\widetilde{A}$-modules

$$u : \widetilde{M} \to j_*(\mathcal{F}).$$

We shall prove that u is an isomorphism. It suffices to prove that $u_{D(f)}$ is an isomorphism for all $f \in A$. Let $s \in \Gamma(D(f) \cap U, \mathcal{F})$. It follows from the first condition of $d(f, U)$ that there is a section $z \in M = \Gamma(U, \mathcal{F})$ and an integer $n \geq 0$ such that z restrict to $f^n s$ on $D(f) \cap U$. That is, we have that $u_{D(f)}(z/f^n) = s$, and we have proved that $u_{D(f)}$ is surjective.

Let $z/f^m \in M_f = \Gamma(D(f), \widetilde{M})$ be in the kernel of $u_{D(f)}$, where $z \in M = \Gamma(U, \mathcal{F})$. Since the image of f in A_f is invertible the restriction of z to $D(f) \cap U$ is consequently equal to zero. It follows from the second condition of $d(f, U)$ that there is an integer $n \geq 0$ such that $f^n z = 0$. Consequently we have that $z/f^n = 0$ in M_f. We have thus proved that $u_{D(f)}$ is injective.

(3.18) Proposition. *Let A be a ring and let $(X, \mathcal{O}_X) = (\mathrm{Spec}(A), \widetilde{A})$. Moreover let U be a compact open subset of X and let $\mathcal{F}$ be an $(\mathcal{O}_X|U)$-modules. Consider the following conditions:*

(1) *The $\mathcal{O}_X$-module $\mathcal{F}$ is coherent.*
(2) *The $\mathcal{O}_X$-module $\mathcal{F}$ is of finite type and coherent.*
(3) *There is a finitely generated A-module M such that $\mathcal{F}$ is isomorphic to $\widetilde{M}|U$.*

Then condition (1) *implies condition* (2), *and condition* (2) *implies condition* (3). *When condition* (3) *is fulfilled we have that* $\mathcal{F}$ *is of finite type.*

Proof. (1)⇒(2) It is clear that the first condition implies the second.

(2)⇒(3) We have that $\mathcal{F}$ is quasi-coherent. Hence it follows from Theorem (3.17) that there is an A-module N such that $\mathcal{F}$ is isomorphic to $\widetilde{N}|U$. for every point $x \in U$ there is an open neighbourhood of x contained in U and a finite collection of sections in $N = \Gamma(U, \mathcal{F})$ that generate the $\mathcal{O}_{X,y}$-modules $\mathcal{F}_y$ for all $y \in U$. Since U is compact we can find elements $f_1, f_2, \dots, f_n$ in A and elements $x_1, x_2, \dots, x_m$ in N such that $N_y = \mathcal{F}_y$ is generated by the classes of $x_1, x_2, \dots, x_m$ for all $y \in U$.

Let M be the submodule of N generated by the elements $x_1, x_2, \dots, x_m$ and let $u : \widetilde{M}|U \to \mathcal{F}$ be the composite of the isomorphism $\widetilde{N}|U \xrightarrow{\sim} \mathcal{F}$ with the homomorphism $\widetilde{M}|U \to \widetilde{N}|U$ obtained from the inclusion of M in N. It follows from Proposition (3.12) that u is injective, and it follows from Proposition (3.12) that u is surjective. Hence condition (3) holds.

When condition (3) holds we have that $\mathcal{F}$ is isomorphic to $\widetilde{M}$ for a finitely generated A-module M. That is, there is a surjection $v : A^n \to M$ for some integer n. It follows from Proposition (3.12) that $\widetilde{u} : \widetilde{A}^n \to \widetilde{M}$ is surjective. Hence $\mathcal{F}$ is of finite type.

(3.19) Exercises.

1. Let $\mathbf{Z}_{(p)}$ be all the rational numbers of the form m/n where n is not divisible by the prime number p. Describe the ringed space $(\operatorname{Spec}(\mathbf{Z}_{(p)}), \widetilde{\mathbf{Z}_{(p)}})$.

2. Describe the ringed space $(\operatorname{Spec}(\mathbf{Z}), \widetilde{\mathbf{Z}})$.

3. Let $K[u, v]$ be the polynomial ring in the variables u, v over the field K, and let $A = K[u, v]/(u^2, uv)$. Describe the ringed space $(\operatorname{Spec}(A), \widetilde{A})$.

4. Let $A = \mathbf{Z}$ and let M be the A-module $\mathbf{Z}/2\mathbf{Z}$. Describe the $(\operatorname{Spec}(\mathbf{Z}), \widetilde{\mathbf{Z}})$-module $\widetilde{M}$.

5. Let A be ring and M an A-module. Show that $\widetilde{M} = (0)$ on $\operatorname{Spec}(A)$ if and only if $\widetilde{M}_x = (0)$ for all points x in $\operatorname{Spec}(A)$.

6. Let $\mathbf{Z}_{(p)}$ be the rational numbers of the form m/n where n is not divisible by the prime number p. Let $\mathcal{F}$ be the *simple* sheaf with fiber $\mathbf{Q}$ on $\operatorname{Spec}(\mathbf{Z}_{(p)})$.

(1) Show that $\mathcal{F}$ is a $\widetilde{\mathbf{Z}_{(p)}}$-module.

(2) Is it true that $\mathcal{F} = \widetilde{M}$ for some $\mathbf{Z}_{(p)}$-module M?

7. Let $\mathbf{Z}_{(p)}$ be the ring of rational numbers of the form m/n where n is not devisible by the prime number p. Moreover let $\mathcal{F}$ be the sheaf on $X = \operatorname{Spec}(\mathbf{Z}_{(p)})$ defined by $\Gamma(X, \mathcal{F}) = \{0\}$ and $\Gamma(\{x_0\}, \mathcal{F}) = \mathbf{Q}$.

(1) Show that $\mathcal{F}$ is a $\widetilde{\mathbf{Z}_{(p)}}$-module.

(2) Is it true that $\mathcal{F} = \widetilde{M}$ for some $\mathbf{Z}_{(p)}$-module M?

8. Is there a ring A such that $\operatorname{Spec}(A)$ consists of two points $\{x, y\}$ and $\widetilde{A}_x = \mathbf{Z}$ and $\widetilde{A}_y = \mathbf{Q}$?

4. Affine schemes.

(4.1) Homomorphisms of locally ringed spaces associated to ring homomorphisms. Let $\varphi : B \to A$ be a homomorphism of rings. We saw in (?) that we have a continous map ${}^a\varphi : \operatorname{Spec}(A) \to \operatorname{Spec}(B)$ of topological spaces. For every $g \in B$ it follows from Proposition (?) that ${}^a\varphi^{-1}(D(g)) = D(\varphi(g))$. It follows from the definition of the sheaves of rings $\widetilde{A}$ and $\widetilde{B}$ that the rings $\Gamma(D(g), \widetilde{B})$ and $\Gamma(D(\varphi(g)), \widetilde{A})$ are canonically identified with B_g respectively $A_{\varphi(g)}$. It follows from (2.1) that the homomorphism φ induces a canonical homomorphism of rings!!

$$\varphi^g : B_g \to A_{\varphi(g)},$$

that is a homomorphism of rings

$$\widetilde{\varphi}_{D(g)} : \Gamma(D(g), \widetilde{B}) \to \Gamma(D(\varphi(g)), \widetilde{A}).$$

It is clear that for all inclusions $D(f) \supseteq D(g)$ of open sets in $\operatorname{Spec}(B)$ we have a commutative diagram

$$\begin{tikzcd}
\Gamma(D(f), \widetilde{B}) \arrow[r, "\widetilde{\varphi}_{D(f)}"] \arrow[d, "(\rho_{\widetilde{B}})_{g,f}"'] & \Gamma(D(\varphi(f)), \widetilde{A}) \arrow[d, "(\rho_{\widetilde{A}})_{\varphi(g),\varphi(f)}"] \\
\Gamma(D(g), \widetilde{B}) \arrow[r, "\widetilde{\varphi}_{D(g)}"'] & \Gamma(D(\varphi(g)), \widetilde{A}).
\end{tikzcd}$$

Since $\Gamma(D(\varphi(g)), \widetilde{A}) = \Gamma({}^a\varphi^{-1}(D(g)), \widetilde{A}) = \Gamma(D(g), ({}^a\varphi)_*\widetilde{A})$ and the sets $D(g)$ with $g \in B$ form a basis for $\operatorname{Spec}(B)$ we obtain a homomorphism of sheaves of rings

$$\widetilde{\varphi} : \widetilde{B} \to ({}^a\varphi)_*(\widetilde{A}).$$

Hence we have a map of ringed spaces!!

$$\Phi = ({}^a\varphi, \widetilde{\varphi}) : (\operatorname{Spec}(A), \widetilde{A}) \to (\operatorname{Spec}(B), \widetilde{B}).$$

It follows from Lemma (?) and (?) that $(\operatorname{Spec}(A), \widetilde{A})$ and $(\operatorname{Spec}(B), \widetilde{B})$ are locally ringed spaces.

We note that the adjoint map

$$\widetilde{\varphi}^\sharp_x : (({}^a\varphi)^*\widetilde{B})_x \to \widetilde{A}_x$$

of $\widetilde{\varphi}$ at the point x on $\operatorname{Spec}(A)$ is the same as the localization

$$\varphi^x : B_{\varphi(x)} \to A_x$$

→ that we obtain from (2.1) with $T = A \setminus \mathfrak{j}_x$ and $S = B \setminus \mathfrak{j}_{\varphi(x)}$. In fact, take an element h in $(({}^a\varphi)^*\widetilde{B})_{\varphi(x)} = B_{\varphi(x)}$. Then $h = g/t$ with g, t in B and $t \notin \mathfrak{j}_{\varphi(x)}$. We have that $D(t)$ is an open neighbourhood of $\varphi(x)$ in $\mathrm{Spec}(B)$ and that $\widetilde{\varphi}_{D(t)} : \Gamma(D(t), \widetilde{B}) \to \Gamma(({}^a\varphi)^{-1}(D(t)), \widetilde{A})$ is the same as the map $\varphi^t : B_t \to A_{\varphi(t)}$. Consequently we have that a section s in $\Gamma(D(t), \widetilde{B})$ represented by g/t^n is mapped to the section $\widetilde{\varphi}(D(t))(g/t^n)$ in $\Gamma(({}^a\varphi)^{-1}(D(t)), \widetilde{A})$ represented by $\varphi(g)/\varphi(t^n)$. In other words $\widetilde{\varphi}_x^\sharp(s) = \varphi(g)/\varphi(t^n)$ in $A_{\varphi(x)}$.

In particular we obtain that $\widetilde{\varphi}_x^\sharp$ is a local map of local rings.

(4.2) Definition. A ringed space $(X, \mathcal{O}_X)$ is an *affine scheme* if it is isomorphic as ringed spaces to $(\mathrm{Spec}(A), \widetilde{A})$ for some ring A. The ring $\Gamma(X, \mathcal{O}_X)$ which is canonically isomorphic to A is called the *coordinate ring* of the affine scheme, and
n is sometimes denoted !!$A(X)$. Sometimes we simply say that $\mathrm{Spec}(A)$ is an *affine scheme*, instead of saying that $(\mathrm{Spec}(A), \widetilde{A})$ is an affine scheme. A *homomorphism of affine schemes* $\Psi = (\psi, \theta) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a local homomorphism of locally ringed spaces.

(4.3) Remark. Let $\varphi : B \to A$ be a homomorphism of rings. We have seen that
n it gives a *local homomorphism*!! $\Phi = ({}^a\varphi, \widetilde{\varphi}) : (\mathrm{Spec}(A), \widetilde{A}) \to (\mathrm{Spec}\, B, \widetilde{B})$ of locally ringed spaces. Note that Φ determines φ uniquely since $\varphi = \Gamma(\widetilde{\varphi}) : \Gamma(\mathrm{Spec}(B), \widetilde{B}) \to \Gamma(\mathrm{Spec}(B), ({}^a\varphi)_*(\widetilde{A}))$ and $\Gamma(\mathrm{Spec}(B), ({}^a\varphi)_*(\widetilde{A})) = \Gamma(\mathrm{Spec}(A), \widetilde{A})$.

(4.4) Theorem. *Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be two affine schemes. A homomorphism of ringed spaces $\Psi = (\psi, \theta) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a homomorphism of affine schemes $\Phi = ({}^a\varphi, \widetilde{\varphi}) : (\mathrm{Spec}(A), \widetilde{A}) \to (\mathrm{Spec}(B), \widetilde{B})$ where $\varphi : B \to A$ is a homomorphism of rings, if and only if the map Ψ is a local homomorphism of locally ringed spaces.*

→ *Proof.* We have already seen in (?) that if Ψ is of the form $\Phi = ({}^a\varphi, \widetilde{\varphi})$ for some ring homomorphism $\varphi : B \to A$ then Ψ is a local homomorphism of locally ringed spaces.

Conversely, assume that $\Psi = (\psi, \theta)$ is a local homomorphism of locally ringed spaces. Let $B = \Gamma(X, \mathcal{O}_X)$ and $A = \Gamma(Y, \mathcal{O}_Y)$. The homomorphism $\theta : \mathcal{O}_Y \to \psi_*\mathcal{O}_X$ gives a canonical homomorphism of rings

$$\varphi = \Gamma(\theta) : \Gamma(Y, \mathcal{O}_Y) \to \Gamma(Y, \psi_*(\mathcal{O}_X))$$

and by the canonical identifications $A = \Gamma(X, \mathcal{O}_X) = \Gamma(Y, \psi_*(\mathcal{O}_X))$ and $B = \Gamma(Y, \mathcal{O}_Y)$ a homomorphism of rings

$$\varphi = \Gamma(\theta) : B \to A.$$

We first note that ψ and ${}^a\varphi$ are the same homomorphism $X \to Y$ of topological spaces. This is because the localization $\theta_x^\sharp : (\psi^*\mathcal{O}_Y)_x = \mathcal{O}_{Y,\psi(x)} \to \mathcal{O}_{X,x}$ of the adjoint map of θ gives the commutative diagram

$$\begin{tikzcd} B = \Gamma(Y, \mathcal{O}_Y) \arrow[r, "\varphi=\Gamma(\theta)"] \arrow[d, "i_B^{\mathfrak{j}_{\psi(x)}}=\rho^Y_{\psi(x)}"'] & \Gamma(X, \mathcal{O}_X) = A \arrow[d, "\rho^X_x = i_A^{\mathfrak{j}_x}"] \\ B_{\psi(x)} = \mathcal{O}_{Y,\psi(x)} = \psi^*(\mathcal{O}_Y)_x \arrow[r, "\theta^\sharp_x"'] & \mathcal{O}_{X,x} = A_x \end{tikzcd}$$

for all $x \in X$. It follows from (?) that the inverse image of $(m_A)_x$ by ρ^X_x is $\mathfrak{j}_x$, and the inverse image of $(m_B)_{\psi(x)}$ by $\rho^Y_{\psi(x)}$ is $\mathfrak{j}_{\psi(x)}$. Since $\theta^\sharp_x$ is local it follows that $\varphi^{-1}(\mathfrak{j}_x) = \mathfrak{j}_{\psi(x)}$, and consequently that ${}^a\varphi(x) = \psi(x)$ and hence that ${}^a\varphi = \psi$. Moreover, since the diagram is commutative, it follows from (2.1) that $\theta^\sharp_x = \varphi^x$ However we saw in (4.1) that ${}^x\varphi = (\widetilde{\varphi}^\sharp)_x$. Hence we have that $\theta^\sharp_x = (\widetilde{\varphi}^\sharp)_x$ for all $x \in X$. It follows from the characterization of sheaves that $\theta^\sharp = \widetilde{\varphi}^\sharp$, and thus by adjunction that $\theta = \widetilde{\varphi}$.

(4.5) Remark. The affine schemes with local homomorphisms form a category.

(4.6) Remark. Let $\varphi : B \to A$ be a homomorphism of rings. Moreover let $g \in B$ and let $f = \varphi(g)$. We saw in (3.1) that we obtain a homomorphism of rings

$$\varphi^g : B_g \to A_f$$

such that $\varphi(h/g^n) = \varphi(h)/f^n$ for all $h \in B$ and all integers $n \geq 0$. Let M be an A-modules. It follows from section (3.1) used to the natural φ-module homomorphism $M_{[\varphi]} \to M$ that we obtain a homomorphism of φ^g-modules

$$(M_{[\varphi]})_g \to M_f,$$

n or equivalently, by restriction of scalars, an isomorphism of B_g-modules!!

$$(u^g)_M : (M_{[\varphi]})_g \to (M_f)_{[\varphi^g]}$$

mapping the element $x/g^n \in (M_{[\varphi]})_g$ to x/f^n for all $x \in M$ and all integers $n \geq 0$. Since g is invertible in B_g it follows from Proposition (?) that we have a natural B_g-module homomorphism $(M_f)_{[\varphi^g]} \to (M_{\varphi]})_g$ and it is easily checked that this homomorphism is the inverse homomorphism to $(u^g)_M$. Consequently $(u^g)_M$ is an isomorphism. When $v : M \to M'$ is a homomorphism of A-modules we obtain a commutative diagram of B_g-modules

$$\begin{tikzcd} (M_{[\varphi]})_g \arrow[r, "(u^g)_M"] \arrow[d, "v_{[\varphi]}"'] & (M_f)_{[\varphi^g]} \arrow[d, "(v^f)_{[\varphi^g]}"] \\ (M'_{[\varphi]})_g \arrow[r, "(u^g)_{M'}"'] & (M'_f)_{[\varphi^g]}. \end{tikzcd}$$

(4.7) Proposition. *Let $\varphi : B \to A$ be a homomorphism of rings and let $\Phi = ({}^a\varphi, \widetilde{\varphi}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be the corresponding homomorphism of affine schemes. Moreover let M be an A-module. Then there is a canonical isomorphism of $\mathcal{O}_Y$-modules*

$$u_M : \widetilde{M_{[\varphi]}} \xrightarrow{\sim} \Phi_*(\widetilde{M}).$$

For every homomorphism $v : M \to M'$ of A-modules we have that

$$\Phi_*(\widetilde{u})u_M = u_{M'}\widetilde{v_{[\varphi]}}$$

where $\widetilde{v_{[\varphi]}}$ is the map associated to the B-modules homomorphism $v_{[\varphi]} : M_{[\varphi]} \to M'_{[\varphi]}$.

When C is an A-algebra the map u_C is a $\mathcal{O}_Y$-algebra homomorphism.

Proof. Let $g \in B$ and let $f = \varphi(g)$. We have identifications $\Gamma(D(g), \widetilde{M_{[\varphi]}}) = (M_{[\varphi]})_g$ and $\Gamma(D(f), \widetilde{M}) = M_f$ as B_g respectively A_f-modules. It follows from Remark (4.6) that we have a ring homomorphism $\varphi^g : B_g \to A_f$ and a B_g-module isomorphism $(u^g)_M : (M_{[\varphi]})_g \to (M_f)_{[\varphi^g]}$ such that $(v^f)_{[\varphi^g]}(u^g)_M = (u^g)_{M'}v_{[\varphi]}$. We consequently obtain an isomorphism of $\Gamma(D(g), \widetilde{B})$-modules

$$\Gamma(D(g), \widetilde{M_{[\varphi]}}) \xrightarrow{\sim} \Gamma({}^a\varphi^{-1}(D(g)), \widetilde{M})_{[\varphi^g]}.$$

These isomorphisms, for varying $g \in B$, clearly define an isomorphism of $\mathcal{O}_Y$-modules $u_M : \widetilde{M_{[\varphi]}} \xrightarrow{\sim} \Phi_*(\widetilde{M})$. The equality $u_{M'}\widetilde{v_{[\varphi]}} = \Phi_*(\widetilde{u})u_M$ follows from the equality $(v^f)_{[\varphi^g]}(u^g)_M = (u^g)_{M'}v_{[\varphi]}$.

The last part of the Proposition is clear.

(4.8) Remark. Let $\varphi : B \to A$ be a homomorphism of rings. Moreover let $\mathfrak{p}$ be a prime ideal of A and let $\mathfrak{q} = \varphi^{-1}(\mathfrak{p})$. We saw in section (3.1) that we obtain a homomorphism of rings

$$\varphi_{\mathfrak{q}} : B_{\mathfrak{q}} \to A_{\mathfrak{p}}$$

such that $\varphi(h/t) = \varphi(h)/\varphi(t)$ for all $h \in B$ and $t \in B \setminus \mathfrak{q}$. Let N be a B-module. It follows from section (3.1) used to the natural φ-module homomorphism $N \to N \otimes_B A$ that we otain a homomorphism of $\varphi_{\mathfrak{q}}$-modules

$$N_{\mathfrak{q}} \to (N \otimes_B A)_{\mathfrak{p}},$$

or equivalently, by extension of scalars, an $A_{\mathfrak{p}}$-module homomorphism

$$u^{\mathfrak{q}} : N_{\mathfrak{q}} \otimes_{B_{\mathfrak{q}}} A_{\mathfrak{p}} \to (N \otimes_B A)_{\mathfrak{p}} \tag{4.8.1}$$

mapping $(y/t) \otimes_{B_{\mathfrak{q}}} (g/s)$ to $(x \otimes_{B_{\mathfrak{q}}} g)/\varphi(t)s$ for all $y \in N$, $s \in A \setminus \mathfrak{p}$ and $t \in B \setminus \mathfrak{p}$. It follows from Proposition (Modules) that we have a natural $A_{\mathfrak{p}}$-module homomorphism $(N \otimes_B A)_{\mathfrak{p}} \to N_{\mathfrak{q}} \otimes_{B_{\mathfrak{q}}} A_{\mathfrak{p}}$ which is easily checked to be the inverse of the map $(u^{\mathfrak{q}})_N$. Hence the homomorphism $(u^{\mathfrak{q}})_N$ is an isomorphism.

When $w : N \to N'$ is a homomorphism of B-modules we obtain a commutative diagram of $A_{\mathfrak{p}}$-modules

$$\begin{tikzcd} N_{\mathfrak{q}} \otimes_{B_{\mathfrak{q}}} A_{\mathfrak{p}} \arrow[r, "(u^{\mathfrak{q}})_N"] \arrow[d, "(w_{\mathfrak{q}} \otimes_{B_{\mathfrak{q}}} \mathrm{id}_{A_{\mathfrak{p}}})"'] & (N \otimes_B A)_{\mathfrak{p}} \arrow[d, "(w \otimes_B \mathrm{id}_A)_{\mathfrak{p}}"] \\ N'_{\mathfrak{q}} \otimes_{B_{\mathfrak{q}}} A_{\mathfrak{p}} \arrow[r, "(u^{\mathfrak{q}})_{N'}"'] & (N' \otimes_B A)_{\mathfrak{p}}. \end{tikzcd}$$

(4.9) Proposition. *Let $\varphi : B \to A$ be a homomorphism of rings and let $\Phi = ({}^a\varphi, \widetilde{\varphi}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be the corresponding homomorphism of affine schemes. Moreover let N be a B-module. Then there is a canonical isomorphism of $\mathcal{O}_X$-modules*

$$u_N : \Phi^*(\widetilde{N}) \xrightarrow{\sim} \widetilde{N \otimes_B} A. \tag{4.9.1}$$

For every homomorphism $w : N \to N'$ of B-modules we have that

$$(w \widetilde{\otimes_B} \mathrm{id}_A) u_N = u_{N'} \Phi^*(w). \tag{4.9.2}$$

When C is a B-algebra we have that u_N is a homomorphism of $\mathcal{O}_X$-algebras.

Proof. We have a natural homomorphism of B-modules $N \to (N \otimes_B A)_{[\varphi]}$ mapping $z \in N$ to $z \otimes_B 1$. In fact for $g \in B$ we have that the image of gz is $gz \otimes_B 1 = z \otimes_B \varphi(g) = \varphi(g)(z \otimes_B 1)$. We obtain a homorphism of $\mathcal{O}_Y$-modules $\widetilde{N} \to (\widetilde{N \otimes_A})_{[\varphi]}$ It follows from Proposition (4.7) that there is a canonical homomorphism of $\mathcal{O}_Y$-modules $\widetilde{N} \to \Phi_*(\widetilde{N \otimes_B} A)$. By adjunction (?) we obtain a homomorphism of $\mathcal{O}_X$-modules $u_N : \Phi^*(\widetilde{N}) \to \widetilde{N \otimes_B} A$. We have thus constructed the map (4.9.1) and it is clear that the equality (4.9.2) holds.

In order to show that the homomorphism u_M is an isomorphism it follows from Theorem (?) that we have to show that the map on stalks $(u_N)_x$ is an isomorphism for all $x \in X$. Let $y = {}^a\varphi(x)$. It follows from the definition of $\Phi^*(\widetilde{N})$ in (?) that we have an isomorphism $\Phi^*(\widetilde{N})_x \xrightarrow{\sim} (({}^a\varphi)^*(\widetilde{N}) \otimes_{\varphi^*(\widetilde{B})} \widetilde{A})_x$, and from Proposition (?) and Proposition (?) we obtain isomorphisms $(({}^a\varphi^*)(\widetilde{N}) \otimes_{\varphi^*(\widetilde{B})} \widetilde{A})_x \xrightarrow{\sim} ({}^a\varphi^*)(\widetilde{N})_x \otimes_{\varphi^*(\widetilde{B})_x} \widetilde{A}_x \xrightarrow{\sim} N_y \otimes_{B_y} A_x$. On the other hand it follows from (?) that $(\widetilde{N \otimes_B} A)_x$ is isomorphic to $(N \otimes_B A)_x$. Using the explicit formulas for adjunction of (?) and (?) we see that u_N induces the isomorphism $N_y \otimes_{B_y} A_x \xrightarrow{\sim} (N \otimes_B A)_x$ of (4.8.1). Hence we have proved that u_N is an isomorphism.

The last part of the Theorem is clear.

(4.10) Exercises.

1. Let A be ring and f an element of A. Show that the ringed space $(\mathrm{Spec}(A_f), \widetilde{A_f})$ is isomorphic to the ringed space $(D(f), \mathcal{O}_{\mathrm{Spec}(A)}|D(f))$.

2. Let A be ring and let $X = \operatorname{Spec}(A)$ and $\mathcal{O}_X = \widetilde{A}$. We say that the ringed space $(X, \mathcal{O}_X)$ is *reduced* if the ideal $\mathfrak{a}(U)$ of nilpotent elements in the ring $\Gamma(U, \mathcal{O}_X)$ is (0) for all open subsets U of X.

(1) Show that the following conditions are equivalent
  (1a) For all open subsets U of X the ring $\Gamma(U, \mathcal{O}_X)$ has no nilpotent elements.
  (1b) For all points $x \in X$ the ring $(\mathcal{O}_X)_x$ has no nilpotent elements.

(2) Show that the restriction $(\rho_{\mathcal{O}_X})^V_U : \Gamma(V, \mathcal{O}_X) \to \Gamma(U, \mathcal{O}_X)$ induces a restriction map $\sigma^V_U : \Gamma(V, \mathcal{O}_X)/\mathfrak{a}(V) \to \Gamma(U, \mathcal{O}_X)/\mathfrak{a}(U)$ for all inclusion $U \subseteq V$ of open subsets of X.

(3) Show that when we let $\mathcal{B}(U) = \Gamma(U, \mathcal{O}_X)/\mathfrak{a}(U)$ for all open subsets and let $(\rho_{\mathcal{B}})^V_U = \sigma^V_U$ for all inclusions $U \subseteq V$ of open subsets of X, then $\mathcal{B}$ with these restrictions is a presheaf of rings on X.

(4) Show that the fiber of $\mathcal{B}$ at x is equal to $A_{\mathfrak{j}_x}/\mathfrak{a}_{\mathfrak{j}_x}$ where $\mathfrak{a}_{\mathfrak{j}_x}$ is the ideal of nilpotent elements in the fiber $A_{\mathfrak{j}_x}$.

(5) Let $\mathcal{C}$ be the associated sheaf of $\mathcal{B}$. Show that $(X, \mathcal{C})$ is an affine scheme isomorphic to the affine scheme $(\operatorname{Spec}(A/\mathfrak{a}), \widetilde{A/\mathfrak{a}})$ where $\mathfrak{a}$ is the ideal of nilpotent elements in A.

(6) Show that the affine scheme $(X, \mathcal{C})$ is reduced.

3. Let A be a ring and let K be a field. Moreover let $X = \operatorname{Spec}(A)$. For every point $x \in X$ we let $\boldsymbol{\kappa}(x) = A_x/\mathfrak{m}_x$ be the residue field in the point x. Show that there is a unique correspondence between homomorphisms $\operatorname{Spec}(K) \to X$ with image x and ring homomorphisms $\boldsymbol{\kappa}(x) \to K$.

4. Let K be a field and let $\varphi : K \to A$ be a K-algebra. Let $X = \operatorname{Spec}(A)$, and for every point $x \in X$ we let $\boldsymbol{\kappa}(x) = A_x/\mathfrak{m}_x$ be the residue field. Assume that the composite map $K \xrightarrow{\varphi} A_x \xrightarrow{u_{\boldsymbol{\kappa}(x)}} \boldsymbol{\kappa}(x)$ is surjective. Show that there is a unique correspondence between the morphisms $\operatorname{Spec}(K[\varepsilon]) \to \operatorname{Spec}(A)$ of affine schemes such that the composite map $\operatorname{Spec}(K) \to \operatorname{Spec}(K[\varepsilon]) \to \operatorname{Spec}(A)$ comes from the map $A \xrightarrow{i^x_A} A_x \xrightarrow{\varphi_{\boldsymbol{\kappa}(x)}} \boldsymbol{\kappa}(x)$, and homomorphisms $\operatorname{Hom}_{\boldsymbol{\kappa}(x)}(\mathfrak{m}_x/\mathfrak{m}_x^2, \boldsymbol{\kappa}(x))$.

5. Let A be a ring, and let $\boldsymbol{K}$ be the category of affine schemes with a fixed homomorphism $(X, \mathcal{O}_X) \to (\operatorname{Spec}(A), \widetilde{A})$ of ringed spaces. Show that the product of $(\operatorname{Spec}(B), \widetilde{B})$ and $(\operatorname{Spec}(C), \widetilde{C})$ in the category $\boldsymbol{K}$ exists and is equal to $(\operatorname{Spec}(B \otimes_A C), \widetilde{B \otimes_A C})$.

Chain conditions

1. Artinian and noetherian modules.

(1.1) Definition. Let A be a ring and M an A-module. The module M is *noetherian* n if every ascending chain !!$M_1 \subseteq M_2 \subseteq \cdots$ of submodules M_n of M is *stable*, that is, there is an n such that $M_n = M_{n+1} = \cdots$.

The A-module M is *artinian* if every descending chain $M_1 \supseteq M_2 \supseteq \cdots$ of submodules M_n of M is *stable*, that is, there is an n such that $M_n = M_{n+1} = \cdots$.

(1.2) Example. K be field and M a finitely generated vector space. Then M → is artinian and noetherian. In fact, it follows from Remark (MODULES 1.26) that when $L \subset L'$ is a strict inclusion of subspaces of M then $\dim(L) < \dim(L')$. Hence there can only be ascending or descending chains of finite length in M.

(1.3) Example. The integers $\mathbf{Z}$ is a noetherian $\mathbf{Z}$-module. This is because every ideal of $\mathbf{Z}$ is of the form (n) for some integer n, and $(m) \subseteq (n)$ means that n divides m. The module $\mathbf{Z}$ is not artinian because $(2) \supset (2^2) \supset (2^3) \supset \cdots$ is an infinite descending chain of ideals.

(1.4) Example. The polynomial ring $\mathbf{Z}[t]$ in the variable t over the integers is not noetherian as an $\mathbf{Z}$ module since it contains the infinite chain $\mathbf{Z} \subset \mathbf{Z} + \mathbf{Z}t \subset \mathbf{Z} + \mathbf{Z}t + \mathbf{Z}t^2 \subset \cdots$ of submodules. It is not artinian either because it contains the infinite chain $(2) \supset (2^2) \supset (2^3) \supset \cdots$ of ideals.

(1.5) Example. Fix a prime number p. Let M be the $\mathbf{Z}$-submodule of $\mathbf{Q}/\mathbf{Z}$ consisting of all the classes of rational numbers m/n such that $p^q m/n \in \mathbf{Z}$ for some positive integer q. That is, the classes of the elements m/n where n is a power of p. We denote by M_n the submodule of M generated by the class of $1/p^n$. Then M_n consists of the p^n elements that are the classes in M of the elements m/p^n for $m = 0, 1, \dots, p^{n-1}$. The modules in the chain $M_1 \subset M_2 \subset M_3 \subset \dots$ are the only proper submodules of M. This is because if L is a proper submodule of M it contains the class of an element m/p^n with $n > 1$, and where p does not divide m. Since p is a prime number that does not divide m there are integers q and r such that $qp^n + rm = 1$. Then $rm/p^n = -qp^n/p^n + 1/p^n = -q + 1/p^n$. Hence the class of $1/p^n$ is in L. It follows that $M_n \subseteq L$, and that $L = M_s$, where s is the largest integer such that $M_s \subseteq L$. Since we have a chain $M_1 \subset M_2 \subset M_3 \subset \cdots$ and the modules M_n are the only proper submodules of M it follows that the $\mathbf{Z}$-module M is artinian but not noetherian.

(1.6) Lemma. *Let A be a ring and M an A-module. The following assertions are equivalent:*

(1) *The module M is noetherian.*
(2) *The collection of submodules of M satisfies the maximum condition.*
(3) *Every submodule of M is finitely generated.*

→ *Proof.* (1) $\Leftrightarrow$ (2) We saw in (TOPOLOGY 1.5) that the assertions (1) and (2) are equivalent.

(2) $\Rightarrow$ (3) Assume that M satisfies the maximum condition and let $L \neq 0$ be a
n submodule of M. Denote by !!$\mathcal{L}$ the collection of non-zero submodules of L that are finitely generated. Then $\mathcal{L}$ is not empty since Ax is in L for all $x \neq 0$ in L. Since M satisfies the maximum condition $\mathcal{L}$ has a maximal element L'. We shall prove that $L = L'$. Assume that $L' \subset L$. Then there is an element $x \in L \setminus L'$. We have that $Ax+L'$ is a finitely generated submodule of L and L' is contained properly in $Ax+L'$. This is impossible because L is maximal in $\mathcal{L}$, and we have obtained a contradiction to the assumption that $L \neq L'$. Hence $L = L'$ and L is finitely generated.

(3) $\Rightarrow$ (1) Assume that every submodule of M is finitely generated. Let $M_1 \subseteq M_2 \subseteq \cdots$ be a chain of submodules of M. Then $L = \cup_{n=0}^{\infty} M_n$ is a submodule of M and thus finitely generated. Each finite set of generators of L must be contained in some M_n. Then $L = M_n = M_{n+1} = \cdots$. That is, the chain $M_1 \subseteq M_2 \subseteq \cdots$ is stable.

(1.7) Proposition. *Let A be a ring and let $0 \to M' \xrightarrow{u} M \xrightarrow{v} M'' \to 0$ be an exact sequence of A-modules. Then:*

(1) *The module M is noetherian if and only if the modules M' and M'' are noetherian.*
(2) *The module M is artinian if and only if the modules M' and M'' are artinian.*

Proof. (1) Assume that M is noetherian. Every chain $M_1'' \subseteq M_2'' \subseteq \cdots$ of submodules of M'' gives rise to a chain $v^{-1}(M_1'') \subseteq v^{-1}(M_2'') \subseteq \cdots$ of submodules of M. Since M is noetherian we have that $v^{-1}(M_n'') = v^{-1}(M_{n+1}'') = \cdots$ for some positive integer n, and thus $M_n'' = M_{n+1}'' = \cdots$. Every chain $M_1' \subseteq M_2' \subseteq \cdots$ of submodules of M' gives rise to a chain $u(M_1') \subseteq u(M_2') \subseteq \cdots$ in M. Since M is noetherian $u(M_n') = u(M_{n+1}') = \cdots$ for some positive integer n, and thus $M_n' = M_2' = \cdots$.

Conversely assume that M' and M'' are noetherian, and let $M_1 \subseteq M_2 \subseteq \cdots$ be a chain of submodules of M. Then $v(M_1) \subseteq v(M_2) \subseteq \cdots$ is a chain of submodules of M'', and $u^{-1}(M_1) \subseteq u^{-1}(M_2) \subseteq \cdots$ is a chain of submodules of M'. Since M' and M'' are noetherian there is a positive integer n such that $v(M_n) = v(M_{n+1}) = \cdots$ and $u^{-1}(M_n) = u^{-1}(M_{n+1}) = \cdots$. However, then $M_n = M_{n+1} = \cdots$ since M_i is completely determined by $v(M_i)$ and $u^{-1}(M_i)$. In fact an element x of M is in M_i if and only if $v(x) \in v(M_i)$ and there is an $x' \in u^{-1}(M_i)$ such that $x - u(x') \in M_i$.

(2) The proof of the second part is analogous to the proof of the first part, with descending chains instead of ascending chains.

(1.8) Corollary. *Let $M_1, M_2, \dots, M_n$ be A-modules.*

(1) *If the modules $M_1, M_2, \dots, M_n$ are noetherian, then the direct sum $\oplus_{i=1}^n M_i$ is noetherian.*
(2) *If the modules $M_1, M_2, \dots, M_n$ are artinian, then the direct sum $\oplus_{i=1}^n M_i$ is artinian.*

Proof. (1) We shall prove the first assertion of the Corollary by induction on n. It holds trivially for $n = 1$. Assume that it holds for $n - 1$. We clearly have a short exact sequence $0 \to M_n \to \oplus_{i=1}^n M_i \to \oplus_{i=1}^{n-1} M_i \to 0$. It follows from the induction hypothesis that $\oplus_{i=1}^{n-1} M_i$ is noetherian. The first part of the Corollary hence follows from the Proposition.

(2) The proof of the second part of the Corollary is similar to the proof of the first part.

n **(1.9) Definition.** Let !!$(0) = M_0 \subset M_1 \subset \cdots \subset M_n = M$ be a chain of submodules of M. We call n the *length of the chain.* A chain is a *refinement* of another chain if we obtain the second by adding modules to the first. We call a chain $(0) = M_0 \subset M_1 \subset \cdots \subset M_n = M$ a *composition series* if the modules $M_1/M_0, M_2/M_1, \dots, M_n/M_{n-1}$ have no proper submodules.

(1.10) Theorem. (The Jordan Theorem) *Let A be a ring and M an A-module that has a composition series. Then all composition series of M have the same length, and every chain can be refined to a composition series.*

n *Proof.* For every submodule L of M that has a composition series we let !$\ell(L)$! be the smallest length of a composition series of L. Let $\ell(M) = n$ and let $(0) = M_0 \subset M_1 \cdots \subset M_n = M$ be a composition series for M.

We shall first show that every submodule L of M has a composition series and that $\ell(L) < \ell(M)$ for all proper submodules L of M. To see this we consider the chain $(0) = L_0 = L \cap M_0 \subseteq L_1 = L \cap M_1 \subseteq \cdots \subseteq L_n = L \cap M_n = L$ of submodules of L. We have that L_i/L_{i-1} has no proper submodules since, by Lemma (MODULES
→ 1.13), we have an injective map $L_i/L_{i-1} \to M_i/M_{i-1}$. Since M_i/M_{i-1} has no proper submodules either $L_i = L_{i-1}$ or $L_i/L_{i-1} \to M_i/M_{i-1}$ is an isomorphism. Hence, removing terms where $L_{i-1} = L_i$ from the chain $(0) = L_0 \subset L_1 \subset \cdots \subset L_n = L$ we obtain a composition series for L. It follows that $\ell(L) \leq \ell(M)$.

We shall show by induction on i that if $\ell(L) = \ell(M)$ then $M_i = L_i$ for $i = 1, 2, \dots, n$. When $\ell(L) = \ell(M)$ all the maps $L_i/L_{i-1} \to M_i/M_{i-1}$ are isomorphisms. In particular $M_1 = L_1$. Assume that $L_{i-1} = M_{i-1}$. Since $L_i/L_{i-1} \to M_i/M_{i-1}$ is an isomorphism we have for each x in M_i that there is an element x_1 in M_{i-1} and an element y in L_i such that $x + x_1 = y$. Then $x = y - x_1$ is in L_i since $-x_1$ is in $M_{i-1} = L_{i-1} \subseteq L_i$. Hence we have that $L_i = M_i$. In particular we have that $L = M$. We have thus shown that when $\ell(L) = \ell(M)$ then $L = M$. Hence when L is properly contained in M we have that $\ell(L) < \ell(M)$.

We shall now prove that all composition series of M have the sema length. Let $K_1 \subset K_2 \subset \cdots \subset K_m = M$ be a chain in M. Then $m \leq \ell(M)$ because $\ell(K_1) <$

$\cdots < \ell(K_m) = \ell(M)$. In particular, if $(0) \subset K_1 \subset K_2 \subset \cdots \subset K_m = M$ is a composition series we must have that $m = \ell(M)$ since $\ell(M)$ is the length of the shortest composition series of M. Hence we have proved that all composition series have the same length.

If $m < \ell(M)$, the chain $K_1 \subset K_2 \subset \cdots \subset K_m$ can not be a composition series. Consequently at least one of the residue modules $K_1/(0), K_2/K_1, \dots, K_m/K_{m-1}$ contains a proper submodule different from (0). Hence we may add one more term to the chain to obtain a chain of length $m+1$. In this way we can add groups in the chain until the chain has length n in which case it is a composition series. Hence every chain can be refined to a composition series.

(1.11) Definition. Let A be a ring and M an A-module. We say that M has *finite length* if it has a composition series. The *length* of M is the common length $\ell(M) = \ell_A(M)$ of all composition series of M.

(1.12) Example. Let K be a field. A finitely dimensional K-vector space M has finite length and $\dim_K(M) = \ell_K(M)$. This is because K has no proper K-submodules, and thus, if $x_1, x_2, \dots, x_n$ is a basis for M, then $Kx_1 \subset Kx_1 + Kx_2 \subset \cdots \subset Kx_1 + Kx_2 + \cdots + Kx_n$ is a composition series.

(1.13) Example. The ring $\mathbf{Z}/6\mathbf{Z}$ has the composition series $(0) \subset \mathbf{2}Z/6\mathbf{Z} \subset \mathbf{Z}/6\mathbf{Z}$. Clearly $(\mathbf{Z}/6\mathbf{Z})/(2\mathbf{Z}/6\mathbf{Z})$ is isomorphic to $\mathbf{Z}/2\mathbf{Z}$. There is another composition series $\{0\} \subset 3\mathbf{Z}/6\mathbf{Z} \subset \mathbf{Z}/6\mathbf{Z}$.

(1.14) Proposition. *Let A be a ring and M an A-module. The length of M is finite if and only if M is an artinian and noetherian A-module.*

→ *Proof.* If M is of finite length it follows from Theorem (1.10) that all chains in M are of finite length. Hence M is artinian and noetherian.

n Conversely, assume that M is artinian and noetherian. Denote by !!$\mathcal{L}$ be the collection of submodules $L \neq (0)$ of M such that there is a chain $L = M_n \subset M_{n-1} \subset \cdots \subset M_1 = M$ for some positive integer n where M_{i-1}/M_i has no proper submodules for each $i = 2, 3, \dots, n$. Then $\mathcal{L}$ is not empty because M belongs to $\mathcal{L}$. Since M is artinian there is a minimal element L' in $\mathcal{L}$. If L' has no proper submodules we have found a composition series $(0) \subseteq L' = M_n \subset M_{n-1} \subset \cdots \subset M_1 = M$ of M and we have proved the Proposition.

We shall show that L' can not have proper submodules. Assume to the contrary
n that L' has proper submodules and let !!$\mathcal{L}'$ be the collection of proper submodules of L'. Since M is noetherian there is a maximal proper submodule M_{n+1} of L. Then L'/M_{n+1} has no proper submodules, and thus $M_{n+1} \subset M_n \subset \cdots \subset M_1 = M$ is a chain such that M_{i-1}/M_i has no proper submodule for $i = 2, 3, \dots, n+1$. Since $M_{n+1} \subset L$ this is impossible since L' is minimal in $\mathcal{L}$. This contradicts the assumption that L' has proper submodules and we have proved the Proposition.

(1.15) Proposition. *Let A be a ring and $0 \to M' \xrightarrow{u} M \xrightarrow{v} M'' \to 0$ an exact* *sequence of A-modules.*

(1) *The A-module M is of finite length if and only if the A-modules M' and M'' are of finite length.*

(2) *When all the modules are of finite length we have that $\ell(M) = \ell(M') + \ell(M'')$.*

Proof. (1) It follows from Proposition (1.7) and Proposition (1.14) that the first part of the Proposition holds.

(2) To prove the second part of the Proposition we take a composition series $0 = M'_0 \subset M'_1 \subset \cdots \subset M'_{n'} = M'$ of M' and a composition series $(0) = M''_0 \subset M''_1 \subset \cdots \subset M''_{n''} = M''$ of M''. Then $(0) = u(M'_0) \subset u(M'_1) \subset \cdots \subset u(M'_{n'}) = u(M') = v^{-1}(0) = v^{-1}(M''_0) \subset v^{-1}(M''_1) \subset \cdots \subset v^{-1}(M''_{n''}) = v^{-1}(M'') = M$ is a composition series of length $n' + n''$ for M. In fact the homomorphism u clearly induces an isomorphism $M'_i/M'_{i-1} \to u(M'_i)/u(M'_{i-1})$, and the homomorphism v clearly induces an isomorphism $v^{-1}(M''_i)/v^{-1}(M''_{i-1}) \to M''_i/M''_{i-1}$.

(1.16) Remark. Let $\varphi : A \to B$ be a surjection of rings and let N be a B-module of finite length. Then we have that $\ell_B(N) = \ell_A(N_{[\varphi]})$. In fact every A-submodule of N is also a B-submodule.

(1.17) Theorem. (The Hölder Theorem) *Let A be a ring and let M be an A module of finite lenght. Two composition series*

$$\{0\} = M_0 \subset M_1 \subset \cdots \subset M_n = M \tag{1.13.1}$$

and

$$\{0\} = M'_0 \subset M'_1 \subset \cdots \subset M'_n = M \tag{1.13.2}$$

of M are equivalent, that is, there is a permutation σ of the numbers $\{1, 2, \ldots, n\}$ such that $M_{\sigma(i)}/M_{\sigma(i)-1}$ is isomorphic to M'_i/M'_{i-1} for $i = 1, 2, \ldots, n$.

Proof. We show the Theorem by induction on $\ell(M)$. It is trivially true when $\ell(M) = 1$. Assume that it holds when $\ell(M) = n - 1$. If $M_{n-1} = M'_{n-1}$ the Theorem holds by the induction hypothesis.

Assume that $M_{n-1} \neq M'_{n-1}$. We choose a composition series $\{0\} = L_0 \subset L_1 \subset \cdots \subset L_{n-2} = M_{n-1} \cap M'_{n-1}$ of $M_{n-1} \cap M'_{n-1}$. Then we obtain two composition series

$$(0) = L_0 \subset L_1 \subset \cdots \subset L_{n-2} = M_{n-1} \cap M'_{n-1} \subset M_{n-1} \subset M_n = M \tag{1.13.3}$$

$$(0) = L_0 \subset L_1 \subset \cdots \subset L_{n-2} = M_{n-1} \cap M'_{n-1} \subset M'_{n-1} \subset M'_n = M \tag{1.13.4}$$

for the module M. Since M/M_{n-1} and M/M'_{n-1} have no proper submodules and $M_{n-1} \neq M'_{n-1}$ it follows from Lemma (MODULES 1.13) that the inclusions $M_{n-1} \subset M$ and $M'_{n-1} \subset M$ induce isomorphisms $M_{n-1}/M_{n-1} \cap M'_{n-1} \to M/M'_{n-1}$ and $M'_{n-1}/M_{n-1} \cap M'_{n-1} \to M/M_{n-1}$. Hence (1.13.3) and (1.13.4) are equivalent composition series. The composition series (1.13.1) and (1.13.3) have M_{n-1} in common. Hence it follows from the induction hypothesis used to the module M_{n-1} that the composition series are equivalent. Similarly the composition series (1.13.2) and (1.13.4) are equivalent since they have M'_{n-1} in common. It follows that (1.13.1) and (1.13.2) are equivalent composition series, as we wanted to prove.

(1.18) Exercises.

1. Let L be the submodule of the $\mathbf{Z}$-module $\mathbf{Z} \times \mathbf{Z}$ generated by the elements $(2, 3)$.

(1) Is the residue module $(\mathbf{Z} \times \mathbf{Z})/L$ noetherian?
(2) Is the residue module $(\mathbf{Z} \times \mathbf{Z})/L$ artinian?

2. Find all composition series for the module $\mathbf{Z}/12\mathbf{Z}$.

3. Let $K[t]$ be the polynomial ring in the variable t over the field K. Find the composition series of the $K[t]$-module $K[t]/(t^3(t + 1)^2)$.

4. Let A be a ring and $0 \to M_0 \to M_1 \to \cdots \to M_n \to 0$ be an exact sequence of A-modules of finite length. Show that $\sum_{i=0}^{n}(-1)^i \ell(M_i) = 0$.

5. Let $0 \xrightarrow{u_{-1}} M_0 \xrightarrow{u_o} M_1 \xrightarrow{u_1} \cdots \xrightarrow{u_{n-1}} M_n \xrightarrow{u_n} 0$ be a complex of modules of finite length.

(1) Show that the A-modules $H_i = \operatorname{Ker}(u_i)/\operatorname{Im}(u_{i-1})$ is of finite length for $i = 0, 1, \dots n$.
(2) Show that $\sum_{i=0}^{n}(-1)^i \ell(M_i) = \sum_{i=0}^{n}(-1)^i \ell(H_i)$.

6. Let A be a ring and $\mathfrak{m}$ a maximal ideal. Moreover let M be a finitely generated A-module.

(1) Show that the A-module $M/\mathfrak{m}M$ is of finite length.
(2) Give an example of a ring A, a prime ideal $\mathfrak{p}$ of A, and a finitely generated A-module M such that $M/\mathfrak{p}M$ is not of finite length.

7. Let M be a noetherian A-module, and let $u : M \to M$ be an A-linear surjective map. Show that u is an isomorphism.

8. Let M be an artinian A-module, and let $u : M \to M$ be an A-linear injective homomorphism. Show that u is an isomorphism.

2. Artinian and noetherian rings.

(2.1) Definition. A ring A is *noetherian*, respectively *artinian*, if it is noetherian, respectively artinian, considered as an A-module. In other words, the ring A is noetherian, respectively artinian, if every chain $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \cdots$ of ideal $\mathfrak{a}_i$ in A is stable, respectively if every chain $\mathfrak{a}_1 \supseteq \mathfrak{a}_2 \supseteq \cdots$ of ideals $\mathfrak{a}_i$ in A is stable.

(2.2) Example. Let $K[t]$ be the polynomial ring in the variable t with coefficients in a field K. Then the residue ring $K[t]/(t^n)$ is artinian and noetherian for all positive integers n. This is because $K[t]/(t^n)$ is a finite dimensional vector space of dimension n.

(2.3) Example. The ring $\mathbf{Z}$ is noetherian, but not artinian. All rings with a finite number of ideals, like $\mathbf{Z}/n\mathbf{Z}$ for $n \in \mathbf{Z}$, and fields are artinian and noetherian.

(2.4) Example. The polynomial ring $A[t_1, t_2, \dots]$ in the variables $t_1, t_2, \dots$ over a ring A is not noetherian since it contains the infinite chain $(t_1) \subset (t_1, t_2) \subset \cdots$ of ideals. It is not artinian either since it contains the infinite chain $(t_1) \supset (t_1^2) \supset (t_1^3) \supset \cdots$.

(2.5) Proposition. *Let A be a ring and let M be a finitely generated A-module.*

(1) *If A is a noetherian ring then M is a noetherian A-module.*
(2) *If A is an artinian ring then M is an artinian A-module.*

Proof. (1) It follows from Proposition (MODULES 1.20) that we have a surjective map $\varphi : A^{\oplus n} \to M$ from the sum of the ring A with itself n times to M. Hence it follows from Proposition (1.7) that M is noetherian.

(2) The proof of the second part is analogous to the proof of the first part.

(2.6) Corollary. *Let $\varphi : A \to B$ be a surjective map from the ring A to a ring B.*

(1) *If the ring A is noetherian then the ring B is noetherian.*
(2) *If the ring A is artinian then the ring B is artinian.*

Proof. (1) Since φ is surjective B is a finitely generated A-module with generator 1. It follows from the Proposition that B is noetherian as an A-module. Then B is clearly noetherian as a B-modules.

(2) The proof of the second part is analogous to the proof of the first part.

(2.7) Proposition. *Let S be a multiplicatively closed subset of a ring A.*

(1) *If A is noetherian then $S^{-1}A$ is noetherian.*
(2) *If A is artinian then $S^{-1}A$ is artinian.*

Proof. (1) It follows from Remark (MODULES 3.13) that every ideal $\mathfrak{b}$ in the localization $S^{-1}A$ satisfies $\varphi_{S^{-1}A}(\mathfrak{b})S^{-1}A = \mathfrak{b}$. Every chain $\mathfrak{b}_1 \subseteq \mathfrak{b}_2 \subseteq \cdots$ of ideals in $S^{-1}A$ therefore gives a chain $\varphi_{S^{-1}A}^{-1}(\mathfrak{b}_1) \subseteq \varphi_{S^{-1}A}^{-1}(\mathfrak{b}_2) \subseteq \cdots$ of ideals in A. Since A is noetherian there is a positive integer n such that $\varphi^{-1}_{S^{-1}A}(\mathfrak{b}_n) = \varphi^{-1}_{S^{-1}A}(\mathfrak{b}_{n+1}) = \cdots$. Consequently we have that $\mathfrak{b}_n = \mathfrak{b}_{n+1} = \cdots$. Hence $S^{-1}A$ is noetherian.

(2) The proof of the second part is analogous to the proof of the first part.

(2.8) Remark. A noetherian ring has only a finite number of minimal prime ideals. This is because $\mathrm{Spec}(A)$ is a noetherian topological space since the descending chains of closed subsets of $\mathrm{Spec}(A)$ correspond to ascending chains of ideals in A by Remark (RINGS 5.2). By Proposition (TOPOLOGY 4.25) $\mathrm{Spec}(A)$ has only a finite number of irreducible components. However, it follows from Proposition (TOPOLOGY 5.13) that the irreducible components of $\mathrm{Spec}(A)$ correspond bijectively to the minimal prime ideals in A.

(2.9) Remark. The radical $\mathfrak{r}_A(0)$ of a noetherian ring A is nilpotent, that is, we have $\mathfrak{r}_A(0)^n = 0$ for some integer n. This follows from Remark (RINGS 4.8) because $\mathfrak{r}_A(0)$ is finitely generated ideal.

(2.10) Theorem. (The Hilbert basis theorem) *Let A be a noetherian ring and B a finitely generated algebra over A. Then B is a noetherian ring.*

Proof. It follows from Proposition (RINGS 3.6) that we have a surjective homomorphism $A[t_1, t_2, \dots, t_n] \to B$ of A-algebras from the polynomial ring $A[t_1, t_2, \dots, t_n]$ in the variables $t_1, t_2, \dots, t_n$ over A. Hence it follows from Corollary (2.6) that is suffices to prove that the polynomial ring $A[t_1, t_2, \dots, t_n]$ is noetherian. If we can prove that the polynomial ring $C[t]$ in one variable t over a noetherian ring C is noetherian, it clearly follows by induction on n that $A[t_1, t_2, \dots, t_n]$ is noetherian. Hence it suffices to prove that $A[t]$ is noetherian.

Let $\mathfrak{b}$ be an ideal in $A[t]$. We shall show that $\mathfrak{b}$ has a finite number of generators. Let $\mathfrak{a}$ be the collection of elements $f \in A$ such that there is a polynomial $f_0 + f_1 t + \cdots + f_{n-1}t^{n-1} + f t^n$ in $\mathfrak{b}$. It is clear that $\mathfrak{a}$ is an ideal in A. Since A is noetherian we can find generators $g_1, g_2, \dots, g_m$ of $\mathfrak{a}$. For every $i = 1, 2, \dots, m$ we can find a polynomial $p_i(t) = g_{i,0} + g_{i,1}t + \cdots + g_{i,d_i-1}t^{d_i-1} + g_i t^{d_i}$ in $\mathfrak{b}$. Let $d = \max_{i=1}^m(d_i)$.

For each polynomial $f(t) = f_0 + f_1 t + \cdots + f_e t^e$ in $\mathfrak{b}$ we can find elements $h_1, h_2, \dots, h_m$ in A such that $f_e = h_1 g_1 + h_2 g_2 + \cdots + h_m g_m$. If $e \geq d$ the polynomial $f(t) = h_1 t^{e-d_1} p_1(t) - h_2 t^{e-d_2} p_2(t) - \cdots - h_m t^{d-d_m} p_m(t)$ is of degree stricly less than e. It follows by descending induction on e that we can find polynomials $h_1(t), h_2(t), \dots, h_m(t)$ such that $g(t) = f(t) - \sum_{i=1}^m h_i(t)p_i(t)$ is of degree strictly less than d. Since $f(t) \in \mathfrak{b}$, and all the polynomials $p_i(t)$ are in $\mathfrak{b}$, we have that $g(t) \in \mathfrak{b}$. Hence $g(t)$ is in the A-module $M = (A + tA + \cdots + t^{d-1}A) \cap \mathfrak{b}$. It follows from Corollary (1.8) and Proposition (1.7) that M is a noetherian module. Hence we can find a finite number of generators $q_1(t), q_2(t), \dots, q_n(t)$ of M. Then $\mathfrak{b}$ will be generated by the polynomials $p_1(t), p_2(t), \dots, p_m(t), q_1(t), q_2(t), \dots, q_n(t)$. Hence $\mathfrak{b}$ is finitely generated as we wanted to prove. Since all ideals $\mathfrak{b}$ of B are finitely generated it follows from Lemma (1.6) that B is noetherian as a module over itself, and hence noetherian.

(2.11) Proposition. *In an artinian ring all the prime ideals are maximal.*

Proof. Let $\mathfrak{p}$ be a prime ideal. We must show that for each element $f \in A \setminus \mathfrak{p}$ we have that $Af + \mathfrak{p} = A$. Since A is artinian the chain $Af + \mathfrak{p} \supseteq Af^2 + \mathfrak{p} \supset \cdots$ must stabilize. Hence there is a positive integer n such that $f^n = gf^{n+1} + h$ for some $g \in A$ and $h \in \mathfrak{p}$. Hence $f^n(1 - gf) \in \mathfrak{p}$. Since $\mathfrak{p}$ is a prime ideal and $f \notin \mathfrak{p}$ we have that $1 - gf \in \mathfrak{p}$. Hence there is an $e \in \mathfrak{p}$ such that $1 - gf = e$. The ideal $Af + \mathfrak{p}$ consequently contains the element $gf - e = 1$ and thus is equal to A is we wanted to prove.

(2.12) Proposition. *Let A be a ring and $\mathfrak{m}_1, \mathfrak{m}_2, \cdots$ different maximal ideals in A. Then $\mathfrak{m}_1\mathfrak{m}_2 \cdots \mathfrak{m}_n$ is a proper submodule of $\mathfrak{m}_1\mathfrak{m}_2 \cdots \mathfrak{m}_{n-1}$.*

Proof. Since the ideals $\mathfrak{m}_i$ are maximal we can for each $i = 1, 2, \ldots, n-1$ find an element $f_i \in \mathfrak{m}_i \setminus \mathfrak{m}_n$. Assume that $\mathfrak{m}_1\mathfrak{m}_2 \cdots \mathfrak{m}_{n-1} = \mathfrak{m}_1\mathfrak{m}_2 \cdots \mathfrak{m}_n$. Then we have that $f_1 f_2 \cdots f_{n-1} \in \mathfrak{m}_1\mathfrak{m}_2 \cdots \mathfrak{m}_{n-1} = \mathfrak{m}_1\mathfrak{m}_2 \cdots \mathfrak{m}_n \subseteq \mathfrak{m}_n$, which is impossible since $\mathfrak{m}_n$ is a prime ideal and $f_i \notin \mathfrak{m}_n$ for $i = 1, 2, \ldots, n-1$. This contradicts the assumption that $\mathfrak{m}_1\mathfrak{m}_2 \cdots \mathfrak{m}_n = \mathfrak{m}_1\mathfrak{m}_2 \cdots \mathfrak{m}_{n-1}$. Hence $\mathfrak{m}_1\mathfrak{m}_2 \cdots \mathfrak{m}_n$ is a proper submodule of $\mathfrak{m}_1\mathfrak{m}_2 \cdots \mathfrak{m}_{n-1}$.

(2.13) Corollary. *An artinian ring has a finite number of maximal ideals.*

Proof. If it had an infinte number of maximal ideals we could find an infinite sequence $\mathfrak{m}_1, \mathfrak{m}_2, \cdots$ of different maximal ideals. Then it follows from the Proposition that we have an infinite chain $\mathfrak{m}_1 \supset \mathfrak{m}_1\mathfrak{m}_2 \supset \cdots$ of ideals in A. This contradicts that A is artinian. Thus A has only a finite number of maximal ideals.

(2.14) Proposition. *In an artinian ring the radical is nilpotent.*

Proof. Since A is artinian the sequence of ideals $\mathfrak{r}_A(0) \supseteq \mathfrak{r}_A(0)^2 \supseteq \cdots$ is stable. Thus there is a positive integer n such that $\mathfrak{a} := \mathfrak{r}_A(0)^n = \mathfrak{r}_A(0)^{n+1} = \cdots$. We shall
n prove that $\mathfrak{a} = 0$. Assume to the contrary that $\mathfrak{a} \neq 0$. Consider the collection !!$\mathcal{B}$ of ideals $\mathfrak{b}$ in A such that $\mathfrak{a}\mathfrak{b} \neq 0$. Then $\mathcal{B}$ is not empty since $\mathfrak{a}$ is in $\mathcal{B}$. Since A is artinian we have that $\mathcal{B}$ contains a minimal element $\mathfrak{c}$. Then there is an $f \in \mathfrak{c}$ such that $\mathfrak{a}f \neq 0$. Since $\mathfrak{c}$ is minimal in $\mathcal{B}$ and $(f) \subseteq \mathfrak{c}$ we must have that $\mathfrak{c} = (f)$. We have that $(f\mathfrak{a})\mathfrak{a} = f\mathfrak{a}^2 = f\mathfrak{a} \neq 0$ and $(f\mathfrak{a}) \subseteq (f) = \mathfrak{c}$. By the minimality of $\mathfrak{c}$ we obtain that $(f\mathfrak{a}) = (f)$. Hence there is an element $g \in \mathfrak{a}$ such that $fg = f$. Hence $f = fg = fg^2 = \cdots$. However, since $g \in \mathfrak{a} \subseteq \mathfrak{r}_A(0)$, we have that $g^n = 0$ for some positive integer n. Thus $f = 0$ which is impossible since $\mathfrak{a}f = \mathfrak{a}\mathfrak{c} \neq 0$. This contradicts the assumption that $\mathfrak{a} \neq 0$. Hence $\mathfrak{a} = 0$ as we wanted to prove.

(2.15) Lemma. *Let A be a ring and let $\mathfrak{m}_1, \mathfrak{m}_2, \ldots, m_n$ be, not necessarily different, maximal ideals in A such that $\mathfrak{m}_1\mathfrak{m}_2 \cdots \mathfrak{m}_n = 0$. Then A is artinian if and only if A is noetherian.*

Proof. We have a chain $A = \mathfrak{m}_0 \supset \mathfrak{m}_1 \supseteq \mathfrak{m}_1\mathfrak{m}_2 \supseteq \mathfrak{m}_1\mathfrak{m}_2\mathfrak{m}_3 \supseteq \cdots \supseteq \mathfrak{m}_1\mathfrak{m}_2 \cdots \mathfrak{m}_n = 0$ of ideals in A. Let $M_i = \mathfrak{m}_1\mathfrak{m}_2 \cdots \mathfrak{m}_{i-1}/\mathfrak{m}_1\mathfrak{m}_2 \cdots \mathfrak{m}_i$ for $i = 1, 2, \ldots, n$. Then each

M_i is an $A/\mathfrak{m}_i$-module, that is, a vector space over $A/\mathfrak{m}_i$. Hence M_i is artinian if and only if it is noetherian. For $i = 1, 2, \dots, n$ we have an exact sequence

$$0 \to \mathfrak{m}_1\mathfrak{m}_2 \cdots \mathfrak{m}_i \to \mathfrak{m}_1\mathfrak{m}_2 \cdots \mathfrak{m}_{i-1} \to M_i \to 0.$$

It follows from Proposition (1.7) that $\mathfrak{m}_1\mathfrak{m}_2 \cdots \mathfrak{m}_i$ and M_i are artinian, respectively noetherian, if and only if $\mathfrak{m}_1\mathfrak{m}_2 \cdots \mathfrak{m}_{i-1}$ is artinian, respectively noetherian. By descending induction on i starting with $M_n = \mathfrak{m}_1\mathfrak{m}_2 \cdots \mathfrak{m}_{n-1}$ we obtain that the module $\mathfrak{m}_1\mathfrak{m}_2 \cdots \mathfrak{m}_i$ is artinian if and only if it is noetherian. For $i = 0$ we obtain that A is artinian if and only if it is noetherian.

(2.16) Remark. Let A be a local noetherian ring with maximal ideal $\mathfrak{m}$, and let $\mathfrak{q}$ be an $\mathfrak{m}$-primary ideal. Then $A/\mathfrak{q}$ is an artinian ring. To show this we first note that $\mathfrak{m} = \mathfrak{r}(\mathfrak{q})$. Since A is noetherian $\mathfrak{m}$ is finitely generated, and thus it follows from Remark (RINGS 4.8) that a power of the maximal ideal in the noetherian local ring $A/\mathfrak{q}$ is zero. Hence it follows from Lemma (2.15) that $A/\mathfrak{q}$ is artinian.

(2.17) Theorem. *A ring is artinian if and only if it noetherian and has dimension* 0.

Proof. When A is artinian it follows from Proposition (2.11) that $\dim(A) = 0$. It follows from Corollary (2.13) that the ring A has a finite number of maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2, \dots, \mathfrak{m}_n$. We have that $\mathfrak{m}_1\mathfrak{m}_2 \cdots \mathfrak{m}_n \subseteq \mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \cdots \cap \mathfrak{m}_n \subseteq \mathfrak{r}_A(0)$. Since $\mathfrak{r}_A(0)$ is nilpotent by Proposition (2.14) it follows from Lemma (2.15) that A is noetherian.

Conversely assume that A is noetherian of dimension 0. Then every prime ideal is maximal, and from Remark (2.8) it follows that A has finitely many maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2, \dots, \mathfrak{m}_n$. Again $\mathfrak{m}_1\mathfrak{m}_2 \cdots \mathfrak{m}_n \subseteq \mathfrak{r}_A(0)$. If follows from Remark (RINGS 4.8) that $\mathfrak{r}_A(0)$ is nilpotent. Hence it follows from Lemma (2.15) that A is artinian.

(2.18) Proposition. *An artinian ring is isomorphic to the direct product of a finite number of local artin rings.*

More precicely, when A is an artinian ring the canonical map $A \to \prod_{x \in \operatorname{Spec}(A)} A_{\mathfrak{j}_x}$ obtained from the localization maps $A \to A_{\mathfrak{j}_x}$ is an isomorphism.

Proof. By Corollary (2.13) we have that $\operatorname{Spec}(A)$ consists of a finite number of points, and by Proposition (2.11) the points are closed. Hence $\operatorname{Spec}(A)$ is a discrete topological space. Since $\mathcal{O}_{\operatorname{Spec}(A)}$ is a sheaf there is an injective map $A = \Gamma(\operatorname{Spec}(A), \mathcal{O}_{\operatorname{Spec}(A)}) \to \prod_{x \in \operatorname{Spec}(A)} A_{\mathfrak{j}_x} = \prod_{x \in \operatorname{Spec}(A)} \mathcal{O}_{\operatorname{Spec}(A),x}$. However each point x is open in $\operatorname{Spec}(A)$. Hence $A_{\mathfrak{j}_x} = \Gamma(\{x\}, \mathcal{O}_{\operatorname{Spec}(A)})$, and $\{x\} \cap \{y\} = \emptyset$ when $x \neq y$. It follows that we can glue any collection of sections $s_x \in \Gamma(\{x\}, \mathcal{O}_{\operatorname{Spec}(A)})$ for $x \in \operatorname{Spec}(A)$ to a section $s \in \Gamma(\operatorname{Spec}(A), \mathcal{O}_{\operatorname{Spec}(A)})$. Hence the map $A \to \prod_{x \in \operatorname{Spec}(A)} \prod A_{\mathfrak{j}_x}$ is also surjective.

(2.19) Exercises.

1. Show that if S is a multiplicatively closed subset of a ring A such that $S^{-1}A$ is noetherian. Then A is not necessarily noetherian.

2. Let $K[t_1, t_2, \dots]$ be the polynomial ring in the infinitely many variables $t_1, t_2, \dots$ over a field K. Morever let $K(t_1, t_2, \dots)$ be the localization of $K[t_1, t_2, \dots]$ in the multiplicatively closed subset of $K[t_1, t_2, \dots]$ consisting of all non-zero elements.

(1) Show that $K(t_1, t_2, \dots)$ is noetherian.
(2) Let $u_1, u_2, \dots$ be independent variables over $K[t_1, t_2, \dots]$. Show that the K-algebra map $K[t_1, t_2, \dots, u_1, u_2, \dots] \to K[t_1, t_2, \dots] \otimes_I KK[t_1, t_2, \dots]$ that sends t_i to $t_i \otimes_K 1$ and u_i to $1 \otimes_K t_i$ is an isomorphism.
(3) Let S be the multiplicatively closed subset of $K[t_1, t_2, \dots, u_1, u_2, \dots]$ consisting of all non-zero products fg with f in $K[t_1, t_2, \dots]$ and g in $K[u_1, u_2, \dots]$. Show that the K-algebra homomorphism of part (3) induces a canonical homomorphism $S^{-1}K[t_1, t_2, \dots, u_1, u_2, \dots] \to K(t_1, t_2, \dots) \otimes_K K(t_1, t_2, \dots)$.
(4) Show that the homomorphism of part (4) is an isomorphism.
(5) Show that the ideal $(t_1 - u_1, t_2 - u_2, \cdots)$ of $K[t_1, t_2, \dots, u_1, u_2, \dots]$ does not intersect S.
(6) Show that $K(t_1, t_2, \dots) \otimes_K K(t_1, t_2, \dots)$ is not Noetherian.

3. Let A a ring. Give an example of a ring A that is not noetherian, but is such that $\operatorname{Spec}(A)$ is noetherian.

4. Let M be a noetherian A-module. Show that the ring $A/\operatorname{Ann}_A(M)$ is noetherian.

5. Prove that there is only a finite number of minimal primes in a noetherian ring A without using properties of the topological space $\operatorname{Spec}(A)$.

6. Let A be a ring. We say that two ideals $\mathfrak{a}$ and $\mathfrak{b}$ in A are *coprime* if $\mathfrak{a} + \mathfrak{b} = A$. Let $\mathfrak{a}_1, \mathfrak{a}_2, \dots, \mathfrak{a}_n$ be ideals of A that are pairwise comprime. We define a map

$$\varphi : A \to \prod_{i=1}^{n} A/\mathfrak{a}_i$$

by $\varphi(f) = (\varphi_{A/\mathfrak{a}_1}(f), \varphi_{A/\mathfrak{a}_2}(f), \dots, \varphi_{A/\mathfrak{a}_n}(f))$ for all $f \in A$.

(1) Show that if $\mathfrak{a}$ and $\mathfrak{b}$ are coprime, then $\mathfrak{a}^m$ and $\mathfrak{b}^n$ are coprime for all positive integers m and n.
(2) Show that for all i the ideals $\mathfrak{a}_i$ and $\cap_{i \neq j} \mathfrak{a}_j$ are coprime.
(3) Show that the homomorphism φ is a ring homomorphism with kernel $\cap_{i=1}^{n} \mathfrak{a}_i$.
(4) Show that the homomorphism φ is surjective.
(5) Use parts (1), (2), (3), and (4) to prove that an artin ring is the direct product of a finite number of artinian rings.

7. Let $\mathfrak{q}$ be a primary ideal in the ring A, and let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals in A such that $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{q}$. Show that either $\mathfrak{a} \subseteq \mathfrak{q}$ or there is a positive integer n such that $\mathfrak{b}^n \subseteq \mathfrak{q}$.

8. Let A and B be noetherian local rings and let $\varphi : A \to B$ be a local homomorphism, that is, we have $\varphi^{-1}(\mathfrak{m}_B) \subseteq \mathfrak{m}_A$. Assume that the following three conditions hold:

(1) The induced map $A/\mathfrak{m}_A \to B/\mathfrak{m}_B$ of residue rings is an isomorphism.
(2) The induced map $\mathfrak{m}_A \to \mathfrak{m}_B/\mathfrak{m}_B^2$ is surjective.
(3) We have that B is a finitely generated A-module via φ.

Show that φ is surjective.

9. Show that whtn $A[t]$ is noetherian then A is noetherian.

10. Let $K[t_1, t_2, \ldots]$ be a polynomial ring over the field K in the independent variables $t_1, t_2, \ldots$ and let A be the residue ring of $K[t_1, t_2, \ldots]$ modulo the ideal generated by the elements $t_i(t_i - 1)$ for $i = 1, 2, \ldots$.

(1) Show that all the prime ideals of A are of the form $(t_1 - \delta_1, t_2 - \delta_2, \ldots)$ where the elements δ_i are either 0 or 1.
(2) Show that all the prime ideals of A are maximal.
(3) Show that for all prime ideal $\mathfrak{p}$ of A we have a canonical ring isomorphism $A_\mathfrak{p} \xrightarrow{\sim} K$.
(4) Show that A is not noetherian, but that $A_\mathfrak{p}$ is noetherian for all prime ideals $\mathfrak{p}$ of A.

3. Modules over noetherian rings.

(3.1) Proposition. *Let A be a noetherian ring and let $M \neq (0)$ be an A-module. Then M has associated prime ideals.*

When M is finitely generated there is a chain

$$)0) = M_n \subset M_{n-1} \subset \cdots \subset M_1 \subset M_0 = M$$

of submodules of M such that each quotient M_{i-1}/M_i is isomorphic to an A-module of the form $A/\mathfrak{p}_i$, where $\mathfrak{p}_i$ is a prime associated to M.

Proof. Let $\mathcal{I}$ be the collection of ideals in A that are annihilators of elements in M. Then $\mathcal{I}$ is not empty because it contains the elements $\mathrm{Ann}(x)$ for all x in M. Since A is noetherian there is a maximal element $\mathfrak{p} = \mathrm{Ann}(x)$ of $\mathcal{I}$. We shall prove that $\mathfrak{p}$ is a prime ideal and thus associated to M. Let let f, g be elements in A such that $fg \in \mathfrak{p}$ and $f \notin \mathfrak{p}$. Then $fx \neq 0$ and $\mathrm{Ann}_A(fx) \supseteq Ag + \mathfrak{p}$. Since $\mathfrak{p}$ is maximal we must have that $\mathfrak{p} = \mathrm{Ann}_A(fx)$ and thus that $g \in \mathfrak{p}$.

To prove the second part we let $\mathcal{L}$ be the collection of submodules of M for which the Proposition holds. Then $\mathcal{L}$ is not empty because it contains the zero module.
→ Since A is noetherian and M is finitely generated it follows from Proposition (2.5) that M is noetherian. Thus there is a maximal element L in $\mathcal{L}$. We shall show that $L = M$. Assume to the contrary that $L \neq M$. Then there is an associated prime ideal $\mathfrak{p}$ of M/L. Let $\mathfrak{p} = \mathrm{Ann}(y)$ for some $y \in M/L$, and denote by x an element of M whose class in M/L is y. We have an isomorphism $A/\mathfrak{p} \to Ay = (Ax + L)/L$. Since the Proposition holds for L it will consequently hold for $Ax + L$. Hence $Ax + L$ is in $\mathcal{L}$, which is impossible since L is maximal in $\mathcal{L}$. This contradicts the assumption that $L \neq M$. Hence we we must have that $L = M$, and the Proposition holds for M.

(3.2) Proposition. *Let A be a noetherian ring and let M be an A-module. An element $f \in A$ is contained in an associated prime ideal if and only if there is an element $x \neq 0$ in M such that $fx = 0$.*

Proof. Let $\mathrm{Ann}(x)$ be an associated prime ideal in A. If $f \in \mathrm{Ann}(x)$ we have that $x \neq 0$ and $fx = 0$.

→ Conversely, assume that $fx = 0$ for some $x \neq 0$. It follows from Proposition (3.1) that Ax has an associated prime ideal $\mathfrak{p}$. Then $\mathfrak{p} = \mathrm{Ann}_A(gx)$ for some $g \in A$, and consequently $\mathfrak{p}$ is associated to M and $f \in \mathfrak{p}$. Thus f is contained in an associated ideal.

(3.3) Proposition. *Let A be a noetherian ring and let M be an A-module.*

(1) *The support of M consists of the prime ideals in A that contain an associated prime.*
(2) *The intersection $\cap_{\mathfrak{p} \in \mathrm{Supp}(M)} \mathfrak{p}$, which is thus the intersection of all associated ideals of M, consists of all elements $f \in A$ such that $f_M : M \to M$ is locally nilpotent.*

Proof. (1) Assume that $\mathfrak{p}$ is in the support of M, that is, we have $M_\mathfrak{p} \neq 0$. Then there is an element $x \in M$ such that $(Ax)_\mathfrak{p} \neq 0$. It follows from Proposition (3.1) that there is a prime ideal $\mathfrak{q}$ that is associated to the A-module $(Ax)_\mathfrak{p}$. Then there is an element $f \in A$ and an element $s \in A \setminus \mathfrak{p}$ such that $\mathfrak{q} = \operatorname{Ann}_A((fx)/s)$. We have that $\mathfrak{p} \supseteq \mathfrak{q}$ because, if $t \in \mathfrak{q} \setminus \mathfrak{p}$, then $(tfx)/s = 0$ and $0 = (1/t)((tfx)/s) = (fx)/s$ in $(Ax)_\mathfrak{p}$, contradicting that $(fx)/s \neq 0$ in $(Ax)_\mathfrak{p}$.

To prove the first part of the Proposition we prove that $\mathfrak{q}$ is an associated prime ideal of M. Let $f_1, f_2, \dots, f_n$ be generators for $\mathfrak{q}$. Since $\mathfrak{q}$ is the anninhilator of the element fx/s in the A-module $(Ax)_\mathfrak{p}$ we can find elements $s_1, s_2, \dots, s_n$ in $A \setminus \mathfrak{p}$ such that $s_i f_i fx = 0$ in M for $i = 1, 2, \dots, n$. Consequently we have an inclusion $\mathfrak{q} \subseteq \operatorname{Ann}_A(s_1 s_2 \cdots s_n fx)$. We shall prove the opposite inclusion. Take an element $g \in \operatorname{Ann}_A(s_1 s_2 \cdots s_n fx)$. Since $s_1 s_2 \cdots s_n gfx = 0$ and $s_1 s_2 \cdots s_n \notin \mathfrak{p}$, we have that $(gfx)/s = 0$ in $(Ax)_\mathfrak{p}$. However, then we have that $g \in \mathfrak{q}$, and we have proved that $\mathfrak{q} = \operatorname{Ann}_A(s_1 s_2 \cdots s_n fx)$. Hence the prime ideal $\mathfrak{q}$ is associated to M. We have proved that every ideal in the support contains an associated prime ideal. In Remark (MODULES 4.13) we saw that every associated prime ideal is contained in the support. Hence every prime ideal that contains an associated ideal is in the support.

(2) Assume that $f \in A$ is not in the intersection of all the prime ideals in the support. Then there is a prime ideal $\mathfrak{p}$ of A with $M_\mathfrak{p} \neq 0$ and $f \notin \mathfrak{p}$. Let $x \in M$ and $s \notin \mathfrak{p}$ be such that $x/s \neq 0$ in $M_\mathfrak{p}$. Since $f \notin \mathfrak{p}$ we have that $f^n x/s \neq 0$ in $M_\mathfrak{p}$, and thus $f^n x \neq 0$ in M for all positive integers n. Consequently f is not locally nilpotent.

Finally let $f \in A$ be an element in the intersection of all the ideals in the support of M. We shall show that f_M is locally nilpotent. Assume to the contrary that f is not locally nilpotent. Then there is an $x \in M$ such that $f^n \notin \operatorname{Ann}(x)$ for all positive integers n. It follows from Proposition (RINGS 4.16) that we can find a prime ideal $\mathfrak{p}$ that contains $\operatorname{Ann}(x)$ but does not contain f. Then we have that $(Ax)_\mathfrak{p} \neq 0$, and thus $\mathfrak{p}$ is contained in the support of M. This contradicts the assumption that f is in the intersection of all ideals in the support. Hence we have proved that f_M is locally nilpotent.

(3.4) Remark. Let A be a noetherian ring and M a finitely generated A-module. It follows from Remark (MODULES 4.8) that the locally nilpotent elements are the elements of $\mathfrak{r}(\operatorname{Ann}(M))$ and hence it follows from Proposition (3.3) that the radical $\mathfrak{r}(\operatorname{Ann}(M))$ of M is equal to the intersection of the prime ideals of the support of M, or equivalently, to the intersection of the associated ideals of M.

In particular we have that $\operatorname{Supp}(M) = V(\operatorname{Ann}(M))$.

(3.5) Proposition. *Let A be a noetherian ring and let M be an A-module. The following assertions are equivalent:*

(1) *The module M has exactly one associated prime ideal.*
(2) *We have that $M \neq 0$ and for every element f in A either f_M is injective or*

locally nilpotent.

When the assertions hold the associated ideal of M consists of the locally nilpotent elements.

Proof. (1) $\Rightarrow$ (2) If there is only one associated prime ideal $\mathfrak{p}$ it follows from Proposition (3.3) that when $f \in \mathfrak{p}$ the map f_M is locally nilpotent. Moreover it follows from Proposition (3.2) that for $f \notin \mathfrak{p}$ the map f_M is injective.

(2) $\Rightarrow$ (1) If f_M is locally nilpotent it follows from Proposition (3.2) that the element $f \in A$ is contained in some associated prime ideal. On the other hand, if f_M is injective, it follows from Proposition (3.2) that f is not contained in any associated ideal. Hence it follows from Proposition (3.3) that the union of the associated prime ideals will be equal to their intersection. Hence there can be only one associated prime ideal for M.

We saw in the proofs of both (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1) that when the assertions of the Proposition holds then the associated ideal consists of the locally nilpotent elements.

(3.6) Corollary. *Let A be a noetherian ring and M a finitely generated A-module. Moreover let L be a submodule of M. The following conditions are equivalent:*

(1) *The module M/L has only one associated ideal.*
(2) *The module L is primary.*

When the conditions hold the associated prime ideal of M/L is the prime ideal belonging to L.

Proof. (1) $\Rightarrow$ (2) Let $\mathfrak{p}$ be the associated prime ideal of M/L. By the Proposition and Remark (MODULES 4.8) we have that $M \neq L$, and that L is primary and $\mathfrak{p}$ is the ideal belonging to L.

(2) $\Rightarrow$ (1) If $M \neq L$ and L is primary it follows from the Proposition that M/L has only one associated ideal.

(3.7) Proposition. *Let A be a noetherian ring and let M be a finitely generated A-module. Then every submodule L of Mcan be written as an intersection $L = L_1 \cap L_2 \cap \cdots \cap L_n$ of submodules L_i of M such that each module L_i is primary.*

Proof. Consider the set $\mathcal{L}$ of submodules L of M that can not be written as $L = L_1 \cap L_2 \cap \cdots \cap L_m$ with all L_i primary. We shall show that $\mathcal{L}$ is empty. Assume to the contrary that it is not empty. Since M is noetherian it follows that $\mathcal{L}$ then has a maximal element L. In particular L is not primary. Thus there is an element $f \in A$ such that the homomorphism $f_{M/L} : M/L \to M/L$ is neither injective nor nilpotent. We therefore obtain a sequence

$$\mathrm{Ker}(f_{M/L}) \subseteq \mathrm{Ker}(f_{M/L}^2) \subseteq \cdots$$

of non-zero proper submodules of M. Since M is noetherian this sequence must stop. Assume that $\mathrm{Ker}(f_{M/L}^r) = \mathrm{Ker}(f_{M/L}^{r+1}) = \cdots$ and let $u = f_{M/L}^r$. We have that $\mathrm{Ker}(u)$ is a proper submodule of M and that $\mathrm{Ker}(u) = \mathrm{Ker}(u^2)$. Consequently $\mathrm{Ker}(u) \cap \mathrm{Im}(u) = (0)$. In particular $\mathrm{Im}(u)$ is different from M/L. Let M_1 and M_2 be the inverse images of $\mathrm{Ker}(u)$ respectively $\mathrm{Im}(u)$ by the canonical map $u_{M/L} : M \to M/L$. Then M_1 and M_2 contain L and are different from L, and $L = M_1 \cap M_2$. By the maximality of L we have that the Proposition holds for M_1 and M_2. Consequently the Proposition holds for L which is impossible since L is in $\mathcal{L}$. This contradicts the assumption that $\mathcal{L}$ is non-empty, and we have proved the Proposition.

(3.8) Proposition. *Let A be a noetherian ring and M a finitely generated A-module. Write $(0) = L_1 \cap L_2 \cap \cdots \cap L_n$ with L_i primary for $i = 1, 2, \ldots, n$, and assume that for each i we have $L_i \not\supseteq \cap_{i \neq j} L_j$. Then the associated primes of M coincide with the primes belonging to the primary modules L_i.*

Proof. We have an injection

$$M \to M/L_1 \oplus M/L_2 \oplus \cdots M/L_n$$

which sends $x \in N$ to $(u_{M/L}(x), u_{M/L}(x), \ldots, u_{M/L}(x))$. It follows from Proposition (MODULES 4.25) that the associated prime ideals of M can be found among the associated primes of $M/L_1, M/L_2, \ldots, M/L_n$. We shall show that the prime ideal $\mathfrak{p}_i$ belonging to L_i is associated to M for $i = 1, 2, \ldots, n$.

We have that $L = L_2 \cap L_3 \cap \cdots \cap L_{i-1} \cap L_{i+1} \cdots \cap L_n \neq (0)$ by assumption. Since $L = L/L \cap L_i$ it follows from Lemma (MODULES 1.13) that we have an injective A-module homomorphism $L \to M/L_i$. It follows from Proposition (3.1) that L has an associated ideal, and from Corollary (3.6) that this ideal must be $\mathfrak{p}_i$. From Proposition (MODULES 4.25) it follows that $\mathfrak{p}_i$ is also associated to M.

(3.9) Proposition. *Let A be a noetherian ring. If A is reduced the associated primes are the minimal prime ideals.*

Proof. It follows from Proposition (3.3) that every prime ideal contains an associated prime. Hence every minimal prime ideal is associated.

Conversely let $\mathfrak{p} = \mathrm{Ann}(f)$ be an associated prime ideal of A. In Remark (2.8) we observed that A has only a finite number of minimal primes $\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_n$. Assume that $\mathfrak{p}$ is not minimal. Then it follows from Proposition (RINGS 4.22) that we can find an element $t \in \mathfrak{p} \setminus \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n$. Then $f\mathfrak{p} = 0$, and consequently $tf = 0$. Thus $f \in \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_n$. However the intersection of the minimal prime ideals is the radical of A and thus $f^n = 0$ for some integer n. Since A is reduced $f = 0$, which is impossible since $\mathfrak{p} = \mathrm{Ann}(f)$. This contradicts the assumption that $\mathfrak{p}$ is not minimal, and we have proved that the associated prime ideals are minimal.

(3.10) Exercises.

1. Find the associated prime ideals of the $\mathbf{Z}$-module $\mathbf{Z}/12\mathbf{Z}$, and write (0) in $\mathbf{Z}/12\mathbf{Z}$ as an intersection of primary modules.

2. Let K be a field and let $K[u, v]$ be the polynomial ring over K in the independent variables u, v.

(1) Find the associated prime ideals of the $K[u, v]$-module $M = K[u, v]/(u^2, uv)$.
(2) Write $(0) \in M$ as an intersection of primary modules.

3. Let A be a noetherian ring and let $\mathfrak{q}$ be a $\mathfrak{p}$-primary ideal. Show that $\mathfrak{p}^n \subseteq \mathfrak{q}$ for some positive integer n.

4. Let A be a ring. An ideal $\mathfrak{a}$ of A is *irreducible* if $\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c}$ implies that $\mathfrak{a} = \mathfrak{b}$ or that $\mathfrak{a} = \mathfrak{c}$.

(1) Show that $\mathfrak{a}$ is irreducible in A if and only if (0) is irreducible in the residue ring $A/\mathfrak{a}$.
(2) Show that $\mathfrak{a}$ is primary in A if and only if (0) is primary in the residue ring $A/\mathfrak{a}$.
(3) Show that when A is a noetherian ring then every ideal in A is the intersection of irreducible ideals.
(4) Assume that A is noetherian the ideal (0) in A is irreducible. Let $fg = 0$ with $g \neq 0$ in A. Let n be such that $\mathrm{Ann}(f^n) = \mathrm{Ann}(f^{n+1} = \cdots$. Show that $(f^n) \cap (g) = 0$.
(5) Show that when A is noetherian then every irreducible ideal is primary.

5. Let A be a noetherian ring. Moreover let $\mathfrak{m}$ be a maximal ideal and $\mathfrak{q}$ and ideal contained in $\mathfrak{m}$. Show that the following assertions are equivalent:

(1) The ideal $\mathfrak{q}$ is $\mathfrak{m}$-primary.
(2) $\mathfrak{r}(\mathfrak{q}) = \mathfrak{m}$.
(3) There is a positive integer n such that $\mathfrak{m}^n \subseteq \mathfrak{q} \subseteq \mathfrak{m}$.

Graded rings and dimension

1. Gradings and filtrations.

n **(1.1) Definition.** A ring A is *graded* if it is the direct sum $!!\oplus_{n=0}^{\infty} A_n$ of subgroups A_n such that $A_m A_n \subseteq A_{m+n}$ for all natural numbers m and n. We say that an
n A-module M is *graded* if it is the direct sum $!\oplus_{n=0}^{\infty} M_n!$ of subgroups M_n such that $A_m M_n \subseteq M_{m+n}$ for all natural numbers m and n.

The elements in A_n and M_n are called *homogeneous* of degree n. When $x \in M$ and $x = \sum_{n=1}^{m} x_n$ with $x_n \in M_n$ we call the elements $x_1, x_2, \dots, x_m$ the *homogeneous components* of x. For all negative integers n we let $A_n = 0$ and $M_n = 0$.

An ideal $\mathfrak{a}$ of A is *homogeneous* if $\mathfrak{a} = \oplus_{n=0}^{\infty} \mathfrak{a}_n$ with $\mathfrak{a}_n \subseteq A_n$.

(1.2) Remark. It follows from the definitions that A_0 is a ring, and that M_n is an A_0-module for each n. We have that A is an A_0-algebra via the inclusion of A_0 in A.

An ideal is homogeneous if and only if it is generated by homogeneous elements.

(1.3) Example. Let A be a ring and $A[t_\alpha]_{\alpha \in I}$ be the polynomial ring in the variables $\{t_\alpha\}_{\alpha \in I}$ for some index set I. Then the elements $t^\mu = \prod_{\alpha \in I} t_\alpha^{\mu(\alpha)}$ with $\mu \in \mathbf{N}^{(I)}$ and $\sum_{\alpha \in I} \mu(\alpha) = n$ generate an A-module $(A[t_\alpha]_{\alpha \in I})_n$, and $A[t_\alpha]_{\alpha \in I}$ is a graded ring with homogeneous elements $(A[t_\alpha]_{\alpha \in I})_n$ of degree n.

Let $\mathfrak{a} \subseteq A[t_\alpha]_{\alpha \in I}$ be an ideal such that for each element $f = \sum_{i=1}^{m} f_i$ in $\mathfrak{a}$ with $f_i \in A_i$ we have that $f_i \in \mathfrak{a}$. Then $\mathfrak{a} = \oplus_{n=0}^{\infty} \mathfrak{a}_n$, with $\mathfrak{a}_n = \mathfrak{a} \cap A_n$, is a graded $A[t_\alpha]_{\alpha \in I}$-module. That is, the ideal $\mathfrak{a}$ is homogeneous. Moreover we have that $A/\mathfrak{a} = \oplus_{n=0}^{\infty} A_i/\mathfrak{a}_i$ is a graded ring.

n **(1.4) Example.** Let A be a ring and $\mathfrak{a}$ an ideal in A. Then the direct sum $!!R_{\mathfrak{a}}(A) = \oplus_{n=0}^{\infty} \mathfrak{a}^n$ of the ideals $\mathfrak{a}^n$, with $\mathfrak{a}^0 = A$, is a graded ring where the multiplication of elements in $\mathfrak{a}^m$ with elements in $\mathfrak{a}^n$ is given by the multiplication in A. The ring $R_{\mathfrak{a}}(A)$ is called the *Rees-ring* of $\mathfrak{a}$, or the *Rees-algebra* of $\mathfrak{a}$ when it is considered as an A_0-algebra.

Let M be an A-module. We have that the direct sum $R_{\mathfrak{a}}(M) = \oplus_{n=0}^{\infty} \mathfrak{a}^n M$ is an $R_{\mathfrak{a}}(A)$-module, where the multiplication of the elements of $\mathfrak{a}^m$ with the elements in $\mathfrak{a}^n M$ is defined by the operation of A on M.

n **(1.5) Example.** Let A be a ring and $\mathfrak{a}$ an ideal in A. The direct sum $!!G_{\mathfrak{a}}(A) = \oplus_{n=0}^{\infty} \mathfrak{a}^n/\mathfrak{a}^{n+1}$ of the $A/\mathfrak{a}$-modules $\mathfrak{a}^n/\mathfrak{a}^{n+1}$, with $\mathfrak{a}^0/\mathfrak{a} = A/\mathfrak{a}$, is a graded ring. To define the multiplication we let $g_m \in \mathfrak{a}^m/\mathfrak{a}^{m+1}$ and $g_n \in \mathfrak{a}^n/\mathfrak{a}^{n+1}$ be the classes of

$f_m \in \mathfrak{a}^m$, respectively $f_n \in \mathfrak{a}^n$. We define $g_m g_n \in \mathfrak{a}^{m+n}/\mathfrak{a}^{m+n+1}$ as the class of the element $f_m f_n \in \mathfrak{a}^{m+n}$. It is clear that the definition is independent of the choice of the representatives f_m and f_n of the classes g_m, respectively g_n. We call the ring $G_{\mathfrak{a}}(A)$ the *graded ring* of $\mathfrak{a}$, or the *graded algebra* of $\mathfrak{a}$ when we consider $G_{\mathfrak{a}}(A)$ as an $A/\mathfrak{a}$-algebra.

Let M be an A-module, and let $G_{\mathfrak{a}}(M) = \oplus_{n=1}^{\infty} \mathfrak{a}^n M/\mathfrak{a}^{n+1}M$. Then $G_{\mathfrak{a}}(M)$ is an $G_{\mathfrak{a}}(A)$-module. To define the operation of $G_{\mathfrak{a}}(A)$ on $G_{\mathfrak{a}}(M)$ we let $g \in \mathfrak{a}^m/\mathfrak{a}^{m+1}$ and $y \in \mathfrak{a}^n M/\mathfrak{a}^{n+1}M$ be the classes of $f \in \mathfrak{a}^m$ respectively of $x \in \mathfrak{a}^n M$. Then we define the product $gy \in \mathfrak{a}^{m+n}M/\mathfrak{a}^{m+n+1}M$ as the class of $fx \in \mathfrak{a}^{m+n}$. It is clear that the product is independent of the choice of the representatives f and x for the classes in $\mathfrak{a}^m/\mathfrak{a}^{m+1}$ respectively $\mathfrak{a}^n M/\mathfrak{a}^{n+1}M$.

(1.6) Proposition. *Let $A = \oplus_{n=0}^{\infty} A_n$ be a graded ring. The following two condition are equivalent:*

(1) *The ring A is noetherian.*
(2) *The ring A_0 is noetherian and A is a finitely generated A_0-algebra.*

Proof. (2) $\Rightarrow$ (1) It follows immediately from The Hilbert Basis Theorem (CHAINS 2.10) that, when condition (2) is fulfilled, then condition (1) is fulfilled.

(1) $\Rightarrow$ (2) We have that the ring A_0 is isomorphic to the residue ring of A modulo the ideal $A_+ = \oplus_{n=1}^{\infty} A_n$. Hence it follows from Corollary (CHAINS 2.6) that A_0 is noetherian.

Since A is noetherian we have that A_+ is a finitely generated A-module. If we, if necessary, take all the homogeneous components of a finite set of generators for A_+, we obtain homogeneous generators $f_1, f_2, \dots, f_m$ of the A-module A_+. Let $f_i \in A_{p_i}$ and write $B = A_0[f_1, f_2, \dots, f_m]$. We shall show that $A = B$. To show that $A = B$ it suffices to show that $A_n \subseteq B$ for all $n \in \mathbf{N}$. This is proved by induction on n. For $n = 0$ it is clear. Assume that $A_{n-1} \subseteq B$. For every element $g \in A_n$ we have that $g = \sum_{i=1}^{m} g_i f_i$ with $g_i \in A_{n-p_i}$. Since $p_i > 0$ for $i = 1, 2, \dots, m$ it follows from the induction assumption that $g_i \in B$ for $i = 1, 2, \dots, m$. Hence we have that $g \in B$ and we have proved that $A_n \subseteq B$. Hence we have that $A = B$, and A is a finitely generated A_0-algebra.

(1.7) Definition. Let A be a ring and $\mathfrak{a}$ an ideal of A. Moreover let M be an A-module. A *filtration* !! $\{M_n\}_{n \in \mathbf{N}}$ of M is a sequence of submodules !!$M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots$ of M. The filtration is an $\mathfrak{a}$-*filtration* if $\mathfrak{a}M_n \subseteq M_{n+1}$ for all n. We say that an $\mathfrak{a}$-filtration is $\mathfrak{a}$-*stable* if $\mathfrak{a}M_n = M_{n+1}$ for all *sufficiently large* n. For all *sufficiently large* n means that there is an integer m such that the property holds for $n \geq m$.

(1.8) Remark. Let M be an A-module and let $\{M_n\}_{n \in \mathbf{N}}$ be an $\mathfrak{a}$-stable filtration, and m an integer such that $\mathfrak{a}M_n = M_{n+1}$ for $n \geq m$. Then we have that $M_{m+n} = \mathfrak{a}^n M_m$ for $n = 0, 1, \dots$.

(1.9) Example. Let A be a ring and $\mathfrak{a}$ an ideal in A. Moreover let M be an $\mathfrak{a}$-module. Then we have a filtration $M = \mathfrak{a}^0 M \supseteq \mathfrak{a}M = \mathfrak{a}^1 M \supseteq \mathfrak{a}^2 M \supseteq \cdots$ of M. This filtration is $\mathfrak{a}$-stable.

(1.10) Example. Let A be a ring and let $\mathfrak{a}$ be an ideal in A. Moreover let M be an A-module and let $\{M_n\}_{n\in\mathbf{N}}$ be an $\mathfrak{a}$-filtration. Then the direct sum $\oplus_{n=0}^{\infty} M_n$ of the A-modules M_n is a graded $R_{\mathfrak{a}}(A) = \oplus_{n=1}^{\infty}\mathfrak{a}^n$-module. The product of $f \in \mathfrak{a}^m$ with $x \in M_n$ is $fx \in \mathfrak{a}^m M_n \subseteq M_{m+n}$.

(1.11) Example. Let A be a ring and $\mathfrak{a}$ an ideal of A. Moreover let M be an A-module, and let $\{M_n\}_{n\in\mathbf{N}}$ be an $\mathfrak{a}$-filtration. Then the direct sum $\oplus_{n=0}^{\infty} M_n/M_{n+1}$ of the $A/\mathfrak{a}$-modules M_n/M_{n+1} is a graded $G_{\mathfrak{a}}(A) = \oplus_{n=0}^{\infty}\mathfrak{a}^n/\mathfrak{a}^{n+1}$-module. The product of the class in $\mathfrak{a}^m/\mathfrak{a}^{m+1}$ of $f \in \mathfrak{a}^m$ and the class in M_n/M_{n+1} of $x \in M_n$ is the class in M_{m+n}/M_{m+n+1} of $fx \in M_{m+n}$. Again the definition of the product is independent of the representatives of the classes of f and x.

When $M_n = \mathfrak{a}^n M$ for $n = 0, 1, \ldots$ we obtain that $\oplus_{n=0}^{\infty}\mathfrak{a}^n M/\mathfrak{a}^{n+1}M$ is the $G_{\mathfrak{a}}(A)$-module $G_{\mathfrak{a}}(M)$ defined in Example (1.5).

(1.12) Lemma. *Let A be a ring and $\mathfrak{a}$ an ideal in A. Moreover let $\{M_n\}_{n\in\mathbf{N}}$ and $\{M'_n\}_{n\in\mathbf{N}}$ be $\mathfrak{a}$-stable filtrations of an A-module M. Then there is a positive integer m such that*

$$M_{m+n} \subseteq M'_n \quad \textit{and} \quad M'_{m+n} \subseteq M_n \quad \textit{for } n = 0, 1, \ldots.$$

Proof. Since $\mathfrak{a}M_n \subseteq M_{n+1}$ and $\mathfrak{a}M'_n \subseteq M'_{n+1}$ we have that $\mathfrak{a}^n M \subseteq M_n$ and $\mathfrak{a}^n M \subseteq M'_n$ for $n = 0, 1, \ldots$. The filtrations $\{M_n\}_{n\in\mathbf{N}}$ and $\{M'_n\}_{n\in\mathbf{N}}$ are $\mathfrak{a}$-stable. Hence we can find natural numbers p, p' in $\mathbf{N}$ such that $M_{n+p} = \mathfrak{a}^n M_p$ and $M'_{n+p'} = \mathfrak{a}^n M'_{p'}$ for $n \geq p + p'$. Let $m = p + p'$. Then we have that $M_{m+n} = \mathfrak{a}^{n+p'} M_p \subseteq \mathfrak{a}^n M \subseteq M'_n$ and $M'_{m+n} = \mathfrak{a}^{n+p} M'_{p'} \subseteq \mathfrak{a}^n M \subseteq M_n$ for $n = 0, 1, 2, \ldots$.

(1.13) Lemma. *Let A be a noetherian ring and let $\mathfrak{a}$ be an ideal in A. Moreover let M be a finitely generated A-module and let $\{M_n\}_{n\in\mathbf{N}}$ be an $\mathfrak{a}$-filtration. The following conditions are equivalent:*

(1) *The graded $R_{\mathfrak{a}}(A)$-module $\oplus_{n=0}^{\infty} M_n$ is finitely generated, where $R_{\mathfrak{a}}(A) = \oplus_{n=0}^{\infty}\mathfrak{a}^n$.*
(2) *The filtration $\{M_n\}_{n\in\mathbf{N}}$ is $\mathfrak{a}$-stable.*

Proof. For each m we have that $K_m = \oplus_{n=0}^{m} M_n$ is an A-submodule of $\oplus_{n=0}^{\infty} M_n$. Since M is a finitely generated A-module and A is noetherian, it follows from Lemma (CHAINS 1.6) that all the A-modules M_n are finitely generated. Hence we have that K_m is a finitely generated A-module. The elements of K_m generate the $R_{\mathfrak{a}}(A)$-submodule

$$L_m = M_0 \oplus M_1 \oplus \cdots \oplus M_m \oplus \mathfrak{a}M_m \oplus \mathfrak{a}^2 M_m \oplus \cdots$$

of $\oplus_{n=0}^{\infty} M_n$. Since K_m is a finitely generated A-module we clearly have that L_m is a finitely generated $R_{\mathfrak{a}}(A)$-module. We have that $L_0 \subseteq L_1 \subseteq L_2 \subseteq \cdots$ and that

$\cup_{n=0}^{\infty} L_n = \oplus_{n=0}^{\infty} M_n$. Since $R_{\mathfrak{a}}(A)$ is noetherian by Lemma (CHAINS 1.6) it follows from Proposition (?) that $\oplus_{n=0}^{\infty} M_n$ is noetherian if and only if there is an m such that $L_m = L_{m+1} = \cdots$. However we clearly have that $L_m = L_{m+1} = \cdots$ if and only if $M_{m+n} = \mathfrak{a}^n M_m$ for $n = 0, 1, \ldots$, that is, if and only if the filtration $\{M_n\}_{n \in \mathbf{N}}$ is $\mathfrak{a}$-stable.

(1.14) Theorem. (Artin-Rees) *Let A be a noetherian ring and $\mathfrak{a}$ an ideal in A. Moreover let M be a finitely generated A-module and let $\{M_n\}_{n \in \mathbf{N}}$ be an $\mathfrak{a}$-stable filtration of M. For every submodule L of M we have that $\{L \cap M_n\}_{n \in \mathbf{N}}$ is an $\mathfrak{a}$-stable filtration of L.*

Proof. We have that $\mathfrak{a}(L \cap M_n) \subseteq \mathfrak{a}L\mathfrak{a}M_n \subseteq L \cap M_{n+1}$. Hence the filtration $\{L \cap M_n\}_{n \in \mathbf{N}}$ of L is an $\mathfrak{a}$-filtration, and consequently $\oplus_{n=0}^{\infty}(L \cap M_n)$ is a $R_{\mathfrak{a}}(A)$-submodule of $\oplus_{n=0}^{\infty} M_n$. Since the filtration $\{M\}_{n \in \mathbf{N}}$ is $\mathfrak{a}$-stable it follows from Lemma (1.13) that $\oplus_{n=0}^{\infty} M_n$ is a finitely generated $R_{\mathfrak{a}}(A)$-module, and from Proposition (1.6) it follows that the ring $R_{\mathfrak{a}}(A)$ is noetherian. Consequently it follows from Lemma (CHAINS 1.6) that $\oplus_{n=0}^{\infty}(L \cap M_n)$ is a finitely generated $R_{\mathfrak{a}}(A)$-module. Using Lemma (1.13) once more we see that the filtration $\{L \cap M_n\}_{n \in \mathbf{N}}$ of L is $\mathfrak{a}$-stable.

(1.15) Theorem. (Krull) *Let A be a noetherian ring and $\mathfrak{a}$ an ideal of A. Moreover let M be a finitely generated A-module. Then the submodule $\cap_{n=1}^{\infty} \mathfrak{a}^n M$ of M consists of the elements $x \in M$ such that $(1+f)x = 0$ for some $f \in \mathfrak{a}$.*

Proof. It is clear that an element $x \in M$ with the property that there is an $f \in \mathfrak{a}$ such that $(1+f)x = 0$ satisfies the equations $x = fx = f^2x = \cdots$. Hence we have an inclusion $x \in \cap_{n=1}^{\infty} \mathfrak{a}^n M$.

To prove the opposite inclusion we let $L = \cap_{n=1}^{\infty} \mathfrak{a}^n M$. It follows from Theorem (1.14) that $\{L \cap \mathfrak{a}^n M\}_{n \in \mathbf{N}}$ is an $\mathfrak{a}$-stable filtration of L. Hence it follows from Lemma (1.12) that we can find a positive integer m such that $L \cap \mathfrak{a}^{m+n} M \subseteq \mathfrak{a}^n L$ for $n = 0, 1, \ldots$. In particular $L \cap \mathfrak{a}^{m+1} M \subseteq \mathfrak{a}L$. Since $L = \cap_{n=1}^{\infty} \mathfrak{a}^n M \subseteq \mathfrak{a}^n M$ for $n = 0, 1, \ldots$ we obtain that $L \subseteq L \cap \mathfrak{a}^{m+1} M \subseteq \mathfrak{a}L$. Hence we have that $L = \mathfrak{a}L$. It follows from Theorem (MODULES 1.27) that there is an element $f \in \mathfrak{a}$ such that $(1+f)L = 0$.

(1.16) Corollary. *Let A be a noetherian local ring with maximal ideal $\mathfrak{m}$. Then $\cap_{n=1}^{\infty} \mathfrak{m}^n = 0$.*

Proof. It follows from the Theorem with $\mathfrak{a} = \mathfrak{m}$ that there is an element $f \in \mathfrak{m}$ such that $(1+f) \cap_{n=1}^{\infty} \mathfrak{m}^n = 0$. However we observed in (RINGS 4.17) that $1+f$ is a unit in A. Hence we have that $\cap_{n=1}^{\infty} \mathfrak{m}^n = 0$.

(1.17) Proposition. *Let A be a noetherian ring and let $\mathfrak{a}$ be an ideal in A. Moreover let M be a finitely generated A-module, and let $\{M_n\}_{n \in \mathbf{N}}$ be an $\mathfrak{a}$-stable filtration of M.*

1. *The ring $G_{\mathfrak{a}}(A) = \oplus_{n=0}^{\infty} \mathfrak{a}^n / \mathfrak{a}^{n+1}$ is noetherian.*
2. *The $G_{\mathfrak{a}}(A)$-module $\oplus_{n=0}^{\infty} M_n / M_{n+1}$ is noetherian.*

Proof. (1) Since the ring A is noetherian we have that $\mathfrak{a}$ is a finitely generated A-module. For every set of generators $f_1, f_2, \dots, f_m$ of $\mathfrak{a}$ we have that $\mathfrak{a}^n/\mathfrak{a}^{n+1}$ is generated as an $A/\mathfrak{a}$-module by the residue classes of the elements $f_{i_1} f_{i_2} \cdots f_{i_n}$ where $i_1, i_2, \dots, i_n$ are, not necessarily different, integers satisfying $1 \leq i_j \leq m$ for $j = 1, 2, \dots, m$. Denote by $g_1, g_2, \dots, g_m$ the classes in $\mathfrak{a}/\mathfrak{a}^2$ of the elements $f_1, f_2, \dots, f_m$. Then we clearly have that $G_\mathfrak{a}(A) = (A/\mathfrak{a})[g_1, g_2, \dots, g_m]$, that is, the $A/\mathfrak{a}$-algebra $G_\mathfrak{a}(A)$ is generated by the elements $g_1, g_2, \dots, g_m$. It follows from the Hilbert Basis Theorem (CHAINS 2.10) that $G_\mathfrak{a}(A)$ is noetherian.

(2) It follows from Proposition (CHAINS 1.7) that M is noetherian. Consequently M_n is a noetherian A-module for all n. It follows that M_n/M_{n+1} is a noetherian $A/\mathfrak{a}$-module for $n = 0, 1, \dots$. Since the filtration $\{M_n\}_{n \in \mathbf{N}}$ is $\mathfrak{a}$-stable it follows from Lemma (1.12) that there is a positive integer m such that $M_{m+n} \subseteq \mathfrak{a}^n M_m$ for $n = 0, 1, \dots$. Hence the $G_\mathfrak{a}(A)$-module $\oplus_{n=0}^\infty M_n/M_{n+1}$ is generated by the elements in $\oplus_{n=0}^m M_n/M_{n+1}$. Since each module M_n/M_{n+1} is noetherian it follows from Lemma (CHAINS 1.6) that $\oplus_{n=0}^m M_n/M_{n+1}$ is a finitely generated $A/\mathfrak{a}$-module, and each collection of generators of $\oplus_{n=0}^m M_n/M_{n+1}$ as a $A/\mathfrak{a}$-module will generate $\oplus_{n=0}^\infty M_n/M_{n+1}$ as a $G_\mathfrak{a}(A)$-module. It follows that $\oplus_{n=0}^\infty M_n/M_{n+1}$ is finitely generated as a $G_\mathfrak{a}(A)$-module, and consequently noetherian.

(1.18) Exercises.

1. Let $A = \oplus_{n=0}^\infty A_n$ be a graded ring and let $M = \oplus_{n=0}^\infty M_n$ be a graded A-modules. A submodule L of M is *graded* if for every element $x = \sum_{n=0}^n x_n$ in L with $x_n \in M_n$ we have that $x_n \in L$.

(1) Show that for every submodule L of M the A-module $\oplus_{n=0}^\infty (L \cap M_n)$ is a graded submodule of M.
(2) Show that if $L = \oplus_{n=0}^\infty L_n$ is a graded submodule of M, then $L_n = L \cap M_n$.
(3) Show that a submodule L of M is a graded submodule of M if and only if L can be generated by homogeneous elements.
(4) Show that for every graded submodule $L = \oplus_{n=0}^\infty L_n$ of M we have that M/L is isomorphic to the graded module $\oplus_{n=0}^\infty M_n/L_n$.

2. Let $A = \oplus_{n=0}^\infty A_n$ and $B = \oplus_{n=0}^\infty$ be graded rings. Moreover let $\varphi : A \to B$ be a ring homomorphism such that $\varphi(A_n) \subseteq B_n$ for all integers n. We say that φ is a *homomorphism of graded rings.*

(1) Show that the kernel of φ is a graded ideal.
(2) Show that the image of φ is a graded ring.

3. Let $A = \oplus_{n=0}^\infty$ be a graded ring, and let $M = \oplus_{n=0}^\infty$ and $N = \oplus_{n=0}^\infty$ be graded A-modules. Moreover let $u : M \to N$ be a homomorphism such that there is an integer m satisfying $u(M_n) \subseteq N_{m+n}$ for all integers n. We say that u is *graded of degree m.*

(1) Show that the kernel of u is a graded submodule of M.
(2) Show that the image of u is a graded submodule of N.

4. Let $S = \oplus_{n=0}^\infty S_n$ and $S_+ = \oplus_{n=1}^\infty S_n$. Moreover let n_0 be a positive integer and assume that we for every integer $n \geq n_0$ have a subgroup $\mathfrak{p}_n$ of S_n. Consider the following three conditions

(a) $S_m \mathfrak{p}_n \subseteq \mathfrak{p}_{m+n}$ for $m \geq 0$ and $n \geq n_0$.
(b) For $m \geq n_0$ and $n \geq n_0$, and for all elements $f \in S_m$ and $g \in S_n$ the relation $fg \in \mathfrak{p}_{m+n}$ implies that either $f \in \mathfrak{p}_m$ or $g \in \mathfrak{p}_n$.
(c) We have that $\mathfrak{p}_n \neq S_n$ for at least one integer $n \geq n_0$.

(1) Show that when $\mathfrak{p}$ is a homogeneous prime ideal in S that does not contain
→ S_+, and such that $\mathfrak{p}_n = \mathfrak{p} \cap S_n$ for $n \geq n_0$, then the conditions (a), (b) and
→ (c) hold.
→ (2) Assume that the conditions (a), (b) and (c) hold. It follows from (c) that there is an $f \in S_d \setminus \mathfrak{p}_d$ for some $d \geq n_0$. Show that for $m \geq n_0$ we have

$$\mathfrak{p}_m = \{x \in S_m : fx \in \mathfrak{p}_{m+d}\}.$$

→ (3) Under the same assumptions as in part (2), we write

$$\mathfrak{p}_m = \{x \in S_m : fx \in \mathfrak{p}_{m+d}\}$$

for all positive integers m. Show that $\mathfrak{p} = \oplus_{n=0}^{\infty} \mathfrak{p}_n$ is a prime ideal.
→ (4) Show that when (a), (b) and (c) hold then there is a unique homogeneous prime ideal $\mathfrak{p}$ that does not contain S_+ and such that $\mathfrak{p}_n = \mathfrak{p} \cap S_n$ for all $n \geq n_0$.

n **5.** Let !!$S = \oplus_{n=0}^{\infty} S_n$ be a graded ring. Moreover let !!$\mathrm{Proj}(S)$ be the homogeneous prime ideals in S that do not contain $S_+ = \oplus_{n=1}^{\infty} S_n$. For every homogeneous ideal
n $\mathfrak{a}$ in S we let !!$V_+(\mathfrak{a})$ be the prime ideals in $\mathrm{Proj}(S)$ that contain $\mathfrak{a}$, and for every
n homogeneous element $f \in S_+$ we let !!$D_+(f)$ be the prime ideals in $\mathrm{Proj}(S)$ that do not contain f.

(1) Show that the sets $V_+(\mathfrak{a})$ for all homogeneous ideals $\mathfrak{a}$ of S are the closed sets of a topology on $\mathrm{Proj}(S)$. This topology we call the *Zariski topology*.
(2) Show that for every homogeneous element $f \in S_+$ the set $D_+(f)$ is open in the Zariski topology, and that the sets $D_+(f)$ for all homogeneous elements $f \in S_+$ form a basis for the topology.
(3) Let $f \in S_d$ be a homogeneous element of S_+. We denote by $S_{(f)}$ the elements g/f^n in the localization S_f of S in the multiplicatively closed set $\{1, f, f^2, \dots\}$ such that $g \in S_{dn}$. Show that $S_{(f)}$ is a ring.
(4) Show that there is a map of sets

$$\psi_f : D_+(f) \to \mathrm{Spec}(S_{(f)})$$

that sends a prime ideal to the set of all elements of the form $g/f^n \in S_{(f)}$ with $g \in \mathfrak{p} \cap S_{dn}$.

(5) Let $\mathfrak{q}_0$ be a prime ideal in $S_{(f)}$. For every positive integer we let

$$\mathfrak{p}_n = \{x \in S_n : x^d/f^n \in \mathfrak{q}_0\}.$$

Show that $\mathfrak{p} = \oplus_{n=0}^{\infty}\mathfrak{p}_n$ is a prime ideal in S that does not contain S_+.

(6) Show that the map ψ_f is a homeomorphism of topological spaces, that is, the map ψ_f is continous and has an inverse that is also continous.

(7) Show that for all homogeneous elements f, g in S_+ there is an inclusion of open sets $D_+(fg) \subseteq D_+(f)$ and a homomorphism $\omega_{fg,f} : S_{(f)} \to S_{(fg)}$ of rings.

(8) Denote by $\iota_{fg,f} : D_+(fg) \to D_+(f)$ the continous map coming from the inclusion $D_+(fg) \subseteq D_+(f)$. Show that $({}^{\mathrm{a}}\omega_{fg,f})(\psi_{fg}) = (\psi_f)(\iota_{fg,f})$.

(9) Define a sheaf of rings $\mathcal{O}_X$ on $X = \mathrm{Proj}(S)$ such that for all homogeneous elements f, g in S_+ there is an isomorphism of ringed spaces $(\psi_f, \theta_f) : (D_+(f), \mathcal{O}_X|D_+(f)) \to (\mathrm{Spec}(S_{(f)}), \mathcal{O}_{\mathrm{Spec}(S_{(f)})})$ such that (ψ_f, θ_f) restricted to $D_+(fg)$ gives the map (ψ_{fg}, θ_{fg}).

2. Hilbert polynomials.

n **(2.1) Definition.** Let A be a ring. An *additive function* $!\lambda = \lambda_A!$ on finitely generated A-modules associates to every finitely generated A-module M an integer $\lambda(M)$ and satisfies the property:

For every exact sequence of finitely generated A-modules

$$0 \to M' \to M \to M'' \to 0$$

we have

$$\lambda(M) = \lambda(M') + \lambda(M'').$$

(2.2) Remark. let $M' = (0)$ and thus $M = M''$. We see that $\lambda((0)) = 0$.

→ **(2.3) Example.** It follows from Proposition (CHAINS 1.15) that when A is an artinian ring the *length* is an additive function on finitely generated A-modules. In particular the vector space dimension is an additive function on finite dimensional vecor spaces.

(2.4) Remark. Let $A = \oplus_{n=0}^{\infty} A_n$ be a graded ring that is finitely generated as an A_0-algebra, and let $M = \oplus_{n=0}^{\infty} M_n$ be a finitely generated A-module. Then each M_n is a finitely generated A_0-module. In fact when we replace, if necessary, a set of generators for the A_0-algebra A, and a set of generators for the A-module M by their homogeneous components, we see that the A_0-algebra A can be generated by a finite set $f_1, f_2, \dots, f_p$ of homogeneous elements of A, respectively that the A-module M can be generated by a finite set $x_1, x_2, \dots, x_q$ of homogeneous elements of M. If $f_i \in A_{m_i}$ for $i = 1, 2, \dots, p$ and $x_i \in M_{n_i}$ for $i = 1, 2, \dots, q$ we clearly have that M_n is generated, as an A_0-module, by the elements $f_{i_1} f_{i_2} \cdots f_{i_r} x_j$ for all collections of integers $i_1, i_2, \dots, i_r$ between 1 and p and and j between 1 and q, and with $m_{i_1} + m_{i_2} + \cdots + m_{i_r} + n_j = n$.

→ In particular, it follows from Proposition (1.6) that for any noetherian graded ring A and finitely generated graded module M, the homogeneous part M_n is finitely generated over A_0 for all n.

(2.5) Definition. Let $A = \oplus_{n=0}^{\infty} A_n$ be a graded ring that is finitely generated as an A_0-algebra, and let $M = \oplus_{n=0}^{\infty} M_n$ be a finitely generated A-module. Moreover let λ be an additive function on finitely generated A_0-modules. The *Poincaré series*
n of the A-module M is the power series!!

$$P_\lambda(M, t) = \sum_{n=0}^{\infty} \lambda(M_n) t^n$$

in the variable t with coefficients in $\mathbf{Z}$.

(2.6) Example. Let $A = K[t_1, t_2, \dots, t_n]$ be the polynomial ring in the variables $t_1, t_2, \dots, t_n$ over a field K. Then $A = \oplus_{i=0}^{\infty} A_i$ where A_i is the vector space of all homogeneous polynomials of degree n. Let $\lambda(M) = \dim_K(M)$ for all finite dimensional vector spaces M over K. Then $P_\lambda(A, t) = \sum_{i=0}^{\infty} \binom{i+n-1}{n-1} t^i = 1/(1-t)^n$.

(2.7) Example. Let $K[t_1, t_2]$ be the polynomial ring in the variables t_1, t_2 over a field K. Moreover let $A = K[t_1, t_2]/(t_1^2, t_1 t_2)$ be the residue ring of the polynomial ring $K[t_1, t_2]$ modulo the ideal $(t_1^2, t_1 t_2)$, and let u and v be the residue classes of t_1, respective t_2 in A. Then $u^2 = 0 = uv$ and we have that $A = K \oplus (Ku + Kv) \oplus Kv^2 \oplus Kv^3 \oplus \cdots$. Hence $P_\lambda(A, t) = 1 + 2t + t^2 + t^3 + \cdots = (1 + t - t^2)/(1 - t)$.

(2.8) Example. Let $K[t_1, t_2]$ be the polynomial ring in the variables t_1, t_2 over a field K. Moreover let $A = K[t_1, t_2]/(t_1^2 + t_2^2)$ be the residue ring of the polynomial ring $K[t_1, t_2]$ modulo the ideal $(t_1^2 + t_2^2)$, and let u and v be the residue classes of t_1 respectively t_2 in A. Then $u^2 + v^2 = 0$ and $A = K \oplus (Ku + Kv) \oplus (Kuv + Kv^2) \oplus (Kuv^2 + Kv^3) \oplus \cdots$. Hence $P_\lambda(A, t) = 1 + 2t + 2t^2 + \cdots = (1 + t)/(1 - t)$.

(2.9) Lemma. *Let $A = \oplus_{n=0}^{\infty} A_n$ be a noetherian graded ring and let $M = \oplus_{n=0}^{n} M_n$ be a finitely generated A-module. Moreover let λ be an additive function on finitely generated A_0-modules. For every homogeneous element $f \in A_m$ with $m > 0$ we have an exact sequence of A-modules*

$$0 \to L \to M \xrightarrow{f_M} M \to N \to 0 \qquad ((.)9.1)$$

where L and N are finitely generated (A/fA)-modules, and

$$(1 - t^m) P_\lambda(M, t) = P_\lambda(N, t) - t^m P_\lambda(L, t). \qquad (2.9.2)$$

Proof. For each integer $n \geq -m$ we have an exact sequence

$$0 \to L_n \to M_n \xrightarrow{f_M} M_{m+n} \to N_{m+n} \to 0 \qquad (2.9.3)$$

where L_n and N_{m+n} are defined as the kernel, respectively the cokernel of the map f_M. Let $L = \oplus_{n=0}^{\infty} L_n$ and $N = \oplus_{n=0}^{\infty} N_n$. Then L and N are A modules, and we have an exact sequence (2.6.1). Since M is noetherian by Lemma (CHAINS 1.6) it follows from Proposition (1.7) that L and N are noetherian A-modules. In particular it follows from Remark (2.4) that L_n and N_n are finitely generated A_0-modules for all n. It follows from (2.9.3) that we have equations

$$\lambda(M_{m+n}) - \lambda(M_n) = \lambda(N_{m+n}) - \lambda(L_n) \quad \text{for } n = -m, -m+1, \dots. \qquad (2.9.4)$$

Multiply both sides of (2.9.4) by t^{m+n} for $n = -m, -m+1, \dots$, and sum the right and left hand sides of the resulting equations. We obtain equation (2.9.2) of the Lemma.

Finally we note that $fL = 0$ and $fN = 0$. Hence L and N are in fact A/fA-modules.

(2.10) Theorem. (Hilbert-Serre) *Let A be a noetherian graded ring, generated as an A_0-module by m homogeneous elements of positive degrees $p_1, p_2, \dots, p_m$. Moreover let M be a finitely generated graded A-module, and λ an additive function on finitely generated A_0-modules. Then*

$$P_\lambda(M,t) = f(t)/\prod_{i=1}^{m}(1-t^{p_i}) \tag{2.10.1}$$

in the ring $\mathbf{Z}[[t]]$ of power series in the variable t over the integers, where $f(t)$ is a polynomial in $\mathbf{Z}[t]$ and $1/(1-t^{p_i}) = 1 + t^{p_i} + t^{2p_i} + \cdots$.

Proof. We prove the Theorem by induction on m. When $m = 0$ we have that $A = A_0$, and since M is finitely generated $M_n = 0$ for all sufficiently large n. Consequently $P_\lambda(M,t)$ is a polynomial when $m = 0$.

Assume that $m > 0$ and that the Theorem holds for $m-1$. Let $f_1, f_2, \dots, f_m$ be homogeneous elements of positive degrees $p_1, p_2, \dots, p_m$ respectively that generate A as an A_0-algebra. It follows from Lemma (2.9) with $f = f_m$ that

$$(1-t^{p_m})P_\lambda(M,t) = P_\lambda(N,t) - t^{p_m}P_\lambda(L,t) \tag{2.10.2}$$

where L and N are (A/f_mA)-modules. We have that the A_0-algebra $A/f_mA = A_0[f_1, f_2, \dots, f_m]/f_mA$ is generated by the residue classes of $f_1, f_2, \dots, f_{m-1}$. It follows from the induction hypothesis that $P_\lambda(N,t) = g(t)/\prod_{i=1}^{m-1}(1-t^{p_i})$ and $P_\lambda(L,t) = h(t)/\prod_{i=1}^{m-1}(1-t^{p_i})$, where $g(t)$ and $h(t)$ are polynomials in $\mathbf{Z}[t]$. Equation (2.10.1) consequently follows from equation (2.10.2).

(2.11) Corollary. *Let A be a noetherian graded ring that is finitely generated as an A_0-algebra by m elements of degree 1. Moreover let M be a finitely generated graded A-module. Write*

$$P_\lambda(M,t) = f(t)/(1-t)^m = g(t)/(1-t)^p$$

where $0 \le p \le m$ and $g(t)$ is a polynomial in $\mathbf{Z}[t]$ with $g(1) \ne 0$. Then there is a polynomial $h(t)$ in $\mathbf{Q}[t]$ of degree $p-1$ such that $\lambda(M_n) = h(n)$ for all sufficiently large n. Here we define the degree of the zero polynomial as -1.

Proof. When $p = 0$, that is, when $(1-t)^m$ divides $f(t)$ we have that $P_\lambda(M,t)$ is a polynomial. Consequently we have that $\lambda(M_n) = 0$ when n is larger than the degree of $P_\lambda(M,t)$. Hence the Corollary holds when $(1-t)^m$ divides $f(t)$.

Assume that $0 < p \le m$. Write $g(t) = \sum_{n=0}^{q} g_n t^n$ with $g_n \in \mathbf{Z}$. Since $1/(1-t)^p = \sum_{n=0}^{\infty}\binom{n+p-1}{p-1}t^n$ in $\mathbf{Z}[[t]]$, we have that

$$g(t)/(1-t)^p = \sum_{n=0}^{\infty}\sum_{i+j=n} g_i\binom{j+p-1}{p-1}t^n.$$

Consequently $\lambda(M_n) = \sum_{i+j=n} g_i \binom{j+p-1}{p-1} = \sum_{i=0}^{q} g_i \binom{n-i+p-1}{p-1}$. We write $\binom{t}{n} = (1/n!)t(t-1)\cdots(t-n+1)$ in $\mathbf{Q}[t]$, and we let $h(t) = \sum_{i=0}^{q} g_i \binom{t-i+p-1}{p-1}$. Then $h(t)$ is a polynomial of degree $p-1$ because the coefficient of t^{p-1} is $(1/(p-1)!)\sum_{i=0}^{q} g_i = (1/(p-1)!)g(1) \neq 0$. Moreover we have that $\lambda(M_n) = h(n)$ when $n \geq q$, and we have proved the Corollary.

(2.12) Definition. Let $A = \oplus_{n=0}^{\infty} A_n$ be a graded noetherian ring which is generated as an A_0-module by elements of degree 1. Moreover let M be a finitely generated A-module, and let λ be an additive function on finitely generated A_0-modules. The polynomial !$h(t)$! in $\mathbf{Q}[t]$ such that $h(n) = \lambda(M_n)$ for all sufficiently large n is called the *Hilbert polynomial* of M with respect to λ. We denote by !!$d_\lambda(M)$ the degree of the Hilbert polynomial. Here we define the degree of the zero polynomial as -1.

(2.13) Example. Let $K[t_1, t_2, \ldots, t_n]$ be the polynomial ring in the variables $t_1, t_2, \ldots, t_n$ over a field K. We saw in Example (2.6) that the Hilbert polynomial $h(t)$ is $\binom{t+n-1}{n-1} = (1/(n-1)!)(t+n-1)(t+n-2)\cdots(t+1)$.

(2.14) Example. Let $A = K[u, v]$ with $u^2 = 0 = uv$ be the ring of Example (2.7). Then the Hilbert polynomial $h(t)$ is equal to 1.

(2.15) Example. Let $A = K[u, v]$ with $u^2 + v^2 = 0$ be the ring of Example (2.8). Then the Hilbert polynomial $h(t)$ is equal to 2.

(2.16) Lemma. *Let A be a noetherian graded ring that is generated as an A_0-module by elements of degree* 1*. Moreover let M be a finitely generated A-module, and let λ be an additive function on finitely generated A_0-modules. For every homogeneous element $f \in A$ of positive degree which is M-regular we have that*

$$d_\lambda(M) = d_\lambda(M/fM) + 1.$$

Proof. Let f be homogeneous of degree $m > 0$. Since f is M-regular the map $f_M : M \to M$ is injective. Hence it follows from the exact sequence (2.9.1) that $L = 0$, and we obtain from equation (2.9.2) that

$$(1 - t^m)P_\lambda(M, t) = P_\lambda(M/fM, t).$$

Write $P_\lambda(M, t) = g(t)/(1-t)^p$ and $P_\lambda(M/fM, t) = h(t)/(1-t)^q$ where $g(t)$ and $h(t)$ are polynomials in $\mathbf{Z}[t]$ with $g(1) \neq 0$ respectively $h(1) \neq 0$. Then $(1-t^m)(1-t)^q g(t) = (1-t)^p h(t)$. Since $1 - t^m = (1-t)(1+t+\cdots+t^{m-1})$ and $(1+t+\cdots+t^{m-1})(1) = m \neq 0$ we have that $p = q+1$. That is, we have $d_\lambda(M) = d_\lambda(M/fM)+1$, and we have proved the Lemma.

(2.17) Exercises.

1. Let $K[u,v]$ be the ring of polynomials in the independent variables u, v with coefficients in a field K. Moreover, let $S = K[u,v]/(u^2, uv^m)$ be the residue ring of $K[u,v]$ modulo the ideal (u^2, uv^m).

(1) Determine the polynomial $g(t)$ in $\mathbf{Z}[t]$ and the non-negative integer p such that

$$P_\lambda(S,t) = g(t)/(1-t)^p$$

and $g(1) \neq 0$, when $\lambda = \dim_K$.

(2) Determine the Hilbert polynomial of S with respect to $\dim_K$.

2. Let $K[u,v]$ be the ring of polynomials in the independent variables u, v with coefficients in a field K. Let $S = K[u,v]/(u^2, v^m)$ be the residue ring of $K[u,v]$ modulo the ideal (u^2, v^m).

(1) Determine the polynomial $g(t)$ in $\mathbf{Z}[t]$ and the non-negative integer p such that

$$P_\lambda(S,t) = g(t)/(1-t)^p$$

and $g(1) \neq 0$, when $\lambda = \dim_K$.

(2) Determine the Hilbert polynomial of S with respect to $\dim_K$.

3. Let $K[t_1, t_2, \dots, t_n]$ be the ring of polynomials in the independent variables $t_1, t_2, \dots, t_n$ over a field K. Moreover, let $f(t_1, t_2, \dots, t_n)$ be a polynomial of degree $d > 0$, and let $S = K[t_1, t_2, \dots, t_n]/(f(t_1, t_2, \dots, t_n))$ be the residue ring of $K[t_1, t_2, \dots, t_n]$ modulo the ideal $(f(t_1, t_2, \dots, t_n))$ generated by $f(t_1, t_2, \dots, t_n)$.

(1) Determine the polynomial $g(t)$ in $\mathbf{Z}[t]$ and the non-negative integer p such that

$$P_\lambda(S,t) = g(t)/(1-t)^p$$

and $g(1) \neq 0$, when $\lambda = \dim_K$.

(2) Determine the Hilbert polynomial of S with respect to $\dim_K$.

4. Let $K[t_0, t_1, \dots, t_n]$ be the ring of polynomials in the independent variables $t_0, t_1, \dots, t_n$ with coefficients in a field K with infinitely many elements. For every point $b = (b_0, b_1, \dots, b_n)$ in the cartesian product K^{n+1} of the field K with itself $n+1$ times, and for every element κ in K we write $\kappa b = (\kappa b_0, \kappa b_1, \dots, \kappa b_n)$. Moreover for every collection of points $a_1, a_2, \dots, a_m$ in K^{n+1} we write

$$\mathfrak{I}(a_1, a_2, \dots, a_m) = \{f \in K[t_0, t_1, \dots, t_n] : \\ f(\kappa a_i) = 0 \text{ for } i = 1, 2, \dots, m \text{ and all } \kappa \in K\}.$$

(1) Show that $\mathfrak{I}(a_1, a_2, \dots, a_m)$ is a homogeneous ideal in $K[t_0, t_1, \dots, t_n]$.

(2) Show that

$$\dim_K(\mathfrak{I}(a_1, a_2, \dots, a_m)) \geq \max(0, \binom{n+d}{d} - m).$$

n (3) Show that for every non-empty collection !!$\mathcal{P}$ of homogeneous polynomials in $K[t_0, t_1, \ldots, t_n]$ of positive degree the subset

$$V(\mathcal{P}) = \{b \in K^{n+1} : f(b) = 0 \text{ for all } f \in \mathcal{P}\}$$

of K^{n+1} is different from K^{n+1}.

(4) Show that we can find points $a_1, a_2, \ldots, a_m$ in K^{n+1} such that

$$\dim_K(\mathfrak{I}(a_1, a_2, \ldots, a_m)) = \max(0, \binom{n+d}{d} - m).$$

(5) Let $S = K[t_0, t_1, \ldots, t_n]/\mathfrak{I}(a_1, a_2, \ldots, a_m)$. Determine the polynomial $g(t)$ in $\mathbf{Z}[t]$ and the non-zero integer p such that

$$P_\lambda(S, t) = g(t)/(1-t)^p$$

and $g(1) \neq 0$, when $\lambda = \dim_K$.

3. Dimension of local rings.

(3.1) Notation. Let $\mathbf{Q}[t]$ be the polynomial ring in the variable t over the rational numbers. For each positive integer m we define a polynomial $\binom{t}{m}$ by

$$\binom{t}{m} = (1/m!)t(t-1)\cdots(t-m+1).$$

We let $\binom{t}{0} = 1$. The polynomial $\binom{t}{m}$ has degree m and the coefficient of t^m is $1/m!$. For all integers n we have that $\binom{t}{m}(n) = \binom{n}{m}$ is an integer, so all the polynomials $\binom{t}{m}$ define a function $\mathbf{Z} \to \mathbf{Z}$.

Define an operator Δ on the collections of functions $h : \mathbf{Z} \to \mathbf{Z}$ by

$$\Delta h(t) = h(t+1) - h(t) \quad \text{for all } f \in \mathbf{Q}[t].$$

Then $\Delta(\binom{t}{m}) = \binom{t}{m-1}$.

(3.2) Remark. The polynomials $\binom{t}{0}, \binom{t}{1}, \dots, \binom{t}{m}$ form a basis for the subspace of the $\mathbf{Q}$-vector space $\mathbf{Q}[t]$ consisting of polynomials of degree at most equal to m.

(3.3) Lemma. *Let $\mathbf{Q}[t]$ be the polynomial ring in the variable t with coefficients in $\mathbf{Q}$.*

(1) *If $f(t) \in \mathbf{Q}[t]$ is a polynomial of degree m such that $f(n) \in \mathbf{Z}$ for all sufficiently large n, there are integers $n_0, n_1, \dots, n_m$ such that*

$$f(t) = n_0\binom{t}{m} + n_1\binom{t}{m-1} + \cdots + n_m.$$

In particular we have that $f(n) \in \mathbf{Z}$ for all integers n.

(2) *Let $h : \mathbf{Z} \to \mathbf{Z}$ be a function such that there is a polynomial $g(t) \in \mathbf{Q}[t]$ of degree $m-1$ with $\Delta h(n) = g(n)$ for all sufficiently large n. Then there is a polynomial $f(t) \in \mathbf{Q}[t]$ of degree m such that $h(n) = f(n)$ for all sufficiently large n.*

Proof. (1) We shall prove assertion (1) by induction on m. When $m = 0$ the assertion clearly holds. Assume that assertion (1) holds for polynomials of degree $m-1$. Since the polynomials $\binom{t}{0}, \binom{t}{1}, \dots, \binom{t}{m}$ form a basis for the vector space over $\mathbf{Q}$ consisting of polynomials of degree at most m there are rational numbers $n_0, n_1, \dots, n_m$ such that $f(t) = n_0\binom{t}{m} + n_1\binom{t}{m-1} + \cdots + n_m$. We shall show that the numbers $n_0, n_1, \dots, n_m$ are integers. We have that $\Delta f(t) = n_0\binom{t}{m-1} + n_1\binom{t}{m-2} + \cdots + n_{m-1}$. Since $\Delta f(t)$ is of degree at most $m-1$ and $\Delta f(n) = f(n+1) - f(n)$ is in $\mathbf{Z}$ for all sufficiently large integers n it follows from the induction hypothesis that $n_0, n_1, \dots, n_{m-1}$ are integers. Since $f(n) \in \mathbf{Z}$ for some n, and $\binom{n}{i}$ is in $\mathbf{Z}$ for $i = 0, 1, \dots,$ it follows that we also have $n_m \in \mathbf{Z}$.

(2) It follows from assertion (1) that $g(t) = n_0\binom{t}{m-1} + n_1\binom{t}{m-2} + \cdots + n_{m-1}$ for some integers $n_0, n_1, \ldots, n_{m-1}$. Let $k(t) = n_0\binom{t}{m} + n_1\binom{t}{m-1} + \cdots + n_{m-1}\binom{t}{1}$. Then $k(t)$ is of degree m and we have that $\Delta k(t) = g(t)$. Consequently there is an integer p such that $\Delta(h-k)(n) = \Delta h(n) - g(n) = g(n) - g(n) = 0$ when $n \geq p$. Hence there is an integer n_m such that $h(n) - k(n) = h(p) - k(p) = n_m$ when $n \geq p$. Let $f(t) = k(t) + n_m$. Then $f(t)$ is of degree m and we have that $f(n) = k(n) + n_m = h(n)$ when $n \geq p$, and we have proved assertion (2).

(3.4) Lemma. *Let A be a noetherian ring and let M be an A-module. Moreover let $\{M_n\}_{n\in\mathbf{N}}$ be a filtration on M such that M_n/M_{n+1} is an A-module of finite length for $n = 0, 1, \ldots$. Then*

(1) *M/M_n is of finite length for $n = 0, 1, \ldots$.*
(2) *For $n = 0, 1, \ldots$ we have*

$$\ell(M_{n-1}/M_n) = \ell(M/M_n) - \ell(M/M_{n-1}) \quad \text{and} \quad \ell(M/M_n) = \sum_{i=1}^{n} \ell(M_{i-1}/M_i).$$

(3) *If there is a polynomial $g(t) \in \mathbf{Q}[t]$ of degree $m-1$ such that $g(n) = \ell(M_n/M_{n+1})$ for all sufficiently large integers n, then there is a polynomial $f(t) \in \mathbf{Q}[t]$ of degree m such that $f(n) = \ell(M/M_n)$ for all sufficiently large n.*

Proof. (1) For each $n \geq 1$ we have an exact sequence

$$0 \to M_{n-1}/M_n \to M/M_n \to M/M_{n-1} \to 0. \tag{3.4.1}$$

It follows from Proposition (CHAINS 1.15) by induction on n, starting with $M_0/M_1 = M/M_1$, that each M/M_n is an A-module of finite length.

(2) The first equality of assertion (2) follows from the exact sequence (3.4.1) and Proposition (CHAINS 1.15), and the second equality follows from the first by induction on n, starting with $n = 1$.

(3) Let $h(n) = \ell(M/M_n)$ for $n = 0, 1, \ldots$, and let $h(n) = 0$ for $n < 0$. It follows from assertion (2) that $\Delta h(n) = g(n)$ for all sufficiently large integers. Assertion (3) consequently follows from assertion (2) of Lemma (3.3).

(3.5) Proposition. *Let A be a noetherian local ring with maximal ideal $\mathfrak{m}$. Moreover let $\mathfrak{q}$ be an $\mathfrak{m}$-primary ideal in A, and let M be a finitely generated A-module with a $\mathfrak{q}$-stable filtration $\{M_n\}_{n\in\mathbf{N}}$. Then*

(1) *The A-module M/M_n has finite length.*
(2) *Let m be the least number of generators for $\mathfrak{q}$. Then there is a polynomial $g(t) \in \mathbf{Q}[t]$ of degree at most m such that $g(n) = \ell(M/M_n)$ for all sufficiently large n.*
(3) *The degree $\deg(g)$ of the polynomial $g(t)$ and the coefficient of $t^{\deg(g)}$ are independent of the filtration $\{M_n\}_{n\in\mathbf{N}}$.*

Proof. (1) It follows from Remark (CHAINS 2.16) that the ring $A/\mathfrak{q}$ is artinian. Moreover it follows from Proposition (1.17) that the ring $G_{\mathfrak{q}}(A) = \oplus_{n=0}^{\infty}\mathfrak{q}^n/\mathfrak{q}^{n+1}$ is noetherian and that $\oplus_{n=0}^{\infty}M_n/M_{n+1}$ is a finitely generated $G_{\mathfrak{q}}(A)$-module. Hence it follows from Remark (2.4) that the modules M_n/M_{n+1} are finitely generated $A/\mathfrak{q}$-modules, and hence of finite length as A-modules. Consequently it follows from Lemma (3.4) that the A-modules M/M_n are of finite length.

(2) The classes in $\mathfrak{q}/\mathfrak{q}^2$ of a set of generators of $\mathfrak{q}$ generate the $A/\mathfrak{q}$-algebra $G_{\mathfrak{a}}(A) = \oplus_{n=0}^{\infty}\mathfrak{q}^n/\mathfrak{q}^{n+1}$. Hence it follows from Corollary (2.11) that there is a polynomial $f(t)$ in $\mathbf{Q}[t]$ of degree at most $m-1$ such that $f(n) = \ell(M_n/M_{n+1})$ for all sufficiently large n. It follows from Lemma (3.4) that there is a polynomial $g(t)$ in $\mathbf{Q}[t]$ of degree at most m such that $g(n) = \ell(M/M_n)$ for all sufficiently large n.

(3) Since $\{\mathfrak{q}^n M\}_{n\in\mathbf{N}}$ is a $\mathfrak{q}$-stable filtration we obtain a polynomial $h(t)$ in $\mathbf{Z}[t]$ such that $h(n) = \ell(M/\mathfrak{q}^n M)$ for all sufficiently large n. It follows from Lemma (1.12) that there is a positive integer m such that $M_{p+n} \subseteq \mathfrak{q}^n M$ and $\mathfrak{q}^{p+n}M \subseteq M_n$ for $n = 0, 1, \ldots$. Then

$$g(p+n) \geq h(n) \quad \text{and} \quad h(p+n) \geq g(n) \tag{3.5.1}$$

for all sufficiently large n. It is easily seen that the equation (3.5.1) for all sufficiently large n implies that $g(t)$ and $h(t)$ have the same degree and that the coefficient of $t^{\deg(g)} = t^{\deg(h)}$ is the same. Hence all $\mathfrak{q}$-stable filtrations have polynomials with the same degrees as $h(t)$ and the same coefficients of $t^{\deg(h)}$ as $h(t)$.

(3.6) Notation. Let A be a noetherian local ring with maximal ideal $\mathfrak{m}$. Moreover let $\mathfrak{q}$ be an $\mathfrak{m}$-primary ideal and M a finitely generated A-module. We denote by $\chi_{\mathfrak{q}}^M(t)$ the polynomial in $\mathbf{Q}[t]$ such that $\chi_{\mathfrak{q}}^M(n) = \ell(M/\mathfrak{q}^n M)$ for all sufficiently large n. When $M = A$ we write $\chi_{\mathfrak{q}}(n) = \chi_{\mathfrak{q}}^A(n)$.

(3.7) Proposition. *Let A be a noetherian local ring and let $\mathfrak{q}$ be an ideal that is primary for the maximal ideal $\mathfrak{m}$ of A. Then*

$$\deg(\chi_q) = \deg(\chi_{\mathfrak{m}}).$$

Proof. It follows from Remark (CHAINS 2.9) that $\mathfrak{m}^m \subseteq \mathfrak{q}$ for some integer m. Hence we have that $\mathfrak{m}^{mn} \subseteq \mathfrak{q}^n \subseteq \mathfrak{m}^n$. It follows that

$$\chi_{\mathfrak{m}}(mn) \geq \chi_{\mathfrak{q}}(n) \geq \chi_{\mathfrak{m}}(n)$$

for all sufficiently large integers n. Since $\chi_{\mathfrak{m}}(t)$ and $\chi_{\mathfrak{q}}(t)$ are polynomials they must have the same degree.

(3.8) Lemma. *Let A be a noetherian local ring and let $\mathfrak{q}$ be an ideal in A that is primary for the maximal ideal $\mathfrak{m}$. Moreover let M be a finitely generated A-module and $f \in A$ an element that is regular for M. Then*

$$\deg(\chi_{\mathfrak{q}}^{(M/fM)}) \leq \deg(\chi_{\mathfrak{q}}^M) - 1.$$

Proof. Let $L = fM$ and let $N = M/fM$. We have a surjective A-module homomorphism $u : M \to N/\mathfrak{q}^n N$ that is the composite of the canonical maps $u_N : M \to N$ and $u_{N/\mathfrak{q}^n N} : N \to N/\mathfrak{q}^n N$. The kernel of u is $fM + \mathfrak{q}^n M = L + \mathfrak{q}^n M$. As we saw in Lemma (MODULES 1.13) the homomorphism u induces a surjective homomorphism $v : M/\mathfrak{q}^n M \to N/\mathfrak{q}^n N$. The kernel of v is $(L + \mathfrak{q}^n M)/\mathfrak{q}^n M$. It follows from Lemma (MODULES 1.13) that we have a canonical isomorphism $(L + \mathfrak{q}^n M)/\mathfrak{q}^n M = L/(L \cap \mathfrak{q}^n M)$. Let $L_n \xrightarrow{\sim} L \cap \mathfrak{q}^n M$. We then have an exact sequence of A-modules

$$0 \to L/L_n \to M/\mathfrak{q}^n M \xrightarrow{v} N/\mathfrak{q}^n N \to 0. \tag{3.8.1}$$

It follows from Theorem (1.14) that the filtration $\{L_n\}_{n\in\mathbf{N}}$ is $\mathfrak{q}$-stable. Hence it follows from Proposition (3.5) that there is a polynomial $g(t) \in \mathbf{Q}[t]$ such that $g(n) = \ell(L/L_n)$ for sufficiently large integers n. It follows from the exact sequence (3.8.1) that we have

$$\chi_{\mathfrak{q}}^N(n) = \chi_{\mathfrak{q}}^M(n) - g(n) \tag{3.8.2}$$

for all sufficiently large integers n. Since f is M-regular the map $M \to L$ that sends x to fx is an isomorphism. Moreover, since $\{L_n\}_{n\in\mathbf{N}}$ is stable and L is isomorphic to M it follows from Proposition (?) that the polynomials $\chi_{\mathfrak{q}}^M(t)$ and $g(n)$ have the same degree and the same leading coefficient of $t^{\deg(g)}$. Hence it follows from (3.8.2) that $\deg(\chi_{\mathfrak{q}}^N) < \deg(\chi_{\mathfrak{q}}^M)$ and we have proved the Lemma.

(3.9) Lemma. *Let A be a local noetherian ring with maximal ideal $\mathfrak{m}$. Then*

$$\dim(A) \le \deg(\chi_{\mathfrak{m}}(t)).$$

In particular we have that $\dim(A)$ *is finite.*

Proof. We show the Lemma by induction on $m = \deg(\chi_{\mathfrak{m}}(t))$. If $m = 0$ then $\ell(A/\mathfrak{m}^n)$ is constant for all sufficiently large integers n. Hence $\mathfrak{m}^n = \mathfrak{m}^{n+1}$ for all sufficiently large integers n. It follows from Nakaymas Lemma (MODULES 1.27) that $\mathfrak{m}^n = 0$ for all sufficiently large integers n. Hence it follows from Remark (CHAINS 2.15) that A is artinian and hence from Proposition (CHAINS 2.17) $\dim(A) = 0$.

Assume that $m > 0$ and that the Lemma holds for $m - 1$. Let $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_p$ be a chain of prime ideals in A. We shall show that $p \le m$. If $p = 0$ there is nothing to prove, so we assume that $p \ge 1$. Then there is an element $f \in \mathfrak{p}_1 \setminus \mathfrak{p}_0$. Let $B = A/\mathfrak{p}_0$ and let g be the residue class of f in B. Then $g \ne 0$, and since B is an integral domain by Proposition (RINGS 4.13) we have that g is not a zero divisor in B. Consequently it follows from Lemma (3.8) that $\deg(\chi_{\mathfrak{m}}^{B/gB}) \le \deg(\chi_{\mathfrak{m}}^B) - 1$.

Let $\mathfrak{M}_B$ be the maximal ideal of the local ring B and let $\mathfrak{m} = \mathfrak{m}_A$. The canonical map $\varphi_B : A \to B$ induces a surjective homomorphism $A/\mathfrak{m}_A^n \to B/\mathfrak{m}_B^n$ for all natural numbers n. Consequently $\ell_A(A/\mathfrak{m}_A^n) \ge \ell_A(B/\mathfrak{m}_B^n)$ for all natural numbers n and we clearly have that $\ell_A(B/\mathfrak{m}_B^n) = \ell_B(B/\mathfrak{m}_B^n)$. It follows that $\deg(\chi_{\mathfrak{m}}^A) \geq \deg(\chi_{\mathfrak{m}}^B)$. We have thus shown that $\deg(\chi_{\mathfrak{m}}^{B/gB}) \leq \deg(\chi_{\mathfrak{m}}^A) - 1 = m - 1$.

It follows from the induction assumption that $\dim(B/gB) \leq \deg(\chi_{\mathfrak{m}}^{B/gB})$, and thus

$$\dim(B/gB) \leq \deg(\chi_{\mathfrak{m}}^A) - 1 = m - 1. \tag{3.9.1}$$

Let $\varphi : A \to B/gB$ be the composite map of the surjection $A \to B$ with the canonical map $\varphi_{B/gB} : B \to B/gB$. The image of the chain $\mathfrak{p}_1 \subset \mathfrak{p}_2 \subset \cdots \subset \mathfrak{p}_p$ by φ is a chain of length $p-1$ in B/gB. Consequently $p - 1 \leq \dim(B/gB)$. Together with the inequality (3.9.1) we obtain that $p \leq m$. Hence $\dim(A) \leq \deg(\chi_{\mathfrak{m}}(t))$ as we wanted to prove.

(3.10) Proposition. *Let A be a noetherian local ring with maximal ideal $\mathfrak{m}$. Then there is an $\mathfrak{m}$-primary ideal generated by $\dim(A)$ elements.*

Proof. Let $m = \dim(A)$. We shall construct elements $f_1, f_2, \dots, f_m$ such that every prime ideal that contains the ideal $(f_1, f_2, \dots, f_n) = \sum_{i=1}^n Af_i$ is of height at least equal to n for $n = 0, 1, \dots, m$. The construction is performed by induction on n. When $n = 0$ there is nothing to prove. Let $n - 1 < m$, and assume that we have constructed $f_1, f_2, \dots, f_{n-1}$ such that every prime ideal that contains $(f_1, f_2, \dots, f_{n-1})$ has height at least equal to $n-1$. If all the prime ideals that contain $(f_1, f_2, \dots, f_{n-1})$ have height at least equal to n we can take f_n to be any element in $(f_1, f_2, \dots, f_{n-1})$. We therefore assume that there is a least one prime ideal of height $n - 1$ that contains $(f_1, f_2, \dots, f_{n-1})$. It follows from Remark (CHAINS 2.8) applied to the ring $A/(f_1, f_2, \dots, f_{n-1})$ that there are finitely many prime ideals that are minimal among those containing $(f_1, f_2, \dots, f_{n-1})$. Let $\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_p$ be those of the minimal prime ideals containing $(f_1, f_2, \dots, f_{n-1})$ that are of height $n - 1$.

Since $\mathrm{ht}(\mathfrak{p}_i) = n - 1 < n \leq m = \dim(A)$, and m is the height of the maximal ideal $\mathfrak{m}$ we have that all the prime ideals $\mathfrak{p}_i$ are different from $\mathfrak{m}$. It therefore follows from Proposition (RINGS 4.22) that $\mathfrak{m} \neq \cup_{i=1}^p \mathfrak{p}_i$. Choose $f_n \in \mathfrak{m} \setminus \cup_{i=1}^p \mathfrak{p}_i$, and let $\mathfrak{q}$ be a prime ideal that contains the ideal $(f_1, f_2, \dots, f_n)$. Then $\mathfrak{q}$ contains a prime ideal $\mathfrak{p}$ that is minimal among the prime ideals that contain $(f_1, f_2 \dots, f_{n-1})$.

If $\mathfrak{p} = \mathfrak{p}_i$ for some i we have that $\mathfrak{q} \supset \mathfrak{p}$ because $f_n \in \mathfrak{q} \setminus \mathfrak{p}$. Hence we have that $\mathrm{ht}(\mathfrak{q}) > \mathrm{ht}(\mathfrak{p}) = \mathrm{ht}(\mathfrak{p}_i) = n - 1$, and thus that $\mathrm{ht}(\mathfrak{q}) \geq n$.

If $\mathfrak{p} \neq \mathfrak{p}_i$ for all i we have that $\mathrm{ht}(\mathfrak{p}) \geq n$ since $\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_p$ are all the prime ideals of height $n - 1$ among the minimal ideals containing $(f_1, f_2, \dots, f_{n-1})$. Hence $\mathrm{ht}(\mathfrak{q}) \geq \mathrm{ht}(\mathfrak{p}) \geq n$. We consequently have constructed elements $f_1, f_2, \dots, f_n$ for $n = 1, 2, \dots, m$ such that all the prime ideals that contain $(f_1, f_2, \dots, f_n)$ have heigth at least equal to n.

It remains to prove that $(f_1, f_2, \dots, f_m)$ is an $\mathfrak{m}$-primary ideal. Every prime ideal containing $(f_1, f_2, \dots, f_m)$ have height at least equal to $m = \dim(A)$, and since $m = \mathrm{ht}(\mathfrak{m})$ we have that $\mathfrak{p} = \mathfrak{m}$. Hence $\mathfrak{m}$ is the only prime ideal that contains $(f_1, f_2, \dots, f_m)$, and consequently $\mathfrak{m}$ is the radical of $(f_1, f_2, \dots, f_m)$. Hence, as we saw in Example (MODULES 4.3), the ideal $(f_1, f_2, \dots, f_m)$ is $\mathfrak{m}$-primary.

(3.11) Lemma. *Let A be a local noetherian ring with maximal ideal $\mathfrak{m}$. Moreover let $f_1, f_2, \dots, f_n$ be elements in $\mathfrak{m}$. We have that the elements $f_1, f_2, \dots, f_n$ generate the ideal $\mathfrak{m}$ if and only if their classes in $\mathfrak{m}/\mathfrak{m}^2$ generate the $A/\mathfrak{m}$-vector space $\mathfrak{m}/\mathfrak{m}^2$.*

In particular the least number of generators of $\mathfrak{m}$ is equal to $\dim_{A/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2)$.

Proof. It is clear that when $f_1, f_2, \dots, f_n$ generate $\mathfrak{m}$, then their residue classes generate $\mathfrak{m}/\mathfrak{m}^2$ as a $A/\mathfrak{m}$-vector space.

Conversely we have that when the residue classes of $f_1, f_2, \dots, f_n$ generate the $A/\mathfrak{m}$-vector space $\mathfrak{m}/\mathfrak{m}^2$ we have that $(f_1, f_2, \dots, f_n) + \mathfrak{m}^2 = \mathfrak{m}$. It follows from Theorem (MODULES 1.27)(3) with $L = (f_1, f_2, \dots, f_n)$ and $\mathfrak{a} = \mathfrak{m}$ that then $(f_1, f_2, \dots, f_n) = \mathfrak{m}$.

(3.12) Theorem. *Let A be a noetherian local ring with maximal ideal $\mathfrak{m}$. The following numbers are equal:*

(1) *The dimension* $\dim(A)$ *of* A.
(2) *The degree* $\deg(\chi_{\mathfrak{m}}(t))$ *of the polynomial* $\chi_{\mathfrak{m}}(t)$.
(3) *The least number of generators of an* $\mathfrak{m}$*-primary ideal.*

Proof. Let m be the least number of generators of an $\mathfrak{m}$-primary ideal. It follows from Proposition (3.5)(2) that $\deg(\chi_{\mathfrak{q}}(t)) \leq m$. Hence it follows from Proposition (3.7) that $\deg(\chi_{\mathfrak{m}}(t)) \leq m$. The inequality $\dim(A) \leq \deg(\chi_{\mathfrak{m}}(t))$ follows from Lemma (3.9). Finally the inequality $m \leq \dim(A)$ follows from Proposition (3.10). Hence $\dim(A) \leq \dim(\chi_{\mathfrak{m}}(t)) \leq m \leq \dim(A)$, and we have proved the Theorem.

(3.13) Corollary. *Let A be a noetherian local ring with maximal ideal $\mathfrak{m}$. Then*

$$\dim(A) \leq \dim_{A/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2)$$

where $\dim_{A/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2)$ is the dimension of the vector space $\mathfrak{m}/\mathfrak{m}^2$ over the field $A/\mathfrak{m}$.

Proof. The Corollary follows immediately from Lemma (3.11) and the Theorem.

(3.14) Corollary. *Let A be a noetherian ring and let $f_1, f_2, \dots, f_m$ be elements of A. Then every prime ideal that is minimal among the associated ideals of the A-module $A/(f_1, f_2, \dots, f_m)$ is of height at most equal to m.*

In particular we have that if f is neither a zero divisor nor a unit, then every prime ideal that is minimal among the prime ideals containing $(f) = Af$ has height 1

Proof. It follows from Proposition (CHAINS 3.3) that every prime ideal of A containing $(f_1, f_2, \dots, f_m)$ contains an associated prime ideal of $B = A/(f_1, f_2, \dots, f_m)$. In particular the ideals that are minimal among those that contain $(f_1, f_2, \dots, f_m)$ are associated ideals of B. It follows from Remark (?) that there is only a finite number of associated ideals of B that are minimal among the associated ideals of B.

Let $\mathfrak{p} = \mathrm{Ann}(g)$ with $g \in B$ be a prime ideal of A that is minimal among the associated prime ideals of B. It follows from Proposition (MODULES 3.11) that $\mathfrak{p}A_{\mathfrak{p}}$ is minimal among the prime ideals that contain $(f_1, f_2, \dots, f_m)A_{\mathfrak{p}}$ and thus the only prime ideal in $A_{\mathfrak{p}}$ containing $(f_1, f_2, \dots, f_m)A_{\mathfrak{p}}$. Hence it follows from the Theorem that $\mathrm{ht}(\mathfrak{p}A_{\mathfrak{p}}) = \dim(A_{\mathfrak{p}}) \leq m$. We obtain that $\mathrm{ht}(\mathfrak{p}) = \mathrm{ht}(\mathfrak{p}A_{\mathfrak{p}}) \leq m$. Hence we have proved the first part of the Corollary.

To prove the last part we use the above observation that the minimal prime ideals $\mathfrak{p}$ in A that contain (f) are associated to $A/(f)$. Hence it follows from the first part that they have height at most equal to 1. If $\mathrm{ht}(\mathfrak{p}) = 0$ we have that $\dim(A_{\mathfrak{p}}) = 0$. It follows from Proposition (CHAINS 2.7) that $A_{\mathfrak{p}}$ is noetherian and hence it follows from Theorem (CHAINS 2.17) that it is artinian. Then we have that $f^m = 0$ in $A_{\mathfrak{p}}$ for some positive integer m, and thus $sf^m = 0$ in A for some $s \notin \mathfrak{p}$. Consequently f is a zero divisor contrary to the assumptions of the Corollary. We have proved the last part of the Corollary.

(3.15) Corollary. *Let A be a noetherian local ring and let $f \in \mathfrak{m}$ be a regular element of A. Then*

$$\dim(A/Af) = \dim(A) - 1.$$

Proof. It follows from the Theorem, from Lemma (3.8), and from Proposition (3.7) that we have an inequality $\dim(A/fA) \leq \dim(A) - 1$.

To prove the opposite inequality we observe that it follows from Proposition (3.10) that, with $\dim(A/fA) = m - 1$, we can find elements $f_1, f_2, \dots, f_{m-1}$ in A whose images by the canonical map $\varphi_{A/fA} : A \to A/fA$ generate an $(\mathfrak{m}/Af)$-primary ideal in A/Af. It is clear that the ideal $(f_1, f_2, \dots, f_{m-1}, f)$ is $\mathfrak{m}$-primary. Hence it follows from the Theorem that $\dim(A) \geq m$. Hence $\dim(A) - 1 \geq m - 1 = \dim(A/fA)$, and we have proved that $\dim(A/fA) = \dim(A) - 1$.

(3.16) Definition. Let A be a local noetherian ring with maximal ideal $\mathfrak{m}$. A *parameter system* for A is a collection of $\dim(A)$ elements that generate an $\mathfrak{m}$-primary ideal.

(3.17) Remark. Let A be a ring and let $f_1, f_2, \dots, f_m$ with $m = \dim(A)$ be elements of A. Write $\mathfrak{q} = (f_1, f_2, \dots, f_m) = \sum_{n=1}^{m} Af_n$. We obtain a map

$$\varphi : (A/\mathfrak{q})[t_1, t_2, \dots, t_m] \to G_{\mathfrak{q}}(A) = \oplus_{n=0}^{\infty} \mathfrak{q}^n/\mathfrak{q}^{n+1} \tag{3.17.1}$$

from the polynomial ring in the varables $t_1, t_2, \dots, t_m$ over $A/\mathfrak{q}$ to the graded ring $G_{\mathfrak{q}}(A)$ by sending the variable t_i to the class in $\mathfrak{q}/\mathfrak{q}^2$ of f_i for $i = 1, 2, \dots, m$. It is clear that the map φ is a surjective map of $(A/\mathfrak{q})$-algebras.

(3.18) Lemma. *Let A be a local noetherian ring with maximal ideal $\mathfrak{m}$. Moreover let $f_1, f_2, \dots, f_m$ be a parameter system for A and let $\mathfrak{q} = (f_1, f_2, \dots, f_m)$. When f is a homogeneous element in the polynomial ring $A[t_1, t_2, \dots, t_m]$ in the variables $t_1, t_2, \dots, t_m$ over A which is in the kernel of the map* (3.17.1)

$$\varphi : (A/\mathfrak{q})[t_1, t_2, \dots, t_m] \to G_{\mathfrak{q}}(A)$$

then f has coefficients in $\mathfrak{m}$.

Proof. Assume that f has a coefficient that is not in $\mathfrak{m}$. It follows from Example (RINGS 2.16) that f is not a zero divisor in $(A/\mathfrak{q})[t_1, t_2, \dots, t_m]$. We obtain from Lemma (2.16) that $d_\ell((A/\mathfrak{q})[t_1, t_2, \dots, t_m]/(f)) = d_\ell((A/\mathfrak{q})[t_1, t_2, \dots, t_m]) - 1$. As we saw in Example (2.13) we have that $d_\ell((A/\mathfrak{q})[t_1, t_2, \dots, t_m]) = m - 1$. Since f is in the kernel of the surjection φ we have that φ induces a surjective map $(A/\mathfrak{q})[t_1, t_2, \dots, t_m]/(f) \to G_\mathfrak{q}(A)$ of graded $(A/\mathfrak{q})$-algebras. Hence we have that $d_\ell(G_\mathfrak{q}(A)) \leq d_\ell((A/\mathfrak{q})[t_1, t_2, \dots, t_m]/(f)) = m - 2$. However it follows from Lemma (3.4)(3) and Theorem (3.12) that $d_\ell(G_\mathfrak{q}(A)) = \deg(\chi_\mathfrak{q}(A)) - 1 = \dim(A) - 1 = m - 1$. This contradicts the assumption that f has a coefficient that is not contained in $\mathfrak{m}$, and we have proved the Lemma.

(3.19) Theorem. *Let A be a noetherian local ring with maximal ideal $\mathfrak{m}$ and let $\boldsymbol{\kappa} = A/\mathfrak{m}$ be the residue field. The following conditions are equivalent:*

(1) *There are generators $f_1, f_2, \dots, f_m$ of $\mathfrak{m}$ with $m = \dim(A)$ such that the homomorphism* (3.17.1)
$$\varphi : \boldsymbol{\kappa}[t_1, t_2, \dots, t_m] \to G_\mathfrak{m}(A)$$
is an isomorphism.

(2) We have an equality $\dim(A) = \dim_{\boldsymbol{\kappa}}(\mathfrak{m}/\mathfrak{m}^2)$.

(3) The maximal ideal $\mathfrak{m}$ can be generated by $\dim(A)$ elements.

When the conditions hold we have that condition (1) holds for all families of generators for $\mathfrak{m}$ with $\dim(A)$ elements.

Proof. (1) $\Rightarrow$ (2) when φ is an isomorphism the $\boldsymbol{\kappa}$-vector space $\boldsymbol{\kappa}t_1 + \boldsymbol{\kappa}t_2 + \dots + \boldsymbol{\kappa}t_m$ is isomorphc to the $\boldsymbol{\kappa}$-vector space $\mathfrak{m}/\mathfrak{m}^2$. It follows that $\dim_{\boldsymbol{\kappa}}(\mathfrak{m}/\mathfrak{m}^2) = m = \dim(A)$.

(2) $\Rightarrow$ (3) Let $g_1, g_2, \dots, g_m$ with $m = \dim(A)$ be elements of $\mathfrak{m}$ whose residue classes in $\mathfrak{m}/\mathfrak{m}^2$ form an $\boldsymbol{\kappa}$-basis. Then it follows from Lemma (3.11) that we have $(g_1, g_2, \dots, g_m) = \mathfrak{m}$.

(3) $\Rightarrow$ (1) When $g_1, g_2, \dots, g_m$ are generators for $\mathfrak{m}$ with $m = \dim(A)$ we have that the homomorphism $\varphi : \boldsymbol{\kappa}[t_1, t_2, \dots, t_m] \to G_\mathfrak{m}(A)$ of (3.17.1) defined by $g_1, g_2, \dots, g_m$ is surjective. It follows from Lemma (3.18) that it is also injective.

The last part of the Theorem was proved in the above three steps.

(3.20) Definition. A local noetherian ring A that satisfies the three conditions of Theorem (3.20) is called a *regular local ring.*

(3.21) Proposition. *Let A be a local regular ring. Then A is an integral domain.*

Proof. Let f, g be non zero elements in A. It follows from Corollary (1.16) that $\cap_{n=0}^\infty \mathfrak{m}^n = 0$. Consequently we can find natural numbers p, q such that $f \in \mathfrak{m}^p \setminus \mathfrak{m}^{p+1}$ and $g \in \mathfrak{m}^q \setminus \mathfrak{m}^{q+1}$. It follows from Theorem (3.19)(1) that $G_\mathfrak{m}(A)$ is an integral domain. Hence the product in $G_\mathfrak{m}(A)$ of the residue classes of f in $\mathfrak{m}^p/\mathfrak{m}^{p+1}$, respectively g in $\mathfrak{m}^q/\mathfrak{m}^{q+1}$ is not zero. That is, the residue class of fg in $\mathfrak{m}^{p+q}/\mathfrak{m}^{p+q+1}$ is not zero. Hence $fg \notin \mathfrak{m}^{p+q+1}$, and we have, in particular, that $fg \neq 0$.

(3.22) Theorem. *Let A be a local noetherian ring with maximal ideal $\mathfrak{m}$.*

(1) *If A is a regular local ring, then every family of generators $f_1, f_2, \dots, f_m$ of $\mathfrak{m}$ with $m = \dim(A)$ is an A-regular sequence, and the ring $A/(f_1, f_2, \dots, f_n)$ is regular of dimension $\dim(A) - n$ for $n = 0, 1, \dots, m$.*

(2) *If $\mathfrak{m}$ is generated by a regular A-sequence we have that A is a regular local ring.*

Proof. (1) We prove by induction on n that the ring $A/(f_1, f_2, \dots, f_n)$ is regular of dimension $\dim(A) - n$. This is trivial when $n = 0$. Assume that it holds for $n - 1$. Since $f_1, f_2, \dots, f_m$ is a family of generators for $\mathfrak{m}$ with the least number of members we have that the classes of $f_n, f_{n+1}, \dots, f_m$ in $A/(f_1, f_2, \dots, f_{n-1})$ are generators with the least number of elements for the maximal ideal of $A/(f_1, f_2, \dots, f_n)$. In particular we have that f_n is not zero in $A/(f_1, f_2, \dots, f_{n-1})$.

It follows from the induction hypothesis that $A/(f_1, f_2, \dots, f_{n-1})$ is regular of dimension $\dim(A) - n + 1$, and hence an integral domain by Proposition (3.22). Then the residue class of f_n in $A/(f_1, f_2, \dots, f_{n-1})$ is not a zero divisor, and it follows from Corollary (3.15) that $\dim(A/(f_1, f_2, \dots, f_n)) = \dim(A/(f_1, f_2, \dots, f_{n-1})) - 1 = \dim(A) - n$. Since the maximal ideal of $A/(f_1, f_2, \dots, f_n)$ can be generated by the residue classes of the $\dim(A) - n = m - n$ elements $f_{n+1}, f_{n+2}, \dots, f_m$ it follows from Theorem (3.20) that $A/(f_1, f_2, \dots, f_n)$ is regular. Hence we have proved assertion (1).

(2) Assume that $g_1, g_2, \dots, g_n$ is an A-regular sequence that generates $\mathfrak{m}$. It follows from Corollary (3.15) by induction on p that $A/(g_1, g_2, \dots, g_p)$ is of dimension $\dim(A) - p$ for $p = 0, 1, \dots, n$. Consequently we have that $\dim(A) \geq n$. It follows from Theorem (3.12) that $\mathfrak{m}$ can not be generated by fewer than $\dim(A)$ elements. Hence $n \geq \dim(A)$ and we have that $\dim(A) = n$. Since $g_1, g_2, \dots, g_n$ is a set of generators of $\mathfrak{m}$ it follows from Theorem (3.19) that A is a regular local ring.

(3.23) Exercises.

1. Let K be a field and let $K[t_1, t_2]$ be the polynomial ring in the independent variables t_1, t_2 with coefficients in the field K. Denote by u and v the residue classes of t_1, respectively t_2, in the residue ring $A = K[t_1, t_2]/(t_1 + t_2)^m$.

(1) Determine the dimension of the local ring $A_{(u,v)}$.

(2) Find a minimal set of generators of an $(u, v)A_{(u,v)}$-primary ideal.

2. Let K be a field and let $K[t_1, t_2]$ be the polynomial ring in the independent variables t_1, t_2 with coefficients in the field K. Denote by u and v the residue classes of t_1, respectively t_2, in ther residue ring $A = K[t_1, t_2]/(t_1^2, t_1 t_2^m)$.

(1) Determine the dimension of the local ring $A_{(u,v)}$.

(2) Find a minimal set of generators of an $(u, v)A_{(u,v)}$-primary ideal.

3. Let K be a field and let $K[t_1, t_2]$ be the polynomial ring in the independent variables t_1, t_2 with coefficients in the field K. Denote by u and v the residue classes of t_1, respectively t_2, in the residue ring $A = K[t_1, t_2]/(t_1^2, t_2^m)$.

(1) Determine the dimension of the local ring $A_{(u,v)}$.

(2) Find a minimal set of generators of an $(u,v)A_{(u,v)}$-primary ideal.

4. Let K be field and let $K[t_1, t_2, \dots]$ be the ring of polynomials in the independent variables $t_1, t_2, \dots$ with coefficients in K. For each positive integer n write $\mathfrak{p}_n = (t_{2^{n-1}}, t_{2^{n-1}+1}, \dots, t_{2^n-1})$, and let $S = K[t_1, t_2, \dots] \setminus \cup_{n=0}^{\infty} \mathfrak{p}_n$.

(1) Show that $\mathfrak{p}_n$ is a prime ideal.

(2) Show that S is a multiplicatively closed set.

(3) Show that $S^{-1}A$ is noetherian. In order to prove this you can use that a ring A is noetherian if every non-zero element is contained in a finite number of maximal ideals only, and if $A_{\mathfrak{m}}$ is noetherian for all maximal ideals $\mathfrak{m}$ of A. To prove the latter statement you can proceed as follows:

(a) Let $\mathfrak{a}$ be a non-zero ideal of A, and let $\mathfrak{m}_1, \mathfrak{m}_2, \dots, \mathfrak{m}_r$ be the maximal ideals containing $\mathfrak{a}$. Choose an element x_0 of $\mathfrak{a}$. We denote by $\mathfrak{m}_1, \mathfrak{m}_2, \dots, \mathfrak{m}_r, \mathfrak{m}_{r+1}, \dots, \mathfrak{m}_s$ be the maximal ideals containing x_0. Show that for $i = r+1, r+2, \dots s$ we have that $\mathfrak{m}_i$ is not contained in $\cup_{j=1}^{r} \mathfrak{m}_j$.

$\rightarrow$ (b) With the notation as in part (a). Choose for $i = r+1, r+2, \dots, s$ an element x_i in $\mathfrak{m}_i \setminus \cup_{j=1}^{r} \mathfrak{m}_j$. Show that $\mathfrak{a}A_{\mathfrak{m}} = (x_0, x_1, \dots, x_s)A_{\mathfrak{m}} = A_{\mathfrak{m}}$ for all maximal ideals of A that are not among the ideals $\mathfrak{m}_1, \mathfrak{m}_2, \dots, \mathfrak{m}_r$.

$\rightarrow$ (c) Choose elements $x_{s+1}, x_{s+2}, \dots, x_t$ whose images in $A_{\mathfrak{m}_i}$ generate $\mathfrak{a}A_{\mathfrak{m}_i}$ for $i = 0, 1, \dots, s$. With the notation as in part (a) and (b), let $\mathfrak{b} = (x_0, x_1, \dots, x_t)$. Show that $\mathfrak{b} = \mathfrak{a}$.

$\rightarrow$ (d) Show that it follows from (a), (b) and (c) that A is noetherian.

(4) Determine the height of $\mathfrak{p}_n S^{-1}A$.

(5) Determine the dimension of $S^{-1}A$.

5. Let A be a ring and let $\mathfrak{p}$ be a prime ideal. Moreover let $\boldsymbol{\kappa}(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$.

(1) Let $\mathfrak{q}$ be a prime ideal in the polynomial ring $A[t]$ in the variable t over A such that $\mathfrak{p} = \mathfrak{q} \cap A$. Show that $\mathfrak{q}A_{\mathfrak{p}}[t]$ is a prime ideal in $A_{\mathfrak{p}}[t]$.

(2) Show that the map that sends $\mathfrak{q}$ to the image of $\mathfrak{q}A_{\mathfrak{p}}[t]$ by the residue map $A_{\mathfrak{p}}[t] \to (A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}})[t] = \boldsymbol{\kappa}(\mathfrak{p})[t]$ gives a bijection between prime ideals $\mathfrak{q}$ in $A[t]$ such that $\mathfrak{p} = \mathfrak{q} \cap A$, and prime ideals in $\boldsymbol{\kappa}(\mathfrak{p})[t]$.

(3) Show that if $\mathfrak{q} \subset \mathfrak{q}'$ are prime ideals in $A[t]$ such that $\mathfrak{p} = \mathfrak{q} \cap A = \mathfrak{q}' \cap A$, then we have $\mathfrak{q} = \mathfrak{p}A[t]$.

6. Let A be a ring and let $A[t]$ be the ring of polynomials in the variable t with coefficients in A.

(1) Show that for every ideal $\mathfrak{a}$ in A we have that $\mathfrak{a}A[t] \cap A = \mathfrak{a}$.

(2) Show that for every prime ideal $\mathfrak{p}$ in A we have that $\mathfrak{p}A[t]$ is a prime ideal in $A[t]$.

(3) Show that $\dim(A) + 1 \leq \dim(A[t])$.

$\rightarrow$ (4) Use Exercise (5) to show that

$$\dim(A[t]) \leq 1 + 2\dim(A).$$

7. Let A be a noetherian ring and let $\mathfrak{p}$ be a prime ideal.

(1) Show that $\mathrm{ht}(\mathfrak{p}A[t]) \geq \mathrm{ht}(\mathfrak{p})$.
(2) Let $m = \mathrm{ht}(\mathfrak{p})$. Show that there are elements $f_1, f_2, \ldots, f_m$ in $\mathfrak{p}$ such that the ideal $(f_1, f_2, \ldots, f_m)A_\mathfrak{p}$ is $\mathfrak{p}A_\mathfrak{p}$-primary.
(3) Show that $\mathfrak{p}$ is minimal among the prime ideals in A that contain the ideal $\mathfrak{a} = (f_1, f_2, \ldots, f_m)$.
(4) Show that $\mathfrak{p}A[t]$ is minimal among the prime ideals in $A[t]$ that contain $\mathfrak{a}A[t]$.
(5) Show that $\mathrm{ht}(\mathfrak{p}A[t]) \leq \mathrm{ht}(\mathfrak{p}) = m$.
(6) Let $\mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \cdots \subset \mathfrak{q}_r$ be prime ideals in $A[t]$, and let $\mathfrak{p}_i = \mathfrak{q}_i \cap A$ for $i = 0, 1, \ldots, r$. Show that $\mathrm{ht}(\mathfrak{p}_s) \geq s$.
(7) Assume that we have proper inclusions $\mathfrak{p}_{s+1} \subset \mathfrak{p}_{s+2} \subset \cdots \subset \mathfrak{p}_r$, and that $\mathfrak{p}_s = \mathfrak{p}_{s+1}$. Show that $r - s - 1 + \mathrm{ht}(\mathfrak{p}_s) \leq \dim(A)$.
(8) Show that
$$\dim(A[t]) = \dim(A) + 1.$$

8. Let K be a field an let $K[u, v]$ be the polynomial ring in the independent variables u, v with coefficients in K. Let A be the localization of the ring $K[u, v]$ in the maximal ideal (u, v). Moreover let B be the localization of the subring $K[u, v, v/u]$ of the ring of fractions of $K[u, v]$ in the multiplicatively closed set $K[u, v] \setminus (u, v)$. Denote by $\mathfrak{m}$ the maximal ideal of A.

(1) Show that $(u, v) = (u, v, v/u)B \cap A$.
(2) Show that $\mathrm{ht}_A(\mathfrak{m}) = 2$.
(3) Show that $\mathrm{ht}_B((u, v, v/u)B) = 2$.
(4) Show that $\dim(B/\mathfrak{m}B) = 1$, and in particular that the strict inequality
$$\mathrm{ht}_B((u, v, v/u)B) < \mathrm{ht}_A(\mathfrak{m}_A) + \dim(B/\mathfrak{m}B)$$
holds.

Flatness

1. Flatness.

(1.1) Setup. Given a ring A and an A–module M. For each prime ideal P of A we write $\kappa(P) = A_P/PA_P$. Let E be a free A–module of rank $r+1$ and $e_0, \dots, e_r$ a basis of E. Denote by $R = \mathrm{Sym}_A(E)$ the symmetric algebra of E over A and write $\mathbf{P}(E) = \mathrm{Proj}(\mathrm{R})$ for the r–dimensional projective space over $\operatorname{Spec} A$.

The particular quotient $A[x]/(x_2)$ we denote by $A[\varepsilon]$ where ε is the class of the variable x over A. Moreover we let $M[\varepsilon] = A[\varepsilon] \otimes_A M$.

(1.2) Definition. Given an A–module M. The module M is *flat* over A if every short exact sequence

$$0 \to N' \to N \to N'' \to 0$$

gives rise to a short exact sequence

$$0 \to M \otimes_A N' \to M \otimes_A N \to M \otimes_A N'' \to 0.$$

(1.3) Definition. Given a morphism $f\colon X \to S$ of schemes and an O_X–module $\mathcal{F}$. We say that *$\mathcal{F}$ is flat over S* if, for every point x of X, we have that $\mathcal{F}_x$ is a flat $\mathrm{O}_{S,f(x)}$–module, where the module structure comes from the map $f^{-1}\,\mathrm{O}_{S,f(x)} \to \mathrm{O}_{X,x}$, or equivalently from the composite map $\mathcal{O}_{S,f(x)} \to (f_*\mathcal{O}_X)_{f(x)} \to \mathcal{O}_{X,x}$. The morphism f *is flat* if O_X is flat over S.

When f is the identity we say that *$\mathcal{F}$ is a flat $\mathcal{O}_S$–module.*

(1.4) Remark. Flatness has the following fundamental properties:

(1) (*Long exact sequences*) We can break long exact sequences into short exact sequences. Hence M is flat over A if and only if every exact sequence

$$\cdots \to N' \to N \to N'' \to \cdots$$

of A–modules gives rise to an exact sequence

$$\cdots \to M \otimes_A N' \to N \otimes_A N \to M \otimes_A N'' \to \cdots.$$

(2) (*Left exactness*) Since the tensor product is right exact ([?], (2.18)) we have that M is flat over A if every injective map $N' \to N$ of A–modules gives rise to an injective map $M \otimes_A N' \to M \otimes_A N''$.

(3) (*Localization*) Let S be a multiplicatively closed subset of A. It follows from the definition of localization that the localization $S^{-1}A$ of A in S, that $S^{-1}A$ is a flat A–module.

(4) (*Base change*) Given a flat A–module N, and let B be an A–algebra. Then $B \otimes_A N$ is a flat B–module. Indeed, for every B–module P we have an isomorphism $P \otimes_B (B \otimes_A N) \cong P \otimes_A N$.

(5) (*Direct sums*) For every set $(N_i)_{i \in I}$ of A–modules and every A–module P we have an isomorphism $P \otimes_A (\oplus_{i\in I} N_i) \cong \oplus_{i \in I}(P \otimes_A N_i)$. Hence $\oplus_{i\in I} N_i$ is exact if and only if it is exact in every factor N_i. We conclude that $\oplus_{i \in I} N_i$ is flat over A if and only if each summand N_i is flat over A. It follows in particular that every free A–module is flat. Moreover, projective A–modules are flat because they are direct summands of free modules.

(1.5) Lemma. *Given an exact sequence*

$$0 \to M \to N \to F \to 0$$

of A–modules, where F is flat. Then the sequence

$$0 \to P \otimes_A M \to P \otimes_A N \to P \otimes_A F \to 0$$

is exact for all A–modules P.

Proof. Write P as a quotient of a free A–module L,

$$0 \to K \to L \to P \to 0.$$

We obtain a commutative diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] \\
 & K \otimes_A M \arrow[r] \arrow[d] & K \otimes_A N \arrow[r] \arrow[d] & K \otimes_A F \arrow[d] \\
0 \arrow[r] & L \otimes_A M \arrow[r] \arrow[d] & L \otimes_A N \arrow[r] \arrow[d] & L \otimes_A F \\
 & P \otimes_A M \arrow[r] \arrow[d] & P \otimes_A N & \\
 & 0 & &
\end{tikzcd}$$

where the upper right vertical map is injective because F is flat, and the middle left horizontal map is injective because L is free. A diagram chase gives that $P \otimes_A M \to P \otimes_A N$ is injective.

(1.6) Proposition. *Given an exact sequence*

$$0 \to F' \to F \to F'' \to 0$$

of A–modules with F'' flat. Then F is flat if and only if F' is flat.

Proof. Given an injective map $M' \to M$. We obtain a commutative diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
0 \arrow[r] & M' \otimes_A F' \arrow[r] \arrow[d] & M' \otimes_A F \arrow[r] \arrow[d] & M' \otimes_A F'' \arrow[r] \arrow[d] & 0. \\
0 \arrow[r] & M \otimes_A F' \arrow[r] & M \otimes_A F \arrow[r] & M \otimes_A F'' \arrow[r] & 0
\end{tikzcd}$$

The rows are exact to the left by Lemma (1.5), and we have injectivity of the top vertical map since F'' is flat. The Proposition follows from a diagram chase.

(1.7) Lemma. *Given an A–module M such that the map*

$$I \otimes_A M \to IM$$

is an isomorphism for all ideals I in A. For every free A–module F and every injective map $K \to F$ of A–modules we have that

$$K \otimes_A M \to F \otimes_A M$$

is injective.

Proof. Since every element in $K \otimes_A M$ is mapped into $F' \otimes_A M$ where F' is a finitely generated free submodule of F we can assume that F is finitely generated.

When the rank of F is 1 the Lemma follows from the assumption. We prove the Lemma by induction on the rank r of F. We have an exact sequence $0 \to F_1 \to F \to A \to 0$, where F_1 is a free rank $r-1$ module. Let $K_1 = K \cap F_1$ and let K_2 be the image of K in A. We obtain a diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & \\
 & K_1 \otimes_A M \arrow[r] \arrow[d] & K \otimes_A M \arrow[r] \arrow[d] & K_2 \otimes_A M \arrow[r] \arrow[d] & 0. \\
0 \arrow[r] & F_1 \otimes_A M \arrow[r] & F \otimes_A M \arrow[r] & A \otimes_A M &
\end{tikzcd}$$

where the right and left top vertical maps are injective by the induction assumption and it follows from Lemma (1.5) that the lower left map is injective because A is free. A diagram chase proves that the middle vertical map is injective.

(1.8) Proposition. *An A–module M is flat if and only if the map*

$$I \otimes_A M \to IM$$

is an isomorphism for all finitely generated ideals I of A.

Proof. If M is flat the tensor product $I \otimes_A M \to M$ of the map $I \to A$ is injective so $I \otimes_A M \to IM$ is an isomorphism.

Conversely, we can assume that $I \otimes_A M \to IM$ is an isomorphism for all ideals I of A. Indeed, every element of $I \otimes_A M$ is contained in $J \otimes_A M$, where J is a finitely generated ideal, and if $J \otimes_A M \to M$ is injective and the element is not zero then it is not mapped to zero by $I \otimes_A M \to M$.

Let $N' \to N$ be an injective map and write N as a quotient $0 \to K \to F \to N \to 0$ of a free A–module F. Let F' be the inverse image of N' in F. Then we have an exact sequence $0 \to K \to F' \to N' \to 0$, and we obtain a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & & \\
K \otimes_A M \arrow[r] \arrow[d] & F' \otimes_A M \arrow[r] \arrow[d] & N' \otimes_A M \arrow[r] \arrow[d] & 0. \\
K \otimes_A M \arrow[r] & F \otimes_A M \arrow[r] & N \otimes_A M &
\end{tikzcd}$$

It follows from Lemma (1.7) that the top vertical map is injective. A diagram chase shows that the right vertical map is injective. Consequently M is flat over A.

(1.9) Remark. It follows from Proposition (1.8) that a module over a principal ideal domain is flat if and only if it does not have torsion.

(1.10) Lemma. *Given a map $\varphi\colon A \to B$ of rings and let N be a B–module. Then N is flat over A if and only if N_Q is flat over A_P for all prime ideals P in A and Q in B such that $\varphi^{-1}(Q) = P$.*

Proof. Assume that N is flat over A. Since B_Q is flat over B the functor that sends an A_P–module F to $B_Q \otimes_B (N \otimes_A F)$ is exact. However $B_Q \otimes_B (N \otimes_A F) = N_Q \otimes_A F = N_Q \otimes_{A_P} F$. Consequently the functor that sends the A_P–module F to the A_P–module $N_Q \otimes_{A_P} F$ is exact, that is, the A_P–module N_Q is flat.

Conversely, assume that N_Q is a flat A_P module for all prime ideals Q in B with $P = \varphi^{-1}(Q)$. The functor that sends an A–module F to the A_P–module F_P is exact by Note (1.4(3)). Consequently the functor that sends the A–module F to the B_Q–module $N_Q \otimes_{A_P} F_P$ is exact. However, we have that $N_Q \otimes_{A_P} F_P = N_Q \otimes_{A_P} (A_P \otimes_A F) = N_Q \otimes_A F$. Hence the functor that sends an A–module F to $N_Q \otimes_A F$ is exact. However, the functor that sends an A–module F to the B–module $N \otimes_A F$ is exact if and only if the functor that sends the A–module F to the B_Q–module $N_Q \otimes_A F$ is exact for all prime ideal Q of B. We thus have that N is a flat A–module.

(1.11) Note. Given a morphism $f\colon X \to S$ of schemes and a quasi–coherent $\mathcal{O}_X$–module $\mathcal{F}$. It follows from Lemma (1.10) that $\mathcal{F}$ is flat over $\operatorname{Spec} A$ if and only if $\mathcal{F}(U)$ is a flat A–module for all open affine subsets U of X.

In particular, if $\mathcal{F}$ is flat over $\operatorname{Spec} A$, and $U_0, \dots, U_r$ is an open affine covering of X, the module $\mathcal{F}(U_{i_0} \cap \dots \cap U_{i_p})$ is flat over A for all $0 \le i_0 < \dots < i_p \le r$, and $\mathcal{F}_{\mathcal{U}}$ is a complex of flat A–modules.

(1.12) Lemma. *Given a regular* ([A], (Theorem 11.22)) *one dimensional ring A and a homomorphism $\varphi\colon A \to B$ into a noetherian ring B. Then B is flat over A if and only if $\varphi^{-1}(Q) = 0$ for all associated prime ideals Q in B.*

In particular, when B is reduced, we have that B is flat over A if and only if $\varphi^{-1}(Q) = 0$ for all minimal primes Q of B.

Proof. Assume that B is flat over A and let Q be a prime ideal in B. If $P = \varphi^{-1}(Q)$ is maximal we have that A_P is a discrete valutation ring ([A] (Proposition 9.2 and Lemma 11.23)). Let $t \in PA_P$ be a generator for the maximal ideal. Since t is not a zero divisor in A_P and B_Q is a flat A_P–module it follows that t is not a zero divisor in B_Q. Consequently Q is not an associated prime in B.

Conversely, assume that $\varphi^{-1}(Q)$ is zero for all associated primes Q of B. It follows from Lemma (1.10) that we must prove that B_R is flat over $A_{\varphi^{-1}(R)}$ for all prime ideals R in B. If $\varphi^{-1}(R) = 0$ we have that $A_{\varphi^{-1}(R)}$ is a field and consequently that B_R is flat. On the other hand, if $P = \varphi^{-1}(R)$ is a maximal ideal we choose a $t \in \varphi^{-1}(R)$ that generates the ideal PA_P. Since A_P is a principal ideal domain it follows from Remark (1.9) that it suffices to show that B_R is a torsion free A_P–module. Since all elements of A_P can be written as a power of t times a unit, this means that it suffices to prove that t is not a zero divisor in B_R. However, if t were a zero divisor in B_R it would be contained in an associated prime ideal Q of B since B is noetherian. This is impossible because $t \neq 0$ and, by assumption, $\varphi^{-1}(Q) = 0$. Hence t is not zero divisor and we have proved the first part of the Proposition.

The last part of the Proposition follows since in a reduced ring the associated primes are the minimal primes. Indeed, on the one hand every prime ideal contains an associated prime so that the minimal primes are associated. Conversely, let Q be an associated prime and $Q_1, \dots, Q_n$ be the minimal primes. Choose a non zero element a such that $aQ = 0$. We have that $Q \subseteq Q_1 \cup \dots \cup Q_n$ because if $b \in Q \setminus Q_1 \cup \dots \cup Q_n$ then $ab = 0$ and thus $a \in Q_1 \cap \dots \cap Q_n = 0$, contrary to the assumption that a is not zero. Hence $Q \subseteq Q_1 \cup \dots \cup Q_n$ and thus $Q \subseteq Q_i$ for some i ([A] (Proposition 1.11)). Hence $Q \subseteq Q_i$ and Q is minimal.

(1.13) Proposition. *Assume that A is a regular ring of dimension one. Given a morphism $f\colon X \to \operatorname{Spec} A$ from a noetherian scheme X. Then f is flat if and only if the associated points of X are mapped to the generic point of* $\operatorname{Spec} A$.

In particular, if X is reduced we have that f is flat if and only if the components of X all dominate $\operatorname{Spec} A$.

Proof. The Proposition is an immediate consequence of Lemma (1.12).

(1.14) Lemma. *Assume that A is noetherian and that M is a finitely generated A–module. Then M is flat if and only if M_P is a free A_P–module for all prime ideals P of A.*

Proof. It follows from Lemma (1.12) that M is flat over A if and only if M_P is flat over A_P for all primes P of A. Since M_P is flat over A_P if M_P is free over A_P it follows that when M_P is a free A_P–module for all prime ideals P of A, we have that M is a flat A–module.

Coversely, assume that M is a flat A–module. Given a prime ideal P of A. The M_P is a flat A_P–module. Since M is finitely generated it follows from Nakayama's Lemma that we can choose a surjection $A_P^n \to M_P$ such that $(\kappa(P))^n \to \kappa(P) \otimes_{A_P} M_P$ is an isomorphism of $\kappa(P)$–vectorspaces. Denote by L the kernel of $A_P^n \to M_P$. Since A is noetherian we have that L a is finitely generated A–module. However, since M is flat, we have that $\kappa(P) \otimes_{A_P} L = 0$. It follows by Nakayamas Lemma that $L = 0$. Consequently we have that the map $A_P^n \to M_P$ is an isomorphism, and that M_P is a free A_P–module.

(1.15) Lemma. *With the notation of Definition (1.9), assume that the A–modules $F^0, F^1, \dots$ of the complex F are flat and that $H^i(F)$ is a flat A–module for $i \geq p$. Then the A–modules $B^i(F)$ and $Z^{i-1}(F)$ are flat for $i \geq p$.*

Proof. We prove the Lemma by descending induction on p. The Lemma holds for $p > r$ since $Z^r = F^r$. Assume that the Lemma holds for $p+1$. By the induction assumption we have that B^{p+1} and Z^p are flat. From the sequence (1.9.2) with $i = p$ and Proposition (1.6) it follows that B^p is flat. Then, from the sequence (1.9.1) with $i = p-1$ and Proposition (1.6) it follows that Z^{p-1} is flat.

(1.16) Theorem. *Given a noetherian scheme S and a morphism $f\colon X \to S$ which is separated of finite type. Let $\mathcal{F}$ be a coherent O_X–module. Then:*

(1) *Assume that $\mathcal{F}$ is flat over S and that $R^i f_* \mathcal{F} = 0$ for $i > 0$. Then $f_*\mathcal{F}$ is a flat $\mathcal{O}_S$–module.*
In particular, if $f_*\mathcal{F}$ is coherent, we have that $f_*\mathcal{F}$ is locally free.

(2) *Assume that $S = \operatorname{Spec} A$ and that X is a closed subscheme of $\mathbf{P}(E)$. If there is an m_0 such that $f_*\mathcal{F}(m)$ is locally free for $m \geq m_0$, we have that $\mathcal{F}$ is flat over* $\operatorname{Spec} A$.

Proof. Both assertions are local on S. Hence we can assume that $S = \operatorname{Spec} A$ in both cases. Then it follows from the equality (1.7.4) that $f_*\mathcal{F} = H^0(X, \mathcal{F})$. Hence $f_*\mathcal{F}$ is a flat $\mathcal{O}_S$–module if and only if $H^0(X, \mathcal{F})$ is flat over A. The last part of (1) consequently follows from the first part of Lemma (1.14).

If $\mathcal{F}$ is flat over $\operatorname{Spec} A$ it follows from Note (1.11) that $\mathcal{F}(U_{i_0} \cap \dots \cap U_{i_p})$ is flat over A, and thus that the complex $\mathcal{F}_{\mathcal{U}}$ consists of flat modules. From the assumption of the Theorem we have that $H^i(\mathcal{F}_{\mathcal{U}}) = H^i(X, \mathcal{F}) = 0$ for $i > 0$. It follows from Lemma (1.15) with $p = 1$ that $Z^0(\mathcal{F}_{\mathcal{U}}) = H^0(X, \mathcal{F})$ is flat, and we have proved the first assertion.

By Assumption we have that $H^0(X, \mathcal{F}(m)) = f_*\mathcal{F}(m)(\operatorname{Spec} A)$ is flat for $m \geq m_0$. Let $N = \oplus_{m \geq m_0} H^0(X, \mathcal{F}(m))$. Then it follows from Setup (2.1) that N is an R/I–module such that $\mathcal{F} = \widetilde{N}$, where $I \subseteq R$ is an ideal defining X in $\mathbf{P}(E)$. We have, with the notation of Setup (2.1) that $\mathcal{F}(U_i) = \widetilde{N}(U_i) = \widetilde{N_{(y_i)}}$, where y_i is the class of e_i in R/I. It therefore suffices to prove that $N_{(y_i)}$ is flat over A. However, the module N is a direct sum of flat A–modules, and thus flat over A. Hence the functor which sends an A–module L to the A–module $N \otimes_A L$ is exact. We consider $N \otimes_A L$ as an R/I–module, via the action of R/I on N. Since $(R/I)_{y_i}$ is flat over R/I for all i we have that the functor that sends an A–module L to the A–module $(R/I)_{y_i} \otimes_{(R/I)} N \otimes_A L$ is exact. Hence $(R/I)_{y_i} \otimes_{(R/I)} N = N_{y_i}$ is a flat A–module. The same is therefore true for the direct summand $N_{(y_i)}$ of degree zero.

(1.17) Lemma. *Given a noetherian integral domain A and an A–algebra B of finite type. Moreover, given a finitely generated B–module N. Then there is a non–zero element $f \in A$ such that N_f is free over A_f.*

Proof. Write $B = A[u_1, \dots, u_h]$. We shall prove the Lemma by induction on h. When $h = 0$ we have that $A = B$. It follows from Lemma (2.6) in the non graded case that we can choose a filtration $N = N_n \supset N_{n-1} \supset \cdots \supset N_0 = 0$ by A–modules such that $N_i/N_{i-1} = A/P_i$, where P_i is a prime ideal in A. Since A is an integral domain we have that the intersection of the non zero primes P_i is not zero. Choose a non zero $f \in A$ in this intersection if there is one non zero prime P_i and let $f = 1$ otherwise. Then $(N_i/N_{i-1})_f$ is zero if P_i is a non zero prime and isomorphic to A_f when $P_i = 0$. Consequently we have that N_f is a free A_f–module.

Assume that $h > 0$ and that the Lemma holds for $h - 1$. Choose generators $n_1, \dots, n_s$ for the B–module N and write $B' = A[u_1, \dots, u_{h-1}]$. Then $B = B'[u_h]$. Moreover, let $N' = B'n_1 + \cdots B'n_s$. We have that N' is a finitely generated B'–module such that $BN' = N$. It follows from the induction assumption used to the A–algebra B' and the B'–module N' that we can find an element $f' \in A$ such that $N'_{f'}$ is a free $A_{f'}$–module. It therefore remains to prove that we can find an element $f'' \in A$ such that $(N/N')_{f''}$ is a free $A_{f''}$–module. To this end we write

$$N'_i = N' + u_h N' + \cdots + u_h^i N'$$

and

$$P_i = \{n \in N' \colon u_h^{i+1} n \in N'_i\}.$$

Clearly N'_i is a B'–submodule of N and P_i a B'–submodule of N'. We obtain a filtration

$$N'_1/N' \subseteq N'_2/N' \subseteq \cdots \subseteq N/N'$$

of N/N' by B'–modules N'_i/N' such that $\cup_i N'_i/N' = N/N'$. The B'–linear homomorphism $N' \to N'_{i+1}$ which sends n to $u_h^{i+1} n$ defines an isomorphism $N'/P_i \to N'_{i+1}/N'_i$ for all i. Since B' is noetherian, the sequence $P_0 \subseteq P_1 \subseteq \cdots \subseteq N'$ must stabilize. That is, among the quotients N'_{i+1}/N'_i there appears only a finite number of B'–modules. It follows from the induction assumption that we can find an element $f'' \in A$ such that all the modules $(N'_{i+1}/N'_i)_{f''}$ are free $A_{f''}$–modules. Hence $(N/N')_{f''}$ is a free $A_{f''}$–module, as we wanted to prove.

(1.18) Proposition. (Generic flatness) *Given a morphism $f: X \to S$ of finite type to a noetherian integral scheme S, and let $\mathcal{F}$ be a coherent O_X–module. Then there is an open dense subset U of S such that $\mathcal{F}_U$ is flat over U.*

Proof. We clearly can assume that S is affine. Since f is of finite type we can cover X with a finite number of open affine subschemes X_i. It follows from Lemma (1.17) that, for each i, there is an open dense affine subset U_i of S such that $(\mathcal{F}|X_i)_{U_i}$ is flat over U_i. We can take U to be the intersection of the sets U_i.

(1.19) Proposition. *Given a morphism $f: X \to S$ finite type to a noetherian scheme S and let $\mathcal{F}$ be a coherent O_X–module. Then S is a finite set theoretic union of locally closed reduced and disjoint subschemes S_i such that $\mathcal{F}_{S_i}$ is flat over S_i.*

Proof. Assume that the Proposition does not hold. Since S is noetherian there is a closed subscheme T of X which is minimal among the closed subschemes for which the Proposition does not hold. Let T' be an irreducible component of T with the reduced scheme structure and let V' be an open subset of T' that does not intersect the other components of T. Then V' is also open in T. It follows from Proposition (1.18) that there is an open non–empty subset V of V' such that $\mathcal{F}_V$ is flat over V. By the induction assumption the complement of V in T has a stratification, and together with V this gives a stratification of T. This contradicts the assumption that T has no stratification and we have proved the Proposition.

(1.20) Proposition. *Assume that A is a regular ring of dimension one. Let x be a closed point in* $\operatorname{Spec} A$ *and Y a closed subscheme of $p^{-1}(\operatorname{Spec} A \setminus x)$ which is flat over* $\operatorname{Spec} A \setminus x$ *and $\overline{Y}$ the scheme theoretic closure of Y in $\mathbf{P}(E)$ Then $\overline{Y}$ is the unique closed subscheme of $\mathbf{P}(E)$ which is flat over* $\operatorname{Spec} A$ *and whose restriction to $p^{-1}(\operatorname{Spec} A \setminus x)$ is equal to Y.*

Proof. Let P be the prime ideal in A corresponding to the point x of $\operatorname{Spec} A$. It clearly suffices to prove the Proposition for an open affine subset $\operatorname{Spec} C$ of $\mathbf{P}(E)$. Let $\varphi: A \to C$ be the homomorphism induced by the projection of $\mathbf{P}(E)$.

We have that $\operatorname{Spec} A \setminus x = \operatorname{Spec} A_t$ where t in P is the generator of PA_P. We have that $\operatorname{Spec} C \cap f^{-1}(\operatorname{Spec} A \setminus x) = \operatorname{Spec} C_{\varphi(t)}$. Let $C_{\varphi(t)} \to B$ define the closed subscheme $Y \cap \operatorname{Spec} C_{\varphi(t)}$ of $\operatorname{Spec} C_{\varphi(t)}$. The closure of $Y \cap \operatorname{Spec} C_{\varphi(t)}$ in $\operatorname{Spec} C$ is defined by the kernel I of the composite map $C \to C_{\varphi(t)} \to B$.

Since A is a principal ideal domain and B is flat, we have that B has no torsion. Hence the submodule C/I of B has no torsion, and thus C/I is flat over A. We have proved that the scheme theoretic closure $\overline{Y}$ of Y is flat over $\operatorname{Spec} A$.

To prove that Y is unique with the given properties we let J be an ideal in C that defines a closed subset which is flat over $\operatorname{Spec} A$ and whose restriction to $\operatorname{Spec} C_{\varphi(t)}$

is Y. That is, the ring C/J is flat over A and has the same image in $C_{\varphi(t)}$ as I. Then $J \subseteq I$. It remains to show that $I \subseteq J$. Let $c \in I$. Since I and J have the same image in $C_{\varphi(t)}$ we have that $t^n c \in J$ for some n. Since C/J is flat over A we have that C/J has no A–torsion. Hence $c \in J$ and we have that $I = J$.

(1.21) Lemma. *Given a ring B and a B–module N. Let F be a B–submodule of N.Denote by $\psi\colon N \to N/F$ the canonical quotient map. Given a homomorphism*

$$\varphi\colon F \to N/F$$

of B–modules.

We define

$$F_\varphi = \{f + \varepsilon n \in N[\varepsilon]\colon f \in F \text{ and } \varphi(f) = -\psi(n)\}.$$

Then:

1. *We have that F_φ is a $B[\varepsilon]$–submodule of $N[\varepsilon]$ whose image in N by the canonical map $\gamma\colon N[\varepsilon] \to N$ is F, and $N[\varepsilon]/F_\varphi$ is a flat $B[\varepsilon]$–module.*
2. *The correspondence which sends φ to F_φ defines a bijection between the set $\mathrm{Hom}_B(F, N/F)$ and the $B[\varepsilon]$–submodules F_ε of $N[\varepsilon]$ whose image by γ is F, and that are such that $N[\varepsilon]/F_\varepsilon$ is flat over $B[\varepsilon]$.*

Proof. It is clear that F_φ is a $B[\varepsilon]$–submodule of $N[\varepsilon]$ and that the image by γ is F. To check that $N[\varepsilon]/F_\varphi$ is flat over $B[\varepsilon]$ it follows from Proposition (1.8) that it suffices to check that the map

$$(\varepsilon) \otimes_{B[\varepsilon]} N[\varepsilon]/F_\varphi \to N[\varepsilon]/F_\varphi \tag{1.21.1}$$

is injective. Denote by $\beta\colon N[\varepsilon] \to N[\varepsilon]/F_\varphi$ the quotient map. Let $f + \varepsilon n \in N[\varepsilon]$ be such that $\varepsilon \otimes \beta(f + \varepsilon n)$ is in the kernel of the map (1.21.1). Then $\varepsilon f \in F_\varphi$ and thus $0 = \psi(f)$ so $f \in F$. Choose $n' \in N$ such that $\varphi(f) = -\psi(n')$. Then $f + \varepsilon n' \in F_\varphi$ and $\varepsilon \otimes \beta(f + \varepsilon n) = \varepsilon \otimes \beta(f) = \varepsilon \otimes \beta(f + \varepsilon n') = 0$. Hence we have proved that the map (1.21.1) is injective.

Conversely, let $F_\varepsilon \in N[\varepsilon]$ be a $B[\varepsilon]$–submodule such that $\gamma(F_\varepsilon) = F$ and such that $N[\varepsilon]/F_\varepsilon$ is flat over $B[\varepsilon]$. It follows from Lemma (1.5) that the sequence

$$0 \to B \otimes_{B[\varepsilon]} F_\varepsilon \to B \otimes_{B[\varepsilon]} N[\varepsilon] = N \to B \otimes_{B[\varepsilon]} N[\varepsilon]/F_\varepsilon \to 0 \tag{1.21.2}$$

is exact. The image of the map $B \otimes_{B[\varepsilon]} F_\varepsilon \to N$ is F, since we have assumed that $\gamma(F_\varepsilon) = F$. Consequently, the middle right map of (1.21.2) induces an isomorphism

$$\rho\colon N/F \to B \otimes_{B[\varepsilon]} N[\varepsilon]/F_\varepsilon.$$

We tensor the exact sequence

$$0 \to B \xrightarrow{\varepsilon} B[\varepsilon] \to B \to 0$$

with $N[\varepsilon]/F_\varepsilon$ over $B[\varepsilon]$ and obtain an exact sequence

$$0 \to N/F \overset{\rho}{\cong} B \otimes_{B[\varepsilon]} N[\varepsilon]/F_\varepsilon \xrightarrow{\delta} N[\varepsilon]/F_\varepsilon = B[\varepsilon] \otimes_{B[\varepsilon]} N[\varepsilon]/F_\varepsilon$$
$$\xrightarrow{\eta} N/F \overset{\rho}{\cong} B \otimes_{B[\varepsilon]} N[\varepsilon]/F_\varepsilon \to 0.$$

Denote by $\beta_\varepsilon \colon N[\varepsilon] \to N[\varepsilon]/F_\varepsilon$ the canonical quotient map and consider N as a submodule of $N[\varepsilon]$. Then $\eta(\beta_\varepsilon|N) = \psi$ and $\delta\psi = \varepsilon(\beta_\varepsilon|N)$.

For $f \in F$ we have that $\eta\beta_\varepsilon(f) = \psi(f) = 0$. Consequently there is a unique element $\psi(n)$ of N/F such that $\delta\psi(n) = \beta_\varepsilon(f)$. We then write $\varphi(f) = \psi(n)$. In this way we define a B–module homomorphism

$$\varphi \colon F \to N/F.$$

It remains to show that $F_\varepsilon = F_\varphi$.

Take $f + \varepsilon n \in F_\varepsilon \subseteq N[\varepsilon]$. Then $f \in F$ because $\gamma(F_\varepsilon) = F$. We have that $0 = \beta_\varepsilon(f + \varepsilon n) = \beta_\varepsilon(f) + \varepsilon\beta_\varepsilon(n)$. Consequently $\beta_\varepsilon(f) = -\varepsilon\beta_\varepsilon(n) = -\delta\psi(n)$, and thus $\varphi(f) = -\psi(n)$, by the definition of φ. Hence $f + \varepsilon n \in F_\varphi$.

Conversely, let $f + \varepsilon n \in F_\varphi$. Then again $f \in F$ and $\varphi(f) = -\psi(n)$, that is $\beta_\varepsilon(f) = -\delta\psi(n)$. We obtain that $\beta_\varepsilon(f + \varepsilon n) = \beta_\varepsilon(f) + \varepsilon\beta_\varepsilon(n) = -\delta\psi(n) + \delta\psi(n) = 0$ so that $f + \varepsilon n \in F_\varepsilon$. We have thus proved that $F_\varepsilon = F_\varphi$.

(1.10) Exercises.
1.

2. Flatness of finitely generated modules.

(2.1) Definition. Let A be a ring. For each prime ideal P in A we write $\kappa(P) = A_P/PA_P$.

The following result is one way of formulating the criterion for flatness by equations (see e.g. [M], Theorem 7.6, p. 49). We shall use this result instead of Lazard's Theorem ([La1], Theorem 1.2, p. 84) asserting that every flat module is the filtering limit of finitely generated free modules. As was observed by Lazard the results are indeed equivalent.

(2.2) Lemma. *Let A be a ring and M an A-module. The following assertions are equivalent:*

(1) *The module M is flat over A.*
(2) *For any finitely presented module N, that is there is an exact sequence $A^m \to A^n \to N \to 0$ of A-modules, the map*

$$\mathrm{Hom}_A(N, A) \otimes_A M \to \mathrm{Hom}_A(N, M) \tag{1.2.1}$$

that sends $u \otimes x$ to the A-linear map sending y to $u(y)x$ is bijective.
(3) *Any A-linear map $N \to M$ from a finitely presented A-module N factors through a finitely generated free A-module.*
(4) *For every A-module homomorphism $u: F \to M$ from a finitely generated free A-module F, and for every element e in the kernel of u, there is a factorization $u = vf$ of u via an A-module homomorphism $f: F \to G$ into a finitely generated free A-module G such that $f(e) = 0$, and an A-module homomorphism $v: G \to M$.*

Proof. For any A-module M the functors $\mathrm{Hom}_A(N, A) \otimes_A M$ and $\mathrm{Hom}_A(N, M)$ are additive and contravariant in N. Since the map (1.2.1) is bijective for $N = A$ it follows that it is bijective for $N = A^n$.

Assume that M is flat over A. Then the two functors are left exact. It follows that the map (1.2.1) is an isomorphism for every finitely presented A-module N. Hence the first assertion implies the second.

Assume that the second assertion holds. Let $u: N \to M$ be an A-linear map from a finitely presented A-module N. Then u is the image by (1.2.1) of an element $\sum_{i=1}^n u_i \otimes x_i$ of $\mathrm{Hom}_A(N, A) \otimes_A M$. Hence u is the composite of the map $N \to A^n$ sending y to $(u_1(y), \dots, u_n(y))$, and the map $A^n \to M$ sending $(a_1, \dots, a_n)$ to $\sum_{i=1}^n a_i x_i$. Hence the third assertion follows from the second.

The fourth assertion follows from the third since F/Ae is finitely presented.

Finally we prove that the last assertion implies the first. We shall show that M is flat over A by showing that the map $I \otimes_A M \to M$ is injective for all ideals I of A. Assume that there is an element $x = \sum_{i=1}^m a_i \otimes x_i$ with $a_i \in I$ and $x_i \in M$ in $I \otimes_A M$ that maps to zero in M. Let $u: F \to M$ be the A-linear homomorphism from the free A-module F with basis $f_1, \dots, f_m$ defined by $u(f_i) = x_i$, and let $y = \sum_{i=1}^m a_i \otimes f_i$.

Then $u(y) = x$, and the image e of y by the map $i\colon I \otimes_A F \to F$ maps to zero by u. Hence the last assertion of the Lemma implies that $u\colon F \to M$ factors via A-module homomorphisms $f\colon F \to G$ and $v\colon G \to M$, where G is a free A-module of finite rank, and where $f(e) = 0$. The map $j\colon I \otimes_A G \to G$ is injective since G is flat over A. We have that $0 = f(e) = fi(y) = j(\mathrm{id}_I \otimes f)(y)$ and consequently that $(\mathrm{id}_I \otimes f)(y) = 0$. Hence we have that $x = (\mathrm{id}_I \otimes u)(y) = (\mathrm{id}_I \otimes v)(\mathrm{id}_I \otimes f)(y) = 0$.

The following two results are well known (see e.g. Matsumura [M], Theorem 7.10, p. 51). We include proofs to show how Lemma (1.2) can be used in this situation instead of the criterion for flatness by equations.

(2.3) Lemma. *Let A be a local ring with maximal ideal P and M a flat A-module. Moreover, let F be a free A-module and $u\colon F \to M$ an A-linear map. If the residue map $u(P)\colon F/PF \to M/PM$ is injective, then the map u is injective.*

Proof. Let e in F be such that $u(e) = 0$. We first prove the Lemma when F is of finite rank. Since M is a flat A-module it follows from Proposition (1.2) that we have a factorization $F \xrightarrow{f} G \xrightarrow{v} M$ of u into A-linear maps, where G is a free A-module of finite rank, and where we have that $f(e) = 0$. Then $u(P)$ factors via $\kappa(P) \otimes_A F \xrightarrow{f(P)} \kappa(P) \otimes_A G \xrightarrow{v(P)} \kappa(P) \otimes_A M$. Since $u(P)$ is injective by assumption, it follows that $f(P)$ is injective. Our claim follows if we show that $F \xrightarrow{f} G$ is injective.

We fix a basis for F and G and let the map f be represented by a matrix. Let n be the rank of F. Since the induced map $f(P)$ is injective, there exist a $(n \times n)$-minor $N(P)$ of the matrix $f(P)$ which is invertible. It follows that the determinant of the corresponding square matrix N of f is invertible since $\det(N) \otimes_A \kappa(P) = \det(N(P))$. Then there exist a matrix N' such that $N'N$ is the identity matrix, and we may construct a map $f' : G \to F$ such that $f'f$ is the identity map. Hence f is injective.

Assume that F has infinite rank. Then the element e is contained in a free A-submodule F' of F of finite rank, which is a direct summand of F. Let $i\colon F' \to F$ be the inclusion. Then $i(P)$ is injective and thus $u(P)i(P) = ui(P)$ is injective. It follows from the first part of the proof that the map $ui\colon F' \to M$ is injective. Hence $ui(e) = u(e) = 0$ implies that $e = 0$ and we have proved the Lemma.

(2.4) Proposition. *Let M be a finitely generated flat A-module. Then M_P is a free A_P-module for all prime ideals P of A.*

Proof. Let P be a prime ideal of A. Then M_P is a flat A_P-module. Since M is finitely generated it follows from Nakayama's Lemma that we can choose a surjection $u\colon A_P^n \to M_P$ such that the residue map $u(P)\colon \kappa(P)^n \to \kappa(P) \otimes_{A_P} M_P$ is an isomorphism of $\kappa(P)$-vector spaces. If follows from Lemma (1.3) that $A_P^n \to M_P$ is injective and hence an isomorphism. Thus M_P is a free A_P-module for all prime ideals P in A.

(2.5) Proposition. *Let M be a finitely generated flat A-module. If there is an integer d such that*

$$d = \dim_{\kappa(P)}(\kappa(P) \otimes_A M) \tag{1.5.1}$$

for all prime ideals P of A we have that M is locally free.

Proof. Let P be a prime ideal of A. Let $m_1, \dots, m_n$ be a generator set for the A-module M, and let F be a free A-module with basis $f_1, \dots, f_d$. Since M is flat it follows from the Proposition that M_Q is a free A_Q-module for all prime ideals Q of A. It follows from (1.5.1) that M_Q is of rank d. In particular there is an isomorphism $u\colon F_P \to M_P$ of A_P modules. Choose elements $g_j = \sum_{i=1}^d a_{j,i} f_i$ in F_P such that $u(g_j) = \frac{m_j}{1}$ for $j = 1, \dots, n$. Let t be a common denominator of the elements $u(f_i)$, and of the coefficients $a_{j,i}$ for $i = 1, \dots, d$ and $j = 1, \dots, n$. Then there is a surjective map $v\colon F_t \to M_t$ of A_t-modules such that the localization of v at P is equal to u. Denote by K the kernel of v. For each prime Q of A we obtain an exact sequence $0 \to K_Q \to F_Q \to M_Q \to 0$ of A_Q-modules. Since F_Q is free of rank d it follows that $K_Q = 0$ for all primes Q of A_t. Consequently we have that $K = 0$. We thus have that M_t is a free A_t-module.

(2.5) Remark. When A is noetherian and M is a finitely generated A-module we have that if M_P is a free A_P-module, then there is an element t in A not in P such that M_t is a free A_t-module. Indeed, in the proof of Corollary (1.5) we constructed a surjective map $v\colon F_t \to M_t$ from a free A_t-module of rank equal to the rank of M_P, whose localization at P is an isomorphism. Hence the localization K_P of the kernel K of v at P is zero. Since A is noetherian by assumption, we have that K is finitely generated and thus we can find an element s in A not contained in P such that $K_s = 0$. It follows that $v_s\colon F_{st} \to M_{st}$ is an isomorphism of A_{st}-modules. In particular it follows from Proposition (1.4) that if M is flat, then M is locally free.

With the following example we will show that when A is not noetherian we can have a finitely generated flat A-module M such that M_P is free for all prime ideals P of A, but where M is not locally free. In particular it follows that condition (1.5.1) is necessary in Corollary (1.5).

(2.) Example. 6Let $B = k[y_1, y_2, \dots]$ be the polynomial ring in the variables $y_1, y_2, \dots$ over the field k, and let A be the residue ring of B by the ideal generated by the polynomials $y_i(y_i - 1)$ for $i = 1, 2, \dots$. Denote by x_i the class of y_i in A. Let P be a prime ideal of A. Then, for each i, the ideal P contains either x_i or $x_i - 1$. It follows that the prime ideals of A are the ideals $(x_1 - \delta_1, x_2 - \delta_2, \dots)$, for all choices of $\delta_1, \delta_2, \dots$, where δ_i, here and below, will take the values 0 and 1. We obtain in particular that $A/P = \kappa(P) = k$.

We note that the ring A is reduced. Indeed, if a polynomial $f(y_1, \dots, y_n)$ in B maps to a nilpotent element in A we must have that $f(\delta_1, \dots, \delta_n) = 0$ for all choices of $\delta_1, \dots, \delta_n$. It is easy to show, by induction on n, that this implies that $f(y_1, \dots, y_n)$ is in the ideal generated by the elements $y_1(y_1 - 1), \dots, y_n(y_n - 1)$. Hence the class of $f(y_1, \dots, y_n)$ in A is zero.

Let $P = (x_1 - \delta_1, x_2 - \delta_2, \dots)$. Then P is a prime ideal of A. For each i we have that $(x_i - \delta_i)(x_i + \delta_i - 1) = 0$, and clearly $x_i + \delta_i - 1 \notin P$. Consequently we have that the class of $x_i - \delta_i$ is zero in A_P. Hence we have that $A_P = \kappa(P) = k$ for all prime ideals P in A. In particular any module M over A is flat.

Fix a prime P of A. We have that the A-module $M = \kappa(P)$ is generated by one element. Moreover we have that $M_P = A_P \otimes_A M = \kappa(P) \otimes_A M = \kappa(P)$, and that $M_Q = A_Q \otimes_A M = \kappa(Q) \otimes_A M = 0$ for all prime ideals Q of A different from P.

In Spec A every non-empty open set contains infinitely many points. Indeed, let A_f be the ring of a non-empty principal open set in Spec A, where f is the residue class of the polynomial $f(y_1, \dots, y_n)$ in A. Then $f(\delta_1, \dots, \delta_n) \neq 0$ for some $\delta_1, \dots, \delta_n$. Then, for all choices of $\delta_{n+1}, \delta_{n+2}, \dots$, the prime $(x_1 - \delta_1, \dots, x_n - \delta_n, x_{n+1} - \delta_{n+1}, \dots)$ is in Spec A_f. Since $M = \kappa(P)$ has fiber k at one point and fiber zero at the remaining points, it follows that $M = \kappa(P)$ can not be locally free.

The condition that M is finitely generated is necessary in Corollary (1.5), even when A is noetherian, as shown by the following example communicated to us by C. Walter.

(2.7) Example. Let $A = \mathbf{Z}$ be the ring of integers and let M be the $\mathbf{Z}$-submodule

$$M = \{x \in \mathbf{Q} \colon v_p(x) \geq -1 \text{ for all primes } p \in \mathbf{Z}\}$$

of the rational numbers $\mathbf{Q}$, where $v_p(x) = d$ if $x = \frac{m}{n} p^d$ with m and n prime to p. If P is a maximal ideal of $\mathbf{Z}$ corresponding to a prime integer p, we have that $M_P = \frac{1}{p}\mathbf{Z}_P$. In particular the $\mathbf{Z}$-module M is a flat. Furthermore we have an isomorphism $\mathbf{Q} \to M \otimes_{\mathbf{Z}} \mathbf{Q} = M_{(0)}$. Hence we have that $\dim_{\kappa(P)}(\kappa(P) \otimes_{\mathbf{Z}} M) = 1$ for all prime ideals P in the ring $\mathbf{Z}$. However we obviously have that $M_n = \{\frac{x}{n^m} \colon x \in M, m \in \mathbf{Z}\}$ is not finitely generated $\mathbf{Z}_n$-module for any non-zero integer n. In particular M is not locally free.

(2.) Example. 8 We shall give another, perhaps more typical, example of a ring A, together with a flat A-module M, such that M_P is a free A_P-module of rank 1 for each prime P of A, but such that M is neither a finite, nor a locally free A-module.

Denote by A the product $\prod_{i \in \mathbf{N}} K_i$ of a field $K = K_i$ for $i \in \mathbf{N}$. Let I be the ideal in A consisting of elements $a = (a_i)_{i \in \mathbf{N}}$ with finite support $\operatorname{Supp}(a) = \{i \colon a_i \neq 0\}$. That is, the ideal I is the direct sum $\oplus_{i \in \mathbf{N}} K_i$ of the field $K = K_i$ for $i \in \mathbf{N}$. Let $M = I \oplus A/I$.

We first show that the ring A is *absolutely flat*, that is all A-modules M are flat. Note that there are no inclusions of prime ideals in A. Indeed, let P be a prime ideal and let a be an element in A not in P. If a is not a unit in A we let b be an element in A having support on the complement of $\operatorname{Supp}(a)$. Then $ab = 0$, and consequently we have b in P. The element $a + b$ is congruent to a modulo P. We have that $a + b$ is a unit in A since $\operatorname{Supp}(a + b) = \mathbf{N}$, hence a is a unit in A/P. Thus A/P is a field and all prime ideals are maximal, and minimal.

The ring A is reduced and consequently any fraction ring of A is reduced. In particular the stalks A_P are reduced for all prime ideals P in A. In our ring A all prime ideals P are minimal, thus we get that $A_P = \kappa(P)$. Consequently any module M is flat over A.

Hence, when we localize the exact sequence

$$0 \to I \to A \to A/I \to 0 \tag{1.8.1}$$

in a prime ideal P of A we see that we either have that I_P is a free A_P-module of rank 1 and $(A/I)_P = 0$, or we have that $I_P = 0$ and $(A/I)_P$ is a free A_P-module of rank 1. In both cases we have that $M_P = A_P \otimes_A M = \kappa(P) \otimes_A M$ is a free $\kappa(P)$-module of rank 1.

We have that I is not a finitely generated A-module, since the elements of I otherwise would have support on a finite subset of $\mathbf{N}$. However I is a quotient of M, so M is not a finitely generated A-module either.

We can tell exactly for which prime ideals P we have that $I_P = 0$. Indeed, it is easily seen that there is an inclusion preserving bijection between ideals in A and filters of $\mathbf{N}$. This correspondence associates to an ideal I of A the ultrafilter consisting of the complement in $\mathbf{N}$ of the support $\operatorname{Supp}(a) = \{i \in \mathbf{N} \colon a_i \neq 0\}$ of the elements $a = (a_i)_{i \in \mathbf{N}}$ of I. Under this correspondance the prime ideals of A correspond to the ultra filters of $\mathbf{N}$. The trivial ultra filters, that is the ultra filters consisting of the sets containing a fixed integer, correspond to the maximal ideals consisting of elements with one fixed coordinate equal to zero.

We have that $I_P = 0$ exactly when P corresponds to a non-trivial ultra filter. Indeed, let $a = (a_i)_{i \in \mathbf{N}}$ be an element of I. Then $ab = 0$ for all elements b in A whose support is in the complement of the support of a. Such an element b has cofinite support, that is, the complement of the support is finite. However, it is easily seen that an ultra filter is non-trivial if and only if it contains the filter of all cofinite sets.

We have that if P is a prime ideal corresponding to a trivial ultra filter, then there exist a f not in P such that $M_f = I_f = A_f$. The module M is however not locally free, that is there exist prime ideals P in A such that M_f is not free for any f not in P. Indeed if there for each prime ideal P exists f_P not in P such that M_{f_P} is free, then there exist prime ideals $P_1, \dots, P_m$ such that $\sum_{i=1}^m a_i f_{P_i} = 1$, with $a_1, \dots, a_m$ in A. We have that $M_P = A_P$ for all prime ideals P in A. It follows that M_{f_P} is a finitely generated A_{f_P}-module. Let $x_1, \dots, x_n$ be elements in M such that the classes of $x_1, \dots, x_n$ generate $M_{f_{P_i}}$ as an $A_{f_{P_i}}$-module for $i = 1, \dots, m$. Then $x_1, \dots, x_n$ generate M as an A-module. In particular we would have that M is finitely generated, which we have seen is not the case. Thus M is not locally free.

(2.10) Exercises.

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Indeks

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