

Gradings of the affine line and its Quot scheme

Gustav Sædén Ståhl
gss@kth.se

December 16, 2011

Abstract

For a field k and a grading of the polynomial ring $k[t]$ with Hilbert function h , we consider the Quot functor Quot_V^h , where $V = \bigoplus_{i=1}^d k[t]$ is a finitely generated and free $k[t]$ -module. The Quot functor parametrizes, for any k -algebra B , homogeneous $B[t]$ -submodules $N \subseteq B \otimes_k V$ such that the graded components of the quotient $(B \otimes_k V)/N$ are locally free B -modules of rank given by h . We find that it is locally representable by a polynomial ring over k in a finite number of variables. Finally, we show that there is a scheme that represents the Quot functor that is both smooth and irreducible.

Contents

1	Introduction	3
1.1	Background	3
1.2	What we will do	5
2	The Hilbert scheme of the affine line	6
2.1	Preliminaries	6
2.2	Grading by $G = \mathbb{Z}$	10
2.3	Grading by $G = \mathbb{Z}/n\mathbb{Z}$	12
3	The Grassmann scheme	17
3.1	The Grassmann functor	17
3.2	The graded Grassmannian	20
4	The Quot scheme of the affine line	21
4.1	Structure theorem of finitely generated graded modules over a graded principal ideal domain	21
4.2	The relative Quot scheme	24
4.2.1	Grading by $G = \mathbb{Z}$	29
4.2.2	Grading by $G = \mathbb{Z}_n$	30
4.3	The Quot scheme	30
	References	38

1 Introduction

In [6] the authors, Haiman and Sturmfels, introduce the multigraded Hilbert scheme by a general construction that, for a ring k , takes a k -algebra B to a set of certain submodules of a graded polynomial ring over B . This construction is also applicable to a generalization of the Hilbert scheme, namely the Quot scheme, which was first introduced by Grothendieck in the context of so called parameter spaces.

1.1 Background

First we will give some motivation to why we study these objects. When we parametrize a line segment in the real plane we assign, to each coordinate on the line, a real number t in some interval $[a, b]$. To generalize this concept we see that we, basically, have a set S containing the structures that we want to parametrize (the set of coordinates) and then we want to find a set M (the interval $[a, b]$) such that any element in M corresponds to an element in S .

The sets M might be a bit complicated and hard to find however. There are two main solutions to this problem. The first is to find *local* parametrizations, where we only consider some specific parts of the structure, one at a time, that might be easier to parametrize, such that they together cover our whole structure. The other possibility is simply to find this more complicated set M . Both these possibilities can be applied in the following example.

Example 1.1. Consider the unit circle $x^2 + y^2 = 1$ in the real plane \mathbb{R}^2 . This can, if we fix a point P on the circle, be parametrized in two similar ways by looking at lines through P .

- (i) The first gives a local parametrization where we remove the point P from our structure. For simplicity we choose $P = (-1, 0)$ and consider lines $y = tx + t$. These lines will intersect the circle in two points, the point P and a point $P_t = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$. Conversely, any point $Q \neq P$ on the circle will correspond to some line of this form for some t . Thus, we have a local parametrization of the circle given by the bijection above between \mathbb{R} and the circle minus the point P . If we instead were to remove another point, say $P' = (1, 0)$ we would get another local parametrization of the circle which together with our previous parametrization covers the entire circle.
- (ii) To get a global parametrization directly we would need to be able to handle the point P as well. Any line through a fixed point is uniquely defined by its slope

and the line that would correspond to the point P would be the line where the slope is infinite, $t = \infty$. It is precisely for these reasons that we have the projective line that parametrizes the lines in the plane that pass through the origin, even the one with infinite slope. Therefore, we have a global parametrization of the unit circle, given by the bijection above, between the projective line over \mathbb{R} , $\mathbb{P}^1\mathbb{R}$, and the unit circle.

Here we got an example of why we introduce the projective line, and more generally the projective space \mathbb{P}^n that parametrizes lines in $(n + 1)$ -dimensions that pass through the origin. We can then go one step further and generalize the projective space by introducing the Grassmannian, often denoted $Gr(r, n)$. This is a space that parametrizes, for any n -dimensional vector space, the r -dimensional subspaces, similarly to the projective space that parametrizes the 1-dimensional subspaces. Indeed, we have that $Gr(1, n) = \mathbb{P}^{n-1}$.

Example 1.2. In an n -dimensional vector space we have that the $(n - 1)$ -dimensional subspaces are uniquely defined by their normal vectors, and vice versa, from which it follows that $Gr(n - 1, n) = Gr(1, n) = \mathbb{P}^{n-1}$. Thus, the first example of a Grassmannian that is not some projective n -space is $Gr(2, 4)$ which consists of the 2-dimensional subspaces, i.e. the planes containing the origin, in a 4-dimensional vector space.

The Grassmannian is usually considered over vector spaces but can also be defined over arbitrary modules where we consider submodules such that the induced quotient module is locally free of rank r . Then we can consider the case when the module that we are interested in has a grading given by some abelian group and we then get the notion of the graded Grassmannian.

The Hilbert and Quot schemes are further generalizations of this graded Grassmannian where we also require the submodules to be modules over a graded polynomial ring.

The concept of parametrization leads us to the theory of representability. A covariant functor F , from a locally small category \mathcal{C} to the category of sets, is said to be representable if it is naturally isomorphic to a functor of the form $\text{Hom}(A, -)$ for some $A \in \mathcal{C}$. Similarly, a contravariant functor F is called representable if it is naturally isomorphic to a functor $\text{Hom}(-, A)$. We will in this paper consider the case where we have functors from the category of k -algebras for some field k to the category of sets. The aim is then to find some k -algebra A that gives us a representation of our functor. Similarly to what we saw in Example 1.1 it is sometimes hard to find the A that represents the functor, sometimes it does not even exist, but we may then be able to find local representations by considering simpler subfunctors of F . Also, by instead considering F as a contravariant functor from the category of schemes over k

to sets it is, sometimes, possible to find a scheme that represents our functor. That is because the category of schemes over k contains the category of affine schemes over k that are equivalent to the category of k -algebras. All the parameter spaces that we mentioned above can be defined as structures that represents some specific functors.

1.2 What we will do

We will in Section 2 start by considering the Hilbert scheme of the affine line $\text{Spec}(k[t])$ for some field k and then, in Section 4, continue by considering a similar construction for the Quot scheme of the affine line. Since the functor Spec gives a one-to-one correspondence between the category of commutative rings and the category of affine schemes, we will be able to work with commutative rings and the theory involving them instead of the more theory demanding scheme theory. For a field k we will with a k -algebra mean what is usually referred to as a commutative unital associative algebra over k . Our method will be to define a specific functor and then show that it is representable, at least locally, by some k -algebra or, equivalently, an affine scheme over k . There are two categories that we will mostly consider, so we give these the following notation.

Notation. The category of sets, where the morphisms are maps between the sets, will be denoted \mathfrak{Set} and the category of k -algebras, where the morphisms are ring homomorphisms, will be denoted \mathfrak{Alg}^k .

First we will consider a grading on the polynomial ring $k[t] = \bigoplus_{g \in G} S_g$ by some abelian group G along with a Hilbert function $h: G \rightarrow \mathbb{N}$. For any k -algebra B , the Hilbert functor, $\text{Hilb}^h: \mathfrak{Alg}^k \rightarrow \mathfrak{Set}$, parametrizes all homogeneous ideals $I \subseteq B \otimes_k k[t] = B[t]$ such that the graded components of the quotient $B[t]/I$ are locally free of finite rank $h(g)$ for all $g \in G$, c.f. Section 2. When $G = \mathbb{Z}/n\mathbb{Z}$ then this functor will be represented by the k -algebra $k[a_1, \dots, a_s]$ together with the universal element

$$(t^m(t^{sn} + a_1 t^{(s-1)n} + \dots + a_s))$$

where the integers m and s are determined by the Hilbert function h .

In Section 3 we will then review the theory of the Grassmannian via the Grassmann functor. For some k -vector space V it is defined, similarly to the Hilbert functor, by taking a k -algebra B to the set of submodules $N \subseteq B \otimes_k V$ such that the quotient $(B \otimes_k V)/N$ is locally free of some given rank. We will show that this functor is locally representable by a polynomial ring. We will also look at the graded Grassmannian where we add a graded structure on the k -vector space V by some abelian group G .

Finally, we look at a generalization of the Hilbert functor by considering, for $V = \bigoplus_{i=1}^d k[t]$, the Quot functor, $\text{Quot}_V^h: \mathcal{A}lg^k \rightarrow \mathcal{S}et$, that parametrizes the homogeneous $B[t]$ -submodules $N \subseteq B \otimes_k V$ such that the graded components of the quotient $(B \otimes_k V)/N$ are locally free of rank $h(g)$ for all $g \in G$. We will show, in Section 4, that the Quot functor is locally representable by polynomial rings over k and universal elements of the same form as in the case with the Hilbert functor. Finally, we show that the Quot functor is, globally, represented by a scheme that is both smooth and irreducible.

I would here like to express my deepest gratitude to my two supervisors, in no particular order other than the alphabetical one, Mats Boij and Roy Skjelnes for all their guidance and support.

2 The Hilbert scheme of the affine line

2.1 Preliminaries

Let k be a field. When one determines a grading of the ring $S = k[t]$ by an abelian group G , i.e. a decomposition $k[t] = \bigoplus_{g \in G} S_g$ where $S_g \cdot S_h \subseteq S_{g+h}$ for any $g, h \in G$, this is equivalent to consider a semi-group homomorphism $\text{deg}: \{1, t, t^2, \dots\} \rightarrow G$. This will be determined by $\text{deg}(t)$ and we can therefore assume, without loss of generality, that G is cyclic, which we will do from now on. We then have two cases, $G = \mathbb{Z}$ or $G = \mathbb{Z}/n\mathbb{Z}$ for some $n \in \mathbb{Z}^+$. The vector space decomposition

$$k[t] = \bigoplus_{r=0}^{\infty} k \cdot t^r$$

has the property that $(k \cdot t^{r_1}) \cdot (k \cdot t^{r_2}) \subseteq k \cdot t^{r_1+r_2}$ so this decomposition gives a grading of $k[t]$ with $G = \mathbb{Z}$ by letting each direct summand be a graded component. Another vector space decomposition of $k[t]$ is

$$k[t] = \bigoplus_{r=0}^{n-1} t^r k[t^n],$$

and since this also has the property that $(t^{r_1} k[t^n]) \cdot (t^{r_2} k[t^n]) \subseteq t^{r_1+r_2} k[t^n]$ we get a grading of $k[t]$ with $G = \mathbb{Z}/n\mathbb{Z}$ by the vector space decomposition when letting each direct summand be a graded component.

Definition 2.1. For any graded ring $S = \bigoplus_{g \in G} S_g$ we define an element $s \in S$ to be a *homogeneous element* if $s \in S_g$ for some $g \in G$. A *homogeneous ideal* is an ideal that is generated by homogeneous elements.

If $I \subseteq \bigoplus_{g \in G} S_g$ is a homogeneous ideal, then S/I is graded by G with

$$S/I = \bigoplus_{g \in G} (S_g + I)/I \cong \bigoplus_{g \in G} S_g/(S_g \cap I) = \bigoplus_{g \in G} S_g/I_g.$$

Definition 2.2. A homogeneous ideal I of a graded k -algebra S , graded by G , $S = \bigoplus_{g \in G} S_g$, is called *admissible* if S_g/I_g is of finite dimension over k for all $g \in G$. An admissible ideal has *Hilbert function* $h_I: G \rightarrow \mathbb{N}$ defined by $h_I(g) = \dim_k(S_g/I_g)$.

When $S = k[t]$ and $h: G \rightarrow \mathbb{N}$ is any function then we want to parametrize all admissible ideals I with Hilbert function $h_I = h$. Since the tensor product is distributive over direct sums we have, for any k -algebra B , an induced grading $B \otimes_k S = B \otimes_k \left(\bigoplus_{g \in G} S_g \right) = \bigoplus_{g \in G} (B \otimes_k S_g)$, where each graded component $B \otimes_k S_g$ is considered as a B -module. Note that $B \otimes_k k[t] = B[t]$.

Definition 2.3. If A is a ring, then the A -module M is *locally free* of rank m if there exist elements $r_1, \dots, r_d \in A$ for some d with $(r_1, \dots, r_d) = A$ such that M_{r_i} is a free A_{r_i} -module of rank m for every i .

Remark 2.4. 1. Note that Definition 2.3 is equivalent to saying that, for any $\mathfrak{p} \in \text{Spec}(A)$, there is some $f \in A \setminus \mathfrak{p}$ such that M_f is a free A_f -module of rank m .

2. Definition 2.3 implies that an A -module M , that is locally free of rank m , has the property that, for any $\mathfrak{p} \in \text{Spec}(A)$, $M_{\mathfrak{p}}$ is a locally free $A_{\mathfrak{p}}$ -module of rank m . However, we do not in general have an equivalence here. If we have, for any $\mathfrak{p} \in \text{Spec}(A)$, that $M_{\mathfrak{p}}$ is a locally free $A_{\mathfrak{p}}$ -module this does not imply that M is locally free, unless we pose some extra requirements, e.g. that M is finitely presented or that the rank is constant for all prime ideals \mathfrak{p} , see [2, Section II.5.2, Theorem 1].

3. A free A -module M is always locally free since $(1) = A$ and when localizing at the element 1 we get that $M_1 = M$ is a free A_1 -module, and $A_1 = A$.

We will now with the following results prove some properties of locally free modules.

Lemma 2.5. *Let G be an abelian group and $h: G \rightarrow \mathbb{N}$ a function such that $\sum_{g \in G} h(g) < \infty$. If $M = \bigoplus_{g \in G} M_g$ is graded module over a ring A and if each graded component M_g is locally free with rank $h(g)$, then M is a locally free A -module of rank $\sum_{g \in G} h(g)$.*

Proof. Take some $\mathfrak{p} \in \text{Spec}(A)$ and consider $M \otimes_A A_{\mathfrak{p}}$. Since any graded component of M is locally free of rank $h(g)$ we have that $M_g \otimes_A A_{\mathfrak{p}} = A_{\mathfrak{p}}^{h(g)}$. This implies that

$$M \otimes_A A_{\mathfrak{p}} = \left(\bigoplus_{g \in G} M_g \right) \otimes_A A_{\mathfrak{p}} = \bigoplus_{g \in G} (M_g \otimes_A A_{\mathfrak{p}}) = \bigoplus_{g \in G} A_{\mathfrak{p}}^{h(g)} = \bigoplus_{i=1}^{\sum_{g \in G} h(g)} A_{\mathfrak{p}}.$$

Hence, the localization of M at any prime ideal $\mathfrak{p} \in \text{Spec}(A)$ is a locally free $A_{\mathfrak{p}}$ -module of rank $\sum_{g \in G} h(g)$ and since that sum is finite and independent of the prime \mathfrak{p} we have, from Remark 2.4, that M is a locally free A -module of rank $\sum_{g \in G} h(g)$. \square

Lemma 2.6. *Let A, B be rings and $\varphi: A \rightarrow B$ a ring homomorphism. If M is a locally free A -module of rank m then $M \otimes_A B$ is a locally free B -module of rank m .*

Proof. Take a prime ideal $\mathfrak{p} \subset B$. Then $\mathfrak{q} = \varphi^{-1}(\mathfrak{p}) \subset A$ is a prime ideal. Therefore, since M is locally free as an A -module it follows that there is some $f \in A \setminus \mathfrak{q}$ such that $M_f = M \otimes_A A_f$ is a free A_f -module of rank m . Letting $g = \varphi(f)$ we get an induced homomorphism $A_f \rightarrow B_g$. Then

$$(M \otimes_A B)_g = (M \otimes_A B) \otimes_B B_g = M \otimes_A (B \otimes_B B_g) = M \otimes_A B_g.$$

Furthermore, we trivially have $B_g = A_f \otimes_{A_f} B_g$ which gives us that

$$(M \otimes_A B)_g = M \otimes_A B_g = M \otimes_A (A_f \otimes_{A_f} B_g) = (M \otimes_A A_f) \otimes_{A_f} B_g = M_f \otimes_{A_f} B_g$$

which is a free B_g -module of rank m , since

$$M_f \otimes_{A_f} B_g = \left(\bigoplus_{i=1}^m A_f \right) \otimes_{A_f} B_g = \bigoplus_{i=1}^m (A_f \otimes_{A_f} B_g) = \bigoplus_{i=1}^m B_g.$$

Hence, for any prime ideal $\mathfrak{p} \subset B$ there exists some $g \in B \setminus \mathfrak{p}$ such that $(M \otimes_A B)_g$ is a free B_g -module of rank m . Thus $M \otimes_A B$ is a locally free B -module. \square

Lemma 2.7. *If R is a ring, M, N are R -modules and $L \subseteq M$ a submodule, then*

$$(M/L) \otimes_R N = (M \otimes_R N) / \text{im}(L \otimes_R N),$$

where $\text{im}(L \otimes_R N)$ denotes the image of the map $\text{incl} \otimes \text{id}: L \otimes_R N \rightarrow M \otimes_R N$.

Proof. This is just the right exactness property of the tensor product. Indeed, consider the short exact sequence

$$0 \rightarrow L \rightarrow M \rightarrow M/L \rightarrow 0.$$

This turns into a right exact sequence when we tensor it with N ,

$$L \otimes_R N \rightarrow M \otimes_R N \rightarrow (M/L) \otimes_R N \rightarrow 0$$

and then we can consider the short exact sequence

$$0 \rightarrow \text{im}(L \otimes_R N) \rightarrow M \otimes_R N \rightarrow (M/L) \otimes_R N \rightarrow 0.$$

Hence we get an isomorphism

$$(M/L) \otimes_R N \cong (M \otimes_R N) / \text{im}(L \otimes_R N).$$

□

Fix a grading of $S = k[t]$ by a group G and consider a Hilbert function $h: G \rightarrow \mathbb{N}$. Then, for any k -algebra B , we define the set

$$\text{Hilb}^h(B) = \left\{ \begin{array}{l} I \text{ is homogeneous and the} \\ I \subseteq B[x] : B\text{-module } (B \otimes_k S_g) / I_g \text{ is locally} \\ \text{free of finite rank } h(g) \text{ for all } g \in G \end{array} \right\}. \quad (1)$$

Proposition 2.8. *Let B_1 and B_2 be k -algebras and $\varphi: B_1 \rightarrow B_2$ a ring homomorphism. Then there is an induced homomorphism $(\varphi \otimes \text{id}): B_1 \otimes_k k[t] \rightarrow B_2 \otimes_k k[t]$. For any $I \in \text{Hilb}^h(B_1)$ we have that*

$$\text{im}(I \otimes_{B_1} B_2) = (\text{incl} \otimes \text{id})(I \otimes_{B_1} B_2) \in \text{Hilb}^h(B_2).$$

Proof. Take an $I \in \text{Hilb}^h(B_1)$, $I \subseteq B_1[t]$, and consider the ideal $\text{im}(I \otimes_{B_1} B_2) \subseteq B_2 \otimes_k k[t] = B_2[t]$. Since the ideal is generated by homogeneous elements it follows that the ideal is homogeneous. By Lemma 2.7 we have that

$$(B_1[t]/I) \otimes_{B_1} B_2 = (B_1[t] \otimes_{B_1} B_2) / \text{im}(I \otimes_{B_1} B_2) = B_2[t] / \text{im}(I \otimes_{B_1} B_2)$$

and that each graded component of this module is a locally free B_2 -module of rank $h(g)$ follows from Lemma 2.6. Hence $\text{im}(I \otimes_{B_1} B_2) = (\text{incl} \otimes \text{id})(I \otimes_{B_1} B_2) \in \text{Hilb}^h(B_2)$. □

Definition 2.9. The *Hilbert functor* $\text{Hilb}^h: \mathfrak{Alg}^k \rightarrow \mathfrak{Set}$ is defined, for any k -algebra B , by $\text{Hilb}^h(B)$ given by (1), and for any ring homomorphism $\varphi: B_1 \rightarrow B_2$, for some k -algebras B_1, B_2 , by $\text{Hilb}^h(\varphi)(I) = \text{im}(I \otimes_{B_1} B_2) = (\text{incl} \otimes \text{id})(I \otimes_{B_1} B_2)$.

Definition 2.10. If A and B are k -algebras, a k -algebra homomorphism $\varphi: A \rightarrow B$ is called an *B -valued point of A* . The set of all such points is, naturally, denoted $\text{Hom}_k(A, B)$.

Our aim is to describe the functor Hilb^h and we will do this by finding a correspondence between the elements of $\text{Hilb}^h(B)$ and the B -valued points of A for some k -algebra A . This gives a natural explanation for the following definition.

Definition 2.11. A functor F from the category of k -algebras to the category of sets, $F: \mathfrak{Alg}^k \rightarrow \mathfrak{Set}$, is *representable* if there is a natural isomorphism $\Phi: \text{Hom}_k(A, -) \rightarrow F$ from the functor $\text{Hom}_k(A, -): \mathfrak{Alg}^k \rightarrow \mathfrak{Set}$ for some k -algebra A . The pair (A, Φ) is called the *representation* of the functor F .

Remark 2.12. By Yoneda's Lemma, see e.g. [4, Lemma VI-1], we have that the set of natural transformations $\Phi: \text{Hom}_k(A, -) \rightarrow F$ are in a one-to-one correspondence with the elements of $F(A)$. The correspondence is given, for any natural transformation $\Phi: \text{Hom}_k(A, -) \rightarrow F$, by $\Phi_A(\text{id}_A) = a \in F(A)$ and conversely, for any $a \in F(A)$ we get a natural transformation $\Phi: \text{Hom}_k(A, -) \rightarrow F$ by $\Phi_B(\varphi) = (F\varphi)(a)$ for any $\varphi \in \text{Hom}_k(A, B)$. Note that the natural transformation Φ that is induced by an element $a \in F(A)$ need not be a natural isomorphism. That is the case if and only if a has the property that, for any B and any $b \in F(B)$, there is a unique morphism $\varphi: A \rightarrow B$ such that $(F\varphi)(a) = b$. If $a \in F(A)$ has this property then it is called a *universal element* of F . Thus, if we have a representation of F given by (A, Φ) we can exchange Φ for $a = \Phi_A(\text{id}_A)$ and call (A, a) a *representation* of F .

2.2 Grading by $G = \mathbb{Z}$

A homogeneous admissible ideal $I \subseteq \bigoplus_{r=0}^{\infty} k \cdot t^r = k[t]$ is, since $k[t]$ is a P.I.D., generated by t^m for some m , i.e. $I = (t^m)$. We have then that

$$I_r = (t^m) \cap k \cdot t^r = \begin{cases} 0 & 0 \leq r < m, \\ k \cdot t^r & r \geq m, \end{cases}$$

which implies that $k[t]/I = \bigoplus_{r=0}^{\infty} k \cdot t^r / I_r = k \oplus k \cdot t \oplus \dots \oplus k \cdot t^{m-1}$ and thus, the only possible Hilbert functions when one has a grading by $G = \mathbb{Z}$ is of the form

$$h_I(r) = \begin{cases} 1 & 0 \leq r < m, \\ 0 & r \geq m, r < 0. \end{cases}$$

Let $h: \mathbb{Z} \rightarrow \mathbb{N}$ be the Hilbert function of some homogeneous admissible ideal $I \subseteq k[t]$. With the grading from $G = \mathbb{Z}$ we have that the graded components of $k[t]$ are of the form $k \cdot t^r$ for $r \in \mathbb{N}$ and therefore the induced grading on $B[t]$, for any k -algebra B , is $B[t] = \bigoplus_{r=0}^{\infty} (B \otimes_k (k \cdot t^r)) = \bigoplus_{r=0}^{\infty} B \cdot t^r$. Thus, we want to study the set

$$\text{Hilb}^h(B) = \left\{ \begin{array}{l} I \text{ is a homogeneous ideal and} \\ I \subseteq B[t] : B \cdot t^r / I_r \text{ is locally free over } B \text{ and} \\ \text{of finite rank } h(r) \text{ for all } r \in \mathbb{Z} \end{array} \right\}.$$

Proposition 2.13. *A homogeneous ideal $I \subseteq \bigoplus_{r=0}^{\infty} B \cdot t^r$ such that $B \cdot t^r / I_r$ is locally free of rank*

$$h(r) = \begin{cases} 1 & 0 \leq r < m, \\ 0 & r \geq m, r < 0, \end{cases}$$

for all $r \in \mathbb{Z}$ is generated by the element t^m .

Proof. For all $r \geq m$ we must have, since $h(r) = 0$, that $B \cdot t^r / I_r = 0$, i.e. $I_r = I \cap (B \cdot t^r) = B \cdot t^r$, which implies that $t^m \cdot B[t] \subseteq I$. Suppose that I is also generated by some other homogenous element $b \cdot t^r$ for some $r < m$ and $b \neq 0$. Then $b \cdot B \cdot t^r \subseteq I_r$. If b is a unit in B then $B \cdot t^r / I_r \subseteq B \cdot t^r / (b \cdot B \cdot t^r) = B \cdot t^r / (B \cdot t^r) = 0$ and since 0 is not a B -module of rank $h(r) = 1$ we have a contradiction. On the other hand, if b is a non-unit, then we get that the B -module $B \cdot t^r / I_r$ is not locally free since it is not faithful, for any possible basis element e we always have $be = 0$ even though $b \neq 0$, also a contradiction. Hence $I = t^m \cdot B[t]$. \square

Thus, we have that $\text{Hilb}^h(B) = \{t^m B[t]\}$. We can now prove the important result of this section.

Theorem 2.14. *Let $S = k[t]$, graded by the abelian group $G = \mathbb{Z}$, i.e. $k[t] = \bigoplus_{r=0}^{\infty} k \cdot t^r$, and fix a Hilbert function $h: \mathbb{Z} \rightarrow \mathbb{N}$ defined by*

$$h(r) = \begin{cases} 1 & 0 \leq r < m, \\ 0 & r \geq m, r < 0. \end{cases}$$

Then the functor Hilb^h is represented by the k -algebra H along with the universal element $t^m H[t]$ where $H = k$.

Proof. Take a k -algebra B . Then it follows directly from Proposition 2.13 that there is a one-to-one correspondence between the sets $\text{Hilb}^h(B)$ and $\text{Hom}_k(H, B)$ since both

sets only contains one element, $\text{Hom}_k(k, B) = \{\text{inclusion}: k \rightarrow B\}$ and $\text{Hilb}^h(B) = \{t^m B[t]\}$. The one-to-one correspondence, given by, say, $\Phi_B: \text{Hom}_k(H, B) \rightarrow \text{Hilb}^h(B)$, is clearly functorial in B which means that we have a representation of the functor Hilb^h given by (H, Φ) with $H = k$ and Φ defined by $\Phi(B) = \Phi_B$. By Remark 2.12 we have that Φ gives us an element $\Phi_H(\text{id}_H) = t^m H[t] \in \text{Hilb}^h(H)$ and we thus have a representation of Hilb^h given by the pair $(H, t^m H[t]) = (k, (t^m))$. \square

2.3 Grading by $G = \mathbb{Z}/n\mathbb{Z}$

Now we consider the case $G = \mathbb{Z}/n\mathbb{Z} = \mathbb{Z}_n$. A homogeneous admissible ideal $I \subseteq \bigoplus_{r=0}^{n-1} t^r k[t^n] = k[t]$ is then generated by homogeneous elements of the form $t^r p(t^n)$ for different $r \in \mathbb{Z}_n$ and polynomials $p(t) \in k[t]$. Since $k[t]$ is Noetherian it follows from [9, Corollary 1] that I is generated by a finite number of homogeneous elements, $t^{r_1} p_1(t^n), \dots, t^{r_d} p_d(t^n)$. Then, since $k[t]$ has the Euclidean algorithm, it follows that I is generated by the greatest common divisor of the homogeneous generators which must be a polynomial of the form $q(t) = t^m (t^{sn} + a_1 t^{(s-1)n} + \dots + a_s)$ for some $m \in \mathbb{Z}_n$, $s \in \mathbb{N}$ and elements $a_1, \dots, a_s \in k$. Hence any homogeneous ideal $I \subseteq k[t]$ is generated by a single homogeneous element. We will now show that there is only one possible form that the Hilbert function of an admissible homogeneous ideal can take.

Proposition 2.15. *If $I \subseteq \bigoplus_{r=0}^{n-1} t^r k[t^n]$ is a homogeneous admissible ideal, $I = (t^m p_0(t^n))$ where $p_0(t^n) = t^{sn} + a_1 t^{(s-1)n} + \dots + a_s$, and $h_I: \mathbb{Z}_n \rightarrow \mathbb{N}$ its associated Hilbert function, then*

$$h_I(r) = \begin{cases} s+1 & 0 \leq r < m, \\ s & m \leq r < n. \end{cases}$$

Proof. We want to find the dimension of the k -vector space $t^r k[t^n]/I_r$ for all r , where $I_r = (t^r k[t^n]) \cap I$. In order to do this we start by describing I_r for all $r \in \mathbb{Z}/n\mathbb{Z}$. Take some $t^r q(t^n) \in t^r k[t^n]$. Then $t^r q(t^n) \in I$ if and only if $t^r q(t^n) = \alpha(t) t^m p_0(t^n)$ for some $\alpha(t) \in k[t]$. We now get two cases.

Case $r < m$. Then we get $q(t^n) = \alpha(t) t^{m-r} p_0(t^n)$, i.e. $\alpha(t) t^{m-r} \in k[t^n]$ which implies that $\alpha(t)$ must be of the form $\alpha(t) = t^{n-(m-r)} \alpha_0(t^n)$ for some $\alpha_0(t^n) \in k[t^n]$. Thus we have that $q(t^n) = \alpha_0(t^n) t^n p_0(t^n)$. On the other hand, if $q(t^n) = \beta(t^n) t^n p_0(t^n)$ for some $\beta(t^n) \in k[t^n]$ then we have that

$$t^r q(t^n) = t^r (\beta(t^n) t^n p_0(t^n)) = (t^{n+r-m} \beta(t^n)) t^m p_0(t^n) \in t^m k[t^n].$$

Hence $I_r = \{t^r q(t^n) \in S_r : t^n p_0(t^n) \mid q(t^n)\} = t^n p_0(t^n) S_r$.

Case $r \geq m$. Then we have that $t^{r-m}q(t^n) = \alpha(t)p_0(t^n)$, i.e. $\alpha(t) \in t^{r-m}k[t^n]$. Thus we have that $\alpha(t) = t^{r-m}\alpha_0(t^n)$ for some $\alpha_0(t^n) \in k[t^n]$. Then we have that $q(t^n) = \alpha_0(t^n)p_0(t^n)$. On the other hand, if $q(t^n) = \beta(t^n)p_0(t^n)$ for some $\beta(t^n) \in k[t^n]$ then

$$t^r q(t^n) = t^r (\beta(t^n)p_0(t^n)) = (t^{r-m}\beta(t^n)) t^m p_0(t^n) \in t^m k[t^n].$$

Hence $I_r = \{t^r q(t^n) \in S_r : p_0(t^n) \mid q(t^n)\} = p_0(t^n)S_r$.

Thus we have that

$$h_I(r) = \dim_k(S_r/I_r) = \begin{cases} \dim_k(S_r/(t^n p_0(t^n)S_r)) = s+1 & 0 \leq r < m, \\ \dim_k(S_r/(p_0(t^n)S_r)) = s & m \leq r < n. \end{cases}$$

□

When $k[t]$ is graded by $\mathbb{Z}/n\mathbb{Z}$ we have that its graded components are of the form $t^r k[t^n]$ from which it follows, for any k -algebra B , that we have the induced grading $B[t] = \bigoplus_{r=0}^{n-1} (B \otimes_k t^r k[t^n]) = \bigoplus_{r=0}^{n-1} t^r B[t^n]$. Hence, we want to consider the set

$$\text{Hilb}^h(B) = \left\{ \begin{array}{l} I \text{ is an homogeneous ideal such that} \\ I \subseteq B[t] : \text{the } B\text{-module } t^r B[t^n]/I_r \text{ is locally free} \\ \text{of finite rank } h(r) \text{ for all } r \in \mathbb{Z}_n \end{array} \right\}.$$

First of all, we show that if the graded components of $B[t]/I$ are locally free of finite rank then they are actually free.

Proposition 2.16. *For any $r \in \mathbb{Z}_n$, if the B -module $t^r B[t^n]/I_r$ is locally free of rank $h(r) = s$, then $t^r B[t^n]/I_r$ is free of rank s (globally) with basis $t^r, t^{r+n}, \dots, t^{r+(s-1)n}$.*

Proof. Since $M = t^r B[t^n]/I_r$ is locally free we have, for any $\mathfrak{p} \in \text{Spec}(B)$, that the $B_{\mathfrak{p}}$ -module $M_{\mathfrak{p}} = (t^r B[t^n]/I_r)_{\mathfrak{p}}$ is free of rank s . We will first determine a basis for this module. Since $B_{\mathfrak{p}}$ is a local ring, with maximal ideal $\mathfrak{m} = \mathfrak{p}B_{\mathfrak{p}}$, we can construct its fraction field $\kappa = B_{\mathfrak{p}}/\mathfrak{m}$, and we have that $M_{\mathfrak{p}}/\mathfrak{m}M_{\mathfrak{p}}$ is a vector space of dimension s over κ . By construction we have that the quotient classes of the elements $\{t^{r+ni}\}_{i \in \mathbb{N}}$ in the vector space $M_{\mathfrak{p}}/\mathfrak{m}M_{\mathfrak{p}}$ generates the vector space over κ .

If the quotient classes of the elements $t^r, t^{r+n}, \dots, t^{r+(s-1)n}$ would not be linearly independent over κ this would imply that we could reduce our generating set $\{t^{r+ni}\}_{i \in \mathbb{N}}$ to a set consisting of less than s elements, a contradiction. Hence it is clear that the s elements $t^r, t^{r+n}, \dots, t^{r+(s-1)n}$ are linearly independent over κ and thus form a basis. From Nakayama's lemma [1, Proposition 2.8] we get that the elements

$t^r, t^{r+n}, \dots, t^{r+(s-1)n}$ form a minimal generating set of $M_{\mathfrak{p}}$, and since the module is free it follows that they are a basis. This is true for any prime \mathfrak{p} and thus it follows, [8, Corollary 1.2], that $t^r, t^{r+n}, \dots, t^{r+(s-1)n}$ is a generating set for M , and thus form a basis with s elements. \square

In the following two propositions we will first show, in Proposition 2.17, that if the graded components of $B[t]/I$ are locally free of finite rank then I_r is principal for any $r \in \mathbb{Z}/n\mathbb{Z}$. After that we will use this result to show, in Proposition 2.18, that I is, in fact, principal.

Proposition 2.17. *For any $r \in \mathbb{Z}_n$, if the B -module $t^r B[t^n]/I_r$ is free of rank s , then there is a unique monic polynomial $G(t) = t^r(t^{sn} + b_1 t^{(s-1)n} + \dots + b_s) \in t^r B[t^n]$ such that $I_r = (G(t))$.*

Proof. Since $t^r B[t^n]/I_r$ is free of rank s we have a basis consisting of the images of the elements $t^r, t^{r+n}, \dots, t^{r+(s-1)n}$ by the quotient map, otherwise we could reduce the generating set $\{t^{r+ni}\}_{i \in \mathbb{N}}$ to a set consisting of less than s elements, a contradiction. We use the notation that we write the image of an element x by the quotient map as x itself. This means that t^{r+sn} can be written, in a unique way, as a linear combination of these elements, i.e. $t^{r+sn} = b_1 t^{r+(s-1)n} + \dots + b_s t^r$.

Hence the polynomial $t^{r+sn} - b_1 t^{r+(s-1)n} - \dots - b_s t^r \in t^r B[t^n]$ is contained in I_r , i.e. $(G(t)) \subseteq I_r$. The uniqueness of $G(t)$ follows from the uniqueness of the linear combination. To show the other inclusion we note that the canonical map $t^r B[t^n]/(G(t)) \rightarrow t^r B[t^n]/I_r$ is a surjection between two free modules of the same rank, so it must be an isomorphism. Hence $I_r = (G(t))$. \square

Proposition 2.18. *For any k -algebra B we have that the set $\text{Hilb}^h(B)$, where*

$$h(r) = \begin{cases} s+1 & 0 \leq r < m \\ s & m \leq r < n, \end{cases}$$

is in a one-to-one correspondence with the set of monic polynomials of the form $t^m(t^{sn} + b_1 t^{(s-1)n} + \dots + b_s)$.

Proof. Take an $I \in \text{Hilb}^h(B)$. Since the graded components of $B[t]/I$ are locally free B -modules it follows from Proposition 2.16 that the graded components are free B -modules. Furthermore, from Proposition 2.17 we have that each I_r is generated by a unique polynomial $p_r(t) = t^r(t^{sn} + b_{1,r} t^{(s-1)n} + \dots + b_{s,r})$ for each r . Now consider the case $r = m$. Then we have that

$$p_m(t) = t^m(t^{sn} + b_{1,m} t^{(s-1)n} + \dots + b_{s,m}) = t^m p(t^n)$$

generates I_m . It follows that $t^d \cdot p_m(t) \in t^{m+d}k[t^n]$ for any $d \in \mathbb{Z}_n$ so we have that

$$\begin{aligned} p_{m+1}(t) &= tp_m(t) = t^{m+1}p(t^n) \in (t^m p(t^n)) \\ &\vdots \\ p_{n-1}(t) &= t^{n-1-m}p_m(t) = t^{n-1}p(t^n) \in (t^m p(t^n)) \\ p_0(t) &= t^{n-m}p_m(t) = t^n p(t^n) \in (t^m p(t^n)) \\ &\vdots \\ p_{m-1}(t) &= t^{n-1}p_m(t) = t^{m-1}t^n p(t^n) \in (t^m p(t^n)). \end{aligned}$$

It follows that $I = (p_m(t)) = (t^m p(t^n))$.

Conversely, if we have a monic polynomial on the form $t^m(t^{sn} + b_1 t^{(s-1)n} + \dots + b_s)$ then it is clear that the ideal generated by it will be an element in $\text{Hilb}^h(B)$. \square

We are now ready to prove the final, and important, results of this section.

Theorem 2.19. *Let $S = k[t]$, graded by the abelian group $G = \mathbb{Z}/n\mathbb{Z}$ and take a Hilbert function*

$$h(r) = \begin{cases} s+1 & 0 \leq r < m, \\ s & m \leq r < n. \end{cases}$$

Then, for any k -algebra B , there is a one-to-one correspondence between the sets $\text{Hilb}^h(B)$ and $\text{Hom}_k(H, B)$, where $H = k[a_1, \dots, a_s]$, given by the map $\Phi_B: \text{Hom}_k(H, B) \rightarrow \text{Hilb}^h(B)$ defined, for any $\varphi \in \text{Hom}_k(H, B)$, by

$$\Phi_B(\varphi) = (t^r(t^{sn} + \varphi(a_1)t^{(s-1)n} + \dots + \varphi(a_s))).$$

Proof. Take a k -algebra B . From Proposition 2.18 we have that

$$\text{Hilb}^h(B) = \{I \subseteq B[t]: I = (t^m p(t^n)) \text{ for some } p(t^n) = t^{sn} + a_1 t^{(s-1)n} + \dots + a_s\}.$$

Therefore it follows that Φ_B is well defined, $\Phi_B(\varphi) \in \text{Hilb}^h(B)$, and that Φ_B has a natural inverse $\Psi_B: \text{Hilb}^h(B) \rightarrow \text{Hom}_k(H, B)$ defined by

$$I = (t^r(t^{sn} + b_1 t^{(s-1)n} + \dots + b_s)) \mapsto (\psi: H \rightarrow B)$$

where ψ is defined by $\psi(a_i) = b_i$ for $i = 1, \dots, s$. We thus have our bijection. \square

Theorem 2.20. *Let $S = k[t]$, graded by the abelian group $G = \mathbb{Z}/n\mathbb{Z}$ and let $h: \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{N}$ be a Hilbert function defined by*

$$h(r) = \begin{cases} s+1 & 0 \leq r < m, \\ s & m \leq r < n. \end{cases}$$

Then the Hilbert functor Hilb^h is represented by the k -algebra $H = k[a_1, \dots, a_s]$ with the natural isomorphism $\Phi: \text{Hom}_k(H, -) \rightarrow \text{Hilb}^h$ defined by $\Phi(B) = \Phi_B: \text{Hom}_k(B) \rightarrow \text{Hilb}^h(B)$ that takes a homomorphism $\varphi: H \rightarrow B$ to the ideal $(t^r(t^{sn} + \varphi(a_1)t^{(s-1)n} + \dots + \varphi(a_s)))$.

Proof. For any k -algebra B we have, from Theorem 2.19, a one-to-one correspondence between the sets $\text{Hilb}^h(B)$ and $\text{Hom}_k(H, B)$. This is clearly functorial in B so this is a natural isomorphism. Thus, we have a representation of the Hilbert functor Hilb^h by (H, Φ) . \square

With the notation from Theorem 2.20 we have that

$$\Phi_H(\text{id}_H) = (t^r(t^{sn} + \text{id}_H(a_1)t^{(s-1)n} + \dots + \text{id}_H(a_s))) = (t^r(t^{sn} + a_1t^{(s-1)n} + \dots + a_s)),$$

so by Yoneda's Lemma it follows that we can write our representation of Hilb^h as $(H, (F(t)))$ where $F(t) = t^m(t^{sn} + a_1t^{(s-1)n} + \dots + a_s)$. All this means is that for any k -algebra B and ideal $I \in \text{Hilb}^h(B)$, there is a unique homomorphism $\varphi: H \rightarrow B$, such that we get a co-Cartesian diagram

$$\begin{array}{ccc} H & \xrightarrow{\varphi} & B \\ \downarrow & \circ & \downarrow \\ H[t]/(F(t)) & \longrightarrow & B[t]/I \end{array}$$

where $B \otimes_H (H[t]/(F(t))) = B[t]/I$.

Example. If $n = 1$ we have a grading over the trivial group $G = 0$. Then we have a Hilbert function $h: 0 \mapsto s$ and Theorem 2.20 tells us that the functor Hilb^h is represented by the ring $k[a_1, \dots, a_s]$. This is, if we instead work in the category of schemes, equivalent to the fact that the Hilbert scheme of s point on the affine line, $\mathbb{A}^1 = \text{Spec}(k[t])$, is the scheme $\mathbb{A}^s = \text{Spec}(k[a_1, \dots, a_s])$, i.e. the affine s -space.

Remark 2.21. Note that the case $G = \mathbb{Z}$ and Theorem 2.14 is just a special case of the grading $G = \mathbb{Z}_n$ and Theorem 2.20 when we let $n \rightarrow \infty$. Indeed, when $n \rightarrow \infty$ we have that $\bigoplus_{r=0}^{n-1} t^r k[t^n] \rightarrow \bigoplus_{r=0}^{\infty} t^r k$, since the powers t^n will grow so big that they will never be attained. Then the homogeneous ideals will be of the form (t^m) for some m and therefore Theorem 2.20 tells us that we have a representation of Hilb^h given by $(k, (t^m))$ which is precisely the result given by Theorem 2.14.

3 The Grassmann scheme

3.1 The Grassmann functor

We will now consider the Grassmann scheme which is the scheme that represents the Grassmann functor. Since this theory can be found in several texts on the subject we will only go through this here in order to find methods that can be applicable for proving the representability of the Quot functor in the next section and so we refer the interested reader to e.g. [5, 8.4] for a more comprehensive study.

Definition 3.1. Let k be a field and V a k -vector space. For any $r \in \mathbb{N}$ the *Grassmann functor* $\text{Grass}_V^r: \mathcal{A}lg^k \rightarrow \mathcal{S}et$ from the category of k -algebras to the category of sets is defined, for any k -algebra B , by

$$\text{Grass}_V^r(B) = \{N \subseteq B \otimes_k V : (B \otimes_k V)/N \text{ is locally free over } B \text{ of rank } r\},$$

and for any homomorphism $\varphi: B_1 \rightarrow B_2$ of R -algebras B_1, B_2 we define $\text{Grass}_V^r(\varphi)(N) = (\text{incl} \otimes \text{id})(I \otimes_{B_1} B_2) = \text{im}(I \otimes_{B_1} B_2)$.

This is similar to the Hilbert functor in the sense that we consider quotients that are locally free, but we do not require the submodules $N \in \text{Grass}_V^r(B)$ to be $B[t]$ -modules which we do, implicitly, for the Hilbert functor when we require $I \in \text{Hilb}^h(B)$ to be an ideal in $B[t]$. In the following we will consider the case when V is the finitely generated k -vector space $V = k^n$ of dimension n .

Example 3.2. First we will give a motivation for the definition of the Grassmann functor. The Grassmannian of a vector space of dimension n , often denoted $Gr(r, n)$, consists of the subspaces of dimension r . We want our version to be the natural extension of this. For this motivation, choose a Noetherian k -algebra B and take some $N \in \text{Grass}_V^r(B)$ and let $P = (B \otimes_k V)/N$. Since B is Noetherian we have that P is a finitely presented and locally free B -module, and this is equivalent to P being projective. Since $\pi: B \otimes_k V \twoheadrightarrow P$ is surjective it therefore splits, locally, and we have that $B \otimes_k V = \ker(\pi) \oplus P$ where $\ker(\pi) = N$. Thus, N is locally a direct summand and this is our natural extension of the notion of subspaces of a vector space.

The Grassmann functor is not that easy to work with, instead we will consider more easily representable subfunctors of this functor. Since our k -vector space $V = k^n$ is finitely generated of dimension n we have that $B \otimes_k V = B^n$. Let us denote the set of canonical basis elements of k^n by X and let $Y \subset X$ consist of r elements. We let k^Y denote the k -vector space with basis Y and $B \otimes_k k^Y = B^Y$ denote the free B -module with the induced basis from Y . Then we have an induced homomorphism $\varphi_Y: B^Y \hookrightarrow B^n$.

Definition 3.3. Let k be a field, $V = k^n$ a k -vector space of dimension n , $r \in \mathbb{N}$ an integer and Y a set consisting of r basis elements of k^n . Then we define the *relative Grassmann functor*, relGrass_V^Y as the subfunctor of Grass_V^r defined by

$$\text{relGrass}_V^Y(B) = \{N \subseteq B^n : B^n/N \text{ is free with basis } Y\}.$$

Lemma 3.4. *If k is a field, $V = k^n$ a k -vector space of dimension n , $r \in \mathbb{N}$ an integer and Y a set consisting of r basis elements of k^n , then the relative Grassmann functor relGrass_V^Y is naturally isomorphic to the functor $G: \mathfrak{Alg}^k \rightarrow \mathfrak{Set}$ defined by $G(B) = \text{Hom}_B(B^n/B^Y, B^Y)$.*

Proof. Fix a k -algebra B . It is clear that $N \in \text{relGrass}_V^Y(B)$ is equivalent to the composition $B^Y \xrightarrow{\varphi_Y} B^n \twoheadrightarrow B^n/N$ being an isomorphism, so we have

$$\text{relGrass}_V^Y(B) = \{N \subseteq B^n : B^Y \hookrightarrow B^n \twoheadrightarrow B^n/N \text{ is an isomorphism}\}.$$

Now, for any $N \in \text{relGrass}_V^Y(B)$ we have an isomorphism $\varphi: B^Y \rightarrow B^n/N$ which implies that the composition $\psi: B^n \twoheadrightarrow B^n/N \xrightarrow{\varphi^{-1}} B^Y$ has kernel $\ker(\psi) = N$ and $\psi \circ \varphi_Y = \text{id}$. Conversely, if we have a homomorphism $\chi: B^n \rightarrow B^Y$ with $\chi \circ \varphi_Y = \text{id}$ then it is clear that $\ker(\chi) \in \text{Grass}_V^Y(B)$. Hence we have that relGrass_V^Y is naturally isomorphic to the functor $F: \mathfrak{Alg}^k \rightarrow \mathfrak{Set}$, defined by

$$F(B) = \{\psi \in \text{Hom}_B(B^n, B^Y) : \psi \circ \varphi_Y = \text{id}\}.$$

Furthermore, since $B^n = B^Y \oplus B^n/B^Y$ we have that maps $\psi: B^n \rightarrow B^Y$ satisfying $\psi \circ \varphi_Y = \text{id}$ are in a one-to-one correspondence with maps $B^n/B^Y \rightarrow B^Y$ (the element in the first coordinate must be mapped to itself and the other coordinate can be mapped to anything). Thus we also have that relGrass_V^Y naturally isomorphic to the functor $G: \mathfrak{Alg}^k \rightarrow \mathfrak{Set}$, defined by $G(B) = \text{Hom}_B(B^n/B^Y, B^Y)$. \square

A useful result in commutative algebra is the following.

Proposition 3.5 ([3, Theorem 10.43]). *If A is a ring and M, N and L are A -modules, then there is a canonical isomorphism*

$$\text{Hom}_A(M \otimes_A N, L) \cong \text{Hom}_A(M, \text{Hom}_A(N, L)).$$

Corollary 3.6. *For A -modules M, E with E being finitely generated and free we have that any homomorphism $M \rightarrow E$ can be canonically identified with a homomorphism $M \otimes_A E^* \rightarrow A$, where E^* denotes the dual of E .*

Proof. Let $N = E$ and $P = A$ in Proposition 3.5. Then we have that

$$\mathrm{Hom}(M \otimes E^*, A) \cong \mathrm{Hom}(M, \mathrm{Hom}(E^*, A)) = \mathrm{Hom}(M, E^{**}) \cong \mathrm{Hom}(M, E),$$

where the last isomorphism comes from the fact that there is a canonical isomorphism $E \cong E^{**}$ when E is finitely generated and free. \square

If we apply Corollary 3.6 to the result of Lemma 3.4 with $B^n/B^Y = M$, $B^Y = E$ and $B = A$ we conclude that the functor $\mathrm{relGrass}_V^Y$ is naturally isomorphic to $H: \mathfrak{Alg}^k \rightarrow \mathfrak{Set}$ defined by

$$H(B) = \mathrm{Hom}_B(B^n/B^Y \otimes (B^Y)^*, B).$$

Since every module homomorphism of a module M into a commutative k -algebra B extends uniquely to an algebra homomorphism of the symmetric algebra over M , denoted $S(M)$, into B we have in particular that any B -module homomorphism $\varphi: B^n/B^Y \otimes (B^Y)^* \rightarrow B$ extends uniquely to a k -algebra homomorphism $\psi: S(B^n/B^Y \otimes (B^Y)^*) \rightarrow B$. Since $B^n/B^Y \cong B^{n-r}$ is free of rank $n-r$ and $(B^Y)^*$ is free of rank r it follows that the tensor product $B^n/B^Y \otimes (B^Y)^*$ is free of rank $r(n-r)$. The symmetric algebra of a free module of rank $r(n-r)$ is the polynomial ring in $r(n-r)$ variables, $B[x_1, \dots, x_{r(n-r)}]$. This must hold for any B and thus, it follows that the functor H is represented by the polynomial ring $k[x_1, \dots, x_{r(n-r)}]$ and we can conclude our main result of this section, for a more detailed proof see [5, Lemma 8.13].

Theorem 3.7. *Let k be a field, $V = k^n$ a finitely generated k -vector space of dimension n , $r \in \mathbb{N}$ an integer and Y a set consisting of r basis elements of k^n . Then the relative Grassmann functor $\mathrm{relGrass}_V^Y$ is represented by $k[x_1, \dots, x_{r(n-r)}]$.*

It is then possible to prove, using these results, that the original functor Grass_V^r is representable in the category of schemes. The proof uses theory involving concepts such as open subfunctors and open coverings of functors. This theory is postponed until Section 4.

Theorem 3.8 ([5, Proposition 8.14]). *If k a field and $V = k^n$ a finitely generated k -vector space of dimension n , then the Grassmann functor Grass_V^r is representable with a scheme that is finitely covered of the affine schemes $\mathbb{A}^{r(n-r)} = \mathrm{Spec}(k[x_1, \dots, x_{r(n-r)}])$.*

Remark 3.9. Another way of realizing the Grassmannian is by considering matrices. Let k be a field and V a finitely generated k -vector space of rank n and suppose that we want, for any k -algebra B , to consider the submodules $N \subseteq B \otimes_k V$ such that the quotients $(B \otimes_k V)/N$ are locally free of rank r say. Then we can consider the set W of $r \times n$ -matrices M over B , modulo multiplication from the left by invertible $r \times r$ -matrices. If $I \subset \{1, 2, \dots, n\}$ is a subset consisting of r elements we let M_I denote the I -th submatrix. If the determinant of M_I is a unit in B , i.e. M_I is non-singular, it follows that the subset of the canonical basis elements $\{e_i\}_{i \in I}$ of $V = k^n$ is a basis for the quotient $(B \otimes_k V)/N$. Furthermore, if the matrix M_I is non-singular we may multiply M with M_I^{-1} to make that submatrix into the identity, and all the other $(n-r)r$ elements are then the coordinates of the quotient. The subset $W_I \subseteq W$ consisting of the $r \times n$ -matrices that have a non-singular I -th submatrix (the equivalence to our relative Grassmann functor) is then realized as an affine $(n-r)r$ -space $\mathbb{A}_B^{(n-r)r} = \text{Spec}(B[x_1, \dots, x_{(n-r)r}])$.

If we let the affine nr -space \mathbb{A}_B^{nr} denote the set of our $r \times n$ -matrices we see that it is therefore covered by affine $(n-r)r$ -spaces $\mathbb{A}_B^{(n-r)r}$ and since their intersection is well behaved (we omit the details) these can be glued into a scheme that represents the Grassmann functor. For a more comprehensive explanation, see [4, III.2.7].

3.2 The graded Grassmannian

Let V now be a finitely generated k -vector space with a grading by an abelian group G , i.e. $V = \bigoplus_{g \in G} V_g$, and let $h: G \rightarrow \mathbb{N}$ be any function. Then we can define the graded Grassmann functor $\text{grGrass}_V^h: \mathfrak{Alg}^k \rightarrow \mathfrak{Set}$ by

$$\text{grGrass}_V^h(B) = \left\{ \begin{array}{l} N \text{ is a homogeneous submodule such} \\ N \subseteq B \otimes V: \text{ that } (B \otimes V_g)/N_g \text{ is a locally free} \\ B\text{-module of rank } h(g) \text{ for all } g \in G \end{array} \right\}.$$

If $N \in \text{grGrass}_V^h(B)$ then N is graded so $N = \bigoplus_{g \in G} N_g$. There is then a canonical map $\text{grGrass}_V^h \rightarrow \prod_{g \in G} \text{Grass}_{V_g}^{h(g)}$ given, for each k -algebra B , by

$$\text{grGrass}_V^h(B) \ni N \mapsto \bigoplus_{g \in G} N_g \in \prod_{g \in G} \text{Grass}_{V_g}^{h(g)}(B).$$

This is a natural isomorphism of functors so we get a decomposition of the graded Grassmannian into a direct product of regular Grassmannians, $\text{grGrass}_V^h \cong \prod_{g \in G} \text{Grass}_{V_g}^{h(g)}$.

4 The Quot scheme of the affine line

4.1 Structure theorem of finitely generated graded modules over a graded principal ideal domain

Definition 4.1. If M is a graded module, then $M(d)$ is the graded module where $M(d)_k = M_{k+d}$.

Definition 4.2. If M and N are graded modules then a *graded homomorphism* $\varphi: M \rightarrow N$ is a homomorphism that has the property that $\varphi(M_k) \subseteq N_k$, i.e. φ preserves the grading of the modules. We will denote the set of graded homomorphism from M to N by $\text{grHom}(M, N)$. A *graded isomorphism* is a graded homomorphism that is an isomorphism. Two modules M and N are *isomorphic as graded modules*, denoted $M \cong_0 N$, if there is a graded isomorphism between them.

Remark 4.3. By definition the zero-homomorphism is graded since the element 0 is homogeneous and is a member of every component.

Lemma 4.4. If $S = k[t] = \bigoplus_{r=0}^{n-1} t^r k[t^n]$, then $S(-d) \cong_0 t^d S = \bigoplus_{r=0}^{n-1} t^{r+d} k[t^n]$.

Proof. The homomorphism which maps $1 \in S(-d)$ to $t^d \in t^d S$ is clearly a graded isomorphism. \square

The following comes from Proposition 5 in [9].

Proposition 4.5. If $\varphi: M \rightarrow N$ is a graded homomorphism then both $\ker(\varphi)$ and $\text{im}(\varphi)$ are homogeneous and the canonical bijection $M/\ker(\varphi) \rightarrow \text{im}(\varphi)$ is a graded isomorphism.

Proof. First of all, $\text{im}(\varphi)$ is generated by the images of the homogeneous generators of M and since φ is graded then the images are also homogeneous. If $x \in \ker(\varphi) \subseteq M$ then x can be written as a sum of its graded components $x = \sum_g x_g$.

Thus $0 = \varphi(x) = \varphi\left(\sum_g x_g\right) = \sum_g \varphi(x_g)$, where the $\varphi(x_g)$ are the graded components of 0, which implies that $\varphi(x_g) = 0$ for all g , or simply $x_g \in \ker(\varphi)$ for all g . That the isomorphism $M/\ker(\varphi) \rightarrow \text{im}(\varphi)$ is graded then follows from the construction of the quotient module and the fact that φ is graded. \square

Definition 4.6. A *graded principal ideal domain* is a principal ideal domain in which every homogeneous ideal is generated by a homogeneous element.

Remark 4.7. 1. Note that the k -algebra $k[t]$ is a graded principal ideal domain when it is graded by \mathbb{Z} or \mathbb{Z}_n .

2. There could also be a different interpretation of a *graded principal ideal domain* than that of Definition 4.6. It could also mean that it simply is a principal ideal domain that is graded, but where every ideal does not need to be generated by a homogeneous element.

The following lemma is inspired by Theorem 12.4 in [3].

Lemma 4.8. *If D is a graded principal ideal domain and M a free graded D -module of rank n then any homogeneous submodule $N \subseteq M$ has the following property: There is some basis y_1, \dots, y_n of M , some $m \in \mathbb{N}$ and homogeneous elements $a_1, \dots, a_m \in D$ with $a_1 \mid a_2 \mid \dots \mid a_m$ such that $a_1 y_1, \dots, a_m y_m$ is a basis of N .*

Proof. If $N = 0$ there is nothing to prove. Suppose $N \neq 0$, then N is free of rank m say (see [3, Theorem 12.4]). For any graded homomorphism $\varphi: M \rightarrow D$ we have, since φ is graded, that $\varphi(N)$ is a homogeneous ideal in D . Since D is a principal ideal domain $\varphi(N)$ is generated by a single element, which we denote by a_φ . Let us consider the set

$$\Sigma = \{(a_\varphi) : \varphi \in \text{grHom}_D(M, D) \text{ so that } a_\varphi \text{ is homogeneous}\},$$

which has a partial ordering by inclusion, \subseteq . It is clear that Σ is non-empty since the zero-homomorphism is graded and its image is generated by the homogeneous element 0. Since D is a P.I.D. it is Noetherian so Σ has a maximal element which corresponding to some a_ν for some graded homomorphism ν , and we let the degree of a_ν be d . The maximality implies that (a_ν) is not properly contained in any other element of Σ . Let $a_1 = a_\nu$. The fact that $a_1 \in \nu(N)$ implies that there is some $y \in N$ such that $\nu(y) = a_1$ and since ν is graded it follows that y has degree d .

Now we choose a basis x_1, \dots, x_n for M and let $\pi_i: M \rightarrow D$ be the projection of M on the i :th coordinate corresponding to this basis. Note that π_i is a graded homomorphism for any i . Since $N \neq 0$ it is clear that there is some i such that $\pi_i(N) \neq 0$, thus it is clear that $a_1 \neq 0$.

Take some $\varphi \in \text{grHom}_D(M, D)$. We will now show that a_1 divides $\varphi(y)$. Since y has degree d and φ is graded we have that $\varphi(y)$ has degree d as well. Let us consider the ideal generated by these two elements, $I = (a_1, \varphi(y))$, which is generated by some element, say b . Since D was a graded principal ideal domain it follows that b is homogeneous of degree d . This implies that $b \in \Sigma$ and thus $a_1 \mid b$ and therefore $a_1 \mid \varphi(y)$.

We can now apply the fact that a_1 divides $\varphi(y)$ to the projections π_i so that $\pi_i(y) = a_1 b_i$ for some b_i . We define

$$y_1 = \sum_{i=1}^n b_i x_i.$$

First we note that $a_1 y_1 = y$ from which it follows that $a_1 = \nu(y) = \nu(a_1 y_1) = a_1 \nu(y_1)$, hence $\nu(y_1) = 1$. That is, y_1 has degree 0. We now claim that $M = Dy_1 \oplus \ker(\nu)$. Take some arbitrary $x \in M$ and write $x = \nu(x)y_1 + (x - \nu(x)y_1)$. Since

$$\begin{aligned} \nu(x - \nu(x)y_1) &= \nu(x) - \nu(x)\nu(y_1) \\ &= \nu(x) - \nu(x) \\ &= 0, \end{aligned}$$

we have that $(x - \nu(x)y_1) \in \ker(\nu)$. Hence $M = Dy_1 + \ker(\nu)$. To show that the sum is direct we have to show that the intersection is trivial. Suppose that $ry_1 \in \ker(\nu)$ for some $r \in D$, then $0 = \nu(ry_1) = r\nu(y_1) = r$, so the intersection is indeed trivial. Hence $M = Dy_1 \oplus \ker(\nu)$.

We now show that $N = Da_1 y_1 \oplus (N \cap \ker(\nu))$. Take some $x \in N$. Then a_1 divides $\nu(x)$, by construction, and therefore we have that $\nu(x) = ba_1$ for some $b \in D$. Write $x = \nu(x)y_1 + (x - \nu(x)y_1) = ba_1 y_1 + (x - \nu(x)y_1)$ and since the second term is an element of N and, similarly to above, an element in the kernel of ν we have that $(x - \nu(x)y_1) \in (N \cap \ker(\nu))$. That the sum is direct follows by the same argumentation as above. Hence $N = Da_1 y_1 \oplus (N \cap \ker(\nu))$.

Now, since $\ker(\nu)$ is free of rank $n - 1$ we have, by induction, that there is a basis of homogeneous elements of degree 0, y_2, \dots, y_n , of $\ker(\nu)$ and homogeneous elements $a_2, \dots, a_m \in D$ with $a_1 \mid \dots \mid a_m$ such that $a_2 y_2, \dots, a_m y_m$ is a basis for $N \cap \ker(\nu)$. Thus, we have that y_1, \dots, y_n is a basis of M and $a_1 y_1, \dots, a_m y_m$ is a basis of N . The only thing we have to show is that a_1 divides a_2 . This we do by considering the graded homomorphism $\varphi: M \rightarrow D$ defined by $\varphi(y_1) = \varphi(y_2) = 1$ and $\varphi(y_3) = \dots = \varphi(y_n) = 0$. Then $a_1 = \varphi(a_1 y_1)$, i.e. $a_1 \in \varphi(N)$ and by maximality of (a_1) it follows that $(a_1) = \varphi(N)$. Since $a_2 = \varphi(a_2 y_2) \in \varphi(N)$ it follows that $a_1 \mid a_2$. \square

We are now ready to prove our structure theorem.

Theorem 4.9. *If M is a finitely generated graded module over a graded principal ideal domain D (graded by an abelian group G), then*

$$M \cong_0 \left(\bigoplus_{k=1}^m D(-g_k)/a_k D \right) \oplus \left(\bigoplus_{k=m+1}^n D(-g_k) \right)$$

for some $g_1, \dots, g_m, g_{m+1}, \dots, g_n \in G$, where $n, m \in \mathbb{N}$, and homogeneous elements $a_1, \dots, a_m \in D$ such that $a_1 \mid \dots \mid a_m$.

Proof. Let $\{\alpha_1, \dots, \alpha_n\}$ be a minimal homogeneous generating set of M , i.e. a set of homogeneous generators of M such that no other homogeneous generating set of M is properly contained in $\{\alpha_1, \dots, \alpha_n\}$, where each α_i is homogeneous of degree $d_i \in G$. That a minimal homogeneous generating set exists follows from the fact that if M is finitely generated and graded then we can choose a finite generating set and each of those generators can be written as a sum of a finite number of homogeneous elements.

We have that the D -module $\bigoplus_{k=1}^n D(-d_k)$ is free of rank n , say with basis x_1, \dots, x_n . Then we have a surjective graded homomorphism $\varphi: \bigoplus_{k=1}^n D(-d_k) \rightarrow M$ defined by $\varphi(x_i) = \alpha_i$ for $i = 1, \dots, n$. Thus

$$\bigoplus_{k=1}^n D(-d_k) / \ker(\varphi) \cong_0 M.$$

From Lemma 4.8 we have that the submodule $\ker(\varphi)$ is free of some rank $m \leq n$, and that we can choose a basis, y_1, \dots, y_n , of $\bigoplus_{k=1}^n D(-d_k)$ such that $a_1 y_1, \dots, a_m y_m$ is a basis of $\ker(\varphi)$ for some homogeneous elements a_1, \dots, a_m of D with $a_i \mid a_{i+1}$. Hence

$$M \cong_0 (D(-d_1)y_1 \oplus \dots \oplus D(-d_n)y_n) / (D(-d_1)a_1y_1 \oplus \dots \oplus D(-d_m)a_my_m).$$

Now we can consider the graded homomorphism

$$\pi: D(-d_1)y_1 \oplus \dots \oplus D(-d_n)y_n \rightarrow D(-d_1)/(a_1) \oplus D(-d_m)/(a_m) \oplus D(-d_{m+1}) \oplus \dots \oplus D(-d_n)$$

that maps $(f_1 y_1, \dots, f_n y_n)$ to $(f_1 + (a_1), \dots, f_m + (a_m), f_{m+1}, \dots, f_n)$. Then we have that $\ker(\pi) = \{(f_1, \dots, f_m, 0, \dots, 0) : f_i \in (a_i)\} = D(-d_1)a_1y_1 \oplus \dots \oplus D(-d_m)a_my_m$. Hence

$$M \cong_0 \left(\bigoplus_{k=1}^m D(-g_k) / a_k D \right) \oplus \left(\bigoplus_{k=m+1}^n D(-g_k) \right).$$

□

4.2 The relative Quot scheme

We will study the Quot scheme in the same way as we did with the Hilbert scheme in Section 2, namely by first defining the Quot functor and then show that it is representable. However to show that it is representable is not so easy, instead we will,

similarly to the Grassmannian in Section 3, first consider specific subfunctors of our Quot functor, so called *relative Quot functors*, and show that these are representable. That result will then be used to show the representability of the Quot functor in the next section.

As before we let k be a field, S be the k -algebra $k[t]$ graded by the abelian group G and h be a Hilbert function $G \rightarrow \mathbb{N}$. For any graded $k[t]$ -module V and k -algebra B we define the set

$$\mathrm{Quot}_V^h(B) = \left\{ \begin{array}{l} N \text{ is a homogeneous } B[t]\text{-submodule such} \\ N \subseteq B \otimes_k V : \text{ that } [(B \otimes_k V)/N]_g \text{ is locally free over } B \\ \text{and of finite rank } h(g) \text{ for all } g \in G \end{array} \right\}. \quad (2)$$

Lemma 4.10. *Let V be a $k[t]$ -module graded by G , B_1, B_2 be k -algebras and $h: G \rightarrow \mathbb{N}$ a Hilbert function. Then, for any $N \in \mathrm{Quot}_V^h(B_1)$ the submodule $\mathrm{im}(N \otimes_{B_1} B_2) = (\mathrm{incl} \otimes \mathrm{id})(N \otimes_{B_1} B_2) \in \mathrm{Quot}_V^h(B_2)$.*

Proof. Take an $N \in \mathrm{Quot}_V^h(B_1)$ and consider $\mathrm{im}(N \otimes_{B_1} B_2) \subseteq B_2 \otimes_k V$. Since this submodule is generated by homogeneous elements it is clearly a homogeneous submodule. By Lemma 2.7 it follows that

$$((B_1 \otimes_k V)/N) \otimes_{B_1} B_2 = ((B_1 \otimes_k V) \otimes_{B_1} B_2) / \mathrm{im}(N \otimes_{B_1} B_2) = (B_2 \otimes_k V) / \mathrm{im}(N \otimes_{B_1} B_2)$$

and that each graded component of this module is a locally free B_2 -module of rank $h(g)$ follows from Lemma 2.6. Hence $\mathrm{im}(N \otimes_{B_1} B_2) = (\mathrm{incl} \otimes \mathrm{id})(N \otimes_{B_1} B_2) \in \mathrm{Quot}_V^h(B_2)$. \square

Definition 4.11. Let V be a $k[t]$ -module graded by G and $h: G \rightarrow \mathbb{N}$ a Hilbert function. The *Quot functor*, $\mathrm{Quot}_V^h: \mathfrak{Alg}^k \rightarrow \mathfrak{Set}$, is defined, for any k -algebra B , by $\mathrm{Quot}_V^h(B)$ given by (2), and for any morphism of k -algebras, $\varphi: B_1 \rightarrow B_2$, by $\mathrm{Quot}_V^h(\varphi)(N) = \mathrm{im}(N \otimes_{B_1} B_2) = (\mathrm{incl} \otimes \mathrm{id})(N \otimes_{B_1} B_2)$.

Remark 4.12. Note that the Quot functor is really a generalization of the Hilbert functor. Indeed, for $V = k[t]$ we have $B \otimes_k k[t] = B[t]$ and the requirement that $N \subseteq B[t]$ is a $B[t]$ -module is equivalent to saying that N is an ideal in $B[t]$. Hence we have

$$\mathrm{Quot}_{k[t]}^h(B) = \left\{ \begin{array}{l} I \text{ is a homogeneous ideal such that} \\ I \subseteq B \otimes_k k[t] : \text{ the } B\text{-module } (B \otimes_k S_g)/I_g \text{ is locally} \\ \text{free of finite rank } h(g) \text{ for all } g \in G \end{array} \right\} = \mathrm{Hilb}^h(B).$$

We will in the following consider the case V being the finitely generated free $k[t]$ -module $V = \bigoplus_{i=1}^d k[t]$ of rank d say.

Notation. When V is the k -vector space $V = \bigoplus_{i=1}^d k[t]$, then we introduce $e_1 = (1, 0, \dots, 0)$, $e_2 = (0, 1, 0, \dots, 0)$, ..., $e_d = (0, 0, \dots, 1)$ to denote the canonical basis in V as a $k[t]$ -module.

Take some k -algebra B . If $N \in \text{Quot}_V^h(B)$ then we have that $(B \otimes_k V)/N$ must be a graded $B[t]$ -module such that each graded component is a locally free B -module of finite rank. If we localize at some prime ideal $\mathfrak{p} \in \text{Spec}(B)$ we have that $[(B \otimes_k V)/N]_{\mathfrak{p}}$ is a $B_{\mathfrak{p}}[t]$ -module such that every graded component $[[(B \otimes_k V)/N]_{\mathfrak{p}}]_g$ is a free $B_{\mathfrak{p}}$ -module of rank $h(g)$ for $g \in G$. Passing to the fraction field of $B_{\mathfrak{p}}$ we can use our structure theorem of finitely generated graded modules over graded P.I.D.s (Theorem 4.9) to see that it will be free with basis $e_1, te_1, \dots, t^{s_1}e_1, e_2, \dots, t^{s_d}e_d$ for some s_1, \dots, s_d . As stated earlier, the Quot functor is quite hard to handle, so instead we will, motivated by what the this calculation, consider more manageable subfunctors. Before we define them we will first add some useful notation.

Notation. Let $X \subseteq V$ be a subset consisting of homogeneous elements of V graded by the abelian group G . We then let X_g denote the subset of X that consists of the elements of degree $g \in G$ and define $x(g) = |X_g|$ to be the number of elements of X of degree g . Also we will let $\chi_1^{(g)}, \chi_2^{(g)}, \dots, \chi_{x(g)}^{(g)}$ denote the elements in X of degree g .

Since we have that the Quot functor is defined with a Hilbert function $h: G \rightarrow \mathbb{N}$ we will choose an $X \subseteq V$ such that $h(g) = x(g)$ for all $g \in G$ and say that X agrees with h .

Definition 4.13. For the k -vector space $V = \bigoplus_{i=1}^d k[t]$ with an induced grading from $k[t] = \bigoplus_{g \in G} S_g$, for some abelian group G , and a Hilbert function $h: G \rightarrow \mathbb{N}$ we define the *relative Quot functor* as the subfunctor of Quot_V^h defined by

$$\text{relQuot}_V^X(B) = \left\{ \begin{array}{l} N \text{ is a homogeneous } B[t]\text{-submodule} \\ N \subseteq B \otimes_k V : \text{ such that } [(B \otimes_k V)/N]_g \text{ is free over } B \\ \text{with basis } X_g \text{ for all } g \in G \end{array} \right\}.$$

Remark 4.14. The reason that we did not have to consider subfunctors when we showed the representability of the Hilbert functor was that subfunctors in that case coincided with the original Hilbert functor. We showed that the module was locally

free implied that the module was actually free (Proposition 2.16). Indeed, when we had a grading by \mathbb{Z}_n and Hilbert function

$$h(r) = \begin{cases} s+1 & 0 \leq r < m \\ s & m \leq r < n \end{cases}$$

for some $m \in \mathbb{Z}_n, s \in \mathbb{N}$, c.f. Section 2, the result was

$$\text{Hilb}^h = \text{relQuot}_{k[t]}^{\{1,t,\dots,t^{s_n-1}\}}.$$

As we remarked earlier, Remark 2.21, the case when we had a grading by \mathbb{Z} was a special case of this when we let $n \rightarrow \infty$.

Notation. If we write a set of basis elements as $X = \{e_1, te_1, \dots, t^{s_1-1}e_1, e_2, te_2, \dots, t^{s_2-1}e_2, \dots, t^{s_d-1}e_d\}$ for some s_1, \dots, s_d then it will be implied that if $s_j = 0$ then there are no basis elements in the e_j -coordinate.

Proposition 4.15. *If $V = \bigoplus_{i=1}^d k[t]$ is a free $k[t]$ -module, graded by $G = \mathbb{Z}$ or \mathbb{Z}_n , of finite rank d and $X = \{e_1, te_1, \dots, t^{s_1-1}e_1, e_2, te_2, \dots, t^{s_d-1}e_d\}$ then there is, for any k -algebra B , a one-to-one correspondence between the set $\text{relQuot}_V^X(B)$ and the set*

$$\left\{ \{p_1(t), \dots, p_d(t)\} : p_j(t) = t^{s_j}e_j - \sum_{r=1}^{x(s_j)} b_r^{(j)} \chi_r^{(s_j)} \text{ for some } b_1^{(j)}, \dots, b_{x(s_j)}^{(j)} \in B \text{ for } j = 1, \dots, d \right\}.$$

Proof. Let us denote the set of monic polynomials above by MP . Fix a k -algebra B and take some $N \in \text{relQuot}_V^X(B)$. Now, consider the image of the element $t^{s_1}e_1 \in B \otimes_k V$ in the quotient $(B \otimes_k V)/N$. Since $(B \otimes_k V)/N$ is a module over the graded ring $B[t]$ we must have that $t^{s_1}e_1 = t^{s_1-1}e_1 \cdot t$, it must respect the grading when we multiply by t . Hence $t^{s_1}e_1$ must be written as a linear combination of the basis elements of the same degree s_1 , i.e. $t^{s_1}e_1 = \sum_{r=1}^{x(s_1)} b_r^{(1)} \chi_r^{(s_1)}$ for some $b_1^{(1)}, \dots, b_{x(s_1)}^{(1)} \in B$. The same argument holds for any s_j . Hence N contains these elements. Furthermore, if N was generated by at least one more non-redundant polynomial then the subsets X_g would no longer be a basis for the quotients graded components $[(B \otimes_k V)/N]_g$. Thus N is generated by precisely the polynomials $t^{s_j}e_j - \sum_{r=1}^{x(s_j)} b_r^{(j)} \chi_r^{(s_j)}$.

Conversely, take an element $\{p_1(t), \dots, p_d(t)\} \in MP$. Then it is clear that the submodule of $B \otimes_k V$ that they generate is an element of $\text{relQuot}_V^X(B)$. Hence we have a natural correspondence given by the correspondence between the submodules and their generators. \square

Example 4.16. To illustrate we give an example of Proposition 4.15. Let $V = k[t]$ and $G = \mathbb{Z}_n$ so that we are back to the setting of Section 2. Then $X = \{1, t, \dots, t^{s-1}\}$ for some s . For simplicity we write $s = s'n + m$ for some integers s', m where m is the smallest non-negative integer possible. Then $\chi_1^s, \dots, \chi_{x(s)}^s$ are simply the elements $t^m, t^{m+n}, \dots, t^{m+(s'-1)n}$, in some order. Proposition 4.15 then states that there is, for any k -algebra B , a one-to-one correspondence between the sets $\text{relQuot}_V^X(B)$ and the set of monic polynomials $t^{s'n+m} + b_1 t^{(s'-1)n+m} + \dots + b_{s'}$, which is precisely the same as what Proposition 2.18 says.

Now we are ready for our main result of this section, namely that the relative Quot functor is representable.

Theorem 4.17. *Let $V = \bigoplus_{i=1}^d k[t]$ be a free $k[t]$ -module, graded by $G = \mathbb{Z}$ or \mathbb{Z}_n , of finite rank d and $X = \{e_1, te_1, \dots, t^{s_1-1}e_1, e_2, te_2, \dots, t^{s_d-1}e_d\}$. Then the relative Quot functor relQuot_V^X is represented by the k -algebra $Q = k \left[a_1^{(1)}, \dots, a_{x(s_1)}^{(1)}, a_1^{(2)}, \dots, a_{x(s_d)}^{(d)} \right]$, i.e. the polynomial ring over k in $\sum_{j=1}^d x(s_j)$ variables, and the natural isomorphism $\Phi: \text{Hom}_k(Q, -) \rightarrow \text{relQuot}_V^X$ given, for any k -algebra B and $\varphi \in \text{Hom}_k(Q, B)$, by*

$$(\varphi: Q \rightarrow B) \mapsto \left(t^{s_1}e_1 - \sum_{r=1}^{x(s_1)} \varphi(a_r^{(1)})\chi_r^{(s_1)}, \dots, t^{s_d}e_d - \sum_{r=1}^{x(s_d)} \varphi(a_r^{(d)})\chi_r^{(s_d)} \right).$$

Proof. Fix a k -algebra B and take some $N \in \text{relQuot}_V^X(B)$. From Proposition 4.15 we have that N is generated by polynomials on the form $p_j(t) = t^{s_j}e_j - \sum_{r=1}^{x(s_j)} a_r^{(j)}\chi_r^{(s_j)}$ for some $a_1^{(j)}, \dots, a_{x(s_j)}^{(j)} \in B$ for $j = 1, \dots, d$. So we can define $\Phi_B: \text{Hom}_k(Q, B) \rightarrow \text{relQuot}_V^X(B)$ by

$$(\varphi: Q \rightarrow B) \mapsto \left(t^{s_1}e_1 - \sum_{r=1}^{x(s_1)} \varphi(a_r^{(1)})\chi_r^{(1)}, \dots, t^{s_d}e_d - \sum_{r=1}^{x(s_d)} \varphi(a_r^{(d)})\chi_r^{(d)} \right).$$

This has a canonical inverse $\Psi_B: \text{relQuot}_V^X(B) \rightarrow \text{Hom}_k(Q, B)$ defined by

$$\left(t^{s_1}e_1 - \sum_{r=1}^{x(s_1)} b_r^{(1)}\chi_r^{(s_1)}, \dots, t^{s_d}e_d - \sum_{r=1}^{x(s_d)} b_r^{(d)}\chi_r^{(s_d)} \right) \mapsto (\varphi: Q \rightarrow B)$$

where φ is defined by $\varphi(a_j^{(i)}) = b_j^{(i)}$ for each $j = 1, \dots, x(s_i)$ and $i = 1, \dots, d$. We thus have an isomorphism between $\text{Hom}_k(Q, B)$ and $\text{relQuot}_V^X(B)$ given by Φ_B and since this is clearly functorial in B we get a natural isomorphism $\Phi: \text{Hom}_k(Q, -) \rightarrow \text{relQuot}_V^X$ given by $\Phi(B) = \Phi_B$ for any k -algebra B . \square

Remark 4.18. Using the notation of Theorem 4.17 we have that

$$\begin{aligned} \Phi_Q(\text{id}_Q) &= \left(t^{s_1} e_1 - \sum_{r=1}^{x(s_1)} \text{id}_Q(a_r^{(1)}) \chi_r^{(s_1)}, \dots, t^{s_d} e_d - \sum_{r=1}^{x(s_d)} \text{id}_Q(a_r^{(d)}) \chi_r^{(s_d)} \right) \\ &= \left(\underbrace{t^{s_1} e_1 - \sum_{r=1}^{x(s_1)} a_r^{(1)} \chi_r^{(s_1)}}_{=F_1(t)}, \dots, \underbrace{t^{s_d} e_d - \sum_{r=1}^{x(s_d)} a_r^{(d)} \chi_r^{(s_d)}}_{=F_d(T)} \right) \in \text{relQuot}_V^X(Q). \end{aligned}$$

So we can then say, by using Yoneda's Lemma, that we can write our representation of relQuot_V^X as the pair $(Q, (F_1(t), \dots, F_d(t)))$.

We will in the following sections give some examples of the use of Theorem 4.17.

4.2.1 Grading by $G = \mathbb{Z}$

Similarly to Section 2 we start by considering the case when $k[t]$ is graded by the abelian group $G = \mathbb{Z}$, $k[t] = \bigoplus_{r=0}^{\infty} k \cdot t^r$.

Example 4.19. First we consider the case when $V = k[t]$, to show the correspondence with the Hilbert scheme. Let $X = \{1, t, \dots, t^{m-1}\}$. Then Theorem 4.17 tells us that relQuot_V^X is represented by $Q = k$ since there are no basis elements in X of degree $(m-1) + 1 = m$. This is precisely what Theorem 2.14 says, that $\text{Hilb}^h = \text{relQuot}_{k[t]}^X$ is represented by $H = k$, where $h(r) = \begin{cases} 1 & 0 \leq r < m, \\ 0 & m \leq r < n \end{cases}$

Example 4.20. Consider the case $V = k[t] \oplus k[t]$ and $X = \{(1, 0), (t, 0), (0, 1), (0, t)\}$. Then we have from Theorem 4.17 that relQuot_V^X is represented by $Q = k$, since there are no basis elements of degree 2.

Example 4.21. Consider the case $V = k[t] \oplus k[t]$ and $X = \{(1, 0), (t, 0), (t^2, 0), (0, 1), (0, t)\}$. Then we have from Theorem 4.17 that relQuot_V^X is represented by $Q = k[a]$, since the element $(0, t^2) = t \cdot (0, t)$ can be written as a linear combination of the one basis element of degree 2, $(0, t^2) = a \cdot (t^2, 0)$ but the element $(t^3, 0) = t \cdot (t^2, 0)$ can not be written as a linear combination of basis elements of degree 3 since there are none so $(t^3, 0) = 0$.

4.2.2 Grading by $G = \mathbb{Z}_n$

Example 4.22. As we did for the case $G = \mathbb{Z}$, we first compare our result to what we got with the Hilbert scheme in Section 2. Let $V = k[t]$ and $X = \{1, t, \dots, t^{sn-1}\}$. Then, from Theorem 4.17, we have that relQuot_V^X is represented by $Q = k[a_1, \dots, a_s]$ since there are s elements in X of degree $(sn - 1) + 1 = sn = 0$, namely $1, t^n, t^{2n}, \dots, t^{sn-1}$. This is the same result that we would have gotten from Theorem 2.20, that says that

$\text{Hilb}^h = \text{relQuot}_V^X$ is represented by $H = k[a_1, \dots, a_s]$, where $h(r) = \begin{cases} s + 1 & 0 \leq r < m, \\ s & m \leq r < n, \end{cases}$
for some $m \in \mathbb{Z}_n$.

Example 4.23. Let $G = \mathbb{Z}_2$ and consider the case $V = k[t] \oplus k[t]$ and $X = \{(1, 0), (t, 0), (t^2, 0), (0, 1), (0, t)\}$. Then we have from Theorem 4.17 that relQuot_V^X is represented by the polynomial ring in five variables, $Q = k[a_1, a_2, \dots, a_5]$. That is because the element $(t^3, 0) = t \cdot (t^2, 0)$ can be written as a linear combination of the two basis elements of degree 3 = 1, namely $(t, 0)$ and $(0, t)$, and the element $(0, t^2) = t \cdot (0, t)$ can be written as a linear combination of the three basis elements of degree 2 = 0, namely $(1, 0), (t^2, 0)$ and $(0, 1)$.

4.3 The Quot scheme

What we will do now is to show that the Quot functor, Quot_V^h , c.f. Definition 4.11, is representable *with a scheme* when V is a finitely generated and free $k[t]$ -module. To do this we will first need to review some theory. This will be more of a summary, for a full treatment of the subject see, e.g., [4, VI.1.1].

We start by widening our definition of a representable functor. Since the category of commutative rings are a dual of only the affine schemes, not all schemes, we would otherwise be too restrictive in our class of representable functors.

Definition 4.24. A functor F from the category of k -algebras to the category of sets, $F: \mathfrak{Alg}^k \rightarrow \mathfrak{Set}$, is *representable by a scheme* if there is a natural isomorphism $\Phi: \text{Hom}_k(-, X) \rightarrow F$ from the functor $\text{Hom}_k(-, X): \mathfrak{Alg}^k \rightarrow \mathfrak{Set}$, which maps a k -algebra B to the set $\text{Hom}_k(\text{Spec}(B), X)$, for some scheme X . As there is no ambiguity with the representation of a functor by a k -algebra, we call also the pair (X, Φ) a *representation* of the functor F .

Remark 4.25. Note that any functor $F: \mathfrak{Alg}^k \rightarrow \mathfrak{Set}$ that is represented by a k -algebra A is also represented (by a scheme) by the affine scheme $\text{Spec}(A)$.

Definition 4.26. Let F, G be functors $\mathfrak{Alg}^k \rightarrow \mathfrak{Set}$, with a natural transformation $\alpha: G \rightarrow F$. Then G is an *open subfunctor* of F if

1. For any k -algebra A , the induced map $G(A) \rightarrow F(A)$ is injective, i.e. G is a subfunctor of F .
2. For any k -algebra A and each element $a \in F(A)$, which by Yoneda's lemma gives a natural transformation $\Phi_a: \text{Hom}_k(A, -) \rightarrow F$, we have that the functor G_{Φ_a} , that is defined by

$$G_{\Phi_a}(B) = \{(g, \varphi) \in G(B) \times \text{Hom}_k(A, B) : \alpha(g) = \Phi(\varphi) \text{ in } F(B)\},$$

is a subfunctor of $\text{Hom}_k(A, -)$ that is representable by an open subscheme of $\text{Spec}(A)$.

Remark 4.27. Note that the functor G_{Φ_a} from Definition 4.26 induces a Cartesian diagram

$$\begin{array}{ccc} G_{\Phi_a} & \longrightarrow & \text{Hom}_k(A, -) \\ \downarrow & & \downarrow \\ G & \longrightarrow & F \end{array}$$

and G is an open subfunctor of F if the upper arrow in the diagram makes G_{Φ_a} into a subfunctor of $\text{Hom}_k(A, -)$ and if G_{Φ_a} is represented by an open subscheme of $\text{Spec}(A)$.

We also need the notion of a *Zariski sheaf* which is a functor that satisfies the sheaf axiom for any open covering of an affine scheme of open affine schemes but will leave the details for the interested reader to find in, e.g., [4, VI.1.1].

Now we state (without proof) the theorem that we will need for the representability of the Quot functor. It can be found, in one form or another, in several introductory texts on the subject of representability of functors, e.g. [4, Theorem VI-14] or [5, Theorem 8.9].

Theorem 4.28. *A functor $F: \mathfrak{Alg}^k \rightarrow \mathfrak{Set}$ is representable by some scheme X if and only if*

1. F is a Zariski sheaf,
2. There exists a collection of open representable subfunctors $\{G_i \rightarrow F\}$ such that, for every field extension K/k , $F(K) = \bigcup_i G_i(K)$.

With the results from the previous section on the relative Quot functor we now have the theory we need to prove that the Quot functor is representable, but first we need a lemma.

Lemma 4.29. *Let $V = \bigoplus_{i=1}^d k[t]$ be a free $k[t]$ -module, graded by $G = \mathbb{Z}$ or \mathbb{Z}_n , of finite rank d , $h: G \rightarrow \mathbb{N}$ a Hilbert function and $X = \{e_1, te_1, \dots, t^{s_1-1}e_1, e_2, te_2, \dots, t^{s_d-1}e_d\}$ a set of basis elements that agree with h . Then the relative Quot functor relQuot_V^X is an open subfunctor of Quot_V^h .*

Proof. That it is a subfunctor follows from the fact that any free module is also locally free. In order to show the other part we will use two particular facts from commutative algebra so we state them here for clarity.

- (i) A square matrix M over a ring R is invertible if and only if its determinant is a unit in R . Hence if we consider M over $R_{\det(M)}$ it is invertible.
- (ii) A localization of a ring R (with respect to the multiplicatively closed set S) has the universal property that if $\varphi: R \rightarrow R'$ is a ring homomorphism that maps the elements of S to units in R' then there is a unique map $S^{-1}R \rightarrow R'$ that factor through φ .

Take a k -algebra A and some $N \in \text{Quot}_V^h(A)$. From Yoneda's lemma this gives us a natural transformation $\Phi: \text{Hom}_k(A, -) \rightarrow \text{Quot}_V^h$ where Φ is defined, for any $\varphi \in \text{Hom}_k(A, B)$, by $\Phi_B(\varphi) = \text{Quot}_V^h(\varphi)(N) = \text{im}(N \otimes_A B)$. We then consider the commutative diagram

$$\begin{array}{ccc} (\text{relQuot}_V^X)_\Phi & \longrightarrow & \text{Hom}_k(A, -) \\ \downarrow & & \downarrow \\ \text{relQuot}_V^X & \longrightarrow & \text{Quot}_V^h \end{array}$$

where $(\text{relQuot}_V^X)_\Phi$ is the functor defined by

$$(\text{relQuot}_V^X)_\Phi(B) = \left\{ (N', \varphi) \in \text{relQuot}_V^X(B) \times \text{Hom}_k(A, B) : N' = \Phi_B(\varphi) = \text{im}(N \otimes_A B) \right\}.$$

Thus, we see that $(\text{relQuot}_V^X)_\Phi(B)$ consists of the ring homomorphisms $\varphi \in \text{Hom}_k(A, B)$ such that $\text{im}(N \otimes_A B)$ is an element of $\text{relQuot}_V^X(B)$, i.e. that the graded components of the quotient $(B \otimes_k V) / \text{im}(N \otimes_A B)$ are free over B with basis X .

Now, we note that we can, for any k -algebra B , visualize the set $\text{Quot}_V^h(B)$ as a set of matrices over B , similarly to what was described in Remark 3.9 at the end of Section 3.1 for the Grassmannian. The matrices will in this case consist of infinitely

many columns corresponding to the fact that $B \otimes_k V = \bigoplus_{i=1}^d B[t]$ will not be finitely generated over B . The number of rows of the matrix will, due to Lemma 2.5, be $\sum_{g \in G} x(g)$. Then, $\text{relQuot}_V^X(B)$ will be equivalent to the set of matrices that have a non-singular I -th submatrix M_I , where I determines the columns of the matrix that corresponds to the basis set X . This means, from fact (i) of commutative algebra stated above, that the determinant of the matrix M_I is a unit in B . In particular we have for the k -algebra A that there is a matrix $M^{(N)}$ that corresponds to the element $N \in \text{relQuot}_V^X(A)$ and we let $\det_N \in A$ denote the value of the determinant of the I -th submatrix $M_I^{(N)}$ where I corresponds to the set X .

Since $\text{im}(N \otimes_A B) \in \text{Quot}_V^h(B)$ it corresponds to a matrix $M^{(\text{im}(N \otimes_A B))}$ over B . Then it follows that $\text{im}(N \otimes_A B) \in \text{relQuot}_V^X(B)$ if and only if the determinant of the I -th submatrix $M_I^{(\text{im}(N \otimes_A B))}$ is a unit in B .

Thus we have that $(\text{relQuot}_V^X)_\Phi$ corresponds to the ring homomorphisms $\varphi: A \rightarrow B$ such that the element $\det_N \in A$ is mapped to a unit in B . From fact (ii) stated above it follows that any such φ factors through a unique $\tilde{\varphi}: A_{\det_N} \rightarrow B$.

Hence $(\text{relQuot}_V^X)_\Phi(B) \cong \text{Hom}_k(A_{\det_N}, B)$ and since this is functorial in B it follows that $(\text{relQuot}_V^X)_\Phi$ is represented by the open affine subscheme $\text{Spec}(A_{\det_N})$ and therefore, relQuot_V^X is an open subfunctor of Quot_V^X . \square

Remark 4.30. Note that we see from the proof of Lemma 4.29 that $(\text{relQuot}_V^X)_\Phi$ is always representable by an affine open subscheme of $\text{Spec}(A)$, which is a much stronger statement than that of an open subfunctor which only requires that $(\text{relQuot}_V^X)_\Phi$ is representable by any open subscheme.

Theorem 4.31. *If $V = \bigoplus_{i=1}^d k[t]$ is a free $k[t]$ -module, graded by $G = \mathbb{Z}$ or \mathbb{Z}_n , of finite rank d and h is a Hilbert function $G \rightarrow \mathbb{N}$, then the Quot functor Quot_V^h is representable with a scheme.*

Proof. That Quot_V^h is a Zariski sheaf follows from the definition, since submodules can be glued together, see [7, Section II.5]. Let X_1, \dots, X_n for some $n \in \mathbb{N}$, where each X_i is of the form

$$X_i = \{e_1, te_1, \dots, t^{s_1^{(i)}-1}e_1, e_2, \dots, t^{s_d^{(i)}-1}e_d\},$$

be those different possible set of basis elements given by h . From Lemma 4.29 it follows that $\text{relQuot}_V^{X_i}$ is an open subfunctor of Quot_V^h for any i and we let $\{\text{relQuot}_V^{X_i} \rightarrow \text{Quot}_V^h\}$ be our collection of open subfunctors. From Theorem 4.17 we know that $\text{relQuot}_V^{X_i}$ is representable for all X_i . That $\text{Quot}_V^h(K) = \bigcup_i \text{relQuot}_V^{X_i}(K)$ for any field K over k , follows directly since, for any $N \in \text{Quot}_V^h(K)$, the K -module $(K \otimes_k V)/N$ is a vector

space so it has a well defined basis which must be given by at least some X_i . Hence, by Theorem 4.28, Quot_V^h is representable. \square

Definition 4.32. The scheme that represents the Quot functor is called the *Quot scheme*.

Finally we prove some geometrical properties of the Quot scheme.

Theorem 4.33. *When $V = \bigoplus_{i=1}^d k[t]$ is a free $k[t]$ -module, graded by $G = \mathbb{Z}$ or \mathbb{Z}_n , of finite rank d and $h: G \rightarrow \mathbb{N}$ a Hilbert function, then the Quot scheme is smooth, connected and, therefore, irreducible.*

Proof. That it is smooth follows from the fact that it is locally representable by a polynomial ring. In order to show that it is connected we need to show that the intersection of two open subfunctors, relQuot_V^X and relQuot_V^Y , is non-empty, where

$$X = \{e_1, te_1, \dots, t^{s_1-1}e_1, e_2, te_2, \dots, t^{s_d-1}e_d\}$$

and

$$Y = \{e_1, te_1, \dots, t^{\sigma_1-1}e_1, e_2, te_2, \dots, t^{\sigma_d-1}e_d\}$$

are two arbitrary sets that agree with h . We therefore have to consider the functor $\text{relQuot}_V^X \cap \text{relQuot}_V^Y$ which is defined, for any k -algebra B , by

$$(\text{relQuot}_V^X \cap \text{relQuot}_V^Y)(B) = \text{relQuot}_V^X(B) \cap \text{relQuot}_V^Y(B).$$

That is to say, we consider the submodules N such that the quotient $(B \otimes_k V)/N$ is free as a B -module with both basis X and Y . We have to show that the set $(\text{relQuot}_V^X \cap \text{relQuot}_V^Y)(B)$ is non-empty. It is enough if we can do this for any k -algebra B and we choose $B = k$.

As before, we let $x(g)$ denote the number of elements in X of degree g and we introduce $y(g)$ to denote the number of elements in Y of degree g . Clearly we must have that $x(g) = y(g)$ for any g since they come from the same Hilbert function. We also let $\chi_1^{(g)}, \dots, \chi_{x(g)}^{(g)}$ denote the elements in X of degree g and we introduce $\xi_1^{(g)}, \dots, \xi_{x(g)}^{(g)}$ to denote the elements in Y of degree g .

Any $N \in \text{relQuot}_V^X(B)$ is, from Proposition 4.15, generated, as a $B[t]$ -module, by the set of elements

$$N_X = \left\{ t^{s_1}e_1 - \sum_{i=1}^{x(s_1)} a_i^{(1)} \chi_i^{(s_1)}, \dots, t^{s_d}e_d - \sum_{i=1}^{x(s_d)} a_i^{(d)} \chi_i^{(s_d)} \right\},$$

for some $a_1^{(1)}, \dots, a_{x(s_d)}^{(d)} \in B$. If $N \in \text{relQuot}_V^Y(B)$ then it must also be generated by the set

$$N_Y = \left\{ t^{\sigma_1} e_1 - \sum_{i=1}^{x(\sigma_1)} b_i^{(1)} \xi_i^{(\sigma_1)}, \dots, t^{\sigma_d} e_d - \sum_{i=1}^{x(\sigma_d)} b_i^{(d)} \xi_i^{(\sigma_d)} \right\},$$

for some $b_1^{(1)}, \dots, b_{x(\sigma_d)}^{(d)} \in B$. Since the elements in N_Y are linearly independent this is true if and only if $t^{\sigma_j} e_j - \sum_{i=1}^{x(\sigma_j)} b_i^{(j)} \xi_i^{(\sigma_j)}$ can be written as a linear combination of the elements in N_X for $j = 1, \dots, d$. Thus we have a system of equations

$$\begin{aligned} t^{\sigma_1} e_1 - \sum_{i=1}^{x(\sigma_1)} b_i^{(1)} \xi_i^{(\sigma_1)} &= p_1^{(1)}(t) \left(t^{s_1} e_1 - \sum_{i=1}^{x(s_1)} a_i^{(1)} \chi_i^{(s_1)} \right) + \dots + p_d^{(1)}(t) \left(t^{s_d} e_d - \sum_{i=1}^{x(s_d)} a_i^{(d)} \chi_i^{(s_d)} \right) \\ &\vdots \\ t^{\sigma_d} e_d - \sum_{i=1}^{x(\sigma_d)} b_i^{(d)} \xi_i^{(\sigma_d)} &= p_1^{(d)}(t) \left(t^{s_1} e_1 - \sum_{i=1}^{x(s_1)} a_i^{(1)} \chi_i^{(s_1)} \right) + \dots + p_d^{(d)}(t) \left(t^{s_d} e_d - \sum_{i=1}^{x(s_d)} a_i^{(d)} \chi_i^{(s_d)} \right) \end{aligned}$$

where $p_1^{(1)}(t), \dots, p_d^{(d)}(t) \in B[t]$. Suppose that $G = \mathbb{Z}_n$, the case $G = \mathbb{Z}$ will follow in the same way and is easier. It is enough to prove that this has a solution in the case when the basis elements in X and Y only differ in the smallest possible way, so we can assume that we have $\sigma_1 = s_1 + r, \sigma_2 = s_2 - r, \sigma_i = s_i$ for $i = 3, \dots, d$ and $r = s_1 - s_2 - nx > 0$ where x is as large as possible, i.e.

$$X = \{e_1, te_1, \dots, t^{s_1-1}e_1, e_2, te_2, \dots, t^{s_2-1}e_2, \dots, t^{s_d-1}e_d\}$$

and

$$Y = \{e_1, te_1, \dots, t^{s_1+r-1}e_1, e_2, te_2, \dots, t^{s_2-r-1}e_2, \dots, t^{s_d-1}e_d\}.$$

Then, for any $g \in G$, we have that X_g and Y_g differ at most with one element. Let $s_1 = q_1 + nm_1$ and $s_2 = q_2 + nm_2$ where $m_1, m_2, q_1, q_2 \in \mathbb{N}$ and $q_1, q_2 < n$. The number of basis elements of the form $t^u e_1$ in X of degree s_2 is equal to $m_1 + \delta_1^X$ where

$$\delta_1^X = \begin{cases} 0 & \text{if } q_1 - q_2 \geq 0, \\ -1 & \text{if } q_1 - q_2 < 0. \end{cases}$$

In the same way we have that the number of basis elements of the form $t^u e_2$ of degree s_1 in X is equal to $m_2 + \delta_2^X$ where

$$\delta_2^X = \begin{cases} 0 & \text{if } q_2 - q_1 \geq 0, \\ -1 & \text{if } q_2 - q_1 < 0. \end{cases}$$

Furthermore, we have that $\sigma_1 = q_1 + r + nm_1$ and $\sigma_2 = q_2 - r + nm_2$. If $q_1 + r < n$ then there are m_1 elements of degree s_1 in Y and if $q_1 + r \geq n$ then there are $m_1 + 1$ elements of degree σ_1 in Y . We therefore define

$$\delta_1^Y = \begin{cases} 0 & \text{if } q_1 + r < n, \\ 1 & \text{if } q_1 + r \geq n, \end{cases}$$

so that there are $m_1 + \delta_1^Y$ elements of degree s_1 in Y . Similarly with σ_2 we define

$$\delta_2^Y = \begin{cases} 0 & \text{if } q_2 - r \geq 0, \\ -1 & \text{if } q_2 - r < 0. \end{cases}$$

Let also $\overline{q_i + r} = q_i + r - n\delta_i^Y$ for $i = 1, 2$.

We only need to find one solution to our system of equations and we will show that we can find one if we choose, for $i, j = 3, \dots, d$,

$$p_j^{(i)}(t) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

as well as $a_r^{(j)} = 0$ and $b_r^{(j)} = 0$ for $r = 1, \dots, x(s_i)$. We also let $p_1^{(1)}(t) = t^r$, $p_2^{(2)}(t) = 0$,

$$p_3^{(1)}(t), \dots, p_d^{(1)}(t) = 0$$

and

$$p_2^{(2)}(t), p_3^{(2)}(t), \dots, p_d^{(2)}(t) = 0.$$

Finally, we let those coefficients, $a_i^{(j)}, b_i^{(j)}$, of the basis elements that are of the form $t^u e_r$ be zero for $r = 3, \dots, d$, $j = 1, 2$. Then, our system of equations is reduced, if we write our basis elements explicitly, to

$$\begin{aligned} t^{\sigma_1} e_1 - \sum_{i=1}^{m_1 + \delta_1^Y} b_i^{(1)} t^{\overline{q_1 + r} + n(i-1)} e_1 - \sum_{i=1}^{m_2 + \delta_2^X + \delta_2^Y} b_{i+m_1+\delta}^{(1)} t^{\overline{q_1 + r} + n(i-1)} e_2 \\ = t^r \left(t^{s_1} e_1 - \sum_{i=1}^{m_1} a_i^{(1)} t^{q_1 + n(i-1)} e_1 - \sum_{i=1}^{m_2 + \delta_1^X} a_{i+m_1}^{(1)} t^{q_1 + n(i-1)} e_2 \right) \\ t^{\sigma_2} e_2 - \sum_{i=1}^{m_1 + \delta_1^X + \delta_1^Y} b_i^{(2)} t^{\overline{q_2 - r} + n(i-1)} e_1 - \sum_{i=1}^{m_2 + \delta_2^Y} b_{i+m_1}^{(2)} t^{\overline{q_2 - r} + n(i-1)} e_2 \\ = p_1^{(2)}(t) \left(t^{s_1} e_1 - \sum_{i=1}^{m_1} a_i^{(1)} t^{q_1 + n(i-1)} e_1 - \sum_{i=1}^{m_2 + \delta_2^Y} a_{i+m_1}^{(1)} t^{q_1 + n(i-1)} e_2 \right). \end{aligned}$$

This might seem quite complicated but by choosing

$$p_1^{(2)}(t) = t^{\sigma_2 - q_1 - (m_2 - 1)n} = t^{q_2 - r + n(m_2 + \delta_2) - q_1 - n(m_2 - 1)} = t^{q_2 - q_1 - r + n(1 + \delta_2)}$$

we can simply read of the solution one coordinate at a time. It follows that there is a solution to our system and that solution gives us a submodule N that is a member of both $\text{relQuot}_V^X(B)$ and $\text{relQuot}_V^Y(B)$. Hence the intersection of any two relative Quot functors is non-empty and it follows that the Quot scheme is connected.

Since any smooth connected scheme is irreducible, it follows that the Quot scheme is irreducible. \square

References

- [1] Michael F. Atiyah and Ian G. Macdonald. *Introduction to Commutative Algebra*. Westview Press, 1969.
- [2] Nicolas Bourbaki. *Commutative algebra*. Springer-Verlag Berlin and Heidelberg, 1998.
- [3] David S. Dummit and Richard M. Foote. *Abstract Algebra*. John Wiley & Sons Inc., Hoboken, NJ, third edition, 2004.
- [4] David Eisenbud and Joe Harris. *The Geometry of Schemes*. Springer-Verlag New York, Inc., 2000.
- [5] Ulrich Görtz and Torsten Wedhorn. *Algebraic geometry I : Schemes With Examples and Exercises*. Vieweg+Teubner Verlag, first edition, 2010.
- [6] Mark Haiman and Bernd Sturmfels. Multigraded Hilbert Schemes. *ArXiv Mathematics e-prints*, January 2002, arXiv:math/0201271.
- [7] Robin Hartshorne. *Algebraic geometry*. Springer, 1977.
- [8] Ernst Kunz. *Introduction to Commutative Algebra and Algebraic Geometry*. Birkhäuser, 1985.
- [9] Marvin Mendelsohn. Graded rings, modules and algebras. Master's thesis, The Massachusetts Institute of Technology, 1970.
- [10] Shigeru Mukai. *An Introduction to Invariants and Moduli*. Cambridge University Press, 2003.