Rees algebras of modules and Quot schemes of points

GUSTAV SÆDÉN STÅHL

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KTH
Institutionen för Matematik
100 44 Stockholm
SWEDEN

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Abstract

This thesis consists of three articles. The first two concern a generalization of Rees algebras of ideals to modules. Paper A shows that the definition of the Rees algebra due to Eisenbud, Huneke and Ulrich has an equivalent, intrinsic, definition in terms of divided powers. In Paper B, we use coherent functors to describe properties of the Rees algebra. In particular, we show that the Rees algebra is induced by a canonical map of coherent functors.

In Paper C, we prove a generalization of Gotzmann's persistence theorem to finite modules. As a consequence, we show that the embedding of the Quot scheme of points into a Grassmannian is given by a single Fitting ideal.

Sammanfattning

Denna licentiatuppsats består av tre artiklar. De två första artiklarna berör en generalisering av Reesalgebran från ideal till moduler. Artikel A visar att definitionen av Reesalgebran som getts av Eisenbud, Huneke och Ulrich är ekvivalent med en intrinsisk definition i termer av dividerade potenser. I Artikel B använder vi koherenta funktorer för att beskriva egenskaper hos Reesalgebran. Speciellt visar vi att Reesalgebran är inducerad av en kanonisk avbildning av koherenta funktorer.

I Artikel C bevisar vi en generalisering av Gotzmanns persistenssats för ändliga moduler. Som en följd av detta visar vi att ekvationerna som definierar inbäddningen av Quotschemat av punkter in i en Grassmannian beskrivs av ett enda Fittingideal.

Contents

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Part I

Introduction and summary

1 Introduction

The results of this thesis relate to a few different areas of algebraic geometry. Therefore, our goal with this introduction is to give a brief description of the various concepts that will appear. This will mostly be done by informal explanations which we hope will motivate why these notions are interesting to study. For more precise statements and definitions we will refer to more detailed texts whenever necessary. In most cases, more details can also be found in the papers that are included in this thesis.

This introduction has been divided into three sections, all of which can be read independently of each other. These sections are not divided in relation to the three articles, but instead by how they relate to each other in the general picture. The first section discusses the Rees algebra and related topics, which are important concepts to both Paper A and Paper B. Secondly, we consider projective resolutions. In the case of modules, projective resolutions give rise to the notion of Fitting ideals which is used in Paper C. By considering projective resolutions of additive functors, we obtain the theory of coherent functors which is fundamental for Paper B. In the last section we look at examples of various moduli spaces, and in particular the Quot scheme, which is an important part of Paper C.

1.1 The Rees algebra

Algebraic geometry is the theory that combines algebra and geometry, that is, it relates algebraic objects (e.g. rings, ideals, modules) to geometric objects (e.g. curves, surfaces, spaces). A nice example of this is the *Rees algebra of an ideal*.

Consider a ring A and an ideal $I \subseteq A$. The Rees algebra of I is an algebraic object defined as the graded ring $\mathcal{R}(I) = \bigoplus_{n=0}^{\infty} I^n$. In algebraic geometry, we relate the ring A to a geometric space, and the graded ring $\mathcal{R}(I)$ is then used to modify this space in a certain way, and this procedure is called *blowing up* the space. We will give a concrete example of this below.

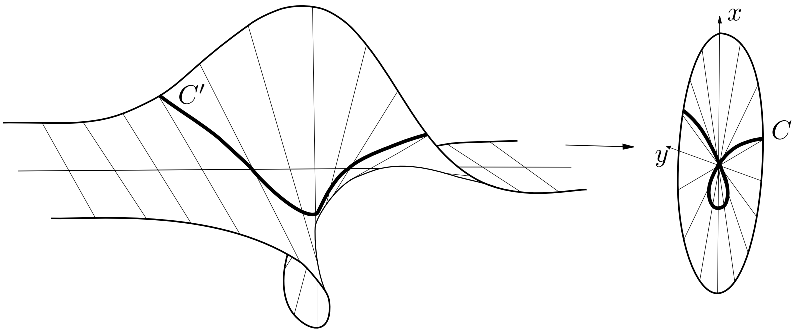
The Rees algebra is named after David Rees who used this object in his papers [Ree55, Ree56a, Ree56b, Ree56c] for studying valuations of ideals. We refer to [EH00] and [BH98] for more modern expositions.

Resolutions of singularities

As mentioned above, even though Rees algebras were first used for studying valuations of ideals, they are perhaps best known for being coordinate rings of blow-ups.

When working with geometric objects such as curves and surfaces, one of the most desirable class of objects are those that are smooth, i.e., without singularities. A singularity of a curve can be thought of as a point where the curve has more than one direction. In many cases, a singularity of a curve can be troublesome, so it is preferable to have methods of resolving the singularity while keeping other properties of the curve. Depending on the type of singularity, different methods can be considered. One of these is the *blow-up* of a space (scheme) in a closed subspace (subscheme). A blow-up of a curve produces a new curve that is *birational* to the original, where singularities might have vanished. We give an example of this below. The blow-up is computed by taking the projective spectrum of the Rees algebra of the corresponding ideal. For this reason, some authors use the term *blow-up algebra* as another name for the Rees algebra. For more detailed explanations, we refer to [EGA_{II}, §8.1], [Har77] and [Kol07].

Consider the equation $y^2 - x^2 - x^3 = 0$ over the complex numbers. This corresponds to a curve C in the affine plane $\mathbb{A}^2 = \operatorname{Spec}(\mathbb{C}[x, y])$. The curve has a singularity in the origin where it has two tangent directions. This can be remedied by blowing up the curve in the origin, and in that way obtaining a new curve C', birational to the original, without the singularity. This can be seen in the following picture, which is a modified version of a figure found in [Har77], where we have rotated the coordinate system for illustrative purposes.



The whole horizontal line in the left figure is projected onto the origin in the figure to the right. We can then think of our original singular curve C as the shadow of the smooth curve C' that lives in a higher dimensional space. In algebraic geometry, this procedure of blowing up the curve is done in the following way. The curve C corresponds the quotient ring $S = \mathbb{C}[x, y]/(y^2 - x^2 - x^3)$ and the origin to the ideal $I = (x, y) \subset S$. Then, the Rees algebra of I is the graded ring

$\mathcal{R}(I) = S[s,t]/(xt - ys)$. Taking the projective spectrum of this gives the blow-up $\operatorname{Bl}_I C = \operatorname{Proj}(\mathcal{R}(I))$ of C defining the curve C' depicted above.

The symmetric algebra

Another algebraic object that is useful in algebraic geometry is the *symmetric algebra of a module*. Similarly to the Rees algebra, this can be defined purely algebraically, but has interesting geometric properties.

For any A-module M, the symmetric algebra of M is a graded A-algebra $\operatorname{Sym}(M)$. It has the following universal property: for any A-algebra B, there is a bijection of sets

$$\operatorname{Hom}_{A\text{-mod}}(M, B) = \operatorname{Hom}_{A\text{-alg}}(\operatorname{Sym}(M), B),$$

where the left hand side denotes the set of A-module homomorphisms $M \to B$ and the right hand side denotes the set of A-algebra homomorphisms $\operatorname{Sym}(M) \to B$.

Example 1.1. If M is free of rank n, then $\operatorname{Sym}(M)$ is the polynomial ring over A in n variables. ▲

To define an object via universal properties can be problematic, as there might not exist an object with the desired properties. However, the symmetric algebra always exists, and there is even an explicit construction of it. Consider the tensor algebra $T(M) = \bigoplus_{n \geq 0} M^{\otimes n}$, where $M^{\otimes n}$ denotes the tensor product of n copies of M. Then, the symmetric algebra is equal to $\operatorname{Sym}(M) = T(M)/I$, where I is the ideal generated by $x \otimes y - y \otimes x$ for all $x, y \in M$. This means that the symmetric algebra is the tensor algebra where we add the relations forcing everything to commute. As a reference, we mention [Eis95] and [Bou70].

In this thesis, we will only consider the algebraic properties of the symmetric algebra, but we mention that it is used in algebraic geometry for defining abelian cones and projective bundles, see, e.g., [EGA_{II}, §1.7 and §4.1] and [GW10].

Using the symmetric algebra, we can give an equivalent definition of the Rees algebra of an ideal. Indeed, we have that the Rees algebra of an ideal $I \subseteq A$ is equal to the image of the induced map $\operatorname{Sym}(I) \to \operatorname{Sym}(A)$.

The Rees algebra of a module

The classical Rees algebra, that we mentioned above, is defined only for *ideals*. An ideal $I \subseteq A$ is a special case of an A-module, that is, an A-module that embeds into A. A main goal of mathematics is to understand the structure of fundamental objects, with an aim to find a universal theory that explains how everything is related. For this reason, we ask if there is some natural way to generalize the Rees algebra to modules, and in that way get a better understanding of the properties of the Rees algebra.

Above, we saw that the Rees algebra of an ideal I can be defined as the image of $\operatorname{Sym}(I) \to \operatorname{Sym}(A)$, that is, it comes from the natural inclusion map $I \to A$.

As the symmetric algebra is defined for any A-module, it seems natural to use the symmetric algebra to try give a definition of the Rees algebra of a *module*. However, a general module M does not naturally embed into A. The next best thing is if M embeds, or naturally maps, into some free module $F = A^n$. Therefore, one can pose the question if there exist a canonical homomorphism from any module to a *free* module. This does not exist in general, but for some types of modules such a map does exist, see, e.g., [KK97].

In [EHU03], the authors presented a new definition of the Rees algebra of a module M that did not use a natural map from the module to a free module. Instead, they used *all* maps from the module to *all* free modules. This was ingenious as it gives a definition that works for any finitely generated module and specializes to the original definition when M is an ideal. Moreover, the definition is functorial. Their definition was the following.

Definition 1.2 ([EHU03, Definition 0.1])**.** Let M be a finitely generated module. Then, the *Rees algebra of* M is the quotient

$$\mathcal{R}(M) = \operatorname{Sym}(M)/\cap_g L_g,$$

where the intersections is taken over all homomorphisms $g\colon M \to E$, where E runs over all free modules, and $L_g = \ker\big(\operatorname{Sym}(g)\colon \operatorname{Sym}(M) \to \operatorname{Sym}(E)\big)$.

However, infinite intersections are often difficult to work with. To calculate this quotient, the authors introduced the notion of a *versal* map $M \to F$, where F is a free module, as a map that any other map $M \to E$, where E is free, factors through. They then proceeded by showing that the Rees algebra of M is equal to the image of the induced homomorphism $\operatorname{Sym}(M) \to \operatorname{Sym}(F)$, which is analogous to $\mathcal{R}(I)$ being equal to the image of the homomorphisms $\operatorname{Sym}(I) \to \operatorname{Sym}(A)$ for an ideal I. Thus, the Rees algebra of a module M is induced by a map from M to a free module F. However, versal maps are neither canonical nor unique.

The algebra of divided powers

Before ending the section about Rees algebras, we mention a third graded A-algebra, namely the *algebra of divided powers*, that will turn out to be related to Rees algebras of modules. The algebra of divided powers can be thought of as a way to make Taylor expansions possible in a ring, even though division is not defined.

Given a module M, we consider the polynomial ring over A in the variables $X(n, x)$ for all $(n, x) \in \mathbb{N} \times M$. Then, the algebra of divided powers $\Gamma_A(M)$ of M is the quotient ring we get when dividing by the ideal generated by the elements

(i) $X(0, x) - 1$,

(ii) $X(n, fx) - f^n X(n, x)$,

(iii) $X(m, x)X(n, x) - \binom{m+n}{m} X(m + n, x)$,

(iv) $X(n, x + y) - \sum_{i+j=n} X(i, x)X(j, y)$, for all $x, y \in M$ and $f \in A$. We write $\Gamma(M) = \Gamma_A(M)$ when it is clear what ring A we consider. The following examples will hopefully justify the name.

Example 1.3. Let $A = \mathbb{Q}$ be the rational numbers, and let $M = \mathbb{Q}$ be the free $\mathbb{Q}$-module of rank 1. Then, the algebra of divided powers of M is

$$\Gamma_{\mathbb{Q}}(\mathbb{Q}) = \mathbb{Q}\left[x, \frac{x^2}{2}, \frac{x^3}{6}, \ldots, \frac{x^n}{n!}, \ldots\right] = \mathbb{Q}[x].$$

If we instead let $A = M = \mathbb{Z}$, then

$$\Gamma_{\mathbb{Z}}(\mathbb{Z}) = \mathbb{Z}\left[x, \frac{x^2}{2}, \frac{x^3}{6}, \ldots, \frac{x^n}{n!}, \ldots\right] \neq \mathbb{Z}[x]. \qquad \blacktriangle$$

Example 1.4. Consider an algebra B over a field $A = k$ of characteristic zero and an ideal $I \subseteq B$. Then I is a k-vector space and $\Gamma_k(I)$ is the k-algebra generated by the elements $\frac{x^n}{n!}$ for all $x \in I$ and all $n \in \mathbb{N}$. ▲

In the examples above we can write $X(n, x) = \frac{x^n}{n!}$. More intricate behaviour appear when working in positive characteristics. For more information, we refer to [Eis95], [Rob63], [Fer98] and [Ryd08].

1.2 Projective resolutions

Mathematical objects are often better understood by understanding how they interact with each other. For instance, when we can embed a scheme X into some projective space, then we automatically know that X comes with many nice properties. Another way to describe an object is as a quotient of two others, e.g., $\mathbb{P}^1 = (\mathbb{A}^2 \setminus 0)/\mathbb{G}_m$. Even stronger is if one can express an object X with a (left) resolution, that is, as an exact sequence

$$\cdots \xrightarrow{f_3} F_3 \xrightarrow{f_2} F_2 \xrightarrow{f_1} F_1 \longrightarrow X \longrightarrow 0,$$

where $F_1, F_2, F_3, \ldots$ are of some particular type. Then it is possible to get information on X from the maps $f_1, f_2, f_3, \ldots$.

We will use these resolutions in the following case. Consider a noetherian ring A. It turns out that any finitely generated A-module M has a *projective* resolution. This means that any module M can be described by a resolution

$$\cdots \longrightarrow F_3 \longrightarrow F_2 \longrightarrow F_1 \longrightarrow M \longrightarrow 0,$$

where $F_1, F_2, F_3, \ldots$ are finitely generated *projective modules.*

The notion of projectivity can be defined for many different mathematical objects and a projective resolution is a useful tool whenever it exists. In the case of finitely generated modules, a projective *module* is a module that is *locally free*, see [Bou61]. In fact, it turns out that every module M even has a *free* resolution. This is a very strong property. For instance, this means that any module is equal to a cokernel of a map $f\colon F_2 \to F_1$ of free modules. After choosing a basis for the free modules, such a map f corresponds to a matrix. Thus, one can in many cases reduce the theory of modules to basic linear algebra. The matrix corresponding to f contains lots of information on M. In particular, the ideals generated by minors of a certain size of this matrix are important invariants. These ideals are called Fitting ideals, named after Hans Fitting who studied these invariants in [Fit36]. As the term invariant suggests, it can be shown that Fitting ideals are independent of the choice of resolution. For more detailed information, we refer to [Eis95].

The main use of Fitting ideals is the following. A module M can be generated by n elements precisely when its n:th Fitting ideal is equal to A. Moreover, a module M is locally free of rank n if and only if its n:th Fitting ideal is equal to A and its $(n-1)$:th Fitting ideal is zero.

Coherent functors

The category $\mathbf{Mod}_A$ of finitely generated A-modules has many nice properties. For instance, it is abelian, which implies that it has both kernels and cokernels. Every module M has a projective resolution, and the category also has a notion of duality, that is, for any module M, we get a dual module $M^* = \mathrm{Hom}_A(M, A)$. However, this dual is not reflexive in the sense that the dual M^{**} of the dual module M^* is in general not isomorphic to M. There is a canonical map $M \to M^{**}$, but this need not be an isomorphism in general. Thus, this is not really a good notion of duality.

Obtaining a better notion of duality for all modules is one of the reasons for enlarging the category that we study and consider more general objects. Concretely, instead of considering the category $\mathbf{Mod}_A$ of finitely generated modules over a noetherian ring A, we look at the category of additive functors $\mathcal{F}\colon \mathbf{Mod}_A \to \mathbf{Mod}_A$.

Example 1.5. Take some A-module M. Then, we have an additive functor $t_M = M \otimes_A (-)$ that sends an A-module N to the A-module $M \otimes_A N$. Another additive functor is $h^M = \mathrm{Hom}_A(M, -)$, that sends an A-module N to the A-module $\mathrm{Hom}_A(M, N)$ of A-module homomorphisms $M \to N$. ▲

The category of finitely generated A-modules can be embedded into the category of additive functors by sending a module M to the functor $t_M = M \otimes_A (-)$.

As noted above, the notion of projectivity can be defined for more general objects than modules. In particular, we can consider projective additive functors. However, unlike the case for modules, not every additive functor has a projective resolution.

It can be shown, for any module M, that the functor $h^M = \mathrm{Hom}_A(M, -)$ is projective. A *coherent functor* is then an additive functor $\mathcal{F}\colon \mathbf{Mod}_A \to \mathbf{Mod}_A$ that has a presentation

$$h^{M_2} \to h^{M_1} \to \mathcal{F} \to 0$$

for some modules M_1 and M_2. A fundamental example of a coherent functor is $t_M = M \otimes_A (-)$ for any module M. This is coherent since any presentation $F_2 \to F_1 \to M \to 0$ of M gives a presentation

$$h^{F_2^*} \to h^{F_1^*} \to t_M \to 0.$$

Thus, the category of finitely generated A-modules actually embeds into the category $\mathcal{C}$ of coherent functors. The theory of coherent functors was first studied by Auslander in [Aus66].

A nice property of the category of coherent functors, that we alluded to earlier, is that it has a reflexive dual. This is stronger than what exists for modules, as that dual is not generally reflexive. Embedding the category of modules into the category of coherent functors, considering a module M as the coherent functor t_M, we are able to use the duality of coherent functors, where the dual of t_M is h^M, to gain new results.

These coherent functors are useful in many aspects of algebraic geometry. For instance, Hall used coherent functors in [Hal12] for giving a nice description of Artin's criteria for algebraicity of a stack. Moreover, coherent functors can help in expressing duality theorems of cohomology of coherent sheaves, which are written in terms of derived categories, by instead saying that we have a simple duality of coherent functors. This example, and more, can be found in Hartshorne's paper [Har98]. It gives a nice exposition on the main properties of coherent functors as well as the history behind them and their applications.

1.3 Moduli spaces

A universal goal in mathematics is to classify all objects of some certain type (e.g. groups, manifolds, schemes). One way of doing this is by finding a parametrization of the objects, so that every object of that type is obtained by varying the parameter. A moduli space is a geometric object that does precisely this. What we mean by this is that a moduli space parametrizing certain objects is a space in which every point in the space corresponds to a unique object. Even more, the geometry of the moduli space gives information of the relations between the objects.

Example 1.6. The moduli space that parametrizes the lines passing through the origin in a two-dimensional space is the projective line, denoted $\mathbb{P}^1$. ▲

In this example, the moduli space is a projective variety, which is one of the simplest geometric objects. However, many moduli spaces do not exist as varieties and, instead, more general spaces must be considered. In this text, we will restrict the theory to *schemes*, but sometimes one has to consider moduli spaces being even more general, such as *algebraic spaces* or *stacks*.

An important aspect of modern algebraic geometry is that schemes can be understood by all the maps into them, see [EGA_{III}, Chapitre 0, §8.1]. Concretely, this means that instead of considering a scheme X over S, we consider the functor

$h_X = \mathrm{Hom}_S(-, X)$ that takes an S-scheme T to the set $\mathrm{Hom}_S(T, X)$ of morphisms $T \to X$. This is similar to what we discussed in the previous section with coherent functors. By Yoneda's lemma, the scheme X can then be recovered from the functor h_X up to isomorphism. Using this functorial approach, it is easy to define different moduli problems: consider a functor $\mathcal{F}$ from the category of schemes to the category of sets and ask if this is equal to the functor h_X for some scheme X. If that is the case, we say that X represents the functor $\mathcal{F}$.

Generalizing our earlier example, the *Grassmannian* is the moduli space that parametrizes n-dimensional subspaces of an m-dimensional space. As we are working in the category of schemes, it is more difficult to see what n-dimensional subspaces of an S-scheme T should correspond to. A scheme is given by both a topological space T and a sheaf of rings $\mathcal{O}_T$. It turns out that the objects corresponding to n-dimensional subspaces are subsheaves $\mathcal{U} \subseteq \mathcal{O}_T^{\oplus m}$ such that the quotients $\mathcal{O}_T^{\oplus m}/\mathcal{U}$ are locally free $\mathcal{O}_T$-modules of rank $m - n$. Thus, we define our moduli problem of parametrizing n-dimensional subspaces as finding a scheme that represents the Grassmann functor that sends an S-scheme T to the set

$$\mathcal{G}rass_{n,m}(T) = \left\{ \mathcal{U} \subseteq \mathcal{O}_T^{\oplus m} \;\middle|\; \begin{array}{l} \mathcal{O}_T^{\oplus m}/\mathcal{U} \text{ is a locally free} \\ \mathcal{O}_T\text{-module of rank } m-n \end{array} \right\}.$$

This functor is representable by a projective scheme $\mathrm{Grass}_{n,m}$ called the Grassmannian, see [GW10]. In particular, the projective line is equal to the scheme $\mathrm{Grass}_{1,2}$.

Hilbert and Quot schemes

As schemes have more structure than just being topological spaces, their subschemes are more intricate objects to parametrize. For this case, we consider the Hilbert functor that parametrizes flat closed subschemes Z of $\mathbb{P}^r$. There are more general Hilbert functors that parametrizes closed subschemes of any scheme, but here we restrict ourselves to only consider the case above. For any S-scheme T, a closed subscheme $Z \subseteq \mathbb{P}^r \times_S T = \mathbb{P}^r_T$ that is flat over T is equivalent to a subsheaf $\mathcal{U} \subseteq \mathcal{O}_{\mathbb{P}^r_T}$ such that the quotient $\mathcal{Q} = \mathcal{O}_{\mathbb{P}^r_T}/\mathcal{U}$ is flat over T. Thus, we construct the Hilbert functor as

$$\mathcal{H}ilb_{\mathbb{P}^r/S}(T) = \left\{ \mathcal{U} \subseteq \mathcal{O}_{\mathbb{P}^r_T} \mid \mathcal{Q} = \mathcal{O}_{\mathbb{P}^r_T}/\mathcal{U} \text{ is flat over } T \right\}.$$

In [FGA, no. 221], Grothendieck proved that this functor is equal to h_X for a projective scheme X which we call the *Hilbert scheme*, and denote by $X = \mathrm{Hilb}_{\mathbb{P}^r/S}$. The name Hilbert scheme arouse from the relation to *Hilbert polynomials*. A Hilbert polynomial of a flat closed subscheme Z over a field is an invariant of Z that gives geometric information of Z, such as its dimension and genus. There is a stratification of the Hilbert functor using these Hilbert polynomials, meaning that we can write

$$\mathcal{H}ilb_{\mathbb{P}^r/S}(T) = \coprod_{P(t) \in \mathbb{Q}[t]} \mathcal{H}ilb^{P(t)}_{\mathbb{P}^r/S}(T),$$

where $\mathcal{H}ilb^{P(t)}_{\mathbb{P}^r/S}(T)$ is the set of all subsheaves $\mathcal{U} \subseteq \mathcal{O}_{\mathbb{P}^r_T}$ such that $\mathcal{Q} = \mathcal{O}_{\mathbb{P}^r_T}/\mathcal{U}$ is flat over T with Hilbert polynomial $P(t)$ on the fibers. To show that the Hilbert scheme exists, it then suffices to show that the there is a scheme $\mathrm{Hilb}^{P(t)}_{\mathbb{P}^r/S}$ representing the functor $\mathcal{H}ilb^{P(t)}_{\mathbb{P}^r/S}$ for every polynomial $P(t)$. Grothendieck proved the existence of the Hilbert scheme corresponding to a Hilbert polynomial $P(t)$ by showing, abstractly, that it embeds as a closed subscheme of a Grassmannian. We refer also to [FGI+05] and [Mum66] for more details. Later, Gotzmann's persistence theorem was used to write down the defining equations of this embedding [Got78].

From the description of the Hilbert scheme, there is a natural generalization where we parametrize quotients of the free module $\mathcal{O}^{\oplus p}_{\mathbb{P}^r}$ of rank $p \geq 1$. Thus, we define the Quot functor as the functor that takes an S-scheme T to the set

$$\mathcal{Q}uot^{P(t)}_{\mathcal{O}^{\oplus p}_{\mathbb{P}^r}/\mathbb{P}^r/S}(T) = \left\{ \mathcal{U} \subseteq \mathcal{O}^{\oplus p}_{\mathbb{P}^r_T} \;\middle|\; \begin{array}{l} \mathcal{Q} = \mathcal{O}^{\oplus p}_{\mathbb{P}^r_T}/\mathcal{U} \text{ is flat over } T \text{ and the fiber } \mathcal{Q}_s \\ \text{has Hilbert polynomial } P(t) \text{ for every } s \in T \end{array} \right\}.$$

Grothendieck proved, again in [FGA, no. 221], that also this functor is representable by a projective scheme $\mathrm{Quot}^{P(t)}_{\mathcal{O}^{\oplus p}_{\mathbb{P}^r}/\mathbb{P}^r/S}$ that we call the *Quot scheme*. Its definition is somewhat complicated but it makes up for it as being a fundamental object in algebraic geometry that generalizes all the moduli spaces that we have encountered in this introduction. Indeed, it is clearly a generalization of the Hilbert scheme, with $\mathrm{Quot}^{P(t)}_{\mathcal{O}_{\mathbb{P}^r}/\mathbb{P}^r/S} = \mathrm{Hilb}^{P(t)}_{\mathbb{P}^r/S}$. Furthermore, with the constant polynomial $P(t) = n$, we have that

$$\mathrm{Quot}^{n}_{\mathcal{O}^{\oplus p}_{\mathbb{P}^0}/\mathbb{P}^0/S} = \mathrm{Grass}_{n,p}\,.$$

The Quot scheme is mainly used for showing existence of many other moduli spaces. Grothendieck's proof of the existence of the Quot scheme was, similarly to the Hilbert scheme, by showing that it, abstractly, embeds as a closed subscheme of a Grassmannian. However, there is no analogue of Gotzmann's persistence theorem for modules, and therefore, unlike the case with the Hilbert scheme, the equations that define this embedding are not known.

The Hilbert scheme parametrizing subschemes with constant Hilbert polynomial $P(t) = n$ is the space that parametrizes zero-dimensional subschemes, that is, points of a scheme. It is therefore called the Hilbert scheme of points. Similarly, the Quot scheme of points is the Quot scheme parametrizing flat quotients with constant Hilbert polynomial $P(t) = n$. This moduli space has been well-studied, see for instance [GLS07]. Recently, Skjelnes proved that the embedding of the Quot scheme of points into a Grassmannian is given by an infinite intersection of closed subschemes defined by certain Fitting ideals [Skj14]. Even more, he proved that only one of these Fitting ideals suffices for determining the underlying topological space. The other ones are a priori needed for the scheme structure. However, he mentions that a generalization of Gotzmann's persistence theorem to modules, with constant Hilbert polynomial, would prove that only one of these Fitting ideals would suffice also for the scheme structure.

2 Summary of results

2.1 Paper A

In Paper A, we study the definition of Rees algebras of finitely generated modules over a noetherian ring due to Eisenbud, Huneke and Ulrich from [EHU03]. We show that this definition, which is in terms of maps to free modules, is equivalent to an intrinsic definition in terms of divided powers. Concretely, we show that the Rees algebra of a module M is equal to the image of a canonical map

$$\mathrm{Sym}(M) \to \Gamma(M^*)^\vee,$$

where

$$\Gamma(M^*)^\vee = \bigoplus_{n=0}^{\infty} \mathrm{Hom}_A\Big(\Gamma^n\big(\mathrm{Hom}_A(M, A)\big), A\Big)$$

denotes the graded dual of the algebra of divided powers of the dual of the module M. Proving this required some other results that we describe below.

To make sense of the graded dual of the algebra of divided powers, we study graded algebras in general. We show that the graded dual of any graded bialgebra is again an algebra in a natural way. In relation, we study colimit preserving functors from the category of modules to the category of (graded) algebras and show that such functors give rise to natural bialgebra structures. From these results, it follows that $\Gamma(M^*)^\vee$ is an algebra. Furthermore, there is a known canonical map $\Gamma(M^*) \to \mathrm{Sym}(M)^\vee$ which is an isomorphism when M is free. Analogously, we show that the canonical map $\mathrm{Sym}(M^*) \to \Gamma(M)^\vee$ is an isomorphism when M is free.

The authors of [EHU03] showed that the Rees algebra of a module M, with a versal map $M \to F$, can more easily be viewed as the image of the induced map $\mathrm{Sym}(M) \to \mathrm{Sym}(F)$. However, as versal maps are not unique, this gives no satisfactory definition of the Rees algebra. We show, for any versal map $M \to F$, that the induced map $\mathrm{Sym}(M) \to \mathrm{Sym}(F)$ has a canonical factorization

$$\mathrm{Sym}(M) \to \Gamma(M^*)^\vee \hookrightarrow \mathrm{Sym}(F),$$

where the second map is injective. Thus, it follows that

$$\mathcal{R}(M) = \mathrm{im}\big(\mathrm{Sym}(M) \to \mathrm{Sym}(F)\big) = \mathrm{im}\big(\mathrm{Sym}(M) \to \Gamma(M^*)^\vee\big).$$

It turns out that this factorization is in fact given by the universal property of the symmetric algebra applied to the canonical module homomorphism $M \to \Gamma(M^*)^\vee$, which sends M to the degree 1 part of $\Gamma(M^*)^\vee$, which is M^{**}.

2.2 Paper B

In Paper B, we continue the work of Paper A and study the functorial properties of Rees algebra of modules. We show that many properties of the Rees algebra are related to the category of coherent functors, rather than the category of modules. In particular, we prove that the Rees algebra of a module M is induced by a canonical map $t_M \to h^{M^*}$ of coherent functors.

From Paper A, we saw that the Rees algebra of M was equal to the image of the canonical map $\mathrm{Sym}(M) \to \Gamma(M^*)^\vee$, induced by the canonical map $M \to M^{**}$. The image M^{tl} of the map $M \to M^{**}$ is called the torsionless quotient of M, and it is natural to try and relate this module with the Rees algebra $\mathcal{R}(M)$ of M. However, it turns out that the torsionless quotient does not give much information on the Rees algebra. Instead, we define the torsionless functor $\mathcal{G}_M$ as the image of the canonical map $t_M \to h^{M^*}$,

$$\mathcal{G}_M = \mathrm{im}\big(t_M \to h^{M^*}\big).$$

This functor is strongly related to the Rees algebra. In particular, we show that if $\mathcal{G}_M \to \mathcal{G}_N$ is injective (resp. surjective), then $\mathcal{R}(M) \to \mathcal{R}(N)$ is injective (resp. surjective).

In [EHU03], it was shown that a map $M \to F$ is versal if and only if the dual $F^* \to M^*$ is surjective. That $F^* \to M^*$ is surjective implies that $M^{**} \to F^{**} = F$ is injective, but this injection is not a sufficient condition for $M \to F$ to be versal. Using the reflexive duality of coherent functors, we give a more descriptive classification of versal maps. Any map $M \to F$, where F is free, induces a factorization $t_M \to h^{M^*} \to t_F$, and we show that $M \to F$ is versal if and only if $h^{M^*} \to t_F$ is injective.

For any module M, there exist a versal map $M \to F$ for some free module F, and there is a natural factorization of this map given by

$$M \twoheadrightarrow M^{tl} \hookrightarrow M^{**} \hookrightarrow F.$$

Working in the category of coherent functors, we can generalize this to the sequence

$$t_M \twoheadrightarrow \mathcal{G}_M \hookrightarrow h^{M^*} \hookrightarrow t_F.$$

Evaluating this sequence at A returns the factorization of modules given above. From Paper A, we also have a factorization

$$\mathrm{Sym}(M) \twoheadrightarrow \mathcal{R}(M) \hookrightarrow \Gamma(M^*)^\vee \hookrightarrow \mathrm{Sym}(F),$$

and by comparing these sequences, we see that there are strong similarities between these different descriptions. Expanding on these similarities, we show that there is a functor Φ from the category of coherent functors to the category of graded algebras giving the following commutative diagram:

$$\begin{array}{ccccc}
M & \longrightarrow & M^{**} & \hookrightarrow & F \\
\big\downarrow{\scriptstyle t_-} & & & & \big\downarrow{\scriptstyle t_-} \\
t_M & \longrightarrow & h^{M^*} & \hookrightarrow & t_F \\
\big\downarrow{\scriptstyle \Phi} & & \big\downarrow{\scriptstyle \Phi} & & \big\downarrow{\scriptstyle \Phi} \\
\mathrm{Sym}(M) & \longrightarrow & \mathcal{Q}(M^*) & \hookrightarrow & \mathrm{Sym}(F)
\end{array}$$

Here, $\mathcal{Q}(M^*)$ denotes the largest subring of $\Gamma(M^*)^\vee$ that is generated in degree 1, that is, $\mathcal{Q}(M^*) = \mathrm{im}\big(\mathrm{Sym}(M^{**}) \to \Gamma(M^*)^\vee\big)$. Thus, the Rees algebra of M is induced by the canonical map $t_M \to h^{M^*}$, with which we mean that

$$\mathcal{R}(M) = \mathrm{im}\Big(\Phi\big(t_M\big) \to \Phi\big(h^{M^*}\big)\Big).$$

There is no functor from the category of modules to the category of graded algebras with this property. Indeed, the functors $t_{M^{**}}$ and h^{M^*} both correspond to the module M^{**}, and we need the higher level structure of coherent functors to distinguish them.

2.3 Paper C

Consider a polynomial ring $S = A[X_0, \ldots, X_r]$ over a noetherian ring A, and a homogeneous ideal $I \subseteq S$, generated in degree d. Write $Q = S/I$ for the quotient. Gotzmann's persistence theorem says that if the graded component Q_t of Q is flat and have a rank given by a certain polynomial growth, for $t = d$ and $t = d + 1$, then it is so for all $t \geq d$ [Got78].

In the case with constant Hilbert polynomial $P(t) = n$, this theorem reduces to the statement that if the graded components of Q are locally free of rank n in degrees d and $d+1$, then Q_{d+s} is locally free of rank n for all $s \geq 0$. In Paper C, we prove a generalization of this result where we replace the ideal I by a submodule $N \subseteq M = \bigoplus_{i=1}^p S$, generated in degree d.

The proof of this result uses Fitting ideals. In particular, we show that if $Q = M/N$ and if Q_d is locally free of rank n, then the $(n-1)$:th Fitting ideals of the graded components Q_{d+s} all coincide for $s \geq 1$. Thus, if the $(n-1)$:th Fitting ideal of Q_{d+1} is equal to zero, then the $(n-1)$:th Fitting ideal of Q_{d+s} is zero for all $s \geq 1$.

Grothendieck proved, in [FGA, no. 221], that the Quot scheme embeds into a Grassmannian. However, this embedding was only given abstractly. In the case with constant Hilbert polynomial $P(t) = n$, we show that this embedding is given by a single Fitting ideal. More precisely, we show the following. Consider a projective scheme $f\colon \mathbb{P}^r \to S = \operatorname{Spec}(A)$, where A is noetherian. For any $d \geq n$, there is an embedding of the Quot scheme $\operatorname{Quot}^n_{\mathcal{O}^{\oplus p}_{\mathbb{P}^r}/\mathbb{P}^r/S}$ into the Grasmannian $G = \operatorname{Grass}_{n,N_d}$, where N_d denotes the rank of the module $f_*\mathcal{O}^{\oplus p}_{\mathbb{P}^r}(d)$ over A. Letting $g\colon G \to S$ denote the structure morphism of G, we have a universal short exact sequence

$$0 \longrightarrow \mathcal{R}_d \longrightarrow g^*f_*\mathcal{O}^{\oplus p}_{\mathbb{P}^r}(d) \longrightarrow \mathcal{E}_d \longrightarrow 0$$

on the Grassmannian. Let $\mathcal{E}_{d+1}$ denote the cokernel of the induced map

$$\mathcal{R}_d \otimes_{\mathcal{O}_G} g^*f_*\mathcal{O}_{\mathbb{P}^r}(1) \to g^*f_*\mathcal{O}^{\oplus p}_{\mathbb{P}^r}(d+1).$$

Then, using the results of Skjelnes [Skj14] we prove that

$$\operatorname{Quot}^n_{\mathcal{O}^{\oplus p}_{\mathbb{P}^r}/\mathbb{P}^r/S} = V\big(\operatorname{Fitt}_{n-1}(\mathcal{E}_{d+1})\big),$$

where $V\big(\operatorname{Fitt}_{n-1}(\mathcal{E}_{d+1})\big)$ denotes the closed subscheme of G defined by the $(n-1)$:th Fitting ideal of the sheaf $\mathcal{E}_{d+1}$.

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Part II

Scientific papers

Paper A

An intrinsic definition of the Rees algebra of a module
Preprint: http://arxiv.org/abs/1402.3219.

Paper B

Rees algebras of modules and coherent functors
Preprint: http://arxiv.org/abs/1409.6464.

Paper C

Gotzmann's persistence theorem for finite modules
Preprint: http://arxiv.org/abs/1411.7940.