SF1544

Övning 3

This övning

- Numerical integration (Numerisk integrering)
- Composite trapezoidal rule (Trapetsmetoden)
- Numerical differentiation (Differensoperator)

Numerisk integrering

Let

$$a = x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

then we approximate

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} w_i f(x_i)$$

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Examples

- Mid point rule $$\int_a^b f(x)dx \approx (b-a)f\left(\frac{a+b}{2}\right)$$

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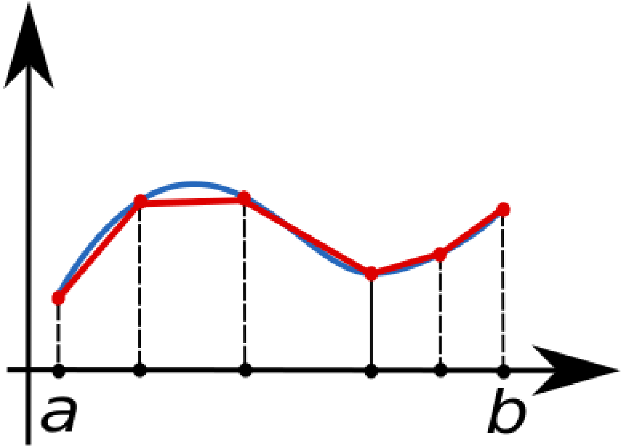
Examples

- Mid point rule $$\int_a^b f(x)dx \approx (b-a)f\left(\frac{a+b}{2}\right)$$
- Trapezoidal rule $$\int_a^b f(x)dx \approx (b-a)\left[\frac{f(a)+f(b)}{2}\right]$$

Illustration of composite trapezoidal



Trapezoidal rule

$$\int_a^b f(x)dx \approx \frac{1}{2}\sum_{i=1}^{N-1}(x_{i+1} - x_i)\left[f(x_{i+1}) + f(x_i)\right]$$

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If $x_{i+1} = x_i + h$ (uniform grid)

$$\int_a^b f(x)dx \approx \frac{h}{2}\left[\sum_{i=1}^{N-1} f(x_{i+1}) + \sum_{i=1}^{N-1} f(x_i)\right]$$

$$\int_a^b f(x)dx \approx h\left[\frac{f(a) + f(b)}{2} + \sum_{i=2}^{N-1} f(x_i)\right]$$

Exercise

Approximate with trapezoidal rule and $n = 5$

$$\int_0^1 x^2 dx$$

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We have $h = 0.25$ and

$$x_1 = 0, \qquad x_2 = 0.25, \qquad x_3 = 0.5, \qquad x_4 = 0.75, \qquad x_5 = 1$$

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$$\int_0^1 x^2 dx \approx 0.25 \left[\frac{0^2 + 2^2}{2} + 0.25^2 + 0.5^2 + 0.75^2 \right] = 0.34375$$

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Approximate with trapezoidal rule and $n = 5$

$$\int_0^1 x^2 dx = \frac{1}{3} = 0.\bar{3}$$

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Matlab implementation

```
function [ res ] = trapez(f, a, b, n )
%TRAPEZ composite trapezoidal rule
    x=linspace(a,b,n);
    res=f(a)+f(b);
    for i=2:n-1
        res=res+2*f(x(i));
    end
    h=x(2)-x(1);
    res=res*h/2;
end
```

MATLAB DEMO

Exercise from the book

6.1 För att mäta bensinförbrukningen vid kallstart av en personbil har man vid förgasaren monterat en genomströmningsmätare. Experiment gav följande värden:

x	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1
B	2.60	2.08	1.72	1.45	1.26	1.13	1.04	0.97	0.92

x är sträckan i mil efter start och B (korrekt avrundat) är momentan bränsleförbrukning i liter/mil. Beräkna bränsleförbrukningen under den första milens körning och gör noggrannhetsbedömning.

Solution

We have $N = 9$ points

$$\int_0^1 B(x)dx \approx h\left[\frac{B(0)+B(1)}{2} + \sum_{i=2}^{8} B(x_i)\right]$$

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and $h = 0.125$,

$$\begin{array}{lllll} x_1 = 0, & x_2 = 0.1250, & x_3 = 0.2500, & x_4 = 0.3750, & x_5 = 0.5000, \\ x_6 = 0.6250, & x_7 = 0.7500, & x_8 = 0.8750 & x_9 = 1.0000 & \end{array}$$

Solution

We have $N = 9$ points

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$$\int_0^1 B(x)dx \approx 0.125\left[\frac{2.60+0.92}{2} + 2.08 + 1.72 + 1.45 + 1.26 \right.$$
$$\left. + 1.13 + 1.04 + 0.97\right] = 1.426$$

Numerical differentiation

Recall

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Then we can approximate for h small

forward difference formula

$$f'(x) \approx \frac{f(x+h) - f(x)}{h} \qquad h \text{ small}$$

Example

Let $f(x) = \log(x)$, approximate $f'(1)$.

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$$f'(1) \approx \frac{log(1.1)}{0.1} = 0.9531$$

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- For $h = 0.01$

$$f'(1) \approx \frac{log(1.01)}{0.01} = 0.9950$$

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- For $h = 0.001$

$$f'(1) \approx \frac{log(1.001)}{0.001} = 0.9995$$

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- For $h = 0.01$
$$f'(1) \approx \frac{log(1.01)}{0.01} = 0.9950$$
- For $h = 0.001$
$$f'(1) \approx \frac{log(1.001)}{0.001} = 0.9995$$

Correct value: $f'(x) = \frac{1}{x}$ and so $f'(1) = 1$

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Prove that

$$f'(x) = \frac{-f(x+2h) + 4f(x+h) - 3f(x)}{2h} + O(h^2)$$

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Solution:

TAYLOR EXPANSION

Taylor expansion

Recall

$$f(x + \Delta x) = f(x) + \Delta x f'(x) + \frac{\Delta x^2}{2} f''(x) + \cdots + f^{(m)}(x) \frac{\Delta x^m}{m!} + O(\Delta x^{m+1})$$

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In a compact form

$$f(x+\Delta x) = \sum_{i=0}^{m} f^{(i)}(x)\frac{\Delta x^i}{i!} + O(\Delta x^{m+1})$$

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TAYLOR EXPANSION

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$$f(x+2h) = f(x) + 2hf'(x) + f''(x)\frac{4h^2}{2} + O(h^3)$$

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$$f(x) = f(x)$$

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- With $\Delta x = 0$ we have

$$f(x) = f(x)$$

With a direct computation

$$\frac{-f(x+2h) + 4f(x+h) - 3f(x)}{2h} = f'(x) + O(h^2)$$