NUMBER OF TRITANGENTS IN A PENCIL OF DEGREE d PLANE CURVES

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1. Problem

We always work over $\mathbb{C}$. Let $C \subset \mathbb{P}^2$ be a degree d plane curve. If C is general, then it has precisely

$$\frac{1}{2}d(d-2)(d-3)(d+3)$$

bitangents. This can be proven by calculating the arithmetic genus of the dual curve of C in two different ways, see Hartshorne Exercise IV.2.3. Further, by a dimension argument, the locus of curves with a tritangent has codimension 1 in the linear system $|\mathcal{O}_{\mathbb{P}^2}(d)|$, so C has no tritangents.

A pencil of degree d plane curves is a degree $(1,d)$-hypersurface in $\mathbb{P}^1 \times \mathbb{P}^2$, viewed as a family of degree d curves via the projection to the first factor. Since curves with a tritangent are codimension 1, a general pencil has finitely many tritangents. We ask more precisely:

> *(*) In a general pencil of degree d curves in $\mathbb{P}^2$, how many curves have a tritangent?*

We will show below that the answer to (*) is

$$(d^2+3d-2)(d-3)(d-4)(d-5)$$

The first values for $d = 6, 7, \ldots$ are

$$312, 1632, 5160, 12720, 26880, 51072, \ldots.$$

The number 312 for sextics is related to a Noether-Lefschetz calculation, see Section 3.

2. Solution

Let f be the equation of a curve of degree d in $\mathbb{P}^2$, and let $L \subset \mathbb{P}^2$ be a fixed line. The line L is a tritangent to $V(f)$ if and only if the restriction $f|_L$ has at least three multiple roots, hence if and only if there exists a subscheme $z = x_1 + x_2 + x_3 \subset L$ (viewed here as a Cartier divisor) such that

$$f|_{2z} = 0 \in H^0(2z, \mathcal{O}_{\mathbb{P}^2}(d)|_{2z}) \tag{1}$$

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with x_1, x_2, x_3 distinct. (If the x_i are not distinct, for example $z = 2x + y$, then the line L would meet the curve $V(f)$ with multiplicity $(4, 2, \ldots)$; since this is a codimension 2 condition in the linear system $|\mathcal{O}_{\mathbb{P}^2}(d)|$ this case will not appear in a general pencil and we will drop the condition on x_i to be distinct below.) The idea to the solution is to interpret (1) as the zero locus of some section on a projective bundle.

Let $G = G(2,3)$ be the Grassmannian of 2-planes in $\mathbb{C}^3$, or equivalently of lines in $\mathbb{P}^2$ (i.e. the dual $\mathbb{P}^2$). On G we have the sequence of vector bundles

$$0 \to U \to \mathcal{O}_G^3 \to Q \to 0$$

where U is the universal rank 2 subbundle, and Q is the 1-dimensional quotient line bundle. The projective bundle

$$L = \mathbb{P}(U)$$

is the universal line over the Grassmannian. Here we use Fulton's notation, i.e. we identify a locally free sheaves $\mathcal{E}$ on a scheme X with the vector bundle $E = \operatorname{Spec} \operatorname{Sym}^\bullet(\mathcal{E}^\vee)$, and the associated projective bundle is

$$\mathbb{P}(\mathcal{E}) = \operatorname{Proj} \operatorname{Sym}^\bullet(\mathcal{E}^\vee).$$

The space of degree d equations on L is $E_d = \mathbb{P}(\operatorname{Sym}^d(U^\vee))$. Over E_d we have a universal scheme

$$Z_d \subset E_d \times_G L \tag{2}$$

whose fiber over a point $[f] \in E_d$ is the subscheme $V(f) \subset L_{\pi([f])}$. The scheme Z_d is cut out by the line bundle

$$\mathcal{O}_{\mathbb{P}(E_d)}(1) \otimes \mathcal{O}_L(d),$$

see also the following remark for a more detailed explanation.

Remark 1. We describe how (2) is cut out by a section of a line bundle. Let X be a scheme with a line bundle $\mathcal{L}$ and let $V = H^0(X, \mathcal{L})$. The evaluation map $V \otimes \mathcal{O}_X \to \mathcal{L}$ induces a morphism

$$\mathcal{O}_X \to V^\vee \otimes \mathcal{L}$$

which in turn defines a morphism of graded $\mathcal{O}_X$-algebras

$$\operatorname{Sym}^\bullet V^\vee \otimes \mathcal{O}_X \to (\operatorname{Sym}^\bullet V^\vee \otimes \mathcal{O}_X)(1) \otimes \mathcal{L}.$$

Applying the $\sim$ functor of the Proj construction to this sequence yields

$$\mathcal{O}_{X \times \mathbb{P}(V)} \to \pi_{\mathbb{P}(V)}^* \mathcal{O}_{\mathbb{P}(V)}(1) \times \pi_X^* \mathcal{L}$$

on the product $X \times \mathbb{P}(V)$, hence a section s of the line bundle on the right hand side. The vanishing locus $Z = V(s)$ is the family of subschemes parametried by V in the sense that the fiber of Z over the point $[f] \in \mathbb{P}(V)$

is precisely the subscheme $V(f)$. The same construction works also when X is taken relative to a base.

In our case, let $\mathcal{L} = \mathcal{O}_L(d)$ and let $\pi : L \to G$ denote the projection. Then

$$\pi_* \mathcal{L} = \operatorname{Sym}^d U^\vee.$$

We let

$$V_d = \pi^* \pi_* \mathcal{L} = \pi^* \operatorname{Sym}^d U^\vee.$$

By adjunction to the identity of $\pi_* \mathcal{L} \to \pi_* \mathcal{L}$ we obtain the evaluation map

$$V_d \to \mathcal{L}$$

hence $\mathcal{O}_L \to V_d^\vee \otimes \mathcal{L}$ which induces the sequence

$$\operatorname{Sym}^\bullet V_d^\vee \to (\operatorname{Sym}^\bullet V_d^\vee)(1) \otimes \mathcal{L}.$$

We have $E_d \times_G L = \mathbb{P}(V_d)$ so this yields a section s_d of $\pi_{E_d}^* \mathcal{O}_{E_d}(1) \otimes \pi_L^* \mathcal{O}_L(d)$ such that the vanishing locus $Z_d = V(s_d)$ has the desired properties. $\square$

We now consider the scheme $E = E_3 = \mathbb{P}(\operatorname{Sym}^d(U^\vee))$, the universal subscheme $Z_0 \subset E \times_G L$ as above (which is a Cartier divisor) and set

$$Z = 2Z_0.$$

Consider the diagram

$$\begin{tikzcd} Z \arrow[r, "j"] \arrow[dr, "p"'] & E \times_G L \arrow[r, "q"] \arrow[d, "\tilde{p}"] & \mathbb{P}^2 \\ & E & \end{tikzcd}$$

where q is the composition of the projection to L, followed by the inclusion $L \subset G \times \mathbb{P}^2$, followed by the projection to $\mathbb{P}^2$. Let $f \in H^0(\mathbb{P}^2, \mathcal{O}(d))$. We consider the restriction to Z, which is

$$\tilde{q}^* f \in H^0(Z, \tilde{q}^* \mathcal{O}_{\mathbb{P}^2}(d))$$

where we let $\tilde{q} = q \circ j$. We pushforward along p and get the section

$$s = p_* \tilde{q}^* f \in H^0(E, p_* \tilde{q}^* \mathcal{O}_{\mathbb{P}^2}(d)).$$

The fiber $p_* \tilde{q}^* \mathcal{O}_{\mathbb{P}^2}(d))$ over a point $[z] \in E$ (corresponding to a subscheme $z \subset \ell$, where $\ell = \pi([z])$) is $H^0(z, \mathcal{O}(d)|_{2z})$. Hence $p_* \tilde{q}^* \mathcal{O}_{\mathbb{P}^2}(d)$ is a vector bundle of rank 6. The section s vanishes at $[z]$ if and only if L meets the curve $C = V(f)$ in the subscheme $2z$.

Finally, we also allow our equation f to vary. Let $W \subset |\mathcal{O}_{\mathbb{P}^2}(d)|$ be a linear subspace and let

$$f \in H^0(\mathbb{P}^2 \times W, \mathcal{O}_{\mathbb{P}^2}(d) \otimes \mathcal{O}_W(1))$$

be the equation cutting out the family of curves parametrized by W. The taking the product of the above discussion with W, we obtain the section

$$s = p_* \tilde{q}^* f \in H^0(E \times W, p_* \tilde{q}^* \mathcal{O}_{\mathbb{P}^2}(d) \otimes \mathcal{O}_W(1)).$$

The vanishing locus of s is the subscheme

$$V(s) = \{([z],[f]) \in E \times W | f|_{2z} = 0\}.$$

Let W be a general 1-dimensional linear system now. Since the class $V(s)$ is given by the top Chern class of the corresponding vector bundle, we have found that the answer to question (*) is given by the following integral:

$$N_d = \int_{E \times W} c_d(p_* \tilde{q}^* \mathcal{O}_{\mathbb{P}^2}(d) \otimes \mathcal{O}_W(1)). \tag{3}$$

We evaluate (3): By Grothendieck-Riemann-Roch we have

$$\mathrm{ch}(p_* \tilde{q}^* \mathcal{O}_{\mathbb{P}^2}(d)) \otimes \mathrm{td}(E \times W) = p_* \left(\mathrm{ch}(\tilde{q}^* \mathcal{O}_{\mathbb{P}^2}(d)) \mathrm{td}(Z) \right).$$

Using that on a projective bundle $\pi(\mathcal{E})$ over a base X we have the exact sequence $0 \to T_{\pi(\mathcal{E})/X} \to T_{\pi(\mathcal{E})} \to \pi^* T_X \to 0$ and since td is multiplicative on short exact sequences we find

$$\frac{\mathrm{td}(Z)}{\mathrm{td}(E \times W)} = \frac{\mathrm{td}(E \times_G L \times W)}{\mathrm{td}(\mathcal{O}(Z)) \mathrm{td}(E \times W)} \Big|_Z = \frac{\mathrm{td}(T_{L/G})}{\mathrm{td}(\mathcal{O}(Z))} \Big|_Z.$$

Hence

$$\mathrm{ch}(p_*(\tilde{q}^* \mathcal{O}_{\mathbb{P}^2}(d) \otimes \mathcal{O}_W(1))) = \mathrm{pr}_* \left(\mathrm{td}(T_{L/G}) \cdot \frac{[Z]}{\mathrm{td}(\mathcal{O}(Z))} \cdot \mathrm{ch}(\mathcal{O}_{\mathbb{P}^2}(d)) \mathrm{ch}(\mathcal{O}_W(1)) \right).$$

We know describe all the rings explicitly. Let $z = c_1(Q)$. Then

$$A^*(G) = \mathbb{Q}[z]/z^3.$$

Let $h = c_1(\mathcal{O}_L(1))$. Then since $c(U) = 1 - z + z^2$ we have

$$A^*(L) = A^*(G)[h]/(h^2 - zh + z^2).$$

Since $\mathcal{O}_L(1) = i^* \mathcal{O}_{\mathbb{P}^2}(1)$, the class h is also the pullback of the hyperplane class from $\mathbb{P}^2$. Similarly, with $t = c_1(\mathcal{O}_{E_d}(1))$ we have

$$A^*(E) = A^*(G)[t]/(t^4 + c_1(\mathrm{Sym}^3(U^\vee))t^3 + \ldots + c_4(\mathrm{Sym}^d(U^\vee)) = 0)$$

Then since Z is cut out by a section of $\mathcal{O}_E(2) \otimes \mathcal{O}_L(6)$ we get

$$[Z] = 2t_1 + 6h.$$

Let also $y = c_1(\mathcal{O}_W(1))$. Finally, by B.5.8 in Fulton, we have

$$c(T_{L/G}) = c(U \otimes \mathcal{O}_L(1)) = 1 + (2h - z)$$

so

$$\mathrm{td}(T_{L/G}) = 1 + \frac{1}{2}(2h - z) + \frac{1}{12}(2h - z)^2.$$

So in conclusion we find

$$\mathrm{ch}(p_*(\tilde{q}^*\mathcal{O}_{\mathbb{P}^2}(d) \otimes \mathcal{O}_W(1))) = \tilde{p}_*(\mathrm{td}(T_{L/G})(1 - e^{-[Z]})e^{dh}e^y).$$

Finally, we go from Chern characters to Chern classes using that a vector bundle E satisfy the universal relation

$$c(E) = \exp\left(\sum_{k\geq 1}(k-1)!(-1)^{k-1}\mathrm{ch}_k(E)\right).$$

The result follows now from a direct calculation. But a more direct way is to observe that the solution N_d is polynomial in d of degree ≤ 5. Hence we only need to evaluate finitely many of the numbers N_d to determine the full answer. This can be done using the Chow SAGE package [1]. The code for this is on the webpage `http://www.math.uni-bonn.de/~georgo/topics.html`.

3. Noether-Lefschetz theory

Let $C \subset \mathbb{P}^1 \times \mathbb{P}^2$ be a general hypersurface on degree $(2, 6)$. We have seen above that the locus of curves in the linear system $|\mathcal{O}_{\mathbb{P}^2}(6)|$ which have a tritangent is a divisor of degree 312. Hence in the family of curves

$$C \to \mathbb{P}^1$$

there are precisely 624 points $t \in \mathbb{P}^1$ such that C_t has a tritangent. This can be interpreted in a different way.

Let X be the double cover of $\mathbb{P}^1 \times \mathbb{P}^2$ branched along C, i.e. if $C = V(f)$, then X is given by the equation $V(y^2 = f) \subset \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(-1, -3))$ where $y \in \mathcal{O}(1, 3)$ is a local coordinate. Then morphism

$$X \to \mathbb{P}^1$$

is a family of K3 surfaces polarized by the pullback H of the hyperplane class from $\mathbb{P}^2$. For a fixed t, the pullback of a bitangent of C_t is a rational curve in $|H_t|$, and every rational curve in $|H_t$ is of this form. We have precisely 324 of these rational curves matching the Yau-Zaslow formula. Over a tritangent L of C_t, the restriction $f|_L$ is a square, so the preimage of L to the K3 surface X_t splits as the union of two smooth rational curves:

$$\pi^{-1}(L) = L_1 \cup L_2.$$

Hence a curve C_t with a tritangent corresponds to K3 surfaces X_t whose Picard group contain the lattice

$$\begin{pmatrix} -2 & 3 \\ 3 & -2 \end{pmatrix}.$$

This lattice is of discriminant $\Delta = -\det = 5$. The number of tritangents in the family $C \to \mathbb{P}^1$ is hence equal to the Noether-Lefschetz number

$$N^X_{\Delta=5}$$

of the family X, see [2] for the notation and reference. Since in each such lattice there are precisely 2 curve classes β with $H \cdot \beta = 1$, we have hence the following Noether-Lefschetz number:

$$N^X_{0,1} = 2 \cdot N^X_{\Delta=5} = 2 \cdot 624 = 1248.$$

This matches perfectly the calculation in [2, Section 6.4] (see the coefficient of $q^{5/4} = q^{\Delta/2m}$ where $\Delta = 5$ is the discriminant and $m = 2$ is the polarization degree).

References

[1] Lehn, Sorger, *Chow - A SAGE library for computations in intersection theory*, `https://www.math.sciences.univ-nantes.fr/~sorger/chow_en.html`

[2] D. Maulik, R. Pandharipande, *Gromov-Witten theory and Noether-Lefschetz theory*, A celebration of algebraic geometry, 469–507, Clay Math. Proc., 18, Amer. Math. Soc., Providence, RI, 2013.