Problem List 2, Topics in Enumerative Geometry

Lecturer: Georg Oberdieck

These question are basic review questions about the material in the course, which together with Problem List 1 may help in preparation for the exam.

1. If a smooth complex variety X is affine, do we have that $A_*(X) = \mathbb{Z}[X]$? If not, provide a counterexample, with proof.

2. Does there exist a non-empty scheme X (of finite type over a field k) such that $A_*(X) = 0$?

3. Determine the Chow rings of $\mathbb{P}^2 \setminus p$ and $\mathbb{P}^2 \setminus C$ where p is a point and C is a non-empty curve in $\mathbb{P}^2$ respectively.

4. Describe the Chow ring of the Hirzebruch surface $S = \mathbb{P}(\mathcal{O}(n) \oplus \mathcal{O})$. What is the canonical class of S, what are the classes of the section at zero and infinity?

5. Let E be a vector bundle of rank 2 on a scheme X. Express $c(\mathrm{Sym}^2 E)$ in terms of the Chern classes of E. Similarly, find $c(\wedge^2 F)$ when F is a vector bundle of rank 3. If G is of arbitrary rank r, find $c_1(\wedge^r G)$.

6. Let S be a smooth projective surface. Write the Todd class of S in terms of the Chern classes of S. Use the Hirzebruch-Riemann-Roch formula to give a formula for $\chi(\mathcal{L})$ where $\mathcal{L}$ is a line bundle on S.

7. Consider a point $p \in \mathbb{P}^2$, a curve $C \subset \mathbb{P}^2$ of degree d and the line bundle $\mathcal{L} = \mathcal{O}_{\mathbb{P}^2}(d)$. Find

(1) $\mathrm{ch}(\mathcal{L})$
(2) $\mathrm{ch}(\mathcal{O}_C)$
(3) $\mathrm{ch}(k_p)$, where k_p is the skyscraper sheaf at p.

Check the HRR theorem in these cases explicitly.

8. Let $\pi : \mathbb{P}(\mathcal{O}(n) \oplus \mathcal{O}) \to \mathbb{P}^1$ be the fibration. Determine the rank and the Chern character of $\pi_* T_\pi$ using the Grothendieck-Riemann-Roch formula. Can you also do it directly?

9. Let $X \subset \mathbb{P}^3 \times \mathbb{P}^1$ be a general hypersurface of degree $(4, 1)$, i.e. a general pencil of quartic surfaces. Let $\pi : X \to \mathbb{P}^1$ be the projection to second factor. Determine the rank and the degree of the Hodge bundle $\pi_* \omega_{X/\mathbb{P}^1}$.