Curve counting on $\mathbb{P}^2 \times E$, Hodge integrals over the elliptic curve and quasi-Jacobi forms

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1 Introduction

1.1 PT invariants

Let at first X be any smooth projective threefold.

Definition 1.1. A stable pair (F, s) on X is a coherent sheaf F on X together with a morphism
$$s\colon \mathcal{O}_X \to F$$
so that

(a) every nontrivial subsheaf $G \subset F$ has one-dimensional support

(b) The section s has zero-dimensional cokernel.

For fixed $n \in \mathbb{Z}$ and $\beta \in H_2(X, \mathbb{Z})$, the stable pairs (F, s) satisfying

$$\chi(F) = n \text{ and } [\mathrm{Supp}(F)] = \beta$$

form a projective moduli space $P_n(X, \beta)$, which carries a virtual class

$$[P_n(X, \beta)]^{vir} \in A_{d_\beta}(P_n(X, \beta))$$

in dimension $d_\beta = \int_\beta c_1(X)$. This construction is carried out in detail in [41]. See also [37, 40] for an introduction to stable pairs.
To define descendent invariants, we fix the projections to the two factors:

$$X \xleftarrow{\pi_1} X \times P_n(X, \beta) \xrightarrow{\pi_2} P_n(X, \beta)$$

and the universal stable pair

$$\mathcal{O}_{X \times P_n(X,\beta)} \to \mathbb{F}_n$$

on $X \times P_n(X, \beta)$. We take:

$$\mathrm{ch}_k(\gamma) = (\pi_2)_*(\mathrm{ch}_k(\mathbb{F}_n) \cdot \pi_1^*(\gamma)) \in H^*(P_n(X, \beta))$$

for $k \geq 0$ and $\gamma \in H^*(X)$. Since all stable pairs are supported on curves, we have
$$\mathrm{ch}_k(\gamma) = 0 \text{ unless } k \geq 2.$$

Using this, we can define the Pandharipande-Thomas descendent series by:

$$\left\langle \mathrm{ch}_{k_1}(\gamma_1) \cdots \mathrm{ch}_{k_l}(\gamma_l) \right\rangle_\beta^{X,\mathrm{PT}} = \sum_{n \in \mathbb{Z}} (-1)^n p^{n - d_\beta/2} \int_{[P_n(X,\beta)]^{vir}} \prod_{i=1}^{l} \mathrm{ch}_{k_i}(\gamma_i),$$

for $k_i \geq 0$ and $\gamma_i \in H^*(X)$. We call an invariant *primary* if $k_i = 2$ for all i.

1.2 The Huang-Katz-Klemm conjecture and the case $X = \mathbb{P}^2 \times E$

We now take X to be *Calabi-Yau* threefold i.e. satisfying

$$\omega_X \cong \mathcal{O}_X \text{ and } H^1(X, \mathcal{O}_X) = 0$$

which also admits an elliptic fibration

$$\pi \colon X \to S$$

to a smooth projective surface i.e. π is flat and proper with integral fibers of arithmetic genus 1. We also assume that the fibration has a section

$$\iota \colon S \to X, \ \pi \circ \iota = \mathrm{id}_S \, .$$

Since $d_\beta = 0$ for any $\beta \in H_2(X, \mathbb{Z})$, the virtual class $[P_n(X, \beta)]^{vir}$ is always in degree 0 and so we do not need to deal with descendents. We define the PT generating series for $H \in H_2(S, \mathbb{Z})$ by:

$$\mathrm{PT}_H(p, q) = \sum_{\substack{\beta \in H_2(S,\mathbb{Z}), \\ \pi_* \beta = H}} q^{(\beta, \sigma_*[S] - \pi^* c_1(N_\sigma)/2)} \langle 1 \rangle_\beta^{X, \mathrm{PT}}$$

with N_σ the normal bundle of the section σ. The series $\mathrm{PT}_0(p, q)$ has been determined by Toda in [44, Thm. 6.9].
The following remarkable conjecture is due to Huang, Katz and Klemm: Introducing the power series

$$\Theta(p, q) := (p^{1/2} - p^{-1/2}) \prod_{k \geq 1} \frac{(1 - q^k p)(1 - q^k p^{-1})}{(1 - q^k)^2}$$

and

$$\eta(q) = q^{1/24} \prod_{k \geq 1} (1 - q^k)$$

we have:

Conjecture A. [1][16] For any effective curve class $H \in H_2(S, \mathbb{Z})$ of arithmetic genus

$$h = 1 + \frac{1}{2}(H^2 + K_S \cdot H)$$

we have:

$$\frac{\mathrm{PT}_H(p, q)}{\mathrm{PT}_0(p, q)} = \frac{1}{\eta(q)^{12 c_1(S) \cdot H}} \sum_{\alpha = (H_1, \ldots, H_k)} \frac{\phi_\alpha(p, q)}{\prod_i \Theta(p^{\mathrm{div}(H_i)}, q)^2}$$

where α runs over the decompositions $H = H_1 + \ldots + H_k$ into effective curve classes, $\mathrm{div}(H_i)$ is the divisibility of H_i in $H^{1,1}(S, \mathbb{Z})$ and ϕ_α are weak Jacobi forms of index $h - 1 + \sum_i \mathrm{div}(H_i)^2$ and weight $6 c_1(S) \cdot H - 2k$.

[1] The conjecture above differs slightly from the presentation of [16]. The precise form was suggested by Georg Oberdieck based on ideas of [29, 31].

For the definition of a weak Jacobi form, we refer to Appendix A.
There has been some progress towards proving this conjecture: Most notably, $\mathrm{PT}_H/\mathrm{PT}_0$ has been proven to satisfy the elliptic transformation law of Jacobi forms of index $h-1$ if H is reduced [31].
One might also ask if this conjecture can be extended to the case where X is not Calabi-Yau. The simplest example of this is perhaps

$$\pi_1\colon X = \mathbb{P}^2 \times E \to \mathbb{P}^2$$

whith π_1 the projection to the first factor.
The study of this example will be the primary focus of this thesis.
In this case, the PT virtual class is not always in degree 0, so we need to deal with descendents as well:

Definition 1.2. For $\gamma_1, ...\gamma_n \in H^*(\mathbb{P}^2 \times E)$, $m \geq 0$ and $k_i \geq 0$ we define:

$$\Big\langle \prod_{i=1}^n \mathrm{ch}_{k_i+2}(\gamma_i) \Big\rangle_m^{\mathbb{P}^2\times E,\mathrm{PT}} := q^{-1/8}\eta(q)^3 \sum_{d\geq 0} q^d \Big\langle \prod_{i=1}^n \mathrm{ch}_{k_i+2}(\gamma_i) \Big\rangle_{(m[\mathbb{P}^1],d)}^{\mathrm{PT}}$$

We will often leave out the subscript m as it is already determined by the degree of the insertions $d_{(m[\mathbb{P}^1],d)} = 3m$.
We propose the following analog of the Huang-Katz-Klemm conjecture:

Conjecture B. For any $m \geq 0$ and insertions $\prod_{i=1}^n \mathrm{ch}_{k_i+2}(H^{n_i}\gamma_i)$ with $\gamma_i \in H^*(E)$ homogeneous and $H = c_1(\mathcal{O}_{\mathbb{P}^2}(1)) \in H^*(\mathbb{P}^2)$ the hyperplane class:

$$\Big\langle \prod_{i=1}^n \mathrm{ch}_{k_i+2}(H^{n_i}\gamma_i) \Big\rangle_m^{\mathbb{P}^2\times E,\mathrm{PT}} = \frac{\phi}{\big(\Theta(p)\Theta(p^2)...\Theta(p^m)\big)^N},$$

where $\phi(p,q)$ is a quasi-Jacobi form in the sense of Appendix A.
We also claim that the index of the entire fraction is $m^2/2$ and its weight equals $\sum_i \big(\deg_{\mathbb{R}}(\gamma_i) + k_i\big) - 3m$.

This is arguably not as restrictive as Conjecture A. The following partial evidence will be the main result of this thesis:

Theorem 1.3. Conjecture B holds for all primary insertions:

(a) in degree $m = 0, 1$

(b) in degree $m = 2$ if $n_i = 2$ for at least one i.

We list some examples which we compute in this thesis:

$$\big\langle \mathrm{ch}_2(H^2\mathrm{pt})\mathrm{ch}_2(H^2)\big\rangle_1^{\mathrm{PT}} = \Theta$$
$$\big\langle \mathrm{ch}_2(H^2\mathrm{pt})\mathrm{ch}_2(H\mathrm{pt})\big\rangle_1^{\mathrm{PT}} = 3D_\tau\Theta = \Theta\Big(\frac{3}{2}A^2 + 6G_2 - \frac{3}{2}\wp\Big)$$

$$\left\langle \mathrm{ch}_2(H\mathrm{pt})^3 \right\rangle_1^{\mathrm{PT}} = \Theta\left(3\frac{D_\tau^2\Theta}{\Theta} + 9\left(\frac{D_\tau\Theta}{\Theta}\right)^2\right)$$

$$\left\langle \mathrm{ch}_2(H^2\mathrm{pt})^3 \right\rangle_2^{\mathrm{PT}} = \Theta^4\left(-\frac{3}{4}A^4 + \frac{9}{2}A^2\wp + \frac{9}{4}\wp^2 + 3A\wp' - 15G_4\right)$$

The third invariant is the only one of these which is not quasi-Jacobi. In particular, one can make N arbitrarily big by adding $\mathrm{ch}_2(\mathrm{pt})$-insertions to the third invariant and using the divisor equation repeatedly.
Furthermore, the numerator may fail to be a weak Jacobi form which is to say that it may involve the two generators A and G_2 defined in Appendix A.
To measure this failure we also propose so called *holomorphic anomaly equations* for $\mathbb{P}^2 \times E$, which control the dependence on A and G_2:

Conjecture C. In the situation of Conjecture B with $n_i \geq 1$:

$$\begin{aligned}
&\frac{d}{dA}\left\langle \prod_{i=1}^n \mathrm{ch}_{k_i+2}(H^{n_i}\gamma_i) \right\rangle_m^{\mathrm{PT}} \\
&= \sum_{i=1}^n \left\langle \prod_{j=1}^n \mathrm{ch}_{k_j-\delta_{i,j}+2}(H^{n_j}\gamma_j)\mathrm{ch}_2(H^2) \right\rangle_m^{\mathrm{PT}} \\
&+ m\sum_{i=1}^n \left\langle \prod_{j=1}^n \mathrm{ch}_{k_j-\delta_{i,j}+2}(H^{n_j+\delta_{i,j}}\gamma_j) \right\rangle_m^{\mathrm{PT}} \\
&+ \sum_{i=1}^n \left(\int_E \gamma_i\right)\left\langle \prod_{j=1}^n \mathrm{ch}_{k_j+\delta_{i,j}+2}(H^{n_j}\gamma_j^{1-\delta_{i,j}}) \right\rangle_m^{\mathrm{PT}}
\end{aligned}$$

and if $\gamma_i \in \mathcal{B} := \{1, \alpha, \beta, \mathrm{pt}\}$ (see Chapter 4) for all i:

$$\begin{aligned}
&\left(\frac{d}{dG_2}\right)_{\mathrm{QJac}}\left\langle \prod_{i=1}^n \mathrm{ch}_{k_i+2}(H^{n_i}\gamma_i) \right\rangle_m^{\mathrm{PT}} \\
=&6\sum_{i=1}^n \frac{1}{k_i+1}\left(\int_E \gamma_i\right)\left\langle \prod_{j=1}^n \mathrm{ch}_{k_j+2}(H^{n_j+\delta_{i,j}}\gamma_j^{1-\delta_{i,j}}) \right\rangle_m^{\mathrm{PT}} \\
&-\sum_{i,j=1}^n \left\langle \prod_{l=1}^n \mathrm{ch}_{k_l-\delta_{i,l}-\delta_{j,l}+2}(H^{n_l+\delta_{i,l}+\delta_{j,l}}\gamma_l) \right\rangle_m^{\mathrm{PT}} \\
&-3\sum_{i=1}^n \sigma_i\delta_{n_i,1} \sum_{\substack{\gamma_a,\gamma_b\in\mathcal{B},\\ \pm\gamma_a\cup\gamma_b=\gamma_i,\\ m_1+m_2=k_i}} \pm\frac{(m_1+a_{\gamma_a})!(m_2+a_{\gamma_b})!}{(k_i+a_{\gamma_i})!} \\
&\left\langle \mathrm{ch}_{m_1+1}(H^2\gamma_a)\mathrm{ch}_{m_2+1}(H^2\gamma_b)\prod_{\substack{j=1\\ j\neq i}}^n \mathrm{ch}_{k_j+2}(H^{n_j}\gamma_j) \right\rangle_m^{\mathrm{PT}}
\end{aligned}$$

$$- 2 \sum_{1 \leq i < j \leq n} \sigma_{i,j} \binom{k_i + k_j + a_{\gamma_i} + a_{\gamma_j}}{k_i + a_{\gamma_i}, k_j + a_{\gamma_j}}$$

$$\left\langle \mathrm{ch}_{k_i+k_j+2}\big(H^{n_i+n_j} \frac{\gamma_i \cup \gamma_j}{\mathrm{pt}}\big) \prod_{\substack{l=1 \\ l \neq i,j}}^{n} \mathrm{ch}_{k_l+2}(H^{n_l}\gamma_l) \right\rangle_m^{\mathrm{PT}}$$

where σ_i (or $\sigma_{i,j}$) is the sign that comes from supercommuting γ_i (or γ_i and γ_j) all the way to the left and $a_\gamma = \deg_{\mathbb{R}}(\gamma) - 1$. Furthermore,

$$\frac{\gamma \cup \gamma'}{\mathrm{pt}} = \begin{cases} \pm 1, & \text{if } \gamma \cup \gamma' = \pm \mathrm{pt}, \\ \gamma, & \text{if } \gamma' = \mathrm{pt}, \\ \gamma', & \text{if } \gamma = \mathrm{pt}, \\ 0, & \text{else} \end{cases}$$

and the binomial coefficient in the last summand is set to zero if one of the two lower numbers is negative.

Remark 1.4. The operators $\frac{d}{dA}$ and $(\frac{d}{dG_2})_{\mathrm{QJac}}$ are just the formal derivatives in the polynomial ring $\mathbb{Q}[\Theta, A, G_2, \wp, \wp', G_4]$ with respect to the two generators. E.g.:

$$\frac{d}{dA}\Big(3A^4 + 18A^2G_2 + 36G_2^2 - 3A\wp' + 15G_4\Big) = 12A^3 + 36AG_2 - 3\wp'$$

Holomorphic anomaly equations have not been considered in PT theory before. Instead we justify both statements using holomorphic anomaly equations in Gromov-Witten theory [30] and the conjectural Gromov-Witten/PT correspondence which is studied in [27] and [32]. See Chapter 2 for details. The proven part of this correspondence gives us:

Theorem 1.5. Conjecture C holds for all primary insertions as in Theorem 1.3 where in addition

(a) $n_i \geq 1$ for all i and

(b) $n_i = 2$ if $\gamma_i \in H^2(E)$.

Remark 1.6. The second holomorphic anomaly equation vanishes if $\gamma_i = H^2$ for all i. Hence any invariant

$$\left\langle \prod_{i=1}^{n} \mathrm{ch}_2(H^2\gamma_i) \right\rangle_m^{\mathrm{PT}}$$

in degree $m = 0, 1, 2$ must be independent of G_2 and also independent of A if $\sum_i \deg_{\mathbb{R}}(\gamma_i) = 2$. See also Remarks 5.7 and 5.16 and Example 5.17.

1.3 Gromov-Witten invariants

In this section we recall the basic definitions of Gromov-Witten invariants. Basic references include [2], [6] and [13].
Throughout this section let X be a complex smooth projective variety.
Recall that for any cohomology class $\beta \in H_2(X, \mathbb{Z})$ and integers $g, n \geq 0$ we can construct the moduli space $\overline{M}_{g,n}(X, \beta)$ whose $\mathbb{C}$-valued points are given by isomorphism classes of *stable* morphisms $f : (C, p_1, ..., p_n) \to X$ with connected domain curve C.
As a moduli stack, it has a universal family of stable maps to X:

$$\begin{tikzcd} U_{g,n}(X,\beta) \arrow[r, "f"] \arrow[d, "\pi"'] & X \\ \overline{M}_{g,n}(X,\beta) \arrow[u, bend right, dotted, "p_i"'] & \end{tikzcd}$$

We will denote $ev_i = f \circ p_i$ from now on. Moreover, using the work of Behrend and Fantechi [3] one can define a virtual fundamental class:

$$[\overline{M}_{g,n}(X,\beta)]^{vir} \in A_{vdim}(\overline{M}_{g,n}(X,\beta))$$

sitting in degree *vdim* with:

$$vdim = d_\beta + (1-g)(\dim X - 3) + n$$

We further define the *psi classes* $\psi_i := c_1(p_i^* \Omega_\pi^1) \in H^{2i}(\overline{M}_{g,n}(X,\beta))$.

Definition 1.7. For $\gamma_1, \cdots, \gamma_n \in H^*(X)$, $k_1, ..., k_n \geq 0$ and $\gamma \in H^*(\overline{M}_{g,n}(X,\beta))$, we define the *connected Gromov-Witten invariant* of genus g to be:

$$\left\langle \gamma \prod_{i=1}^n \tau_{k_i}(\gamma_i) \right\rangle_{g,\beta}^X := \int_{[\overline{M}_{g,n}(X,\beta)]^{vir}} \gamma \prod_{i=1}^n \psi_i^{k_i} ev_i^*(\gamma_i)$$

If $2g - 2 + n > 0$, the associated *Gromov-Witten class* is the pushforward

$$I_{g,\beta}^X\left(\prod_{i=1}^n \tau_{k_i}(\gamma_i) \right) := p_*\left([\overline{M}_{g,n}(X,\beta)]^{vir} \cap \prod_{i=1}^n \psi_i^{k_i} ev_i^*(\gamma_i) \right) \in H^*(\overline{M}_{g,n})$$

along the forgetful map $p \colon \overline{M}_{g,n}(X,\beta) \to \overline{M}_{g,n} := \overline{M}_{g,n}(\mathrm{pt}, 0)$. In case X is a threefold, we define the following formal power series:

$$\left\langle \prod_{i=1}^n \tau_{k_i}(\gamma_i) \right\rangle_{\beta}^X := \sum_{g \geq 0} (-1)^{g-1+d_\beta} z^{2g-2+d_\beta} \left\langle \prod_{i=1}^n \tau_{k_i}(\gamma_i) \right\rangle_{g,\beta}^X \in \mathbb{Q}((z))$$

To define disconnected Gromov-Witten invariants, let

$$\overline{M}^\bullet_{g,n}(X,\beta)$$

be the moduli space of stable maps $f\colon (C, p_1, ..., p_n) \to X$ of genus $g \in \mathbb{Z}$ and class β with possibly disconnected domain so that all connected components which are contracted by f are of genus at most 1^2. Likewise, we have psi classes and a virtual fundamental class

$$[\overline{M}^{\bullet}_{g,n}(X,\beta)]^{vir} \in A_{vdim}(\overline{M}^{\bullet}_{g,n}(X,\beta)) \otimes \mathbb{Q}$$

in the same degree $vdim$ as before, which allows us to define:

Definition 1.8. For $\gamma_1, \cdots, \gamma_n \in H^*(X)$ and $k_1, ..., k_n \geq 0$ we define the *disconnected Gromov-Witten invariant* of genus $g \in \mathbb{Z}$ to be:

$$\left\langle \prod_{i=1}^n \tau_{k_i}(\gamma_i) \right\rangle_{g,\beta}^{X,\bullet} := \int_{[\overline{M}^{\bullet}_{g,n}(X,\beta)]^{vir}} \prod_{i=1}^n \psi_i^{k_i} ev_i^*(\gamma_i)$$

In case X is a threefold, we set:

$$\left\langle \prod_{i=1}^n \tau_{k_i}(\gamma_i) \right\rangle_{\beta}^{X,\bullet} := \sum_{g\in\mathbb{Z}} (-1)^{g-1+d_\beta} z^{2g-2+d_\beta} \left\langle \prod_{i=1}^n \tau_{k_i}(\gamma_i) \right\rangle_{g,\beta}^{X,\bullet}$$

1.4 The conjecture in Gromov-Witten theory

Definition 1.9. For any choice of cohomology classes $\gamma_1, ..., \gamma_n \in H^*(\mathbb{P}^2 \times E)$ and $k_i \geq 0$, $m \geq 0$, we define

$$\left\langle \prod_{i=1}^n \tau_{k_i}(\gamma_i) \right\rangle_m := \sum_{d\geq 0} q^d \left\langle \prod_{i=1}^n \tau_{k_i}(\gamma_i) \right\rangle_{(m[\mathbb{P}^1],d)}^{\mathbb{P}^2\times E}$$

and for disconnected invariants:

$$\left\langle \prod_{i=1}^n \tau_{k_i}(\gamma_i) \right\rangle_m^{\bullet} := q^{-1/8}\eta(q)^3 \sum_{d\geq 0} q^d \left\langle \prod_{i=1}^n \tau_{k_i}(\gamma_i) \right\rangle_{(m[\mathbb{P}^1],d)}^{\mathbb{P}^2\times E,\bullet}$$

As for PT invariants, we may dispense with the subscript m since it is determined by the insertions.

Using this, we can restate Conjecture B in Gromov-Witten theory:

Conjecture D. For any $m \geq 0$, $n_i \geq 0$ and insertions $\prod_{i=1}^n \tau_{k_i}(H^{n_i}\gamma_i)$ with $\gamma_i \in H^*(E)$ homogeneous:

$$\left\langle \prod_{i=1}^n \tau_{k_i}(H^{n_i}\gamma_i) \right\rangle_m^{\mathbb{P}^2\times E,\bullet} = \frac{\phi}{\left(\Theta(z)\Theta(2z)...\Theta(mz)\right)^N},$$

[2]The usual convention is to have no contracted components at all. We only stray from this to be compatible with [27]. In Lemma 5.1 we see that this is already automatic for $X = \mathbb{P}^2 \times E$.

where $\phi(z)$ is in $\mathrm{QJac}[z,z^{-1}]$ of degree at most $\sum_i k_i$ in z and Θ is as in Appendix A.
We also claim that the index of the entire fraction is $m^2/2$ and its weight equals $\sum_i \deg_{\mathbb{R}}(\gamma_i) - 3m$.

We have the following Theorem will be used later to prove Theorem 1.3:

Theorem 1.10. Conjecture D holds for all:

(a) invariants in degree $m=0$,

(b) primary invariants in degree $m=1$,

(c) primary invariants in degree $m=2$ if $n_i=2$ for at least one i.

We will also give examples of descendent invariants - all of which satisfy Conjecture D. Here is a list of some invariants that will be computed in Chapter 5:

$$
\begin{aligned}
&\big\langle \tau_0(H^2\mathrm{pt})\tau_0(H^2)\big\rangle_1^\bullet = \Theta \\
&\big\langle \tau_0(H^2\mathrm{pt})\tau_0(H\mathrm{pt})\big\rangle_1^\bullet = \Theta\big(\frac{1}{2}A^2 + 2G_2 - \frac{1}{2}\wp\big) \\
&\big\langle \tau_0(H^2\mathrm{pt})\tau_2(1)\big\rangle_1^\bullet = \Theta\big(9 - 8zA - \frac{1}{2}z^2(-3A^2 + 18G_2 + 3\wp)\big) \\
&\big\langle \tau_0(H^2\mathrm{pt})^3\big\rangle_2^\bullet = \Theta^4\Big(-\frac{3}{4}A^4 + \frac{9}{2}A^2\wp + \frac{9}{4}\wp^2 + 3A\wp' - 15G_4\Big) \\
&\big\langle \tau_0(H^2\mathrm{pt})^2\tau_1(H^2)\big\rangle_2^\bullet = z\Theta^4\Big(-A^3 + 3A\wp + \wp'\Big)
\end{aligned}
$$

Overview. In chapter 2, we will review the Gromov-Witten/PT correspondence for toric threefolds discovered in [27, Sec. 0.6] and show how one can derive Conjecture C from their formulas and Proposition 2.4. Next, we review basic material on the localization formula and the Gromov-Witten theory of the elliptic curve in order to set up our proof of Theorem 1.10 in Chapter 5. Appendix A introduces quasi-Jacobi and quasi-modular forms and Appendix B deals with some of the vertex terms that arise in the localization formula in degree $m=2$. Appendix C gives a complete list of all Gromov-Witten invariants on $\mathbb{P}^2\times E$ which are computed here.

2 The Gromov-Witten/PT correspondence and holomorphic anomaly equations

In this section, we review Gromov-Witten/PT correspondence for toric threefolds presented in [27, Sec. 0.6] and use it to derive the formulas of Conjecture C. A proof of this correspondence in the case $X=\mathbb{P}^2\times E$ would therefore yield a proof of Conjecture 4.5.

2.1 The Gromov-Witten/PT correspondence

In the last two decades, Rahul Pandharipande, with various changing coauthors, has published a series of papers addressing the relationship between Gromov-Witten and PT invariants in ever increasing detail. The most recent of these is [27], where the following formalism appears:
Let X be any smooth projective threefold. For $\gamma \in H^{\geq 2}(X)$ and $k \geq 1$, we introduce formal insertions $\mathfrak{a}_k(\gamma)$ using the recursive rule

$$\tau_0(\gamma) = \mathfrak{a}_1(\gamma) + \frac{1}{24}\int_X \gamma c_2$$

and for $k \geq 2$:

$$\begin{aligned}\tau_{k-1}(\gamma) =& \frac{z^{k-1}}{k!}\mathfrak{a}_k(\gamma) - \frac{z^{k-2}}{(k-1)!}\Big(\sum_{i=1}^{k-1}\frac{1}{i}\Big)\mathfrak{a}_{k-1}(\gamma\cdot c_1)\\ &+ \frac{z^{k-3}}{(k-2)!}\Big(\sum_{i=1}^{k-2}\frac{1}{i^2} + \sum_{1\leq i<j\leq k-2}\frac{1}{ij}\Big)\mathfrak{a}_{k-2}(\gamma\cdot c_1^2)\end{aligned}$$

where $c_i = c_i(TX)$ and z is the formal variable in Definition 1.7.
We will use the convention:

$$\mathfrak{a}_{k_1}...\mathfrak{a}_{k_l}(\gamma) := \sum_i \mathfrak{a}_{k_1}(\gamma_1^i)...\mathfrak{a}_{k_l}(\gamma_l^i)$$

where

$$\sum_i \gamma_1^i \otimes ... \otimes \gamma_l^i = \Delta \cdot ev_1^*(\gamma) \in H^*(X^l)$$

is the Künneth decomposition and Δ is the small diagonal in $H^*(X^l)$.
We also have to change the PT-insertions slightly to make the correspondence more natural:

$$\widetilde{\mathrm{ch}}_k(\gamma) = \mathrm{ch}_k(\gamma) + \frac{1}{24}\mathrm{ch}_{k-2}(\gamma\cdot c_2)$$

The series

$$\Big\langle \widetilde{\mathrm{ch}}_{k_1}(\gamma_1)\cdots\widetilde{\mathrm{ch}}_{k_l}(\gamma_l)\Big\rangle_\beta^{X,\mathrm{PT}}$$

is then defined by multilinearity.

Using this convention, we can define the following operator on PT insertions:
For $\gamma_1, ..., \gamma_l \in H^*(X)$ and $k_1, ..., k_n \geq 0$:

$$\mathfrak{C}^\bullet\Big(\prod_{i=1}^{l}\widetilde{\mathrm{ch}}_{k_i}(\gamma_i)\Big) = \sum_{P \text{ partition of } \{1,...,l\}}\prod_{S\in P}\mathfrak{C}^\circ\Big(\prod_{i\in S}\widetilde{\mathrm{ch}}_{k_i}(\gamma_i)\Big),$$

where the set partitions are unordered and no member is allowed to be empty. The $\mathfrak{C}^\circ$ terms are defined in the following way:

(1)

$$\mathfrak{C}^\circ\Big(\widetilde{\mathrm{ch}}_{k_1+2}(\gamma)\Big) = \frac{1}{(k_1+1)!}\mathfrak{a}_{k_1+1}(\gamma) + \frac{z^{-1}}{2\cdot k_1!}\sum_{m_1+m_2=k_1-1}\mathfrak{a}_{m_1}\mathfrak{a}_{m_2}(\gamma\cdot c_1)$$

$$+\frac{z^{-2}}{2\cdot k_1!}\sum_{m_1+m_2=k_1-2}\mathfrak{a}_{m_1}\mathfrak{a}_{m_2}(\gamma\cdot c_1^2)+\frac{z^{-2}}{6\cdot(k_1-1)!}\sum_{m_1+m_2+m_3=k_1-3}\mathfrak{a}_{m_1}\mathfrak{a}_{m_2}\mathfrak{a}_{m_3}(\gamma\cdot c_1^2),$$

(2)

$$\mathfrak{C}^\circ\Big(\widetilde{\mathrm{ch}}_{k_1+2}(\gamma)\widetilde{\mathrm{ch}}_{k_2+2}(\gamma')\Big) = -\frac{z^{-1}}{k_1!k_2!}\mathfrak{a}_{k_1+k_2}(\gamma\gamma') - \frac{z^{-2}}{k_1!k_2!}\mathfrak{a}_{k_1+k_2-1}(\gamma\gamma'\cdot c_1)$$

$$-\frac{z^{-2}}{2\cdot k_1!k_2!}\sum_{m_1+m_2=k_1+k_2-2}\max(\max(k_1,k_2),\max(m_1+1,m_2+1)\mathfrak{a}_{m_1}\mathfrak{a}_{m_2}(\gamma\gamma'\cdot c_1),$$

(3)

$$\mathfrak{C}^\circ\Big(\widetilde{\mathrm{ch}}_{k_1+2}(\gamma)\widetilde{\mathrm{ch}}_{k_2+2}(\gamma')\widetilde{\mathrm{ch}}_{k_3+2}(\gamma'')\Big) = \frac{z^{-2}|k|}{k_1!k_2!k_3!}\mathfrak{a}_{|k|-1}(\gamma\gamma'\gamma''),\ |k| = k_1+k_2+k_3$$

and

$$\mathfrak{C}^\circ\Big(\prod_{i=1}^{l}\widetilde{\mathrm{ch}}_{k_i+2}(\gamma_i)\Big) = 0 \text{ for } l\geq 4. \tag{4}$$

Here, all occurences of $\mathfrak{a}_{\leq 0}$ are set to zero.

Theorem 2.1. [27, Thm. 6] Let X be toric and $\beta\in H_2(X,\mathbb{Z})$ be a curve class with $d_\beta = \int_\beta c_1(X)$. For all insertions

$$\prod_{i=1}^{l}\widetilde{\mathrm{ch}}_{k_i}(\gamma_i)$$

satisfying for all i:

$$k_i\geq 3, \deg_{\mathbb{R}}(\gamma_i)\geq 2 \text{ or } k_i = 2, \deg_{\mathbb{R}}(\gamma_i)\geq 4, \tag{5}$$

we have

$$\Big\langle\prod_{i=1}^{l}\widetilde{\mathrm{ch}}_{k_i}(\gamma_i)\Big\rangle_\beta^{X,\mathrm{PT}} = \Big\langle\mathfrak{C}^\circ\Big(\prod_{i=1}^{l}\widetilde{\mathrm{ch}}_{k_i}(\gamma_i)\Big)\Big\rangle_\beta^{X,GW,\bullet}$$

after the variable change $p=e^z$.

Note that the above variable change only makes sense in context of:

Conjecture E. [41] For any X and insertions $\gamma_i \in H^*(X)$ and $k_i \geq 0$, the series

$$\left\langle \mathrm{ch}_{k_1}(\gamma_1) \cdots \mathrm{ch}_{k_l}(\gamma_l) \right\rangle_\beta^{X,PT}$$

is the Laurent expansion of a rational function in p.

The conjecture was proven to be true for a large variety of examples including all toric threefolds and $X = \mathbb{P}^2 \times E$ [38, 39]. It was conjectured in [32] that a similar formalism should hold for any threefold X and all insertions. The following conjecture encompasses all that we will need:

Conjecture F. For all Gromov-Witten insertions $\prod_{i=1}^n \tau_{k_i}(\gamma_i)$ we have:

(a) [24, Conj. 4] Let $\overline{M}'_{g,n}(X,\beta)$ be the moduli space of stable maps with *no* contracted components. The corresponding Gromov-Witten invariant

$$\left\langle \prod_{i=1}^n \tau_{k_i}(\gamma_i) \right\rangle'_\beta \in \mathbb{Q}[[z]]$$

is an element of $\mathbb{Q}(e^z)[z, z^{-1}]$ of degree at most $\sum_i k_i$ in z, whose z-coefficients are (universal) $\mathbb{Q}$-linear combinations of PT Invariants for the curve class β. The coefficient of $z^{\sum_i k_i}$ is:

$$\left\langle \prod_{i=1}^n \mathrm{ch}_{k_i+2}(\gamma_i) \right\rangle_\beta^{X,\mathrm{PT}}$$

and all lower coefficients consist of PT invariants with a strictly lower sum $\sum_i k_i$.

(b) If we have $\deg_{\mathbb{R}}(\gamma_i) \geq 2$ and $\gamma_i \cdot c_2 = 0$ for all i, then the top two z-coefficients are:

$$\left\langle \prod_{i=1}^n \tau_{k_i}(\gamma_i) \right\rangle_\beta^\bullet = z^{\sum_i k_i} \left\langle \prod_{i=1}^n \mathrm{ch}_{k_i+2}(\gamma_i) \right\rangle_\beta^{\mathrm{PT}} \tag{6}$$

$$+ z^{\sum_i k_i - 1} \Bigg[-\sum_{i=1}^n \Big(\sum_{l=1}^{k_i} \frac{1}{l} \Big) \left\langle \prod_{j=1}^n \mathrm{ch}_{k_j - \delta_{i,j}+2}(\gamma_j \cdot c_1^{\delta_{i,j}}) \right\rangle_\beta^{\mathrm{PT}}$$

$$-\frac{1}{2} \sum_{i=1}^n \left\langle \prod_{j=1}^n \mathrm{ch}_{k_j+2}(\gamma_j)^{1-\delta_{i,j}} \Big(\sum_{\substack{m_1+m_2=k_i-1,\\ m_1,m_2>0}} \frac{m_1! m_2!}{k_i!} \mathrm{ch}_{m_1+1}\mathrm{ch}_{m_2+1}(\gamma_i \cdot c_1) \Big)^{\delta_{i,j}} \right\rangle_\beta^{\mathrm{PT}}$$

$$+ \sum_{1 \leq i < j \leq n} \frac{(k_i+k_j)!}{k_i! k_j!} \sigma_{i,j} \left\langle \prod_{\substack{l=1\\ l \neq i,j}} \mathrm{ch}_{k_l+2}(\gamma_l) \mathrm{ch}_{k_i+k_j+1}(\gamma_i \gamma_j) \right\rangle_\beta^{\mathrm{PT}} \Bigg]$$

here $\sigma_{i,j}$ is the sign that ensues from supercommuting γ_i and γ_j all the way to the right.[3]

In particular, part (a) implies the equivalence of Conjectures B and D since the difference between the two disconnected moduli spaces is minor. The following special case of Conjecture F has already been shown:

Theorem 2.2. For $X = \mathbb{P}^2 \times E$, part (a) holds for all primary insertions. Furthermore, for any insertion of the form $\prod_{i=1}^n \tau_{k_i}(H^{n_i}\gamma_i)$ with $n_i \geq 1$ for any i and $n_i = 2$ if $k_i > 0$, we have:

$$\left\langle \prod_{i=1}^n \tau_{k_i}(H^{n_i}\gamma_i) \right\rangle_m^{\mathbb{P}^2\times E,\bullet} = z^{\sum_i k_i} \left\langle \prod_{i=1}^n \mathrm{ch}_{k_i+2}(H^{n_i}\gamma_i) \right\rangle_m^{\mathbb{P}^2\times E,\mathrm{PT}}$$

Proof. In the second case it is easy to see that the Gromov-Witten invariant does not change if it is integrated over $\overline{M}'_{g,n}(\mathbb{P}^2 \times E, (m,d))$, which is the convention taken in [38, 39]. The result then follows from the Gromov-Witten/PT correspondence for a surface times elliptic curve proven in [38, Prop. 20] and the degree constraint of the correspondence matrix in [39, Prop. 24]. $\square$

And hence we obtain:

Corollary 2.3. Theorem 1.3 follows from Theorem 1.10.

$\square$

2.2 Holomorphic anomaly equations

There seem to be no natural holomorphic anomaly equations for the generators A and G_2 on the Gromov-Witten side of $X = \mathbb{P}^2 \times E$. However, we have the following:

Proposition 2.4. For all $\gamma_i \in H^*(E)$ homogeneous, $n_i \geq 1$ and $k_i \geq 0$, we

[3]Thanks to Alexei Oblomkov for confirming that formulas (1) - (3) should hold for any X and γ_i with $\deg_{\mathbb{R}}(\gamma_i) \geq 2$ up to correction terms consisting of $\mathfrak{a}$-insertions times z^i with $i \leq -2$. From this, we can deduce equation (6)

have:

(7)
$$\left(\frac{d}{dG_2}\right)_P \left\langle \prod_{i=1}^n \tau_{k_i}(H^{n_i}\gamma_i) \right\rangle_m^\bullet = -2z^2 \sum_{j=1}^n \left\langle \prod_{i=1}^n \tau_{k_i-\delta_{ij}}(H^{n_i}\gamma_i)\tau_0(H^2) \right\rangle_m^\bullet$$
$$+ 2 \sum_{\substack{1\le i<j\le n \\ k_i=k_j=0}} \operatorname{sgn}(i,j)\delta_{n_i+n_j,2}\left(\int_E \gamma_i\gamma_j\right)\left\langle \prod_{\substack{l=1\\ l\neq i,j}}^n \tau_{k_l}(H^{n_l}\gamma_l)\tau_0(H^2)\right\rangle_m^\bullet$$
$$- z^2m^2\left\langle \prod_{i=1}^n \tau_{k_i}(H^{n_i}\gamma_i)\right\rangle_m^\bullet - 2mz^2\sum_{i=1}^n \left\langle \prod_{j=1}^n \tau_{k_j-\delta ij}(H^{n_j+\delta_{ij}}\gamma_j)\right\rangle_m^\bullet$$
$$- z^2 \sum_{i,j=1}^n \left\langle \prod_{l=1}^n \tau_{k_l-\delta_{il}-\delta_{jl}}(H^{n_l+\delta_{il}+\delta_{jl}}\gamma_l)\right\rangle_m^\bullet$$
$$- 2\sum_{i=1}^n \left(\int_E \gamma_i\right)\left\langle \prod_{j=1}^n \tau_{k_j+\delta_{ij}}(H^{n_j}\gamma_j^{1-\delta_{ij}})\right\rangle_m^\bullet,$$

where $\operatorname{sgn}(i,j)$ is the sign that arises from supercommuting the ith and jth insertions all the way to the right (or the left).

Remark 2.5. The invariant
$$\left\langle \prod_{i=1}^n \tau_{k_i}(H^{n_i}\gamma_i)\right\rangle_m^{\mathbb{P}^2\times E,\bullet}$$
is an element of $\mathrm{QMod}((z))$ as shown in [30, Cor. 2] and the derivative $(\frac{d}{dG_2})_P$ is the coefficient-wise formal derivative with respect to the generator G_2 of $\mathrm{QMod} = \mathbb{Q}[G_2, G_4, G_6]$.

Proof. This formula follows from [30, Conj. B, Cor. 2] and [29, Lemma* 8]. The prefactor of $q^{-1/8}\eta(q)^3$ gets rid of the contributions from unstable genus 1 components mapping with degree $(0, d)$ (we show this in Lemma 5.1). Hence our conventions are compatible with the ones used in [29, Sec. 3.2]. We get the rest from the divisor and string equations - however the string equation takes on a strange form in our case:
$$\left\langle \prod_{i=1}^n \tau_{k_i}(H^{n_i}\gamma_i)\tau_0(1)\right\rangle_m^\bullet = \sum_{i=1}^n \left\langle \prod_{j=1}^n \tau_{k_j-\delta ij}(H^{n_j}\gamma_j)\right\rangle_m^\bullet$$
$$- z^{-2} \sum_{\substack{1\le i<j\le n \\ k_i=k_j=0}} \operatorname{sgn}(i,j)\left(\int_E \gamma_i\gamma_j\right)\delta_{n_1+n_2,2}\left\langle \prod_{\substack{l=1\\ l\neq i,j}}^n \tau_{k_l}(H^{n_l}\gamma_l)\right\rangle_m^\bullet.$$

On the other hand, the divisor equation for $\tau_0(H)$ happens to retain its usual form. □

Assuming conjecture F from now on, we can give the reason why the weight and index in Conjectures B and D can only be as claimed. If we assume that any PT invariant is in QJac, then the weight follows directly from Theorem 4.4 and if $n_i \geq 1$ for all i, then we get the index in the following way:
In the ring $\mathrm{QJac}[z, z^{-1}, \Theta(z)^{-1}, \Theta(2z)^{-1}, ...]$, we have:

$$\left(\frac{d}{dG_2}\right)_P = -2z^2 ind - 2z\frac{d}{dA} + \left(\frac{d}{dG_2}\right)_{\mathrm{QJac}},$$

where ind multiplies a form with its index.
Both sides in Proposition 2.4 are Laurent polynomials in z over $\mathbb{Q}(e^z)[[q]]$ of degree at most $\sum_i k_i + 2$. The leading coefficients are:

$$-2ind\left\langle \prod_{i=1}^n \mathrm{ch}_{k_i+2}(H^{n_i}\gamma_i)\right\rangle_\beta = -m^2\left\langle \prod_{i=1}^n \mathrm{ch}_{k_i+2}(H^{n_i}\gamma_i)\right\rangle_\beta$$

and hence the index is $m^2/2$ as claimed before.
Using this same idea, we now derive the holomorphic anomaly equations for $\mathbb{P}^2 \times E$:
Since all coefficients in the Laurent polynomial

$$\left\langle \prod_{i=1}^n \tau_{k_i}(H^{n_i}\gamma_i)\right\rangle_m^\bullet \in \mathrm{QJac}[z, z^{-1}]$$

have the same index $m^2/2$, the first term in the left hand side of Proposition 2.4 i.e.:

$$\begin{aligned}&-2z^2 ind\left\langle \prod_{i=1}^n \tau_{k_i}(H^{n_i}\gamma_i)\right\rangle_m^\bullet - 2z\frac{d}{dA}\left\langle \prod_{i=1}^n \tau_{k_i}(H^{n_i}\gamma_i)\right\rangle_m^\bullet \\ &+ \left(\frac{d}{dG_2}\right)_{\mathrm{QJac}}\left\langle \prod_{i=1}^n \tau_{k_i}(H^{n_i}\gamma_i)\right\rangle_m^\bullet\end{aligned}$$

cancels exactly with the third term on the right hand side. Now both sides are only of degree $\sum_i k_i + 1$ and the top coefficients give the holomorphic anomaly equation for $\frac{d}{dA}$. For the second equation, we have to look at the coefficient of $z^{\sum_i k_i}$, which on the left hand side is

$$\left(\frac{d}{dG_2}\right)_{\mathrm{QJac}}\left\langle \prod_{i=1}^n \mathrm{ch}_{k_i}(H^{n_i}\gamma_i)\right\rangle_m^{\mathrm{PT}} - 2z\frac{d}{dA}\left(\left[z^{\sum_i k_i - 1}\right]\left\langle \prod_{i=1}^n \tau_{k_i-2}(H^{n_i}\gamma_i)\right\rangle_m^\bullet\right)$$

Using our equation for $\frac{d}{dA}$ and (6), we can compute the second term as well as the right hand side. Since this is a tedious and straightforward computation, we do not spell it out here.
In particular, Theorem 1.5 is shown using the above discussion and Theorem 2.2.

3 Localization and the Gromov-Witten classes of $\mathbb{P}^n$

The goal of this chapter is to give an algorithm for computing the Gromov-Witten classes of $\mathbb{P}^n$, which will be of great importance in the discussion that follows. To this end we give a short introduction to equivariant cohomology and the localization formula.

3.1 Equivariant cohomology

We recall the definitions and basic properties of equivariant cohomology that are needed in this thesis. We mostly follow Lecture one in [1]. More details can be found in [11].

Definition 3.1. Let X be a topological space and T a topological group acting continuously on X. The equivariant cohomology ring of X is defined to be

$$H_T^*(X) := H^*(\mathbb{E}\mathrm{T} \times_{\mathrm{T}} X; \mathbb{Q})$$

where we take $\mathbb{E}\mathrm{T}$ to be a weakly contractible space with a free T-action on it and $\mathbb{E}\mathrm{T} \times_{\mathrm{T}} X$ is the quotient of $\mathbb{E}\mathrm{T} \times X$ by the equivalence relation $(e, tx) \sim (et, x)$ for any $x \in X, e \in \mathbb{E}\mathrm{T}, t \in T$.

Remark 3.2. (a) If the T-action on X is itself free, then we have $H_T^*(X) = H^*(X/T)$. Indeed, the above definition just reduces to this case since X and $\mathbb{E}\mathrm{T} \times X$ have the same weak homotopy type and T acts freely on $\mathbb{E}\mathrm{T} \times X$ via the diagonal action.

(b) If the T-action on X is trivial, then

$$H_T^*(X) = H^*((\mathbb{E}\mathrm{T}/\mathrm{T}) \times X; \mathbb{Q}) = H_T^*(\mathrm{pt}) \otimes_{\mathbb{Q}} H^*(X)$$

(c) One can also define equivariant Chern classes for an equivariant complex vector bundle E on X. Indeed, $\mathbb{E}\mathrm{T} \times_{\mathrm{T}} E \to \mathbb{E}\mathrm{T} \times_{\mathrm{T}} X$ is again a vector bundle of the same rank, so we can simply take

$$c_i^T(E) := c_i(\mathbb{E}\mathrm{T} \times_{\mathrm{T}} E) \in H^*(\mathbb{E}\mathrm{T} \times_{\mathrm{T}} X) = H_T^*(X)$$

(d) In general, there is a natural morphism $H_T^*(X) \longrightarrow H^*(X)$, which is induced by the composite

$$\{t\} \times X \hookrightarrow \mathbb{E}\mathrm{T} \times X \longrightarrow \mathbb{E}\mathrm{T} \times_{\mathrm{T}} X$$

for some point $t \in \mathbb{E}\mathrm{T}$. This morphism sends $c_i^T(E)$ to $c_i(E)$ for any equivariant vector bundle E on X.

(e) If T is a linear algebraic group and X is a DM-stack, there is a notion of equivariant Chow rings as well. The key idea is to approximate $\mathbb{ET}$ by schemes and then replace "H^*" by "A^*" in the above definition. For more details, see [8]

From now on, we will only be concerned with X a smooth complex projective variety and $T = (\mathbb{C}^*)^n$ a complex torus acting on it algebraically. We have

$$H_T^*(\mathrm{pt}) = \mathbb{Q}[\alpha_0, ..., \alpha_n],$$

where $\alpha_i = c_1^T(L_{-e_i})$ with the T-representation $L_{-e_i} = \mathbb{C}$ given by:

$$\begin{aligned} T \times L_{-e_i} &\longrightarrow L_{-e_i} \\ (t_1, ..., t_n) \cdot x &= t_i^{-1} x, \end{aligned}$$

which can be regarded as an equivariant line bundle over the point. In this case, the following remarkable theorem holds

Theorem (localization theorem)**.** [12] Let X be be a DM-stack with a given T-equivariant perfect obstruction theory. Let $\{X_i\}_i$ be the connected components of the fixed locus X^T. Then the resulting equivariant virtual fundamental class $[X]^{vir,T}$ is given by

$$[X]^{vir,T} = \sum_i \frac{[X_i]^{vir,T}}{c_{top}^T(N_i)} \in H_T^*(X) \otimes_{\mathbb{Q}} \mathbb{Q}(\alpha_1, ..., \alpha_n)$$

where N_i is the equivariant normal bundle of X_i in X and $[X_i]^{vir,T}$ are the induced virtual fundamental classes.

This theorem also holds in the equivariant Chow ring and is one of the main techniques to explicitly compute Gromov-Witten invariants.
Indeed, if we want to compute an integral of the form

$$\int_{[X]^{vir}} \gamma$$

for some $\gamma \in H^*(X)$, we need only lift γ to a class $\gamma^T \in H_T^*(X)$ which restricts to γ under the canonical map $H_T^*(X) \to H^*(X)$ and we get:

$$\int_{[X]^{vir}} \gamma = \int_{[X]^{vir,T}} \gamma^T = \sum_i \int_{[X_i]^{vir,T}} \frac{\gamma^T|_{X_i}}{e^T(N_i)}$$

if the obstruction theory on X is T-equivariant.
The integrals on the right are usually easier to compute and since the left hand side is independent of the equivariant parameters α_i, we do not alter it by specializing the α_i on the right.

3.2 Gromov-Witten classes of $\mathbb{P}^m$

We now apply the localization theorem to $X = \overline{M}_{g,n}(\mathbb{P}^m, d) := \overline{M}_{g,n}(\mathbb{P}^m, d[\mathbb{P}^1])$ to find an explicit formula for the Gromov-Witten classes of $\mathbb{P}^m$. An action of $T = (\mathbb{C}^*)^{m+1}$ on X is induced by the following action of T on $\mathbb{P}^m$:

$$T \times \mathbb{P}^m \longrightarrow \mathbb{P}^m,$$
$$(t_0, ..., t_m) \cdot [x_0 : .. : x_m] = [t_0 x_0 : ... : t_m x_m]$$

First, we must identify the fixed locus X^T of this action and its connected components.

Lemma 3.3. The T-fixed points of this action on $\mathbb{P}^n$ are precisely the points of the form $P_i = [e_i]$ where e_i is the i-th standard basis vector in $\mathbb{C}^{m+1}$. Furthermore, any T-invariant closed subcurve of $\mathbb{P}^m$ is a union of linear subspaces of the form $\{[se_i + te_j] \mid [s : t] \in \mathbb{P}^1\}$

Proof. The claim about fixed points is clear. Also observe that for any point $x = [x_0 : ... : x_m] \in \mathbb{P}^m$ we have $\overline{T.x} \cong \mathbb{P}^l$ where $l = \#\{i \mid x_i \neq 0\} - 1$. The claim about subcurves immediately follows from this. $\square$

We now have to identify the connected components of X^T. Much like the stratification of $\overline{M}_{g,n}$, they are in bijection with a certain set of graphs, however some care must be taken in order to not confuse them with stable graphs.

Definition 3.4. A *decorated graph* on $\overline{M}_{g,n}(\mathbb{P}^m, d)$ is a tuple

$$\Gamma = (V, E, v_1, ..., v_n, g, d, \mu)$$

consisting of

1. an underlying connected Graph with vertex set V and undirected edges E between them. Any two vertices can have multiple edges in common.
2. marked vertices $v_1, ..., v_n \in V$
3. a function $g : V \to \mathbb{N}_{\geq 0}$ such that

$$\sum_{v \in V} g(v) + h^1(\Gamma) = \sum_{v \in V} g(v) + |E| - |V| + 1 = g$$

4. a function $d : E \to \mathbb{N}_{\geq 1}$ such that

$$\sum_{e \in E} d(e) = d$$

5. an function $\mu : V \to \{0, 1, ..., m\}$ turning (V, E) into an $m+1$-colored graph, i.e.: $\mu(v) \neq \mu(v')$ for any two vertices v and v' connected by at least one edge. In particular, this means that Γ cannot have any loops.

For any decorated graph we set $\mathbb{A}_\Gamma := \#\mathrm{Aut}(\Gamma) \prod_{e \in E} d(e)$

In analogy with stable graphs, we can define a morphism for any decorated graph whose image is the associated component of X^T. The domain of the map we denote by

$$\overline{M}_\Gamma := \prod_{v \in V(\Gamma)} \overline{M}_{g_v, val(v)}$$

where we simply set $\overline{M}_{g,n} = \mathrm{pt}$ if $2g - 2 + n \leq 0$ and $val(v)$ is the number of edges or markings incident to v. The morphism

$$\zeta_\Gamma \colon \overline{M}_\Gamma \longrightarrow \overline{M}_{g,n}(\mathbb{P}^m, d)$$

is given by considering each vertex v as a contracted component of genus g_v if it is stable or as a point if it is unstable - its image will be the fixed point $P_{\mu(v)}$. The edges correspond to non-contracted components of genus 0 in our domain curve that touch the two components corresponding to its two vertices. If e connects v and v', then the map on e is given by

$$\begin{aligned} \mathbb{P}^1 &\to \mathbb{P}^m, \\ [s : t] &\mapsto [s^{d(e)} e_{\mu(v)} + t^{d(e)} e_{\mu(v')}] \end{aligned}$$

and is hence of degree $d(e)$ onto its image curve. Finally, the n vertices correspond to the n markings.
Indeed, it is not difficult to see that any component of X^T comes from a decorated graph and the map ζ_Γ is easily seen to have degree $\mathbb{A}_\Gamma$. In order to present the formula for the Euler class $e(N_\Gamma^{vir})$ of the virtual normal bundle associated to Γ, we need to introduce the *lambda classes*:

Definition 3.5. For $\pi \colon \overline{M}_{g,n+1} \to \overline{M}_{g,n}$ the universal curve, we write

$$\lambda_i = c_i(\mathbb{E})$$

for the chern classes of the rank g *Hodge bundle* $\mathbb{E} = \pi_* \Omega^1_\pi$. Also denote:

$$\mathbb{E}^\vee(x) = \sum_{i=0}^{g} (-1)^i \lambda_i x^{g-i},$$

We will also consider $\lambda_i \in H^*(\overline{M}_{g,n}(\mathbb{P}^m, d))$ via pullback along the forgetful map.

Proposition 3.6. Let Γ be a decorated graph on $\overline{M}_{g,n}(\mathbb{P}^m, d)$.

$$
\begin{aligned}
e(N_\Gamma^{vir})^{-1} = & \prod_{\text{stable flag } F} \Big(\frac{1}{\omega_F - \psi_F} \prod_{\nu \neq \mu(F)} (\alpha_{\mu(F)} - \alpha_\nu) \Big) \\
& \prod_{v \text{ stable}} \prod_{\nu \neq \mu(v)} \frac{\mathbb{E}^\vee(\alpha_{\mu(v)} - \alpha_\nu)}{\alpha_{\mu(v)} - \alpha_\nu} \\
& \prod_{\substack{e \text{ any edge} \\ \text{connecting } v_1, v_2}} \frac{(-1)^{d(e)} d(e)^{2d(e)}}{\big(d(e)!\big)^2 (\alpha_{\mu(v_1)} - \alpha_{\mu(v_2)})^{2d(e)}} \\
& \prod_{\substack{a+b=d(e) \\ \nu \neq \mu(v_1), \mu(v_2)}} \frac{1}{\frac{a}{d(e)}\alpha_{\mu(v_1)} + \frac{b}{d(e)}\alpha_{\mu(v_2)} - \alpha_\nu} \\
& \prod_{\substack{v \text{ vertex} \\ \text{of type 1}}} \frac{\prod_{\nu \neq \mu(v)} (\alpha_{\mu(v)} - \alpha_\nu)}{\omega_{F_{v,1}} + \omega_{F_{v,2}}} \prod_{\substack{v \text{ vertex} \\ \text{of type 2}}} \omega_F
\end{aligned} \tag{8}
$$

where a *flag* F is a pair $F = (v, e)$ of a vertex v with an incident edge e and $\mu(F) := \mu(v)$. If v' is the other vertex incident to e, we write $\omega_F := \frac{\alpha_{\mu(v)} - \alpha_{\mu(v')}}{d(e)}$. We call the flag stable if v is a stable vertex in the graph Γ. Furthermore, we call a vertex v *of type 1* if it is of genus 0 with no markings and incident to precisely two flags $F_{v,1}$ and $F_{v,2}$. We say v is *of type 2* if it is of genus 0 with no markings and incident to only one flag.

Remark 3.7. If we adopt the conventions:

$$\frac{1}{(a-\psi_1)(b-\psi_2)} = \frac{1}{a+b} \text{ and } \frac{1}{a-\psi_1} = 1 \text{ on } \overline{M}_{0,2}$$

and

$$\frac{1}{a-\psi_1} = a \text{ on } \overline{M}_{0,1}$$

then we can simplify the formula:

$$
\begin{aligned}
e(N_\Gamma^{vir})^{-1} = & \prod_{\text{any flag } F} \Big(\frac{1}{\omega_F - \psi_F} \prod_{\nu \neq \mu(F)} (\alpha_{\mu(F)} - \alpha_\nu) \Big) \\
& \prod_{v \text{ vertex}} \prod_{\nu \neq \mu(v)} \frac{\mathbb{E}^\vee(\alpha_{\mu(v)} - \alpha_\nu)}{\alpha_{\mu(v)} - \alpha_\nu} \\
& \prod_{\substack{e \text{ any edge} \\ \text{connecting } v_1, v_2}} \frac{(-1)^{d(e)} d(e)^{2d(e)}}{\big(d(e)!\big)^2 (\alpha_{v_1} - \alpha_{v_2})^{2d(e)}} \prod_{\substack{a+b=d(e) \\ k \neq i,j}} \frac{1}{\frac{a}{d(e)}\alpha_{v_1} + \frac{b}{d(e)}\alpha_{v_2} - \alpha_k}
\end{aligned}
$$

In fact, most of the decorated graphs that we will encounter only have tubes of degree 1. If this is the case, we can simplify the formula even further:

$$e(N_\Gamma^{vir})^{-1} = \prod_{\text{any flag } F} \frac{1}{\omega_F - \psi_F} \prod_{v \text{ vertex}} \prod_{\nu \neq \mu(v)} \frac{\mathbb{E}^\vee(\alpha_{\mu(v)} - \alpha_\nu)}{\alpha_{\mu(v)} - \alpha_\nu}.$$

For a proof of the above statements, see [12] and [13, Ch. 27]. A more careful treatment justifying the conventions in Remark 3.7 and which also deals with the case of general toric varieties can be found in [23].
Computing Gromov-Witten Classes of $\mathbb{P}^m$ means evaluating pushforwards of the form:

$$\pi_*\bigg([X]^{vir}\prod_{i=1}^{n}\psi_i^{k_i}ev_i^*(H^{n_i})\bigg),$$

where

$$\pi\colon \overline{M}_{g,n}(\mathbb{P}^m,d)\to\overline{M}_{g,n}$$

is the stabilization map and $H = c_1(\mathcal{O}_{\mathbb{P}^m}(1))$ is the hyperplane class. We therefore need to find lifts of ψ_i and H as discussed in the previous section. This is particularly easy to do for H:
The natural surjection $(\mathbb{C}^{m+1})^*\otimes\mathcal{O}_{\mathbb{P}^m}\to\mathcal{O}_{\mathbb{P}^m}(1)$, where T acts on $\mathbb{C}^{m+1}$ with the standard multiplication, turns $\mathcal{O}_{\mathbb{P}^m}(1)$ into an equivariant line bundle, so we can define $H^T = c_1^T(\mathcal{O}_{\mathbb{P}^m}(1))$. We thus see $\iota^*(H^T) = \alpha_i$ under the natural inclusion $\iota_i\colon P_i\hookrightarrow\mathbb{P}^m$. However, a standard trick is to simplify the localization formula by taking a different lift of H:

$$H_i \coloneqq H^T-\alpha_i,$$

which then vanishes on P_i. For any decorated graph Γ, we easily obtain

$$\zeta_\Gamma^*(ev_j^*(H_i)) = \alpha_{v_j}-\alpha_i,$$

which means that the contribution of Γ to the localization formula vanishes if $\mu(v_j)=i$.
We lift $\psi_i^T = c_1^T(T_{C,p_i})$ by taking the T action induced by the differential of the map to $\mathbb{P}^n$. We see:

$$\zeta_\Gamma^*((\psi_i^T)^k) = \psi_i^k, \tag{9}$$

if p_i lives on a stable component of Γ or

$$\zeta_\Gamma^*((\psi_i^T)^k) = \frac{\alpha_{\mu(v')}-\alpha_{\mu(v)}}{d(e)},$$

if p_i is the only marking on a tube e which has v' as its second vertex. Using the convention

$$\frac{\psi_2^k}{a-\psi_1} = (-a)^k \text{ on } \overline{M}_{0,2}$$

we can even write (9) for all Γ (at least after cupping with $e(N_\Gamma^{vir})^{-1}$).
We obtain:

$$I_{g,d}^{\mathbb{P}^m,T}\bigg(\prod_{i=1}^{n}\tau_{k_i}\big(\prod_{j=1}^{n_i}H_{l_{i,j}}\big)\bigg) = \sum_{\Gamma\text{ decorated}}\frac{1}{\mathbb{A}_\Gamma}(\xi_{\Gamma'}\pi_\Gamma)_*\bigg(\frac{\prod_{i=1}^{n}\psi_i^{k_i}\prod_{j=1}^{n_i}(\alpha_{\mu(i)}-\alpha_{l_j})}{e(N_\Gamma)}\bigg)$$

For any choice of $l_{i,j} = 0, ..., m$. Here, Γ' is the stabilization of Γ and $\xi_{\Gamma'}$ is the map onto the corresponding boundary stratum:

$$\xi_{\Gamma'} \colon \overline{M}_\Gamma = \prod_{v \in V(\Gamma')} \overline{M}_{g(v),n(v)} \to \overline{M}_{g,n},$$

see [43] for a treatment of such boundary strata. The map

$$\pi_\Gamma \colon \overline{M}_\Gamma \to \overline{M}_{\Gamma'}$$

can be described as first forgetting the factors corresponding to contracted components and then forgetting the contracted markings on the remaining factors. Finally, we make a substitution $\alpha_i = a_i$ for integer a_i so that no denominator in $e(N_\Gamma)^{-1}$ vanishes. This means that for all $a, b \in \mathbb{N}_0$ with $a + b \leq d$ and pairwise distinct i, j, k, we must have:

$$a \cdot a_i + b \cdot a_j \neq (a+b) \cdot a_k. \tag{10}$$

If $d = 1$, then we can just set $a_i = i$ and if $d > 1$, the uniqueness of the d-adic expansion implies that $a_i = d^i$ also satisfies (10). Indeed, we can unify both cases in the substitution

$$a_i = \sum_{j=0}^{i-1} d^j,$$

which is $\frac{d^i-1}{d-1}$ if $d > 1$ and hence also satisfies (10). Occasionally, it will be better to do a different substitution. See Section 5.4 for an example of this. We conclude:

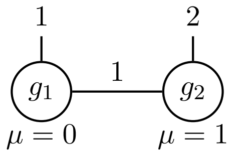
$$I_{g,d}^{\mathbb{P}^m}\left(\prod_{i=1}^n \tau_{k_i}(H^{n_i})\right) =$$
$$\left\{ \sum_{\Gamma \text{ decorated}} \frac{1}{\mathbb{A}_\Gamma} (\xi_{\Gamma'}\pi_\Gamma)_* \left(\frac{\prod_{i=1}^n \psi_i^{k_i} \prod_{j=1}^{n_i} (\alpha_{\mu(i)} - \alpha_{l_j})}{e(N_\Gamma)} \right)\Bigg|_{\alpha_i = a_i} \right\}_e,$$

where e is the expected cohomology degree $2\big(g-1-(m+1)d+\sum_i (k_i+n_i)\big)$ and $l_{i,j} = 0, ...m$ arbitrary.

Example 3.8. To illustrate this, we compute

$$I_{g,1}^{\mathbb{P}^2}(\tau_0(H^2), \tau_0(H^2)) \in H^{2g}(\overline{M}_{g,2}).$$

By lifting $H^2 = H_1 H_2$ for the first and $H^2 = H_0 H_2$ for the second insertion, we ensure that the only decorated graphs Γ_{g_1,g_2} that contribute here are of the form:



for different values of g_1 and g_2 with $g_1 + g_2 = g$. The expression on $\overline{M}_\Gamma$ arising in the localization formula is:

$$\left(\mathbb{E}^\vee(\alpha_0-\alpha_1)\mathbb{E}^\vee(\alpha_0-\alpha_2)\frac{1}{\alpha_0-\alpha_1-\psi_3}\right)\boxtimes\left(\mathbb{E}^\vee(\alpha_1-\alpha_2)\mathbb{E}^\vee(\alpha_1-\alpha_0)\frac{1}{\alpha_1-\alpha_0-\psi_4}\right),$$

where 3 and 4 are the half edges building the edge of Γ. We can make the substitution

$$\alpha_0 = 0, \alpha_1 = 1, \alpha_2 = x \text{ for arbitrary } x$$

to get:

$$\left(\mathbb{E}^\vee(-1)\mathbb{E}^\vee(-x)\frac{1}{-1-\psi_3}\right)\boxtimes\left(\mathbb{E}^\vee(1-x)\mathbb{E}^\vee(1)\frac{1}{1-\psi_4}\right).$$

In case $g_1 = 0$ or $g_2 = 0$ one vertex becomes unstable and has to be contracted. This gives us three summands:

(11)

$$I^{\mathbb{P}^2}_{g,1}(\tau_0(H^2),\tau_0(H^2)) = \left\{\mathbb{E}^\vee(-1)\mathbb{E}^\vee(-x)\frac{1}{-1-\psi_2} + \mathbb{E}^\vee(1-x)\mathbb{E}^\vee(1)\frac{1}{1-\psi_1}\right.$$

$$\left. + \sum_{\substack{g_1+g_2=g\\ g_1,g_2>0}} \xi_{{\Gamma'}_{g_1,g_2}*}\left[\left(\mathbb{E}^\vee(-1)\mathbb{E}^\vee(-x)\frac{1}{-1-\psi_3}\right)\boxtimes\left(\mathbb{E}^\vee(1-x)\mathbb{E}^\vee(1)\frac{1}{1-\psi_4}\right)\right]\right\}_{2g}$$

One can now stabilize to $x = 0$ or $x = 1$ to compute this for any given g. However, it is rather striking that the above expression is apparently independent of x (note the similarity with [10, Sec. 2.1]).

4 The Gromov-Witten theory of the elliptic curve

In the coming sections, we will often refer to Pixton's Bachelors thesis [42], so we only collect a few facts that are not mentioned there:
Throughout the rest of the thesis, let E be the elliptic curve.
We fix a basis $\{1, \alpha, \beta, \mathrm{pt}\}$ of $H^*(E)$ consisting of $1 \in H^0(X)$, $\alpha \in H^{1,0}(E)$, $\beta \in H^{0,1}(E)$ so that $\int_E \alpha \cup \beta = 1$ and the point class $\mathrm{pt} \in H^2(E)$.
The virtual dimension of $\overline{M}_{g,n}(E, d)$ is independent of d, hence we can define

Definition 4.1. For classes $\gamma_i \in H^*(E)$, $k_i \geq 0$ and $\gamma \in H^*(\overline{M}_{g,n})$, we define:

$$I^E_g(\gamma_1, ..., \gamma_n) := \sum_{d\geq 0} q^d I^E_{g,d}(\gamma_1, ..., \gamma_n)$$

and

$$\left\langle \gamma \prod_{i=1}^{n} \tau_{k_i}(\gamma_i) \right\rangle_g^E := \sum_d q^d \left\langle \gamma \prod_{i=1}^{n} \tau_{k_i}(\gamma_i) \right\rangle_{g,d}^E.$$

We sometimes leave out the subscript g since the genus is already determined by the degree of the insertions.
Also note that

$$\left\langle \gamma \prod_{i=1}^{n} \tau_{k_i}(\gamma_i) \right\rangle_g = \int_{\overline{M}_{g,n}} \gamma \prod_{i=1}^{n} \psi_i^{k_i} I_{g,d}^E(\gamma_1, ...\gamma_n),$$

because of the fact that any map $\mathbb{P}^1 \to E$ is already constant.

In the next chapter we will need the following facts:

Proposition 4.2. The following identities of Gromov-Witten classes hold:

(a) For genus $g > 0$

$$\left\langle \lambda_g \prod_{i=1}^{n} \tau_{k_i}(\gamma_i) \right\rangle_g^E = 0$$

(b) For classes $\gamma_1, ...\gamma_n \in \{1, \alpha, \beta, \mathrm{pt}\}$:

$$I_g(\gamma_1, ..., \gamma_n) = 0$$

unless

$$\#\{i \mid \gamma_i = \alpha\} = \#\{i \mid \gamma_i = \beta\}$$

(c) For any number n of insertions and $1 \leq i < j \leq n$:

$$I_g(..., \alpha, ..., \beta, ...,) = -I_g(..., \beta, ..., \alpha, ...),$$

where α and β are in ith or jth place and there are even insertions everywhere else.

(d) If we have at least two insertions:

$$\begin{aligned} &I_g(\alpha, \mathrm{pt}, ..., \mathrm{pt}, \beta) + I_g(\mathrm{pt}, \alpha, ..., \mathrm{pt}, \beta) + ... + I_g(\mathrm{pt}, ..., \mathrm{pt}, \alpha, \beta) \\ &= I_g(\mathrm{pt}, ..., \mathrm{pt}, 1) \end{aligned}$$

(e) For four or more insertions:

$$\begin{aligned} &I_g(\alpha, \mathrm{pt}, ..., \mathrm{pt}, \beta, \alpha, \beta) + I_g(\mathrm{pt}, \alpha, ..., \mathrm{pt}, \beta, \alpha, \beta) + ... + I_g(\mathrm{pt}, ..., \mathrm{pt}, \alpha, \beta, \alpha, \beta) \\ &- I_g(\alpha, \mathrm{pt}, ..., \mathrm{pt}, \beta, \beta, \alpha) - I_g(\mathrm{pt}, \alpha, ..., \mathrm{pt}, \beta, \beta, \alpha) - ... - I_g(\mathrm{pt}, ..., \mathrm{pt}, \alpha, \beta, \beta, \alpha) \\ &= I_g(\alpha, \mathrm{pt}, ..., \mathrm{pt}, \beta, 1, \mathrm{pt}) + I_g(\mathrm{pt}, \alpha, ..., \mathrm{pt}, \beta, 1, \mathrm{pt}) + ... + I_g(\mathrm{pt}, ..., \mathrm{pt}, \alpha, \beta, 1, \mathrm{pt}) \\ &+ I_g(\alpha, \mathrm{pt}, ..., \mathrm{pt}, \beta, \mathrm{pt}, 1) + I_g(\mathrm{pt}, \alpha, ..., \mathrm{pt}, \beta, \mathrm{pt}, 1) + ... + I_g(\mathrm{pt}, ..., \mathrm{pt}, \alpha, \beta, \mathrm{pt}, 1) \\ &- I_g(\mathrm{pt}, ..., \mathrm{pt}, 1, 1, \mathrm{pt}) - I_g(\mathrm{pt}, ..., \mathrm{pt}, 1, \mathrm{pt}, 1) + I_g(\mathrm{pt}, ..., \mathrm{pt}, \alpha, \beta) - I_g(\mathrm{pt}, ..., \mathrm{pt}, \beta, \alpha) \end{aligned}$$

(f) All classes of the form $I_g(\gamma_1, \gamma_2)$ are linear combinations of $I_g(\mathrm{pt}, 1)$ and $I_g(\mathrm{pt}, \mathrm{pt})$. In fact:

$$I_g(\mathrm{pt}, 1) = I_g(1, \mathrm{pt}) = I_g(\alpha, \beta) = -I_g(\beta, \alpha).$$

(g) The following three equations hold:

$$\begin{aligned} I_g(\alpha, \beta, \mathrm{pt}) &= \frac{I_g(1, \mathrm{pt}, \mathrm{pt}) + I_g(\mathrm{pt}, 1, \mathrm{pt}) - I_g(\mathrm{pt}, \mathrm{pt}, 1)}{2} \\ I_g(\alpha, \mathrm{pt}, \beta) &= \frac{I_g(1, \mathrm{pt}, \mathrm{pt}) + I_g(\mathrm{pt}, \mathrm{pt}, 1) - I_g(\mathrm{pt}, 1, \mathrm{pt})}{2} \\ I_g(\mathrm{pt}, \alpha, \beta) &= \frac{I_g(\mathrm{pt}, 1, \mathrm{pt}) + I_g(\mathrm{pt}, \mathrm{pt}, 1) - I_g(1, \mathrm{pt}, \mathrm{pt})}{2}. \end{aligned}$$

In particular, all classes $I_g(\gamma_1, \gamma_2, \gamma_3)$ are linear combinations of classes with only even insertions.

(h) For four insertions, we get:

$$\begin{aligned} I_g(\alpha, \beta, \alpha, \beta) - I_g(\alpha, \beta, \beta, \alpha) &= -I_g(\mathrm{pt}, \mathrm{pt}, 1, 1) - I_g(1, 1, \mathrm{pt}, \mathrm{pt}) \\ &+ \frac{I_g(\mathrm{pt}, 1, \mathrm{pt}, 1) + I_g(\mathrm{pt}, 1, 1, \mathrm{pt})}{2} + \frac{I_g(1, \mathrm{pt}, \mathrm{pt}, 1) + I_g(1, \mathrm{pt}, 1, \mathrm{pt})}{2} \end{aligned}$$

Remark 4.3. (a) Proposition 4.2 an easy application of the techniques of [36].

(b) Identities (c) - (h) can also be pulled back along forgetful maps

$$\overline{M}_{g,n+m} \to \overline{M}_{g,n}$$

to yield the same identities with more 1-insertions. For example, (f) implies that

$$I_g(1, ..., 1, \mathrm{pt}, 1, ..., 1) \in H^{2g}(\overline{M}_{g,n})$$

is independent of the location of "pt".

Proof. For part (a), see [42, Lemma 4.4.1] and part (b) was shown in [17, Sec. 3]. Part (c) can be shown using the monodromy invariance of E: Both follow immediately from (b) by applying the transformation

$$\alpha \mapsto \alpha + \beta, \beta \mapsto \beta$$

to the class

$$I_g(..., \alpha, ..., \alpha, ...) = 0.$$

For part (d), note that E acts freely on $\overline{M}_{g,n}(E, d)$ by translating the image of the first marked point. Hence:

$$\overline{M}_{g,n}(E, d) = ev_1^{-1}(0_E) \times E.$$

It is easy to show that the virtual fundamental class is pulled back from the first factor, which yields

$$I_g(\gamma'\pi_1^*(\gamma)) = 0$$

for $\gamma \in H^*(ev_1^{-1}(0_E))$ and $\gamma' \in H^{\leq 1}(\overline{M}_{g,n}(E,d))$. We also have the commutative diagram

$$\begin{array}{ccc} \overline{M}_{g,n}(E,d) & \xrightarrow{ev} & E^n \\ \Big\downarrow{\scriptstyle \pi_1} & & \Big\downarrow{\scriptstyle p} \\ ev_i^{-1}(0_E) & \xrightarrow{ev_{\neq 1}} & E^{n-1} \end{array}$$

where p subtracts the first coordinate from all other coordinates and then forgets it. Thus, all insertions that are pulled back from p are also pulled back from $ev_i^{-1}(0_E)$. One sees that (d) is precisely the fact

$$0 = I_g\big(ev_1^*(\alpha)ev^*p^*(\mathrm{pt} \boxtimes ... \boxtimes \mathrm{pt} \boxtimes \beta)\big)$$

and (f) follows directly from this and (c). To get (g), use the equivariance of Gromov-Witten classes under the symmetric group to obtain two more linear equations out of (d) that can then be solved. Part (e) is simply (d) pulled back under the self-node:

$$\overline{M}_{g-1,n+2} \to \overline{M}_{g,n}.$$

where the last two markings are glued together. Part (h) follows from (e) and (g). $\square$

It is a well-known fact that

$$\left\langle \gamma \prod_{i=1}^{n} \tau_{k_i}(\gamma_i) \right\rangle_g^E$$

is always quasi-modular of weight $2g-2+\sum_i \deg_{\mathbb{R}}(\gamma_i)$ and [42] summarizes the algorithm for computing these invariants.
There is also the following stronger fact that we will need:

Theorem 4.4. [30, Cor. 1] For $n \geq 1$:

$$I_g(\gamma_1, ...\gamma_n) \in H^*(\overline{M}_{g,n}) \otimes \mathrm{QMod}_{2g-2+\sum_i \deg_{\mathbb{R}}(\gamma_i)}$$

The dependence on G_2 is also known:
Let $\iota \colon \overline{M}_{g-1,n+2} \to \overline{M}_{g,n}$ be the map corresponding to the self-node stratum and for $g = g_1 + g_2$ and $\{1, ..., n\} = S_1 \sqcup S_2$ let:

$$j \colon \overline{M}_{g_1, S_1 \sqcup \{\bullet\}} \times \overline{M}_{g_2, S_2 \sqcup \{\bullet\}} \to \overline{M}_{g,n}$$

be the map gluing the two $\bullet$'s.

Theorem 4.5. [30, Thm. 3]

$$\begin{aligned}\frac{d}{dG_2} I_g(\gamma_1, ..., \gamma_n) &= \iota_* I_{g-1}(\gamma_1, ..., \gamma_n, 1, 1) \\ &\quad + \sum_{\substack{g=g_1+g_2 \\ \{1,...,n\}=S_1 \sqcup S_2}} \sigma_{S_1,S_2} j_*(I_{g_1}(\gamma_{S_1}, 1) \boxtimes I_{g_2}(\gamma_{S_2}, 1)) \\ &\quad - 2\sum_{i=1}^n \Big(\int_E \gamma_i \Big) \psi_i \cdot I_g(\gamma_1, ..., \gamma_{i-1}, 1, \gamma_{i+1}, ..., \gamma_n),\end{aligned}$$

where $\frac{d}{dG_2}$ is the formal derivative with respect to G_2 in the polynomial ring $H^*(\overline{M}_{g,n})[G_2, G_4, G_6]$ and σ_{S_1,S_2} is the sign that comes from supercommuting the γ_i.

We call this the *holomorphic anomaly equation* of I_g^E. It measures the failure of $I_g(\gamma_1, ..., \gamma_n)$ to be an honest modular form.

5 The Gromov-Witten theory of $\mathbb{P}^2 \times E$

5.1 Computation scheme

Theorem 1.10 concerns the disconnected Gromov-Witten theory. However, we will find it easier to compute the connected Gromov-Witten theory, so we must first express the disconnected in terms of the connected invariants:

Lemma 5.1. For $\gamma_i \in H^*(\mathbb{P}^2 \times E)$ and $k_i \geq 0$ for all i:

$$\Big\langle \prod_{i=1}^n \tau_{k_i}(\gamma_i) \Big\rangle_m^\bullet = \sum_{\substack{P \text{ partition of } \{1,...,n\}, \\ m = \sum_{I \in P} m_I}} \operatorname{sgn}(P) \prod_{I \in P} \Big\langle \prod_{i \in I} \tau_{k_i}(\gamma_i) \Big\rangle_{m_I},$$

where the unordered partition P has no empty members.
The sign $\operatorname{sgn}(P)$ comes from supercommuting the classes $\gamma_1, ..., \gamma_n$ into products over the members of P.

Proof. This is similar to [42, Prop. 3.1.1]. The moduli space $\overline{M}^\bullet_{g,n}(\mathbb{P}^2 \times E, (\beta, d))$ is a disjoint union of components of the form

$$\prod_{I \in P} \overline{M}_{g_I, I}(\mathbb{P}^2 \times E, (\beta_I, d_I)) / \operatorname{Aut}(P, (m_I, d_I))$$

where P is a partition of the set of markings $\{1, ..., n\}$ with possibly some empty parts and $\operatorname{Aut}(P, (m_I, d_I)_I)$ is the group permuting empty parts with the same genus and curve classes. The g_I, m_I and d_I are chosen so that:

1. $\sum_{I \in P} (g_I - 1) = g - 1$,

2. $\sum_{I\in P} m_I = m$ and

3. $\sum_{I\in P} d_I = d$.

It is important to note that our condition $(m_I, d_I) \neq (0,0)$ for $g_I \geq 2$ is already automatic. Indeed, we see that

$$\overline{M}_{g,n}(\mathbb{P}^2 \times E, (0,0)) = \overline{M}_{g,n} \times \mathbb{P}^2 \times E$$

with virtual class equal to:

$$\begin{aligned}&[\overline{M}_{g,n}(\mathbb{P}^2 \times E, (0,0))]^{vir} = e(\mathbb{E}^\vee \boxtimes (T\mathbb{P}^2 \oplus TE)) \cap [\overline{M}_{g,n} \times \mathbb{P}^2 \times E]\\ &= (-1)^g \lambda_g \Big(\lambda_g^2 - 3H\lambda_g\lambda_{g-1} + 3H^2(\lambda_{g-1}^2 + \lambda_g\lambda_{g-2})\Big) \cap [\overline{M}_{g,n} \times \mathbb{P}^2 \times E],\end{aligned}$$

where $c_i = c_i(T\mathbb{P}^2)$. Using the two identities

$$\lambda_g^2 = \delta_{g0} \text{ and } \lambda_{g-1}^2 = 2\lambda_g\lambda_{g-2} \text{ for } g \geq 2$$

derived from the Mumford relation (first shown in [28])

$$\mathbb{E}^\vee(1)\mathbb{E}^\vee(-1) = (-1)^g \tag{12}$$

we see that this indeed vanishes if $g \geq 2$. Noting that any connected invariant with no insertions has to be of degree 0 in $\mathbb{P}^2$, we only have to show that such connected components give us the prefactor $q^{1/8}\eta(q)^{-3}$:
Indeed:

$$\langle 1 \rangle_{g,(0,d)}^{\mathbb{P}^2\times E} = 3\langle \lambda_{g-1}^2 \rangle_{g,d}^E + 3\langle \lambda_g\lambda_{g-2}\rangle_{g,d}^E$$

Using Proposition 4.2 (a), we see that this vanishes for $g \neq 1$ and we get:

$$3\langle 1 \rangle_{1,(0,d)}^E = 3\sum_{h|d} \frac{1}{h}$$

if $d > 0$ and 0 if $d = 0$.
Thus we get a factor of

$$\sum_{n\geq 0} \frac{1}{n!}\Big(3\sum_{d>0} (\sum_{h|d}\frac{1}{h})q^d\Big)^n = \prod_{k\geq 1}(1-q^k)^{-3}$$

in addition to those factors permitted on the right hand side of the claimed formula. □

Remark 5.2. Using this, we can already establish the degree 0 case. Indeed,

$$\begin{aligned}&[\overline{M}_{g,n}(\mathbb{P}^2 \times E, (0,d))]^{vir}\\ &= \Big(\lambda_g^2 - 3\lambda_g\lambda_{g-1}H + 3H^2(\lambda_{g-1}^2 + \lambda_g\lambda_{g-2})\Big) \cap [\overline{M}_{g,n}(E,d)]^{vir}\end{aligned}$$

and so Proposition 4.2 (a) gives us:

$$
\begin{aligned}
&\left\langle \prod_{i=1}^{n} \tau_{k_i}(H^{n_i}\gamma_i) \right\rangle_0^{\mathbb{P}^2\times E} \\
&= -z^{-2}\delta_{\sum_i n_i,2} \left\langle \prod_{i=1}^{n} \tau_{k_i}(\gamma_i) \right\rangle_{g=0}^{E} + 3\delta_{\sum_i n_i,0} \left\langle \prod_{i=1}^{n} \tau_{k_i}(\gamma_i) \right\rangle_{g=1}^{E} \\
&= -z^{-2}\delta_{\sum_i n_i,2} \left(\int_E \prod_{i=1}^{n} \gamma_i \right) \binom{n-3}{k_1,...k_n} + 3\delta_{\sum_i n_i,0} \left\langle \prod_{i=1}^{n} \tau_{k_i}(\gamma_i) \right\rangle_{g=1}^{E}
\end{aligned}
$$

which is in QMod$[z^{-1}]$ and using Lemma 5.1, this establishes the degree 0 case of Theorem 1.10. In fact, it is easy to see that the second term is a multiple of $D_\tau^m G_2$, where $m = (\sum_i \deg_{\mathbb{C}}(\gamma_i)) - 1$.

To compute higher degree invariants we need to break up the connected theory of $\mathbb{P}^2 \times E$ into the connected theories of its two factors:

Proposition 5.3. For all $n_1, ..., n_l \geq 0$, $\gamma_1, ..., \gamma_l \in H^*(E)$:

$$
\begin{aligned}
&\left\langle \prod_{i=1}^{l} \tau_{k_i}(H^{n_i}\gamma_i) \right\rangle_m^{\mathbb{P}^2\times E} \\
&= \sum_{g\geq 0} (-1)^{g-1+3m} z^{2g-2+3m} \int_{\overline{M}_{g,n}} I_{g,m}^{\mathbb{P}^2}\Big(\prod_{i=1}^{l} \tau_{k_i}(H^{n_i})\Big) \cup I_g^E(\gamma_1, ..., \gamma_l)
\end{aligned}
$$

The summands with $g = 0$ and $n = 0, 1$ have to be treated on their own:

$$
\langle \tau_k(H^n\gamma) \rangle_{0,\beta}^{\mathbb{P}^2\times E} = \left(\int_E \gamma \right) \langle \tau_k(H^n) \rangle_{0,m}^{\mathbb{P}^2}
$$

and

$$
\langle \tau_{k_1}(H^{n_1}\gamma_1)\tau_{k_2}(H^{n_2}\gamma_2) \rangle_{0,m}^{\mathbb{P}^2\times E} = \left(\int_E \gamma_1\gamma_2 \right) \langle \tau_{k_1}(H^{n_1})\tau_{k_2}(H^{n_2}) \rangle_{0,m}^{\mathbb{P}^2}.
$$

Proof. [4] Fixing the following (stabilization) maps:

$$
\overline{M}_{g,n}(\mathbb{P}^2 \times E, (m,d)) \xrightarrow{p} \overline{M}_{g,n}(\mathbb{P}^2, m) \xrightarrow{s} \overline{M}_{g,n} \xleftarrow{e} \overline{M}_{g,n}(E, d)
$$

We consider the fiber diagram:

$$
\begin{tikzcd}
\overline{M}_{g,n}(\mathbb{P}^2 \times E, (m,d)) \arrow[r, "h"] & P \arrow[r] \arrow[d, "q"] & \overline{M}_{g,n}(\mathbb{P}^2, m) \times \overline{M}_{g,n}(E, d) \arrow[d, "(\mathrm{id},e)"] \\
& \overline{M}_{g,n}(\mathbb{P}^2, m) \arrow[r, "\gamma=(\mathrm{id},s)"] \arrow[d] & \overline{M}_{g,n}(\mathbb{P}^2, m) \times \overline{M}_{g,n} \arrow[d] \\
& \overline{M}_{g,n} \arrow[r, "\Delta"] & \overline{M}_{g,n} \times \overline{M}_{g,n}
\end{tikzcd}
$$

[4]This proof was suggested by Georg Oberdieck

where Δ is the inclusion of the diagonal, P is the fiber product and h is the morphism induced by the projections of $\mathbb{P}^2 \times E$ to the respective factors. By the product formula [4, Thm. 1] we have

$$h_*[\overline{M}_{g,n}(\mathbb{P}^2 \times E, (m,d))]^{vir} = \Delta^!\Big([\overline{M}_{g,n}(\mathbb{P}^2, m)]^{vir} \times [\overline{M}_{g,n}(E,d)]^{vir}\Big).$$

Taking the cup product with $\prod_i ev_i^*(H^{n_i}\gamma_i)$ and pushing forward by q yields

$$\begin{aligned}
&p_*\big([\overline{M}_{g,n}(\mathbb{P}^2 \times E, (m,d))]^{vir} \prod_i ev_i^*(H^{n_i}\gamma_i)\big)\\
&= q_*\Delta^!\Bigg(\Big([\overline{M}_{g,n}(\mathbb{P}^2, m)]^{vir} \prod_i ev_i^*(H^{n_i})\Big) \times \Big([\overline{M}_{g,n}(E,d)]^{vir} \prod_i ev_i^*(\gamma_i)\Big)\Bigg)\\
&= q_*\gamma^!\Bigg(\Big([\overline{M}_{g,n}(\mathbb{P}^2, m)]^{vir} \prod_i ev_i^*(H^{n_i})\Big) \times \Big([\overline{M}_{g,n}(E,d)]^{vir} \prod_i ev_i^*(\gamma_i)\Big)\Bigg)\\
&= \gamma^*\Bigg(\Big([\overline{M}_{g,n}(\mathbb{P}^2, m)]^{vir} \prod_i ev_i^*(H^{n_i})\Big) \times e_*\Big([\overline{M}_{g,n}(E,d)]^{vir} \prod_i ev_i^*(\gamma_i)\Big)\Bigg)\\
&= [\overline{M}_{g,n}(\mathbb{P}^2, m)]^{vir} \prod_i ev_i^*(H^{n_i}) \cap s^*\big(e_*[\overline{M}_{g,n}(E,d)]^{vir} \prod_i ev_i^*(\gamma_i)\big)
\end{aligned}$$

Since any map $\mathbb{P}^1 \to \mathbb{P}^2 \times E$ is already constant in E, we see $p^*(\psi_i) = \psi_i$ for any i. Hence we can finish the proof by multiplying with $\prod_i \psi_i^{k_i}$ and pushing forward along s. □

The algorithm for computing Gromov-Witten invariants on $\mathbb{P}^2 \times E$ is now clear:

1. Compute Gromov-Witten classes on $\mathbb{P}^2$ as a sum over certain stable graphs Γ (c.f. Chapter 3)
2. use the splitting and reduction axioms to pull back the Gromov-Witten classes of E to the $\overline{M}_\Gamma$'s and integrate
3. use Lemma 5.1 to compute the disconnected invariants.

In fact, the first two steps amount to directly applying localization to $\mathbb{P}^2 \times E$ with the torus acting only on the first factor.
We now take a closer look at the vertex terms arising in step 2 - they contain all of the nontrivial information:

5.2 Hodge integrals over the elliptic curve

We use the following convention:

Definition 5.4. For given classes $\gamma_i, \delta_j \in H^*(E)$, $a, b \in \mathbb{Q}$, $x_i \in \mathbb{Q} \setminus \{0\}$ and $k_j \geq 0$, the corresponding *Hodge integral over* E is given by:

$$\begin{aligned}
&\left\langle \mathbb{E}^\vee(a)\mathbb{E}^\vee(b) \prod_{i=1}^{n} \frac{\gamma_i}{x_i - \psi_i} \prod_{j=1}^{m} \tau_{k_j}(\delta_j) \right\rangle \\
&:= \sum_{g \geq 0} (-1)^g z^{2g} \left\langle \mathbb{E}^\vee(a)\mathbb{E}^\vee(b) \prod_{i=1}^{n} \frac{\gamma_i}{x_i - \psi_i} \prod_{j=1}^{m} \tau_{k_j}(\delta_j) \right\rangle_g^E .
\end{aligned}$$

where we use the expansion

$$\frac{1}{x - \psi} = \sum_{i=0}^{\infty} x^{-i-1} \psi^i.$$

The possibly unstable terms in genus 0 are defined by using the formal expansion

$$\frac{1}{x - \psi} = \sum_{i \in \mathbb{Z}} x^{-i-1} \psi^i$$

and then applying the machinery of negative descendents as outlined in [42]. More specifically, this means:

$$\begin{aligned}
\langle \mathbb{E}^\vee(a)\mathbb{E}^\vee(b) \frac{\gamma}{x - \psi_1} \rangle_0^E &= x \int_E \gamma \\
\langle \mathbb{E}^\vee(a)\mathbb{E}^\vee(b) \frac{\gamma_1}{x - \psi_1} \tau_k(\gamma_2) \rangle_0^E &= (-x)^k \int_E \gamma_1 \cup \gamma_2 \\
\langle \mathbb{E}^\vee(a)\mathbb{E}^\vee(b) \frac{\gamma_1}{x - \psi_1} \frac{\gamma_2}{y - \psi_2} \rangle_0^E &= \frac{1}{x + y} \int_E \gamma_1 \cup \gamma_2,
\end{aligned}$$

where the last term is artificially set to 0 if $x = -y$.

Remark 5.5. (a) There is a useful trick to simplify these integrals: In the infinite sums above, only those terms that have the right cohomology degree *vdim* will give a nonzero contribution. Hence, multiplying every cohomology class γ by $c^{\deg_{\mathbb{C}}(\gamma)}$ will give an overall factor of c^{vdim} and otherwise leaves the invariant unchanged. In other words:

$$\begin{aligned}
&\left\langle \mathbb{E}^\vee(a)\mathbb{E}^\vee(b) \prod_{i=1}^{n} \frac{\gamma_i}{x_i - \psi_i} \prod_{j=1}^{m} \tau_{k_j}(\delta_j) \right\rangle \\
&c^d \left\langle \mathbb{E}^\vee(\frac{a}{c})\mathbb{E}^\vee(\frac{b}{c}) \prod_{i=1}^{n} \frac{\gamma_i}{\frac{x_i}{c} - \psi_i} \prod_{j=1}^{m} \tau_{k_j}(\delta_j) \right\rangle
\end{aligned}$$

with

$$d = \sum_i (\deg_{\mathbb{C}}(\gamma_i) - 2) + \sum_j (\deg_{\mathbb{C}}(\delta_j) + k_j - 1) + 2$$

and c is any nonzero number. In principal, this allows us to always reduce to the case $a = 1$.

(b) Coming back to localization, we see that the conventions in definition 5.4 exactly match the unstable terms in chapter 3. In fact, the Gromov-Witten brackets of $\mathbb{P}^2 \times E$ will now be graph-sums with vertex terms looking like:

$$\left\langle \mathbb{E}^\vee(a)\mathbb{E}^\vee(b) \prod_{i=1}^{n} \frac{\gamma_i}{\frac{a}{\mu_i} - \psi_i} \prod_{j=n+1}^{n+m} \frac{\delta_j}{\frac{b}{\nu_j} - \psi_j} \prod_{k=m+n+1}^{l} \tau_{l_k}(\epsilon_k) \right\rangle,$$

where μ, ν are partitions consisting of the degrees of the various tubes coming out of our vertex and $\gamma_i, \delta_j, \epsilon_k \in H^*(E)$ come from the the diagonal classes in the splitting and reduction axioms for E. Note that we can apply these axioms directly to the original (possibly unstable) decorated graph (without genera specified on the vertices). Furthermore,

$$(a, b) = (1, 1 - x), (-1, -x), (x, x - 1)$$

for some x chosen so that no denominator vanishes in the localization formula. Recall from Chapter 3 that $x = m + 1$ always accomplishes this, where m is the degree of the curve class in $\mathbb{P}^2$.

5.3 Computations in degree 1

First, we look at the Hodge integrals arising in degree 1 - a small subset of which we will actually compute. In this section, we always denote:

$$F(x; \gamma; \tau_{k_1}(\gamma_1)...\tau_{k_n}(\gamma_n)) := \langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x) \frac{\gamma}{1 - \psi_1} \tau_{k_1}(\gamma_1)...\tau_{k_n}(\gamma_n)\rangle^E$$

and $\widetilde{\Theta} := \Theta(z)/z$.

Proposition 5.6. For all $\gamma, \gamma_1, ..., \gamma_n \in H^*(E)$ homogenous classes and $k_1, ..., k_n \geq 0$ there is a polynomial $p \in \mathrm{QMod}[[z]][t]$ of degree at most

$$\deg_{\mathbb{C}}(\gamma) - 1 + \sum_i (\deg_{\mathbb{C}}(\gamma_i) + k_i)$$

such that

$$F(x; \gamma; \tau_{k_1}(\gamma_1)...\tau_{k_n}(\gamma_n)) = \widetilde{\Theta}^x p(x)$$

In particular

$$F(x) := F(x; \mathrm{pt}; \emptyset) = \langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x) \frac{\mathrm{pt}}{1 - \psi_1} \rangle = \widetilde{\Theta}^x.$$

Proof. We first show the special case $n = 0$ by computing the two invariants $\langle \tau_0(H^2\mathrm{pt})\tau_0(H^2) \rangle_1^{\mathbb{P}^2 \times E}$ and $\langle \mathbb{E}^\vee(x)\tau_0(H\mathrm{pt}) \rangle_1^{\mathbb{P}^1 \times E}$.
For the first invariant, we use Example 3.8 to see that

$$\langle \tau_0(H^2\mathrm{pt})\tau_0(H^2) \rangle^{\mathbb{P}^2 \times E}$$

$$= z\langle \mathbb{E}^\vee(-1)\mathbb{E}^\vee(-x)\frac{1}{-1-\psi_1}\tau_0(\mathrm{pt})\rangle^E \langle \mathbb{E}^\vee(1-x)\mathbb{E}^\vee(1)\frac{\mathrm{pt}}{1-\psi_1}\rangle^E$$
$$= zF(x)F(1-x),$$

where we used the divisor equation as well as the scaling trick from the previous section to get rid of the sign in the first factor. Note here that the divisor equation even holds on the level of cycles.
Note that this holds for any x with the left hand side constant in x.
Similarly, applying the localization formula to the lift H_1 of H, we get:

$$\begin{aligned}\langle \mathbb{E}^\vee(x)\tau_0(H\mathrm{pt})\rangle_1^{\mathbb{P}^1\times E} &= \langle \mathbb{E}^\vee(-1)\mathbb{E}^\vee(x)\frac{1}{-1-\psi_1}\tau_0(\mathrm{pt})\rangle^E \langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\frac{\mathrm{pt}}{1-\psi_1}\rangle^E \\ &= F(-x)F(x),\end{aligned}$$

where we made the substitution $\alpha_i = 0, \alpha_1 = 1$.
The left hand side is also independent of x. Indeed, the virtual dimension of the corresponding moduli space is $g+2$, so $\mathbb{E}^\vee(x) = (-1)^g\lambda_g$ up to terms that yield 0 in the bracket. Proposition 4.2 (a) implies $F(0) = 1$, and so we see:

$$F(x+1) = F(x+1)F(-x)F(x) = I \cdot F(x),$$

where $I = z^{-1}\langle \tau_0(H^2\mathrm{pt})\tau_0(H^2)\rangle^{\mathbb{P}^2\times E}$. We therefore obtain

$$F(x) = I^x$$

for every integer x. Noting that the z-coefficients of both sides are polynomials in x, the equation indeed holds for arbitrary x. To determine I, we observe that the Mumford relation (12) and the Bloch-Okounkov formula [42, Prop. 3.2.3] give us the case $x = -1$:

$$F(-1) = \langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(-1)\frac{\mathrm{pt}}{1-\psi_1}\rangle = \sum_{g\geq 0} z^{2g}\langle \tau_{2g-2}(\mathrm{pt})\rangle_g^E = \widetilde{\Theta}^{-1}.$$

and so $I = \widetilde{\Theta}$.
For the general F, we compute the equivariant bracket

$$\langle \mathbb{E}^\vee(x)\tau_0(H_1\gamma)\prod_{i=1}^n \tau_{k_i}(H_0\gamma_i)\rangle_{\beta=1}^{\mathbb{P}^1\times E,T}$$

and then substitute $\alpha_0 = 0, \alpha_1 = 1$. In the sum $\mathbb{E}^\vee(x) = \sum_{i=0}^g(-1)^i\lambda_i x^{g-i}$ only the terms for $i \geq \deg_{\mathbb{C}}(\gamma) - 1 + \sum_i(\deg_{\mathbb{C}}(\gamma_i) + k_i)$ can give a nonzero contribution. Hence, the above invariant is a polynomial in x of at most that degree. Applying the localization formula again, we obtain:

$$\langle \mathbb{E}^\vee(x)\tau_0(H_1\gamma)\prod_{i=1}^n \tau_{k_i}(H_0\gamma_i)\rangle_{\beta=1}^{\mathbb{P}^1\times E,T}|_{\alpha_i=i}$$

$$
\begin{aligned}
&= \pm\big\langle \mathbb{E}^\vee(-1)\mathbb{E}^\vee(x)\frac{\gamma'}{-1-\psi_1}\tau_0(\gamma)\big\rangle\big\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\frac{\gamma}{1-\psi_1}\prod_{i=1}^n \tau_{k_i}(\gamma_i)\big\rangle \\
&= \widetilde{\Theta}^{-x}F(x;\gamma;\tau_{k_1}(\gamma_1)...\tau_{k_n}(\gamma_n))
\end{aligned}
$$

if $\gamma = 1, \alpha, \beta$ and $\gamma' = \mathrm{pt}, \beta, \alpha$ respectively in those cases. In the last step we have used Proposition 4.2 (f). The ± 1 in the second step is the sign coming from the splitting axiom.
This completes the proof in case $\gamma \in H^{\leq 1}(E)$. If $\gamma = \mathrm{pt}$, the above computation yields:

$$
\begin{aligned}
&- D_\tau(\widetilde{\Theta}^{-x})F(x;1;\tau_{k_1}(\gamma_1)...\tau_{k_n}(\gamma_n)) + \widetilde{\Theta}^{-x}F(x;\mathrm{pt};\tau_{k_1}(\gamma_1)...\tau_{k_n}(\gamma_n)) \\
&= x\frac{D_\tau\Theta}{\Theta}\widetilde{\Theta}^{-x}F(x;1;\tau_{k_1}(\gamma_1)...\tau_{k_n}(\gamma_n)) + \widetilde{\Theta}^{-x}F(x;\mathrm{pt};\tau_{k_1}(\gamma_1)...\tau_{k_n}(\gamma_n)),
\end{aligned}
$$

which concludes the proof. □

Remark 5.7. (a) The above proof was inspired by [10], where a similar exponential behavior is shown for certain integrals on $\overline{M}_{g,n}$.

(b) Note that we just computed the first nontrivial invariant on $\mathbb{P}^2 \times E$:

$$
\langle \tau_0(H^2\mathrm{pt})\tau_0(H^2)\rangle^{\mathbb{P}^2\times E} = \Theta
$$

(c) It is somewhat difficult to compute any one of the F's by hand and we will see some examples later. However, the following conjecture fits all the available data:

Conjecture G. For any given $k_1, ..., k_n \geq 0$ and cohomology classes $\gamma, \gamma_1, ..., \gamma_n \in H^*(E)$ the polynomial p with

$$
F(x;\gamma;\tau_{k_1}(\gamma_1)...\tau_{k_n}(\gamma_n)) = \widetilde{\Theta}^x p(x)
$$

has coefficients in $\mathbb{Q}[A, \wp, \wp', G_2, G_4, z]$ with weight $k = \sum_i \deg_{\mathbb{R}} \gamma_i$ and degree in z at most $\sum_i k_i$.

Theorem 5.8. Theorem 1.10 holds in degree 1. Furthermore, any (connected or disconnected) primary invariant is Θ times a polynomial in expressions of the form $\frac{D_\tau^n\Theta}{\Theta}$.

Proof. Using Lemma 5.1, it suffices to show the connected case using localization. Every decorated graph in degree one consists of two vertices connected by one edge and if we take the specialization $\alpha_i = i$, the contribution of this graph is (up to a constant factor):

$$
z\left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\frac{\gamma}{1-\psi_1}\prod_{i=1}^n \tau_0(\gamma_i)\right\rangle\left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(1-x)\frac{\gamma'}{1-\psi_1}\prod_{i=1}^n \tau_0(\gamma_i')\right\rangle
$$

for $x = 2, 1, 1/2$ after an application of the scaling trick. Assuming that all γ's are $1, \alpha, \beta$ or pt, we note that no more than two odd classes can occur in each bracket. Otherwise, we would have two $\tau_0(\alpha)$ or two $\tau_0(\beta)$ which can then be supercommuted - this yields the same invariant multiplied by -1, so such an invariant must have already been zero.
After applying the string and divisor equations and Proposition 4.2 (g), we see that this is a multiple of

$$zD_\tau^n(\widetilde{\Theta}^x)D_\tau^m(\widetilde{\Theta}^{1-x}),$$

which is just Θ times a polynomial in the $\frac{D_\tau^l\widetilde{\Theta}}{\widetilde{\Theta}} = \frac{D_\tau^l\Theta}{\Theta}$ and hence of index $1/2$. □

Below we have the list of all primary invariants with even classes that cannot simplified using Proposition 4.2 or the string and divisor equations:

$$\begin{aligned}
&\langle \tau_0(H^2\mathrm{pt})\tau_0(H^2)\rangle = \Theta \\
&\langle \tau_0(H^2\mathrm{pt})\tau_0(H\mathrm{pt})\rangle = 3D_\tau\Theta \\
&\langle \tau_0(H\mathrm{pt})^2\tau_0(H^2)\rangle = 4D_\tau\Theta \\
&\langle \tau_0(H\mathrm{pt})^3\rangle = \Theta\left(3\frac{D_\tau^2\Theta}{\Theta} + 9\left(\frac{D_\tau\Theta}{\Theta}\right)^2\right)
\end{aligned}$$

Remark 5.9. (a) One can compute all primary invariants with odd insertions from the above list.
Indeed, if a given invariant has two odd insertions, we can use Proposition 4.2 (d) to reduce to the case where these are $\tau_0(\alpha H^2)\tau_0(\beta H^2)$. For degree reasons, the invariant must then be a multiple of $D_\tau^n\Theta$. We cannot have six or more odd insertions as one sees from localization, so we are left with the case of four odd insertions. Then there must be an α and a β that have the same power of H next to them. We can then apply Proposition 4.2 (e) to reduce to the case of only two odd insertions.

(b) Here are some descendent invariants which can be computed with the above method and the dilaton equation:

$$\begin{aligned}
&\langle \tau_1(H^2\mathrm{pt})\rangle = z\Theta A \\
&\langle \tau_0(H^2\mathrm{pt})\tau_1(H)\rangle = \Theta(2zA - 3) \\
&\langle \tau_0(H\mathrm{pt})^2\tau_1(H)\rangle = z\left(2D_zD_\tau\Theta + 6\frac{D_\tau\Theta D_z\Theta}{\Theta}\right) - 6D_\tau\Theta
\end{aligned}$$

In order to compute more descendent invariants, we have to compute more Hodge integrals:

Proposition 5.10. We have:

$$F(x;\mathrm{pt};\tau_1(\mathrm{pt})) = \widetilde{\Theta}^x(ax+bx^2)$$

for

$$\begin{aligned} a &= \Big(\frac{1}{2}zD_z - 1\Big)\frac{D_\tau\Theta}{\Theta} \\ b &= z\Big(2AG_2 + \frac{\wp'}{6} + \frac{A^3}{3}\Big) + a \end{aligned}$$

Proof. From Proposition 4.2 we know that this vanishes at $x=0$ hence the claim holds for some a and b which we now determine. Checking $x=-1$, we see:

$$\begin{aligned} \frac{b-a}{\widetilde{\Theta}} &= \Big\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(-1)\frac{\mathrm{pt}}{1-\psi_1}\tau_1(\mathrm{pt})\Big\rangle \\ &= \sum_{g\geq 0} z^{2g}\Big\langle \frac{\mathrm{pt}}{1-\psi_1}\tau_1(\mathrm{pt})\Big\rangle_g \\ &= \sum_{g\geq 0} z^{2g}\big\langle \tau_{2g-3}(\mathrm{pt})\tau_1(\mathrm{pt})\big\rangle^\bullet \\ &= \sum_{g\geq 0} z^{2g}[z_1^{2g-2}z_2^2]\left[\frac{1}{\Theta(z_1+z_2)}\Big(A(z_1)+A(z_2)\Big)\right] \\ &= z^2[z_2^2]\left[\left(\frac{1}{\Theta(z)} - \frac{A(z)}{\Theta(z)}z_2 + \frac{A(z)^2+2G_2+\wp(z)}{\Theta(z)}\frac{z_2^2}{2}\right.\right. \\ &\quad \left.\left. + \frac{\wp'-A^3-3A(2G_2+\wp)}{\Theta}(z)\frac{z_2^3}{6}\right)\Big(A(z)+\frac{1}{z_2}-2z_2G_2\Big)\right] \\ &= \frac{z^2}{6}\frac{12AG_2+\wp'+2A^3}{\Theta}, \end{aligned}$$

where we used the Bloch-Okounkov formula [42, Prop. 3.2.3].
In order to get more information, we use the degeneration

$$E \rightsquigarrow \mathbb{P}^1 \cup \mathbb{P}^1,$$

where 0 of the first $\mathbb{P}^1$ is glued to ∞ of the second $\mathbb{P}^1$ and vice versa.
Using the discussion in the first few pages of [30, Sec. 1.4], we can express our invariant as a sum over weighted graphs with certain relative invariants as vertex terms, i.e.:

$$\begin{aligned} &\Big\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\frac{\mathrm{pt}}{1-\psi_1}\tau_1(\mathrm{pt})\Big\rangle^E \\ &= \sum_{d\geq 0} q^d \sum_{\sum_{i=1}^l b_i+2a\leq 2d} \frac{1}{l!}(-xz^2)^{l+1}a^2\prod_{i=1}^l b_i \end{aligned}$$

$$\big\langle (b_i)_i, a \big| \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\frac{\mathrm{pt}}{1-\psi_1}\big|(b_i)_i, a\big\rangle \big\langle a\big|\mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\tau_1(\mathrm{pt})\big|a\big\rangle$$

$$\#\Big\{(a_i>0, x_1>0, x_2>0)\Big|\sum_{i=1}^{l} a_i b_i + (x_1+x_2)a = 2d, a_i \text{ even}, x_i \text{ odd}\Big\}$$

$$+\,2\sum_{d\geq 0} q^d \sum_{\substack{\sum_{i=1}^{l} b_i + a+b+c \leq 2d \\ a=b+c}} \frac{1}{l!}\frac{1}{2}(-xz^2)^{l+2} abc \prod_{i=1}^{l} b_i$$

$$\big\langle (b_i)_i, a \big| \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\frac{\mathrm{pt}}{1-\psi_1}\big|(b_i)_i, b, c\big\rangle \big\langle a\big|\mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\tau_1(\mathrm{pt})\big|b,c\big\rangle$$

$$\#\Big\{(a_i>0, x_1>0, x_2>0, x_3>0)\Big|\sum_{i=1}^{l} a_i b_i + x_1 a + x_2 b + x_3 c = 2d, a_i \text{ even}, x_i \text{ odd}\Big\},$$

where we used the notations

$$\big\langle \mu\big|\mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\cdots\big|\nu\big\rangle = \sum_{g\geq 0}(-1)^g z^{2g}\big\langle \mu\big|\mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\cdots\big|\nu\big\rangle_g^{\mathbb{P}^1/\{0,\infty\}}$$

with ordered ramification profile and

$$\big\langle (b_i)_i, a \big| \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\frac{\mathrm{pt}}{1-\psi_1}\big|(b_i)_i, b, c\big\rangle = \big\langle b_1,...,b_l, a\big|\mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\frac{\mathrm{pt}}{1-\psi_1}\big|b_1,...,b_l,b,c\big\rangle$$

and the fact that for dimension reasons:

$$\big\langle \mu\big|\mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\tau_1(\mathrm{pt})|\nu\big\rangle = 0$$

unless $l(\mu)+l(\nu)\leq 3$.
Using Lemmas 5.11 and 5.13, we obtain:

$$(x+1)\sum_{d\geq 0} q^d \sum_{\sum_{i=1}^{l} b_i + 2a \leq 2d} \frac{1}{l!}(-xz^2)^{l+1} S(z)^x a^2 S(az)^2 \frac{S'(az)}{S(az)} \prod_{i=1}^{l} b_i S(b_i z)^2$$

$$\#\Big\{(a_i>0, x_1>0, x_2>0)\Big|\sum_{i=1}^{l} a_i b_i + (x_1+x_2)a = 2d, a_i \text{ even}, x_i \text{ odd}\Big\}$$

$$+\sum_{d\geq 0} q^d \sum_{\substack{\sum_{i=1}^{l} b_i + a+b+c \leq 2d \\ a=b+c}} \frac{1}{l!}(-xz^2)^{l+2} S(z)^x abc S(az)S(bz)S(cz) \prod_{i=1}^{l} \big(b_i S(b_i z)\big)^2$$

$$\#\{(a_i>0, x_1>0, x_2>0, x_3>0)|\sum_{i=1}^{l} a_i b_i + x_1 a + x_2 b + x_3 c = 2d, a_i \text{ even}, x_i \text{ odd}\}$$

$$= -x(x+1)z^2 \sum_{d\geq 0} q^d \sum_{\sum_{i=1}^{l} c_i + b + c = 2d} \frac{1}{l!}(-x)^l S(z)^x$$

$$\sum_{\substack{e|b,c\\ b/e,\ c/e \text{ odd}}} e^2 S(ez)^2 \frac{S'(ez)}{S(ez)} \prod_{i=1}^{l} \sum_{\substack{f|b_i\\ b_i/f \text{ even}}} \frac{(2\sinh(fz/2))^2}{f}$$

$$+ 8x^2 z \sum_{d\geq 0} q^d \sum_{\sum_{i=1}^{l} c_i+e+f+g=2d} \frac{1}{l!}(-x)^l S(z)^x \prod_{i=1}^{l} \sum_{\substack{h|c_i\\ c_i/h \text{ even}}} \frac{(2\sinh(hz/2))^2}{h}$$

$$\sum_{\substack{h_1|f,\ h_2|g,\ (h_1+h_2)|e\\ e/(h_1+h_2),\ f/h_1,\ g/h_2 \text{ odd}}} \sinh((h_1+h_2)z/2)\sinh(h_1 z/2)\sinh(h_2 z/2)$$

Now using equation (14) in Appendix A, we see that our above expression is of the form

$$\widetilde{\Theta}^x(-x(x+1)U + x^2 V),$$

where

$$U = z^2 \sum_d q^d \sum_{a+b=2d} \sum_{\substack{h|a,b\\ h/a,\ h/b \text{ odd}}} h^2 S(hz)^2 \frac{S'(hz)}{S(hz)}$$

and from the first calculation of this proof we know that V must be $z(2AG_2 + \frac{\wp'}{6} + \frac{A^3}{3})$. Hence it remains to identify U:

$$\begin{aligned}
U &= z^2 \sum_d q^d \sum_{a+b=2d} \sum_{\substack{h|a,b\\ h/a,h/b \text{ odd}}} h^2 S(hz)^2 \frac{S'(hz)}{S(hz)} \\
&= z^2 \sum_d q^d \sum_{h|d} dh S(hz)^2 \frac{S'(hz)}{S(hz)} \\
&= z^2 \sum_d q^d \sum_{h|d} dh (\frac{\sinh(hz/2)}{hz/2})^2 (\frac{1}{2}\frac{\cosh(hz/2)}{\sinh(hz/2)} - 1) \\
&= \sum_d q^d \sum_{h|d} (\frac{1}{2} z D_z - 1)\frac{d}{h}(2\sinh(hz/2))^2 \\
&= -(\frac{1}{2} z D_z - 1)\frac{D_\tau \Theta}{\Theta}
\end{aligned}$$

where we used (15) in appendix A. This concludes the proof with exception of the two Lemmas that we used. □

Lemma 5.11. For any $d \geq 0$ and ordered partitions $(a_i)_{i=1}^n$ and $(b_i)_{i=1}^m$ of d:

$$\left\langle (a_i)_i | \mathbb{E}^\vee(1)\mathbb{E}^\vee(x) \frac{\mathrm{pt}}{1-\psi_1} | (b_i)_i \right\rangle_d^{\mathbb{P}^1/\{0,\infty\}} = S(z)^x \prod_{i=1}^{n} S(a_i z) \prod_{i=1} S(b_i z)$$

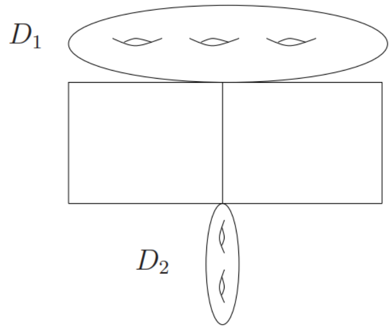


Figure 1: Fixed locus in rubber localization[5]

for $S(z) = \frac{\sinh(z/2)}{z/2}$.

Proof. Using the Mumford relation (12) and [34, (3.11)], we see that the claim holds for $x = -1$. Hence, we only need to show that $F(x)$ is proportional to $S(z)^x$ in x. To this end, we consider the invariant:

$$\Big\langle (a_1, H), ..., (a_n, H) \Big|\ \mathbb{E}^\vee(x)\ \Big|(b_1, H), ..., (b_m, H)\Big\rangle_{g,(d,1)}^{\mathbb{P}^1\times\mathbb{P}^1/\{0,\infty\}\times\mathbb{P}^1,\sim}$$

for fixed d and g which is rubber in the first factor. In fact, this invariant is independent of x as $(-1)^g\lambda_g$ is the only summand of $\mathbb{E}^\vee(x)$ that gives a nonzero contribution. We compute it using localization in the second variable and lift all hyperplane classes to H_0. The torus action will be the same as in Chapter 3.

It is not difficult to see that the only fixed loci which give a contribution consist of a tube with degree one in the second factor and two curves D_1 and D_2, where D_1 has genus g_1 and maps of degree d onto $\mathbb{P}^1 \times \{P_1\}$ and D_2 is a curve of genus g_2 with a constant map to $\mathbb{P}^1 \times \{P_0\}$ and empty ramification profile. See Figure 1. By specializing $\alpha_0 = 0$ and $\alpha_1 = 1$, we get:

$$\begin{aligned}
&-\sum_{g_1+g_2=g} \Big\langle (a_i)|\mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\frac{1}{1-\psi_1}|(b_i)\Big\rangle_{g_1,d}^{\mathbb{P}^1/\{0,\infty\},\sim} \\
&\Big\langle \emptyset|\mathbb{E}^\vee(-1)\mathbb{E}^\vee(x)\frac{1}{-1-\psi_1}|\emptyset\Big\rangle_{g_2,0}^{\mathbb{P}^1/\{0,\infty\},\sim} \\
&= \sum_{g_1+g_2=g} \langle (a_i)|\mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\frac{1}{1-\psi_1}|(b_i)\rangle_{g_1,d}^{\mathbb{P}^1/\{0,\infty\},\sim} \\
&\langle \emptyset|\mathbb{E}^\vee(1)\mathbb{E}^\vee(-x)\frac{1}{1-\psi_1}|\emptyset\rangle_{g_2,0}^{\mathbb{P}^1/\{0,\infty\},\sim} \\
&= \sum_{g_1+g_2=g} \langle (a_i)|\mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\frac{\mathrm{pt}}{1-\psi_1}|(b_i)\rangle_{g_1,d}^{\mathbb{P}^1/\{0,\infty\}}
\end{aligned}$$

[5]this is Figure 2 from [26]

$$\langle\emptyset|\mathbb{E}^\vee(1)\mathbb{E}^\vee(-x)\frac{\mathrm{pt}}{1-\psi_1}|\emptyset\rangle^{\mathbb{P}^1/\{0,\infty\}}_{g_2,0}$$
$$= \sum_{g_1+g_2=g} \langle(a_i)|\mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\frac{\mathrm{pt}}{1-\psi_1}|(b_i)\rangle^{\mathbb{P}^1/\{0,\infty\}}_{g_1,d},$$
$$\int_{\overline{M}_{g_2,1}} \mathbb{E}^\vee(0)\mathbb{E}^\vee(1)\mathbb{E}^\vee(-x)\frac{1}{1-\psi_1}$$

where we used rigidification [25, Lemma 2] in the second equality. The last Hodge integral was implicitly computed as $g_x(z)$ in the proof of [10, Prop. 3]:

$$\sum_{g\geq 0}(-1)^g z^{2g}\int_{\overline{M}_{g,1}} \mathbb{E}^\vee(0)\mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\frac{1}{1-\psi_1} = S(z)^x$$

and so $F(x)S(z)^{-x}$ is constant in x, which concludes the proof. ☐

Remark 5.12. (a) One can give a second proof of $F(x) = \widetilde{\Theta}^x$ by using degeneration, Lemma 5.11 and equation (14) in Appendix A.

(b) The argument used here is a slight generalization of the proof of [26, Lemma 27]. It seems that one should be able to compute all Hodge integrals over the tube $\mathbb{P}^1$ in a similar way. Then one could use degeneration to determine all Hodge integrals over E with only even insertions and use the methods of [36] for the odd insertions. This may be investigated in future work.

Lemma 5.13. We have:

$$\big\langle d|\mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\tau_1(\mathrm{pt})|d\rangle_d = (x+1)d\frac{S'(dz)}{S(dz)},$$
$$\big\langle a_1,a_2|\mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\tau_1(\mathrm{pt})|d\big\rangle_d = 1$$

where $a_1+a_2=d$ and the ramification profiles are ordered.

Proof. The second invariant is easy to compute:

$$\big\langle a_1,a_2|\lambda_g^2\tau_1(\mathrm{pt})|d\big\rangle^{\mathbb{P}^1/\{0,\infty\}}_{g,d} = \delta_{g,0}\big\langle a_1,a_2|\tau_1(\mathrm{pt})|d\big\rangle^{\mathbb{P}^1/\{0,\infty\}}_{0,d} = \delta_{g,0}$$

where the last equality follows from [35, (3.11)]. For the first invariant we use rigidification [25, Lemma 2] and the rubber dilaton equation [25, Sec. 1.5.4] to see

$$\big\langle d|\lambda_g\lambda_{g-1}\tau_1(\mathrm{pt})|d\big\rangle_{g,d} = -2g\big\langle d|\lambda_g\lambda_{g-1}|d\big\rangle^{\mathbb{P}^1/\{0,\infty\}}_{g,d},$$

and the last line is $(-1)^g B_{2g}\frac{d^{2g}}{(2g)!}$ by [18, Prop. 9]. Using the generating series

$$\sum_{g\geq 0}\frac{B_g}{g!}z^g = \frac{z}{e^z-1}$$

the rest of the claim now follows. ☐

Using this, we can compute another special case of F:

Proposition 5.14. We have:

$$F(x;\mathrm{pt};\tau_2(1)) = \widetilde{\Theta}^x(1+bx+cx^2)$$

for

$$\begin{aligned} b &= 2 - \frac{3}{2}zA - \frac{1}{2}z^2(\wp + 4G_2) \\ c &= 1 - \frac{3}{2}zA + \frac{1}{2}z^2(A^2 - 2G_2) \end{aligned}$$

Proof. Using the cycle-valued holomorphic anomaly equation in Theorem 4.5, we see:

$$\begin{aligned} &\left(\frac{d}{dG_2}\right)_P F(x;\mathrm{pt};\tau_1(\mathrm{pt})) \\ &= -z^2 F(x;\mathrm{pt};\tau_1(\mathrm{pt})) - 2z^2 F(x;\mathrm{pt};\tau_0(\mathrm{pt})) + 2\widetilde{\Theta}^x \langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\tau_0(\mathrm{pt})\rangle \\ &- 2\big(F(x;\mathrm{pt};\tau_2(1)) + F(x;1;\tau_1(\mathrm{pt})) - \langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\tau_0(\mathrm{pt})\rangle\big). \end{aligned}$$

Using Proposition 4.2 (b) and

$$\begin{aligned} \langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\tau_0(\mathrm{pt})\rangle_g^E &= \langle (x^2\lambda_g\lambda_{g-2} + x\lambda_{g-1}^2 + \lambda_{g-2}\lambda_g)\tau_0(\mathrm{pt})\rangle \\ &= x\delta_{g,1}\langle \tau_0(\mathrm{pt})\rangle_1 = x\delta_{g,1}G_2 \end{aligned}$$

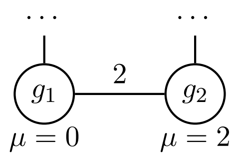
we see that every term is already determined by Propositions 5.6 and 5.10 except $F(x;\mathrm{pt};\tau_2(1))$, which can be computed in this way. □

Here is a list of some descendent invariants that can be computed using these Hodge integrals:

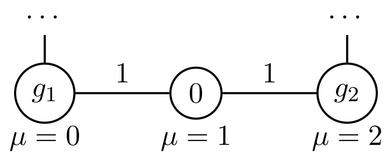
$$\begin{aligned} &\big\langle \tau_2(H\mathrm{pt})\big\rangle = \frac{3}{2}\Theta\big(-3zA + z^2(A^2 - 2G_2)\big) \\ &\big\langle \tau_0(H^2\mathrm{pt})\tau_2(1)\big\rangle = \Theta\big(9 - 8zA - \frac{1}{2}z^2(-3A^2 + 18G_2 + 3\wp)\big) \\ &\big\langle \tau_0(H^2\mathrm{pt})\tau_1(\mathrm{pt})\big\rangle = \Theta\big(-9(\frac{1}{2}A^2 + 2G_2 - \frac{1}{2}\wp) + z(\frac{3}{2}A^3 - \frac{9}{2}A\wp - \frac{3}{2}\wp')\big) \end{aligned}$$

5.4 Computations in degree 2

It turns out that the degree 2 case is not as easily computable as the degree 1 case. In fact, only invariants with at least one insertion $\tau_0(H^2\gamma)$ are accessible with our techniques. The reason for this is that we need to know more Hodge integrals than we can compute using the relations that localization gives us - except if we take the specialization $\alpha_i = i$ as in the degree 1 case. However, this might cause some denominators in (8) to become zero. More specifically, this happens for the contributions of the following two graphs:



and



where "$\cdots$" indicates the presence of an arbitrary number of markings. Both graphs can be avoided if one of the insertions is $\tau_0(H^2\gamma)$ since we can lift H^2 to H_0H_2 and therefore force all contributing graphs to have a vertex with $\mu = 1$ and a marking on it.
If this is not the case, we are dealing with one of the following three invariants (up to using the divisor equation):

$$\begin{gathered}\big\langle \tau_0(H\mathrm{pt})^6 \big\rangle, \\ \big\langle \tau_0(H\mathrm{pt})^{9-m-n}\tau_0(H^m\alpha)\tau_0(H^n\beta) \big\rangle, \\ \big\langle \tau_0(H\mathrm{pt})^{10}\tau_0(\alpha)\tau_0(\beta)\tau_0(H\alpha)\tau_0(H\beta) \big\rangle\end{gathered}$$

where $m, n = 0, 1$.
Using the argument of Remark 5.9, we can express the second and third invariant in terms of the first, which is hence the only unknown primary invariant.
We now compute the vertex terms for all other primary invariants:

Proposition 5.15. We have

$$\big\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(-1)\frac{\mathrm{pt}}{1-\psi_1}\frac{1}{-1-\psi_2} \big\rangle = 0,$$

$$X := \big\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(-1)\frac{\mathrm{pt}}{1-\psi_1}\frac{\mathrm{pt}}{-1-\psi_2} \big\rangle = \Theta^{-2} - (2G_2 + \wp),$$

$$Y := \big\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(2)\frac{\mathrm{pt}}{2-\psi_1}\frac{1}{1-\psi_2} \big\rangle = \frac{1}{3}\widetilde{\Theta}^{9/2},$$

$$Z := \big\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(2)\frac{\mathrm{pt}}{2-\psi_1}\frac{\mathrm{pt}}{1-\psi_2} \big\rangle = \widetilde{\Theta}^{9/2}X = \widetilde{\Theta}^{9/2}\Big(\Theta^{-2} - (2G_2 + \wp)\Big),$$

$$R := \big\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(-1)\frac{\mathrm{pt}}{1-\psi_1}\frac{1}{1-\psi_1} \big\rangle = -\frac{z}{\Theta^4\wp'},$$

$$S := \big\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(-1)\frac{\mathrm{pt}}{1-\psi_1}\frac{\mathrm{pt}}{1-\psi_2} \big\rangle = -\Theta^{-2} - \frac{2A}{\Theta^4\wp'},$$

$$T := \big\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(-1)\frac{\mathrm{pt}}{1/2-\psi_1} \big\rangle = -\frac{z}{\Theta^4\wp'},$$

$$U := \left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(2)\frac{\mathrm{pt}}{1-\psi_1}\frac{1}{1-\psi_2}\right\rangle = \frac{\Theta^8\wp}{2z^6},$$
$$V := \left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(2)\frac{\mathrm{pt}}{1-\psi_1}\frac{\mathrm{pt}}{1-\psi_2}\right\rangle = \frac{\Theta^4}{2z^6} - \frac{\Theta^8}{z^6}\Big(2G_2\wp + \frac{1}{2}\wp^2 + \frac{5}{6}G_4\Big),$$
$$W := \left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(2)\frac{\mathrm{pt}}{1/2-\psi_1}\right\rangle = \frac{\Theta^8}{z^5}\Big[z\Big(\frac{3}{4}G_2\wp + \frac{3}{16}\wp^2 + \frac{5}{16}G_4\Big) + \frac{3}{8}A\wp + \frac{1}{32}\wp'\Big]$$

Proof. Recall that the constant coefficient of the first invariant is set to zero. The scaling trick applied to the factor -1 yields:

$$-\left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(-1)\frac{\mathrm{pt}}{-1-\psi_1}\frac{1}{1-\psi_2}\right\rangle$$

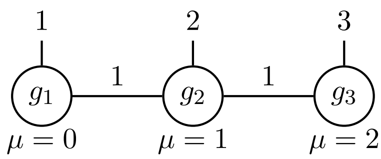
which by Proposition 4.2 (f) is the same invariant but with negative sign, hence this must have been zero. We can compute X using the Mumford relation (12) and [42]. Let $F_n(z_1, ..., z_n)$ denote the n-point correlation function:

$$\begin{aligned}
X &= -\sum_{g\geq 0} z^{2g}\Big\langle \frac{\mathrm{pt}}{1-\psi_1}\frac{\mathrm{pt}}{1+\psi_2}\Big\rangle_g = -\sum_{g\geq 0} z^{2g}\Bigg[\Big\langle \frac{\mathrm{pt}}{1-\psi_1}\frac{\mathrm{pt}}{1+\psi_2}\Big\rangle_g^\bullet \\
&- \sum_{g_1+g_2=g+1}\Big\langle \frac{\mathrm{pt}}{1-\psi_1}\Big\rangle_{g_1}^\bullet \Big\langle \frac{\mathrm{pt}}{1+\psi_2}\Big\rangle_{g_2}^\bullet\Bigg] = F_2(z,-z) - F_1(z)F_1(-z) \\
&= \lim_{a\to 0} F_2(z, a-z) - F_1(z)F_1(-z) = \lim_{a\to 0}\frac{A(z)+A(a-z)}{\Theta(a)} + \Theta^{-2} \\
&= -(2G_2+\wp) + \Theta^{-2}.
\end{aligned}$$

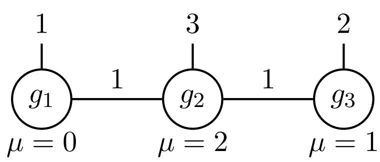
We can also compute R, S and T in a similar way.
Next, we have apply the localization formula to an invariant which is obviously zero:

$$\big\langle \tau_0(H^3\mathrm{pt})\tau_0(H^3)\tau_0(H^3)\big\rangle$$

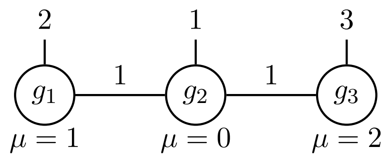
We lift it to $\langle \tau_0(H_1^2H_2\mathrm{pt})\tau_0(H_0^2H_2)\tau_0(H_0^2H_1)\rangle$, hence the only graphs contributing in localization are Γ_1:



as well as Γ_2:



and Γ_3:



Using the usual computational tricks, the three graphs produce the following contributions:

$$
\begin{aligned}
0 = &\Big\langle \mathbb{E}^\vee(-1)\mathbb{E}^\vee(-2)\frac{1}{-1-\psi_1}\tau_0(\mathrm{pt})\Big\rangle\Big\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(-1)\frac{\mathrm{pt}}{1-\psi_1}\frac{1}{-1-\psi_2}\tau_0(1)\Big\rangle \\
&\Big\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(2)\frac{\mathrm{pt}}{1-\psi_1}\tau_0(1)\Big\rangle + \Big\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(-1)\frac{\mathrm{pt}}{1-\psi_1}\tau_0(1)\Big\rangle \\
&\Big\langle \mathbb{E}^\vee(-1)\mathbb{E}^\vee(-2)\frac{1}{-1-\psi_1}\frac{1}{-2-\psi_2}\tau_0(\mathrm{pt})\Big\rangle\Big\langle \mathbb{E}^\vee(2)\mathbb{E}^\vee(1)\frac{\mathrm{pt}}{2-\psi_1}\tau_0(1)\Big\rangle \\
&+ \Big\langle \mathbb{E}^\vee(-1)\mathbb{E}^\vee(-2)\frac{1}{-2-\psi_1}\tau_0(\mathrm{pt})\Big\rangle\Big\langle \mathbb{E}^\vee(2)\mathbb{E}^\vee(1)\frac{\mathrm{pt}}{2-\psi_1}\frac{1}{1-\psi_2}\tau_0(1)\Big\rangle \\
&\Big\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(-1)\frac{\mathrm{pt}}{-1-\psi_1}\tau_0(1)\Big\rangle \\
&= -\widetilde{\Theta}^4 + \frac{3}{2}\widetilde{\Theta}^{-1}\widetilde{\Theta}^{1/2}Y + \frac{3}{2}\widetilde{\Theta}^{1/2}\widetilde{\Theta}^{-1}Y \\
&= -\widetilde{\Theta}^4 + 3Y\widetilde{\Theta}^{-1/2}
\end{aligned}
$$

determining Y. Similarly, to find out Z one computes the invariant

$$\langle \tau_0(H_1^2 H_2 \mathrm{pt})\tau_0(H_0 H_2 \mathrm{pt})\tau_0(H_0 H_1)\rangle$$

In this case we have to consider the same graphs as before. However, each graph now carries two point insertions from the elliptic curve which increases the number of nontrivial summands coming from the splitting axiom. The contribution of Γ_1 is:

$$
\begin{aligned}
&-\langle \mathbb{E}^\vee(-1)\mathbb{E}^\vee(-2)\frac{\mathrm{pt}}{-1-\psi_1}\tau_0(\mathrm{pt})\rangle\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(-1)\frac{1}{1-\psi_1}\frac{1}{-1-\psi_2}\tau_0(\mathrm{pt})\rangle \\
&\langle \mathbb{E}^\vee(2)\mathbb{E}^\vee(1)\frac{\mathrm{pt}}{1-\psi_1}\tau_0(1)\rangle - \langle \mathbb{E}^\vee(-1)\mathbb{E}^\vee(-2)\frac{1}{-1-\psi_1}\tau_0(\mathrm{pt})\rangle \\
&\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(-1)\frac{\mathrm{pt}}{1-\psi_1}\frac{1}{-1-\psi_2}\tau_0(\mathrm{pt})\rangle\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(2)\frac{\mathrm{pt}}{1-\psi_1}\tau_0(1)\rangle \\
&= -D_\tau(\widetilde{\Theta}^2)\widetilde{\Theta}^2 + \widetilde{\Theta}^4 X
\end{aligned}
$$

and similarly one can see that Γ_2 yields $-\frac{3}{2}\widetilde{\Theta}^{-1/2}Z$ and Γ_3 gives $\widetilde{\Theta}^{-1/2}(D_\tau Y + \frac{1}{2}Z) - \frac{3}{2}D_\tau(\widetilde{\Theta}^{-1})\widetilde{\Theta}^{1/2}Y$. This gives:

$$0 = \widetilde{\Theta}^4 X - \widetilde{\Theta}^{-1/2}Z,$$

which yields Z as claimed.
To compute U, V and W we have to use different techniques. See Appendix B for details. $\square$

Now we can finish the proof of Theorem 1.10:

Proof of the degree 2 part of Theorem 1.10. It is not difficult to see that Proposition 5.15 indeed gives all vertex terms up to using Proposition 4.2 and some terms that can be computed from Proposition 5.6.
This means that every such invariant is a polynomial in these Hodge integrals and hence of the claimed form.
For the index statement note that each of the above vertex terms becomes homogeneous with respect to index if we take the corresponding disconnected invariant. In fact, these will be the vertex terms in the disconnected localization formula and one can check explicitly that every graph yields a multiple of exactly Θ^4. $\square$

Remark 5.16. Note that we can avoid $\Theta(2z)^{-1}$ whenever we have at least three insertions of the form $\tau_0(\gamma H^2)$. Indeed, we can lift these insertions so that each corresponding marking has to lie over a different fixed point in $\mathbb{P}^2$, so only the vertex terms X, Y and Z contribute.
Without any computation we get

$$\left\langle \tau_0(H^2\mathrm{pt})\tau_0(H^2)^4 \right\rangle = -\Theta^4$$

since the left hand side must be in $\mathrm{QJac}[\Theta^{-1}]_{-4,2} = \mathbb{Q}\cdot\Theta^4$ and has z^4-coefficient:

$$-\left\langle \tau_0(H^2\mathrm{pt})\tau_0(H^2)^4 \right\rangle_0^{\mathbb{P}^2\times E} = -\left\langle \tau_0(H^2)^5 \right\rangle_0^{\mathbb{P}^2} = -1$$

It is also not difficult to compute the invariant directly.

Example 5.17. We compute the invariant $\langle \tau_0(H^2\mathrm{pt})^3\rangle_2$ by lifting it to:

$$\left\langle \tau_0(H_1H_2\mathrm{pt})\tau_0(H_0H_2\mathrm{pt})\tau_0(H_1H_2\mathrm{pt}) \right\rangle$$

so that only the graphs $\Gamma_1, \Gamma_2, \Gamma_3$ as depicted in the proof of Proposition 5.15 contribute in the localization formula. Each one now contributes 4 summands with the total sum being:

$$\sum_{a,b=0,1} \Bigg[\left\langle \mathbb{E}^\vee(-1)\mathbb{E}^\vee(-2)\frac{\mathrm{pt}^a}{-1-\psi_1}\tau_0(\mathrm{pt}) \right\rangle \left\langle \mathbb{E}^\vee(-1)\mathbb{E}^\vee(1)\frac{\mathrm{pt}^{1-a}}{1-\psi_1}\frac{\mathrm{pt}^{1-b}}{-1-\psi_2}\tau_0(\mathrm{pt}) \right\rangle$$

$$\left\langle \mathbb{E}^\vee(2)\mathbb{E}^\vee(1)\frac{\mathrm{pt}^b}{1-\psi_1}\tau_0(\mathrm{pt}) \right\rangle + \left\langle \mathbb{E}^\vee(-1)\mathbb{E}^\vee(-2)\frac{\mathrm{pt}^a}{-2-\psi_1}\tau_0(\mathrm{pt}) \right\rangle$$

$$\left\langle \mathbb{E}^\vee(2)\mathbb{E}^\vee(1)\frac{\mathrm{pt}^{1-a}}{2-\psi_1}\frac{\mathrm{pt}^{1-b}}{1-\psi_2}\tau_0(\mathrm{pt}) \right\rangle \left\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(-1)\frac{\mathrm{pt}^b}{-1-\psi_1}\tau_0(\mathrm{pt}) \right\rangle$$

$$
\begin{aligned}
&+ \big\langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(-1)\frac{\mathrm{pt}^a}{1-\psi_1}\tau_0(\mathrm{pt})\big\rangle \big\langle \mathbb{E}^\vee(-1)\mathbb{E}^\vee(-2)\frac{\mathrm{pt}^{1-a}}{-1-\psi_1}\frac{\mathrm{pt}^{1-b}}{-2-\psi_2}\tau_0(\mathrm{pt})\big\rangle \\
&\big\langle \mathbb{E}^\vee(2)\mathbb{E}^\vee(1)\frac{\mathrm{pt}^b}{2-\psi_1}\tau_0(\mathrm{pt})\big\rangle\Bigg] \\
&= \sum_{a,b=0,1}\Bigg[(-1)^a D_\tau^a(\widetilde{\Theta}^2)D_\tau^b(\widetilde{\Theta}^2)\Big(\delta_{a,0}\delta_{b,0}D_\tau X - \delta_{b,1}\delta_{a,0}X + \delta_{a,1}\delta_{b,0}X - \delta_{a,1}\delta_{b,1}\Big) \\
&+ (-1)^{a+b}2^a D_\tau^a(\widetilde{\Theta}^{1/2})D_\tau^b(\widetilde{\Theta}^{-1})\Big(\delta_{a,0}\delta_{b,0}D_\tau Z + \delta_{a,0}\delta_{b,1}(Z + D_\tau Y) \\
&+ \delta_{a,1}\delta_{b,0}(\frac{1}{2}Z + D_\tau Y) + \delta_{a,1}\delta_{b,1}\frac{3}{2}Y\Big) \\
&+ (-1)^{a+b}2^b D_\tau^a(\widetilde{\Theta}^{-1})D_\tau^b(\widetilde{\Theta}^{1/2})\Big(\delta_{a,0}\delta_{b,0}D_\tau Z + \delta_{a,1}\delta_{b,0}(Z + D_\tau Y) \\
&+ \delta_{a,0}\delta_{b,1}(\frac{1}{2}Z + D_\tau Y) + \delta_{a,1}\delta_{b,1}\frac{3}{2}Y\Big)\Bigg] \\
&= \widetilde{\Theta}^4 D_\tau X + 4\widetilde{\Theta}^4(\frac{D_\tau\Theta}{\Theta})^2 - 4\widetilde{\Theta}^4\frac{D_\tau\Theta}{\Theta}X + 2\widetilde{\Theta}^{-1/2}D_\tau Z \\
&+ \frac{D_\tau\Theta}{\Theta}\widetilde{\Theta}^{-1/2}Z - 3(\frac{D_\tau\Theta}{\Theta})^2\widetilde{\Theta}^{-1/2}Y \\
&= \widetilde{\Theta}^4\Big(\frac{3}{4}A^4 - \frac{9}{2}A^2\wp - \frac{9}{4}\wp^2 - 3A\wp' + 15G_4\Big)
\end{aligned}
$$

Because of the special way we sum over g, we have to multiply this by $-z^4$ and obtain

$$
\big\langle \tau_0(H^2\mathrm{pt})^3\big\rangle = \Theta^4\Big(-\frac{3}{4}A^4 + \frac{9}{2}A^2\wp + \frac{9}{4}\wp^2 + 3A\wp' - 15G_4\Big)
$$

which is independent of G_2 as we saw in Remark 1.6. Using Theorem 1.5, one can also deduce:

$$
\begin{aligned}
\big\langle \tau_0(H^2\mathrm{pt})^2\tau_1(H^2)\big\rangle &= z\frac{1}{3}\frac{d}{dA}\Bigg[\Theta^4\Big(-\frac{3}{4}A^4 + \frac{9}{2}A^2\wp + \frac{9}{4}\wp^2 + 3A\wp' - 15G_4\Big)\Bigg] \\
&= z\Theta^4\Big(-A^3 + 3A\wp + \wp'\Big)
\end{aligned}
$$

which we also could have computed directly.

Appendices

A Quasi-Modular and quasi-Jacobi forms

We collect the essentials that are important for this thesis. For more on quasi-modular forms, see in particular [5, 19] and [9] for Jacobi forms. Quasi-Jacobi forms were first introduced in [22]. See also [14, Sec. 2] and [15, Sec. 2].

A.1 Quasi-modular forms

The ring of quasi-modular forms $\mathrm{QMod} = \mathbb{Q}[G_2, G_4, G_6]$ is a free polynomial ring generated by Eisenstein series

$$G_k = -\frac{B_k}{2k} + \sum_{n\geq 1}\left(\sum_{d|n} d^{k-1}\right)q^n$$

for even $k \geq 2$ with B_k the kth Bernoulli number. We can define Bernoulli numbers in terms of the generating series

$$\sum_{k\geq 0}\frac{B_k}{k!}z^k = \frac{z}{e^z-1},$$

which is the convention that is used throughout this thesis.
QMod has a natural grading by weight

$$\mathrm{QMod} = \bigoplus_{k\geq 0}\mathrm{QMod}_k$$

so that $G_k \in \mathrm{QMod}_k$.
It is also worth mentioning that QMod is stable under the derivation

$$D_\tau := q\frac{d}{dq},$$

which does not hold for the subalgebra $\mathrm{Mod} = \mathbb{Q}[G_4, G_6]$ of modular forms.

A.2 Quasi-Jacobi forms

Much like quasi-modular forms, quasi-Jacobi forms can also be defined using certain transformation properties. However, we will not need this and only treat them as a certain kind of power series in two variables:
The most important example is

$$\Theta(z) = (e^{z/2} - e^{-z/2})\prod_{k\geq 1}\frac{(1-q^k e^z)(1-q^k e^{-z})}{(1-q^k)^2} = ze^{-2\sum_{k\geq 1} G_{2k}\frac{z^{2k}}{(2k)!}}. \tag{13}$$

Using this, we can also write Θ^n as a power series in q:

$$\Theta(z)^n = (2\sinh(z/2))^n \sum_{d\geq 0} q^d \sum_{\substack{\sum_{i=1}^l c_i = d,\\ c_i>0}} \frac{(-n)^l}{l!} \prod_{i=1}^{l} \sum_{h|c_i} \frac{(2\sinh(hz/2))^2}{h}, \tag{14}$$

which we use in Section 5.3.
This can be derived from the expansion

$$\log(\Theta) = \log(z) + \log(\frac{\sinh(z/2)}{z/2}) - \sum_{d>0} q^d \sum_{h|d} \frac{(2\sinh(hz/2))^2}{h}, \tag{15}$$

which follows directly from (13). All other generators of QJac can be defined in terms of Θ. We denote

$$A = \frac{D_z\Theta}{\Theta} = \frac{1}{z} - 2\sum_{k\geq 1} G_{2k} \frac{z^{2k-1}}{(2k-1)!}, \tag{16}$$

where $D_z = \frac{d}{dz}$ and $\wp$ the Weistrass $\wp$-function, which we can write as

$$\wp = -2G_2 - D_z A.$$

We also denote $\wp' = D_z \wp$.

Definition A.1. We define the ring of *quasi-Jacobi forms* as the subring

$$\mathrm{QJac} \subset \mathbb{Q}[\Theta, A, G_2, \wp, \wp', G_4]$$

of power series in z and $q = e^{2\pi i\tau}$ which are holomorphic as functions $(z,\tau) \in \mathbb{C} \times \mathbb{H} \to \mathbb{H}$. The ring is doubly graded

$$\mathrm{QJac} = \bigoplus_{k,m} \mathrm{QJac}_{k,m}$$

by *weight* k and *index* m, which is specified on generators as follows:

Form	weight	index
Θ	-1	1/2
A	1	0
G_2	2	0
$\wp$	2	0
$\wp'$	3	0
G_4	4	0

The ring of *weak Jacobi forms*(of half-integral index) is the subring

$$\widetilde{\mathrm{Jac}} = \mathbb{Q}[\Theta, \Theta^2\wp, \Theta^3\wp', G_4, G_6] \subset \mathrm{QJac}$$

which also inherits a double grading.

Remark A.2. (a) One can show that the ring $\mathbb{Q}[\Theta, A, G_2, \wp, \wp', G_4]$ is a free polynomial algebra, see [15, Prop. 21]. Furthermore, by writing $\frac{d}{dA}$ and $\frac{d}{dG_2}$ for the formal derivatives in this polynomial ring:

$$\widetilde{\mathrm{Jac}} = \left\{ \phi \in \mathrm{QJac} \;\middle|\; \frac{d}{dA}\phi = \frac{d}{dG_2}\phi = 0 \right\}$$

(b) One can show

$$\mathrm{QJac}_{*,0} = \mathrm{QMod}.$$

Indeed, the Weierstrass equation tells us $G_6 \in \mathrm{QJac}_{6,0}$ by

$$G_6 = \frac{12}{7}\wp^3 - \frac{3}{7}(\wp')^2 - \frac{60}{7}\wp G_4$$

and so we get "$\supset$".

(c) The weight of a given quasi-Jacobi form $\phi \in \mathrm{QJac}_{k,m}$ is easy to see from the Taylor expansion in z. Indeed, we always have:

$$\phi(z) = \sum_{g \geq 0} a_g z^g$$

with $a_g \in \mathrm{QMod}_{g+k}$ (c.f. [9, Thm. 3.1]). The index m cannot be obtained a similar way since it is more closely connected with the Fourier expansion of ϕ i.e. in terms of $p = e^z$. We will not need this, but we note that any one of the five generators $\Theta, A, G_2, G_4, \wp, \wp'$ only depends on q and e^z.

(d) We sometimes adjoin z and z^{-1} to QJac. Because of the previous remark, z is algebraically independent over QJac and one can extend the double-grading to $\mathrm{QJac}[z, z^{-1}]$:

Form	weight	index
Θ	-1	1/2
A	1	0
G_2	2	0
$\wp$	2	0
$\wp'$	3	0
G_4	4	0
z	-1	0
z^{-1}	1	0

(e) QJac is closed under D_z and $D_\tau = q\frac{d}{dq}$ as well as the formal derivatives $\frac{d}{dA}$ and $\frac{d}{dG_2}$, which have grading $(1, 0)$, $(2, 0)$, $(-1, 0)$ and $(-2, 0)$

respectively. In particular, $D_\tau\Theta, D_z\Theta \in \mathrm{QJac}$. There is also a notion of Hecke-operators on Jacobi forms, one of which is

$$\phi(z) \mapsto \phi(n \cdot z)$$

for $n \geq 0$ (see [9]), which also extends to quasi-Jacobi forms. For example:

$$\Theta(2z) = -\Theta^4\wp'$$

B More Hodge integrals over the elliptic curve

In this section, we will compute the following Hodge integrals:

$$\begin{aligned}
F(x) &:= \langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\frac{1}{1-\psi_1}\frac{\mathrm{pt}}{1-\psi_2}\rangle \\
G(x) &:= \langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\frac{\mathrm{pt}}{1-\psi_1}\frac{\mathrm{pt}}{1-\psi_2}\rangle \\
H(x) &:= \langle \mathbb{E}^\vee(1)\mathbb{E}^\vee(x)\frac{\mathrm{pt}}{1/2-\psi_1}\rangle
\end{aligned}$$

for $x \in \mathbb{Z}$.

Proposition B.1. F, G and H admit the following formulas:

(a) For $x = 0$: $F(0) = \frac{1}{2}$, $G(0) = 0$, $H(0) = \frac{1}{2}$.

(b) For $x > 0$ we have:

$$\begin{aligned}
F(x) &= 4^{x-1/2}\frac{(x-1)!^2}{(2x-1)!}\frac{\Theta^{4x}}{z^{2x+2}}a(x) \\
G(x) &= \frac{1}{x}\frac{\Theta^{2x}}{z^{2x+2}} - \frac{\Theta^{4x}}{z^{2x+2}}4^{x+1/2}\frac{(x-1)!^2}{(2x-1)!}\Big[G_2a(x) + b(x)\Big] \\
H(x) &= \frac{\Theta^{4x}}{z^{2x+1}}\Big[(A + 2zG_2)a(x) + z2b(x) + 2c(x)\Big]
\end{aligned}$$

for quasi-Jacobi forms $a(x), b(x), c(x) \in \mathbb{Q}[\wp, \wp', G_4]$ of weights $2x-2$, $2x$ and $2x-1$ respectively. They are characterized by

$$a(1) = \frac{1}{4},\ b(1) = \frac{1}{8}\wp,\ c(1) = 0$$

and the recursive formula:

$$\begin{aligned}
a(x) &= \frac{1}{2x}Da(x-1) + \frac{4x-3}{2x}\wp a(x-1) + \frac{1}{x}b(x-1) \\
b(x) &= \frac{1}{2}\wp a(x) - \frac{d}{dz}c(x) + (x-\frac{1}{2})c(x-1)\wp'
\end{aligned}$$

$$c(x) = \frac{1}{2x} Dc(x-1) + \frac{4x-3}{2x} \wp c(x-1) + \frac{1}{8x(x-1/2)} \frac{d}{dz} a(x)$$

for all $x \geq 2$, where D is the operator

$$D = -D_\tau + A\frac{d}{dz} - 2G_2 wt$$

of degree 2. It is easily checked that D indeed preserves $\mathbb{Q}[\wp, \wp', G_4]$.

(c) Likewise, formulas for $x < 0$ are given by:

$$F(x) = \frac{1}{2} z^{-2(x+1)} \Theta(2z)^{2x+1} \Theta(z)^{-4(x+1)} \Big[zd(x) - \frac{1}{2} e(x) \Big]$$
$$G(x) = \frac{1}{x} z^{-2(x+1)} \Theta(z)^{2x} + z^{-2(x+1)} \Theta(2z)^{2x+1} \Theta(z)^{-4(x+1)} \Big[Ad(x) + G_2 e(x) + f(x) \Big]$$
$$H(x) = 4^{-x-3/2} \frac{(-x-1)!(-x)!}{(-2x-1)!} z^{-2x-1} \Theta(2z)^{2x+1} \Theta(z)^{-4(x+1)} d(x)$$

where again $d(x), e(x), f(x) \in \mathbb{Q}[\wp, \wp', G_4]$ are of weights $-4(x+1)$, $-4x-5$ and $-4x-3$ and determined by

$$d(-1) = 2, \; e(-1) = 0, \; f(-1) = 0$$

and the recursive formula:

$$\begin{aligned} d(x) &= D(x) d(x+1) \\ e(x) &= -\frac{2}{x} \wp' d(x+1) + D(x) e(x+1) \\ f(x) &= D(x) f(x+1) - \frac{1}{x} \wp \wp' d(x+1) \end{aligned}$$

which holds for $x < -1$. Here, $D(x)$ is the operator

$$D(x) = \frac{1}{x} \wp' \frac{d}{dz} + 2\wp'' \frac{x+3/2}{x}$$

Sketch of proof. The idea is to use certain tautological relations on $\overline{M}_{g,2}$ much like in the proof of Proposition 5.6. In our case, these relations come from $\overline{M}_{g,2}(\mathbb{P}^1, 2)$. More specifically, we look at the bundle

$$E(n) := R^1 \pi_* f^* \mathcal{O}_{\mathbb{P}^1}(n)$$

for $n = 0, -1$, which comes from $\mathbb{P}^1$ via the maps:

$$\begin{array}{ccc} U_{g,2}(\mathbb{P}^1, 2) & \xrightarrow{f} & \mathbb{P}^1 \\ \big\downarrow{\scriptstyle \pi} & & \\ \overline{M}_{g,2}(\mathbb{P}^1, 2) & & \end{array}$$

For any integer $x \in \mathbb{Z}$, we can turn $\mathcal{O}_{\mathbb{P}^1}(n)$ into an equivariant bundle so that $\mathcal{O}_{\mathbb{P}^1}(n)|_{P_0}$ has weight $n\alpha_0 + x\alpha_1$ and $\mathcal{O}_{\mathbb{P}^1}(n)|_{P_1}$ has weight $n\alpha_1 + x\alpha_1$ - we do this by taking the canonical T action and tensoring with $\mathcal{O}_{\mathbb{P}^1}$ equipped with a constant action. Here, we always use the torus action introduced in Section 3.2. Using Riemann-Roch, we see that $E(n)$ also becomes an equivariant bundle of rank g if $n = 0$ and rank $g + 1$ if $n = -1$ so we can consider the following expressions:

$$\int_{[\overline{M}_{g,2}(\mathbb{P}^1,2)]^{vir,T}} ev_1^*(H_1^{3+n}) ev_2^*(H_0) e^T(E(n)) p^* I_g^E(\mathrm{pt}, 1)$$
$$\int_{[\overline{M}_{g,2}(\mathbb{P}^1,2)]^{vir,T}} ev_1^*(H_1^{2+n}) \psi_2 ev_2^*(H_0) e^T(E(n)) p^* I_g^E(\mathrm{pt}, 1)$$
$$\int_{[\overline{M}_{g,2}(\mathbb{P}^1,2)]^{vir,T}} ev_1^*(H_1^{2+n}) ev_2^*(H_1) e^T(E(n)) p^* I_g^E(\mathrm{pt}, \mathrm{pt})$$

where the Gromov-Witten classes of E are pulled back along the natural map $p\colon \overline{M}_{g,2}(\mathbb{P}^1, 2) \to \overline{M}_{g,2}$. For degree reasons, these are rational numbers that are independent of α_0 and α_1 and hence x, so we can specialize to $\alpha_0 = 0, \alpha_1 = 1$ and compute these expressions using localization. Note that the Euler class $e^T(E(n))$ can be computed using the normalization sequence on every fixed locus (c.f. [10, Ch. 2]). Using [42] and the Mumford relation (12) we can deduce

$$F(-1) = \frac{z}{\Theta(2z)}, G(-1) = -\Theta^{-2} + \frac{2A}{\Theta(2z)}, H(-1) = \frac{z}{\Theta(2z)}$$

and hence compute the three integrals for $x = -1$ and all g. After summing over g in the manner of Definition 5.4, the independence of x then yields the following two systems of linear equations:

$$\begin{aligned}
\mathrm{I}:& F(-x)\Big[x\widetilde{\Theta}^{2x-2} - x(x-1)z^2G(x-1)\Big] \\
&+ G(-x)\Big[x(x-1)z^2F(x-1)\Big] \\
&+ H(-x)\Big[-4(x-\frac{1}{2})H(x-1)\Big] = -(x-1)\widetilde{\Theta}^{-2x}F(x-1) \\
\mathrm{II}:& F(-x)\Big[x(x-1)z^3D_zG(x-1) - 2x\widetilde{\Theta}^{2x-2}((x-1)zA - x)\Big] \\
&+ G(-x)\Big[-x(x-1)z^3D_zF(x-1)\Big] \\
&+ H(-x)\Big[4(x-\frac{1}{2})(zD_zH(x-1) - H(x-1))\Big] \\
&= (x-1)\widetilde{\Theta}^{-2x}zD_zF(x-1) + \frac{1}{x\widetilde{\Theta}(2z)} \\
\mathrm{III}:& F(-x)\Big[4x(x-1)\frac{D_\tau\Theta}{\Theta}\widetilde{\Theta}^{2x-2} - 2x(x-1)z^2D_\tau G(x-1)\Big]
\end{aligned}$$

$$
\begin{aligned}
&+ G(-x)\Big[-x\widetilde{\Theta}^{2x-2} + x(x-1)z^2(G(x-1)+2D_\tau F(x-1))\Big]\\
&+ H(-x)\Big[-8(x-\frac{1}{2})D_\tau H(x-1)\Big]\\
&= -(x-1)\widetilde{\Theta}^{-2x}(G(x-1)+2D_\tau F(x-1)) + \frac{1}{\Theta(z)^2} - 2\frac{A(z)}{\Theta(2z)}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{IV}: &F(x)\Big[\widetilde{\Theta}^{-2x} + xz^2G(-x)\Big]\\
&+ G(x)\Big[-xz^2F(-x)\Big]\\
&+ H(x)\Big[-4H(-x)\Big] = -\widetilde{\Theta}^{2x}F(-x)\\
\mathrm{V}: &F(x)\Big[-xz^3D_zG(-x) - 2\widetilde{\Theta}^{-2x}(-xzA+x-1)\Big]\\
&+ G(x)\Big[xz^3D_zF(-x)\Big]\\
&+ H(x)\Big[4(zD_zH(-x)-H(-x))\Big] = \widetilde{\Theta}^{2x}zD_zF(-x)\\
\mathrm{VI}: &F(x)\Big[2xz^2D_\tau G(-x) - 4x\frac{D_\tau\Theta}{\Theta}\widetilde{\Theta}^{-2x}\Big]\\
&+ G(x)\Big[-\widetilde{\Theta}^{-2x} - xz^2(G(-x)+2D_\tau F(-x))\Big]\\
&+ H(x)\Big[-8D_\tau H(-x)\Big] = -\widetilde{\Theta}^{2x}(G(-x)+2D_\tau F(-x))
\end{aligned}
$$

The derivatives D_z and D_τ come from the dilaton and divisor equations respectively (note that both hold on the level of cycles). After specializing all equations to $z=0$, one sees that the first system has determinant $2x^2(x-\frac{1}{2})$ and the second one has determinant 2. Hence $F(x)$, $G(x)$ and $H(x)$ are determined for integer x by these equations and the values for $x = 0, -1$. Now one simply inserts the claimed formulas for $x < 0$ and $x > 0$ into these equations and shows inductively that they satisfy I through VI. □

Remark B.2. (a) In particular, we get

$$
F(1) = \frac{\Theta^4}{2z^4},\ G(1) = \frac{\Theta^2}{z^4} - \frac{\Theta^4}{z^4}(2G_2+\wp),\ H(1) = \frac{\Theta^4}{4z^3}(z(2G_2+\wp)+A)
$$

and

$$
\begin{aligned}
F(2) &= \frac{\Theta^8\wp}{2z^6},\ G(2) = \frac{\Theta^4}{2z^6} - \frac{\Theta^8}{z^6}(2G_2\wp + \frac{1}{2}\wp^2 + \frac{5}{6}G_4),\\
H(2) &= \frac{\Theta^8}{32z^5}(z(24G_2\wp + 6\wp^2 + 10G_4) + 12A\wp + \wp')
\end{aligned}
$$

(b) Note that G determines F because of Theorem 4.5.

(c) The tautological relations here are the ones used by Okounkov and Pandharipande to prove the Mariño-Vafa formula in [33].
The triple Hodge integrals there satisfy linear equations where the matrix entries are determined by the ELSV-formula [7] and the operator formalism of [35]. This allows them to write down an exact formula for essentially all triple Hodge integrals.
In our case, the matrix entries are given in terms of the same invariants that one tries to constrain, which creates an awkward recursive behavior.
Whether one can use a similar strategy to compute all Hodge integrals over E remains to be seen.

C List of invariants

We list here all connected degree $m > 0$ Gromov-Witten invariants on $\mathbb{P}^2 \times E$ which were computed in this thesis. For the $m = 0$ case see Remark 5.2.

$$\langle \tau_0(H^2\mathrm{pt})\tau_0(H^2)\rangle_1 = \Theta$$
$$\langle \tau_0(H^2\mathrm{pt})\tau_0(H\mathrm{pt})\rangle_1 = 3D_\tau\Theta$$
$$\langle \tau_0(H\mathrm{pt})^2\tau_0(H^2)\rangle_1 = 4D_\tau\Theta$$
$$\langle \tau_0(H\mathrm{pt})^3\rangle_1 = \Theta\left(3\frac{D_\tau^2\Theta}{\Theta} + 9(\frac{D_\tau\Theta}{\Theta})^2\right)$$
$$\langle \tau_1(H^2\mathrm{pt})\rangle_1 = z\Theta A$$
$$\langle \tau_0(H^2\mathrm{pt})\tau_1(H)\rangle_1 = \Theta(2zA - 3)$$
$$\langle \tau_0(H\mathrm{pt})^2\tau_1(H)\rangle_1 = z\left(2D_zD_\tau\Theta + 6\frac{D_\tau\Theta D_z\Theta}{\Theta}\right) - 6D_\tau\Theta$$
$$\langle \tau_2(H\mathrm{pt})\rangle_1 = \frac{3}{2}\Theta\big(-3zA + z^2(A^2 - 2G_2)\big)$$
$$\langle \tau_0(H^2\mathrm{pt})\tau_2(1)\rangle_1 = \Theta\big(9 - 8zA - \frac{1}{2}z^2(-3A^2 + 18G_2 + 3\wp)\big)$$
$$\langle \tau_0(H^2\mathrm{pt})\tau_1(\mathrm{pt})\rangle_1 = \Theta\big(-9(\frac{1}{2}A^2 + 2G_2 - \frac{1}{2}\wp) + z(\frac{3}{2}A^3 - \frac{9}{2}A\wp - \frac{3}{2}\wp')\big)$$
$$\langle \tau_0(H^2\mathrm{pt})\tau_0(H^2)^4\rangle_2 = -\Theta^4$$
$$\langle \tau_0(H^2\mathrm{pt})^3\rangle_2 = \Theta^4\Big(-\frac{3}{4}A^4 + \frac{9}{2}A^2\wp + \frac{9}{4}\wp^2 + 3A\wp' - 15G_4\Big)$$
$$\langle \tau_0(H^2\mathrm{pt})^2\tau_1(H^2)\rangle_2 = z\Theta^4\Big(-A^3 + 3A\wp + \wp'\Big)$$

References

[1] D. Anderson, *Introduction to Equivariant Cohomology in Algebraic Geometry (IMPANGA 2010)*, arXiv:1112.1421.

[2] K. Behrend, *Gromov-Witten invariants in algebraic geometry*, Invent. Math. **127** (1997), 601–617.

[3] K. Behrend and B. Fantechi, *The intrinsic normal cone*, Invent. Math. **128** (1997), 45-88.

[4] K. Behrend, *The product formula for Gromov–Witten invariants*, J. Algebraic Geom. 8 (1999), no. **3**, 529–541.

[5] S. Bloch and A. Okounkov, *The character of the infinite wedge representation*, Adv. Math. **149** (2000), no. 1, 1–60.

[6] D. Cox and S. Katz, *Mirror Symmetry and Algebraic Geometry*, Amer. Math. Soc. (1999).

[7] T Ekedahl, S Lando, M Shapiro, A Vainshtein, *Hurwitz numbers and intersections on moduli spaces of curves*, Invent. Math. **146** (2001) 297–327

[8] D. Edidin and W. Graham, *Equivariant intersection theory (With an Appendix by Angelo Vistoli: The Chow ring of M2)*, Invent. Math **131** (1998), 595–634.

[9] M. Eichler and D. Zagier, *The theory of Jacobi forms*, volume 55 of Progress in Mathematics, Birkh¨auser Boston Inc., Boston, MA, 1985.

[10] C. Faber and R. Pandharipande, *Hodge integrals and Gromov–Witten theory*, Invent. Math. **139** (2000), no. 1, 173–199.

[11] W. Fulton, *Equivariant Cohomology in Algebraic Geometry*, `https://people.math.osu.edu/anderson.2804/eilenberg/`.

[12] T. Graber and R. Pandharipande, *Localization of virtual classes*, Invent. Math **135** (1999), 487-518.

[13] K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. Thomas, C. Vafa, R. Vakil and E. Zaslow *Mirror symmetry*, Clay Mathematics Monographs, vol. 1, American Mathematical Society, Providence, RI, 2003.

[14] J. van Ittersum, *The Bloch–Okounkov theorem for congruence subgroups and Taylor coefficients of quasi-Jacobi forms*, arXiv:2102.12964.

[15] J. van Ittersum, G. Oberdieck, A. Pixton, *Gromov-Witten theory of K3 surfaces and a Kaneko-Zagier equation for Jacobi forms*,arXiv:2007.03489.

[16] M. Huang, S. Katz, A. Klemm, *Topological string on elliptic CY 3-folds and the ring of Jacobi forms*, J. High Energy Phys. 2015, no. **10**, 125.

[17] F. Janda, *Gromov-Witten theory of target curves and the tautological ring*, Michigan Math. J. **66** (2017), no. 4, 683–698.

[18] F. Janda, R. Pandharipande, A. Pixton and D. Zvonkine, *Double ramification cycles on the moduli spaces of curves*, Publ. Math. Inst. Hautes Etudes Sci. **125**(1) (2017), 221–266.

[19] M. Kaneko and D. Zagier, *A generalized Jacobi theta function and quasimodular forms* in The moduli space of curves, (Texel Island, 1994), 165–172, Progr. Math., **129**, Birkh¨auser Boston, Boston, MA, 1995.

[20] J. Li, Stable morphisms to singular schemes and relative stable morphisms, J. Differential Geom. **55** (2001), no. 3, 509–578.

[21] J. Li, A degeneration formula for Gromov-Witten invariants, J. Differential Geom. **60** (2002), no. 2, 199–293.

[22] A. Libgober, *Elliptic genera, real algebraic varieties and quasi-Jacobi forms*, Topology of stratified spaces, 95–120, Math. Sci. Res. Inst. Publ., **58**, Cambridge Univ. Press, Cambridge, 2011.

[23] C. Liu, *Localization in Gromov–Witten theory and orbifold Gromov–Witten theory*, Handbook of Moduli, vol II, 353-425, Internat. Press and Higher Education Press (2013).

[24] D. Maulik, N. Nekrasov, A. Okounkov, R. Pandharipande, *Gromov-Witten theory and DonaldsonThomas theory II.*, Compos. Math. **142**, 1286–1304 (2006).

[25] D. Maulik and R. Pandharipande, *A topological view of Gromov-Witten theory*, Topology **45**, 887–918, 2006.

[26] D. Maulik, R. Pandharipande and R. Thomas, *Curves on K3 surfaces and modular forms*, J. Topol. **3** (2010), 937–996.

[27] M. Moreira, A. Oblomkov, A. Okounkov, R. Pandharipande, *Virasoro constraints for stable pairs on toric 3-folds*, arXiv:2008.12514.

[28] D. Mumford, *Towards an enumerative geometry of the moduli space of curves*, in Arithmetic and Geometry (M. Artin and J. Tate, eds.), Part II, Birkhäuser, 1983, 271-328.

[29] G. Oberdieck and A. Pixton, *Gromov–Witten theory of elliptic fibrations: Jacobi forms and holomorphic anomaly equations*, Geom. Topol. **23** (2019), no. 3, 1415–1489.

[30] G. Oberdieck and A. Pixton, *Holomorphic anomaly equations and the Igusa cusp form conjecture*, Invent. Math **213** (2018), no. 2, 507–587.

[31] G. Oberdieck and J. Shen, *Curve counting on elliptic Calabi-Yau threefolds via derived categories*, J. Eur. Math. Soc. **22** (2020) 967.

[32] A. Oblomkov, A. Okounkov, R. Pandharipande, *GW/PT descendent correspondence via vertex operators*, Comm. Math. Phys. **374** (2020), no. 3, 1321–1359.

[33] A. Okounkov and R. Pandharipande, *Hodge integrals and invariants of the unknot*, Geom. Top. **8** (2004), 675-699.

[34] A. Okounkov and R. Pandharipande, *Gromov-Witten theory, Hurwitz theory, and completed cycles*, Ann. of Math. (2) **163** (2006), no. 2, 517–560.

[35] A. Okounkov and R. Pandharipande, *The equivariant Gromov-Witten theory of* $\mathbb{P}^1$, Ann. of Math., **163** (2006), 561–605.

[36] A. Okounkov and R. Pandharipande, *Virasoro constraints for target curves*, Invent. Math. **163** (2006), 47–108.

[37] R. Pandharipande, *Descendents for stable pairs on 3-folds*, Modern Geometry: A celebration of the work of Simon Donaldson, Proc. Sympos. Pure Math. **99** (2018), 251–288.

[38] R. Pandharipande and A. Pixton, *Gromov-Witten/Pairs correspondence for the quintic threefold*, J. Amer. Math. Soc. **30** (2017), 389–449.

[39] R. Pandharipande and A. Pixton, *Gromov-Witten/Pairs descendent correspondence for toric 3-folds*, Geom. Topol. **18** (2014), 2747–2821.

[40] R. Pandharipande and R. P. Thomas, *13/2 ways of counting curves*, in Moduli spaces, 282–333, LMS Lecture Note Ser. **411**, Cambridge Univ. Press, Cambridge, 2014.

[41] R. Pandharipande and R. P. Thomas, *Curve counting via stable pairs in the derived category*, Invent. Math. **178** (2009), 407–447.

[42] A. Pixton, *Gromov–Witten theory of an elliptic curve and quasimodular forms*, Senior Thesis, Princeton University (2009).

[43] J. Schmitt, *The moduli space of curves*, Lecture notes, `https://www.math.uni-bonn.de/~schmitt/ModCurves/Script.pdf`.

[44] Y. Toda, *Stability conditions and curve counting invariants on Calabi–Yau 3-folds*, Kyoto J. Math. **52** (2012), no. 1, 1–50.