On Noether-Lefschetz Theory on Compactifications of the Moduli Space of K3 Surfaces

Jonas Baltes

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Advisor: Prof. Dr. Georg Oberdieck

Second Advisor: Prof. Dr. Daniel Huybrechts
MATHEMATISCHES INSTITUT

MATHEMATISCH-NATURWISSENSCHAFTLICHE FAKULTÄT DER

RHEINISCHEN FRIEDRICH-WILHELMS-UNIVERSITÄT BONN

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1. Introduction

In this thesis we are going to generalize a classical result of Borcherds. The goal is to count line bundles of a fixed degree in families and degenerations of elliptic K3 surfaces.

Let $\mathcal{M}_l$ be the moduli space of quasi-polarized K3 surfaces (X, L) of degree $L^2 = l$. We can define the Noether-Lefschetz divisors $NL_d \subset \mathcal{M}$ for every $d < 0$ which consist set-theoretically of all quasi-polarized K3 surfaces (X, L) whose Picard group contains elements

$$v \in \mathrm{Pic}(X)$$

such that the intersection product admits

$$\begin{aligned} v^2 &= 2d, \\ v \cdot L &= 0. \end{aligned} \tag{1.0.1}$$

Given a quasi-polarized family of K3 surfaces $X \to C$ over a curve C, this allows us to count line bundles as above for all surfaces of the family at once: Firstly, we map C to the moduli space $\mathcal{M}_l$ by just taking the isomorphism class of the corresponding surface for every point. Then we can calculate the intersection

$$NL_d \cdot C$$

which computes the number of line bundles v that satisfy (1.0.1) with a multiplicity. As it turns out, there are many relations between the intersections $(NL_d \cdot C)_d$, which have been investigated by Borcherds: Taking into account the Hodge bundle $\lambda \in \mathrm{Pic}(\mathcal{M}_l)$ we get:

Theorem. *The generating series*

$$C \cdot \phi = C \cdot \lambda + \sum_{n>0} C \cdot NL_{-n} q^n \tag{1.0.2}$$

is a modular form of weight $\frac{21}{2}$ for a subgroup of $SL(2, \mathbb{Z})$.

As the space of such modular forms is finite dimensional, knowing finitely many of the intersection products amounts to knowing all of them. For example in [MP07] they calculated them for generic families of K3 surfaces of small degree: In the degree $d = 2$ case a general K3 surface is a double cover of the projective space along a sextic curve. Taking a generic hypersurface of type $(6, 2)$ in

$$\mathbb{P}^2 \times \mathbb{P}^1$$

one can construct a family of K3 surfaces by taking the double cover with branch locus this surface. Then

$$C \cdot \phi = \frac{1}{1024}(U^{21} - 12U^{17}V^4 - 402U^{13}V^8 - 572U^9V^{12} - 39U^5V^{16})$$

for

$$U = \sum_{n\in\mathbb{Z}} q^{n^2/4}, \quad V = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2/4}.$$

In this thesis we will investigate this behavior for *elliptic* K3 surfaces and extend the theory to degenerations.

The plan is as follows:
In Section 2 we will introduce *elliptic* K3 surfaces which carry two distinguished divisors: a section s and a fiber f. The span $\langle s, f\rangle_{\mathbb{Z}}$ is a lattice with intersection form

$$U = \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}.$$

As it turns out, every K3 surface that has a sublattice U is elliptic. Therefore we can form the moduli space of such K3 surfaces by allowing only U-quasi-polarized surfaces, i.e. a certain lattice polarization. This is explained in detail in Section 3. Again, we can define the Noether-Lefschetz divisors similar to the classical case. By Borcherds result, one obtains that the generating series (1.0.2) is a modular form of weight 10 for the full modular group $\mathrm{SL}(2,\mathbb{Z})$, i.e. an integral multiple of the Eisenstein series E_{10}.

We would like to generalize this result to *degenerations* of K3 surfaces. These are introduced in Section 4. Instead of allowing only families of K3 surfaces, we allow families $X \to C$ such that there are finitely many points $p \in C$ where the surface X_p is not necessarily K3. Under mild assumptions on the degeneration, one can classify them into three classes: type I (i.e. K3 surfaces), type II and type III. In this thesis only type II degenerations are treated. Unfortunately, the period map

$$C \dashrightarrow \mathcal{M}$$

does not necessarily extend to the whole of the curve C as the space $\mathcal{M}$ is not compact.

To avoid this circumstance, we introduce compactification of the moduli space in Section 5. One choice is the *Baily-Borel compactification* $\mathcal{M}^{\mathrm{bb}}$. This space adds three boundary components: One for type III degenerations and two for type II degenerations. As these components have high codimension, the space is singular and we have to take another compactification into consideration: The *toroidal compactification* $\mathcal{M}^{\mathrm{Tor}}_{\Sigma}$ which depends on the choice of a fan Σ. It comes equipped with a map $\mathcal{M}_\Sigma^{\mathrm{Tor}} \to \mathcal{M}^{\mathrm{bb}}$. Therefore the boundary has three components as well, two corresponding to type II and one for type III degenerations. From an explicit description of the type II boundary - which is given in Section 5 - it follows that these type II components do not depend on the choice of fan Σ.

For any degeneration $X \to C$ the period map extends to a map

$$C \to \mathcal{M}_\Sigma^{\mathrm{Tor}}.$$

We can take the closure $\overline{NL}_d$ of the Noether-Lefschetz divisor in the space $\mathcal{M}_\Sigma^{\mathrm{Tor}}$ and an extension $\overline{\lambda}$ of λ. In Section 8 we will prove the following main theorem. It was proven for one of the type II components by François Greer in [Gre18] and we will follow his argument with slight modifications to allow both type II components.

Theorem. *Let $X \to C$ be a degeneration of K3 surfaces, such that $C \to \mathcal{M}^{Tor}$ only meets the boundary in type II components. Then the generating series*

$$C \cdot \phi = C \cdot \overline{\lambda} + \sum_{n>0} C \cdot \overline{NL}_{-n} q^n$$

is an element of

$$\mathbb{Z}E_{10} \oplus \frac{1}{480}\mathbb{Z}DE_8.$$

where DE_8 is the derivation of the Eisenstein series E_8, i.e. a quasi-modular form of weight 10 for the full modular group $\mathrm{SL}(2,\mathbb{Z})$.

In Section 9, we will calculate the modular form for a specific example similar to the one presented before. As explained in Section 2, a double cover of $\mathbb{F}_4 \xrightarrow{p} \mathbb{P}^1$ over the section times a generic element in $\mathcal{O}_{\mathbb{F}_4}(3) \otimes p^*\mathcal{O}(12)$ produces an elliptic K3 surface. Taking certain quadratic pencils creates degenerations

$$X \to \mathbb{P}^1$$

of type II. We will compute that - under certain conditions on the pencil - we get

$$\begin{aligned} \mathbb{P}^1 \cdot \bar{\lambda} + \sum_{\substack{n\in\mathbb{Z} \\ n>0}} \mathbb{P}^1 \cdot \overline{\mathrm{NL}}_{-n} q^n &= -E_{10} - \frac{263}{480} DE_8 \\ &= -1 + 144q^1 + 67578q^2 + 3470244q^3 + \dots \end{aligned}$$

2. Elliptic K3 Surfaces

2.1. Basics on K3 Surfaces

Definition 2.1. A *K3 surface* is a smooth complex surface X such that

$$\omega_X = \mathcal{O}_X$$

and

$$H^1(X, \mathcal{O}_X) = \{0\}.$$

Remark 2.2. As we will see later on, we impose some conditions on these K3 surfaces such that we are mainly concerned with algebraic K3 surfaces.

Next, we will analyse the structure of the second integral cohomology which comes equipped with an intersection form from the map $H^2(X, \mathbb{Z}) \times H^2(X, \mathbb{Z}) \to H^4(X, \mathbb{Z}) \cong \mathbb{Z}$.

Proposition 2.3 (Huybrechts[Huy16])**.** *A complex K3 surface is Kähler and the Hodge diamond of a K3 surface is given by*

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1. \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

□

Theorem 2.4 (Huybrechts [Huy16])**.** *For a complex K3 surface X, the lattice $H^2(X, \mathbb{Z})$ is isomorphic to*

$$\Lambda_{3,19} = E_8(-1)^2 \oplus U^3.$$

□

As we will see in Section 3 these two facts give rise to a simple classification of K3 surfaces. Later on, we are interested in counting certain line bundles on K3 surfaces. Doing this amounts to knowing the Hodge decomposition as can be seen from this lemma:

Lemma 2.5 (Huybrechts [Huy16])**.** *For a complex K3 surface we have the following canonical isomorphism*

$$\mathrm{Pic}(X) \cong H^{1,1}(X) \cap H^2(X, \mathbb{Z})$$

which is induced by the first Chern class of a line bundle and respects the intersection pairing.

Remark 2.6. By the last lemma we can identify line bundles L with their first Chern class $c_1(L) \in H^2(X, \mathbb{Q})$. Therefore we use the notation $L \in H^2(X, \mathbb{Q})$ from now on as no distinction is needed for our purposes.

2.2. Elliptic Surfaces

In this section we introduce a special class of K3 surfaces, namely elliptic ones. This will later on allow us to pick *two* line bundles - the fiber and the section class - which simplifies the constructions in Section 8. Here we mainly follow [Mir89].

2.2.1. Elliptic K3 Surfaces

Definition 2.7. An *elliptic surface* is a complex surface X together with an *elliptic fibration*, i.e. a holomorphic map $p : X \to C$ to a smooth curve C, such that

- the general fiber of p is a smooth connected curve and has genus 1,
- every fiber is irreducible and
- the map p admits a section $s : C \to X$.

By [Huy16], a surjective map $X \to C$ from a K3 surface to a smooth curve C can only exist if $C \cong \mathbb{P}^1$. Hence, we define further:

Definition 2.8 ([CD07])**.** An *elliptic K3 surface* is a K3 surface X that admits an elliptic fibration $p : X \to \mathbb{P}^1$.

A simple calculation determines the lattice generated by a fiber and the section:

Lemma 2.9. *Let $X \xrightarrow{p} \mathbb{P}^1$ be a elliptic K3 surface. Then the lattice $L = \langle f, s \rangle_{\mathbb{Z}}$, where f is a fiber and s is the section, has the intersection form*

$$\begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}.$$

Proof. A fiber and the section meet transversally at one point. Hence $f \cdot s = 1$. On the other hand any two distinct fibers are linearly equivalent as they are just the pullback of different points of $\mathbb{P}^1$. Hence $f \cdot f = 0$. As any section is isomorphic to $\mathbb{P}^1$, we get by the adjunction formula

$$-2 = 2g - 2 = s \cdot (s + \omega_X) = s \cdot s,$$

where ω_X is the canonical sheaf, which is trivial. □

Remark 2.10. Substituting s by $s+f$, one sees that this lattice is isomorphic to the standard two-dimensional indefinite unimodular lattice U (see appendix A). On the other hand, a deeper result shows, that if there exists a line bundle L with $L.L = 0$, then X has an elliptic fibration, i.e. a map as above but without a section. Moreover every K3 surface that admits an injection $U \hookrightarrow \mathrm{Pic}(X)$ is elliptic.

Our goal is now, to construct elliptic K3 surfaces from rational surfaces. Recall that we defined $\mathcal{O}_{\mathbb{F}_4}(a,b) = \mathcal{O}_{\mathbb{F}_4}(a) \otimes p^*\mathcal{O}_{\mathbb{P}^1}(b)$ for the Hirzebruch surface $\mathbb{F}_4 \xrightarrow{p} \mathbb{P}^1$ in appendix D.

Lemma 2.11. *Any smooth double cover X of $\mathbb{F}_4 = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}_X(-4)) \xrightarrow{p} \mathbb{P}^1$, whose associated branch locus belongs to $\mathcal{O}_{\mathbb{F}_4}(4,12)$, is a K3 surface.*

Proof. Denote by $f : X \to \mathbb{F}_4$ the double cover. As shown in appendix D, $\omega_{\mathbb{F}_4} = \mathcal{O}_{\mathbb{F}_4}(-2,-6)$. From the standard theory of double covers we get

$$\omega_X = f^*(\omega_{\mathbb{F}_4} \otimes \mathcal{O}_{\mathbb{F}_4}(2,6)).$$

Hence

$$\omega_X = f^*\mathcal{O}_{\mathbb{F}_4} = \mathcal{O}_X.$$

Furthermore any double cover is a finite morphism. Hence we can compute the cohomology in the easier space $\mathbb{F}_4$. First we compute $f_*\mathcal{O}_X = \mathcal{O}_{\mathbb{F}_4} \oplus \mathcal{O}_{\mathbb{F}_4}(-2,-6)$. Then

$$H^1(X,\mathcal{O}_X) = H^1(\mathbb{F}_4, f_*\mathcal{O}_X) = H^1(\mathbb{F}_4,\mathcal{O}_{\mathbb{F}_4}) \oplus H^1(\mathbb{F}_4,\mathcal{O}_{\mathbb{F}_4}(-2,-6)).$$

Using Serre duality, we get for the second term

$$H^1(\mathbb{F}_4,\mathcal{O}_{\mathbb{F}_4}(-2,-6)) \cong H^1(\mathbb{F}_4,\mathcal{O}_{\mathbb{F}_4})^*.$$

Again, by appendix D this cohomology group is 0, hence $H^1(X,\mathcal{O}_X) = 0$ and X is a K3 surface as claimed. $\square$

An elliptic fibration is now constructed by specifying the branch locus further: Recall from appendix D that $Z \in \mathcal{O}_{\mathbb{F}_4}(1,0)$ is chosen such that $V(Z)$ is the section.

Corollary 2.12. *Let $X \xrightarrow{p} \mathbb{F}_4$ be a double cover of $\mathbb{F}_4 \xrightarrow{\pi} \mathbb{P}^1$ that is associated to a section of the form*

$$Z \cdot h,$$

where the vanishing locus of the irreducible $h \in \mathcal{O}_{\mathbb{F}_4}(3,12)$ is smooth and disjoint from the one of Z. Then $X \xrightarrow{\pi\circ p} \mathbb{P}^1$ is an elliptic K3 surface.

Proof. By the foregoing lemma, X is a K3 surface. The only thing we need to show is the ellipticity. We know that $V(Z) \subset \mathbb{F}_4$ is a section of the ruling, and by the assumptions $p^{-1}(V(Z)) \in X$ maps isomorphically to its image in $\mathbb{F}_4$. Hence $S = p^{-1}(V(Z))$ is a section of $X \xrightarrow{p} \mathbb{P}^1$. On the other hand, let $F \overset{i}{\hookrightarrow} X$ be a generic smooth fiber of p. Then by locality of the construction of the double cover, F is a double cover of $\mathbb{P}^1$ (as X is a double cover of the $\mathbb{P}^1$-bundle $\mathbb{F}_4$) with corresponding equation in $i^*\mathcal{O}_X(4)$. Then by the Hurwitz equation, we get

$$2g - 2 = 2(2g' - 2) + 4 = 0.$$

Therefore the genus of F is 1, i.e. an elliptic curve. □

This construction even exists in greater generality, as is shown in the next chapter.

2.2.2. Weierstraß Fibrations

Definition 2.13. A map $p : X \to C$ from a surface X to a curve C is called a *Weierstraß fibration* if

- p is flat and proper,
- every geometric fiber has arithmetic genus 1,
- the general fiber is smooth and
- there is a given section that does not hit possible singularities of the fibers.

For elliptic curves E it is well known that they admit a *Weierstraß form* determined by two numbers $a, b \in \mathbb{C}$, i.e.

$$E \cong \mathrm{V}(y^2 = x^3 + ax + b).$$

It turns out, that there are sections of line bundles A, B on C, that mimic this behavior pointwise, i.e. give Weierstraß data for the fiber X_c for every $c \in C$.

Theorem 2.14 (Miranda[Mir89])**.** *For a Weierstraß fibration $p : X \to C$ with section $s : C \to X$, the sheaf $\mathbb{L} = R^1p_*\mathcal{O}_X \cong s^*\Omega^{-1}_{X/C}$ is a line bundle and p is totally determined by giving two sections $A \in H^0(C, \mathbb{L}^4)$ and $B \in H^0(C, \mathbb{L}^6)$.*

Moreover every Weierstraß fibration has an explicit description in two different ways:

Lemma 2.15 (Miranda[Mir89])**.** *Let $p: X \to C$ be a Weierstraß fibration with fundamental line bundle $\mathbb{L}$ and Weierstraß data $A \in H^0(C, \mathbb{L}^4), B \in H^0(C, \mathbb{L}^6)$. Then $X \to C$ is isomorphic to the divisor*

$$Y^2Z = X^3 + AXZ^2 + BZ^3$$

in the $\mathbb{P}^2$-bundle $W = \mathbb{P}(\mathcal{O} \oplus \mathbb{L}^{-2} \oplus \mathbb{L}^{-3}) \xrightarrow{\pi} C$, where $Z \in H^0(W, \mathcal{O}_B(1)), X \in H^0(W, \mathcal{O}_B(1) \otimes \pi^*\mathbb{L}^2)$ and $X \in H^0(W, \mathcal{O}_B(1) \otimes \pi^*\mathbb{L}^3)$.

Therefore we get as a corollary the second description by looking at the double cover that is induced by the double cover $E = \mathrm{V}(y^2 = x^3 + ax + b) \to \mathbb{P}^1$, which just omits the y-coordinate.

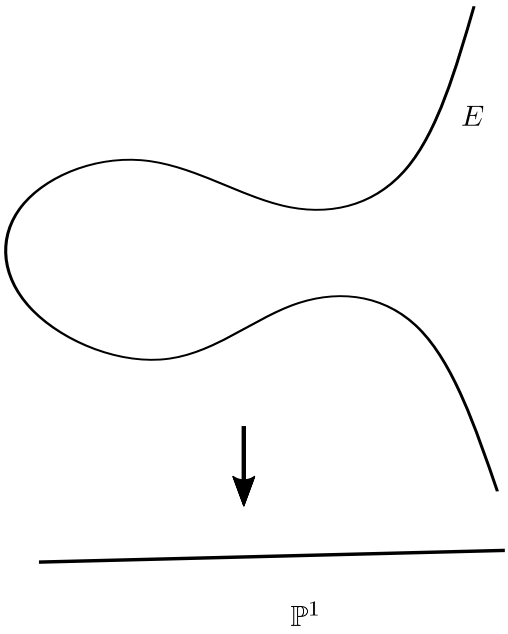

Figure 1: Projection of an elliptic curve to projective space.

Hence we have, with the same notation as above:

Lemma 2.16. *Let $\mathbb{F} = \mathbb{P}(\mathcal{O} \oplus \mathbb{L}^{-2}) \to C$. Then X is isomorphic to the double cover of $\mathbb{F}$ over the curve*

$$Z \cdot (X^3 + AXZ^2 + BZ^3).$$

The following Lemma analyses the fundamental line bundle $\mathbb{L}$ further:

Proposition 2.17. *Let $X \xrightarrow{p} C$ be a Weierstraß fibration. Then*

$$\omega_X = p^*(\omega_C \otimes \mathbb{L}).$$

Proof. See [Mir89]. □

Lemma 2.18. *An elliptic surface X over $\mathbb{P}^1$ that is a Weierstraß fibration with fundamental line bundle $\mathbb{L}$ is a K3 surface if and only if $\mathbb{L} \cong \mathcal{O}_{\mathbb{P}^1}(2)$.*

Proof. It is obvious from the proposition, that if X is a K3 surface then $\mathbb{L} = \mathcal{O}(2)$ as $\omega_{\mathbb{P}^1} = \mathcal{O}(-2)$. On the other hand, if $\mathbb{L} = \mathcal{O}(2)$, we get that

$$\omega_X = \mathcal{O}_X.$$

But by 2.16, we get that X has the structure as in 2.11. Hence X is a K3 surface. □

Remark 2.19. Assume that X is not a product of curves. Then as is shown in [Mir89], the irregularity q of $X \to C$ for any Weierstraß fibration is given by $g(C)$, the geometric genus $p_g = g(C) + \deg \mathbb{L} - 1$. The plurigenera P_n for a Weierstraß fibration over $\mathbb{P}^1$ are given by 0 if $\deg \mathbb{L} \leq 1$ and $P_n \geq 1$ else. On the other hand, Castelnovu's rationality theorem states

$$(p_g, q, P_2) = (0, 0, 0) \Longleftrightarrow X \text{ is rational.}$$

Hence by the above, a Weierstraß fibration $X \to \mathbb{P}^1$ is rational if and only if

$$\mathbb{L} = \mathcal{O}_{\mathbb{P}^1}(1).$$

By the canonical bundle formula, we moreover get

$$\omega_X = p^* \mathcal{O}_{\mathbb{P}^1}(-1),$$

i.e. the class is the negative of the class of an elliptic fiber E.

3. Moduli of K3 Surfaces

In this section, we recall the construction of the moduli space of elliptic K3 surfaces.

3.1. The Period Map

Definition 3.1. A smooth, proper, surjective map $X \xrightarrow{f} C$ between two complex manifolds is called *family of K3 surfaces*, if the fibers $f^{-1}(t)$ are K3 surfaces for every $t \in C$.

In the following we are only interested in the case where C is a curve. As follows from the theorems on the Hodge structure (see e.g. Appendix B), we know that $H^{2,0}(X_t) \perp H^{1,1}(X_t)$ and $H^{2,0}(X_t) = \overline{H^{0,2}(X_t)}$. Thus, by just knowing the 1-dimensional space $H^{2,0}(X_t)$, we can recover the full Hodge structure on $H^2(X_t, \mathbb{Z}) \cong \Lambda_{3,19}$. Hence, a K3 surface determines a well-defined element in $\mathbb{P}(\Lambda_{3,19} \otimes \mathbb{C})$ after choosing an isomorphism $H^2(X_t, \mathbb{Z}) \cong \Lambda_{3,19}$. As we know that $\langle x, x \rangle = 0$ and $\langle x, \overline{x} \rangle > 0$ for every $x \in H^{2,0}(X_t)$(see appendix B) this gives rise to the following definition:

Definition 3.2. The space

$$\mathcal{D} = \mathbb{P}(x \in \Lambda_{3,19} \otimes \mathbb{C} \mid \langle x, x \rangle = 0 \wedge \langle x, \overline{x} \rangle > 0)$$

is called *period domain.*

Now, we would like to construct a map from a family of K3 surfaces to this space. Unfortunately this depends on the chosen isomorphism $H^2(X_t, \mathbb{Z}) \cong \Lambda_{3,19}$. But in the case that the base curve C is simply connected we get:

Proposition 3.3. *Let $X \xrightarrow{f} C$ be a family of K3 surfaces with C a simply connected curve. Moreover let $\phi : H^2(X_0, \mathbb{Z}) \xrightarrow{\sim} \Lambda_{3,19}$ be an isomorphism for a basepoint $0 \in C$. Then there is an isomorphism $\phi_c : H^2(X_c, \mathbb{Z}) \cong \Lambda_{3,19}$ for every $c \in C$ which leads to a well defined holomorphic map $C \to \mathcal{D}$ given by*

$$c \to [\phi_c(H^{2,0}(X_c))] \in \mathcal{D}$$

for every $c \in C$.

For the proof, we need the two following facts:

Lemma 3.4. *Let $f : X \to C$ be a family of K3 surfaces. Then there is a natural holomorphic injection*

$$f_* \Omega^2_{X/C} \hookrightarrow R^2 f_* \underline{\mathbb{C}} \otimes \mathcal{O}_C$$

of vector bundles, which corresponds to

$$H^{2,0}(X_t) \hookrightarrow H^2(X_t, \mathbb{C})$$

in every fiber. $\square$

Proof. See [Huy16]. $\square$

Lemma 3.5 (Ehresmann [Dun18])**.** *Let $f : M \to N$ be a smooth, proper, submersive map between two manifolds M, N. Then f is a locally trivial fibration.*

Proof of Propostion 3.3. Due to the preceding lemma, we have that the sheaf $R^2 f_* \underline{\mathbb{Z}}$ is a local system. As C is simply connected the local system is constant by [ZS10]. Hence $R^2 f_* \underline{\mathbb{Z}} \cong \underline{\Lambda_{3,19}}$, where the latter is considered as a constant system. As the stalk of $R^2 f_* \underline{\mathbb{Z}}$ at a point $c \in C$ is isomorphic to $H^2(X_c, \mathbb{Z})$, we get a well defined isomorphism $\phi_c : H^2(X_c, \mathbb{Z}) \to \Lambda_{3,19}$ for every cohomology of the fiber. Hence the map

$$\begin{aligned} C &\to \mathcal{D} \\ c &\mapsto \phi_c(H^{2,0}(X_c)) \end{aligned}$$

is continuous. The holomorphicity follows directly from Lemma 3.4. $\square$

If C is not necessarily simply connected we can choose different paths γ_1, γ_2 in C and take the corresponding isomorphisms ϕ_1, ϕ_2, that we get from the locally constant system $R^2 f_*(X, \mathbb{Z})$. They only differ just by some automorphism of $\Lambda_{3,19}$. On the other hand there is a group action $\mathcal{O}(\Lambda_{3,19}) \times \mathcal{D} \to \mathcal{D}$ on the period domain. Therefore taking the orbit space $\mathcal{O}(\Lambda_{3,19}) \backslash \mathcal{D}$ we get the following well-defined map for any curve C

$$\begin{aligned} C &\to \mathcal{O}(\Lambda_{3,19}) \backslash \mathcal{D} \\ c &\mapsto \phi_c(H^{2,0}(X_c)) \end{aligned}$$

where $\phi_c : H^2(X_c) \to \Lambda_{3,19}$ is *any* isomorphism.

Unfortunately this space is not Hausdorff. But this can be avoided as shown in the next section.

3.2. Polarized K3 Surfaces

Definition 3.6. Let L be a lattice and $L' \overset{i}{\hookrightarrow} L$ a sublattice. The embedding is called *primitive*, if $\operatorname{coker}(i)$ is torsion free.

Definition 3.7. A *quasi-polarized* K3 surface of degree $2d$ is a tuple (X, L), where X is a K3 surface and L is primitive line bundle, such that it is nef, big and satisfies $L^2 = 2d$. If moreover L is ample, then (X, L) is called *polarized.*

Remark 3.8. By definition a line bundle L is nef if and only if $L \cdot C \geq 0$ for any curve $C \subset X$. It is called big if $L^2 > 0$. As is shown in [MP07] any K3 surface X with an quasi-polarization is algebraic.

As our main interest lies in algebraic elliptic K3 surfaces, which contain fiber and section cycles, we see that the Picard group of every such surface contains a sublattice L with intersection form

$$\begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}$$

(see Lemma 2.9), which is equivalent to the standart unimodular lattice with intersection form

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We now explain a polarization with respect to an arbitrary lattice, where in our case only the hyperbolic lattice U above will be considered afterwards.

Definition 3.9 ([KMPS10])**.** Let $\Lambda \subset \Lambda_{3,19}$ be a fixed primitive sublattice. The tuple $(X, j : \Lambda \hookrightarrow \mathrm{Pic}(X))$ is called a Λ-*quasi-polarized* K3 surface if j is a primitive emdedding

$$\Lambda \hookrightarrow \mathrm{Pic}(X)$$

such that there exists an isomorphism $H^2(X, \mathbb{Z}) \to \Lambda_{3,19}$ which restricts to the inclusion j on the sublattice Λ. Furthermore $\Lambda \subset \mathrm{Pic}(X)$ has to contain a quasi polarization, i.e. a line bundle that is nef and big. If it is moreover ample, (X, j) is called Λ-*polarization.*

Definition 3.10. A family of U-polarized K3 surfaces is a family of K3 surfaces $X \to C$ together with a primitive sublattice $Z \subset \mathrm{Pic}(X)$, such that

$$U \cong Z|_{X_t} \subset \mathrm{Pic}(X_t)$$

is a U-quasi-polarization for every K3 surface X_t and there is a line bundle $L \in Z$ that is nef and big on every fiber.

Definition 3.11. Two Λ-quasi-polarized K3 surfaces X, X' are said to be isomorphic, if there exists an isomorphism of surfaces $\phi : X \to X'$, such that

$$\begin{tikzcd} \Lambda \arrow[r, hook] \arrow[d, equal] & \mathrm{Pic}(X') \arrow[d, "\phi^*"] \\ \Lambda \arrow[r, hook] & \mathrm{Pic}(X), \end{tikzcd}$$

commutes.

Accordingly, we define isomorphisms of K3 surfaces

Definition 3.12. Two elliptic K3 surfaces $(X, p, s), (X', p', s')$ with p, p' the fibrations and s, s' the sections are isomorphic, if there exists an isomorphism of surfaces $\phi : X \to X'$, such that $\phi(s) = s'$ and

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, "p"'] & & X' \arrow[dl, "p'"] \\ & \mathbb{P}^1 & \end{tikzcd}$$

commutes.

Denote by $U_{p,s} \subset \mathrm{Pic}(X)$ the lattice generated by the section and a fiber of the fibration p. Then [CD07] shows, that there is a one-to-one correspondence

$$\begin{aligned} \{\text{elliptic fibrations}\,(X,p,s)\,\text{on}\,X\} \Longleftrightarrow \qquad & \{U\text{-quasi-polarizations of}\,X\} \\ (X,p,s) \mapsto \qquad & (U_{p,s} \hookrightarrow \mathrm{Pic}(X)). \end{aligned}$$

Furthermore the mapping takes isomorphic elliptic K3 surfaces to isomorphic U-quasi-polarizations. Modulo isomorphism, the map is still one-to-one. Hence, we can construct our moduli space of elliptic K3's as a moduli space of U-polarized K3 surfaces as in [Dol95].

3.2.1. Moduli of Polarized K3 Surfaces

In this section the moduli space for U-polarized K3 surfaces is constructed. For a detailed description, see [Dol95]. By Lemma A.9 U is a direct summand of the lattice $H^2(X, \mathbb{Z})$. Furthermore by A.10, the embedding of U is unique up to automorphism, hence we can assume that U is the first component of

$$H^2(X, \mathbb{Z}) \cong \Lambda_{3,19} = E_8(-1)^2 \oplus U^3$$

modulo automorphism of $H^2(X, \mathbb{Z})$. As $U \hookrightarrow \mathrm{Pic}(X)$ and $H^{2,0}(X) \perp \mathrm{Pic}(X)$, we know that the image of the period map lies in

$$\mathbb{P}(U^\perp) \cap \mathcal{D} \cong \mathbb{P}(E_8(-1)^2 \oplus U^2) \cap \mathcal{D}.$$

As $\Lambda_{2,18} := E_8(-1)^2 \oplus U^2$ is an unimodular lattice of signature $(2, 18)$, we get that every automorphism of $U^\perp$ extends to an automorphism of $H^2(X, \mathbb{Z})$, just by acting as the identity on U. The following theorem is now the starting point for the Noether-Lefschetz theory.

Theorem 3.13 ([Dol95]). *The space*

$$\mathcal{M} = O(\Lambda_{2,18})\backslash\mathbb{P}\{x \in \Lambda_{2,18} \otimes \mathbb{C} | \langle x, x\rangle = 0, \langle x, \bar{x}\rangle > 0\}$$

is a coarse moduli space of U-quasi-polarized K3 surfaces.

Proof. See [Dol95]. □

Similarly, in the case of quasi-polarized K3 surfaces (X, L) of a fixed degree $L.L = d$, we get that up to automorphism $L = e + df$, where $e, f \in U \subset \Lambda_{3,19}$ are the standard vectors of the lattice U in a fixed U-component of $\Lambda_{3,19}$. The image of the period map then lies in

$$\Lambda_d := L^\perp \cong E_8(-1)^2 \oplus U^2 \oplus \mathbb{Z}(-d)$$

and the following theorem holds:

Theorem 3.14 ([Huy16]). *The space*

$$\mathcal{M}_d = \tilde{O}(\Lambda_{3,19})\backslash\mathbb{P}\{x \in \Lambda_d \otimes \mathbb{C} | \langle x, x\rangle = 0, \langle x, \bar{x}\rangle > 0\},$$

where

$$\tilde{O}(\Lambda_{3,19}) = \{g|_{\Lambda_d} \mid g \in O(\Lambda_{3,19}),\, g(e + df) = e + df\}$$

is a coarse moduli space for quasi-polarized K3 surfaces.

The huge advantage of these spaces in contrast to the space $O(\Lambda_{3,19})\backslash\mathcal{D}$ is, that the following holds:

Theorem 3.15. *The spaces $\mathcal{M}$ and $\mathcal{M}_d$ are Hausdorff and admit the structure of a quasi-projective variety.*

Proof. See [Huy16]. □

Due to Borel, also the following holds:

Theorem 3.16 (Borel, Theorem 3.10 [B+72]). *Let V be an algebraic variety and $V \xrightarrow{f} \mathcal{M}$ a holomorphic map between analytic spaces. Then f is also algebraic.*

Hence the period map $C \to \mathcal{M}$ is even a morphism between algebraic spaces.

4. Degenerations

Definition 4.1. A proper, surjective map $f : X \to C$, where X is a smooth threefold and C a curve, is called a *degeneration of K3 surfaces*, if there is a finite subset $\{p_i\}_i \subset C$ such that

$$f|_{X\backslash f^{-1}(\{p_i\})} : X\backslash f^{-1}(\{p_i\}) \to C\backslash\{p_i\}$$

is a family of K3 surfaces. A U-quasi-polarization is given by a primitive sublattice $Z \subset \mathrm{Pic}(X)$ such that this is a U-quasi-polarization for the family $f|_{X\backslash f^{-1}(\{p_i\})}$.

Unfortunately, the fibers that are non-K3 can be arbitrarily bad, e.g. singular and reducible. As an example, see the construction of such degenerations in the Sections 4.3.1 and 4.3.2. But, as it will turn out, a mild local condition can improve the situation, such that we get a complete description of all possible fibers.

4.1. Kulikov Degenerations

This section will loosely follow [Fri84] and [Bru15]. We will impose a local condition for degenerations, which will allow us to classify those completely. Denote by $\Delta \subset \mathbb{C}$ the unit disc $\Delta = \{z \in \mathbb{C} \,|\, z\bar{z} < 1\}$.

Definition 4.2. A degeneration $X \to \Delta$ of K3 surfaces is called a *Kulikov degeneration*, if

- $X \to \Delta$ is *semistable*, i.e. X is smooth, X_t is a K3 surface for all $t \neq 0$, and X_0 is a reduced normal crossing divisor,
- $\omega_X = \mathcal{O}_X$.

Remark 4.3. As cited in [HT15], this condition is rather mild. It can be shown that for any degeneration of K3 surfaces there exists a base change $\Delta \to \Delta$, $t \to t^n$ for some n, such that the resulting space X' admits a birational morphism $X'' \to X'$, such that X'' is semistable.
Moreover if $X \to \Delta$ is any semistable degeneration of K3 surfaces, such that the components of the central fiber are Kähler, then there is a birational morphism $X' \to X$, such that X' is a Kulikov degeneration and $X' \to X$ is an isomorphism outside the central fibers.

To state the classification, we need to define the monodromy map N: Fix a $0 \neq t \in \Delta$. By the assumption and the Ehresmann lemma 3.5, the degeneration is topologically locally trivial outside the central fiber. Therefore we can define

$$T : H^2(X_t, \mathbb{Z}) \to H^2(X_t, \mathbb{Z})$$

to be the map, that is induced by a path $\gamma : I \to X$, that starts and ends in t and goes around $0 \in \Delta$ counterclockwise once. As $\Delta^* = \Delta\backslash\{0\}$ is homotopy equivalent to S^1, and hence $\pi_1(\Delta\backslash 0) = \mathbb{Z}$, this map is well-defined. If a degeneration is semistable then T is unipotent (of index at most 3), as is shown in [PS00]. Hence we define

$$N = \log T = \sum_{m-1}^{i=1} (-1)^{i+1}(T - Id)^i.$$

For better readability and to match the notation in later on sections we denote by $\langle x, y\rangle$ the cup product between two cohomology classes $x, y \in H^*(X)$. As the isomorphism T respects the cup product, we get

$$\langle Nx, y\rangle = -\langle \log(T^{-1})x, y\rangle.$$

But $\langle (T^{-1} - Id)x, y\rangle = \langle x, Ty\rangle - \langle x, y\rangle = \langle x, (T - Id)y\rangle$. Therefore $\langle \log(T^{-1})x, y\rangle = \langle x, \log(T)y\rangle$ and

$$\langle Nx, y\rangle = -\langle x, Ny\rangle$$

is skew-symmetric.

Theorem 4.4 ([Fri84]). *Let $X \to \Delta$ be a Kulikov degeneration. Then three cases can occur:*

- *$N = 0$: The central fiber X_0 is a K3 surface, (Type I)*
- *$N^2 = 0, N \neq 0$: $X_0 = Y_1 \cup ... \cup Y_n$, where Y_1, Y_n are rational, $Y_i (1 < i < n)$ is elliptic ruled, and $Y_i \cap Y_j = D_{ij}$ is an elliptic curve, (Type II)*
- *$N^3 = 0, N^2 \neq 0$: $X_0 = \cup_i Y_i$ is a union of rational surfaces and $D_{ij} = Y_i \cap Y_j$ are cycles of rational curves. The dual graph of X_0 is a triangulation of the sphere S^2. (Type III)*

Remark 4.5. In the above theorem, the dual graph Γ of $X_0 = \cup Y_i$ is a simplicial complex constructed as follows: The vertices are the components Y_i, and the k-simplex $\langle Y_{i_1}, ..., Y_{i_k}\rangle$ belongs to Γ if and only if $Y_{i_1} \cap \ldots \cap Y_{i_k} \neq \emptyset$.

As our main interest lies in Type II degenerations, we make the following simplifying assumption:

Definition 4.6. A Kulikov degeneration of Type II is called *short*, if the central fiber only has two components $X = V_1 \cup_E V_2$. Furthermore it is *d-semistable* if the normal bundles satisfy

$$N_{E/V_1} \otimes N_{E/V_2} = \mathcal{O}_E.$$

Remark 4.7. As shown in [Fri84], every surface with normal crossings $X = V_1 \cup_E V_2$ - with an elliptic curve E - that is d-semistable is a central fiber of a degeneration.

Moreover d-semistability is immediate for Kulikov degenerations, it even suffices that X is a simple normal crossing:

$$N_{E/V_1} = \mathcal{O}_{V_1}(E)|_E = \mathcal{O}_X(V_2)|_{V_1}|_E = \mathcal{O}_X(V_2)|_E$$

and

$$N_{E/V_2} = \mathcal{O}_X(V_1)|_E$$

analogously. Thus, the product satisfies

$$\begin{aligned} N_{E/V_1} \otimes N_{E/V_2} &= (\mathcal{O}_X(V_1) \otimes \mathcal{O}_X(V_2))|_E \\ &= \mathcal{O}_X(V)|_E \\ &= \mathcal{O}_E. \end{aligned}$$

Proposition 4.8. *A short degeneration satisfies*

$$E \in |-\omega_{V_i}| \qquad \textit{for } i = 1, 2.$$

If it is furthermore d-semistable, we get

$$\omega_{V_1}^2 + \omega_{V_2}^2 = 0.$$

Proof. By the adjunction formula we get

$$\omega_{V_1} = (\omega_X + \mathcal{O}(V_1))|_{V_1}.$$

But on the other hand

$$\mathcal{O}_X = \mathcal{O}_X(X_0) = \mathcal{O}_X(V_1) + \mathcal{O}_X(V_2).$$

as X_0 is a fiber. Hence

$$\omega_{V_1} = (\omega_X - \mathcal{O}_{V_2})|_{V_1} = -\mathcal{O}_{V_1}(E).$$

If $X \to \Delta$ is moreover d-semistable, then

$$\deg N_{E/V_1} = \deg \mathcal{O}_{V_1}(E)|_E = (E.E)_{V_1} = \omega_{V_1}^2.$$

But the degree is additive, therefore

$$0 = \deg \mathcal{O}_E = N_{E/V_1} \otimes N_{E/V_2} = \omega_{V_1}^2 + \omega_{V_2}^2.$$

□

4.2. Hodge Theory of Kulikov Degenerations

In this section, we are going to analyze the mixed Hodge structures that we associate to the central fiber:

- The mixed Hodge structure of Deligne of the central fiber X_0(see Appendix B)
- The limit mixed Hodge structure associated to a degeneration, only depending on a punctured disc around the central fiber.

Again, fix some $0 \neq t \in \Delta$. By means of the nilpotent N, there is a natural way to define a weight filtration $W = (W_i)_i$ on $H^n(X_t, \mathbb{Q})$, which in our case of degenerations with $N^2 = 0$ and $n = 2$ is simply given by

$$0 \subset W_0 = 0 \subset W_1 = \operatorname{Im} N \subset W_2 = \operatorname{Ker} N \subset W_3 = H^2(X_t, \mathbb{Q}) \subset H^2(X_t, \mathbb{Q})$$

and for $N^3 = 0$

$$0 \subset W_0 = \operatorname{Im} N^2 \subset W_1 = W_0 \subset W_2 = \operatorname{Ker} N^2 \subset W_3 = W_2 \subset W_4 = H^2(X_t, \mathbb{Q})$$

for $H^2(X, \mathbb{Q})$.

As explained in [SZ85], there is an increasing filtration F on $H^2(X_t, \mathbb{Q})$ constructed as follows: Let

$$\overline{\mathcal{D}} = \mathbb{P}(\Lambda_{3,19} \otimes \mathbb{C}).$$

Now let $X \to \Delta^*$ and the universal cover $\mathbb{H} \xrightarrow{p} \Delta^*$ be given by $x \to \exp(2\pi i x)$. Hence, we get the period map

$$F : \mathbb{H} \to \mathcal{D}$$

as $\mathbb{H}$ is simply connected. Now, we define a map

$$\begin{aligned} \tilde{F} : \mathbb{H} &\to \overline{\mathcal{D}} \\ \tau &\mapsto \exp(-\tau N) F(\tau). \end{aligned}$$

As $F(z+1) = TF(z)$, we get

$$\begin{aligned} \tilde{F}(\tau + n) &= \exp(-\tau N - nN) F(\tau + n) \\ &= \exp(-\tau N) \circ \exp(-nN) T^n F(\tau) \\ &= \exp(-\tau N) \circ T^{-n} T^n F(\tau) \\ &= \exp(-\tau N) F(\tau) \\ &= \tilde{F}(\tau), \end{aligned}$$

we get that $\tilde{F}$ is invariant under translation in $\mathbb{Z}$ and hence descends to a map

$$\overline{F} : \Delta^* \to \overline{\mathcal{D}}.$$

By the nilpotent orbit theorem (see e.g.[SZ85]), we get that $\overline{F}$ even extends holomorphically to

$$\Delta \to \overline{\mathcal{D}}.$$

Hence

$$\overline{F}(0) \in \overline{\mathcal{D}}$$

yields a decreasing filtration $F = (F_q)_q \subset H^2(X_t, \mathbb{C})$ by setting

$$\begin{aligned} H^{2,0} &= \langle \overline{F}(0) \rangle_{\mathbb{C}} \\ H^{0,2} &= \overline{H^{2,0}} \\ H^{1,1} &= (H^{2,0})^{\perp} \end{aligned}$$

analogously to the usual case and taking the corresponding decreasing filtration, see Appendix B.

These filtrations satisfy the following theorem, see e.g. [PS00]:

Theorem 4.9. *The filtrations W, F on $H^2(X_t, \mathbb{Q})$ as above, yield a mixed Hodge structure $(H^2(X_t; \mathbb{Q}), F, W)$, called the* limit mixed Hodge structure H^2_∞. *Moreover the map $N : H^2(X_t, \mathbb{Q}) \to H^2(X_t, \mathbb{Q})$ is a map of mixed Hodge structures of weight -2.*

By a theorem of Schmid, there is a retraction $c : X \times [0, 1] \to X$:

Proposition 4.10 (Schmid, as stated in [PS00])**.** *Let $p : X \to \Delta$ be a semistable degeneration. Then there is a retraction*

$$c : X \times [0, 1] \to X$$

to X_0, such that

$$\begin{tikzcd} X \times [0,1] \arrow[r] \arrow[d, "p \times id"] & X \arrow[d, "p"] \\ \Delta \times [0,1] \arrow[r] & \Delta \end{tikzcd}$$

commutes, where the lower horizontal is the radial projection, i.e. $(x, t) \mapsto (1 - t)x$. Moreover

$$c_t = c|_{X_t \times 1} : X_t \to X_0$$

is an isormorphism outside the singularities of X_0. This map is called the Clemens map.

There is a useful tool to compare the limiting Hodge structure H^2_∞, to the mixed Hodge structure naturally associated to the central fiber (see Appendix B).

Theorem 4.11 (Clemens-Schmid exact sequence, [Fri84]). *Let $X \to \Delta$ be a Kulikov degeneration. The sequence*

$$0 \to H^0(X_t, \mathbb{Q}) \to H_4(X_0, \mathbb{Q}) \xrightarrow{\mu} H^2(X_0, \mathbb{Q}) \xrightarrow{c^*} H^2_\infty \xrightarrow{N} H^2_\infty$$

is an exact sequence of mixed Hodge structures, where the Hodge structure on $H_4(X_0, \mathbb{Q})$ comes from the natural one of $H^4(X_0, \mathbb{Q})$ pushed forward by the duality $H_4(X_0, \mathbb{Q}) \cong H^4(X_0, \mathbb{Q})^*$ and the one on $H^2(X_0, \mathbb{Q})$ is the one of Deligne. Moreover μ is of weight 6 *and c of weight* 0.

Remark 4.12. As the Clemens map is a retraction - even of the whole space - to X_0, we get that $H^2(X_0, \mathbb{Q}) \xrightarrow{c^*} H^2_\infty$ factors as

$$H^2(X_0, \mathbb{Q}) \xrightarrow{c^*, \cong} H^2(X, \mathbb{Q}) \xrightarrow{\text{incl}^*} H^2(X_t, \mathbb{Q}) = H^2_\infty.$$

On the other hand, as c is a deformation retraction we get that

$$H^2(X_0, \mathbb{Q}) \xrightarrow{c^*, \cong} H^2(X, \mathbb{Q})$$

is the inverse of

$$H^2(X, \mathbb{Q}) \xrightarrow{\text{incl}^*} H^2(X_0, \mathbb{Q}).$$

Hence, exactness of the Clemens-Schmid-sequence implies, that for every cycle $f \in H^2(X_t, \mathbb{Q})$ we have $Nf = 0$ (i.e. $Tf = f$) if and only if there is a $F \in H^2(X, \mathbb{Q})$ such that $F|_{X_t} = f$. This is called *invariant cycle theorem.*

Setting 4.13. *Let from now on $X \to \Delta$ be a Kulikov degeneration of Type II with short central fiber $X_0 = V_1 \cup_E V_2$.*

Lemma 4.14. *For $X \to \Delta$ as above, we have* $\dim H^2(X_0, \mathbb{Q}) = 21$.

Proof. As V_1, V_2 are rational they satisfy $\dim H^1(V_i, \mathbb{Q}) = \dim H^3(V_i, \mathbb{Q}) = 0$. Moreover $\dim H^2(V_i, \mathbb{Q}) = 10 - \omega^2_{V_i}$. From the Mayer-Vietoris sequence on cohomology we obtain

$$0 \to H^1(E, \mathbb{Q}) \xrightarrow{\alpha_1} H^2(X_0, \mathbb{Q}) \xrightarrow{\alpha_2} H^2(V_1, \mathbb{Q}) \oplus H^2(V_2, \mathbb{Q}) \xrightarrow{\alpha_3} H^2(E, \mathbb{Q}) \to 0.$$

As E is elliptic, we have $\dim H^1(X, \mathbb{Q}) = 2$ and $\dim H^2(E, \mathbb{Q}) = 1$. Hence

$$\begin{aligned} \dim\ \operatorname{Im} \alpha_2 &= \dim \operatorname{Ker}(\alpha_3) = 20 - \omega^2_{V_1} - \omega^2_{V_2} - 1 = 19, \\ \dim \operatorname{Ker}(\alpha_2) &= 2. \end{aligned}$$

Therefore $\dim H^2(X_0, \mathbb{Q}) = 21$. □

Next, we want to calculate the graded pieces of the limiting mixed Hodge structure H^2_∞. To do so, we first define an important element of $H^2(X_0, \mathbb{Z})$: Let

$$\mathcal{E} = \mathcal{O}_X(V_1)|_{X_0} \in H^2(X_0, \mathbb{Z}).$$

This is well defined as $X_0 = V_1 \cup V_2$ is a normal crossing divisor. Furthermore, $\mathcal{E} \in H^2(X_0, \mathbb{Z})$ satisfies $\mathcal{E}|_{V_1} = \mathcal{O}_{V_1}(-E)$ and $\mathcal{E}|_{V_2} = \mathcal{O}_{V_2}(E)$, as $\mathcal{O}_{V_1}(-E)|_E + \mathcal{O}_{V_2}(E)|_E = \mathcal{O}_E$ by d-semistability. By construction it maps to $\tilde{\mathcal{E}} = \mathcal{O}_{V_1}(E) - \mathcal{O}_{V_2}(E) \in H^2(V_1, \mathbb{Z}) \oplus H^2(V_2, \mathbb{Z})$ under the first map in the following Mayer-Vietoris sequence:

$$H^2(X_0, \mathbb{Q}) \to H^2(V_1, \mathbb{Q}) \oplus H^2(V_2, \mathbb{Q}) \to H^2(E, \mathbb{Q}).$$

Lemma 4.15 (Friedman[Fri84]). *Let $\tilde{\mathcal{E}} = \mathcal{O}_{V_1}(E) - \mathcal{O}_{V_2}(E) \in H^2(V_1, \mathbb{Z}) \oplus H^2(V_2, \mathbb{Z})$. Then the Clemens Schmid exact sequence is exact over $\mathbb{Z}$. Furthermore we have*

$$W_1 H^2_\infty \cong W_1 H^2(X_0) \cong H^1(E, \mathbb{Z})$$

and

$$\mathrm{Gr}_2 H^2_\infty \cong \tilde{\mathcal{E}}^\perp / \mathbb{Z}\tilde{\mathcal{E}}$$

as a quotient of a sublattice in $H^2(V_1, \mathbb{Z}) \oplus H^2(V_2, \mathbb{Z})$. Moreover this lattice has signature $(1, 17)$.

Proof. For the first statement see [Fri84]. For the other statements, we analyse the weight filtration of the Clemens Schmid exact sequence: First observe that

$$H^4(X_0, \mathbb{Q}) = H^4(V_1, \mathbb{Q}) \oplus H^4(V_2, \mathbb{Q})$$

carries a pure Hodge structure, and hence the dual Hodge structure on $H_4(X_0, \mathbb{Q})$ is pure of weight -4 by definition. Therefore the weight filtration is given by

$$0 = W_{-5} \subset W_{-4} = H_4(X_0, \mathbb{Q}) = W_{-3} = \ldots = W_4.$$

Hence:

$$
\begin{tikzcd}
W_{-5} & 0 \arrow[ddddr] & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
W_{-1} & H_4(X_0,\mathbb{Z}) & 0 \arrow[r] & 0 & 0 \\
W_0 & H_4(X_0,\mathbb{Z}) & 0 \arrow[r] & 0 & 0 \\
W_1 & H_4(X_0,\mathbb{Z}) & H^1(E,\mathbb{Z}) \arrow[r] & \operatorname{Im} N \arrow[uur] & \operatorname{Im} N \\
W_2 & H_4(X_0,\mathbb{Z}) & H^2(X_0,\mathbb{Z}) \arrow[r] & \operatorname{Ker} N \arrow[uur] & \operatorname{Ker} N \\
W_3 & H_4(X_0,\mathbb{Z}) & H^2(X_0,\mathbb{Z}) \arrow[r] & H^2(X_t,\mathbb{Z}) \arrow[uur] & H^2(X_t,\mathbb{Z}) \\
W_4 & H_4(X_0,\mathbb{Z}) & H^2(X_0,\mathbb{Z}) \arrow[r] & H^2(X_t,\mathbb{Z}) \arrow[uur] & H^2(X_t,\mathbb{Z})
\end{tikzcd}
$$

Therefore $W_1 H^2_\infty \cong H^1(E,\mathbb{Z})$ is immediate. Let $\mathcal{E} = \mathcal{O}_X(V_1)|_{X_0}$. This Cartier divisor obviously extends to the whole of X by taking $\mathcal{O}_X(V_1)$. By the discussion of Remark 4.12, we get that $\mathcal{E} \in \operatorname{Ker}(c^*)$ as $\mathcal{O}_X(V_1)|_{X_t} = 0$ for $t \neq 0$. So $\mathcal{E} \in \operatorname{Im}(H^2(X_0,\mathbb{Z}) \to H^2_\infty)$ by exactness.

Now, we will show that the image is spanned by that element. By Mayer Vietoris, we get that $\dim H_4(X_0,\mathbb{Q}) = 2$. But the Clemens Schmid exact sequence shows, that $0 \to \mathbb{Q} \cong H^0(X_0,\mathbb{Q}) \to H_4(X_0,\mathbb{Q}) \to H^2(X_0,\mathbb{Q})$ is exact and hence the dimension of the image must be 1.

Looking at the Mayer Vietoris sequence, we get the following exact sequence

$$0 \to H^1(E,\mathbb{Z}) \to H^2(X_0) \to \tilde{\mathcal{E}}^\perp \to 0$$

as the elements (f_1, f_2) in $H^2(V_1,\mathbb{Z}) \oplus H^2(V_2,\mathbb{Z})$ that come from $H^2(X_0,\mathbb{Z})$ are exactly those, that satisfy $f_1|_E = f_2|_E$, i.e. are othorgonal to $\tilde{\mathcal{E}}$. Taking the weight filtration, we get the following commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[r] & H^1(E,\mathbb{Z}) \arrow[r] \arrow[d] & \operatorname{Im} N \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{E}\mathbb{Z} \arrow[r] & H^2(X_0,\mathbb{Z}) \arrow[r] \arrow[d, two heads] & \operatorname{Ker} N \arrow[r] \arrow[d] & 0 \\
 & & \tilde{\mathcal{E}}^{\perp} & 0 &
\end{tikzcd}$$

Therefore $\operatorname{Ker} N \cong H^2(X_0,\mathbb{Z})/\mathcal{E}\mathbb{Z}$. But as the diagram commutes, we get

$$\mathrm{Gr}_2 H^2_{\infty} = \operatorname{Ker} N/\operatorname{Im} N \cong (H^2(X_0,\mathbb{Z})/\mathcal{E}\mathbb{Z})/H^1(E,\mathbb{Z}) \cong \tilde{\mathcal{E}}^{\perp}/\tilde{\mathcal{E}}\mathbb{Z}. \tag{4.15.1}$$

The isomorphism is induced by taking an element $\alpha \in \tilde{\mathcal{E}}$ of which we take a preimage $\bar{\alpha} \in H^2(X_0,\mathbb{Z})$. The resulting element is then $c^*(\bar{\alpha})$. As the cup product is natural with respect to continuous maps we get the following diagram

$$\begin{tikzcd}
(H^2(V_1) \oplus H^2(V_2)) \times (H^2(V_1) \oplus H^2(V_2)) \arrow[d] & H^2(X_0) \times H^2(X_0) \arrow[l] \arrow[r, "c^* \times c^*"] \arrow[d, "\cup"] & \operatorname{Ker} N \times \operatorname{Ker} N \arrow[d, "\cup"] \\
H^4(V_1) \oplus H^4(V_2) & H^4(X_0) \arrow[l, "\cong"'] \arrow[r, "(a,b)\to a+b"] & H^4(X_t) \cong \mathbb{Z}
\end{tikzcd}$$

Now we want to show that the isomorphism (4.15.1) is even an isomorphism of lattices. Let $(A,B) = ((a,a'),(b,b')) \in \tilde{\mathcal{E}}^{\perp} \times \tilde{\mathcal{E}}^{\perp}$. By the above diagram, we know that taking the cup product of A and B and pulling it back to $H^4(X_t) \cong \mathbb{Z}$ is the same as just pushing A and B forward via the isomorphism (4.15.1)(which is just the upper row) and then taking the cup product. But

$$\mathbb{Z} \times \mathbb{Z} \cong H^4(V_1,\mathbb{Z}) \times H^4(V_2,\mathbb{Z}) \to H^4(X_t,\mathbb{Z}) \cong \mathbb{Z}$$

is just taking the sum of the two elements, and hence the isomorphism (4.15.1) is an isomorphism *of lattices*.
By the Hodge index theorem, we have that the signature of $H^2(V_1,\mathbb{Q}) \oplus H^2(V_2,\mathbb{Q})$ is $(2,n)$ as V_1, V_2 are rational and hence $H^2(V_i,\mathbb{Z}) = \mathrm{Pic}(V_i)$. As we observed earlier $\dim H^2(V_1,\mathbb{Q}) + \dim H^2(V_2,\mathbb{Q}) = 20$ and hence the signature is $(2,18)$. Therefore by linear algebra it follows that $\tilde{\mathcal{E}}^{\perp}/\mathbb{Z}\tilde{\mathcal{E}}$ has signature $(1,17)$ as $\tilde{\mathcal{E}} \cdot \tilde{\mathcal{E}} = 0$. $\square$

4.3. Examples

4.3.1. A D_{16}^{+} Degeneration

Let $X \xrightarrow{p} \mathbb{F}_4$ be the double cover of the Hirzebruch surface $\mathbb{F}_4 \xrightarrow{\pi} \mathbb{P}^1$ considered in Corollary 2.12 with the same notations. As we saw, it is determined by an irreducible section of $h \in \mathcal{O}_{\mathbb{F}_4}(3, 12)$ with smooth vanishing loci disjoint from $V(Z)$. We now want to alter h in two ways:

- $h \rightsquigarrow f^2 g$ with $f, g \in \mathcal{O}_{\mathbb{F}_4}(1, 4)$,
- $h \rightsquigarrow Zg$ with $g \in \mathcal{O}_{\mathbb{F}_4}(2, 12)$

such that in the first case the vanishing loci of f, g are disjoint from $V(Z)$. This is possible, as $f \cdot Z = 0 = g \cdot Z$.
The following example was suggested in [Bru15]:
Example 4.16 ($h \rightsquigarrow f^2 g$). Let f, g, h be chosen generically as above.

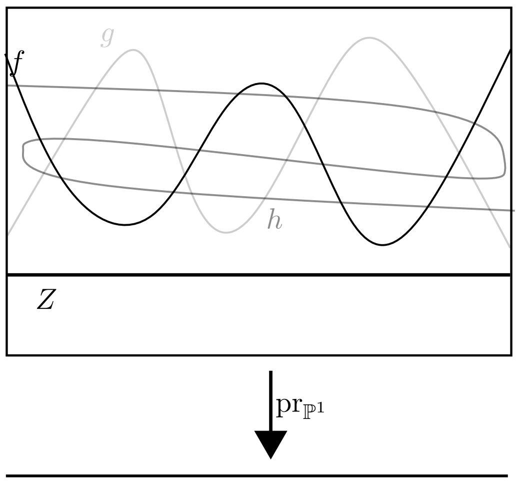

Figure 2: f, g, h in $\mathbb{F}_4$.

Now consider the double cover X of

$$\mathbb{F}_4 \times \mathbb{A}^1 \tag{4.16.1}$$

defined by the equation

$$Z \cdot f^2 g + t^2 Z \cdot h, \tag{4.16.2}$$

where t is the standard coordinate of $\mathbb{A}^1$. Again, due to the genericity of f, g, h, we can assume that for every fixed $t \neq 0$ in a neighborhood U of 0, the equation (4.16.2) is irreducible and has smooth vanishing locus disjoint from $V(Z)$. Hence by Corollary 2.12, for every $0 \neq t \in U$, the fiber X_t of

$$X \to \mathbb{F}_4 \times \mathbb{A}^1 \to \mathbb{A}^1$$

is an elliptic K3 surface.

Remark 4.17. Unfortunately, this is not a Kulikov model as the central fiber of the degeneration above is irreducible *and* singular.

Proposition 4.18. *Blowing up $X \to \mathbb{A}^1$ along the subvariety V defined as the vanishing locus of $t = f = 0$, we get a Kulikov model $\tilde{X}$ with central fiber*

$$\tilde{X}_0 = \mathbb{F}_2 \cup_E \mathrm{Bl}_{16}\mathbb{F}_n,$$

for some $n \in \mathbb{N}$, where E is an elliptic curve.

Proof. As the subvariety $V \xrightarrow{i} X$ lies completely in the central fiber X_0, the blow up does not effect the surfaces X_t for $t \neq 0$. Therefore the other fibers remain elliptic K3 surfaces.
As a first step, we show that one component is indeed isomorphic to $\mathbb{F}_2$. Let $I \subset \mathcal{O}_X$ be the ideal sheaf defined by V. Moreover let $J = i^{-1}J \cdot \mathcal{O}_{X_0}$. Furthermore denote by $\mathrm{Bl}_J X_0 \to X_0$ the blow up along J. By [Har13], we get the following commutative diagram

$$\begin{tikzcd}
\mathrm{Bl}_J X_0 \arrow[r, hook] \arrow[d] & \tilde{X} \arrow[d] \\
X_0 \arrow[r, hook] & X,
\end{tikzcd}$$

where also the upper horizontal arrow is a closed immersion. Therefore it suffices to compute $\mathrm{Bl}_J X_0$. Taking a local chart of $\mathbb{F}_4$, we see that locally

$$X = V(w^2 = Z \cdot f^2 g + t^2 Z \cdot h)$$

and

$$I = \langle w, f, t \rangle_{\mathcal{O}_X},$$

where w is the extra coordinate coming from the double cover. Hence, locally

$$X_0 = V(w^2 = Z \cdot f^2 g) \qquad \text{and} \qquad J = \langle w, f \rangle_{\mathcal{O}_{X_0}}.$$

Taking coordinates U, V, such that

$$f = Uw,$$
$$w = Vf$$

we see that the blowup satisfies the following description

$$\begin{aligned} w^2 &= U^2 w^2 \cdot Zg & &\Leftrightarrow w^2(1 - U^2 \cdot Zg) = 0 & &\overset{w\neq 0}{\Longleftrightarrow} U^2 \cdot Zg = 1, \\ V^2 f^2 &= Zf^2 g & &\Leftrightarrow f^2(V^2 - Zg) = 0 & &\overset{f\neq 0}{\Longleftrightarrow} V^2 = Zg. \end{aligned}$$

Hence, it is just the double cover of $\mathbb{F}_4$ along Zg. The induced map $\mathrm{Bl}_J X_0 \to \mathbb{P}^1$ admits a section s_{Bl}, just by the $V(Z)$ component of the branching locus of the cover. Denote by F a fiber of this map. Again, using Hurwitz, we get

$$2g - 2 = 2(2g' - 2) + 2 = -2$$

as Zg has precisely two zeroes on a fiber of $\mathbb{F}_4$. Therefore $g = 0$ and $F \cong \mathbb{P}^1$. From the standard theory of double covers, we get the following formula for the canonical sheaf

$$\omega_{\mathrm{Bl}X_0} = p^*\omega_{\mathbb{F}_4} \otimes \mathcal{O}(V(gZ)) = p^*(-2s_{\mathbb{F}} - 6f_{\mathbb{F}} + 1s_{\mathbb{F}} + 2f_{\mathbb{F}}) = p^*(-1s_{\mathbb{F}} - 4f_{\mathbb{F}}) = -2s_{\mathrm{Bl}} - 4f_{\mathrm{Bl}}$$

with $\mathrm{Bl}X_0 \xrightarrow{p} \mathbb{F}_4$, and $s_{\mathbb{F}}, f_{\mathbb{F}}, s_{\mathrm{Bl}}, f_{\mathrm{Bl}}$ denote the section and fiber in the corresponding spaces. As the only Hirzebruch surface with such a canonical sheaf is $\mathbb{F}_2$, we are done. Now, we want to calculate the exceptional divisor Y: By the adjunction formula, we get that

$$2g(V(f)) - 2 = (\mathcal{O}_{\mathbb{F}_4}(1,4) \otimes \mathcal{O}_{\mathbb{F}_4}(-2,-6)) \cdot \mathcal{O}_{\mathbb{F}_4}(1,4) = -2$$

in X_0. Hence, $g(V(f)) = 0$ and $V(f) \cong \mathbb{P}^1$. Therefore, if Y denotes the exceptional divisor, we get a map

$$Y \to \mathbb{P}^1.$$

Next, we calculate the fibers of this map: A local computation shows that

$$Y = V(w^2 = f^2 \cdot Zg + t^2 Zh) \subset \mathbb{P}(\mathcal{O}(w) \oplus \mathcal{O}(f) \oplus \mathcal{O}(t))$$

where by abuse of notation the line bundles denote the pullback to $V(f)$. Thus, for every point $p \in V(f)$ there are three cases:

1 $Zh(p) \neq 0 \neq Zg(p)$

2 $Zh(p) = 0$ and $Zg(p) \neq 0$

3 $Zg(p) = 0$ and $Zh(p) \neq 0$.

By genericity of f, g, h, the case where everything is zero cannot happen. In case 1 a fiber is:

$$\mathbb{P}^1 \cong V(w^2 = f^2 + t^2) \subset \mathbb{P}^2.$$

In case 2:

$$\mathbb{P}^1 \cup_p \mathbb{P}^1 \cong V(w^2 = f^2) \subset \mathbb{P}^2.$$

In case 3:

$$\mathbb{P}^1 \cup_p \mathbb{P}^1 \cong V(w^2 = t^2) \subset \mathbb{P}^2.$$

Calculating the occurrences of case 2 and 3 we get the number by forming the intersection product:

$$f \cdot h + f \cdot g = 12 + 4 = 16.$$

As we will see later in Lemma 4.21, the degeneration is indeed a Kulikov model, and hence E is rational. So the minimal model must indeed be $\mathbb{F}_n (n \neq 1)$ or $\mathbb{P}^2$. But by $\omega_{\mathbb{F}_2}^2 + \omega_Y^2 = 0$, i.e. $\dim H^2(Y) = 10 + \omega_{\mathbb{F}_2}^2 = 18$ we have $Y = \mathrm{Bl}_{16}\mathbb{F}_n$ or $\mathrm{Bl}_{17}\mathbb{P}^2 = \mathrm{Bl}_{16}\mathbb{F}_1$. So indeed

$$Y \cong \mathrm{Bl}_{16}\mathbb{F}_n.$$

Next, we analyse the elliptic curve along which both surfaces are glued: This is just the preimage of $V(f)$ of the map

$$\mathbb{F}_2 \to X_0$$

i.e. even the preimage of $V(f)$ of the double cover

$$\mathbb{F}_2 \to \mathbb{F}_4$$

that we constructed above. But $f \in \mathcal{O}_{\mathbb{F}_4}(1,4)$, hence - as the double cover has ramification loci $V(Zg)$ - we get that the pullback of a section is twice a section in $\mathbb{F}_2$ and the pullback of a fiber is just a fiber. Therefore

$$\mathcal{O}_{\mathbb{F}_2}(E) = 2s_{\mathbb{F}_2} + 4f_{\mathbb{F}_2} = -\omega_{\mathbb{F}_2}.$$

$\square$

Remark 4.19. A local computation shows that the resulting space $\tilde{X}$ is smooth. To verify that it is a Kulikov model, we need the following lemma.

Lemma 4.20. *Let $X \to \Delta^*$ be the restriction of an family of algebraic K3 surfaces. Then*

$$\omega_X = \mathcal{O}_X.$$

Proof. We have $\mathcal{O}_X(X_t) = \mathcal{O}_X$, as it is a pullback from the space Δ^*, which has trivial holomorphic Picard group by [For12], as it is a non-compact Riemann surface. Hence, by adjunction

$$0 = \omega_{X_t} = \omega_X|_{X_t}$$

for all t. Hence, also $\dim H^0(X_t, \omega_X|_{X_t}) = 1$ is constant for all t. Therefore, by Grauerts semi-continuity theorem for complex proper maps (see e.g.[BHPVdV15]), we get that

$$p_*\omega_{\tilde{X}}$$

is a line bundle on Δ^*. On the other hand, we get a canonical map

$$p^*p_*\omega_X \to \omega_X.$$

This is surjective on Δ^*: Fix some $t \in \Delta^*$. Then

$$(p_*\omega_X)_t = \omega_{X_t}$$

just by the definition, where the latter denotes the stalk around X_t. On the other hand, let $\eta \in X_t$ be the generic point. Then

$$\omega_{X_t} = \omega_X|_\eta = \omega_X|_{X_t}|_\eta = \mathcal{O}_{X_t}|_\eta.$$

But the image of

$$\mathcal{O}_{X_t}|_\eta \to \mathcal{O}_{X_t,t}$$

meets a generator of the stalk $\mathcal{O}_{X_t,t}$ as X_t is irreducible. Therefore

$$p^*p_*\omega_X \to \omega_X$$

is surjective and hence an isomorphism by [Har13]. But as we saw earlier, $\mathrm{Pic}(\Delta^*) = \{0\}$ and hence

$$\omega_X = p^*p_*\omega_X = p^*\mathcal{O}_{\Delta^*} = \mathcal{O}_X.$$

□

Lemma 4.21. *The space $\tilde{X}$ is Calabi-Yau when restricted to the neighborhood U of the central fiber, i.e. $\omega_{\tilde{X}} = \mathcal{O}_{\tilde{X}}$ on U.*

Proof. As above $\mathcal{O}_{\tilde{X}}(\tilde{X}_t) = \mathcal{O}_{\tilde{X}}$, as it is a pullback from the affine space. Hence, by adjunction and as X_t is a K3 surface

$$0 = \omega_{\tilde{X}_t} = \omega_{\tilde{X}}|_{\tilde{X}_t}$$

for all $0 \neq t \in U$. After shrinking we may assume that $U = \Delta^*$. By the lemma, the canonical sheaf is trivial outside the central fiber. Denote by X^1, X^2 the two components of the central fiber. Then,

$$\omega_{\tilde{X}} = r_1\mathcal{O}_{\tilde{X}}(X^1) + r_2\mathcal{O}_{\tilde{X}}(X^2).$$

By adjunction we get

$$\omega_{X^1} - \mathcal{O}_{\tilde{X}}(X^1)|_{X^1} = \omega_{\tilde{X}}|_{X^1}.$$

Putting the last two equations together, this leads to

$$\begin{aligned}\omega_{X^1} - \mathcal{O}_{\tilde{X}}(X^1)|_{X^1} &= r_1\mathcal{O}_{\tilde{X}}(X^1)|_{X^1} + r_2\mathcal{O}_{\tilde{X}}(X^2)|_{X^1} \\ &= r_1\mathcal{O}_{\tilde{X}}(X^1)|_{X^1} + r_2\mathcal{O}_{X^1}(E) \\ &= -r_1\mathcal{O}_{\tilde{X}}(X^2)|_{X^1} + r_2\mathcal{O}_{X^1}(E) \\ &= (r_2 - r_1)\mathcal{O}_{X^1}(E).\end{aligned}$$

As $\mathcal{O}_{\tilde{X}} = \mathcal{O}_{\tilde{X}}(\tilde{X}_0) = \mathcal{O}_{\tilde{X}}(X^1) + \mathcal{O}_{\tilde{X}}(X^2)$, we have

$$\mathcal{O}_{\tilde{X}}(X^1)|_{X^1} = -\mathcal{O}_{\tilde{X}}(X^2)|_{X^1} = -\mathcal{O}_{X^1}(E).$$

But as shown above $\omega_{X^1} = -\mathcal{O}_{X^1}(E)$ as $X^1 = \mathbb{F}_2$. All together we get

$$0 = \omega_{X^1} - \mathcal{O}_{\tilde{X}}(X^1)|_{X^1} = (r_2 - r_1)\mathcal{O}_{X^1}(E).$$

Hence, as $\mathcal{O}_{X^1}(E) \neq 0$, we have $r_2 = r_1$ and hence

$$\omega_{\tilde{X}} = \mathcal{O}_{\tilde{X}},$$

as $\mathcal{O}_{\tilde{X}} = \mathcal{O}_{\tilde{X}}(X^1) + \mathcal{O}_{\tilde{X}}(X^2)$. □

Remark 4.22. As we will see in Section 6 want to U-polarize the degeneration. If we have such a polarization U, we have that $U|_{X_0} \subset \mathrm{Pic}(X_0)$. We then define

$$\mathrm{Gr}_2 H_\infty^{\mathrm{pol}} = (\mathrm{N} \cap U|_{X_0}{}^{\perp})/\mathrm{Im}N.$$

For the well-definedness of this construction, see Section 6.

Proposition 4.23. *The degeneration can be polarized, such that indeed*

$$\mathrm{Gr}_2 H_\infty^{pol} = D_{16}^{+}.$$

Proof. Let

$$\mathrm{pr}_{\mathbb{F}_4} : \tilde{X} \to \mathbb{F}_4.$$

We define the polarization by specifying two divisors in X:

$$D_1 = \mathrm{pr}_{\mathbb{F}_4}^* \mathcal{O}_{\mathbb{F}_4}(1,0)$$

and

$$D_2 = \mathrm{pr}_{\mathbb{F}_4}^* \mathcal{O}_{\mathbb{F}_4}(0,1).$$

By construction it is apparent that this is a U-quasi-polarization for X, as $D_1|_{X_t}$ is just the section of X_t and $D_2|_{X_t}$ is a fiber. Recalling from above,

$$p : \tilde{X} \to X \to \mathbb{F}_4 \times \mathbb{A}^1$$

is the blowup of the double cover. On the other hand, we observe that by construction:

$$D_2|_{X_0} = p^{-1}(p,0) = f_1 + f_2$$

where f_1, f_2 are fibers of $\mathbb{F}_2, \mathrm{Bl}_{16}\mathbb{F}_n$ (as a fiber of X_0, meets the base of the blow up in one point). Doing the same for D_1, we get

$$D_1|_{X_0} = \mathrm{pr}_{\mathbb{F}_4}^{-1}(V(Z)) \cap X_0.$$

But this is just the section of $s_1 \subset \mathbb{F}_2$, as it does not meet the base of the blow up. By the invariant cycle theorem, we have

$$\langle D_1|_{X_0}, D_2|_{X_0}\rangle_{\mathbb{Z}} \subset \mathrm{Ker}\, N.$$

Therefore we can polarize with respect to the lattice

$$U = \langle f_1 + f_2, s_1\rangle_{\mathbb{Z}} \subset H^2(\mathbb{F}_2, \mathbb{Z}) \oplus H^2(\mathrm{Bl}_{16}\mathbb{F}_n, \mathbb{Z}).$$

If we fix the notation

$$s_i, f_i \qquad (i = 1, 2)$$

for the section and the fiber class in the two spaces, and

$$e_i \qquad (i = 1, \dots, 16)$$

the classes of the exceptional divisors in $\mathrm{Bl}_{16}\mathbb{F}_n$. Then

$$\mathcal{E} = 2s_1 + 4f_1 - 2s_2 - (2+n)f_2 + \sum e_i.$$

On the other hand, we observe that

$$\begin{aligned} \alpha_0 &= f_2 + e_1 + e_2 \\ \alpha_i &= e_i - e_{i+1} \qquad i = 1, \dots, 15 \end{aligned}$$

are roots, that satisfy

$$\begin{aligned} \alpha_i \cdot \mathcal{E} &= 0, \\ \alpha_i \cdot (f_1 + f_2) &= 0, \\ \alpha_i \cdot s_1 &= 0, \end{aligned}$$

for all i. Hence, they define elements in $\mathrm{Gr}_2^{\mathrm{pol}} H^2_\infty$. The set (α_i) is linearly independent in $H^2(\mathbb{F}_2, \mathbb{Z}) \oplus H^2(\mathrm{Bl}_{16}\mathbb{F}_2, \mathbb{Z})$. But as $\mathcal{E}$ contains a factor s_1, that is not contained in any α_i, we get that

$$R = \langle \alpha_i \rangle_{\mathbb{Z}} \subset (U, \mathcal{E})^\perp / \mathcal{E}\mathbb{Z}$$

is a free sublattice of dimension 16. On the other hand, we observe that

$$\begin{aligned} \alpha_0 \cdot \alpha_2 &= -1 \\ \alpha_0 \cdot \alpha_i &= 0 && (i \neq 2) \\ \alpha_i \cdot \alpha_{i+1} &= -1 && (i > 0) \\ \alpha_i \cdot \alpha_j &= 0 && (0 < i < j-1). \end{aligned}$$

Hence, this sublattice corresponds to a Dynkin diagram of Type $D_{16}(-1)$.

But $E_8(-1)^2$ cannot contain such a sublattice: All roots in $E_8(-1)^2$ are of the form $(w, 0)$ or $(0, w')$, but as R has dimension 16, the roots α_i cannot lie completely in one component E_8. Hence, we cannot have a chain $\alpha_1, \ldots, \alpha_{15}$, that have intersection $\alpha_i \cdot \alpha_{i+1} = -1$. Therefore, as the lattice $\mathrm{Gr}_2 H_\infty^{\mathrm{pol}}$ is even, non-degenerate and unimodular (see Section 6), we get by appendix A, that

$$\mathrm{Gr}_2 H_\infty^{\mathrm{pol}} = D_{16}^+(-1).$$

□

Example 4.24 ($h \rightsquigarrow Zg$). As it will turn out, the model we obtain will not be Kulikov. But in a way it is similar to a Type II degeneration. Define X to be the double cover of $\mathbb{F}_4 \times \mathbb{A}^1$ defined by

$$Z^2 \cdot g + t \cdot Z \cdot h.$$

As in the above case, we blow up the singular locus, which is given by

$$V = V(Z, t).$$

Similar to the above, we see that the resulting central fiber V_1 has one component which is a double cover

$$V_1 \to \mathbb{F}_4 \tag{4.24.1}$$

defined by g. But as g is generic and the intersection is $\mathcal{O}(0,1) \cdot Z^2 = 2$, we get that fiberwise V_1 is just a double cover of $\mathbb{P}^1$ ramified over 2 points, i.e. by Hurwitz we get that the resulting space is isomorphic to $\mathbb{P}^1$, as

$$2g - 2 = 2(2g' - 2) + 2 = -2.$$

Hence, by Appendix D, V_1 is a Hirzebruch surface. As above the second component is isomorphic to

$$V(w^2 = Z^2 g + tZh) \subset \mathbb{P}(\mathcal{O}(w)|_{V(Z,t)} \oplus \mathcal{O}(Z)|_{V(Z,t)} \oplus \mathcal{O}(t)|_{V(Z,t)}).$$

Calculating this locally, every fiber of the projection to $V(Z, t)$ is isomorphic to $\mathbb{P}^1$, as

$$w^2 = Z^2 + Zg$$

and

$$w^2 = Zg$$

define a $\mathbb{P}^1$ in $\mathbb{P}^2$. On the other hand $h = 0$ cannot happen over $V(Z, t)$, as the intersection product $h \cdot Z = 0$. Therefore it is also an Hirzebruch surface. By the dimension formula $H^2(X_0, \mathbb{Q}) = 21$ from Section 4.2 for degenerations with trivial canonical bundle, we see that this model cannot be Kulikov. But a local computation again shows, that X is smooth around the origin, and that it is semi-stable. Hence,

$$X_0 = V_1 \cup_E V_2.$$

But E is just the preimage of $V(Z)$ in V_1. As the intersection $g \cdot Z = 4$, we get that E is indeed an elliptic curve by Hurwitz's theorem. From writing down the Clemens-Schmid sequence as before, we get an exact sequence

$$0 \to H^1(E, \mathbb{Z}) \to W_1 H^2_\infty \to 0.$$

Hence, W_1 is two-dimensional as it only happens in the Type II case.[1]

4.3.2. A $E_8(-1)^2$ Degeneration

We want to construct a degeneration $X : X_t \rightsquigarrow V_1 \cup_E V_2$ to a union of elliptic surfaces, that respects the elliptic fibration.
To do this, we first construct a degeneration of the curve $\mathbb{P}^1 \rightsquigarrow \mathbb{P}^1 \cup \mathbb{P}^1$, which will be the base of the fibration. Fix a point $z \in \mathbb{P}^1$. Let $p = (z, 0) \in \mathbb{P}^1 \times \mathbb{A}^1$. Then

$$T = \mathrm{Bl}_p \mathbb{P}^1 \times \mathbb{A}^1 \xrightarrow{pr_2} \mathbb{A}^1$$

is a degeneration of $\mathbb{P}^1$, such that $T_t = \mathbb{P}^1$ for all $t \neq 0$. To compute the central fiber, we observe that

$$(\mathbb{P}^1 \times \mathbb{A}^1)_0 = \mathbb{P}^1,$$

and the point $z \in \mathbb{P}^1$ is blown up to the exceptional divisor $E \cong \mathbb{P}^1$ as it is a degeneration of surfaces. I.e.

$$T_0 = \mathbb{P}^1 \cup_z E.$$

Denote by $pr_1 : T \to \mathbb{P}^1$ the projection to the first component. We now construct a fundamental line bundle: Let

$$\mathbb{L} = pr_1^* \mathcal{O}_{\mathbb{P}^1}(2) \otimes \mathcal{O}_T(-E).$$

Then

$$\mathbb{L}|_{T_t} = \mathcal{O}_{\mathbb{P}^1}(2)$$

for $t \neq 0$ as $\mathbb{P}^1 \cong T_t \hookrightarrow T \xrightarrow{pr_1} \mathbb{P}^1$ is an isomorphism and $\mathcal{O}(E)|_{T_t} = 0$. On the other hand, for $t = 0$, we get

$$\mathbb{L}|_E = pr_1^* \mathcal{O}_{\mathbb{P}^1}(2)|_E \otimes \mathcal{O}_T(-E)|_E.$$

[1] In the Type III case W_1 is only one dimensional.

But the map $E \xrightarrow{pr_1} \mathbb{P}^1$ is just constant, hence $pr_1^*\mathcal{O}_{\mathbb{P}^1}(2)|_E$ is trivial. Furthermore $\deg \mathcal{O}_T(-E)|_E = -E^2 = 1$. So,

$$\mathbb{L}|_E = \mathcal{O}_{\mathbb{P}^1}(1).$$

Computing it for the other component:

$$\mathbb{L}|_{\mathbb{P}^1} = pr_1^*\mathcal{O}_{\mathbb{P}^1}(2)|_{\mathbb{P}^1} \otimes \mathcal{O}_T(-E)|_{\mathbb{P}^1}.$$

Again, as in the first case $pr_1^*\mathcal{O}_{\mathbb{P}^1}(2)|_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(2)$. And $\deg \mathcal{O}_T(-E)|_{\mathbb{P}^1} = -E.\mathbb{P}^1 = -1$, as both meet transversally. Hence

$$\mathbb{L}|_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(1).$$

As was shown in Section 2.2, if we take a Weierstraß fibration corresponding to $\mathbb{L}$, we get a degeneration

$$X : X_t \rightsquigarrow V_1 \cup_E V_2.$$

If each X_t is smooth (which can be achieved by choosing suitable Weierstraß data, c.f.[Gre18]), then X_t is a K3 surface and V_1, V_2 are elliptic rational surfaces. Moreover

$$\begin{array}{ccccc} X_t & \longrightarrow & X & \longleftarrow & X_0 \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{P}^1 & \longrightarrow & T & \longleftarrow & \mathbb{P}^1 \cup_z \mathbb{P}^1 \end{array}$$

commutes. Now, as shown in Remark 2.19, $\omega_{V_1} = -\mathcal{O}_{V_1}(E)$, where E is a fiber. Hence, as in the D_{16}^+ case, we can assume that $\omega_X = -\mathcal{O}_X(V_1) + c\mathcal{O}_X(V_2)$. Hence by adjunction

$$\omega_{V_1} = (\mathcal{O}_X(V_1) + \omega_X)|_{V_1} = c\mathcal{O}_X(V_2)|_{V_1} = c\mathcal{O}_{V_1}(E)$$

where E is the fiber along which we glue. But as every fiber is the same in the Picard group, we get that $c = -1$ by assumption. Hence

$$\omega_X = -\mathcal{O}_X(V_1) - \mathcal{O}_X(V_2) = \mathcal{O}_X$$

as $\mathcal{O}_X(V_1) + \mathcal{O}_X(V_2) = \mathcal{O}_X$. Thus, this a Kulikov model of Type II.

Remark 4.25. Observe, that $\mathbb{P}^1 \times \mathbb{A}^1 \subset \mathbb{F}_n$ for every n. Moreover when restricting to a neighborhood of the origin, this is a local model for any ruled surface. Hence the above local construction corresponds to taking any ruled surface, blowing it up at a point and then constructing a Weierstraß model. An analysis of these degeneration of K3 surfaces can be found in [Gre18].

Proposition 4.26. *The degeneration can be polarized such that*

$$\mathrm{Gr}_2^{pol}(H^2_\infty) \cong E_8(-1)^2.$$

Proof. Let

$$\mathrm{pr}_{\mathbb{P}^1} : X \to T \to \mathbb{P}^1.$$

We define

$$D_1 = \mathrm{pr}_{\mathbb{P}^1}^* \mathcal{O}_{\mathbb{P}^1}(1),$$

i.e. $D_1|_{X_t}$ is just the class of a fiber of the elliptic fibration for $t \neq 0$. For $t = 0$, we can just assume that $D_1|_{X_0}$ is just a fiber in the first component of $X_0 = V_1 \cup_E V_2$. From the local description in Section 2.2, we see that X is a double cover of

$$\mathbb{F} = \mathbb{P}(\mathcal{O}_T \oplus \mathbb{L})$$

with ramification loci

$$Z \cdot (X^3 + AXZ^2 + BZ^3)$$

where Z is the canonical section in $\mathbb{F}$. Hence, the section S in X is just the preimage of $V(Z)$ in X. Define

$$D_2 = \mathcal{O}(S).$$

Again, by the invariant cycle theorem, we get that

$$s = S \cap X_0 \in \operatorname{Ker} N.$$

But this is just the union of two sections s_0, s_1 in V_i meeting in a point on the double curve. It is obvious that

$$\langle D_1, D_2 \rangle_{\mathbb{Z}} \subset \operatorname{Pic}(X)$$

defines a U-quasi-polarization, as on every fiber X_t both are just the section and the fiber of the elliptic fibration for X_t. Recalling from Section 4.1,

$$\mathrm{Gr}_2 = \mathcal{E}^{\perp}/\mathcal{E}\mathbb{Z} \subset H^2(V_1, \mathbb{Z}) \oplus H^2(V_2, \mathbb{Z})/\mathcal{E}\mathbb{Z}$$

with $\mathcal{E} = E_1 - E_2$ where E_i is the cohomology class of the double curve restricted to the space V_i, i.e. in our case just a fiber of the elliptic fibration. But with our polarization ($s = s_1 + s_2, f_0 = E_1$), we get

$$\mathrm{Gr}_2 H_{\infty}^{\mathrm{pol}} = (\mathcal{E}, s, E_1)^{\perp}/\mathcal{E}\mathbb{Z}.$$

Let $\gamma \in H^2(V_i)$, such that $\gamma \cdot E_i = 0$ and $\gamma \cdot s_i = 0$. Denote $H^2_{\mathrm{prim}}(V_i)$ the space of all such γ satisfying the above. Then

$$H^2_{\mathrm{prim}}(V_1) \oplus H^2_{\mathrm{prim}}(V_2) \hookrightarrow \mathrm{Gr}_2 H_{\infty}^{\mathrm{pol}}$$

is well-defined. Moreover it is injective: If $\gamma = (\gamma_1, \gamma_2) \in \mathcal{E}\mathbb{Z}$, then

$$0 = \gamma \cdot s_1 = (aE_1, -aE_2) \cdot s_1 = a.$$

Therefore $\gamma = 0$. On the other hand, it is surjective: Let $\gamma = (\gamma_1, \gamma_2) \in H_{\infty}^{\mathrm{pol}}$ be given. Then

$$\begin{aligned} 0 =& \gamma \cdot \mathcal{E} = \gamma_1 \cdot E_1 - \gamma_2 \cdot E_2, \\ 0 =& \gamma \cdot E_1 = \gamma_1 \cdot E_1, \end{aligned}$$

implies that also $\gamma_2 \cdot E_2 = 0$. Let $a = \gamma_1 \cdot s_1$. Then, by $0 = \gamma \cdot s = \gamma_1 \cdot s_1 + \gamma_2 \cdot s_2$, we get

$$\begin{aligned}(\gamma - a\mathcal{E}) \cdot s_1 &= \gamma_1 \cdot s_1 - aE_1 \cdot s_1 = 0 \\ (\gamma - a\mathcal{E}) \cdot s_2 &= \gamma_2 \cdot s_2 + aE_1 \cdot s_1 = \gamma_2 \cdot s_2 - aE_2 \cdot s_2 = 0,\end{aligned}$$

as $E_i \cdot s_i = 1$. Hence, surjectivity is shown. So

$$\mathrm{Gr} H_\infty^{\mathrm{pol}} = E_8(-1) \oplus E_8(-1),$$

as it is non-irreducible, unimodular and negative definite. $\square$

5. Compactifications of the Moduli Space of Elliptic K3s

As the period map cannot be extended for generations for which the central fiber is non-K3, we want to compactify the moduli space that we obtained in Section 3. We will do this in two ways: The Baily-Borel compactification and the Mumford Toroidal compactification. As it turns out those are espacially handy for Type II degenerations. The first one distinguishes the different degenerations by the second graded piece of the limit Hodge structure and the j-invariant of the double curve, whereas the toroidal compactification is finer: it will classify the degenerations up to the whole mixed Hodge structure.

Throughout the whole section, we specify this setting:

Setting 5.1. *Let $\Lambda = \Lambda_{2,18}$ and $\Lambda_{\mathbb{C}}$ its complexification. Denote by $\Omega = \mathbb{P}(\Lambda_{\mathbb{C}})$ and let $\mathcal{D}$ be one of the connected components of $\{z \in \mathbb{P}(\Lambda_{\mathbb{C}}) \,|\, z^2 = 0,\, z\overline{z} > 0\}$[2]. By $\Gamma \subset O(\Lambda)$ denote the subset of its othorgonal group that leaves the components fixed. If we fix a subspace $J \subset \Lambda_{\mathbb{C}}$, we denote by $\Gamma_J \subset \Gamma$ its stabilizer, and by Γ^J the group that acts as the identity on J.*

Remark 5.2. In contrast to Section 3, we use only one component. But this does not change the resulting space, as here we take Γ as the subset that respects the component. Thus, the orbit spaces $\mathcal{D}/\Gamma$ in both chapters are isomorphic.

Remark 5.3. The two components of $\mathcal{D}$ can be specified by first taking an affinization by taking two variables f_1, f_2 of $\mathbb{C}^{20} \cong \Lambda_{\mathbb{C}}$ and specifying a sign of

$$\Im \frac{f_1}{f_0}$$

for $f_0 \neq 0$.

As the signature of Λ is $(2, 18)$, every isotropic[3] subspace has dimension ≤ 2. Before we construct the general compactification, we stick with a special case that is of most interest for us, as it corresponds to Type II degenerations. The following lemma will be useful later on, to classify all boundary components of the compactification.

Lemma 5.4 ([Bru15]). *There are exactly two Γ-orbits J, J' of isotropic planes in Λ, corresponding to*

- $J^{\perp}/J \cong E_8(-1) \oplus E_8(-1)$ *and*
- $J'^{\perp}/J' \cong D_{16}^{+}(-1)$.

[2] The two components get interchanged by taking the complex conjugate.
[3] A subspace is isotropic if $a \cdot a = 0$ for every $a \in J$

Moreover, there is only one such orbit of isotropic lines.

Remark 5.5. From standard lattice theory, we get that

$$E_8(-1)^2 \oplus U^2 \cong D_{16}^+(-1) \oplus U^2.$$

Thus, up to isomorphism we may assume, that

$$J = \mathbb{Z}f_1 \oplus \mathbb{Z}f_2 \subset E_8(-1)^2 \oplus U^2 = \Lambda$$

where f_i is the first standard coordinate of U, see Appendix A, as this clearly satisfies $J^\perp/J \cong E_8(-1) \oplus E_8(-1)$. In the same manner, $J' = \mathbb{Z}f_1 \oplus \mathbb{Z}f_2 \subset D_{16}^+(-1) \oplus U^2 = \Lambda$.

5.1. The Period Domain as Siegel Domain of the third Kind

In order to construct the compactifications, we embed $\mathcal{D}$ into a larger space, which will be shown now. We follow closely [Kon99].
Fix a rational isotropic sublattice J of $\Lambda_{\mathbb{C}}$. Then let $N_J \subset O(\Lambda \otimes \mathbb{R})$ the subset of the orthogonal group of the real lattice preserving J. Furthermore set

$$W_J = \{x \in \mathrm{Rad}(N_J) \,|\, (x - \mathrm{id})^n = 0 \text{ for some } n)\},$$

i.e. the unipotent elements of the radical of N_J. Moreover denote by U_J the center of W_J, in particular it is abelian. First we will investigate these groups further for the case $\dim J = 2$. By Remark 5.5, we can assume that

$$\Lambda = U \oplus U \oplus J^\perp/J,$$

where both U have basis f_i, s_i for both components and $J = f_1\mathbb{C} + f_2\mathbb{C}$. Consequently, we may assume that a basis of $\Lambda_{\mathbb{C}}$ is given by

$$f_1, f_2, w_1, \dots w_{16}, s_1, s_2, \tag{5.5.1}$$

where (w_i) is a basis for $J^\perp/J$. Denote the corresponding coordinates by t_i for $i = 1, \dots, 20$. Hence, in this representation, the bilinear form of the lattice looks like

$$A = \begin{pmatrix} 0 & 0 & I \\ 0 & L & 0 \\ I & 0 & 0 \end{pmatrix},$$

where L is the matrix of $J^\perp/J$ and I is the identity. As $g \in N_J$ preserves J, it also preserves $J^\perp$, as

$$j \cdot g(v) = g^{-1}(j) \cdot v = 0$$

for every $j \in J, v \in J^\perp$. Therefore $g \in N_J$ has a matrix representation

$$B = \begin{pmatrix} U & V & W \\ 0 & X & Y \\ 0 & 0 & Z \end{pmatrix},$$

with U, V, W, X, Y, Z matrices of the corresponding size. As any g respects the pairing of the lattice, we have that

$$A = B^T AB = \begin{pmatrix} 0 & 0 & U^T Z \\ 0 & X^T LX & V^T Z + X^T LY \\ Z^T U & Y^T LX + Y^T V & W^T U + Y^T LY + Z^T W \end{pmatrix}.$$

I.e. these are exactly those g, that satisfy

$$\begin{aligned} U^T Z &= I, \\ X^T LX &= L, \\ V^T Z + X^T LY &= 0, \\ W^T Z + Y^T LY + Z^T W &= 0, \end{aligned} \tag{5.5.2}$$

and moreover respect the component of Ω. As pointed out in Remark 5.3, the last condition is equivalent to $\Im \frac{t_1}{t_2} > 0$, i.e. U has to preserve this condition, which is equivalent to $\det U > 0$.

Now we will analyse its unipotent radical: By [Kon99] the unipotent radical is the normal subgroup consisting of those block matrices with trivial diagonal blocks:

$$W_J = \left\{ \begin{pmatrix} I & V & W \\ 0 & I & Y \\ 0 & 0 & I \end{pmatrix} \mid LY + V^T = W + W^T + Y^T LY = 0 \right\}. \tag{5.5.3}$$

Furthermore, any such matrix as above, with $V \neq 0 \neq Y$ does not commute with every element in W_J, as for two choices (V, W, Y), (V', W', Y') and corresponding matrices B, B' as above, we get that

$$B \cdot B' = \begin{pmatrix} I & V + V' & W + W' + VY' \\ 0 & I & Y + Y' \\ 0 & 0 & I \end{pmatrix}.$$

Hence, by symmetry, these commute if and only if

$$VY' = V'Y \tag{5.5.4}$$

for every choice of V', Y' as above. But for a given Y', we can arrange $V' = -(L^{-1}Y')^T$ and $W' = \frac{-1}{2} Y'^T LY'$. Then this element is contained in W_J. Thus, (5.5.4) holds for all such choices if and only if $V' = Y' = 0$. But then the condition (5.5.3) simply reads

$$W^T = -W,$$

i.e. the centralizer of W_J is given by

$$U_J = \left\{ \begin{pmatrix} I & 0 & W \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \middle| W = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} \text{ and } a \in \mathbb{R} \right\}.$$

For the next theorem, we need a small lemma:

Lemma 5.6. *Let $\Lambda = U \oplus J^{\perp}/J$. Then there is no element $z \in \Lambda$ such that*

$$z^2 = 0$$

and

$$z\overline{z} > 0.$$

Proof. As the signature of the lattice is $(1, 17)$, we can assume that the pairing is induced by the pairing

$$\begin{pmatrix} 1 & 0 \\ 0 & -I_{17} \end{pmatrix}$$

in $\mathbb{R}^{18}$. Hence, for coordinates $t = (t_1, t_2, \ldots, t_{18})$, we get that

$$0 = t^2 = t_1^2 - \sum_{i>1} t_i^2$$

and therefore

$$\sum_{i>1} t_i^2 = t_1^2.$$

On the other hand, $z\overline{z} > 0$ yields

$$\sum_{i>1} |t_i|^2 < |t_1|^2,$$

which is a contradiction to the triangle inequality. $\square$

Theorem 5.7. *Let $\mathcal{D}_J = U_J \cdot \mathcal{D} \subset \Omega$. Then*

$$\mathcal{D}_J \cong (U_J \otimes \mathbb{C}) \times \mathbb{C}^k \times F,$$

such that $F = \mathbb{H}$ is the upper half plane in $\mathbb{C}$ if J is a plane and $F = \{point\}$ otherwise. Furthermore the isomorphism is equivariant with respect to the action of U_J and

$$\mathcal{D} = \{(z, w, \tau) \in \mathcal{D}_J \,|\, \Im z - h_\tau(w, w) \in C_J\}$$

for a quasi-hermitian form h_τ (which depends real analytically on τ) and C_J a self-dual cone in U_J.[4]

Proof. For a proof in the one dimensional case, see [Kon99]. Choose a basis for Λ as in Lemma 5.4, i.e.

$$\Lambda = J^{\perp}/J \oplus U \oplus U$$

and $J = f_1\mathbb{C} \oplus f_2\mathbb{C}$, where f_i, s_i is the standard basis for one of the U-components. Next we take the projective coordinates $[t_i]$ from (5.5.1). Denote $t_0 = (t_3, \ldots, t_{18})$ the part

[4] Here self-dual means, that there is a positive definite form on U_F, such that C_J is self dual with respect to that form.

that comes from $J^\perp/J$. Let $q : J^\perp/J \times J^\perp/J \to \mathbb{C}$ be the induced pairing. An element t is contained in $\mathcal{D}$, iff

$$2t_1t_{19} + 2t_2t_{20} + q(t_0, t_0) \quad = 0 \tag{5.7.1}$$
$$2\Re t_1\overline{t_{19}} + 2\Re t_2\overline{t_{20}} + q(t_0, \overline{t_0}) \quad > 0. \tag{5.7.2}$$

But if the first condition is satisfied, the second one simplifies to

$$2\Im t_1 \Im t_{19} + 2\Im t_2 \Im t_{20} + q(\Im t_0, \Im t_0) > 0.$$

By the foregoing lemma, we see that $t_{20} = 0$ cannot happen, as this would yield the element $((t_1, t_{19}), t_0) \in U \oplus J^\perp/J$ satisfying the assumption of the lemma, which is a contradiction. Hence we can assume that $t_{20} = 1$ by taking the affinization. We choose a component of $\mathcal{D}$, such that $\Im t_1 > 0$. Then the above condition simplifies to

$$2\Im t_1 \Im t_{19} + q(\Im t_0, \Im t_0) > 0. \tag{5.7.3}$$

From (5.7.1), we get that t_2 is uniquely determined by the other coordinates (as $t_{20} = 1$)[5] and hence

$$\mathcal{D} \hookrightarrow \mathbb{C} \times J^\perp/J \times \mathbb{H}.$$

On the other hand, as we have seen before, $U_J \cong \mathbb{R}$ by identifying

$$a \mapsto \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}.$$

By checking the explicit description of U_J, one sees, that U_J acts on

$$\mathbb{C} \times J^\perp/J \times \mathbb{H}$$

just by translation in the first coordinate. Therefore we can identify the above equivariantly with

$$(U_J \otimes \mathbb{C}) \times J^\perp/J \times \mathbb{H}.$$

On the other hand, for every pair $(w, \tau) \in J^\perp/J \times \mathbb{H}$, we can find an element $z \in \mathbb{C}$, such that $(z, w, \tau) \in \mathcal{D}$. And hence

$$\mathcal{D}_J = (U_J \otimes \mathbb{C}) \times J^\perp/J \times \mathbb{H}.$$

For the statement on the hermitian form, see [Kon99]. As the only self-dual cone in $\mathbb{R}$ is $\mathbb{R}^+$, we get

$$C_J = \mathbb{R}^+ \subset \mathbb{R} = U_J.$$

$\square$

[5] I.e. $t_2 = -q(t_0, t_0) - 2t_1t_{19}$.

5.2. Baily-Borel Compactification

In this section, we will construct the *Baily-Borel* compactification, which is highly singular since the boundary has large codimension. The presented material follows [Bru15]. Let

$$\hat{\mathcal{D}} = \text{closure of } \mathcal{D} \subset \mathcal{D} \cup \{z \mid z^2 = 0\} \subset \{z \in \mathbb{P}(\Lambda_{\mathbb{C}}) \mid z^2 = 0\} = \Omega.$$

Clearly $\mathcal{D} \subset \hat{\mathcal{D}}$. We then define $\mathcal{D}^*$ to be the union of $\mathcal{D}$ and the interior of the images π_I of isotropic subspaces defined over $\mathbb{Q}$ inside $\hat{\mathcal{D}}$, i.e.

$$\mathcal{D}^* = \mathcal{D} \sqcup \bigsqcup_I \pi_I.$$

The topology near the boundary components must of course be defined suitably, but for our case this is not of any interest. It is important to note, that the topology in the boundary coincides with analytical one, obtained from $\hat{\mathcal{D}}$.
As Γ takes isotropic subspaces to isotropic subspaces, one gets a natural Γ-action on $\mathcal{D}^*$.

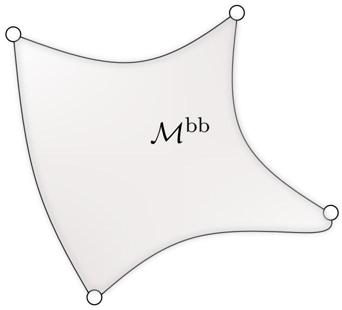

Figure 3: The Baily-Borel compactification with lines (points) the boundary components corresponding to isotropic planes (lines)

Definition 5.8. We define the Baily-Borel compactification as

$$\mathcal{M}^{\mathrm{bb}} = \Gamma \backslash \mathcal{D}^*.$$

As every component of $\mathcal{D}^*$ is contained in $\mathbb{P}(\Lambda_{\mathbb{C}})$, there is a tautological choice for a line bundle $\mathbb{L}$ on $\mathcal{D}^*$ as is explained in [L$^+$03]:

$$\mathbb{L} = \{(z, x) \mid z \in \mathcal{D}^*, x \in \Lambda_{\mathbb{C}} \text{ s.t. } [x] = z \in \mathbb{P}(\Lambda_{\mathbb{C}}) \text{ or } x = 0\}.$$

This line bundle admits a natural action of Γ, that is compatible with the action on $\mathcal{D}^*$. Hence it descends to the Baily-Borel compactification.

Definition 5.9. The line bundle $\mathbb{L}^{\mathrm{bb}}$ constructed above is called *Hodge line bundle.*

Corollary 5.10. *The compactification set-theoretically looks like*

$$\mathcal{M}^{bb} = \mathcal{M} \sqcup \mathbb{H}/\mathrm{SL}(2,\mathbb{Z}) \cup \mathbb{H}/\mathrm{SL}(2,\mathbb{Z}) \cup \{p\},$$

where the one dimensional boundary components represent the two orbits of isotropic planes and the distinguished point corresponds to the isotropic line.

Proof. Let $I = f_1\mathbb{C} + f_2\mathbb{C}$ be an isotropic plane with f_i rational. Then the interior of image of I in $\hat{\mathcal{D}}$ is isomorphic to the upper half plane: As we have seen in the previous section we can take an affinisation of $\hat{\mathcal{D}}$ by t_0 without losing information. Then the two components are distinguished specifying w.l.o.g. that $\Im t_1 > 0$. Hence, the interior of the image is

$$\pi_I = \mathbb{H} \subset \mathbb{P}^1 = \mathbb{P}(J).$$

Moreover Γ interchanges all isotropic planes of one orbit. Consequently, we can assume that for one such orbit, there is only one component, and Γ acts on $\mathbb{H}$ via Γ_J, i.e. those elements $g \in \Gamma$ that satisfy $g(I) = I$. Hence, as $\det g|_I = \pm 1$. But as g has to fix $\mathbb{H}$, we get $\det g|_I = 1$. Hence, Γ acts on $\mathbb{H}$ as a subgroup of $\mathrm{SL}(2,\mathbb{Z})$.
But from the discussion in the previous section, we see that any such element $g \in \mathrm{SL}(2,\mathbb{Z})$ extends to an element $g' \in \Gamma_J$ and so, the action is via the full group $\mathrm{SL}(2,\mathbb{Z})$. Thus the corollary follows by Lemma 5.4. □

Although we worked in the analytic category, we get that

Theorem 5.11 ([L$^+$03])**.** *The space $\mathcal{M}^{bb}$ admits the structure of a normal analytic ringed space.*

5.3. Constrution of the Toroidal Compactification

The toroidal construction in our case is rather simple. Fix one isotropic line J. Let C_J be as in the previous section. Again, we are following [Bru15].

Definition 5.12. An *admissible* fan of C_J is given by a fan σ of U_J consisting of rational cones, that subdivide C_J, such that

- $N_J \cap \Gamma$ acts on Σ.
- The stabilizer of each cone $\sigma \in \Sigma$ is finite.
- There are only finitely many N_J-orbits in Σ.

Remark 5.13. In general, one has to fix such admissible fans for every isotropic subspace, but in our case, there is only one choice of fan for isotropic planes, as for planes, U_J is one-dimensional.

First we describe the construction locally, over a boundary component. Fix an isotropic subspace J. Then as in the last section, we get a representation

$$\mathcal{D} \hookrightarrow (U_J \otimes \mathbb{C}) \times \mathbb{C}^k \times F.$$

As U_J is clearly finite dimensional, we get that $U_J \otimes \mathbb{C} \cong \mathbb{C}^n$ for some n. Hence, $U_J \otimes \mathbb{C}/(U_J \cap \Gamma) \cong \mathbb{C}^n/\mathbb{Z}^n$, which in turn is isomorphic to $(\mathbb{C}^*)^n$ via the isomorphism

$$(z_j)_j \mapsto (\exp(2\pi i j))_j$$

As a self-dual cone does not contain any straight line, we get that $(\mathbb{C}^*)^n \subset TV(\Sigma)$, see Appendix C. Therefore there is an embedding

$$(U_J \otimes \mathbb{C})/(U_J \cap \Gamma) \hookrightarrow TV(\Sigma).$$

By this inclusion, we get

$$\mathcal{D}_J/(U_J \cap \Gamma) \hookrightarrow TV(\Sigma) \times \mathbb{C}^k \times F. \tag{5.13.1}$$

We then define $\mathcal{D}_J^{\mathrm{Tor}}$ as the interior of the closure of $\mathcal{D}/(U_J \cap \Gamma) \subset TV(\Sigma) \times \mathbb{C}^k \times F$. Moreover this set comes equipped with an action of N_J. Hence, we get the orbitspace

$$\mathcal{D}_J^{\mathrm{Tor}}/\Gamma_J.$$

for $\Gamma_J = N_J \cap \Gamma$. Denote by $\partial\mathcal{D}_J^{\mathrm{Tor}}/\Gamma_J = (\mathcal{D}_J^{\mathrm{Tor}}/\Gamma_J)\backslash(\mathcal{D}/\Gamma_J)$. Then we can form the union

$$\mathcal{D}^{\mathrm{Tor}} = \bigsqcup_J \mathcal{D}_J^{\mathrm{Tor}}$$

where the union is over every isotropic subspace J. Then there is an obvious equivalence relation R on

$$\bigsqcup_J \mathcal{D}/(U_J \cap \Gamma) \subset \bigsqcup_J \mathcal{D}_J^{\mathrm{Tor}},$$

which is induced by the action of Γ, i.e. $R \subset \bigsqcup_J \mathcal{D}/(U_J \cap \Gamma) \times \bigsqcup_J \mathcal{D}/(U_J \cap \Gamma)$. Denote by

$$\overline{R} \subset \bigsqcup_J \mathcal{D}_J^{\mathrm{Tor}}$$

its closure.

The *toroidal compactification* $\mathcal{M}^{\mathrm{Tor}}$ is defined as

$$\mathcal{M}^{\mathrm{Tor}} = \mathcal{D}^{\mathrm{Tor}}/\overline{R}.$$

The topology near the boundary components is given by the one induced by the fiber bundle (5.13.1). By Lemma 5.4 and the discussion in [L+03] it follows that set-theoretically

$$\mathcal{M}^{\mathrm{Tor}} = \mathcal{M} \sqcup \mathcal{D}_I^{\mathrm{Tor}}/\Gamma_I \sqcup \mathcal{D}_J^{\mathrm{Tor}}/\Gamma_J \sqcup \mathcal{D}_K^{\mathrm{Tor}}/\Gamma_K$$

where I, J are isotropic planes corresponding to $I^\perp/I = E_8(-1)^2$, $J^\perp/J = D_{16}^+(-1)$ and K is an isotropic line. We get:

Theorem 5.14 (Mumford [AMS+10])**.** *The toroidal compactification $\mathcal{M}_\Sigma^{Tor}$ is a compact algebraic space. Moreover there is a proper map to the Baily-Borel compactification, which is simply given by the identity on $\mathcal{M}$ and on the boundary of Type II, it is just induced by the projection $\mathcal{D}_J/(U_J \cap \Gamma) \subset TV(\Sigma) \times \mathbb{C}^k \times \mathbb{H} \to \mathbb{H} \to \mathbb{H}/\mathrm{SL}(2,\mathbb{Z})$.*

Remark 5.15. In [AMS+10] it is moreover shown, that if one allows neat $\Gamma' \subset \Gamma$ the resulting toroidal compactification $\mathcal{M}^{\mathrm{Tor}'}$ is smooth. As there exist neat normal subgroups of finite index, one has a group action $G \times \mathcal{M}^{\mathrm{Tor}'} \to \mathcal{M}^{\mathrm{Tor}'}$ with G finite, such that the orbit space is equal to $\mathcal{M}^{\mathrm{Tor}}$. The same also holds for the moduli space $\mathcal{M}$ itself, see e.g. [Bru04].

Regarding degenerations, we get the following useful lemma:

Theorem 5.16 (Mumford [AMS+10])**.** *For a map $f : \Delta^* \to \Gamma\backslash\mathcal{D}$ it is equivalent:*[6]

- *f extends to a map $f : \Delta \to \Gamma\backslash\mathcal{D}^{Tor}$*
- *There is a map $f^0 : \Delta \to \mathcal{D}_J^{Tor}$ for some J, which induces f on Δ^*.*

5.4. Type II Toroidal Boundary

We will now analyse the boundary that corresponds to the isotropic *planes*, following [Kon99] and [Bru15]. As we will see later on, this corresponds to exactly the Type II degenerations. Fix a isotropic plane J, which corresponds to either $E_8(-1)^2$ or $D_{16}^+(-1)$ and denote by $q : J^\perp/J \times J^\perp/J \to \mathbb{C}$ the corresponding bilinear pairing. As we saw in the previous sections, we get the map

$$\mathcal{D} \subset (U_J \otimes \mathbb{C}) \times \mathbb{C}^{16} \times \mathbb{H},$$

with $(z, w, \tau) \in \mathcal{D}$ if and only if

$$\Im z \Im \tau + q(\Im w, \Im w) > 0,$$

see equation (5.7.3). As in the example in Appendix C, we see that $TV(\mathbb{R}^+) = \mathbb{C}$, hence the construction in the previous section yields:

$$\mathcal{D}/(U_J \cap \Gamma_J) \subset \mathbb{C} \times J^\perp/J \times \mathbb{H},^{7}$$

where the first coordinate is the filling of $(U_J \otimes \mathbb{C})/(U_J \cap \Gamma_J)$. Nonetheless, we mean by a triple $(z, w, \tau) \in \mathcal{D}$ an element in the coordinates of $U_J \otimes \mathbb{C} \times J^\perp/J \times \mathbb{H}$. Observing

[6]Here $\Delta^* = \{z \in \mathbb{C} \mid 0 < |z| < 1\}$.
[7]It is important to note here that the first coordinate $\mathbb{C}$ is not equal to $U_J \otimes \mathbb{C}$, as it is the filing of $\mathbb{C}^* = \mathbb{C}/\mathbb{Z}$.

that $(z, w, \tau) \in \mathcal{D}$ if and only if $\Im z > -q(\Im w, \Im w)/\Im\tau$ it follows that the interior of the closure is given set-theoretically by

$$\mathcal{D}/(U_J \cap \Gamma) \sqcup 0 \times J^\perp/J \times \mathbb{H},$$

as $\Im z \gg 0$ implies $|\exp(2\pi i z)| \ll 1$. Moreover for fixed $w \in J^\perp/J, \tau \in \mathbb{H}$ we have that

$$(z, w, \tau) \in \mathcal{D}_J^{\mathrm{Tor}} := \overline{\mathcal{D}/(U_J \cap \Gamma)}^\circ \subset \mathbb{C} \times J^\perp/J \times \mathbb{H} = TV(\mathbb{R}^+) \times J^\perp/J \times \mathbb{H}$$

if and only if $z \in \{c \in \mathbb{C} \,|\, |c| < \exp(-q(\Im w, \Im w)/\Im\tau)\}$. Hence, we get that $\mathcal{D}_J^{\mathrm{Tor}}$ is a $\Delta = \{z \,|\, |z| < 1\}$-bundle, as q is negative-definite.

Now we will analyse the effect of Γ_J on $\mathcal{D}_J^{\mathrm{Tor}}$. From our description, we saw that $t_{20} = 1$

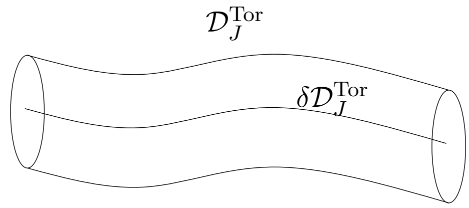

Figure 4: $\mathcal{D}_J^{\mathrm{Tor}}$ as a Δ-bundle

and $t_2 = -q(w, w) - 2z\tau$. Hence

$$g = \begin{pmatrix} U & V & W \\ 0 & X & Y \\ 0 & 0 & Z \end{pmatrix} \in N_J$$

acts on (z, w, τ) as

$$g \cdot (z, w, \tau) = \begin{pmatrix} z + \frac{1}{c\tau+d}(-cq(w, w) + v_1 w + w_1\tau + w_2) \\ \frac{1}{c\tau+d}(Xw + Y \begin{pmatrix} 1 \\ \tau \end{pmatrix}) \\ \frac{a\tau+b}{c\tau+d} \end{pmatrix}, \tag{5.16.1}$$

where $Z = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $U = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$, v_1 is the first row of V and (w_1, w_2) is the first row of W. The division by $c\tau + d$ comes from the affinization by the coordinate t_{20}.

As we are only concerned with the boundary, we will only analyze the second and the third component further. Clearly, the action of Γ_J preserves the boundary. By (5.5.2), we see that $\det Z = \det U > 0$, but it is invertible over $\mathbb{Z}$, hence $\det U = \det Z = 1$ and $Z \in \mathrm{SL}(2, \mathbb{Z})$. On the other hand, clearly $X \in O(J^\perp/J)$. Therefore by the previous calculation, we conclude that g acts on (z, w, τ) as $Z \cdot \tau$ in the last component, where this is the usual $\mathrm{SL}(2, \mathbb{Z})$-action on the upper half plane.

Moreover any Y gives rise to an element of W_J. On the boundary, g acts as

$$(0, w, \tau) \mapsto (0, \frac{1}{c\tau + d}(Xw + Y \begin{pmatrix} 1 \\ \tau \end{pmatrix}), Z\tau). \tag{5.16.2}$$

If we first quotient out $G = (W_J \cap \Gamma)/(U_J \cap \Gamma)$, which is determined by Y-component, we get that

$$\partial\mathcal{D}_J^{\mathrm{Tor}}/G = 0 \times \mathcal{E} \otimes J^{\perp}/J,$$

as any such $g \in W_J$ simply acts by translation by $Y \begin{pmatrix} 1 \\ \tau \end{pmatrix}$ in the second variable. Here $\mathcal{E} \to \mathbb{H}$ is the universal elliptic curve, i.e.

$$(\mathbb{C} \times \mathbb{H})/((z,\tau) \sim (z + n + m\tau, \tau) \text{ for every n, m}).$$

Hence the fiber $\mathcal{E}_\tau$ is just an elliptic curve isomorphic to

$$\mathcal{E}_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}).$$

There is a natural $\mathrm{SL}(2,\mathbb{Z})$ action on this space, namely

$$g \cdot (z,\tau) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (z,\tau) = ((c\tau + d)^{-1} z, g\tau),$$

which induces isomorphisms

$$\mathcal{E}_\tau \xrightarrow{\cong, g} \mathcal{E}_{g\tau}.$$

Calculating

$$\partial\mathcal{D}_J^{\mathrm{Tor}}/\Gamma_J$$

we see that the effect of Z in (5.16.2) on the middle component of

$$\partial\mathcal{D}_J^{\mathrm{Tor}}/(W_J \cap \Gamma) = 0 \times \mathcal{E} \otimes J^{\perp}/J$$

is just multiplication by $(c\tau + d)^{-1}$. Hence, N_J acts on

$$\partial\mathcal{D}_J^{\mathrm{Tor}}/(W_J \cap \Gamma) = 0 \times \mathcal{E} \otimes J^{\perp}/J$$

just as $O(J^{\perp}/J) \times \mathrm{SL}(2,\mathbb{Z})$. Hence we get:

Theorem 5.17 ([Bru15]). *Let J be an isotropic plane. Then the corresponding boundary component is isomorphic to*

$$\partial\mathcal{D}_J^{Tor}/\Gamma_J = ((\mathcal{E} \otimes J^{\perp}/J)/(O(J^{\perp}/J) \times \mathrm{SL}(2,\mathbb{Z}))$$

independent of the chosen fan Σ. In particular a fiber of τ of the projection map to $\mathbb{H}/\mathrm{SL}(2,\mathbb{Z})$ has the form

$$Z_\tau = (J^{\perp}/J \otimes (\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})))/O_E(J^{\perp}/J). \tag{5.17.1}$$

where

$$O_E(J^{\perp}/J) = O(J^{\perp}/J) \times \mathrm{Aut}(E, 0).$$

Proof. Only the last statement needs further explanation. By construction

$$Z_\tau = (J^\perp/J \otimes (\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})))/G$$

with

$$G = \{(g,h) \in O(J^\perp/J) \times \mathrm{SL}(2,\mathbb{Z}) \,|\, h(\tau) = \tau\}.$$

But in [Hai08] it is shown that

$$\{h \in \mathrm{SL}(2,\mathbb{Z}) \,|\, h(\tau) = \tau\} = \mathrm{Aut}(E,0),$$

which consists just of multiplication maps. $\square$

Remark 5.18. By [Hai08], $\mathrm{Aut}(E) = \{\pm 1\}$ for $\tau \neq i, e^{2\pi i/3} \in \mathbb{H}/\mathrm{SL}(2,\mathbb{Z})$. Thus, in these cases

$$Z_\tau = (J^\perp/J \otimes (\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})))/O(J^\perp/J),$$

as multiplication by ± 1 on $J^\perp/J \otimes (\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z}))$ is also induced by $-id \in O(J^\perp/J)$. In the other two special cases the group $\mathrm{Aut}(E)$ still remains finite by [Hai08].

Remark 5.19. As we will see later on, for our purpose it is necessary to map the above isomorphically to

$$((\mathcal{E} \otimes (J^\perp/J)^*)/(O(J^\perp/J) \times \mathrm{SL}(2,\mathbb{Z})) \cong \mathrm{Hom}(J^\perp/J, \mathcal{E})/(O(J^\perp/J) \times \mathrm{SL}(2,\mathbb{Z}))$$

via the canonical isomorphism $J^\perp/J \cong (J^\perp/J)^*$ induced by the pairing. The fiber of the theorem above is given by

$$Z_\tau = \mathrm{Aut}(E)\backslash \mathrm{Hom}(J^\perp/J, E_\tau)/O(J^\perp/J)$$

which we - by abuse of notation - abbreviate by

$$O_E(J^\perp/J)\backslash \mathrm{Hom}(J^\perp/J, E_\tau).$$

In [L$^+$03], Looijenga considered the intermediate group $U_J \cap \Gamma \subset \Gamma^J \subset \Gamma_J$ that consists of the elements that are the identity on J. I.e. with the notation from before, these are the matrices of the form

$$\begin{pmatrix} I & V & W \\ 0 & X & Y \\ 0 & 0 & I \end{pmatrix} \in \Gamma_J.$$

In the paper he showed, that the boundary is of the form

$$O(J^\perp/J)\backslash(J^\perp/J \otimes E_\tau)$$

in every fiber of the projection to $\mathbb{H}$, which also directly follows from our description above. The following theorem is shown in [L$^+$03]:

Theorem 5.20. *The map*

$$\mathcal{D}_J^{Tor}/\Gamma^J \to \partial\mathcal{D}_J^{Tor}/\Gamma^J$$

is a disc bundle and the pullback of the euler class e of that bundle to a fiber as above is nontrivial and invariant under the $O_E(J^\perp/J)$-action.

Next, we will analyze the closure of certain divisors: Let $v \in \Lambda = U \oplus U \oplus J^\perp/J$. Then $v = u + u' + j$ in the respective components. Here $u = (u_1, u_2), u' = (u'_1, u'_2)$. We want to analyze the closure of

$$v^\perp \subset \mathcal{D}/(U_J \cap \Gamma)$$

in $\mathcal{D}_J^{Tor}$.
Let $(z, w, \tau) \in \mathcal{D}$ as before. Then

$$(z, w, \tau) \cdot v = 0$$

is equivalent to

$$0 = \tau u_1 + z u_2 + (-q(w,w) - 2\tau z)u'_1 + u'_2 + q(w, j).$$

Now assume $(u_2, u'_1) \neq (0, 0)$. Then

$$z = \frac{q(w,w)u'_1 - u'_2 - q(w,j) - \tau u_1}{u_2 - 2\tau u'_1}.$$

Therefore

$$\begin{aligned}\Im z = \;& \tfrac{1}{|u_2 - 2\tau u'_1|^2}(u_2 u'_1 q(w,w) + 2\Im\tau u_1'^2 q(w,w) - u_2 u'_2 - 2u'_1 u'_2 \Im\tau - u_2(w, \Im j) \\ & +2u'_1(\Im\tau(w, \Re j) + \Re\tau q(w, \Im j)) - \Im\tau u_1 u_2).\end{aligned}$$

Hence

$$\Im z \in \mathcal{O}(\frac{q(w,j)}{|u_2 - 2\tau u'_1|^2}).$$

Suppose that a point (j, τ) is in the closure. That means, there is a sequence $\alpha_n = (z_n, j_n, \tau_n) \in \mathcal{D}$ such that

$$\begin{aligned}\Im z_n &\to \infty \\ j_n &\to j \\ \tau_n &\to \tau \in \mathbb{H}.\end{aligned}$$

But clearly the last two equations contradict the first one as the bound (5.4) shows, that $\Im z_n \ll \infty$. Hence for $(u_2, u'_1) \neq 0$, we have

$$\overline{v^\perp} \cap \partial\mathcal{D}_J^{\mathrm{Tor}} = \emptyset.$$

So observe for the case $(u_2, u'_1) = 0$: The equation reads

$$0 = \tau u_1 + u'_2 + q(w, j)$$

or more explicitly

$$q(w,j) = -\tau u_1 - u_2' \tag{5.20.1}$$

and z is arbitrary. Hence

$$\overline{v^\perp} \cap \partial\mathcal{D}_J^{\mathrm{Tor}} = \{(j,\tau) \,|\, q(w,j) = -\tau u_1 - u_2'\}.$$

In this case, it is important to notice that all these points are of the form (j,τ) with $q(w,j) = 0 \in J^\perp/J \otimes \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ in $\mathcal{D}_J^{\mathrm{Tor}}/\Gamma_J = ((\mathcal{E} \otimes J^\perp/J)/(O(J^\perp/J) \times \mathrm{SL}(2,\mathbb{Z}))$. Summarizing:

Proposition 5.21. *With the notation as above*

$$\overline{\bigcup_{v^2=-n} v^\perp} \cap \partial\mathcal{D}_J^{Tor}/\Gamma_J = \{(w,\tau) \,|\, q(w,j) = 0 \in \mathcal{E}_\tau, q(j,j) = -n\}.$$

Proof. This follows directly from the above and the fact, that

$$((u_1,u_2),(u_1',u_2'),j)^2 = q(j,j)$$

for $u_2 = u_1' = 0$. □

Remark 5.22. I.e. the restriction of $\overline{\bigcup_{v^2=-n} v^\perp} \subset \mathcal{M}^{\mathrm{Tor}}$ to a fiber

$$Z_\tau = O_E(J^\perp/J)\backslash \mathrm{Hom}(J^\perp/J, E_\tau)$$

is given by

$$\bigcup_{v^2=-2n} v^\perp$$

where $v^\perp = \{f \in \mathrm{Hom}(J^\perp/J, E_\tau) \,|\, f(v) = 0 \in E_\tau\}$. This is clearly a divisor, as the union is finite, as $J^\perp/J$ is negative-definite.

Remark 5.23. As we see from equation (5.20.1), if $(u_2,u_1') = (0,0)$ then $v^\perp$ is cut out by a single equation, i.e. the closure we computed above is indeed an analytic divisor. In the other case this also follows, as the closure does not meet the boundary and even by the same argument, for every point in the boundary there is a small neighborhood which is disjoint to the closure.

5.5. Topology of Orbitspaces of Finite Group Actions

In the last section we computed the boundary structure of the toroidal compactification. It turned out that fiberwise it is an orbitspace of a finite group action. Our goal is now to analyze its homology. Therefore we have to recall some general theorems about orbitspaces of finite group actions:

Theorem 5.24 ([Loo76] and [Gre18]). *Let R be an irreducible root system and L the corresponding root lattice. Then for any elliptic curve E the space*

$$\mathcal{W}(L)\backslash \mathrm{Hom}(L, E)$$

is a weighted projective space. Here $\mathcal{W}(L)$ denotes the Weyl group.

Theorem 5.25 ([Ill83]). *Let G be a compact Lie group and M be a smooth manifold with a G-action. Then there exists an equivariant simplicial complex K and a smooth equivariant triangulation $h : K \to M$.*

Remark 5.26. This theorem therefore applies also to finite groups G, for which Illman found an independent proof in [Ill78].

Theorem 5.27 ([Bre72]). *Let M be space with a G-group action and $K \to M$ be a regular G-equivariant triangulation. Denote by $\pi : M \to M/G$ the usual projection. Then there exists a transfer map $\tau : H_i(M/G, k) \to H_i(M, k)^G$ such that $\pi_*|_{H_i(M,k)^G}$ and τ are inverse to each other, whenever the field k has characteristic* 0.

Remark 5.28 ([Bre72]). Bredon proves that the above condition of the triangulation being regular can be achieved for any G-equivariant triangulation, by just passing to the second barycentric subdivision.

Hence, we get the following corollary:

Corollary 5.29. *For any smooth manifold M with a G-action, where G is a finite group, the projection $\pi : M \to M/G$ induces an isomorphism*

$$H_i(M, \mathbb{Q})^G \to H_i(M/G, \mathbb{Q}).$$

Remark 5.30. By duality this result also applies to cohomology.

In the following we are only concerned with rational homology. Recall that we defined

$$O_E(L) = O(L) \times \mathrm{Aut}(E).$$

Lemma 5.31. *For both lattices $L = E_8(-1)^2$ or $D_{16}^+(-1)$ we define the variety $V = O_E(L)\backslash \mathrm{Hom}(L, E)$ for an arbitrary elliptic curve E. Then this space satisfies*

$$H_1(V) = \{0\} \qquad \text{and} \qquad H_2(V) \cong \mathbb{Q}.$$

Moreover, the second homology is generated by the element α, which is a pushforward under the map $H_2(E^{16}) \cong H_2(\mathrm{Hom}(L, E)) \to H_2(V)$ of

$$\overline{\alpha} = \sum_{g \in O(L)} g([E_0])$$

where $E_0 = E \times 0^{15}$.

Proof. Case $E_8(-1)^2$:
By Corollary 5.29 we get that

$$H_i(V) \cong H_i(\mathrm{Hom}(L,E)))^{O_E(L)}.$$

As L is a direct summand the Weyl group $\mathcal{W}$ acts on both summands separately. Hence we have

$$\mathcal{W}(L)\backslash(\mathrm{Hom}(L,E)) \cong \mathcal{W}(E_8(-1))\backslash\mathrm{Hom}(E_8(-1),E))\times\mathcal{W}(E_8(-1))\backslash(\mathrm{Hom}(E_8(-1),E))).$$

By Theorem 5.24 we get that both factors are isomorphic weighted projective spaces $W\mathbb{P}_i$(i = 1,2). Therefore, as the homology of weighted projective spaces satisfies

$$\begin{aligned} H_0(W\mathbb{P},\mathbb{Q}) &\cong \mathbb{Q}, \\ H_1(W\mathbb{P},\mathbb{Q}) &\cong 0, \\ H_2(W\mathbb{P},\mathbb{Q}) &\cong \mathbb{Q}, \end{aligned}$$

applying Künneth's theorem yields

$$H_1(\mathcal{W}(L)\backslash(\mathrm{Hom}(L,E))) \cong 0$$

and

$$\begin{aligned} &H_2(\mathcal{W}(L)\backslash(\mathrm{Hom}(L,E))) \\ &\cong H_2(W\mathbb{P}_1,\mathbb{Q}) \otimes H_0(W\mathbb{P}_2,\mathbb{Q}) \oplus H_2(W\mathbb{P}_2,\mathbb{Q}) \otimes H_0(W\mathbb{P}_1,\mathbb{Q}) \\ &\cong \mathbb{Q}^2. \end{aligned} \tag{5.31.1}$$

Furthermore the injective map

$$H_i(V) \cong H_i(\mathrm{Hom}(L,E))^{O_E(L)} \to H_i(\mathrm{Hom}(L,E))^{\mathcal{W}(L)}$$

yields $H_1(V,\mathbb{Q}) \cong 0$.

Clearly the map $t : L \to L$ which interchanges the two factors is an element of $O(E_8(-1)^2)$ but not of $\mathcal{W}(E_8(-1)^2)$ as every reflection leaves the two components invariant. By (5.31.1) we see, that only the elements on the diagnal of $\mathbb{Q}^2 \cong H_2(\mathcal{W}(L)\backslash(\mathrm{Hom}(L,E)))$ are invariant under the action of t. Hence $\dim(H_2(\mathrm{Hom}(L,E))^{\mathcal{W}(L)}) \leq 1$.
Case $D_{16}^+(-1)$:
Recalling the construction of D_{16}^+ we get that the integral part of the lattice is equal to its root lattice. Looking at the definition, we see that this is isomorphic to D_{16}. Now take any $\mathbb{Z}$-basis $\{e_i\}_{1\leq i\leq 16}$ of D_{16}^+. As seen in Appendix A we can assume that $e_{16} = (1/2)^{16}$ is one of them, and all other vectors are integral. As one easily sees $\{e_1,\ldots,e_{15},2e_{16}\}$ is a $\mathbb{Z}$-basis for D_{16}. Hence, if we identify $\mathrm{Hom}(D_{16}^+,E)$ and $\mathrm{Hom}(D_{16},E)$ via the basis given above with E^{16}, the map

$$E^{16} \cong \mathrm{Hom}(D_{16}^+,E) \xrightarrow{incl^*} \mathrm{Hom}(D_{16},E) \cong E^{16} \tag{5.31.2}$$

is just multiplication by 2 in the last component. But looking at the usual CW-structure of an elliptic curve, we see that multiplication by 2 induces multiplication by 2 on homology on $H_1(E)$ and multiplication by 4 on $H_2(E)$. By Künneth's theorem we know that $\mathrm{incl}^* : H^2(\mathrm{Hom}(D_{16}^+, E)) \xrightarrow{incl^*} H_2(\mathrm{Hom}(D_{16}, E))$ is a bijection.
On the other hand, let $g \in \mathcal{W}(D_{16})$ be given. By the construction of the Weyl group g is a composition of reflections at roots. But those reflections extend to the whole lattice D_{16}^+. Hence we have a natural map $\mathcal{W}(D_{16}) \to \mathcal{W}(D_{16}^+)$ which leaves the action on $D_{16} \subset D_{16}^+$ invariant. Forming the square

$$\begin{array}{ccc} \mathrm{Hom}(D_{16}^+, E) & \xrightarrow{g^*} & \mathrm{Hom}(D_{16}^+, E) \\ \Big\downarrow{\scriptstyle incl^*} & & \Big\downarrow{\scriptstyle incl^*} \\ \mathrm{Hom}(D_{16}, E) & \xrightarrow{g^*} & \mathrm{Hom}(D_{16}, E) \end{array}$$

we see that it is commutative. Namely let $f \in \mathrm{Hom}(D_{16}^+, E)$. Then

$$g^* \circ incl^*(f) = g^*(f|_{D_{16}}) = f|_{D_{16}} \circ g = f \circ g|_{D_{16}} = incl^* \circ g^*(f).$$

Applying homology to (5.5), we get the commutative diagram

$$\begin{array}{ccc} H_2(\mathrm{Hom}(D_{16}^+, E)) & \xrightarrow{g^*} & H_2(\mathrm{Hom}(D_{16}^+, E)) \\ \cong\Big\downarrow{\scriptstyle incl^*} & & \cong\Big\downarrow{\scriptstyle incl^*} \\ H_2(\mathrm{Hom}(D_{16}, E)) & \xrightarrow{g^*} & H_2(\mathrm{Hom}(D_{16}, E)). \end{array}$$

Consequently $H_2(\mathrm{Hom}(D_{16}^+, E))^{\mathcal{W}(D_{16}^+)} \cong H_2(\mathrm{Hom}(D_{16}, E))^{\mathcal{W}(D_{16})}$. But the latter space satisfies the conditions of Theorem 5.24, hence is a weighted projective space. Again, by taking the injective homomorphism

$$H_i(V) \cong H_i(\mathrm{Hom}(D_{16}, E))^{O_E(D_{16}^+)} \to H_i(\mathrm{Hom}(D_{16}, E))^{\mathcal{W}(D_{16}^+)},$$

we see that $\dim H_1(V) = 0$ and $\dim H_2(V) \leq 1$.
$\underline{\dim H_2(V) = 1:}$
By the above, we only need to show that there is one element of the desired form in $H_2(\mathrm{Hom}(L, E))$, that is invariant under the group action of $O_E(L)$. First, fix a basis e_i of L. Under this identification we get that

$$\mathrm{Hom}(L, E) \cong E^{16}.$$

Let $E_0 = E \times 0^{15}$. Then

$$\overline{\alpha} = \sum_{g \in O(L)} [g(E_0)]$$

is clearly an element in $H_2(\mathrm{Hom}(L,E))$ that is invariant under the orthogonal group. But as it consists of a sum of fundamental classes, it is also invariant under complex (hence orientation preserving) automorphisms of E. Therefore $\overline{\alpha}$ is invariant under the group action $O_E(L)$. By Corollary 5.29, we get that $\pi_*\overline{\alpha} \neq 0$ if and only if $\overline{\alpha} \neq 0$. But as we will see later in the proof of 8.15, the intersection of $\overline{\alpha}$ with a certain divisor is non-zero, hence $\overline{\alpha}$ is non-zero. Consequently we get that

$$\alpha = p_*\overline{\alpha}$$

$\mathbb{Q}$-linearly spans the whole space $H_2(V,\mathbb{Q})$. □

Remark 5.32. Regarding the proof, it also follows that

$$H_1(V) = \{0\} \qquad \text{and} \qquad H_2(V) \cong \mathbb{Q}$$

for L as in the Lemma and

$$V = O(L)\backslash \mathrm{Hom}(L,E).$$

5.6. Topology near the Boundary

In this section we want to show that every fundamental class of degenerations of Type II in $\mathcal{M}^{\mathrm{Tor}}$ splits, i.e.

$$\alpha_C = (\pi_0)_*\alpha_0 + \sum_{1\leq i\leq n} (\pi_i)_*\alpha_i$$

where $\alpha_0 \in H_2(\mathcal{M},\mathbb{Q})$ and $\alpha_i \in H_2(Z_i,\mathbb{Q})$ where $Z_1,\dots,Z_n$ are fibers of the boundary of $\mathcal{M}^{\mathrm{Tor}} \to \mathcal{M}^{\mathrm{bb}}$ and π_i are the corresponding inclusions.

Firstly, we recall two lemmas proven by Looijenga in [L+03].

Lemma 5.33. *Let $J \subset \Lambda$ be an isotropic subspace. Then with the notation from Section 5.2, there is a neighborhood $U \subset \mathcal{D}\cup\pi_J \subset \mathcal{D}^*$ of π_J such that the image of U in $\Gamma_J\backslash\mathcal{D}_J^{Tor}$ maps isomorphically to a neighborhood of the boundary $\mathbb{H}/\mathrm{SL}(2,\mathbb{Z}) = \Gamma_J\backslash\pi_J \subset \mathcal{M}^{bb}$ in the Baily-Borel compactification.*

The following proposition was proven in [Gre18] for the boundary component $E_8(-1)^2$. We closely follow the proof with minor changes, that also allow the D_{16}^+ case.

Proposition 5.34. *Let $C \to \mathcal{M}^{Tor}$ be a continuous map from a topological space C to the toroidal compactification that meets the boundary only in the finitely many points in the type II components. Then for an arbitrary $\alpha \in H_2(C)$ we get that the pushforward $\alpha_C \in H_2(\mathcal{M}^{Tor},\mathbb{Q})$ decomposes as*

$$\alpha_C = (\pi_0)_*\alpha_0 + \sum_{1\leq i\leq n} (\pi_i)_*\alpha_i$$

where $\alpha_0 \in H_2(\Gamma\backslash\mathcal{D}, \mathbb{Q})$ and $\alpha_i \in H_2(Z_i, \mathbb{Q})$ where $Z_1, \dots, Z_n$ are fibers from the boundary of $\mathcal{M}^{Tor} \to \mathcal{M}^{bb}$ and π_i are the corresponding inclusions.

Remark 5.35. The main tool of the proof is the Euler class Looijenga constructed. But this class only is defined for $\Gamma^J\backslash\mathcal{D}$. Therefore we have to compare the homology of this space with the quotient space by the full group Γ_J.

Proof. We want to use the Mayer-Vietoris sequence, with $V = \mathcal{M} \subset \mathcal{M}^{\mathrm{Tor}}$ and U a suitable neighborhood of the points meeting the boundary.
Construction of U:
Denote by p_i, q_j the points of $C \subset \mathcal{M}^{\mathrm{Tor}}$ meeting the $L = E_8(-1)^2$, respectively the $L = D_{16}^+(-1)$ boundary component $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$. Let

$$\Gamma^J\backslash\mathcal{D} \to \Gamma^J\backslash\partial\mathcal{D}_J^{\mathrm{Tor}}$$

be the Δ^*-bundle corresponding to an isotropic lattice J with $J^\perp/J = L$.[8] Denote

$$p_{bb} : \Gamma^J\backslash\mathcal{D}^{\mathrm{Tor,J}} \to \pi_J = \mathbb{H}$$

the map that is the projection to the boundary component from the description in equation (5.13.1), i.e. induced by

$$\mathbb{C} \times J^\perp/J \times \mathbb{H} \to \mathbb{H}.$$

For every i, j choose a representative of p_i, q_j in $\mathbb{H}$, which by abuse of notation we denote by the same symbol. Let $U_i \subset \pi_J$ be a small neighborhood of $p_{bb}(p_i) \subset \pi_J$, such that all such U_i's are disjoint and contractible. Then

$$\tilde{U} = p_{bb}^{-1}(\bigcup_i U_i)$$

contracts to a disjoint union of Δ-bundles B_i over $Z_i = J^\perp/J \otimes E_{\tau_i} = p_{bb}^{-1}(\tau_i) \cap \partial\mathcal{D}^{\mathrm{Tor,J}}$ for $\tau_i = p_{bb}(p_i)$ by Theorem 5.20. Moreover by Lemma 5.33, we can arrange that the image V_i of this bundle in $\Gamma_J\backslash\mathcal{D}^{\mathrm{Tor,J}}$ - after a possible modification in each fibre[9]- maps injectively into $\mathcal{M}^{\mathrm{Tor}}$. Doing the same for the other component, and possibly after another modification, we can assume, that $\tilde{U}$ retracts to a disjoint union of such bundles for every point p_i and q_j as above. Let U be the image of $\tilde{U}$ in $\mathcal{M}^{\mathrm{Tor}}$. By the lemma, the image of $\tilde{U}$ in $\Gamma_J\backslash\mathcal{D}$ maps isomorphically to U.
Mayer-Vietoris sequence:
The Mayer-Vietoris sequence reads

$$H_2(U, \mathbb{Q}) \oplus H_2(V, \mathbb{Q}) \to H_2(U \cup V, \mathbb{Q}) \to H_1(U \cap V, \mathbb{Q}).$$

As C maps into $U \cup V$, it suffices to show that $H_1(U \cap V, \mathbb{Q}) = 0$, because then the homology class splits even as an element of $H_2(U \cup V, \mathbb{Q})$ and pushing forward gives the

[8]Observe that we work with the intermediate group Γ^J and not Γ_J.
[9]For example by replacing Δ with $\{c \,|\, |c| < \epsilon\}$ for suitable ϵ's in each fibre.

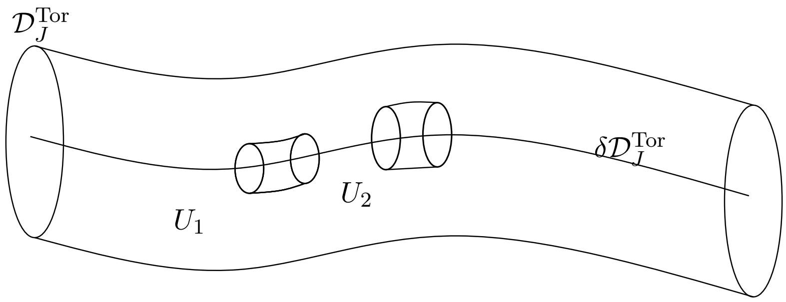

Figure 5: Neighborhoods U_i constructed above for every τ_i.

desired element. But $U \cap V$ is just U without the boundary. To conclude, we only need to show $H_1(V_i^\circ, \mathbb{Q}) = 0$ for all i, where V_i° is V_i without the boundary.
By the construction, one sees that V_i° is just the image of B_i°, which by definition is B_i without the zero section. But the Euler class $e_i \in H^2(Z_i, \mathbb{Z})$ of the Δ^*-bundles is non-zero by the Theorem 5.20. Hence the Gysin-sequence is exact and reads

$$H_2(B_i^\circ, \mathbb{Q}) \to H_2(Z_i, \mathbb{Q}) \xrightarrow{\cap e_i} H_0(Z_i, \mathbb{Q}) \to H_1(B_i^\circ, \mathbb{Q}) \to H_1(Z_i, \mathbb{Q}).$$

Mapping to the image in $\Gamma_J \backslash \mathcal{D}^{\mathrm{Tor,J}}$, we get the commutative diagram

$$\begin{array}{ccccccc}
H_2(Z_i, \mathbb{Q}) & \longrightarrow & H_0(Z_i, \mathbb{Q}) & \longrightarrow & H_1(B_i^\circ, \mathbb{Q}) & \longrightarrow & H_1(Z_i, \mathbb{Q}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_2(Y_i, \mathbb{Q}) & \longrightarrow & H_0(Y_i, \mathbb{Q}) & \longrightarrow & H_1(V_i^\circ, \mathbb{Q}) & \longrightarrow & H_1(Y_i, \mathbb{Q})
\end{array}$$

where $Y_i \cong O_E(J^\perp/J) \backslash (J^\perp/J \otimes E_\tau)$ is the image in $\mathcal{M}^{\mathrm{Tor}}$. The first box commutes, as $Z_i \to Y_i$ is just the map

$$O(J^\perp/J) \backslash (J^\perp/J \otimes E_\tau) \to O_E(J^\perp/J) \backslash (J^\perp/J \otimes E_\tau)$$

and hence, is the quotient by a finite group. But the Euler class $e_i \in H^2(Z_i, \mathbb{Q})$ is invariant under the group action by the theorem and hence pulls back from a non-zero element $e_i' \in H_2(Y_i, \mathbb{Q})$. Then the push-pull formula shows commutativity. But by Lemma 5.31 we have

$$\begin{aligned}
H_0(Y_i, \mathbb{Q}) &= \mathbb{Q}, \\
H_1(Y_i, \mathbb{Q}) &= 0, \\
H_2(Y_i, \mathbb{Q}) &= \mathbb{Q}, \\
H_0(Z_i, \mathbb{Q}) &= \mathbb{Q}, \\
H_1(Z_i, \mathbb{Q}) &= 0, \\
H_2(Z_i, \mathbb{Q}) &= \mathbb{Q},
\end{aligned}$$

and the vertical maps between the homology of the spaces induced by $Z_i \to Y_i$ is an isomorphism by Corollary 5.29.

But by the universal coefficient theorem $H^2(Y_i, \mathbb{Q}) \cong \mathrm{Hom}(H_2(Y_i, \mathbb{Q}), \mathbb{Q})$. As $e'_i \neq 0$, we see that $H_2(Y_i, \mathbb{Q}) \xrightarrow{\cap e'_i} H_2(Y_i, \mathbb{Q})$ is an isomorphism and the same holds for Z_i.

To conclude, we have the following diagram

$$\begin{tikzcd}
\mathbb{Q} \arrow[r, "\cong"] \arrow[d, "\cong"] & \mathbb{Q} \arrow[r] \arrow[d, "\cong"] & H_1(B_i^\circ, \mathbb{Q}) \arrow[r] \arrow[d] & 0 \arrow[d] \\
\mathbb{Q} \arrow[r, "\cong"] & \mathbb{Q} \arrow[r] & H_1(V_i^\circ, \mathbb{Q}) \arrow[r] & 0
\end{tikzcd}$$

with the upper row being exact. Therefore $H_1(B_i^\circ, \mathbb{Q}) = 0$. On the other hand, $V_i^\circ = G \backslash B_i^\circ$ with G a finite group: it is just the action of $\begin{pmatrix} T^{-1T} & 0 & 0. \\ 0 & I & 0 \\ 0 & 0 & T \end{pmatrix}$ with T preserving τ, i.e. it is finite. Therefore every element $y \in H_1(V_i^\circ, \mathbb{Q})$ has a preimage $x \in H_1(B_i^\circ, \mathbb{Q})$ by Corollary 5.29. Hence $y = 0$ by a diagram chase. $\square$

6. Extension of the Period Map for Degenerations

In this section we will study, in which way the period map

$$C \dashrightarrow \mathcal{M}^{\mathrm{Tor}}$$

extends, where C is a degeneration of K3 surfaces.
In the last section, we saw that $\mathcal{M}^{\mathrm{Tor}}$ is proper over $\mathbb{C}$, as it is compact in the analytic setting. Therefore we get the following standard extension theorem:

Lemma 6.1. *Let $X \to C$ be a U-quasi-polarized degeneration with C a smooth curve. Then the period map extends to an algebraic map*

$$C \to \mathcal{M}^{Tor}.$$

Proof. Let $p \in C$ be one of the finitely many points in C where the period map is not defined. As C is smooth, $\mathcal{O}_{C,p}$ is a discrete valuation ring. Let η be the generic point of C. Hence, we have a map

$$\mathrm{Spec}(\mathcal{O}_{C,\eta}) \to \mathcal{M}^{\mathrm{Tor}}$$

and

$$\mathrm{Spec}(\mathcal{O}_{C,\eta}) \to \mathrm{Spec}(\mathcal{O}_{C,p}).$$

Then

$$\begin{tikzcd} \mathrm{Spec}(\mathcal{O}_{C,\eta}) \arrow[r] \arrow[d] & \mathcal{M}^{\mathrm{Tor}} \arrow[d] \\ \mathrm{Spec}(\mathcal{O}_{C,p}) \arrow[r] & \mathrm{Spec}(\mathbb{C}) \end{tikzcd}$$

commutes. By the properness of $\mathcal{M}^{\mathrm{Tor}} \to \mathrm{Spec}(\mathbb{C})$, we get a map $\mathrm{Spec}(\mathcal{O}_{C,p}) \to \mathcal{M}^{\mathrm{Tor}}$, that commutes with the diagram. Via this map we can extend

$$C \to \mathcal{M}^{\mathrm{Tor}}.$$

□

By means of this extension, we also get an extension

$$C \to \mathcal{M}^{\mathrm{bb}}.$$

By a theorem in [CEZG+14], the boundary components of $\mathcal{M}^{\mathrm{bb}}$ each parametrize the pure Hodge structures on $\mathrm{Gr}_1(H^2_\infty)$. In the case of short type II degenerations $X_t \rightsquigarrow V_1 \cup_E V_2$ we have seen that

$$\mathrm{Gr}_1(H^2_\infty) \cong H^1(E, \mathbb{Q})$$

even as Hodge structures. But the Hodge structure in this case is determined by τ for $E = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$. The extension is then simply given by $\tau \in \mathrm{SL}(2, \mathbb{Z})\backslash\mathbb{H}$, see [Gre18].

Extension for Type II degenerations

Let $X \to C$ be a Kulikov degeneration of Type II with short degenerated fibers. I.e.

$$X_0 = V_1 \cup_E V_2.$$

Then $E = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$. Let $N = \log T$ be the logarithm of the monodromy as in Section 4.2. Then $N^2 = 0$, as it is of type two. Let $J = \operatorname{Im} N$. This subspace is clearly generated rationally. As N is skew-symmetric with respect to the cup product, we get that J is even isotropic, as

$$Nx \cap Ny = -x \cap N^2 y = 0$$

for any $x, y \in H^2(X_t, \mathbb{C})$. Moreover

$$\operatorname{Ker} N = (\operatorname{Im} N)^{\perp}$$

as $0 = Nx \cap y = -x \cap Ny$. As we saw in Section 4.2, $H^2_{\infty} \cong H^2(X_t, \mathbb{C})$ for some $t \in C$ nearby the fiber. Denote by $Z \subset \operatorname{Pic}(X)$ the U-polarization. We polarize

$$\mathrm{Gr}_2 H^2_{\infty} = \operatorname{Ker} N / \operatorname{Im} N$$

by $Z|_{X_t}$. This is possible, as by the invariant cycle theorem $U \cong Z|_{X_t} \subset \operatorname{Ker} N$. On the other hand, $U \cap \mathrm{Im} N = \{0\}$: Assume the contrary, i.e. there is a $0 \neq j \in U \cap \mathrm{Im} N$. As $\mathrm{Im} N$ is isotropic, we have that $j \cdot j = 0$. But $U \subset \operatorname{Ker} N = (\operatorname{Im} N)^{\perp}$. Therefore $U \perp j$. But this cannot happen, as U has signature $(1, 1)$. Hence,

$$\mathrm{Gr}_2^{\mathrm{pol}} H^2_{\infty} = (\operatorname{Ker} N \cap U)/\operatorname{Im} N$$

is well-defined.

By Appendix A, we see that $U^{\perp}$ is an unimodular even lattice of signature $(2, 18)$, i.e. it is isomorphic to $\Lambda_{2,18}$. Hence, we get an isotropic plane $J = \operatorname{Im} N \subset \Lambda_{2,18}$, such that

$$\mathrm{Gr}_2^{\mathrm{pol}} H^2_{\infty} = J^{\perp}/J$$

is even, unimodular and of signature $(0, 16)$. Hence by Appendix A it is isomorphic to one of the following lattices:

$$E_8(-1)^2 \quad \text{or} \quad D_{16}^{+}(-1).$$

By a theorem of Deligne, as stated in [Gre18], we have for our case:

Theorem 6.2 ([Gre18]). *There is a group-isomorphism*

$$\mathrm{Ext}_{MHS}(\mathrm{Gr}_2 H^2_{\infty}, \mathrm{Gr}_1 H^2_{\infty}) \to \mathrm{Hom}(\mathrm{Gr}_2 H^2_{\infty}, E) = J^{\perp}/J \otimes E.$$

Recalling the construction of the isomorphism

$$\mathcal{E}^{\perp}/\mathcal{E}\mathbb{Z} \cong \mathrm{Gr}_2 H^2_{\infty}$$

from Section 4.2, we see that it is constructed by taking an element

$$(a,b) \in \mathcal{E}^{\perp} \subset H^2(V_1,\mathbb{Z}) \oplus H^2(V_2,\mathbb{Z}),$$

pulling it back to $H^2(V_1 \cup_E V_2, \mathbb{Z})$ by Mayer-Vietoris and then applying the Clemens map. But as $Z|_{X_t}$ extends to X, we get that the $Z|_{X_t}$ corresponds to $Z|_{X_0}$ under the above isomorphism by the invariant cycle theorem.

Let $(a,b) \in \mathcal{E}^{\perp} \subset H^2(V_1,\mathbb{Z}) \oplus H^2(V_2,\mathbb{Z})$. Then

$$a|_E - b|_E \in \mathrm{Pic}_0(E) = J(E) \cong E,$$

as by definition $0 = \mathcal{E} \cdot (a,b) = \deg a - \deg b$. Moreover this map factors through $\mathcal{E}\mathbb{Z}$, as

$$(\mathcal{O}_{V_1}(E), -\mathcal{O}_{V_2}(E)) \mapsto (N_{E/V_1} \otimes N_{E/V_1}) = 0$$

by d-semistability. Hence we get:

Theorem 6.3 ([Fri84])**.** *There is a morphism*

$$\mathrm{Gr}_2 H^2_{\infty} \to J(E) \cong E$$

where $J(E)$ is the Jacobian of E. It is given by

$$l = (l_1, l_2) \mapsto l_1|_E - l_2|_E \in \mathrm{Pic}_0(E). \tag{6.3.1}$$

Recall that for a degeneration as above, we get that

$$\mathcal{M}^{\mathrm{Tor}} \to \mathcal{M}^{\mathrm{bb}}$$

has the fiber

$$Z_\tau = \mathcal{O}_E(J^{\perp}/J)\backslash \mathrm{Hom}(J^{\perp}/J, E_\tau)$$

over the type II boundary component corresponding to J. As Friedman proved, the extension of the period map in our Type II case for a degeneration

$$X_t \rightsquigarrow V_1 \cup_{\mathcal{E}_\tau} V_2$$

is then given by the point in Z_τ corresponding to the map in (6.3.1).

Geometric interpretation of extension of the period map for non-Kulikov models

As previously stated, any local degeneration $X \to \Delta$ can be transformed to a Kulikov model by successively taking an n-fold cover and then taking an birational morphism, that just alters the central fiber. But any extension of

$$\Delta^* \to \mathcal{M}^{\mathrm{Tor}}$$

of the n-fold covering determines the extension of the original family, as this is just a topological property. Hence, for any degneration $X \to C$, the degenerated fibers get mapped to the point corresponding to the constructed Kulikov model.

Remark 6.4. By taking an n-fold cover of Δ, the logarithm of the monodromy operator changes only by a scalar multiple, i.e. $N' = nN$, as $T' = T^n$. Therefore one can directly spot which Type the corresponding Kulikov model will be, just by taking the monodromy of the original family. I.e. the non-Kulikov example in Section 4.3 maps to the type II boundary as well.

Remark 6.5. Let $f : X \to C$ be a semistable degeneration. As was stated in Section 4.1 there is a birational morphism $X' \to X$ that is an isomorphism outside the degenerated fibers. Therefore the period map does not change if we replace $X \to C$ by $X' \to C$.

Period map for K3 surfaces with ADE singularities

Let $X \to \Delta$ be a degeneration of K3 surfaces, such that the central fiber is an irreducible surface that has only ADE singularities. Then, by [AHVAV17], there is an $n \in \mathbb{N}$, such that the n-fold cover

$$p : t \to t^n$$

the resulting family $\tilde{X} = X \times_p \Delta$ admits a simultaneous resolution, i.e. there is a birational morphism $Y \to \tilde{X}$, such that $Y_t \to \tilde{X}_t$ is the minimal resolution for all t. But by Lemma 4.20 the resulting canonical sheaf of Y is trivial outside the central fiber. But as the central fiber is irreducible, and $\mathcal{O}_Y(Y_0) = \mathcal{O}_Y$, we get that Y is a Kulikov model. Hence by the classification Y_0 is a K3 surface. By the last paragraph, the period point of an extension

$$\Delta \to \mathcal{M}^{\mathrm{Tor}}$$

is given by the one of Y_0, i.e. the minimal resolution of X_0.

7. Modular Forms and Quasi-Modular Forms

7.1. Modular Forms

In this section we give a brief overview of modular forms by following [Zag08].

Let $\mathbb{H} = \{z \in \mathbb{C} \,|\, \Im z > 0\}$ be the upper half plane. Recall the classical group action of $\mathrm{SL}(2, \mathbb{Z})$ on $\mathbb{H}$ which is given by sending $h \in \mathbb{H}$ via $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to

$$\gamma \cdot h = \frac{ah + b}{ch + d}.$$

Definition 7.1. A holomorphic map $f : \mathbb{H} \to \mathbb{C}$ is called a *modular function of weight* k, if

$$f(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot h) = (ch + d)^k f(h).$$

Remark 7.2. Analysing this behaviour for $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, we see that

$$f(z) = f(z + 1).$$

Considering the covering map

$$\mathbb{H} \xrightarrow{z \to e^{2\pi i z} = q} \Delta^* = \{q \in \mathbb{C} \,|\, 0 < q < 1\}$$

we see that f factors as a holomorphic map

$$f : \mathbb{H} \xrightarrow{z \to q} \Delta^* \to \mathbb{C}.$$

Therefore the map has the following form

$$f = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i z n} = \sum_{n=-\infty}^{\infty} a_n q^n.$$

Definition 7.3. In the above setting, a modular function that satisfies $a_n = 0$ for all $n < 0$ is called *modular form.*

7.2. Examples

Of big importance for our constructed modular forms is the following theorem, taken from [Zag08]:

Theorem 7.4. *The space M_k of weight k modular forms is a finite dimensional vectorspace. Moreover the dimensions satisfy*

$$\dim M_8 = \dim M_{10} = 1.$$

Therefore we only need to construct one modular form in the whole space.

Example 7.5. For $k \geq 2$ the *Eisenstein*-modular forms are defined in the following way:

$$E_k(z) = \frac{1}{2} \sum_{\substack{c,d \in \mathbb{Z} \\ \gcd(c,d)=1}} \frac{1}{(cz+d)^k}.$$

As is shown [Zag08] they satisfy

$$\begin{aligned} E_8 &= 1 + 480q + 61920q^2 + 1050240q^3 + \dots \\ E_{10} &= 1 - 264q - 135432q^2 - 5196576q^3 - \dots. \end{aligned}$$

Moreover, one can show (see e.g. [Zag08]) that for $n > 2$

$$E_n = 1 + a_1(q + \sum_{i=2}^{\infty} c_n)$$

with $c_n \in \mathbb{Z}$.

Next, we will see another way of constructing modular forms, namely from lattices.

Definition 7.6. Let L be an even positive definite lattice. The *Theta series* of the lattice is defined by

$$\Theta = \sum_{x \in L} q^{\frac{1}{2}x^2} = \sum_{n>0} R_n q^{\frac{n}{2}},$$

where $R_n = \sharp\{x \in L,\, x^2 = n\}$.

Of special importance for elliptic K3s are the lattices D_{16}^+ and $E_8 \oplus E_8$, for which the following theorem holds:

Theorem 7.7 ([Zag08])**.** *Let L be a unimodular even lattice of dimension $2m$. Then its theta function is a modular form of weight m.*

Example 7.8. As is shown in Appendix A, both lattices D_{16}^+ and $E_8 \oplus E_8$ are even, unimodular and positive definite. Therefore there is only one element whose self intersection is zero. This yields

$$\Theta_{D_{16}^+} = \Theta_{E_8 \oplus E_8} = E_8,$$

as $\dim M_8 = 1$.

7.3. Quasi-Modular Forms

Definition 7.9. A function $F : \mathbb{H} \to \mathbb{C}$ is called an *almost holomorphic modular form*, if it can be expressed as

$$F(z) = \sum_{r=0}^{p} f_r(z) \cdot (-4\pi y)^r$$

where f_r is a holomorphic function and $y = \Im z$, such that it transforms appropriately, i.e.

$$F(\gamma \cdot z) = (cz + d)^k F(z)$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$. A holomorphic map $f : \mathbb{H} \to \mathbb{C}$ is called *quasi-modular form of weight* k, if there is an almost holomorphic modular form F of weight k as above such that $f = f_0$.

Proposition 7.10. *The differential* $Df(z) = \frac{1}{2\pi i} \frac{\partial}{\partial z} f(z) = \sum_{i=0}^{\infty} n a_n q^n$ *of a holomorphic modular form* $f = \sum a_n q^n$ *of weight* k *is a quasi-modular form of weight* $k + 2$.

Proof. See [BvdGHZ08]. □

Example 7.11. By the foregoing proposition DE_8 is a quasi-modular form and by the discussion of the last section

$$DE_8(q) = 480(q + \sum_{n \geq 2} c_n q^n)$$

with $c_n \in \mathbb{Z}$. In particular $DE_8/480$ is integral, has no constant term and the factor of q^1 is 1.

8. Quasi-Modular Forms from Degenerations

8.1. Heegner Divisors and Noether-Lefschetz Divisors

In this section we introduce divisors that represent elements in the Picard group of K3 surfaces. We follow the description of [MP07].

Let

$$\mathcal{D} = \quad \mathbb{P}\{x \in \Lambda_{2,18} \otimes \mathbb{C} | \, \langle x, x \rangle = 0, \langle x, \bar{x} \rangle > 0\}.$$

and $\Gamma = O(\Lambda_{2,18})$. Consequently $\Gamma \backslash \mathcal{D}$ is the moduli space for U-quasi-polarized, hence elliptic, K3 surfaces from Section 3. For a fixed $n \in \mathbb{Z}$, let $I_n = \{v \in \Lambda_{2,18} \,|\, v^2 = 2n\}$. This set is clearly preserved by Γ. As $\Lambda_{2,18}$ is unimodular, the description of the Heegner divisors from [MP07] simplifies to

Definition 8.1. Let $n \neq 0$, then define the *Noether-Lefschetz* or *Heegner* divisor as

$$\mathrm{NL}_n = \sum_{v \in I_n} v^{\perp}.$$

Clearly the sum is Γ-invariant and by Appendix A their is only one Γ-orbit of v's such that $v^2 = 2n$. Therefore, by [BvdGHZ08] the divisor above descends to an algebraic divisor $NL_d \in \mathrm{Pic}(\Gamma \backslash \mathcal{D}) = \mathrm{Pic}(\mathcal{M})$.

Remark 8.2. A point $p \in \mathcal{M}$ is contained in the Noether-Lefschetz divisor if and only if the corresponding K3 surface X has a Cartier divisor $v \in \mathrm{Pic}(X)$, such that $v^2 = 2n$ and v is orthogonal to the polarization.

Moreover the canonical line bundle $\mathcal{O}(-1) = \{([x], y) \in \mathcal{D} \times \Lambda_{2,18} \otimes \mathbb{C} \,|\, [y] = [x]\}$ admits an obvious action of Γ. This action is equivariant with respect to the projection $\mathcal{O}(-1) \to \mathcal{D}$. Hence it also descends to a line bundle ν on

$$\mathcal{M} = \Gamma \backslash \mathcal{D}.$$

Definition 8.3. Define the *Hodge line bundle* as the inverse

$$\lambda = \nu^*.$$

In our case of $\mathcal{M}^{\mathrm{Tor}}$, we have that $\mathcal{M} \subset \mathcal{M}^{\mathrm{Tor}}$ as a dense open subset and the complement is of codimension 1. Hence the following definition constructs another divisor

Definition 8.4. The closure $\overline{\mathrm{NL}}_n$ of NL_n in $\mathcal{M}^{\mathrm{Tor}}$ is called the *completed Noether Lefschetz divisor.*

Denote by $\partial_{E_8(-1)^2}\mathcal{M}^{\mathrm{Tor}}$ and $\partial_{D_{16}^+(-1)}\mathcal{M}^{\mathrm{Tor}}$ the corresponding boundary components.

Remark 8.5. Recall that in Remark 5.22, we proved that

$$\overline{\mathrm{NL}}_n \cap Z_\tau = \bigcup_{v^2=-n} v^\perp \subset \mathrm{Hom}(J^\perp/J, E_\tau)/O_E(J^\perp/J) = Z_\tau.$$

Moreover in Section 5, we saw that ν - and hence λ - extend naturally to a line bundle on $\mathcal{M}^{\mathrm{bb}}$.

Chern Classes of Heegner Divisors

In the following, we want to regard these divisors as cohomology classes. This is done as follows (for a more detailed description, see [Bru04]): Let $\mathcal{M}' = \Gamma'\backslash\mathcal{D}$ be the quotient of a normal neat subgroup $\Gamma' \subset \Gamma$ of finite index. As we have seen in Remark 5.15 this space is smooth and we have a group action $G \curvearrowright \mathcal{M}'$ of a finite group $G \cong \Gamma/\Gamma'$ such that the orbit space is $\mathcal{M}$. Denote $p : \mathcal{M}' \to \mathcal{M}$ the projection.
As $\mathcal{M}'$ is smooth, we get the Chern class of $NL'_d = p^*NL_d \subset \Gamma'\backslash\mathcal{D}$:

$$c_1(NL'_d) \in H^2(\Gamma'\backslash\mathcal{D}, \mathbb{Z}).$$

But by construction this is invariant under the G action, as NL_d is Γ invariant. By Theorem 5.29, this class is the pull back of an unique element

$$c(NL_d) \in H^2(\mathcal{M}, \mathbb{Q}).$$

The same construction works for the Noether-Lefschetz divisors in $\mathcal{M}^{\mathrm{Tor}}$ by replacing $\mathcal{M}$ with $\mathcal{M}^{\mathrm{Tor}}$ in the above. Denote by $p : \mathcal{M}^{\mathrm{Tor}'} \to \mathcal{M}^{\mathrm{Tor}}$ the corresponding projection map.
Again, as $\mathcal{M}^{\mathrm{Tor}}$ is not smooth, we have to show that the intersection product in $\mathcal{M}^{\mathrm{Tor}}$ is the same as the product in (co)homology. Let $\alpha \in A^1(\mathcal{M}^{\mathrm{Tor}})$. By [Ful13], $\alpha = p_*\alpha'$ with $\alpha' = \frac{1}{|G|}p^*\alpha \in A^1(\mathcal{M}^{\mathrm{Tor}'})$ in the smooth space. Then for the cycle map $cl : A^1(X) \to H_2(X)$ as in [Ful13]:

$$\begin{aligned}
\overline{NL} \cap \alpha &= p_*(p^*\overline{NL} \cap \alpha') \\
&= p_*(cl(p^*\overline{NL}) \cap cl(\alpha')) \\
&= p_*(c_1(p^*\overline{NL}) \cap cl(\alpha')) \\
&= p_*(p^*c(\overline{NL}) \cap cl(\alpha')) \qquad \text{by definition as } p^*\overline{NL} = \overline{NL}' \\
&= c(\overline{NL}) \cap p_*cl(\alpha') \\
&= c(\overline{NL}) \cap cl(\frac{1}{|G|}p_*p^*\alpha) \\
&= c(\overline{NL}) \cap |G|\frac{1}{|G|}cl(\alpha) \\
&= c(\overline{NL}) \cap cl(\alpha).
\end{aligned}$$

As $\mathcal{M}^{\mathrm{Tor}}$ is compact $cl(\alpha)$ is just the fundamental class of α. Therefore computing the intersection in $\mathcal{M}^{\mathrm{Tor}}$ in (co)homology is the same as the algebraic intersection product. By abuse of notation we denote $\overline{NL}_d$ also for the cohomology class $c(\overline{NL}_d)$ from above.

Intersection with a Degeneration

Now, we can define the intersection of a family and a degeneration with the Noether-Lefschetz divisors: Let $f : X \to C$ be a degeneration. Then the period map extends to a map

$$C \to \mathcal{M}^{\mathrm{Tor}}.$$

Because of properness the image is a closed curve $\tilde{C} \subset \mathcal{M}^{\mathrm{Tor}}$. Thus, we can define the intersection as

$$C \cdot \overline{NL}_d := \tilde{C} \cdot \overline{NL}_d,$$

which is just the usual algebraic intersection product in $\mathcal{M}^{\mathrm{Tor}}$. By the above, we have that

$$C \cdot \overline{NL}_d = [C] \cap c(\overline{NL}_d),$$

where $[C] \in H_2(\mathcal{M}^{\mathrm{Tor}}, \mathbb{Z})$ is the push forward of the fundamental class of C.

8.2. Borcherds Results - Modularity for Type I Degenerations

Here, we will present the theorem of Borcherds, which relates the different intersection products with Noether-Lefschetz divisors. This is the main building block of the results, which are presented in the subsequent sections.
We need the following definition of a generating series:

Definition 8.6. Let

$$\Phi(q) = \lambda \cdot q^0 + \sum_{n \in \mathbb{Z},\, n>0} \mathrm{NL}_{-n} q^n \in \mathrm{Pic}(\mathcal{M})[[q]]$$

be a formal power series.

Remark 8.7. Here we heavily use that $\Lambda_{2,18}$ is unimodular, which simplifies all constructions in [MP07]. In the general case, the generating series is an element of $\mathrm{Pic}(\mathcal{M})[[q]] \otimes \mathbb{C}[\Lambda^* \backslash \Lambda]$.

Now, we come to the main result:

Theorem 8.8: Borcherds, MacGraw [MP07]

The generating function $\Phi(q)$ is an element of

$$\mathrm{Pic}(\mathcal{M}) \otimes_{\mathbb{Z}} M_{10},$$

where M_{10} denotes the weight 10 modular forms.

Therefore as an corollary, we get:

Corollary 8.9 (Modularity for Type I degenerations)**.** *Let $\alpha \in H_2(\mathcal{M})$. Then*

$$\alpha \cap \lambda + \sum_{n\in\mathbb{Z},\, n>0} \alpha \cap NL_{-n} q^n$$

is a modular form of weight 10*. In particular, for every Type I degeneration over a curve C, we get that*

$$C \cdot \lambda + \sum_{n\in\mathbb{Z},\, n>0} C \cdot NL_{-n} q^n$$

is a modular form of weight 10*.*

8.3. Main Theorem - Quasi-Modularity for Type II Degenerations

In this section, we want to prove the following theorem: Denote by $\bar{\lambda}$ the pull back of the Hodge line bundle of $\mathcal{M}^{\mathrm{bb}}$ to $\mathcal{M}^{\mathrm{Tor}}$. Then:

Theorem 8.10: Main theorem
Quasi-Modularity for Degenerations of Type II

Let

$$\Phi(q) = \bar{\lambda} \cdot q^0 + \sum_{n\in\mathbb{Z},\, n>0} \overline{\mathrm{NL}}_{-n} q^n \in \mathrm{Pic}(\mathcal{M}^{\mathrm{Tor}})[[q]]$$

and $X \to C$ be a degeneration such that the period map extends to the Type II boundary component. Denote by α the fundamental class of C in the toroidal compactification $\mathcal{M}^{\mathrm{Tor}}$. Then

$$\alpha \cap \Phi(q) = \alpha \cap \bar{\lambda} \cdot q^0 + \sum_{n\in\mathbb{Z},\, n>0} \alpha \cap \overline{\mathrm{NL}}_{-n} q^n \tag{8.10.1}$$

is a quasi-modular form of weight 10 and is an element of

$$\mathbb{Z}E_{10} \oplus \frac{1}{480}\mathbb{Z}DE_8.$$

Remark 8.11. The theorem has been proven for the case that the degeneration only meets the $E_8(-1)^2$ boundary by François Greer in [Gre18]. In the following we will mimic his prove for the general case.
The general idea is as follows: We decompose the homology class of the curve into a boundary component α and one that is supported on the interior. To the latter, Theorem 8.2 of Borcherd applies. As it will turn out, the intersection numbers $\overline{NL_d} \cdot \alpha$ are quadratic in d. The next lemma will investigate those further.

Throughout this whole subsection let $L = D_{16}^+(-1)$ or $E_8(-1)^2$ unless otherwise stated, and $\langle -, - \rangle$ the corresponding intersection pairing.

Lemma 8.12. *Let $L = E_8, E_8^2, D_{16}$ or D_{16}^+. Then every quadratic form[10] q that is invariant under the orthogonal group of the lattice, is a multiple of the pairing of the lattice.*

Remark 8.13. The proof of the lemma even generalizes to irreducible root systems with associated Dynkin diagram only having simple edges. I.e. there is a generating set of the lattice, that consists of roots and every two such roots v_0, v_n are connected by a chain of roots $v_0 \to v_1 \to \ldots \to v_n$ that have intersection $\langle v_i, v_{i+1} \rangle = 1$ and every two roots that have intersection $\neq 1$ are orthogonal.

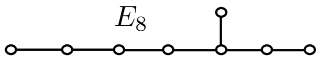

Figure 6: E_8-Dynkin diagram

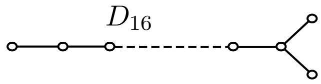

Figure 7: D_{16}-Dynkin diagram

Proof. The root lattices E_8, D_{16}:
It is important to note, that in this case, i.e. the case of root lattices, it suffices to have invariance under the Weyl group.
As is shown in Appendix A, E_8 and D_{16} satisfy the property from the remark. Fix such a generating set. For convenience, we work with the associated bilinear form $(-,-)$ that is clearly also invariant. Let v, v' be two roots of the generating set that have intersection $\langle v, v' \rangle = 1$. There are two elements in the Weyl group, i.e. the group generated by reflections at roots, that we need: the reflection at the corresponding roots $s_v, s_{v'}$. They satisfy

$$\begin{aligned} s_v(v) &= -v, \\ s_{v'}(v) &= v - v', \\ s_v(v') &= v' - v. \end{aligned}$$

Therefore by invariance

$$(v, v') = (s_v(v), s_v(v')) = -(v, v' - v) = (v, v) - (v, v').$$

[10]A quadratic form is a map $q : L \to \mathbb{Z}$ that is equal to (v, v) for a symmetric bilinear form (v, v').

Thus $(v, v') = \frac{1}{2}(v, v)$. Then

$$(v, v) = (s_{v'}(v), s_{v'}(v)) = (v - v', v - v') = (v, v) - 2(v, v') + (v', v').$$

Consequently $(v', v') = 2(v, v') = (v, v)$. On the other hand, let $\langle v, v' \rangle = 0$, then

$$\begin{aligned} s_v(v) &= -v, \\ s_v(v') &= v', \\ s_v^2 &= \mathrm{id}. \end{aligned}$$

Hence, we get

$$-(v, v') = (s_v(v), v')) = (v, s_v(v')) = (v, v').$$

It follows that $(v, v') = 0$. Therefore on the whole generating set the bilinear form (for a fixed root v_0 in the generating set) satisfies

$$2(v, v') = (v_0, v_0)\langle v, v' \rangle$$

and hence the identity holds on the whole lattice.

The lattice $E_8 \oplus E_8$:

Let

$$\begin{aligned} (-,-)_1 &= (-,-)|_{(E_8 \times 0) \times (E_8 \times 0)}, \\ (-,-)_2 &= (-,-)|_{(0 \times E_8) \times (0 \times E_8)}, \\ (-,-)_3 &= (-,-)|_{(0 \times E_8) \times (E_8 \times 0)}, \\ (-,-)_4 &= (-,-)|_{(E_8 \times 0) \times (0 \times E_8)}. \end{aligned}$$

Then

$$(-,-)_1 + (-,-)_2 + (-,-)_3 + (-,-)_4 = (-,-)$$

by bilinearity. From now on, by abuse of notation, we interpret these bilinear maps as maps $(-,-)_i : E_8 \times E_8 \to \mathbb{Z}$.

As for any $g \in O(E_8)$, $g \times \mathrm{id} \in O(E_8 \oplus E_8)$, we get that $(-,-)_1$ is also invariant under the orthogonal group of E_8. Hence, by the above, $(-,-)_1 = c_1\langle -,- \rangle_{E_8}$ and by symmetry $(-,-)_2 = c_2\langle -,- \rangle_{E_8}$. But on the other hand, $g \times \mathrm{id}$ shows, that

$$(a, b)_3 = (g(a), b)$$

and by symmetry also $(a, b)_3 = (a, g(b))$. Hence

$$(a, b)_3 = (g(a), g(b)).$$

Therefore $(a, b)_3 = c_3\langle a, b \rangle_{E_8}$. But let $s_v \in O(E_8)$ be the reflection at a root v. Then $s_v(v) = -v$. Therefore

$$(v, v)_3 = (-v, v)_3 = -(v, v)_3$$

and c_3 must be zero. By symmetry, also $c_4 = 0$. Hence

$$(-,-) = c_1(-,-)_1 + c_2(-,-)_2.$$

But as

$$\begin{aligned} T : E_8 \oplus E_8 &\to E_8 \oplus E_8 \\ a \oplus b &\mapsto b \oplus a \end{aligned}$$

is an element in $O(E_8 \oplus E_8)$, we get that

$$(v, v')_1 = (v \oplus 0, v' \oplus 0) = (0 \oplus v, 0 \oplus v') = (v, v')_2.$$

Take a root $v \in E_8$. Then by the above

$$2c_1 = c_1\langle v, v\rangle_{E_8} = (v, v)_1 = (v, v)_2 = 2c_2.$$

Thus, $c_1 = c_2$ and

$$(-,-) = (-,-)_1 + (-,-)_2 = c(\langle -,-\rangle_{E_8\times 0} + \langle -,-\rangle_{0\times E_8}) = c\langle -,-\rangle_{E_8\times E_8}.$$

The lattice D_{16}^+:
By Appendix A, $D_{16} \subset D_{16}^+ \subset \mathbb{Q}^{16}$ both $\mathbb{Q}$-linearly span the whole space $\mathbb{Q}^{16}$ and get the intersection pairing from the canonical one in $\mathbb{Q}^{16}$. On the other hand, every element $g \in \mathcal{W}(D_{16})$ extends to an automorphism of D_{16}^+, as every reflection does so. Thus,

$$(-,-)|_{D_{16}}$$

is invariant under the Weyl group action. By the proof above, we get that

$$(-,-)|_{D_{16}} = c\langle -,-\rangle_{D_{16}}. \tag{8.13.1}$$

But as $D_{16} \subset D_{16}^+$ both $\mathbb{Q}$-linearly span the same subspace in $\mathbb{Q}^{16}$, the only bilinear form extending $\langle -,-\rangle_{D_{16}}$ on D_{16}^+ is the intersection pairing of D_{16}^+ itself. Hence

$$(-,-) = c\langle -,-\rangle_{D_{16}^+}.$$

□

Lemma 8.14. *Let E be an elliptic curve, $0 \neq a \in \mathbb{Z}$. Then there are exactly a^2 elements $e \in E$, that satisfy*

$$ae = 0 \in E.$$

Proof. It is well known, that E is isomorphic to $\mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$ for some $\tau \in \mathbb{H}$. Hence $x + y\tau \in E$ satisfies

$$ax + ay\tau = 0 \in E$$

if and only if

$$ax + iay \in \mathbb{Z} \oplus \tau\mathbb{Z}.$$

But this is the case if and only if

$$\begin{aligned} ax \in \mathbb{Z} \Leftrightarrow x \in \frac{1}{a}\mathbb{Z} \qquad \text{and} \\ ay \in \tau\mathbb{Z} \Leftrightarrow y \in \frac{\tau}{a}\mathbb{Z}. \end{aligned}$$

As every $x + \tau y$ has precisely one representative with $0 \leq x, y < 1$, all solutions are given by the pairs

$$\{x + y\tau \mid x = \frac{m}{a}, y = \frac{n}{a}, 0 \leq m, n < a\}$$

which are precisely a^2-many.

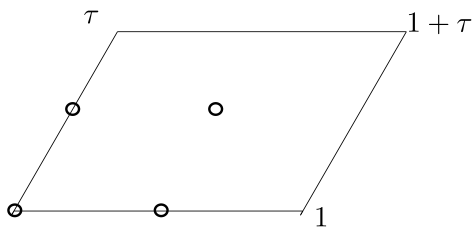

Figure 8: Solutions on elliptic curve with $a = 2$.

□

The following lemma is the main component of the proof of the main theorem. It follows closely the exposition in [Gre18].

Lemma 8.15. *Let* $\alpha \in H_2(O_E(L)\backslash\mathrm{Hom}(L, E), \mathbb{Q})$ *and* $N_n = \cup_{v^2=-2n} v^{\perp}$ *the restriction of the Noether Lefschetz divisor as in Section 8.1. Then*

$$\sum_{n>0} (N_n \cap \alpha)\, q^n = c \cdot DE_8$$

with a constant $c \in \mathbb{Q}$.

Proof. By Lemma 5.31 we know that $\dim H_2(O_E(L)\backslash\mathrm{Hom}(L, E), \mathbb{Q}) = 1$ and that α is a multiple of the pushforward of

$$\overline{\alpha} = \sum_{g \in O(L)} [g(E_0)]$$

where $E_0 = E \times 0^{15} \subset E^{16}$.

To simplify the calculation we first compute the cap product in $\mathrm{Hom}(L, E) \cong E^{16}$ with the pullback of $\cup_{v^2=-2d} v^\perp$ which is just $\cup_{v^2=-2d} v^\perp \subset E^{16}$. But

$$[g(E_0)] \cap v^\perp = [E_0] \cap g^{-1}(v)^\perp$$

as g induces a homeomorhism $E^{16} \to E^{16}$. Hence, we have to count the intersections of the elliptic curve E_0 with $v^\perp$. To do this, we have to analyze the morphism $E^{16} \cong \mathrm{Hom}(L, E)$ again. It was chosen in such a way, that we pick generators $g_1, \ldots, g_{16}$ of L, and $(e_1, \ldots, e_{16}) \in E^{16}$ is sent to

$$\sum a_i g_i \mapsto \sum a_i e_i \in E.$$

Let $v = \sum v_i g_i$ with $v_i \in \mathbb{Z}$. Hence $e \in E_0$ is contained in $v^\perp$, if and only if

$$v_0 e = 0 \in E.$$

As $v_0 \in \mathbb{Z}$, that is the case for exactly v_0^2-many points in E_0 by the foregoing lemma if $v_0 \neq 0$. On the other hand, if $v_0 = 0$, then $E_0 \subset v^\perp$. We can assume without loss of generality that $v_1 \neq 0$, as $v \neq 0$. Hence forming the diagram

$$\begin{tikzcd} E \times 0^{15} \arrow[r] \arrow[d, "\mathrm{incl}"] & E \times E \times 0^{14} \arrow[d, "\mathrm{incl}"] \\ E^{16} \arrow[r, "\mathrm{id}"] & E^{16} \end{tikzcd}$$

we get by the push-pull formula

$$\mathrm{incl}_*([E \times 0^{14}] \cap v^\perp|_{E \times E \times 0^{14}}) = [E \times 0^{14}] \cap v^\perp.$$

But $v^\perp|_{E \times E \times 0^{14}} = E \times \{e \in E \mid v_1 e = 0\}$. By the foregoing lemma this is just $E \times \{v_1^2 -$ points$\}$. But

$$(E \times 0) \cap (E \times \{v_1^2 - \text{points}\}) = 0,$$

as cohomologically $E \times 0$ can be moved to $E \times \{p\}$, for an arbitrary point p, as it is a fiber of the projection $E \times E \xrightarrow{pr_2} E$.
Hence we get that

$$E_0 \cap v^\perp = v_0^2$$

for every $v \in L$. Thus it is a quadratic form in v, that comes from a bilinear mapping. So even

$$\overline{\alpha} \cap v^\perp$$

satisfies the same. But as we saw

$$\overline{\alpha} \cap g(v)^\perp = g^{-1}(\overline{\alpha}) \cap v^\perp = \overline{\alpha} \cap v^\perp$$

as $\overline{\alpha}$ is invariant under the $O(L)$-action by construction. Hence, $\overline{\alpha} \cap v^{\perp}$ is a quadratic form coming from a bilinear form, that is invariant under the orthogonal group of the lattice. By Lemma 8.12,

$$\overline{\alpha} \cap v^{\perp} = c \cdot \langle v, v \rangle$$

for our two choices of L. By the push-pull formula for the map $p : \mathrm{Hom}(L, E) \to O_E(L)\backslash\mathrm{Hom}(L, E)$ we get

$$c \sum_{v^2=-2d} v^2 = p_*(\overline{\alpha} \cap \bigcup_{v^2=-2d} v^{\perp}) = \alpha \cap \bigcup_{v^2=-2d} v^{\perp}.$$

Taking the generating series of this expression yields

$$\sum_{n \in \mathbb{Z}_{>0}} (\alpha \cap \bigcup_{v^2=-2n} v^{\perp}) q^n = c \sum_{n \in \mathbb{Z}_{>0}} \sum_{v^2=-2n} -2nq^n = -cD\Theta_L = -cDE_8$$

by Theorem 7.7. So it is a quasi-modular form of weight 10. □

Remark 8.16. Recall from Section 5, for any continous map $C \to \mathcal{M}^{\mathrm{Tor}}$ that meets the boundary in finitely many Type II points and for every $\alpha \in H_2(C, \mathbb{Q})$ the pushforward $\alpha_C \in H_2(\mathcal{M}^{\mathrm{Tor}}, \mathbb{Q})$ of α decomposes as

$$\alpha_C = (\pi_0)_* \alpha_0 + \sum_{1 \leq i \leq n} (\pi_i)_* \alpha_i$$

where $\alpha_0 \in H_2(\Gamma\backslash\mathcal{D}, \mathbb{Q})$ and $\alpha_i \in H_2(Z_i, \mathbb{Q})$ where Z_i is a fiber from the boundary of $\mathcal{M}^{\mathrm{Tor}} \to \mathcal{M}^{\mathrm{bb}}$ and π_i are the corresponding inclusions.

The following proves the main theorem under more general assumptions:

Theorem 8.17. *Let $f : C \to \mathcal{M}^{Tor}$ be any continuous map from a topological space C to the toroidal compactification. Assume moreover that the map meets the boundary only in finitely many points in the Type II components. Let $\alpha_C \in H_2(C, \mathbb{Z})$ be any homology class and $\alpha = f_* \alpha_C$ the pushforward in $\mathcal{M}^{Tor}$. Then*

$$\alpha \cap \Phi(q) \in \mathbb{Z}E_{10} \oplus \frac{1}{480} \mathbb{Z}DE_8.$$

Proof. By the foregoing lemma, we may assume that $\alpha = (\pi_0)_* \alpha_0 + \sum_i (\pi_i)_* \alpha_i$ as above. But by Borcherds result 8.9 and the push-pull formula on homology we get

$$\begin{aligned}
(\pi_0)_* \alpha_0 \cap \Phi(q) = &(\pi_0)_* (\alpha_0 \cap \pi_0^* \Phi(q)) \\
= &(\pi_0)_* (\alpha_0 \cap (\pi_0^* \bar{\lambda} \cdot q^0 + \sum_{\substack{n \in \mathbb{Z} \\ n > 0}} \pi_0^* \overline{\mathrm{NL}}_{-n} q^n)) \\
= &(\pi_0)_* (\alpha_0 \cap (\lambda \cdot q^0 + \sum_{\substack{n \in \mathbb{Z} \\ n > 0}} \mathrm{NL}_{-n} q^n)) \in \mathbb{Q}E_{10}.
\end{aligned}$$

In the same way, we see that

$$(\pi_i)_*\alpha_i \cap \Phi(q) = (\pi_i)_*(\alpha_i \cap (\pi_i^*\bar{\lambda} \cdot q^0 + \sum_{\substack{n\in\mathbb{Z}\\ n>0}} \pi_i^*\overline{\mathrm{NL}}_{-n}q^n)).$$

But by Lemma 8.15, we get that

$$\sum_{n\in\mathbb{Z},\, n>0} \alpha_i \cap \pi_i^*\overline{\mathrm{NL}}_{-n}q^n \in \mathbb{Q}DE_8.$$

As λ is the pullback of the Hodge line bundle of $\mathcal{M}^{\mathrm{bb}}$, we get that

$$\alpha_i \cap \pi_i^*\bar{\lambda} = 0,$$

as $\pi_i^*\bar{\lambda} = 0$, because π_i is just the inclusion of a fiber of $\mathcal{M}^{\mathrm{Tor}} \to \mathcal{M}^{\mathrm{bb}}$. Hence

$$\alpha \cap \Phi(q) \in \mathbb{Q}E_{10} \oplus \mathbb{Q}DE_8.$$

By construction every intersection product $\alpha \cap \overline{\mathrm{NL}}_{-n}$ is integral. Therefore we get

$$\alpha \cap \Phi(q) \in \mathbb{Z}E_{10} \oplus \frac{1}{480}\mathbb{Z}DE_8,$$

as

$$cE_{10} + c'DE_8 = c + (-264c + 480c')q + \ldots$$

shows that $c \in \mathbb{Z}$ and $c' \in \frac{1}{480}\mathbb{Z}$. □

Remark 8.18. By Example 7.11, we see that even every quasi-modular form in $\frac{1}{480}\mathbb{Z}DE_8$ is integral.

Structure of the Hodge Bundle

Let $X \xrightarrow{f} C$ be a *family* of K3 surfaces and $C \xrightarrow{p} \mathcal{M}$ the period map. Then by construction and the structure of

$$f_*\Omega^2_{X/C} \hookrightarrow R^2f_*\underline{\mathbb{C}} \otimes \mathcal{O}_C$$

which is just

$$H^{2,0}(X_t) \hookrightarrow H^2(X_t, \mathbb{C})$$

in every fiber, one sees that

$$p^*\lambda \cong (f_*\Omega^2_{X/C})^* = (f_*\omega_{X/C})^*.$$

If f is a semi-stable degeneration then even

$$p^*\overline{\lambda} \cong (f_*\omega_{X/C})^*,$$

where $\omega_{X/C}$ is the relative dualizing sheaf, as is shown in [Fuj03]. But by a theorem in [CD14] $f_*\omega_{X/C}$ is nef, i.e. it has non-negative degree. Concluding:

Corollary 8.19. *Let* $X \xrightarrow{f} C$ *be a degeneration as in the main theorem 8.3 that is semi-stable. Then*

$$C \cdot \Phi(q) = C \cdot \bar{\lambda} \cdot q^0 + \sum_{n \in \mathbb{Z},\, n>0} C \cdot \overline{NL}_{-n} q^n$$

is an element of

$$\mathbb{Z}_{\leq 0} E_{10} + \frac{1}{480} \mathbb{Z} DE_8.$$

Proof. The coefficient of the constant term of $C \cdot \Phi$ is $C \cdot \overline{\lambda}$. But this is also the coefficient of E_{10}. Then

$$C \cdot \overline{\lambda} = \deg_C \overline{\lambda} = -\deg_C f_* \omega_{X/C} \leq 0$$

by nefness. □

9. Calculation of Noether-Lefschetz Numbers

Degeneration associated to the generic pencil of $\mathcal{O}_{\mathbb{F}_4}(3,12)$

The general construction

In this section we will construct a degeneration as in Section 4.3.1. I.e. let

$$P[\lambda,\mu] = \lambda L + \mu L' \in \mathbb{P}(H^0(\mathcal{O}_{\mathbb{F}_4}(3,12)))$$

be a generic Lefschetz pencil in $\mathcal{O}_{\mathbb{F}_4}(3,12)$. The degeneration $X \to \mathbb{P}^1$ is then defined as the double cover of

$$\mathbb{F}_4 \times \mathbb{P}^1$$

along

$$(Z \cdot P([\lambda^2,\mu^2]), [\lambda,\mu]).$$

As we have seen in Sections 4.3.1 and 2.2 the generic element of this degeneration is indeed a K3 surface.

The singular fibers

Next, we want to examine which singular fibers can occur. Recall, that in a generic pencil, there is an open dense subset of $\mathbb{P}^1$ such that the fibers over these points are smooth and irreducible by Bertini's theorem.

By Appendix D, we have

$$H^0(\mathbb{F}_4, \mathcal{O}(a,b)) \cong H^0(\mathbb{P}^1, \pi_*\mathcal{O}(a,b))$$

for $a \geq 0$, where $\pi : \mathbb{F}_4 \to \mathbb{P}^1$ is the ruling. On the other hand, by [Har13], we have

$$\pi_*\mathcal{O}(a,0) = S^a(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-4)) = \bigoplus_{i=0,\dots,a} \mathcal{O}(-4i).$$

Hence, we get

$$\begin{aligned} H^0(\pi_*\mathcal{O}(a,b)) &= H^0(\pi_*(\mathcal{O}(a,0)\otimes\pi^*\mathcal{O}_{\mathbb{P}^1}(b))) \\ &= H^0(\pi_*\mathcal{O}(a,0)\otimes\mathcal{O}_{\mathbb{P}^1}(b)) \\ &= H^0(\bigoplus_{i=0,\dots,a}\mathcal{O}_{\mathbb{P}^1}(-4i)\otimes\mathcal{O}_{\mathbb{P}^1}(b)) \\ &= H^0(\bigoplus_{i=0,\dots,a}\mathcal{O}_{\mathbb{P}^1}(-4i+b)) \\ &= \bigoplus_{i=0,\dots,a} H^0(\mathcal{O}_{\mathbb{P}^1}(-4i+b)). \end{aligned}$$

As $\dim H^0(\mathcal{O}_{\mathbb{P}^1}(a)) = a+1$ for $a \geq 0$ and 0 otherwise, we can compute all possible dimensions that can occur: See table 1.

We now want to analyze, which splittings

$$h \rightsquigarrow f\cdot g$$

can occur in a generic pencil. If this happens then

$$\begin{aligned} f &\in \mathcal{O}(a,b), \\ g &\in \mathcal{O}(a',b'), \end{aligned}$$

with $a+a'=3, b+b'=12$. Hence, all elements of this form for fixed a,b form a subset of $\mathcal{O}(3,12)$ with dimension

$$\dim \leq \dim \mathbb{P}H^0(\mathcal{O}(a,b)) + \dim \mathbb{P}H^0(\mathcal{O}(a',b')).$$

By analyzing the table 1, one sees that the codimension in $\mathbb{P}H^0(\mathcal{O}(3,12))$ is always greater than 2, except for the case

$$\begin{aligned} f &\in \mathcal{O}(1,0), \\ g &\in \mathcal{O}(2,12), \end{aligned}$$

which was the the degeneration we analyzed in Section 4.3.1.

Number of singular curves - Göttsche conjecture

Following the idea of [KST11], we compute the number of nodal fibers of the pencil. Suppose, there are only ordinary double points as singularities, i.e. singularities that analytically look like $xy=0$ in $\mathbb{C}^2$. Denote by g the arithmetic genus of the curve in the pencil, which satisfies

$$2g-2 = \mathcal{O}(3,12)\cdot(\mathcal{O}(3,12)+\omega_{\mathbb{F}_4}) = 18.$$

Table 1: Dimensions of line bundles

L	$\dim \mathbb{P}H^0(L)$	L	$\dim \mathbb{P}H^0(L)$
$\mathcal{O}(0,0)$	0	$\mathcal{O}(2,6)$	9
$\mathcal{O}(1,0)$	0	$\mathcal{O}(3,6)$	9
$\mathcal{O}(2,0)$	0	$\mathcal{O}(0,7)$	7
$\mathcal{O}(3,0)$	0	$\mathcal{O}(1,7)$	11
$\mathcal{O}(0,1)$	1	$\mathcal{O}(2,7)$	11
$\mathcal{O}(1,1)$	1	$\mathcal{O}(3,7)$	11
$\mathcal{O}(2,1)$	1	$\mathcal{O}(0,8)$	8
$\mathcal{O}(3,1)$	1	$\mathcal{O}(1,8)$	13
$\mathcal{O}(0,2)$	2	$\mathcal{O}(2,8)$	14
$\mathcal{O}(1,2)$	2	$\mathcal{O}(3,8)$	14
$\mathcal{O}(2,2)$	2	$\mathcal{O}(0,9)$	9
$\mathcal{O}(3,2)$	2	$\mathcal{O}(1,9)$	15
$\mathcal{O}(0,3)$	3	$\mathcal{O}(2,9)$	17
$\mathcal{O}(1,3)$	3	$\mathcal{O}(3,9)$	17
$\mathcal{O}(2,3)$	3	$\mathcal{O}(0,10)$	10
$\mathcal{O}(3,3)$	3	$\mathcal{O}(1,10)$	17
$\mathcal{O}(0,4)$	4	$\mathcal{O}(2,10)$	20
$\mathcal{O}(1,4)$	5	$\mathcal{O}(3,10)$	20
$\mathcal{O}(2,4)$	5	$\mathcal{O}(0,11)$	11
$\mathcal{O}(3,4)$	5	$\mathcal{O}(1,11)$	19
$\mathcal{O}(0,5)$	5	$\mathcal{O}(2,11)$	23
$\mathcal{O}(1,5)$	7	$\mathcal{O}(3,11)$	23
$\mathcal{O}(2,5)$	7	$\mathcal{O}(0,12)$	12
$\mathcal{O}(3,5)$	7	$\mathcal{O}(1,12)$	21
$\mathcal{O}(0,6)$	6	$\mathcal{O}(2,12)$	26
$\mathcal{O}(1,6)$	9	$\mathcal{O}(3,12)$	27

By [KST11], we have that the Euler characteristic $\chi(C) = 2-2g$ for smooth curves and $\chi(C) = 2-2g+n$ for curves with n nodes, i.e

$$\chi(C) - (2-2g)\overbrace{\chi(pt)}^{=1} = n.$$

Note, that this is not necessarily true for reducible curves of the form $C = C_1 \cup_{p_1,\ldots,p_n} C_2$. Now suppose the pencil meets a degeneration into $f \cdot g$ with $f \in \mathcal{O}(1,0), g \in \mathcal{O}(2,12)$ *once* and all other curves are irreducible (which is the case for the generic pencil, as we will see soon). Generically those f and g meet in 4 points transversally. By the additivity of the Euler characteristic

$$\chi(C) = \chi(C_1) + \chi(C_2) - \chi(\bigcup_i p_i) = \chi(C_1) + \chi(C_2) - 4.$$

Computing the Euler characteristic via the adjunction formula, we get

$$\chi(C_1) + \chi(C_2) = -(-2+12) = -10.$$

Therefore for this curve

$$\chi(C) - \overbrace{(2-2g)}^{=-18}\chi(pt) = -10 - 4 - (-18) = 4.$$

Thus, in this case it equals the number of nodel points. Now let $\mathcal{C} \to \mathbb{P}^1$ be the universal curve of the pencil, which is isomorphic to the blowup of $\mathbb{F}_4$ in $c_1(\mathcal{O}(3,12))^2 = 36$ points. Hence by the additivity of the Euler characteristic we get

$$\begin{aligned}
&\#(\text{number of nodal singularities}) \\
=&\#(\text{number of nodal singularities in irreducible fibers}) + 4 \\
=&\chi(\mathbb{F}_4) + 36 - (2-2g)\chi(\mathbb{P}^1) = 4 + 36 + 18 \cdot 2 = 76.
\end{aligned}$$

Intersection with $\overline{NL}_1$

Let $A, B \in \mathcal{O}(3,12)$ be chosen generically such that the related pencil P

$$\lambda A + \nu B$$

has only nodal singularities. Then the quadratic pencil

$$\lambda^2 A + \nu^2 B$$

contains every curve from above twice, except the ones A and B. As the pencil P is chosen generically, we know that the reducible fibers are of the form

$$f \cdot g$$

for $f \in \mathcal{O}(1,0), g \in \mathcal{O}(2,12)$. But all those elements define a hypersurface of degree one, hence P contains only one such element. By the Göttsche formula above, we know that the quadratic pencil hence contains $2 \cdot 76 = 152$ nodal singularities, where $152 - 2 \cdot 4 = 144$ lie in irreducible curves, and the other 8 nodes are contained in the two reducible ones. Observing the dimensions of $\mathcal{O}(1,0)$, we see that $f = Z$ is the section. For a generic g, f and g intersect in 4 points. Hence, this curve contains 4 nodes.

Now, we want to calculate the intersection with the Noether-Lefschetz divisor.

Lemma 9.1. *Let X be an elliptic K3 surface that is a Weierstraß model. Then there is no $v \in Pic(X)$ such that $v \cdot f = 0$, $v \cdot s = 0$ and $v^2 = -2$, where f is a fiber and s the section.*

Proof. As we saw in Section 2.2, every such K3 surface is a double cover $X \xrightarrow{c} \mathbb{F}_4 \xrightarrow{p} \mathbb{P}^1$ over the curve

$$Z \cdot (X^3 + AXZ^2 + BZ^3)$$

with $A \in p^*\mathcal{O}_{\mathbb{P}^1}(8), b \in p^*\mathcal{O}_{\mathbb{P}^1}(12)$. It is immediate, that $c^* f_{\mathbb{F}_4} = f_X$ and $c^* s_{\mathbb{F}_4} = 2s_X$. But on $\mathbb{F}_4$, the line bundle

$$a := s + 5f$$

is very ample by[Har13]. On the other hand, $X \to \mathbb{F}_4$ is a finite morphism. Hence c^*a is ample as well. But c^*a is contained in the span of s_X, f_X. Therefore

$$c^*a \cdot v = 0$$

implies that v is *not* effective. On the other hand, the Riemann Roch formula states

$$h^0(X,v) - h^1(X,v) + h^2(X,v) = \chi(v) = \chi(\mathcal{O}) + \frac{1}{2}v^2 = 1 - 0 + 1 - 1 = 1.$$

Hence, either $h^0(X,v) \neq 0$ or $h^2(X,v) \neq 0$. The first one would be a contradiction to v being non-effective. Consequently $0 \neq h^2(X,v) = h^0(X,-v)$, where the equality comes from Serre-duality. Therefore $-v$ is effective. But

$$0 < (-v) \cdot c^*a = -(v \cdot c^*a) = 0,$$

which is a contradiction. Hence no such line bundle can exist. □

Theorem 9.2. *Let $X \to \mathbb{P}^1$ be the degeneration as above. Furthermore assume, that the period map $\mathbb{P}^1 \to \mathcal{M}^{Tor}$ satisfies*

$$\overline{NL}_1 \cap \delta\mathcal{M}^{Tor} \cap \operatorname{Im}\mathbb{P}^1 = \emptyset$$

and the image of the period map intersects the Noether Lefschetz divisor $\overline{NL}_1$ transversally. Then

$$\mathbb{P}^1 \cdot \overline{NL}_1 = 144.$$

Proof. Every nodal curve in the pencil yields an A^1 singularity in the resulting family. Resolving this singularity produces one -2 curve per singularity. By the assumptions and the foregoing lemma, we get that

$$\mathbb{P}^1 \cdot \overline{NL}_1 = \#(\text{number of nodal points in irreducible fibers}) = 152 - 2 \cdot 4,$$

as there are two reducible fibers with 4 nodal points each, whose period point lies in the boundary. $\square$

Intersection with $\overline{\lambda}$

To compute the degree of $\overline{\lambda}$, we first observe that composing $f : X \to \mathbb{P}^1$ with the blow up $h : Y \to X$ as in Example 4.24 we get a semi-stable model $F : Y \to \mathbb{P}^1$ and the degree of the Hodge bundle does not change. By [Kle80]

$$\omega_{Y/\mathbb{P}^1} = \omega_Y \otimes F^* \omega_{\mathbb{P}^1}^{-1}$$

and for $p : \mathbb{F}_4 \times \mathbb{P}^1 \to \mathbb{P}^1$ the canonical projection

$$\omega_{\mathbb{F}_4 \times \mathbb{P}^1/\mathbb{P}^1} = \omega_{\mathbb{F}_4 \times \mathbb{P}^1} \otimes p^* \omega_{\mathbb{P}^1}^{-1}.$$

Moreover for every line bundle L

$$F_*(L^* \otimes \omega_{X/\mathbb{P}^1}) \cong (R^2 F_* L)^*$$

and the same holds for $p : \mathbb{F}_4 \times \mathbb{P}^1 \to \mathbb{P}^1$. The map $F : Y \to \mathbb{P}^1$ factors as

$$F : Y \xrightarrow{h} X \xrightarrow{g} \mathbb{F}_4 \times \mathbb{P}^1 \xrightarrow{p} \mathbb{P}^1.$$

As we have seen in Example 4.24, the fibers of the blowup $Y \to X$ are either a point or $\mathbb{P}^1$. Hence by Grauerts theorem

$$R^i h_* \mathcal{O}_Y = 0$$

for $i > \dim \mathbb{P}^1 = 1$ as the map is projective since X is. Therefore

$$H^2(h^{-1}(U), \mathcal{O}_Y) = H^2(U, h_* \mathcal{O}_Y).$$

Let $V \subset \mathbb{P}^1$ be an open subset. Then

$$H^2(F^{-1}(V), \mathcal{O}_Y) = H^2(h^{-1}(f^{-1}(V)), \mathcal{O}_Y) = H^2(f^{-1}(V), h_* \mathcal{O}_Y).$$

Since $h_* \mathcal{O}_Y = \mathcal{O}_X$ as $h : Y \to X$ is a blow up, it follows that

$$R^2 F_* \mathcal{O}_Y \cong R^2 f_* \mathcal{O}_X.$$

The map $g : X \to \mathbb{F}_4 \times \mathbb{P}^1$ is affine by construction and therefore

$$H^2(f^{-1}(V), \mathcal{O}_X) = H^2(g^{-1}(p^{-1}(V)), \mathcal{O}_X) = H^2(p^{-1}, g_* \mathcal{O}_X)$$

and

$$R^2 f_* \mathcal{O}_X = R^2 p_*(g_* \mathcal{O}_X).$$

For easier notation, we abbreviate $\mathcal{O}(a,b,c) := \mathcal{O}_{\mathbb{F}_4}(b,c) \otimes p^* \mathcal{O}_{\mathbb{P}^1}(a)$ on $\mathbb{F}_4 \times \mathbb{P}^1$. Then by construction

$$g_* \mathcal{O}_X = \mathcal{O}_{\mathbb{F}_4 \times \mathbb{P}^1} \oplus \mathcal{O}(-1,-2,-6).$$

Using duality for this sheaf and $\omega_{\mathbb{F}_4 \times \mathbb{P}^1/\mathbb{P}^1} = \mathcal{O}(0,-2,-6)$, we get

$$R^2 p_*(\mathcal{O}(0,0,0) \oplus \mathcal{O}(-1,-2,-6)) = (p_* \mathcal{O}(0,-2,-6) \oplus p_*(1,0,0))^*.$$

In total

$$F_* \omega_{Y/\mathbb{P}^1} = p_* \mathcal{O}(0,-2,-6) \oplus p_*(1,0,0).$$

But using the Künneth formula for sheaf cohomology, we get for every open $V \subset \mathbb{P}^1$ that naturally

$$\mathcal{O}(0,0,0)(p^{-1}(V)) \cong \mathcal{O}_{\mathbb{P}^1}(V),$$

and therefore

$$p_* \mathcal{O}(0,0,0) = \mathcal{O}_{\mathbb{P}^1}.$$

On the other hand

$$\mathcal{O}(0,-2,-6)(p^{-1}(V)) = \{0\}$$

as $\mathcal{O}_{\mathbb{F}_4}(-2,-6)$ has no global sections. Thus,

$$p_* \mathcal{O}(0,-2,-6) = 0.$$

Therefore by the projection formula $F_* \omega_{Y/\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(1)$.

Remark 9.3. As the degeneration $Y \to \mathbb{P}^1$ is not semistable (because of the irreducible nodal fibers), we can not apply the direct formula for $p^* \overline{\lambda}$. We will proceed as in [MP07]: Let $\alpha : C_{144} \to \mathbb{P}^1$ be the double cover over the points that correspond to the irreducible nodal fibers. Denote by $\tilde{Y} \xrightarrow{\alpha'} Y$ the pull back of Y by this map. As explained in [MP07] this space admits a small resolution of singularities *of the fibers*. The resulting degenerations are denoted $F' : Y' \to C_{144}$, which is semistable, and $\tilde{F} : \tilde{Y} \to C_{144}$.

Theorem 9.4. *The degeneration $Y \to \mathbb{P}^1$ admits*

$$\deg_{\mathbb{P}^1} \overline{\lambda} = -1.$$

Proof. By theorem 4.4. in [Fuj03], we have

$$\tilde{F}_* \omega_{\tilde{Y}/C_{144}} = \alpha^* F_* \omega_{Y/\mathbb{P}^1}.$$

As the resolution $r : Y' \to \tilde{Y}$ is a resolution of the singularities of the fibers, the dimension of the fibers is at most 1. Again by Grauert's theorem and duality as above, we can conclude that

$$F'_* \omega_{Y'/C_{144}} = \tilde{F}_* \omega_{\tilde{Y}/C_{144}}.$$

As $Y' \to C_{144}$ is semistable $F'_*\omega_{Y'/C_{144}}$ is the inverse of the Hodge bundle corresponding to this degeneration. Therefore by the computation before, we get

$$\deg_{C_{144}} \overline{\lambda} = -2 \deg_{\mathbb{P}^1} F_*\omega_{Y/\mathbb{P}^1} = -2 \deg_{\mathbb{P}^1} \mathcal{O}(1) = -2,$$

as $\alpha : C_{144} \to \mathbb{P}^1$ is a double cover. But on the other hand

$$-2 = \deg_{C_{144}} \overline{\lambda} = 2 \deg_{\mathbb{P}^1} \overline{\lambda}$$

since the map $C_{144} \to \mathcal{M}^{\mathrm{bb}}$ factors as

$$C_{144} \xrightarrow{\alpha} \mathbb{P}^1 \to \mathcal{M}^{\mathrm{bb}}.$$

Therefore

$$\deg_{\mathbb{P}^1} \overline{\lambda} = -1.$$

$\square$

Remark 9.5. By Remark 4.3, every semistable degeneration of K3 surfaces can be transformed into a Kulikov model by a birational morphism $X' \to X$ that only changes the degenerated fibers. Thus, replacing X with X' does not change the Noether-Lefschetz numbers and the following theorem holds for X', too.

Theorem 9.6. *The generating series for the degeneration $X \to \mathbb{P}^1$, with the assumptions as in Theorem 9.2, is given by*

$$\begin{aligned}\mathbb{P}^1 \cdot \bar{\lambda} + \sum_{n \in \mathbb{Z},\, n>0} \mathbb{P}^1 \cdot \overline{NL}_{-n} q^n &= -E_{10} - \frac{263}{480} DE_8 \\ &= -1 + 144q^1 + 67578q^2 + 3470244q^3 + \dots\end{aligned}$$

Proof. As we have seen in Section 6, the period map of this degeneration extends to the type II boundary. Thus, the theorem directly follows from the calculations and the main theorem, since

$$aE_{10} + bDE_8 = a + (480b - 264a)q^1 + \dots$$

$\square$

10. Outlook

To summarize this thesis, we found that the generating series of Noether-Lefschetz numbers is a quasi-modular form for type II degenerations. We needed to construct the toroidal compactification, which had a dependence on some fan Σ. But in our case, the calculations were invariant under choosing different Σ. Naturally, one would like to understand this behavior in the type III case, too:
In this case the construction of the boundary is simpler, as it is even a tube domain: One can embed the period domain

$$\mathcal{D} \subset U_J \otimes \mathbb{C} \cong \mathbb{C}^{18}$$

for J an isotropic line. But then the construction of the boundary components depends on the chosen fan. A question is how to relate

$$C \cdot \overline{NL}_d^{\Sigma}$$

for different choices of Σ and moreover if the generating series considered in this thesis is still a quasi-modular form.
As is shown in [ABE20], there are two 'natural' choices for such fans: The ramification fan Σ^{ram} and the rational curve divisor fan Σ^{rcd}. The resulting toroidal compactifications are normalizations of stable pair KSBA compactifications of $\mathcal{M}$. These compactifications hence admit a modular description and this could yield to an different viewpoint on the intersection products.
In [BZ19], Jan Hendrik Bruinier took a different approach, which is related more directly to the original construction of Borcherds: He showed that for a special subgroup $\Gamma_L \subset O(\Lambda_{2,18})$ specific divisors Z_n can be defined on the toroidal compactification of $\Gamma_L \backslash \mathcal{D}$: They are the closures of the Noether-Lefschetz divisors plus some boundary divisors with a given multiplicity. Unfortunately it is hard to compute the multiplicities. The main result of the paper is:

$$\sum_i Z_i q^i \in M_{10} \otimes CH^1(\overline{\Gamma_L \backslash \mathcal{D}}^{\mathrm{Tor}}).$$

Naturally, one would like to know if this theorem also extends to the moduli space of K3 surfaces.

A. Some Lattice Theory

Definition A.1. A *lattice* L is a free finitely generated $\mathbb{Z}$-module, together with a bilinear symmetric pairing $\langle -, - \rangle : L \times L \to \mathbb{Z}$. It is called *even* if $\langle v, v \rangle \in 2\mathbb{Z}$ for every $v \in L$. The signature of a lattice is defined in the same way as for symmetric bilinear forms.

Most lattices that naturally arise come from discrete subspaces of $\mathbb{R}^n$:

Example A.2. The lattice D_n is defined as a subspace of $\mathbb{R}^n$ in the following way

$$D_n = \{(a_i)_{1 \leq i \leq n} \mid \sum a_i \equiv 0 \bmod 2\}.$$

The intersection is the standard one from $\mathbb{R}^n$ restricted to the subspace. In the same way we create a bigger lattice, which set-theoretically is defined by

$$E_n = D_n \cup (D_n + (\frac{1}{2}, \dots, \frac{1}{2})).$$

for $n = 8, 16$. Again, the intersection pairing is given by restriction. Clearly, they are all even and non-degenerate.

An important subclass of lattices is defined as follows:

Definition A.3. A *root* is an element $v \in L$, such that

$$\langle v, v \rangle = \pm 2.$$

The *root lattice* of L is the subspace R spanned by all roots. If $R = L$, we also call L a root lattice. For each root v, we define the corresponding reflection

$$s_v(v') = v' - \frac{\langle v, v' \rangle}{\langle v, v \rangle} v.$$

The group $\mathcal{W}(L)$ generated by these reflections is called the *Weylgroup*.

Example A.4. Clearly, D_n is a root lattice, as the elements $e_i = (0, \dots, 0, \pm 1, \pm 1, 0, \dots 0)$ span the whole lattice. For the lattices E_8 and E_{16} we will have to differentiate: Clearly E_8 is spanned by D_8 and $(\frac{1}{2}, \dots, \frac{1}{2})$. But the latter is also a root. So E_8 is a root lattice as well. On the other hand, any element in $(\frac{1}{2}, \dots, \frac{1}{2}) + D_{16}$ has square ≥ 4. Therefore the sublattice $D_{16} \subset E_{16}$ is the root lattice.
Therefore, from now on the lattice E_{16} is called D_{16}^+.

Definition A.5. For a lattice L, we define the *discriminant* Δ. Therefore fix a basis e_i. Let $A = (a_{ij})_{i,j}$ be the corresponding matrix of the bilinear form. Then

$$\Delta = -\det A.$$

If $\Delta = \pm 1$ the lattice is called *unimodular*.

The following example is taken from [NS].

Example A.6. A generating set of E_8 is given by the rows of the following matrix

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

A computation of the corresponding matrix shows, that E_8 is indeed unimodular.
A generating set for D_{16}^{+} is given in a similar fashion, and again one shows that D_{16}^{+} is unimodular. It is important to note, that a generating set for D_{16} is given by the same matrix, but the last row constists of the vector $(1, \dots, 1)$.

The importance of these lattices is given by the following theorem:

Theorem A.7. *[CS13] Up to automorphism there are 2 unimodular even non-degenerate lattices, namely the irreducible lattice D_{16}^{+} and reducible one $E_8 \oplus E_8$.*

Another important example is given by the following:

Example A.8. Let $U = \mathbb{Z}^2$ and the intersection matrix is given by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It is unimodular as well, but not non-degenerate. The basis is usually denoted by f, s.

As is taken from [Huy16]:

Lemma A.9. *Let Λ be any lattice and $U \hookrightarrow \Lambda$ an inclusion. Then*

$$\Lambda = U \oplus U^{\perp}.$$

Theorem A.10. *Let $\Lambda_{3,19} = U^3 \oplus E_8(-1)^2$. Then any primitive embedding*

$$U \hookrightarrow \Lambda_{3,19}$$

is unique up to isomorphism.

Theorem A.11. *Let Λ be an unimodular, even lattice of signature (n_+, n_-) with $1 < n_\pm$. Then any element $x \in \Lambda$ with $x^2 = 2d$ is unique up to isomorphism.*

Remark A.12. This in particular applies to the two lattices

$$\begin{aligned} \Lambda_{3,19} &= U^3 \oplus E_8 \oplus E_8 \\ \Lambda_{2,18} &= U^2 \oplus E_8 \oplus E_8. \end{aligned}$$

Remark A.13. Let L be an arbitrary lattice. Then $-L$ or $L(-1)$ denotes the same lattice, but the intersection pairing is replaced by its negative.

B. Some Hodge Theory

In this section, we briefly recall Hodge structures on cohomology and introduce mixed Hodge structures.

Following [Huy16], let V be a free $\mathbb{Z}$-module of finite rank, and denote by $V_{\mathbb{Q}}$ and $V_{\mathbb{C}}$ the tensor product of V with the respective field.

Definition B.1. A *pure Hodge structure* of weight n of V (or $V_{\mathbb{Q}}$) is given by vector spaces $(V^{p,q})_{p+q=n}$, such that

$$V_{\mathbb{C}} = \bigoplus_{p+q=n} V^{p,q}$$

and $\overline{V^{p,q}} = V^{q,p}$. A morphism of weight k of Hodge structures H_1, H_2 is given by a morphism $f : H_1 \to H_2$, such that $f(H_1^{p,q}) \subset H_2^{p+k,q+k}$.

Remark B.2. This definition is equivalent to giving a decreasing filtration $(F^i)_i \subset V_{\mathbb{C}}$

$$V_{\mathbb{C}} \supset F^0 \supset \ldots \supset F^n = 0$$

such that $F^p \oplus \overline{F^q} = V_{\mathbb{C}}$ for all $p + q = n + 1$.
We get F^p from $V^{q,p}$ by just setting $F^i = \bigoplus_{p+q=n, p \geq i} V^{p,q}$.

Definition B.3 ([PS00])**.** A *mixed Hodge structure* on the free $\mathbb{Z}$-module V of finite rank is given by

- An increasing (*weight*) filtration $W = (W_i)_i \subset V_{\mathbb{Q}}$ and
- an decreasing (*Hodge*) filtration $F = (F^j)_j \subset V_{\mathbb{C}}$,

such that F induces a pure Hodge structure of weight n on $Gr_n W := W_n/W_{n-1}$ in the obvious manner. A morphism of mixed Hodge structures of weight $2l$ between H_1, H_2 is given by a $\mathbb{Q}$-linear morphism $f : H_1 \to H_2$, such that

$$f(W_i H_1) \subset W_{i+2l} H_2,$$
$$f(F^i H_1) \subset F^{i+l} H_2.$$

Remark B.4. It follows that a map of mixed Hodge structures defines a morphism of pure Hodge structures of weight $2l$ on the graded pieces $Gr_i H_1 \to Gr_{i+2l} H_2$.

Remark B.5. It is well known that on smooth Kähler manifolds, e.g. smooth varieties (over $\mathbb{C}$), a pure Hodge structure of weight n on the rational cohomology $H^n(X, \mathbb{Q})$ is simply given by the Hodge decomposition.
The situation is more involved, if we regard singular varieties over $\mathbb{C}$. A theorem by Deligne states, that the rational cohomology groups $H^n(X, \mathbb{Q})$ can be equipped with

a natural mixed hodge structure, such that for every morphism of algebraic varieties $f : X \to Y$, the map

$$f^* : H^n(Y, \mathbb{Q}) \to H^n(X, \mathbb{Q})$$

is a map of mixed Hodge structures of weight 0.

Remark B.6. In the case of K3 surfaces X, we get the Hodge decomposition

$$H^2(X, \mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X).$$

Moreover by the Hodge-Riemann bilinear relations, we get that the usual pairing

$$\cap : H^2(X, \mathbb{C}) \times H^2(X, \mathbb{C}) \to H^4(X, \mathbb{C}) \cong \mathbb{C}, ^{11}$$

satisfies

- $x \cap x = 0$ and
- $x \cap \overline{x} > 0$

for all $x \in H^{2,0}(X)$, see e.g. [Huy16].

The following example is taken from [PS00].

Example B.7. Let $X = V_1 \cup_E V_2$ be a complex complete surface, with V_i smooth and normal crossings. Then we obtain the following sequence

$$H^1(V_1, \mathbb{Q}) \oplus H^1(V_2, \mathbb{Q}) \to H^1(E, \mathbb{Q}) \xrightarrow{\delta} H^2(X, \mathbb{Q}).$$

The weight filtration for the mixed Hodge structure is then given by

$$0 \subset W_1 = \mathrm{Im}(\delta) \subset W_2 = H^2(X, \mathbb{Q}).$$

Applied to our main case, where E is an elliptic curve and V_i is rational, we obtain

$$0 \subset W_1 = \mathrm{Im}(\delta) \cong H^1(E) \subset W_2 = H^2(X, \mathbb{Q})$$

as $H^1(V_1, \mathbb{Q}) = 0$. Moreover in [PS00], it is shown that the Hodge structure on $Gr_1 = W_1$ is given by the Hodge structure of $H^1(E)$ coming from the variety E.

C. Toric Varieties

The following is mainly taken from [CLS11] and [Bru15]. It is a short introduction to toric varieties and embeddings of tori into these spaces.

[11] The last isomorphism is canonical, as we can specify that the fundamental class of X is mapped to 1.

Definition C.1. The n-dimensional *torus* T is defined as the group variety

$$T = (\mathbb{C}^*)^n$$

where the group structure is given by the componentwise multiplication.

Remark C.2. The tori from above are affine and isomorphic to

$$\mathrm{Spec}(x_1^{\pm}, \dots, x_n^{\pm}).$$

There are two canonical lattices, which we will define next:

Definition C.3. Let T be a torus. The group of characters M is given by

$$M = \mathrm{Hom}(T, \mathbb{C}^*).$$ [12]

The group of one-parameter subgroups is given by

$$N = \mathrm{Hom}(\mathbb{C}^*, T).$$

Remark C.4. One can show, that every character χ of a torus $T = (\mathbb{C}^*)^n$ is given by

$$\chi(x_1, \dots, x_n) = x_1^{a_1} \cdot \dots \cdot x_n^{a_n},$$

where $(a_1, \dots, a_n) \in \mathbb{Z}^n$. Hence $M \cong \mathbb{Z}^n$. On the other hand every one parameter subgroup is given by

$$z \mapsto (z^{b_1}, \dots, z^{b_n}).$$

And hence, $N \cong \mathbb{Z}^n$ as well. Moreover we get a natural pairing

$$M \times N \to \mathrm{Hom}(\mathbb{C}^*, \mathbb{C}^*) \cong \mathbb{Z}$$

which turns out to be perfect, see [CLS11].

The building blocks of toric varieties are the affine ones:

Definition C.5. A *toric variety* is a variety V, that contains a torus $T \subset V$ as a dense open subset and the action $T \times T \to T$ given by the group structure extends to an action

$$T \times V \to V.$$

It is called affine if V is affine as a scheme.

[12] Here Hom denotes the group homomorphisms that are also morphisms of varieties.

Proposition C.6 ([Bru15]). *Let $\sigma \subset N \otimes \mathbb{R}$ be a rational polyhedral cone, i.e. it is a cone, that has a finite generating set which is rational. Then there is a one-to-one correspondence between the affine toric varieties and rational polyhedral cones $\sigma \subset N \otimes R$ given by*

$$\sigma \leftrightarrow TV(\sigma) = \mathit{Spec}(\mathbb{C}[\sigma^\vee \cap M]),$$

where $\mathbb{C}[\sigma^\vee \cap M]$ is the algebra generated by $\sigma^\vee \cap M$ and $\sigma^\vee \subset M \otimes \mathbb{R}$ is the dual cone, i.e.

$$\sigma^\vee = \{m \in M \otimes \mathbb{R} \,|\, (m, \sigma) \geq 0\}.^{13}$$

An important example is given by the following, which is handy for the Type II cusps in the toroidal compactification.

Example C.7. Let $T = \mathbb{C}^*$ and hence $N, M = \mathbb{Z}$. The cone $\mathbb{R}^+ \subset \mathbb{R} = N \otimes \mathbb{R}$ has dual $\mathbb{R}^+$ and

$$\mathrm{Spec}(\mathbb{C}[\mathbb{R}^+ \cap M]) = \mathrm{Spec}(\mathbb{C}[\mathbb{N}]) = \mathrm{Spec}(\mathbb{C}[t]) \cong \mathbb{C},$$

which clearly contains $\mathbb{C}^*$ in the obvious way.

Remark C.8. Let σ be a rational cone in $N \otimes \mathbb{R}$, and $\sigma' \subset \sigma$ be a face. Then

$$TV(\sigma) \subset TV(\sigma').$$

By the forgoing remark, we can glue certain affine toric varieties together, if we impose some conditions:

Definition C.9. Let Σ be a collection of cones in $N \otimes \mathbb{R}$. It is called a *fan*

- if $\sigma_1, \sigma_2 \in \Sigma$, then $\sigma_1 \cap \sigma_2 \in \Sigma$ and
- f σ_1 is a face of $\sigma_2 \in \Sigma$, then $\sigma_1 \in \Sigma$.

As it turns out, we get

Proposition C.10. *Let Σ be a fan of $N \otimes \mathbb{R}$. Then, glueing the affine toric varieties $TV(\sigma)$ for all $\sigma \in \Sigma$ as indicated, produces a toric variety $TV(\Sigma)$. Moreover we get a bijection between normal toric varieties and fans in $N \otimes \mathbb{R}$:*

$$\Sigma \leftrightarrow TV(\Sigma).$$

Proof. See [Bru15]. □

Remark C.11. If σ does not contain a straight line, which in our case is true, then $(\mathbb{C}^*)^n \subset V$ for $n = \dim N$, see [CLS11].

[13] Here the pairing is the canonical one defined above.

D. Hirzebruch Surfaces

Here we recall some basic facts and theorems about ruled surfaces and in particular Hirzebruch surfaces. This section mainly follows [Har13].

Definition D.1. A surface X is called *ruled* surface, if there is a morphism

$$\pi : X \to C$$

to a curve C, together with a section $s : C \to X$, such that every fiber is isomorphic to $\mathbb{P}^1$.

Remark D.2. One can show that the existence of a section follows from the other conditions.

Proposition D.3. *Every ruled surface $X \to C$ is of the form*

$$\mathbb{P}(\mathcal{E}) \to C$$

where $\mathcal{E}$ is a locally free sheaf of rank 2 on C.

Definition D.4. A ruled surface $X \to \mathbb{P}^1$ is called *Hirzebruch surface.*

Remark D.5. By a standard theorem every coherent locally free sheaf $\mathcal{E}$ on $\mathbb{P}^1$ is of the form $\bigoplus_{1 \leq i \leq m} \mathcal{O}(n_i)$. Hence every Hirzebruch surface is isomorphic to

$$\mathbb{P}(\mathcal{O}(n) \oplus \mathcal{O}(m)) \cong \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(m-n)).$$

Without loss of generality we may assume $m - n \leq 0$. Define

$$\mathbb{F}_n = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-n)) \to \mathbb{P}^1$$

for $n \geq 0$.

Theorem D.6. *Let $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n)) \xrightarrow{\pi} \mathbb{P}^1$ and $\mathcal{O}_X(1)$ the relative invertible sheaf. Then there is a section $s : \mathbb{P}^1 \to X$. By abuse of notation, also denote the image of the section by s. Then*

$$\mathcal{O}_X(s) = \mathcal{O}_X(1).$$

Denote by f a fiber. We then have $\mathcal{O}(f) = \pi^*\mathcal{O}(1)$,

$$Pic(X) = \langle f, s \rangle_{\mathbb{Z}}$$

and the intersection form is given by

$$\begin{pmatrix} 0 & 1 \\ 1 & -n \end{pmatrix}.$$

Moreover the canonical bundle is given by

$$\omega_X = -2s - (2+n)f.$$

Calculating the cohomology we get

Proposition D.7. *Let D be a divisor on $\mathbb{F}_n \xrightarrow{\pi} \mathbb{P}^1$ with $D \cdot f \geq 0$. Then*

$$H^i(\mathbb{F}_n, \mathcal{O}(D)) \cong H^i(\mathbb{P}^1, \pi_* \mathcal{O}(D)).$$

Remark D.8. One can moreover show that $\pi_* \mathcal{O}_{\mathbb{F}_4} = \mathcal{O}_{\mathbb{P}^1}$. Hence

$$H^i(\mathbb{F}_n, \mathcal{O}) = 0$$

for $i > 0$.

Example D.9. We get that

$$\begin{aligned} \mathbb{F}_0 &= \mathbb{P}^1 \times \mathbb{P}^1 \\ \mathbb{F}_1 &= \mathrm{Bl}_p \mathbb{P}^2. \end{aligned}$$

Moreover, except for $n = 1$, $\mathbb{F}_n$ is a minimal model by the Enrique-Kodaira classification of surfaces.

Next we will fix some notation:

Setting D.10. *Let $\mathbb{F}_n$ be the Hirzebruch surface. We then denote by*

$$Z \in \mathcal{O}_{\mathbb{F}_4}(1)$$

the element that cuts out the section, i.e. $V(Z) = s$.
Moreover for simplicity we denote

$$\mathcal{O}_X(a, b) = \mathcal{O}_X(a) \otimes \pi^* \mathcal{O}_{\mathbb{P}^1}(b)$$

for every Hirzebruch surface $X = \mathbb{F}_n$.

List of Figures

References

[AB19] Kenneth Ascher and Dori Bejleri. Compact moduli of elliptic k3 surfaces. *arXiv preprint arXiv:1902.10686*, 2019.

[ABE20] Valery Alexeev, Adrian Brunyate, and Philip Engel. Compactifications of moduli of elliptic k3 surfaces: stable pair and toroidal. *arXiv preprint arXiv:2002.07127*, 2020.

[AHVAV17] Asher Auel, Brendan Hassett, Anthony Várilly-Alvarado, and Bianca Viray. *Brauer groups and obstruction problems: moduli spaces and arithmetic*, volume 320. Birkhäuser, 2017.

[AMS+10] Avner Ash, David Mumford, Peter Scholze, Michael Rapoport, and Yungsheng Tai. *Smooth compactifications of locally symmetric varieties*. Cambridge University Press, 2010.

[B+72] Armand Borel et al. Some metric properties of arithmetic quotients of symmetric spaces and an extension theorem. *Journal of Differential Geometry*, 6(4):543–560, 1972.

[BHPVdV15] Wolf Barth, Klaus Hulek, Chris Peters, and Antonius Van de Ven. *Compact complex surfaces*, volume 4. Springer, 2015.

[Bre72] Glen E. Bredon. *Introduction to compact transformation groups*. Academic press, 1972.

[Bru04] Jan H. Bruinier. *Borcherds products on O(2, l) and Chern classes of Heegner divisors*. Springer, 2004.

[Bru15] Adrian Brunyate. *A modular compactification of the space of elliptic K3 surfaces*. PhD thesis, University of Georgia, 2015.

[BvdGHZ08] Jan H. Bruinier, Gerard van der Geer, Günter Harder, and Don Zagier. *The 1-2-3 of modular forms: lectures at a summer school in Nordfjordeid, Norway*. Springer Science & Business Media, 2008.

[BZ19] Jan H. Bruinier and Shaul Zemel. Special cycles on toroidal compactifications of orthogonal shimura varieties. *arXiv preprint arXiv:1912.11825*, 2019.

[CD07] Adrian Clingher and Charles F. Doran. On k3 surfaces with large complex structure. *Advances in Mathematics*, 215(2):504–539, Nov 2007.

[CD14] Fabrizio Catanese and Michael Dettweiler. The direct image of the relative dualizing sheaf needs not be semiample. *Comptes Rendus Mathematique*, 352(3):241–244, 2014.

[CEZG+14] Eduardo Cattani, Fouad El Zein, Phillip A. Griffiths, et al. *Hodge Theory (MN-49)*. Princeton University Press, 2014.

[CLS11] David A. Cox, John B. Little, and Henry K. Schenck. *Toric varieties*. American Mathematical Soc., 2011.

[CS13] John H. Conway and Neil J. A. Sloane. *Sphere packings, lattices and groups*, volume 290. Springer Science & Business Media, 2013.

[DK07] Igor V. Dolgachev and Shigeyuki Kondō. Moduli of k3 surfaces and complex ball quotients. In *Arithmetic and geometry around hypergeometric functions*, pages 43–100. Springer, 2007.

[Dol95] Igor V. Dolgachev. Mirror symmetry for lattice polarized k3 surfaces. *arXiv preprint alg-geom/9502005*, 1995.

[Dun18] Bjørn I. Dundas. *A Short Course in Differential Topology*. Cambridge Mathematical Textbooks. Cambridge University Press, 2018.

[For12] Otto Forster. *Lectures on Riemann surfaces*, volume 81. Springer Science & Business Media, 2012.

[Fri84] Robert Friedman. A new proof of the global torelli theorem for k3 surfaces. *Annals of Mathematics*, pages 237–269, 1984.

[FS86] Robert Friedman and Francesco Scattone. Type iii degenerations of k3 surfaces. *Inventiones mathematicae*, 83(1):1–39, 1986.

[Fuj03] Osamu Fujino. A canonical bundle formula for certain algebraic fiber spaces and its applications. *Nagoya Mathematical Journal*, 172:129–171, 2003.

[Ful13] William Fulton. *Intersection theory*, volume 2. Springer Science & Business Media, 2013.

[Gre18] François Greer. Quasi-modular forms from mixed noether-lefschetz theory. *arXiv preprint arXiv:1809.06945*, 2018.

[Hai08] Richard Hain. Lectures on moduli spaces of elliptic curves. *arXiv preprint arXiv:0812.1803*, 2008.

[Har13] Robin Hartshorne. *Algebraic geometry*, volume 52. Springer Science & Business Media, 2013.

[HT15] Andrew Harder and Alan Thompson. The geometry and moduli of k3 surfaces. In *Calabi-Yau varieties: arithmetic, geometry and physics*, pages 3–43. Springer, 2015.

[Huy16] Daniel Huybrechts. *Lectures on K3 surfaces*. Cambridge University Press, 2016.

[Ill78] Sören Illman. Smooth equivariant triangulations of g-manifolds for g a finite group. *Mathematische Annalen*, 233(3):199–220, 1978.

[Ill83] Sören Illman. The equivariant triangulation theorem for actions of compact lie groups. *Mathematische Annalen*, 262(4):487–501, 1983.

[Kle80] Steven L. Kleiman. Relative duality for quasi-coherent sheaves. *Compositio Mathematica*, 41(1):39–60, 1980.

[KMPS10] Albrecht Klemm, Davesh Maulik, Rahul Pandharipande, and Emanuel Scheidegger. Noether-lefschetz theory and the yau-zaslow conjecture. *Journal of the American Mathematical Society*, 23(4):1013–1040, 2010.

[Kob12] Neal I. Koblitz. *Introduction to elliptic curves and modular forms*, volume 97. Springer Science & Business Media, 2012.

[Kon99] Shigeyuki Kondo. On the kodaira dimension of the moduli space of k3 surfaces ii. *Compositio Mathematica*, 116(2):111–118, 1999.

[KSB88] Janos Kollár and Nicholas Shepherd-Barron. Threefolds and deformations of surface singularities. *Inventiones mathematicae*, 91:299–338, 06 1988.

[KST11] Martijn Kool, Vivek Shende, and Richard P. Thomas. A short proof of the göttsche conjecture. *Geometry & Topology*, 15(1):397–406, 2011.

[L+03] Eduard Looijenga et al. Compactifications defined by arrangements, ii: Locally symmetric varieties of type iv. *Duke Mathematical Journal*, 119(3):527–588, 2003.

[Loo76] Eduard Looijenga. Root systems and elliptic curves. *Inventiones mathematicae*, 38(1):17–32, 1976.

[Mir89] Rick Miranda. *The basic theory of elliptic surfaces.* ETS, 1989.

[MP07] Davesh Maulik and Rahul Pandharipande. Gromov-witten theory and noether-lefschetz theory. *arXiv preprint arXiv:0705.1653*, 2007.

[NS] Gabriele Nebe and Neil J. A. Sloane. A catalogue of lattices. `http://www.math.rwth-aachen.de/~Gabriele.Nebe/LATTICES/E8.html`. Accessed: 2020-02-17.

[PS00] Aleksei N. Parshin and Igor R.s Shafarevich. *Algebraic Geometry III, Complex Algebraic Varieties. Algebraic curves and their Jacobians*, volume 102. Berlin: Georg Reimer, 1892-, 2000.

[SZ85] Joseph Steenbrink and Steven Zucker. Variation of mixed hodge structure. i. *Inventiones mathematicae*, 80(3):489–542, 1985.

[Zag08] Don Zagier. Elliptic modular forms and their applications. In *The 1-2-3 of modular forms*, pages 1–103. Springer, 2008.

[ZS10] Fouad E. Zein and Jawad Snoussi. Local systems and constructible sheaves. In Fouad El Zein, Alexandru I. Suciu, Meral Tosun, A. Muhammed Uludağ, and Sergey Yuzvinsky, editors, *Arrangements, Local Systems and Singularities*, pages 111–153, Basel, 2010. Birkhäuser Basel.