Gromov-Witten Theory of Elliptic Curves and K3 Surfaces

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Born February 17, 1997 in Madrid, Spain

August 14, 2021

Master's Thesis Mathematics

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Acknowledgements

Before anything else, I would like to thanks Prof. Dr. Georg Oberdieck and Dr. Johannes Schmitt, my advisors, for all the help, support, and patient during this year. It would have been impossible to finish this master thesis without your advice and supervision. Thank a lot for all the mathematics I have learned with you.

I would like to thank the German Academic Exchange Service (DAAD) for their economic support and assistance throughout my Master's studies, via the scholarship "Study Scholarships for Graduates of All Disciplines".

Special thanks to all the friends I have done at Bonn for their companionship, specially during the pandemic.

Finally, I must express my very profound gratitude to my family for all the support and care they have sent me from Spain during my Master's studies.

Abstract

In this master thesis we carry out the underlying theoretical study for computing Gromov-Witten invariants on Elliptic curves and $K3$ surfaces, and we develop an implementation of algorithms in SageMath. For elliptic curves, we follow the algorithm developed by A. Pixton in [45] and by A. Okounkov and R. Pandharipande in [43]. In the case of the $K3$ surface, we restrict our study to primitive classes following the algorithm described in [36]. In addition, we give a short guide for using the implemented code and we illustrate some computations by means of the performance of some examples.

Contents

Notation 1

1 Introduction 2

2 Gromov-Witten Theory of elliptic curves 6
- 2.1 Connected and disconnected generating series 7
- 2.2 Stationary invariants 9
- 2.3 Non stationary invariants 13
- 2.4 Hodge insertions 20
- 2.5 Invariants with tautological classes 21
- 2.6 Summary of the algorithm 35

3 Gromov-Witten Theory of K3 surfaces 37
- 3.1 K3 surfaces and elliptically fibered K3 fibers 38
- 3.2 Gromov-Witten invariants on K3 surfaces 40
- 3.3 Degeneration Formula 49
- 3.4 Product formula 64
- 3.5 Localization on $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1/0,\infty,1)$ 66
- 3.6 Rubber calculus 75
- 3.7 Summary of the algorithm 81

4 Guide for the code and examples of computations 84
- 4.1 Guide for the elliptic curve algorithm 84
- 4.2 Guide for the $K3$ surface algorithm 89

5 Conclusions and further considerations 96

References 97

Appendix A Introduction to Gromov-Witten Theory 101
- A.1 The moduli space of stable curves and the tautological ring 101
- A.2 The moduli spaces of stable maps and Gromov-Witten Theory 109
- A.3 The moduli space of relative stable maps and non-rigid targets 120

Notation

We briefly introduce the main notation and terminology to be used throughout this master thesis.

We denote by $\overline{\mathcal{M}}_{g,n}$ and $\mathfrak{M}_{g,n}$ the moduli space of stable and prestable curves respectively. In addition, given a stable graph Γ, $\overline{\mathcal{M}}_\Gamma$ denotes the product $\prod_{v\in V(\Gamma)} \overline{\mathcal{M}}_{g(v),n(v)}$ and $\xi_\Gamma : \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{g,n}$ denotes the gluing morphism. The tautological ring is denoted by $RH^*(\overline{\mathcal{M}}_{g,n})$ and the corresponding decorated stratum class by $[\Gamma, \alpha]$.

In Gromov-Witten theory, for X a nonsingular projective variety and $\beta \in H_2(X)$, we denote the moduli space of stable map by $\overline{\mathcal{M}}_{g,n}(X, \beta)$. We denote by ev_i : $\overline{\mathcal{M}}_{g,n}(X, \beta) \to X$ the evaluation morphisms, and ρ and $\overline{\rho}$ denote the projections to $\overline{\mathcal{M}}_{g,n}$ and $\mathfrak{M}_{g,n}$, respectively. The Gromov-Witten classes are denoted by

$$I^X_{g,n,\beta} := PD \circ \rho_* \left(\left[\overline{\mathcal{M}}_{g,n}(X, \beta)\right]^{\mathrm{vir}} \frown \prod_{i=1}^n \mathrm{ev}_i^*(\gamma_i) \right)$$

for $\gamma_1, \ldots, \gamma_n \in H^*(X)$. Given a tautological class μ, we denote by $\langle \mu; \gamma_1, \ldots, \gamma_n\rangle^X_{g,n,\beta}$ the Gromov-Witten invariant

$$\int_{\overline{\mathcal{M}}_{g,n}} \mu \, I^X_{g,n,\beta}(\gamma_1, \ldots, \gamma_n) = \int_{\left[\overline{\mathcal{M}}_{g,n}(X,\beta)\right]^{\mathrm{vir}}} \rho^*(\mu) \prod_{i=1}^n \mathrm{ev}_i^*(\gamma_i).$$

The descendent classes $\psi_i^{k_i} \mathrm{ev}_i^*(\gamma_i) \in H^*(\overline{\mathcal{M}}_{g,n}(X, \beta)$ are denoted by $\tau_{k_i}(\gamma_i)$, and $\langle \mu; \tau_{k_1}(\gamma_1) \cdots \tau_{k_n}(\gamma_n)\rangle^X_{g,n,\beta}$ denotes the descendent invariant defined as

$$\int_{\left[\overline{\mathcal{M}}_{g,n}(X,\beta)\right]^{\mathrm{vir}}} \rho^*(\mu) \prod_{i=1}^n \tau_{k_i}(\gamma_i).$$

The disconnected versions of $\overline{\mathcal{M}}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}(X, \beta)$ are denoted by $\overline{\mathcal{M}}^\bullet_{g,n}$ and $\overline{\mathcal{M}}_{g,n}(X, \beta)^\bullet$, respectively. The corresponding disconnected invariants will be denoted by $\langle - \rangle^\bullet$.

In the frame of the moduli space of relative stable maps, let D be a smooth divisor of X and μ a length m partition of $\int_\beta D$. We denote by $\overline{\mathcal{M}}_{g,n}(X/D, \mu, \beta)$ the moduli space of relative stable maps, and by ev_i^D the respective relative evaluation map. Let $\Delta = \mathbb{P}(N^\vee_{X/D} \oplus \mathcal{O}_X)$, we denote by $X[k]$ the k–degeneration of X and by $\Delta[k]$ the k copies of Δ inside $X[k]$. We denote by $\varepsilon : X[k] \to X$ to the projection contracting $\Delta[k]$ to D fiber-wise.

Let D be a smooth projective variety and $X = \mathbb{P}(\mathcal{L} \oplus \mathcal{O}_D)$ for $\mathcal{L}$ a line bundle on D. For $\beta \in H_(X)$ and for μ and ν partitions of $\int_\beta D_0$ and $\int_\beta D_\infty$, $\overline{\mathcal{M}}_{g,n}(X/D_0, D_\infty, \mu, \nu, \beta)^\sim$ denotes the corresponding moduli space of non rigid stable maps.

For a detail introduction of the above concepts we refer to Appendix A.

1 Introduction

The theoretical framework of this master thesis is Gromov-Witten theory. So, it seems natural to start this introduction devoting some words to motivate the role of this theory in Algebraic Geometry and, more precisely, in Enumerative Geometry. Enumerative Geometry studies, among others, problems as counting subvarieties inside a given variety. For example, one can ask how many lines are contained in a cubic surface or on a generic quintic threefold (27 and 2785, respectively). However, in general, the computation of these integers is remarkable hard. On could say that Gromov-Witten theory arises for approaching this type of problems from a different perspective. For example, instead of counting the number of lines inside a quintic threefold, Gromov-Witten theory focuses on maps from genus 0 curves to the threefold. The idea behind Gromov-Witten theory is thus to construct an space collecting all those maps to afterward apply intersection theory to reach the answer. Roughly speaking, the Gromow Witten invariants are intersection numbers over this space. These invariants are one of the main objects to study inside Gromov-Witten theory, and an important tool inside the field of Enumerative Geometry.

In this master thesis we study the Gromov-Witten theory of elliptic curves and K3 surfaces. The main goal is to give a complete presentation of an algorithm for computing their Gromov-Witten invariants and to develop a SageMath implementation for it.

The algorithm for $K3$ surfaces will require the previous computation of the invariants on elliptic curves. As a consequence, the first treated case will be the case of the elliptic curve. For this purpose, let E be an elliptic curve, i.e., a smooth genus 1 projective curve. The geometric intuition behind the invariants in this setting is to count some covers of the elliptic curve by some other curve. Moreover, this study, as mentioned above, will help to compute invariants on other varieties as $K3$ surfaces. Furthermore, the product formula (see Subsection 3.4), proven by K. Behrend in [5], allows to compute invariants on products by the invariants on each factor of the product. In particular, one can compute invariants on $\mathbb{P}^1 \times E$ by means of invariants on E using the product formula and Theorem 1 in [14].

Let us go into the details a bit on how to deal with these invariants. Effective curve classes correspond to the classes $d[E]$ for d a non-negative integer. As a consequence, we can gather the Gromov-Witten invariants in a generating series as follows

$$\langle \mu; \gamma_1, \ldots, \gamma_n \rangle_{g,n}^E = \sum_{d \geq 0} \langle \mu; \gamma_1, \ldots, \gamma_n \rangle_{g,n,d[E]}^E q^d.$$

Thus, instead of computing Gromov-Witten invariants over E for a particular class β, we will directly determine these generating series. More precisely, we will see that these generating series are quasimodular forms. The idea then is to first use the Gromov-Witten axioms (see Appendix A.2) to reduce the computation of these invariants to the determination of descendent invariants of the form

$$\langle \tau_{k_1}(\gamma_1)\cdots\tau_{k_n}(\gamma_n)\rangle_{g,n}^{E}.$$

The algorithm for computing these descendent invariants was developed by A. Pixton in [45] and by A. Okounkov and R. Pandharipande in [43]. This algorithm is structured in three reduction steps: First, the invariants are written by means of disconnected invariants (see Subsection 2.1). These invariants are defined analogously to the Gromov-Witten invariants but for the moduli space of possibly disconnected stable maps (see Appendix A.2). Then, for computing the disconnected invariants, the main tool is the Virasoro operators. These operators were introduced in the frame of nonsingular curves in [43] by A. Okounkov and R. Pandharipande (see Subsection 2.3). Using these operators the invariants can be written by means of stationary invariants (invariants where all evaluation classes are the class of a point; see Subsection 2.2). Finally, the computation of stationary invariants is solved in [43] (see Subsection 2.2).

Let us now deal with the second part of this work, namely, Gromov-Witten invariants of $K3$ surfaces. In the case of the $K3$ surface, we can associate enumerative interpretations to some Gromov-Witten invariants. However, a problem with the virtual fundamental class arises: the virtual fundamental class vanishes for $\beta \neq 0$; here β represents the effective curve class. Nevertheless, this difficulty is overcome with the definition of the reduced virtual class (see Subsection 3.2)

$$\left[\overline{\mathcal{M}}_{g,n}(S,\beta)\right]^{\text{red}} \in H_{2(g+n)}(\overline{\mathcal{M}}_{g,n}(S,\beta)),$$

where S is a $K3$ surface. For the construction of this class we refer to [9] and [38]. Using the reduced virtual class we can define again reduced Gromov-Witten invariants. As expected, we can find enumerative interpretations to these rational numbers. For example, consider the invariant

$$N_g(h) = \langle 1; \mathbf{p}, \ldots, \mathbf{p}\rangle_{g,g,\beta}^{S}$$

where $\mathbf{p}$ is the class of a point and β is a primitive class with $\langle \beta, \beta\rangle = 2h-2$. It is known that these invariants coincide with the number of genus g curves in S passing through g generic points and with h nodes. These invariants can be computed through the Yau Zaslow formula, first proven by Bryan and Leung in [9].

In contrast to the elliptic curve case, an algorithm for computation of Gromov-Witten invariants is not known for all possible choices of the effective curve class β. However, for the case where β is primitive such algorithm was provided by D. Maulik, R. Pandharipande, and R. P. Thomas in [36]. Let β_h be a primitive effective curve class where h denotes the non-negative integer given by $\langle \beta, \beta\rangle = 2h-2$. By deformation invariance, the invariants with class β_h only depend on h. As a result we can specialize to an elliptic K3 surface with section and consider the invariants in the generating series

$$\langle \mu; \gamma_1, \ldots, \gamma_n \rangle^S_{g,n} = \sum_{h \geq 0} \langle \mu; \gamma_1, \ldots, \gamma_n \rangle^S_{g,n,\beta_h} q^{h-1}.$$

where β_h is taken to be the sum of the section class plus h copies of the fiber class. Again, we will focus on the computation of these generating series. In [36], D. Maulik, R. Pandharipande, and R. P. Thomas proved that these invariants lie in $\frac{1}{\Delta(q)}\mathbf{QM}$ where $\Delta(q)$ denotes the discriminant quasimodular form and $\mathbf{QM}$ denotes the algebra of quasimodular forms. To prove these results, the authors gave an explicit algorithm for computing these invariants. Our goal is to carry out the theoretical study of this algorithm. Some of the most important tools that we used are the degeneration formula, proven by J. Li in [34], the product formula, and the virtual localization formula, first introduced in the virtual frame by T. Graber and R. Pandharipande in [19]. These three results allow us to reduce the computation of the invariants on $K3$ surfaces to invariants on elliptic curves.

The algorithms discussed above have been implemented in SageMath (see [48]) as part of this thesis. The implemented code computes Gromov-Witten invariants on the elliptic curve and $K3$ surfaces. Moreover, the program can also be used to compute relative invariants on $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1 \times E/E, (1), s+hf)$ and $\overline{\mathcal{M}}_{g,n}(S/E, (1), s+hf)$, and the n–point correlation function. Other implemented functions allow us to check some conjectures on the invariants for base cases, as we will see in Section 4. In the implementation, the manipulation of tautological classes is required in different steps of the algorithms. The implementation of the tools required for dealing with the tautological classes was done by V. Delecroix, J. Schmitt, and J. van Zelm in the SageMath package `admcycles`. In [12] the authors give a short introduction on how to use the code. As a consequence, our program highly relies on `admcycles` to deal with tautological classes.

Summarizing, in this master thesis we analyze the required theory for computing the Gromov-Witten invariants of elliptic curves, and of $K3$ surfaces. As a consequence, we outline the algorithms for computing Gromov-Witten invariants on elliptic curves and, in the case of $K3$ surfaces, for primitive classes. Moreover, we have developed an SageMath implementation of both algorithms and we have given a brief introduction of how to use it.

The master thesis is structured in 4 sections. Section 2 is devoted to the theoretical study of the algorithm for computing invariants on elliptic curves. Subsections 2.1, 2.2, 2.3, and 2.4 are devoted to compute descendent invariants while Subsection 2.5 shows how to reduce Gromov-Witten invariants with tautological classes to descendent invariants with the help of the Gromov-Witten axioms. Finally, in Subsection 2.6 we give a summary of the algorithm developed in the previous subsections. Section 3 is devoted to the Gromov-Witten theory of $K3$ surfaces. In its first subsection we introduce some properties of $K3$ surfaces that are needed for the algorithm. In Subsection 3.2, we introduce the reduced Gromov-Witten invariant and we begin the study of the algorithm. In Subsections 3.3, 3.4, 3.5, and 3.6, we reduce the computation of the invariants to the elliptic curve case. Again, the last subsection of Section 3 summarizes the algorithm developed along the section. In Section 4, we present a brief guide on how to use the main functions of the program implemented, and we illustrate it by some examples. In Section 5 we briefly expose the conclusions and future lines of the thesis. In addition to the above sections, in Appendix A we give an introduction to Gromov-Witten theory and some important notions that will appear all along this thesis as the tautological ring, the Gromov-Witten axioms, or the moduli space of relative stable maps.

2 Gromov-Witten Theory of elliptic curves

In this section, we assume that X is an elliptic curve that we denote by E, i.e. E is a smooth projective genus 1 curve. We are interested in computing the Gromov-Witten invariants of the form

$$\langle \mu; \gamma_1, \ldots, \gamma_n \rangle^E_{g,n,\beta} := \int_{\overline{\mathcal{M}}_{g,n}} \mu \, I^E_{g,n,\beta}(\gamma_1, \ldots, \gamma_n) = \int_{\left[\overline{\mathcal{M}}_{g,n}(E,\beta)\right]^{\mathrm{vir}}} \rho^*(\mu) \prod_{i=1}^{n} \mathrm{ev}_i^*(\gamma_i)$$

for $g, n \geq 0$, $\beta \in H_2(E)$, $\mu \in RH^*(\overline{\mathcal{M}}_{g,n})$, and $\gamma_1, \ldots, \gamma_n \in H^*(E)$. However, it is important to observe that, since we have fixed X to be an elliptic curve E, we have an explicit description of the cohomology and homology rings of E. In particular, this implies that $H_2(E)$ is generated by $[E]$ and by the effectivity axiom (see Appendix 3.2), the possible choices of β are $d[E]$ for $d \in \mathbb{N}$. Therefore, given $\mu \in RH^*(\overline{\mathcal{M}}_{g,n})$ and $\gamma_1, \ldots, \gamma_n \in H^*(E)$, we can combine all the invariants $\langle \mu; \gamma_1, \ldots, \gamma_n \rangle^E_{g,n,d[E]}$ into a generating series, namely,

$$\langle \mu; \gamma_1, \ldots, \gamma_n \rangle^E_{g,n} = \sum_{d \geq 0} \langle \mu; \gamma_1, \ldots, \gamma_n \rangle_{g,n,d[E]} \, q^d. \tag{1}$$

In general, the virtual dimension of each of the terms of the generating series might change since the class β changes. As a result, it might happen that all the terms are zero except one by the degree axiom. However, in the case of the elliptic curve it holds that $\int_\beta c_1(\omega_E) = 2g - 2 = 0$, and hence the virtual dimension stays constant along the generating series.

The main goal of this section is to study the algorithm for computing these generating series. Note that if we have computed the generating series, we can compute invariants for $d \geq 0$ just by extracting the corresponding coefficient of the series. Thus, the curve class can be omitted as part of the input of the algorithm, leaving only the tautological class, and the evaluation classes, as the only data appearing in the input. In addition, taking into account Theorem A.1.3, we can assume that the tautological class is a decorated stratum class $[\Gamma, \mu]$.

For the evaluation classes, we have a Hodge structure on the cohomology of E , i.e., there exists a basis $\mathcal{B} = \{1, \alpha, \beta, \omega\}$ of $H^*(E)$ where $1 = [E] \in H^0(E)$, $\alpha \in H^{10}(E)$ and $\beta \in H^{01}(E)$ with $H^1(E) = H^{10}(E) \oplus H^{01}(E)$, and $\omega = [\mathrm{pt}] \in H^2(E)$ (we recall that pt denotes a point), satisfying $\alpha\beta = \omega$, $\alpha^2 = 0$ and $\beta^2 = 0$. By the linearity axiom (see Appendix A.2), we may assume that the evaluation classes γ_i lie in the basis $\mathcal{B}$.

As a consequence of the theoretical study of the computation of these invariants, we will see in Corollary 2.5.3 that the invariants $\langle \mu; \gamma_1, \ldots, \gamma_n \rangle^E_{g,n}$ are quasimodular forms. We recall that the algebra of quasimodular forms **QM** is generated by the Einsentein series E_2, E_4, and E_6 (we refer to [50] for an introduction to quasimodular forms). As a result, the output of our algorithm will be a polynomial in these three quasimodular forms or possible normalizations of them.

Summarizing, the final goal of this section is to compute the invariant $\langle [\Gamma, \mu] \, ; \gamma_1, \ldots, \gamma_n \rangle$, as a polynomial in E_2, E_4, and E_6, from the knowledge of a decorated stratum class $[\Gamma, \mu]$ and $\gamma_1, \ldots, \gamma_n \in \mathcal{B}$. For this purpose, this section is structured in 6 subsections. The first four subsections will be devoted to the computation of invariants of the form

$$\langle \lambda_{l_1} \cdots \lambda_{l_m} ; \tau_{k_1}(\gamma_1) \cdots \tau_{k_n}(\gamma_n) \rangle^E_{g,n} := \sum_{d \geq 0} \int_{\left[\overline{\mathcal{M}}_{g,n}(E,d)\right]^{\text{vir}}} \lambda_{l_1} \cdots \lambda_{l_m} \tau_{k_1}(\gamma_1) \cdots \tau_{k_n}(\gamma_n) q^d,$$

where $\tau_{k_i}(\gamma_i) = \psi_i^{k_i} \mathrm{ev}_i^*(\gamma_i)$. As commented in the introduction, we refer to these invariants as descendent invariants. More concretely, in Subsection 2.1 we express these connected invariants by means of the disconnected ones. In Subsection 2.2 we study the case in which all evaluation classes are ω and we have no λ classes. These invariants are called stationary invariants. In Subsection 2.3 we reduce the computations of the descendent invariants without Hodge classes to stationary invariants, and in Subsection 2.4 we deal with the λ classes. For these subsections, we have mainly followed the references [43], [44], [45] and [49]. In Subsection 2.5 we will see how to reduce the computation of the invariants $\langle [\Gamma, \mu] \, ; \gamma_1, \ldots, \gamma_n \rangle$ to the ones studied in the first part of the section using the Gromov-Witten Axioms. Subsection 2.6 consists in a summary of the reduction steps of the algorithm studied throughout the previous subsections of this section.

2.1 Connected and disconnected generating series

We start this subsection recalling that, as it is done for $\overline{\mathcal{M}}_{g,n}$, one can define ψ, the λ classes on $\overline{\mathcal{M}}_{g,n}(X, \beta)$, and the descendent classes $\tau_{k_i}(\gamma_i) = \psi_i^{k_i} \mathrm{ev}_i^*(\gamma_i)$ for $\gamma_i \in H^*(X)$ (see Appendix A.2). Furthermore, as commented above, the task to be accomplished in the next subsections is to compute invariants with these classes, i.e. invariants of the form

$$\langle \Lambda ; \tau_{k_1}(\gamma_1) \cdots \tau_{k_n}(\gamma_n) \rangle^E_{g,n,d[E]} := \int_{\left[\overline{\mathcal{M}}_{g,n}(E,\beta)\right]^{\text{vir}}} \Lambda \tau_{k_1}(\gamma_1) \cdots \tau_{k_n}(\gamma_n),$$

for $\Lambda = \lambda_{l_1} \cdots \lambda_{l_m}$ and $\gamma_1, \ldots, \gamma_n \in H^*(E)$. Using the effectivity axiom (see Appendix 3.2), and that E has genus 1, we can gather all these invariants in the generating series

$$\langle \Lambda ; \tau_{k_1}(\gamma_1) \cdots \tau_{k_n}(\gamma_n) \rangle^E_{g,n} := \sum_{d \geq 0} \int_{\left[\overline{\mathcal{M}}_{g,n}(E,d)\right]^{\text{vir}}} \Lambda \tau_{k_1}(\gamma_1) \cdots \tau_{k_n}(\gamma_n) q^d. \tag{2}$$

We will refer to these invariants as descendent invariants. Moreover, from the degree axiom, it holds that

$$2(2(g-1)+n) = \sum_{j=1}^{m} 2l_j + \sum_{i=1}^{n} 2k_i + \deg(\gamma_i).$$

Thus, the genus g can be expressed in terms of n and the degree of Λ and the descendent classes. We will omit the genus from our invariants since we will assume that it is fixed as the degree axiom indicates. Moreover, during this section, all invariants will be taken over E. As a result we will omit E from the notation and we will denote by $\langle \Lambda; \tau_{k_1}(\gamma_1)\cdots\tau_{k_n}(\gamma_n)\rangle$ the descendent invariants defined in equation (2)

The idea for computing the generating series (2) is to express it by means of a disconnected analogous generating series (see Appendix A.2 for an introduction to disconnected Gromov-Witten invariants). Subsections 2.2, 2.3, and 2.4 will focus on solving the disconnected case. To find this relation between connected and disconnected invariants we first define the disconnected generating series as:

$$\langle \Lambda; \tau_{k_1}(\gamma_1)\cdots\tau_{k_n}(\gamma_n)\rangle^\bullet := \prod_{k\geq 1}(1-q^k)\sum_{d\geq 0}\int_{\left[\overline{\mathcal{M}}_{g,n}(E,d)^\bullet\right]^{\mathrm{vir}}} \Lambda\tau_{k_1}(\gamma_1)\cdots\tau_{k_n}(\gamma_n)q^d \tag{3}$$

with $\Lambda = \lambda_{l_1}\cdots\lambda_{l_m}$, where $\overline{\mathcal{M}}_{g,n}(E,d)^\bullet$ denotes the moduli space of stable maps with possibly disconnected domain; at the end of the Appendix A.2 the spaces $\overline{\mathcal{M}}_{g,n}^\bullet$ and $\overline{\mathcal{M}}_{g,n}(X,\beta)^\bullet$ are introduced.

One can check that this disconnected versions of $\overline{\mathcal{M}}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}(E,d)$ are disjoint union of product of the respective connected moduli spaces. As a result, the fundamental classes and virtual fundamental classes of $\overline{\mathcal{M}}_{g,n}^\bullet$ and $\overline{\mathcal{M}}_{g,n}(X,\beta)^\bullet$ split as sum of products of the respective classes in the connected case. As a consequence, the disconnected invariants can be expressed as sum of products of connected invariants. This formula is described in the following proposition (see [45] Proposition 3.1.1.).

Proposition 2.1.1. *Let $I = \lambda_{l_1}\cdots\lambda_{l_m}\tau_{k_1}(\gamma_1)\cdots\tau_{k_n}(\gamma_n)$ then*

$$\langle I\rangle^\bullet = \sum_{\{1,\ldots,n\}=\sqcup_{J\in S}J}\mathrm{sign}(S)\sum_{\sum_{J\in S}b_{J,j}=l_j}\prod_{J\in S}\langle \lambda_{b_{J,1}}\cdots\lambda_{b_{J,m}};\tau_{k_i}(\gamma_i) : i\in J\rangle. \tag{4}$$

The first sum is indexed over all possible partitions of $\{1,\ldots,n\}$ while the second sum is indexed over the partitions of length $J\in S$ of each integer l_j corresponding to the Hodge index of λ_{l_j}. For each partition S of $\{1,\ldots,n\}$, sign(S) *is the sign corresponding to ordering the descendent classes of I such that the descendent classes associated to each subset of the partition are together. In other words,* sign(S) *is the sign of a permutation mapping $(1,\ldots,n)$ to $(J : J\in S)$.*

Note that by the degree axiom, either disconnected or connected invariants with an odd number of odd descendent classes are zero. As a result, the choice of order of each partition S does not change the sign of the partition.

Proposition 2.1.1 allows us to compute disconnected invariants in terms of connected invariants. However, we are interested in writing connected invariants in terms of disconnected invariants since these are the ones we we will actually compute. This can be done using recursively Proposition 2.1.1 as follows:

- If $n = 1$, equation (4) states that the disconnected and connected invariants coincide.
- If $n > 1$, the sum of right hand side of equation (4) splits as the term corresponding to the partition $S = \{\{1, \ldots, n\}\}$ which coincides with the connected invariant of the insertion, and the sum over non trivial partitions of $\{1, \ldots, n\}$, that we will denote by C_I. Thus, one gets that

$$\langle I \rangle = \langle I \rangle^\bullet - C_I.$$

Every invariant in C_I has at most $n-1$ descendent classes and, hence, we can apply recursively the same argument, expressing C_I in terms of disconnected invariants

For example, for $n = 2$ we have that

$$\langle \tau_{k_1}(\gamma_1)\tau_{k_2}(\gamma_2)\rangle = \langle \tau_{k_1}(\gamma_1)\tau_{k_2}(\gamma_2)\rangle^\bullet - \langle \tau_{k_1}(\gamma_1)\rangle^\bullet \langle \tau_{k_2})(\gamma_2)\rangle^\bullet,$$

and for $n = 3$ the formula is

$$\begin{aligned}\langle \tau_{k_1}(\gamma_1)\tau_{k_2}(\gamma_2)\tau_{k_3}(\gamma_3)\rangle = \; & \langle \tau_{k_1}(\gamma_1)\tau_{k_2}(\gamma_2)\tau_{k_3}(\gamma_3)\rangle^\bullet - \langle \tau_{k_1}(\gamma_1)\tau_{k_2}(\gamma_2)\rangle^\bullet \langle \tau_{k_3}(\gamma_3)\rangle^\bullet \\ & -\langle \tau_{k_1}(\gamma_1)\rangle^\bullet \langle \tau_{k_2}(\gamma_2)\tau_{k_3}(\gamma_3)\rangle^\bullet - \langle \tau_{k_1}(\gamma_1)\tau_{k_3}(\gamma_3)\rangle^\bullet \langle \tau_{k_2}(\gamma_2)\rangle^\bullet \\ & -\langle \tau_{k_1}(\gamma_1)\rangle^\bullet \langle \tau_{k_2})(\gamma_2)\rangle^\bullet \langle \tau_{k_3}(\gamma_3)\rangle^\bullet.\end{aligned}$$

In the following subsections we will see how to compute the invariants (3) proving that they are quasimodular forms. As a result, taking into account Proposition 2.1.1, the connected invariants will also be quasimodular forms.

2.2 Stationary invariants

The main idea behind the algorithm for computing disconnected Gromov-Witten invariants over a elliptic curve is to reduce the computation to the case of stationary invariants. In this subsection we will see how to determine this type of invariants. In particular, Theorem 2.2.1 allows us to identify stationary invariants with the coefficients of the n–point correlation function. We present several ways of computing this function in Theorems 2.2.2 and 2.2.3. Moreover, we will conclude that stationary invariants are quasimodular form and we will compute them as polynomials in the normalization of the Eisenstein series G_2, G_4, and G_6. The main references of this subsection are [44] and [45]; for the results related to the n–point correlation function see [8] and [49].

Definition 2.2.1. *An stationary invariant is a disconnected invariant without Hodge classes where all the evaluation classes are* $\omega = [\mathrm{pt}]$, *i. e. an invariant of the form*

$$\langle \tau_{k_1}(\omega) \cdots \tau_{k_n}(\omega)\rangle^\bullet.$$

These invariants are the basis of our algorithm. As mentioned above, this subsection will focus on computing these invariants, while the rest of the section will be devoted to express more general types of invariants in terms of stationary invariants. As a consequence, the quasimodularity of the invariants (1) will derive from the quasimodularity of the stationary invariants that we will show in this subsection as an outcome of Theorem 2.2.1. In order to state this result we have to define first the n–point correlation function.

Following the notation of [8], let $\mathcal{P}$ denote the set of all integer partitions and let $f : \mathcal{P} \to \mathbb{Q}$ be an arbitrary function. We define the bracket

$$\langle f \rangle_q = \frac{\displaystyle\sum_{\lambda\in\mathcal{P}} f(\lambda) q^{|\lambda|}}{\displaystyle\sum_{\lambda\in\mathcal{P}} q^{|\lambda|}}$$

where $\mu = (\mu_1 \geq \mu_2 \geq \cdots)$ and $|\mu| = \sum_i \mu_i$.

Definition 2.2.2. *For $n \geq 1$ the n–point correlation function F_n is defined as*

$$F_n := \left\langle \prod_{k=1}^{n} \sum_{i\geq 1} e^{(\mu_i - i + \frac{1}{2})z_k} \right\rangle_q = \frac{\displaystyle\sum_{\lambda\in\mathcal{P}} \prod_{k=1}^{n} \sum_{i\geq 1} e^{(\mu_i - i+\frac{1}{2})z_k} q^{|\lambda|}}{\displaystyle\sum_{\lambda\in\mathcal{P}} q^{|\lambda|}}.$$

For $n = 0$ we fix $F_0 = 1$.

One can show, using that $\sum_{i\geq 1} \mathrm{e}^{(\mu_i - i+\frac{1}{2})z}$ admits a meromorphic expansion of the form $\frac{1}{z} + O(z)$ around zero, that $F_n(z_1, \ldots, z_n) \in \frac{1}{z_1\cdots z_n}\mathbb{Q}[[q]][z_1, \ldots, z_n]$. We will be interested in the coefficients of this power expansion.

The definition of the n–point correlation function might seem distant from the notion of stationary invariants. The next theorem is the central result of this subsection and relates both notions (see [44] Theorem 5 or [45] Theorem 3.2.2):

Theorem 2.2.1. *For $k_1, \ldots, k_n \in \mathbb{N}$, it holds that*

$$\langle \tau_{k_1}(\omega) \cdots \tau_{k_n}(\omega) \rangle^\bullet = [z_1^{k_1+1}, \ldots, z_n^{k_n+1}] F_n(z_1, \ldots, z_n),$$

where $[z_1^{k_1}, \ldots, z_n^{k_n}]F_n$ represents the coefficient of F_n corresponding to the term $z_1^{k_1} \cdots z_n^{k_n}$ as an element in $\frac{1}{z_1\cdots z_n}\mathbb{Q}[[q]][z_1, \ldots, z_n]$.

Hence, this theorem allows us to compute stationary invariants by determining the power expansion of F_n as an element in $\frac{1}{z_1\cdots z_n}\mathbb{Q}[[q]][z_1, \ldots, z_n]$. So, the next step is to study different ways of computing F_n. Moreover, we will see that its coefficients are quasimodular form and, hence, we are interested in computing the coefficients as a polynomial in the three quasimodular forms that generates **QM**; in the sequel we will denote by **QM** the algebra of the quasimodular forms. In the following we introduce these three generators of **QM** (see [50] for more details). We denote by E_k the k–Eisenstein series defined as

$$E_k = 1 - \frac{2k}{B_k} \sum_{k \geq 1} \sigma_{k-1}(n) q^n$$

where B_K is a Bernoulli number and $\sigma_k(n) = \sum_{d|n} d^k$. Then, it is known that $\mathbf{QM} = \mathbb{Q}[E_2, E_4, E_6]$. We will be interested in two other normalizations of these three quasimodular forms, in particular we will consider

$$G_k = -\frac{B_k}{2k} E_k \quad \text{and} \quad C_k = -\frac{B_k}{k\, k!} E_k$$

So we want to express the coefficients of F_n as polynomials in G_2, G_4, and G_6. To do so, the genus 1 theta function will play a crucial role.

Let Θ be the genus 1 theta function normalized such that $\Theta^{'}(0) = 1$, i.e.

$$\Theta(z) = (e^{\frac{z}{2}} - e^{-\frac{z}{2}}) \prod_{k \geq 1} \frac{(1 - q^k e^z)(1 - q^k e^{-z})}{(1 - q^k)^2}.$$

In Theorems 2.2.2 and 2.2.3 we will see how to compute F_n in two different ways using Θ. In this sense, we are interested in a proper expression for the power expansion of Θ in which the coefficients are elements in **QM**. The following two propositions provide us two different ways of reaching this expression (see [49] equality (14), and [45] equality (4.3) respectively).

Proposition 2.2.1. *Let $D = q\frac{\partial}{\partial q}$ be the differentiation operator $D : \mathbf{QM} \to \mathbf{QM}$. D acts on the generators of* **QM** *by*

$$D(E_2) = \frac{E_2^2 - E_4}{12}, \quad D(E_4) = \frac{E_2 E_4 - E_6}{3}, \quad D(E_6) = \frac{E_2 E_6 - E_4^2}{2}.$$

Then, $\Theta = \sum_{i \geq 0} H_i(q) z^{i+1}$ where $H_i(q)$ is the quasimodular form defined inductively by

$$H_0 = 1, \quad H_1 = 0, \quad H_i = \frac{1}{4i(i+1)}(8D(H_{i-2}) + E_2 H_{i-2}) \text{ for } i > 1.$$

Proposition 2.2.2. *For $k \geq 2$, let $C_k := -\dfrac{B_k}{k \cdot K!} E_k$ be a normalization of the Eisenstein series. Then,*

$$\Theta(z) = z\, e^{-\sum_{k\geq 1} C_{2k} z^{2k}}.$$

The drawback of Proposition 2.2.2 is that the coefficients of the power expansions are polynomials in the variables C_{2k}, and not in G_2, G_4, and G_6. However, this is solved using the following recursion expression for G_{2k} in terms of C_2, C_4, and C_6:

$$G_{2k} = 6\frac{(2k-2)!}{(k-3)(2k+1)}\sum_{l\geq 2}^{k-1}\frac{G_{2j}G_{2(k-j)}}{(2j-2)!(2(k-j)-2)!}$$

We recall that $C_{2k} = \frac{k!}{2}G_{2k}$.

After the study of Θ, we can state properly the results that allow us to compute F_n. Theorems 2.2.2 and 2.2.3, proven in [8] and [49] respectively, give us two explicit formulas for determining the n–point correlation function in term of Θ.

Theorem 2.2.2. *For $n > 0$ it holds that*

$$F_n(z_1,\dots,z_n) := \sum_{\sigma\in S_n}\sigma\left(\frac{\det\left(\frac{\Theta^{(j-i+1)}(z_1+\cdots+z_{n-j})}{(j-i+1)!}\right)_{1\leq i,j\leq n}}{\Theta(z_1)\Theta(z_1+z_2)\cdots\Theta(z_1+\cdots+z_n)}\right) \tag{5}$$

where the sum is indexed over the set S_n of the permutations of $\{1,\dots,n\}$, each permutation acts on the term of the sum by permuting the indexes of $z_1,\dots,z_n$, and $\frac{1}{k!}$ is zero for $k<0$.

Theorem 2.2.3. *For $n > 0$, the n–point correlation functions satisfy the following recursion:*

$$\Theta(z_1+\cdots+z_n)F_n(z_1,\dots,z_n) = \sum_{I\subsetneq\{1,\dots,n\}}(-1)^{n-1-|I|}\Theta^{(n-|I|)}\left(\sum_{i\in I}z_i\right)F_{|I|}(\{z_i\}_{i\in I}). \tag{6}$$

For $n=0$, we set $F_0 = 0$.

For example, using these theorems, one can check that

$$F_1(z) = \frac{1}{\Theta(Z)} = \frac{1}{z}\left(\sum_{k\geq 0}H_i(q)z^k\right)^{-1} = \frac{1}{z}e^{\sum_{k\geq 1}C_{2k}z^{2k}}$$

One can extract from this expression some particular invariants as:

$$\langle\tau_0(\omega)\rangle^\bullet = G_2,\quad \langle\tau_2(\omega)\rangle^\bullet = \frac{G_2^2}{2}+\frac{G_4}{12},\ \text{and}\ \langle\tau_4(\omega)\rangle^\bullet = \frac{G_2^3}{6}+\frac{G_2G_4}{12}+\frac{G_6}{360}.$$

As a result, we can prove the quasimodularity of stationary invariants.

Corollary 2.2.1. *For every $k_1,\dots,k_n$ non negative integers, $\langle\tau_{k_1}(\omega)\cdots\tau_{k_n}(\omega)\rangle^\bullet$ and $\langle\tau_{k_1}(\omega)\cdots\tau_{k_n}(\omega)\rangle$ are quasimodular forms.*

Proof. Using Proposition 2.1.1, it is enough to check the statement for disconnected invariants. Then, Proposition 2.2.1, together with Theorem 2.2.3, implies that the coefficients of the power expansion of F_n are quasimodular forms. Hence, now, the proof follows from Theorem 2.2.1. □

2.3 Non stationary invariants

After studying the stationary invariants the next step is to analyze the case where other evaluation classes are allowed. This subsection will be devoted to answer this question. More concretely, we will see how to express the invariants $\langle \tau_{k_1}(\gamma_1)\cdots\tau_{k_n}(\gamma_n)\rangle^\bullet$, for $\gamma_i \in \mathcal{B} = \{1, \alpha, \beta, \omega\}$, in terms of stationary invariants. The main tool for achieving this goal is the Virasoro Operators, introduced in [43] for relative and absolute Gromov-Witten invariants over nonsingular curves. We will follow this reference to introduce the operators in a more general frame. Then, we will restrict the definition, and the main results, to the case of the elliptic curve following Section 3.3 of [45], giving more explicit formulas for the Virasoro operators.

Coming back to a more general setting, let X be a nonsingular projective curve of genus g over $\mathbb{C}$. Let $\{1, \alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g, \omega\}$ be a basis of $H^*(X)$ where $\omega \in H^2(X)$ is the Poincaré dual of a point, $\alpha_i\beta_j = \delta_{ij}\omega$ and $\alpha_i^2 = \beta_i^2 = 0$. Let $q_1, \ldots, q_m$ be distinct points in X and $\eta^1, \ldots, \eta^m$ be partitions of d. In this section we will consider disconnected Gromov-Witten invariants of X relative to $q_1, \ldots, q_m$ in the moduli space $\overline{\mathcal{M}}_{g,n}(X/q_i, \eta^1, \ldots, \eta^m)$, that we will denote by $\langle I; \eta^1, \ldots, \eta^m\rangle^\bullet$. We can recover the absolute Gromov-Witten theory with the case $m = 0$. We fix X^* to be $X^* = X\backslash\{q_1, \ldots, q_m\}$ which implies that $\mathcal{X}(X^*) = 2 - 2g - m$.

For each descendent $\tau_{k_0}(1)$, $\tau_{k_i}(\alpha_i)$, $\tau_{k_j}(\beta_j)$ and $\tau_{k_1}(\omega)$, we introduce the formal variables $t^0_{k_0}$, $s^i_{k_i}$, $\overline{s}^j_{k_j}$ and $t^1_{k_1}$ respectively, with $s^i_{k_i}\overline{s}^j_{k_j} = -\overline{s}^j_{k_j}s^i_{k_i}$. The idea is to, using these variables, encode all possible disconnected invariants in a generating series and then find some operators annihilating this series. To do so, we denote by ξ the formal sum

$$\xi := \sum_{k\geq 0}(t^0_k\tau_k(1) + t^1_k\tau_k(\omega)) + \sum_{i=1}^{g}\sum_{k\geq 0}(s^i_k\tau_k(\alpha_i) + \overline{s}^i_k\tau_k(\beta_i)),$$

i.e., ξ is the formal sum of all descendent classes together with their respective variables. Now, let $Z_d[\eta^1, \ldots, \eta^m] \in \mathbb{Q}[[t^0_{k_0}, s^i_{k_i}, \overline{s}^j_{k_j}, t^1_{k_1}]]$ be the generating series

$$Z_d[\eta^1, \ldots, \eta^m] = \sum_{n\geq 0}\frac{1}{n!}\langle \xi^n; \eta^1, \ldots, \eta^m\rangle^\bullet$$

where $\langle \xi^n; \eta^1, \ldots, \eta^m\rangle^\bullet$ is expanded linearly in the variables $t^0_{k_0}, s^i_{k_i}, \overline{s}^j_{k_j}, t^1_{k_1}$. The reason behind defining $Z_d[\eta^1, \ldots, \eta^m]$ is that its terms encode every possible disconnected invariant $\langle \tau_{k_1}(\gamma_1)\cdots\tau_{k_n}(\gamma_n); \eta^1, \ldots, \eta^m\rangle^\bullet$. Using these notions, we can finally introduce the Virasoro operators.

Definition 2.3.1. *For $k > 0$ we define the Virasoro operators L_k, for the relative theory* *of X, as the differential operators acting on $\mathbb{Q}[[t^0_{k_0}, s^i_{k_i}, \overline{s}^j_{k_j}, t^1_{k_1}]]$ as follows:*

$$\begin{aligned}
L_k = \; & -(k+1)!\frac{\partial}{\partial t^0_{k+1}} - \mathcal{X}(X^*)(k+1)!\left(\sum_{r=1}^{k+1}\frac{1}{r}\right)\frac{\partial}{\partial t^1_k} + \sum_{l\geq 0}\left(\frac{(l+k)!}{(l-1)!}t^0_l\frac{\partial}{\partial t^0_{k+l}}\right. \\
& + \frac{(l+k+1)!}{l!}t^1_l\frac{\partial}{\partial t^1_{k+l}} + \left.\sum_{i=1}^{g}\left(\frac{(l+k+1)!}{(l)!}s^i_l\frac{\partial}{\partial s^i_{k+l}} + \frac{(l+k)!}{(l-1)!}\overline{s}^i_l\frac{\partial}{\partial \overline{s}^i_{k+l}}\right)\right) \\
& + \mathcal{X}(X^*)\sum_{l\geq 0}\frac{(l+k)!}{(l-1)!}\left(\sum_{r=l}^{k+l}\frac{1}{r}\right)t^0_l\frac{\partial}{\partial t^1_{k+l-1}} + \frac{\mathcal{X}(X^*)}{2}\sum_{l\geq 0}^{k-2}(k-l-1)!(l+1)!\frac{\partial}{\partial t^1_l}\frac{\partial}{\partial t^1_{k-l-2}}.
\end{aligned}$$

For $k=-1$ and $k=0$ we define the Virasoro operators *as*

$$\begin{aligned}
L_{-1} = \; & -\frac{\partial}{\partial t^0_0} + \sum_{l\geq 0}\left(t^0_{l+1}\frac{\partial}{\partial t^0_l} + t^1_{l+1}\frac{\partial}{\partial t^1_l} + \sum_{i=1}^{g}\left(s^i_{l+1}\frac{\partial}{\partial s^i_l} + \overline{s}^i_{l+1}\frac{\partial}{\partial \overline{s}^i_l}\right)\right) + t^0_0 t^1_0 + \sum_{i=1}^{g} s^i_0\overline{s}^i_0, \\
L_0 = \; & -\frac{\partial}{\partial t^0_1} - \mathcal{X}(X^*)\frac{\partial}{\partial t^1_0} + \sum_{l\geq 0}\left(l t^0_l\frac{\partial}{\partial t^0_l} + (l+1)!t^1_l\frac{\partial}{\partial t^1_l} + \sum_{i=1}^{g}\left((l+1)!s^i_l\frac{\partial}{\partial s^i_l} + l!\overline{s}^i_l\frac{\partial}{\partial \overline{s}^i_l}\right)\right) \\
& + \mathcal{X}(X^*)\sum_{l\geq 0} t^0_{l+1}\frac{\partial}{\partial t^1_l} + \frac{\mathcal{X}(X^*)}{2}t^0_0 t^0_0.
\end{aligned}$$

We will see how to use these operators to erase from the invariants the descendent classes $\tau_k(1)$. However, to rise to the stationary case, we need also some analogous operators for dealing with the descendent classes with evaluation classes α_i or β_j. In this sense, we introduce the differential operators D^i_k and $\overline{D}^i_k$ for $k\geq -1$, as follows:

$$\begin{aligned}
D^i_k = \; & -(k+1)!\frac{\partial}{\partial s^i_{k+1}} + \textstyle\sum_{l\geq 0}(\frac{(l+k)!}{(l-1)!}t^0_l\frac{\partial}{\partial s^i_{l+k}} + \frac{(l+k+1)!}{l!}\overline{s}^i_l\frac{\partial}{\partial t^1_{l+k}}, \\
\overline{D}^i_k = \; & -(k+1)!\frac{\partial}{\partial \overline{s}^i_{k+1}} + \textstyle\sum_{l\geq 0}(\frac{(l+k)!}{(l-1)!}t^0_l\frac{\partial}{\partial \overline{s}^i_{l+k}} - \frac{(l+k+1)!}{l!}s^i_l\frac{\partial}{\partial t^1_{l+k}}.
\end{aligned}$$

The potential of all these operators for our algorithm lies in the following theorems (see [43] Theorems 3 and 4).

Theorem 2.3.1. *For all $k\geq -1$, it holds that $L_k(Z_d[\eta^1,\ldots,\eta^m]) = 0$.*

Theorem 2.3.2. *For all $k\geq -1$ and for $i=1,\ldots,g$, it holds that*

$$D^i_k(Z_d[\eta^1,\ldots,\eta^m]) = \overline{D}^i_k(Z_d[\eta^1,\ldots,\eta^m]) = 0.$$

These are results allow us to remove non stationary descendent classes from our invariant to reach the stationary case. However, from the formulas of the operators and $Z_d[\eta^1,\ldots,\eta^m]$ it might not be clear how to use Theorems 2.3.1 and 2.3.2 for these purpose. Let us clarify this through an example.

Consider the Gromov-Witten class $I = \tau_k(1)\tau_{k_2}(\omega)\cdots\tau_{k_n}(\omega)$ with $k \geq 2$, and let $I^{'} = \tau_{k_2}(\omega)\cdots\tau_{k_n}(\omega)$. The idea is to express $\langle I\rangle^{\bullet}$ in terms of stationary invariants. To do so, we have to remove the descendent class $\tau_k(1)$, and thus, we will use the Virasoro operator L_{k-1}. First of all, notice that $\langle I\rangle^{\bullet}$ is the coefficient of the term $t_k^0 t_{k_2}^1\cdots t_{k_n}^1$ in $Z_d[\eta^1,\ldots,\eta^m]$. After applying L_{k-1} to $Z_d[\eta^1,\ldots,\eta^m]$, the term $t_k^0 t_{k_2}^1\cdots t_{k_n}^1$ is mapped to the term $t_{k_2}^1\cdots t_{k_n}^1$, since we have a term $\frac{\partial}{\partial t_k^0}$ in the expression of L_{k-1}. So the next question to ask is which terms of $Z_d[\eta^1,\ldots,\eta^m]$ contribute to the term $t_{k_2}^1\cdots t_{k_n}^1$ after applying L_{k-1}. The derivatives that appear in the expression of L_k are

$$\frac{\partial}{\partial t_{k-1}^0},\ \frac{\partial}{\partial t_{k-1}^1},\ t_l^0\frac{\partial}{\partial t_{l+k-1}^0},\ t_l^1\frac{\partial}{\partial t_{k+l-1}^1},\ s_l^i\frac{\partial}{\partial s_{k+l-1}^i},\ \overline{s}_l^i\frac{\partial}{\partial \overline{s}_{k+l-1}^i},\ t_l^0\frac{\partial}{\partial t_{k+l-2}^1},\ \frac{\partial}{\partial t_l^1}\frac{\partial}{\partial t_{k-l-3}^1}.$$

From this list, those that can contribute to the term $t_{k_2}^1\cdots t_{k_n}^1$ are

$$\frac{\partial}{\partial t_{k-1}^0},\frac{\partial}{\partial t_{k-1}^1},t_l^1\frac{\partial}{\partial t_{k+l-1}^1},\ \text{and}\ \frac{\partial}{\partial t_l^1}\frac{\partial}{\partial t_{k-l-3}^1}.$$

Hence, we get that the terms of $Z_d[\eta^1,\ldots,\eta^m]$, contributing to the coefficient of $t_{k_2}^1\cdots t_{k_n}^1$ after applying L_{k-1}, are $t_k^0 t_{k_2}^1\cdots t_{k_n}^1$, $t_{k-1}^1 t_{k_2}^1\cdots t_{k_n}^1$, $t_{k_i+k-1}^1 t_{k_2}^1\cdots t_{k_n}^1$ for $i \in \{2,\ldots,n\}$ and $t_l^1 t_{k-l-3}^1 t_{k_2}^1\cdots t_{k_n}^1$ for $l \in \{0,\ldots,k-3\}$. Thus, we get that the coefficient of the term $t_{k_2}^1\cdots t_{k_n}^1$ of $L_{k-1}(Z_d[\eta^1,\ldots,\eta^m])$ is

$$\begin{aligned}&-k!\langle I\rangle^{\bullet} - \mathcal{X}(X^*)(k)!\left(\sum_{r=1}^{k}\frac{1}{r}\right)\langle\tau_{k-1}(\omega)I^{'}\rangle^{\bullet}\\&+\frac{\mathcal{X}(X^*)}{2}\sum_{l=0}^{k-3}(l+1)!(k-l-3)!\langle\tau_{k_l}(\omega)\tau_{k-l-3}(\omega)I^{'}\rangle^{\bullet}\\&+\sum_{i=2}^{n}\frac{(l+k)!}{l!}\langle\tau_{k_i+k-1}(\omega)\tau_{k_2}(\omega)\cdots\tau_{k_{i-1}}(\omega)\tau_{k_{i+1}}(\omega)\cdots\tau_{k_2}(\omega)\rangle^{\bullet}.\end{aligned}$$

Now, using Theorem 2.3.1, we get that the above expression is zero, and hence we obtain

$$\begin{aligned}\langle I\rangle^{\bullet} = &-\mathcal{X}(X^*)\left(\sum_{r=1}^{k}\frac{1}{r}\right)\langle\tau_{k-1}(\omega)I^{'}\rangle^{\bullet}\\&+\frac{\mathcal{X}(X^*)}{2}\sum_{l=0}^{k-3}\frac{(l+1)!(k-l-2)!}{k!}\langle\tau_{k_l}(\omega)\tau_{k-l-3}(\omega)I^{'}\rangle^{\bullet}\\&+\sum_{i=2}^{n}\binom{l+k}{k}\langle\tau_{k_i+k-1}(\omega)\tau_{k_2}(\omega)\cdots\tau_{k_{i-1}}(\omega)\tau_{k_{i+1}}(\omega)\cdots\tau_{k_2}(\omega)\rangle^{\bullet}.\end{aligned} \tag{7}$$

This last equation finishes our computation since every invariant appearing in the left hand side of (7) is stationary.

Exactly the same argument can be done with the operators D_k^i and $\overline{D}_k^i$ to erase, from the insertion, the odd descendent classes.

Now that we have introduced the Virasoro operators for relative invariants over nonsingular curves, we can restrict this study to the case we are interested in, namely, the absolute Gromov-Witten Theory of elliptic curves. In particular, we are interested in finding a suitable expression of the operators. The idea is to rewrite these operators as operators acting on the descendent classes instead of on the formal variables $t_{k_0}^0$, $s_{k_i}^i$, $\overline{s}_{k_j}^j$, and $t_{k_1}^1$. In order to do that, first, we have to introduce some notation.

Let $\mathcal{A}$ be the supercommutative (i.e. $xy = (-1)^{\deg(x)\deg(y)}yx$) grade polynomial algebra over $\mathbb{Q}$ on the formal symbols $\tau_k(\gamma)$ where $k \geq 0$, $\gamma \in H^{p_\gamma q_\gamma}(E)$ and the grading of $\tau_k(\gamma)$ is $2k - 2 + p_\gamma + q_\gamma$. Since we are also interested in invariants with λ classes and Chern characters on the moduli space of stable maps, we will denote $\mathcal{A}'$ to the algebra resulting of adjoining to $\mathcal{A}$ the formal symbols λ_k and ch_k with grading $2k$. Gromov-Witten invariants of an elliptic curve, see (2) and (3), can be seen as linear maps:

$$\langle \cdot \rangle, \langle \cdot \rangle^\bullet : \quad \mathcal{A}' \quad \longrightarrow \quad \mathbb{Q}[[q]]$$

We recall that the genus g is avoided in our notation, since it can be computed from the cohomological degree of the insertion. However, we will see that the Virasoro operators contain first order derivatives in the formal symbols $\tau_k(\gamma)$. This means that the operator might change the genus of the invariants. To deal with this difficulty, we will add to our algebra the variables $\hbar, \hbar^{-1}$ where the grading of $\hbar$ is $2(\dim(E) - 3)$ and $\langle I\hbar^{g-1}\rangle = \langle I\rangle_g$ (resp. in the disconnected case). The idea is to rewrite the Virasoro operators as a family of linear operators acting on $\mathcal{A}[\hbar, \hbar^{-1}]$.

Since we are interested in absolute Gromov-Witten invariants, we fix $m = 0$. As a result we have that $\mathcal{X}(E^*) = \mathcal{X}(E) = 0$ and, hence, simplifying the expression of L_k. The goal is to visualize the operators L_k, D_k and $\overline{D}_k$ (we simplify the notation D_k^i since there is only one possible i) as the linear operators acting on $\mathcal{A}[\hbar, \hbar^{-1}]$. The idea is to take the dual point of view of these operators. As it can be seen in the previous example, the derivative $t_l^1 \frac{\partial}{\partial t_{l+k}^1}$ means that we erased from our insertion $\tau_l(\omega)$ and we add $\tau_{k+l}(\omega)$. Hence, we can rewrite the operators L_k, D_k and $\overline{D}_k$ as operators

$$V_k, W_k, \overline{W}_k : \mathcal{A}[\hbar, \hbar^{-1}] \longrightarrow \mathcal{A}[\hbar, \hbar^{-1}].$$

respectively defined as follows:

$$V_k = -\tau_{k+1}(1) + \sum_{l\geq 0, \gamma\in\mathcal{B}} \binom{k+l+p_\gamma}{k+1} \tau_{k+l}(\gamma)\frac{\partial}{\partial \tau_l(\gamma)}, \text{ for } k \geq 0. \tag{8}$$

$$V_{-1} = -\tau_0(1) + \sum_{l\geq 1, \gamma\in\mathcal{B}} \tau_{l-1}(\gamma)\frac{\partial}{\partial \tau_l(\gamma)} + \hbar\left(\frac{\partial}{\partial \tau_0(1)}\frac{\partial}{\partial \tau_0(\omega)} + \frac{\partial}{\partial \tau_0(\beta)}\frac{\partial}{\partial \tau_0(\alpha)}\right). \tag{9}$$

$$W_k = -\tau_{k+1}(\beta) + \sum_{l\geq 0, \gamma\in\mathcal{B}} \binom{k+l+p_\gamma}{k+1} \tau_{k+l}(\beta\gamma) \frac{\partial}{\partial \tau_l(\gamma)}, \text{ for } k \geq -1. \quad (10)$$

$$\overline{W}_k = -\tau_{k+1}(\alpha) + \sum_{l\geq 0, \gamma\in\mathcal{B}} \binom{k+l+p_\gamma}{k+1} \tau_{k+l}(\alpha\gamma) \frac{\partial}{\partial \tau_l(\gamma)}, \text{ for } k \geq -1. \quad (11)$$

Remark 2.3.1. *Recall that $\mathcal{A}[\hbar, \hbar^{-1}]$ is a supercommutative graded algebra. This means that $\frac{\partial}{\partial \tau_l(\gamma)}$ does not satisfy the usual Leibniz law. We need to add some signs:*

$$\frac{\partial(xy)}{\partial \tau_l(\gamma)} = \frac{\partial(x)}{\partial \tau_l(\gamma)} y + (-1)^{|x||\tau_l(\gamma)|} x \frac{\partial(y)}{\partial \tau_l(\gamma)}$$

where $x, y \in \mathcal{A}[\hbar, \hbar^{-1}]$ and $|x|$ denotes its grading as element of $\mathcal{A}[\hbar, \hbar^{-1}]$.

Before continuing with the study of the operators we open a brief parenthesis to deal with an index problem. From the formulas of W_{-1} and $\overline{W}_{-1}$ we might get descendent symbols of the form $\tau_{-1}(\gamma)$. For example, consider $I = \tau_0(\beta)\tau_0(1)$, then applying $\overline{W}_{-1}$ to I we will get terms of the form $\tau_{-1}(\omega)\tau_0(1)$ and $\tau_{-1}(\omega)\tau_0(\beta)$. Recall that $\tau_k(\gamma) = \psi_i^k \mathrm{ev}_i^*(\gamma)$, so it doesn't make sense to have descendent classes of the form $\tau_{-1}(\gamma)$. However, we will deal with these negative descendent symbols as operators on $\mathcal{A}[\hbar, \hbar^{-1}]$ as follows.

Definition 2.3.2. *Let $k < 0$ and $\gamma \in \mathcal{B}$, Let $\gamma' \in \mathcal{B}$ be such that $\gamma\gamma' = \epsilon\omega$ with $\epsilon = \pm 1$. We define the formal descendent symbol $\tau_k(\gamma)$ as the linear operator acting on $\mathcal{A}$ by*

- *If $(k, \gamma) \neq (-2, \omega)$, $\tau_k(\gamma) = \left((-1)^k \epsilon \frac{\partial}{\partial \tau_{-k-1}(\gamma')}\right) \hbar$.*
- *If $(k, \gamma) = (-2, \omega)$, $\tau_{-2}(\omega) = \left(1 - \frac{\partial}{\partial \tau_1(1)}\right) \hbar$.*

Remark 2.3.2.

- *Using the previous definition, Theorem 2.2.1 can be rewritten in a more general frame, including negative descendent symbols, as:*

$$F_n(z_1, \ldots, z_n) = \sum_{k_1,\ldots,k_n\in\mathbb{Z}} \langle \tau_{k_1}(\omega) \cdots \tau_{k_n}(\omega)\rangle^\bullet z_1^{k_1+1} \cdots z_n^{k_n+1}.$$

- *Using these negative descendent symbols, we can rewrite V_k, W_k and $\overline{W}_k$ as:*

$$V_k = \sum_{i\in\mathbb{Z}} (-1)^i \binom{i}{k+1} (\tau_i(1)\tau_{k-1-i}(\omega) - \tau_i(\alpha)\tau_{k-1-i}(\beta)).$$

$$W_k = \sum_{i\in\mathbb{Z}} (-1)^i \binom{i}{k+1} \tau_i(\beta)\tau_{k-1-i}(\omega).$$

$$\overline{W}_k = \sum_{i\in\mathbb{Z}} (-1)^i \binom{i}{k+1} \tau_i(\alpha)\tau_{k-1-i}(\omega).$$

Now that we have translated the operators, introduced in Def. 2.3.1, to the elliptic curve case, in terms of the algebra $\mathcal{A}[\hbar, \hbar^{-1}]$, the analogous results to Theorems 2.3.1 and 2.3.2 can be derived.

Corollary 2.3.1. *For all $k \geq -1$ and $I \in \mathcal{A}[\hbar, \hbar^{-1}]$, it holds that $\langle V_k(I)\rangle^\bullet = \langle W_k(I)\rangle^\bullet = \langle \overline{W}_k(I)\rangle^\bullet = 0$*

As we illustrate above for Theorems 2.3.1 and 2.3.2, we develop an example on how the algorithm should go using Corollary 2.3.1.

Consider the Gromov-Witten class $I = \tau_{k_1}(1)\tau_{k_2}(\alpha)\tau_{k_3}(\beta)\tau_{k_4}(\omega)$ with $k_1 \geq 1$. In order to deal with the first descendent class of I we will apply V_{k_1-1} to $I_1 = \tau_{k_2}(\alpha)\tau_{k_3}(\beta)\tau_{k_4}(\omega)$:

$$\begin{aligned} V_{k_1-1}(I_1) = & -\tau_{k_1}(1)I_1 + \sum_{l\geq 0, \gamma\in\mathcal{B}} \binom{k+l+p_\gamma}{k+1}\tau_{k+l}(\gamma)\frac{\partial I_1}{\partial \tau_l(\gamma)} \\ = & -I + \binom{k_1+k_2}{k_1}\tau_{k_1+k_2-1}(\alpha)\tau_{k_3}(\beta)\tau_{k_4}(\omega) - \binom{k_1+k_3-1}{k_1}\tau_{k_1+k_3-1}(\beta)\tau_{k_3}(\alpha)\tau_{k_4}(\omega) \\ & + \binom{k_1+k_4}{k_1}\tau_{k_1+k_4-1}(\omega)\tau_{k_2}(\alpha)\tau_{k_3}(\beta). \end{aligned}$$

Thus, using Corollary 2.3.1, we get that

$$\begin{aligned} \langle I\rangle^\bullet = & \binom{k_1+k_2}{k_1}\langle\tau_{k_1+k_2-1}(\alpha)\tau_{k_3}(\beta)\tau_{k_4}(\omega)\rangle^\bullet - \binom{k_1+k_3-1}{k_1}\langle\tau_{k_1+k_3-1}(\beta)\tau_{k_3}(\alpha)\tau_{k_4}(\omega)\rangle^\bullet \\ & + \binom{k_1+k_4}{k_1}\langle\tau_{k_1+k_4-1}(\omega)\tau_{k_2}(\alpha)\tau_{k_3}(\beta)\rangle^\bullet. \end{aligned}$$

Hence, the computation of $\langle I\rangle^\bullet$ has been reduced to determine

$$\langle\tau_{k_1+k_2-1}(\alpha)\tau_{k_3}(\beta)\tau_{k_4}(\omega)\rangle^\bullet, \langle\tau_{k_1+k_3-1}(\beta)\tau_{k_3}(\alpha)\tau_{k_4}(\omega)\rangle^\bullet, \langle\tau_{k_1+k_4-1}(\omega)\tau_{k_2}(\alpha)\tau_{k_3}(\beta)\rangle^\bullet,$$

invariants with less non-stationary descendent classes than the initial one. We compute these three invariants:

- Applying $\overline{W}_{k_1+k_2-2}$ to $\tau_{k_3}(\beta)\tau_{k_4}(\omega)$ we get

$$\langle\tau_{k_1+k_2-1}(\alpha)\tau_{k_3}(\beta)\tau_{k_4}(\omega)\rangle^\bullet = \binom{k_1+k_2+k_3-1}{k_1+k_2-1}\langle\tau_{k_1+k_2+k_3-2}(\omega)\tau_{k_4}(\omega)\rangle^\bullet.$$

- Applying $W_{k_1+k_3-2}$ to $\tau_{k_3}(\alpha)\tau_{k_4}(\omega)$ we get

$$\langle\tau_{k_1+k_3-1}(\beta)\tau_{k_3}(\alpha)\tau_{k_4}(\omega)\rangle^\bullet = \binom{k_1+k_2+k_3-1}{k_1+k_3-1}\langle\tau_{k_1+k_2+k_3-2}(\omega)\tau_{k_4}(\omega)\rangle^\bullet.$$

- Applying $\overline{W}_{k_2-1}$ to $\tau_{k_1+k_4-1}(\omega)\tau_{k_3}(\beta)$ we get

$$\langle \tau_{k_1+k_4-1}(\omega)\tau_{k_2}(\alpha)\tau_{k_3}(\beta)\rangle^{\bullet} = \binom{k_2+k_3}{k_2}\langle \tau_{k_2+k_3-1}(\omega)\tau_{k_1+k_4-1}(\omega)\rangle^{\bullet}.$$

Finally, using the notation $\binom{k_1+\cdots+k_n}{k_1,\ldots,k_n} = \frac{(k_1+\cdots+k_n)!}{k_1!\cdots k_n!}$ we conclude that

$$\begin{aligned}\langle I\rangle^{\bullet} = &\binom{k_1+k_2+k_3}{k_1,k_2,k_3}\langle \tau_{k_1+k_2+k_3-2}(\omega)\tau_{k_4}(\omega)\rangle^{\bullet}\\ &+\binom{k_2+k_3}{k_2,k_3}\binom{k_1+k_2}{k_1,k_2}\langle \tau_{k_2+k_3-1}(\omega)\tau_{k_1+k_4-1}(\omega)\rangle^{\bullet}.\end{aligned}$$

Following the example, the idea of the algorithm is to apply recursively the operators V_k, W_k, and $\overline{W}_k$ to reduce the computation of disconnected invariants without Hodge classes to the stationary case. However, one can also encode this recursion in the following formula (see [43] Theorem 2 or [45] Proposition 3.3.2.).

Proposition 2.3.1. *Let $I = \tau_{k_1}(\gamma_1)\cdots\tau_{k_n}(\gamma_n)$, then it holds that*

$$\langle I\rangle^{\bullet} = \sum_{\substack{\{1,\ldots,n\} = \sqcup_{J\in S} J\\ \cup_{i\in J}\gamma_i = \pm\omega}} \operatorname{sign}(S)\left\langle \prod_{J=\{i_1,\ldots,i_m\}\in S}\binom{k_{i_1}+\cdots+k_{i_m}}{k_{i_1},\ldots,k_{i_m}}\tau_{1+\sum_{i\in J}(k_i-1)}(\omega)\right\rangle^{\bullet}$$

where the first sum is indexed over all partitions S of $\{1,\ldots,n\}$ such that for each $J\in S$ $\prod_{i\in J}\gamma_i = \pm\omega$, and where $\operatorname{sign}(S)$ is taken as the sign resulting of reordering the descendent classes such, that for every $J\in S$, all descendent classes whose indexes are in J are together. In other words, $\operatorname{sign}(S)$ is the sign of a permutation σ mapping $(1,\ldots,n)$ to $(J : J\in S)$.

As a consequence of this proposition, the same exact formula can be stated for connected invariants. More precisely, one has the following corollary.

Corollary 2.3.2. *Let $I = \tau_{k_1}(\gamma_1)\cdots\tau_{k_n}(\gamma_n)$, it holds that*

$$\langle I\rangle = \sum_{\substack{\{1,\ldots,n\} = \sqcup_{J\in S} J\\ \cup_{i\in J}\gamma_i = \pm\omega}} \operatorname{sign}(S)\left\langle \prod_{J=\{i_1,\ldots,i_m\}\in S}\binom{k_{i_1}+\cdots+k_{i_m}}{k_{i_1},\ldots,k_{i_m}}\tau_{1+\sum_{i\in J}(k_i-1)}(\omega)\right\rangle$$

where the first sum is indexed over all partitions S of $\{1,\ldots,n\}$ such that each $J\in S$ satisfies $\prod_{i\in J}\gamma_i = \pm\omega$.

Before finishing the subsection, one can conclude, as a consequence of the previous study, the quasimodularity of the these invariants.

Corollary 2.3.3. *For $\gamma_1,\ldots,\gamma_n\in H^*(E)$ and $k_1,\ldots,k_n\in\mathbb{N}$, it holds that*

$$\langle \tau_{k_1}(\gamma_1)\cdots\tau_{k_n}(\gamma_n)\rangle \text{ and } \langle \tau_{k_1}(\gamma_1)\cdots\tau_{k_n}(\gamma_n)\rangle^{\bullet}$$

are quasimodular forms.

2.4 Hodge insertions

The work done so far allows us to compute invariants of the form $\langle \tau_{k_1}(\gamma_1)\cdots\tau_{k_n}(\gamma_n)\rangle^\bullet$. However, we will be also interested in invariants of the form $\langle \Lambda; \tau_{k_1}(\gamma_1)\cdots\tau_{k_n}(\gamma_n)\rangle^\bullet$ where Λ is a product of λ classes. Our interest on these invariants arises from applying the localization formula, where the Euler class of the Hodge bundle will appear.

We note that for the forgetful morphism $\pi : \overline{\mathcal{M}}_{g,n}(E,\beta) \to \overline{\mathcal{M}}_{g,n}$, $\pi^*(\lambda_k) = \lambda_k$ holds. So, one has that the invariant $\langle \Lambda; \tau_{k_1}(\gamma_1)\cdots\tau_{k_n}(\gamma_n)\rangle^\bullet$ is a particular case of the more general case treated on Subsection 2.5. However, the expression of λ classes as tautological class is remarkably complicated. In this subsection we develop another method for computing these invariants using the same idea as before, namely: we will find linear operators Y_k acting on $\mathcal{A}'[\hbar,\hbar^{-1}]$ and satisfying an analogous result to Corollary 2.3.1. In [15] a method for expressing Hodge integrals in terms of descendent integrals is presented for the general case; it is valid for non-singular projective varieties. As we saw in Subsection 2.3, while explaining the results of [43], a generating series, gathering all the possible Hodge integrals, and a family of operators where the generating series vanishes have been introduced (see [15] Theorem 1). However, in this subsection, we will follow [45], where the dual point of view to these operators is presented for the specific case of the elliptic curve.

The algorithm will erase, one by one, the Chern characters in the insertion until there is no Hodge insertion. Thus, we will work with Chern characters. For an introduction to Chern characters we refer to [16] Chapter 3. The first step to be performed is to express every λ class in term of Chern characters using the formula

$$1 + \lambda_1 + \cdots + \lambda_g = e^{\sum_{k\geq 1}(k-1)!\,\mathrm{ch}_k t^k}.$$

For $k \geq 1$, we define the operators $Y_k : \mathcal{A}'[\hbar,\hbar^{-1}] \longrightarrow \mathcal{A}'[\hbar,\hbar^{-1}]$ by

$$\begin{aligned} Y_k = \quad & -\mathrm{ch}_k + \frac{B_{k+1}}{(k+1)!}(\tau_{k+1}(1) - \sum_{\substack{l\geq 0\\ \gamma\in\mathcal{B}}} \tau_{k+l}(\gamma)\frac{\partial}{\partial \tau_l(\gamma)} \\ & +\hbar^{-1}\sum_{i=0}^{k-1}(-1)^i(\tau_i(1)\tau_{k-1-i}(\omega) - \tau_i(\alpha)\tau_{k-1-i}(\beta)) \end{aligned}$$

where B_{k+1} is the Bernoulli number defined by $\sum_{k\geq 0} B_n z^n = \frac{z}{e^z-1}$. Note that, since ch_k and B_{k+1} are zero for k even (see [39] Corollary 5.3.), so is Y_k.

As for the Virasoro operators, the main result for Y_k is its vanishing with respect $\langle\cdot\rangle^\bullet$ (see [45] Proposition 3.4.1. or [15] Proposition 2).

Proposition 2.4.1. *For all $I \in \mathcal{A}'[\hbar,\hbar^{-1}]$ and $k \geq 1$, it holds that $\langle Y_k(I)\rangle^\bullet = 0$.*

Again, the strength of this result lies in the first term of Y_k that, as with the Virasoro operators, allows as to erase inductively the Chern classes from our insertion.

Let us first rewrite Y_k using negative descendent symbols such that, as in Proposition 2.3.1, we can find a nice formula for the invariants.

For $k \geq 1$, we denote by ϖ the operator

$$\varpi = \hbar^{-1} \sum_{i \in \mathbb{Z}} (-1)^i (\tau_i(1)\tau_{k-1-i}(\omega) - \tau_i(\alpha)\tau_{k-1-i}(\beta)).$$

Then, Y_k can be expressed as

$$Y_k = -\mathrm{ch}_k + \frac{B_{k+1}}{(k+1)!} \varpi_k.$$

Hence, using Proposition 2.4.1, we get the following formula for invariants with Chern characters.

Corollary 2.4.1. *For any $k_1, \ldots, k_m \geq 1$ and $I \in \mathcal{A}$ it holds that*

$$\langle \mathrm{ch}_{k_1} \cdots \mathrm{ch}_{k_m}; I \rangle^\bullet = \left\langle \prod_{i=1}^{m} \left(\frac{B_{k+1}}{(k+1)!} \varpi_{k_i} \right) I \right\rangle^\bullet .$$

Again, we conclude the subsection observing that, as a consequence of Corollary 2.4.1 and Proposition 2.1.1, the invariants $\langle \Lambda; \tau_{k_1}(\gamma_1) \cdots \tau_{k_n}(\gamma_n) \rangle$ and $\langle \Lambda; \tau_{k_1}(\gamma_1) \cdots \tau_{k_n}(\gamma_n) \rangle^\bullet$ are quasimodular forms.

2.5 Invariants with tautological classes

So far, we have shown how to compute invariants of the form:

$$\langle \Lambda; \tau_{k_1}(\gamma_1) \cdots \tau_{k_n}(\gamma_n) \rangle = \sum_{d \geq 0} \langle \Lambda; \tau_{k_1}(\gamma_1) \cdots \tau_{k_n}(\gamma_n) \rangle^E_{g,n,d} \, q^d$$

for $\gamma_i \in H^*(E)$, Λ being a product of λ classes, as well as for the analogous disconnected invariants. In particular, we have seen that they are quasimodular form and we know how to express them in terms of the three generators of **QM**. However, our goal is to determine invariants of the form

$$\langle \mu; \gamma_1, \ldots, \gamma_n \rangle = \sum_{k \geq 0} \int_{\overline{\mathcal{M}}_{g,n}} \mu I_{g,n,d}(\gamma_1, \ldots, \gamma_n) q^k$$

where $\mu \in RH^*(\overline{\mathcal{M}}_{g,n})$ and $\gamma_i \in H^*(E)$. This subsection will be devoted to reduce the computation of these invariants to the ones we have already studied. For an introduction to the tautological ring we refer to Appendix A.1. There, Theorem A.1.3 provides an explicit list of generators of the tautological ring through the notion of decorated stratum classes (see Definition A.1.10). As a result, we can focus our study on invariants with a decorated stratum class $[\Gamma, \alpha]$. The strategy will be to first study the case when Γ is a trivial graph. Afterwards, the case where Γ is a tree will be faced. Here, the splitting axiom will be the key of the argument. Finally, we will solve the general case to reduce, via the reduction axiom, the problem to the previous cases. Note that previously we omitted the genus from our notation since it was assumed to satisfy the degree axiom. However, in this subsection we need to specify the genus in order to fix the stable graph of the decorated stratum class. Thus, we include the genus in our notation again, and our invariants might be directly zero by the degree axiom.

Before starting with the analysis of these cases, it is important to observe that the arguments of this subsection will be applied, not for the generating series $\langle \mu; \gamma_1, \ldots, \gamma_n \rangle$ but for concrete invariants for a fixed $\beta \in H_2(E)$. Nevertheless, one can easily check that all the results and formulas will hold for the generating series too. Moreover, most of the arguments of this subsection perfectly work for every projective non-singular variety X. However, at some point of the exposed argument in this subsection we will need to pullback of ψ classes on $\overline{\mathcal{M}}_{g,n}$ to $\overline{\mathcal{M}}_{g,n}(E, \beta)$. In the case of the elliptic curve, we will see that this pullback coincides with the ψ classes on $\overline{\mathcal{M}}_{g,n}(E, \beta)$. However, this does not hold in general. In Proposition A.2.1 we state the general statement for $\overline{\mathcal{M}}_{g,n}(X, \beta)$. This is the only point where our argument will differ from the general case. In this sense, we will apply the splitting and reduction axioms in the most general frame. Nevertheless, it is important to keep in mind that that we will apply these formulas to E. For example, the diagonal class plays a fundamental role in these axioms and in this case its expression is

$$[\Delta] = 1 \otimes \omega - \alpha \otimes \beta + \beta \otimes \alpha + \omega \otimes 1.$$

As commented above, our first task is to solve the case when Γ is a trivial graph. Let Γ be the trivial graph of genus g and n legs. This means that the decorated stratum class is of the form $\alpha = \kappa_{a_1}^{e_1} \cdots \kappa_{a_l}^{e_l} \psi_1^{m_1} \cdots \psi_n^{m_n}$. We will show how to express the invariant

$$\int_{\overline{\mathcal{M}}_{g,n}} \alpha I_{g,n,\beta}(\gamma_1, \ldots, \gamma_n)$$

in terms of the invariants

$$\langle \tau_{m'_1}(\gamma'_1) \cdots \tau_{m'_{n'}}(\gamma'_{n'}) \rangle^E_{g',n',\beta'},$$

which we know how to compute. We will structure the argument in three reduction steps:

- First, we will study the case where α has not kappa classes. The idea will be to use Proposition A.2.1 to pullback the invariant to $\overline{\mathcal{M}}_{g,n}(X, \beta)$.
- Secondly, we will focus on the case where the only κ–class in α is κ_{a_1}. Using the fundamental class axiom we will be able to reduce this case to the previous one.

- The last step will be to erase the κ–classes recursively using the forgetful morphism until the previous case is reached.

Let us assume first that $\alpha = \psi_1^{m_1} \cdots \psi_n^{m_n}$. One has that

$$\langle \alpha; \gamma_1 \ldots, \gamma_n \rangle^E_{g,n,\beta} := \int_{\overline{\mathcal{M}}_{g,n}} \alpha I_{g,n,\beta}(\gamma_1, \ldots, \gamma_n) = \int_{\left[\overline{\mathcal{M}}_{g,n}(E,\beta)\right]^{\mathrm{vir}}} \rho^*(\alpha) \prod_{i=1}^{n} \mathrm{ev}_i^*(\gamma_i). \quad (12)$$

Hence, if we can express $\rho^*(\psi_i)$ in terms of the ψ–classes in $\overline{\mathcal{M}}_{g,n}(E,\beta)$ we will be able to compute the invariant using the invariants studied in Subsection 2.3. In Appendix A.2, Proposition A.2.1, the relation among these ψ–classes is stated. Applying this result to the case of the elliptic curve we get that $[E_i] = 0$ and, hence,

$$\rho^*(\psi_i) = \psi_i. \quad (13)$$

To proof that E_i is empty we use the following fact, consequence of Hurwitz's Theorem (see [22] Chapter 4.2.):

> *Let $f : X \to Y$ be a finite separable morphism between two complete non-singular curves over an algebraically closed field with genus g_X and g_Y, respectively. Then, if $g_X < g_Y$, f is constant.*

Now, let $f : C \to E$ be a stable map, with a genus 0 component C_i, whose special points are p_i and a node. Then, since C_i has genus 0 and E has genus 1, $f(C_i)$ is a point and hence, f is not stable. Thus E_i is empty.

As a consequence of (13) and (12), we get the following result.

Proposition 2.5.1. *For $m_1, \ldots, m_n \geq 0$ and $\gamma_1, \ldots, \gamma_n \in H^*(E)$, it holds that*

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{m_1} \cdots \psi_n^{m_n} I_{g,n,\beta}(\gamma_1, \ldots, \gamma_n) = \langle \tau_{m_1}(\gamma_1) \cdots \tau_{m_n}(\gamma_n) \rangle^E_{g,n,\beta}$$

This result concludes the case in which $\alpha = \psi_1^{m_1} \cdots \psi_n^{m_n}$. Before dealing with the next case, it is important to remark that this is the only argument where we will use that E is an elliptic curve. Continuing to the next step, we consider now that α has one κ–class, i.e. $\alpha = \kappa_a \psi_1^{m_1} \cdots \psi_n^{m_n}$ for $a, m_1, \ldots, m_n \geq 0$. The argument for this case will differ from the previous one. This time, instead of looking at $\rho^*(\alpha)$, we will pullback our classes through the forgetful morphism of $\overline{\mathcal{M}}_{g,n}$. Hence, the fundamental class axiom will play a crucial role. However, this will not be the only Gromov-Witten axiom that we will use. As we will see, the splitting axiom will also appear in our reasoning. Nevertheless, before going into the details of the argument, we will need first the following results about the pullbacks of ψ–classes (see [47] Section 6.3. or [27] Lemma 1.2.6., Lemma 1.3.1, and Lemma 2.2.3.):

Proposition 2.5.2. *Let Γ be a stable graph of genus g and n markings and consider the gluing morphism*

$$\xi_\Gamma : \overline{\mathcal{M}}_\Gamma \longrightarrow \overline{\mathcal{M}}_{g,n}.$$

Then,

$$\xi_\Gamma^*(\psi_i) = \pi_v^*(\psi_h),$$

where h is the i–th leg of Γ corresponding to the marked point p_i, v is the vertix of Γ where the h lies, and π_v is the projection from $\overline{\mathcal{M}}_\Gamma$ to $\overline{\mathcal{M}}_{g(v),n(v)}$.

Proposition 2.5.3. *Let π denote the forgetful morphism arriving at $\overline{\mathcal{M}}_{g,n}$ forgetting the $n+1$ marked point. Let p_i be the section associated to the i–th marked points from the universal curve. Then,*

$$\pi^*(\psi_i) = \psi_i - (p_i)_*([\overline{\mathcal{M}}_{g,n}]) \text{ and } \pi^*(\kappa_a) = \kappa_a - \psi_{n+1}^a.$$

Now we can prove the result that reduces this case to the previous one.

Proposition 2.5.4. *For $a, m_1, \dots, m_m \geq 0$ and $\gamma_1, \dots, \gamma_n \in H^*(X)$, it holds that*

$$\int_{\overline{\mathcal{M}}_{g,n}} \kappa_a \psi_1^{m_1} \cdots \psi_n^{m_n} I_{g,n,\beta}(\gamma_1, \dots, \gamma_n) = \int_{\overline{\mathcal{M}}_{g,n+1}} \psi_{n+1}^{a+1} \psi_1^{m_1} \cdots \psi_n^{m_n} I_{g,n+1,\beta}(\gamma_1, \dots, \gamma_n, 1).$$

Proof. By definition $\kappa_a = \pi_*(\psi_{n+1}^{a+1})$, so using the projection formula we get

$$\int_{\overline{\mathcal{M}}_{g,n}} \kappa_a \psi_1^{m_1} \cdots \psi_n^{m_n} I_{g,n,\beta}(\gamma_1, \dots, \gamma_n) = \int_{\overline{\mathcal{M}}_{g,n+1}} \psi_{n+1}^{a+1} \pi^*(\psi_1^{m_1} \cdots \psi_n^{m_n} I_{g,n,\beta}(\gamma_1, \dots, \gamma_n)). \quad (14)$$

Using the fundamental class axiom and Proposition 2.5.3, we get that

$$\pi^*(\psi_1^{m_1} \cdots \psi_n^{m_n} I_{g,n,\beta}(\gamma_1, \dots, \gamma_n)) = \prod_{i=1}^n \left(\psi_i - p_{i*}\left([\overline{\mathcal{M}}_{g,n}]\right)\right)^{m_i} I_{g,n+1,\beta}(\gamma_1, \dots, \gamma_n, 1).$$

This expression can be expanded and rewritten as

$$\pi^*(\psi_1^{m_1} \cdots \psi_n^{m_n} I_{g,n,\beta}(\gamma_1, \dots, \gamma_n)) = \left(\psi_1^{m_1} \cdots \psi_n^{m_n} + \sum_{i=1}^n p_{i*}\left([\overline{\mathcal{M}}_{g,n}]\right) A_i\right) I_{g,n+1,\beta}(\gamma_1, \dots, \gamma_n, 1),$$

for certain classes A_i. Substituting this last expression in (14), one deduces that it is enough to check that

$$\int_{\overline{\mathcal{M}}_{g,n+1}} p_{i*}\left([\overline{\mathcal{M}}_{g,n}]\right) \psi_{n+1}^{a+1} A_i I_{g,n+1,\beta}(\gamma_1, \dots, \gamma_n, 1) = 0$$

for every $i \in \{1, \dots, n\}$.

We recall that the section p_i of forgetful morphism corresponds to the gluing morphism $\xi_i := \xi_{\Gamma_i}$ where Γ_i is the graph given by Appendix A.1 Figure 14. Hence, we get that

$$\int_{\overline{\mathcal{M}}_{g,n+1}} p_{i*}\left([\overline{\mathcal{M}}_{g,n}]\right)\psi_{n+1}^{a+1}A_i I_{g,n+1,\beta}(\gamma_1,\ldots,\gamma_n,1) = \int_{\overline{\mathcal{M}}_{\Gamma_i}} \xi_i^*\left(\psi_{n+1}^{a+1}A_i I_{g,n+1,\beta}(\gamma_1,\ldots,\gamma_n,1)\right)$$

Writing $\xi_i^*(A_i)$ as $A_{i,1}\otimes A_{i,2}$ in the cohomology of $\overline{\mathcal{M}}_{\Gamma_i} = \overline{\mathcal{M}}_{g,n}\times\overline{\mathcal{M}}_{0,3}$, using Proposition 2.5.2, and the splitting axiom, we get that

$$\begin{aligned}
&\int_{\overline{\mathcal{M}}_{\Gamma_i}} \xi_i^*\left(\psi_{n+1}^{a+1}A_i I_{g,n+1,\beta}(\gamma_1,\ldots,\gamma_n,1)\right) = \\
&\sum_{\substack{\beta=\beta_1+\beta_2\\ k,l}} \int_{\overline{\mathcal{M}}_{\Gamma_i}} \left(A_{i,1}I_{g,n,\beta_1}(\gamma_1,\ldots,\not{\gamma_i},\ldots,\gamma_n,T^k)\right)\otimes\left(\psi_{n+1}^{a+1}A_{i,2}I_{0,3,\beta 2}(\gamma_i,1,T^l)\right) = \\
&\sum_{\substack{\beta=\beta_1+\beta_2\\ k,l}} \int_{\overline{\mathcal{M}}_{g,n}} A_{i,1}I_{g,n,\beta_1}(\gamma_1,\ldots,\not{\gamma_i},\ldots,\gamma_n,T^k)\int_{\overline{\mathcal{M}}_{0,3}} \psi_{n+1}^{a+1}A_{i,2}I_{0,3,\beta 2}(\gamma_i,1,T^l).
\end{aligned}$$

Now, since $\overline{\mathcal{M}}_{0,3} = \{\mathrm{pt}\}$ and the degree of $\psi_{n+1}^{a+1}A_{i,2}I_{0,3,\beta 2}(\gamma_i,1,T^l)$ is greater that 0 since $a\geq 0$, we get that

$$\int_{\overline{\mathcal{M}}_{0,3}} \psi_{n+1}^{a+1}A_{i,2}I_{0,3,\beta 2}(\gamma_i,1,T^l) = 0,$$

and, hence,

$$\int_{\overline{\mathcal{M}}_{g,n+1}} p_{i*}\left([\overline{\mathcal{M}}_{g,n}]\right)\psi_{n+1}^{a+1}A_i I_{g,n+1,\beta}(\gamma_1,\ldots,\gamma_n,1) = 0.$$

□

Proposition 2.5.4 shows how to reduce the case where α has one κ–class to the case where it has only ψ–classes. The next step consists in solving the general case where $\alpha = \kappa_{a_1}^{e_1}\cdots\kappa_{a_l}^{e_l}\psi_1^{m_1}\cdots\psi_n^{m_n}$. The answer to this case is gather in the next theorem which proof provides in fact an effective method for computing the invariants.

Theorem 2.5.1. *For $\alpha = \kappa_{a_1}^{e_1}\cdots\kappa_{a_l}^{e_l}\psi_1^{m_1}\cdots\psi_n^{m_n}$, the invariant*

$$\int_{\overline{\mathcal{M}}_{g,n}} \alpha I_{g,n,\beta}(\gamma_1,\ldots,\gamma_n)$$

can be expressed in terms of invariants of the form $\langle\tau_{k_1}(\gamma_1)\cdots\tau_{k_{n'}}(\gamma_{n'})\rangle_{g,n',\beta}^E$.

Proof. For $\alpha = \psi_1^{m_1} \cdots \psi_n^{m_n}$ the proof follows from Proposition 2.5.1 . More precisely, the idea is to recursively reduce the computation of $\alpha = \kappa_{a_1}^{e_1} \cdots \kappa_{a_l}^{e_l} \psi_1^{m_1} \cdots \psi_n^{m_n}$ to the case without κ–classes. We will argue by recursion on $E_\alpha = \sum_{k=1}^{l} e_k$, writing the invariant

$$\int_{\overline{\mathcal{M}}_{g,n}} \alpha I_{g,n,\beta}(\gamma_1, \ldots, \gamma_n)$$

in terms of invariants whose respective tautological classes α' satisfy $E_{\alpha'} < E_\alpha$. The base case in which $E = 1$ corresponds to Proposition 2.5.4. So, let us focus on the recursion step. Let $\alpha = \kappa_{a_1}^{e_1} \cdots \kappa_{a_l}^{e_l} \psi_1^{m_1} \cdots \psi_n^{m_n}$ with $E_\alpha > 1$. Then

$$\int_{\overline{\mathcal{M}}_{g,n}} \kappa_{a_1}^{e_1} \cdots \kappa_{a_l}^{e_l} \psi_1^{m_1} \cdots \psi_n^{m_n} I_{g,n,\beta}(\gamma_1, \ldots, \gamma_n) =$$

$$\int_{\overline{\mathcal{M}}_{g,n}} \pi_*(\psi_{n+1}^{a_1+1}) \kappa_{a_1}^{e_1-1} \cdots \kappa_{a_l}^{e_l} \psi_1^{m_1} \cdots \psi_n^{m_n} I_{g,n,\beta}(\gamma_1, \ldots, \gamma_n) =$$

$$\int_{\overline{\mathcal{M}}_{g,n+1}} \psi_{n+1}^{a_1+1} \pi^* \left(\kappa_{a_1}^{e_1-1} \cdots \kappa_{a_l}^{e_l} \psi_1^{m_1} \cdots \psi_n^{m_n} I_{g,n,\beta}(\gamma_1, \ldots, \gamma_n) \right).$$

Note that, as in the proof of Proposition 2.5.4, the class $p_{i*}([\overline{\mathcal{M}}_{g,n}])$ appearing from $\pi^*(\psi_i)$, leads to an invariant that is zero by dimension reasons. Now, using Proposition 2.5.3 and the fundamental class axiom, we deduce that the previous expression is equal to

$$\int_{\overline{\mathcal{M}}_{g,n+1}} \psi_{n+1}^{a_1+1} \left(\kappa_{a_1} - \psi_{n+1}^{a_1+1} \right)^{e_1-1} \cdots \left(\kappa_{a_l} - \psi_{n+1}^{a_l+1} \right)^{e_l} \psi_1^{m_1} \cdots \psi_n^{m_n} I_{g,n,\beta}(\gamma_1, \ldots, \gamma_n). \quad (15)$$

The term $\left(\kappa_{a_1} - \psi_{n+1}^{a_1+1} \right)$ of this expression appears up to $e_1 - 1$. Hence, (15) can be expanded as a sum of invariants whose κ–classes are at most $\kappa_{a_1}^{e_1-1} \cdots \kappa_{a_l}^{e_l}$. Thus, we can apply recursion to (15). This ends the proof. □

As we have remarked before, the proof of the previous theorem provides an effective way of computing these invariants. Thus, this result concludes our first situation in which the stable graph of the decorated stratum class is trivial. However, before moving to the general case, let us illustrate this by an example where we show how the recursion used in the proof of Theorem 2.5.1 works. Consider the invariant

$$\int_{\overline{\mathcal{M}}_{g,n}} \kappa_{a_1} \kappa_{a_2} \Psi I_{g,n,\beta}(\gamma_1, \ldots, \gamma_n)$$

where $\Psi = \psi_1^{m_1} \cdots \psi_n^{m_n}$. Following the proof of Theorem 2.5.1, first we have to get rid of the first κ–class pulling back through the forgetful morphism:

$$\begin{aligned} \int_{\overline{\mathcal{M}}_{g,n}} \kappa_{a_1} \kappa_{a_2} \Psi I_{g,n,\beta}(\gamma_1, \ldots, \gamma_n) &= \int_{\overline{\mathcal{M}}_{g,n+1}} \psi_{n+1}^{a_1+1} \pi^*(\kappa_{a_2} \Psi I_{g,n,\beta}(\gamma_1, \ldots, \gamma_n)) \\ &= \int_{\overline{\mathcal{M}}_{g,n+1}} \psi_{n+1}^{a_1+1} \left(\kappa_{a_2} - \psi_{n+1}^{a_2} \right) \Psi I_{g,n+1,\beta}(\gamma_1, \ldots, \gamma_n, 1). \end{aligned}$$

Thus, we have reduced the computation to determine the invariants

$$\int_{\overline{\mathcal{M}}_{g,n+1}} \kappa_{a_2}\Psi\psi_{n+1}^{a_1+1} I_{g,n+1,\beta}(\gamma_1,\ldots,\gamma_n,1) \text{ and } \int_{\overline{\mathcal{M}}_{g,n+1}} \Psi\psi_{n+1}^{a_1+a_2+1} I_{g,n+1,\beta}(\gamma_1,\ldots,\gamma_n,1),$$

both with less κ–classes than the initial one. Using Corollary 2.5.1, the second invariant is

$$\int_{\overline{\mathcal{M}}_{g,n+1}} \Psi\psi_{n+1}^{a_1+a_2+1} I_{g,n+1,\beta}(\gamma_1,\ldots,\gamma_n,1) = \langle \tau_{m_1}(\gamma_1)\cdots\tau_{m_n}(\gamma_n)\tau_{a_1+a_2+1}(1)\rangle_{g,n+1,\beta}^{E}.$$

Applying Proposition 2.5.4 to the first invariant, we get

$$\int_{\overline{\mathcal{M}}_{g,n+1}} \kappa_{a_2}\Psi\psi_{n+1}^{a_1+1} I_{g,n+1,\beta}(\gamma_1,\ldots,\gamma_n,1) = \langle \tau_{m_1}(\gamma_1)\cdots\tau_{m_n}(\gamma_n)\tau_{a_1+1}(1)\tau_{a_2+1}(1)\rangle_{g,n+2,\beta}^{E}.$$

Gathering all these computations together we get that

$$\begin{aligned}\int_{\overline{\mathcal{M}}_{g,n}} \kappa_{a_1}\kappa_{a_2}\Psi I_{g,n,\beta}(\gamma_1,\ldots,\gamma_n) = \; & \langle \tau_{m_1}(\gamma_1)\cdots\tau_{m_n}(\gamma_n)\tau_{a_1+1}(1)\tau_{a_2+1}(1)\rangle_{g,n+2,\beta}^{E} \\ & -\langle \tau_{m_1}(\gamma_1)\cdots\tau_{m_n}(\gamma_n)\tau_{a_1+a_2+1}(1)\rangle_{g,n+1,\beta}^{E}.\end{aligned}$$

As a result, in terms of the generating series, we get that

$$\begin{aligned}\langle \kappa_{a_1}\kappa_{a_2}\Psi;\gamma_1,\ldots,\gamma_n\rangle = \; & \langle \tau_{m_1}(\gamma_1)\cdots\tau_{m_n}(\gamma_n)\tau_{a_1+1}(1)\tau_{a_2+1}(1)\rangle \\ & -\langle \tau_{m_1}(\gamma_1)\cdots\tau_{m_n}(\gamma_n)\tau_{a_1+a_2+1}(1)\rangle.\end{aligned}$$

Once we have solved the trivial graph case, we can focus on the general case in which we have decorated stratum class over any stable graph. We will prove that one can express the general case in terms of the trivial graph case. The main idea to do so is to use the splitting axiom and the reduction axiom. More precisely, let Γ be a stable graph of genus g and n legs, and let $[\Gamma,\alpha]$ be a decorated stratum class. By definition, $[\Gamma,\alpha] = \xi_{\Gamma *}(\alpha)$ and hence,

$$\int_{\overline{\mathcal{M}}_{g,n}} [\Gamma,\alpha]\, I_{g,n,\beta}(\gamma_1,\ldots,\gamma_n) = \int_{\overline{\mathcal{M}}_{\Gamma}} \alpha\xi_{\Gamma}^{*}\left(I_{g,n,\beta}(\gamma_1,\ldots,\gamma_n)\right). \tag{16}$$

Using the Künneth formula on $\overline{\mathcal{M}}_{\Gamma}$, one deduces that $\xi_{\Gamma}^{*}\left(I_{g,n,\beta}(\gamma_1,\ldots,\gamma_n)\right) = \prod_{v\in V(\Gamma)}\pi_v^{*}(I_v)$ for some classes I_v. Thus we have that

$$\int_{\overline{\mathcal{M}}_{g,n}} [\Gamma,\alpha]\, I_{g,n,\beta}(\gamma_1,\ldots,\gamma_n) = \prod_{v\in V(\Gamma)} \int_{\overline{\mathcal{M}}_{g(v),n(v)}} \alpha_v I_v.$$

So the goal will be to find a nice expression of each of the I_v in terms of Gromov-Witten classes at $\overline{\mathcal{M}}_{g(v),n(v)}$. However, a priori we only know how Gromov-Witten classes behave under pullbacks through gluing morphisms involved in the reduction and splitting axiom. As we will see, this difficulty will be solved by factorizing ξ_Γ through a sequence of morphisms where we can apply our axioms.

Intuitively, the idea is to use the reduction axiom to break all the loops of our graph and reach the situation where the graph is a tree. Then, the splitting axiom will split the tree into the trivial graphs corresponding to each of the vertices of the graph. Thus, first we will deal with the case where the stable graph is a tree. However, before going into the details of the result answering this case, let us see an example that illustrate how the argument will go.

Let Γ be the stable graph of genus $g = g_1 + g_2 + g_3$ and 3 legs given by Figure 1.

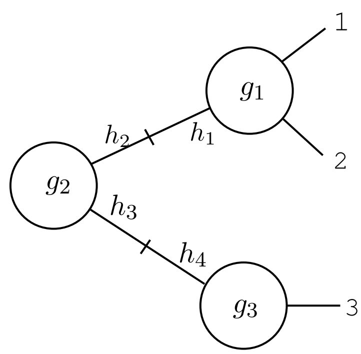


Figure 1: The stable graph Γ of genus $g_1 + g_2 + g_3$ and 3 legs with 3 vertices $\{v_1, v_2.v_3\}$ of genus g_i respectively, 2 edges, and 3 legs, two lying on v_1 and one lying on v_3.

Consider the gluing morphism $\xi_\Gamma : \overline{\mathcal{M}}_\Gamma = \overline{\mathcal{M}}_{g_1,3} \times \overline{\mathcal{M}}_{g_2,2} \times \overline{\mathcal{M}}_{g_3,2} \longrightarrow \overline{\mathcal{M}}_{g,3}$, and the decorated stratum class $[\Gamma, \alpha]$ where $\alpha = \alpha_1 \otimes \alpha_2 \otimes \alpha_3$. Then, for $\gamma_1, \gamma_2, \gamma_3 \in H^*(E)$ we will find a nice expression of $\xi_\Gamma^* \left(I_{g,3,\beta}(\gamma_1, \gamma_2, \gamma_3)\right)$. To do so, we will use the stable graphs Γ_1 and Γ_2 of Figure 2.

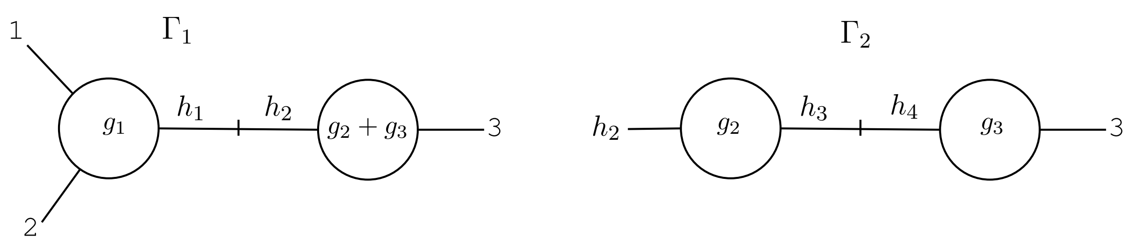


Figure 2: The stable graphs Γ_1, Γ_2 of genus g, $g - g_1 = g_2 + g_3$, and 3, 2 legs, respectively.

Note that Γ_2 is the stable subgraph of Γ obtanied after erasing the vertex v_1. On the other hand, Γ_1 arises from contracting the vertices v_2 and v_3 of Γ to one vertex.

As a result, we have that ξ_Γ factors through

$$\begin{array}{ccc}
\overline{\mathcal{M}}_\Gamma = \overline{\mathcal{M}}_{g_1,3} \times \overline{\mathcal{M}}_{g_2,2} \times \overline{\mathcal{M}}_{g_3,2} & \xrightarrow{\xi_\Gamma} & \overline{\mathcal{M}}_{g,3} \\
& \searrow^{\mathrm{Id}\times\xi_{\Gamma_1}} & \uparrow{\scriptstyle \xi_{\Gamma_2}} \\
& & \overline{\mathcal{M}}_{\Gamma_2} = \overline{\mathcal{M}}_{g_1,3} \times \overline{\mathcal{M}}_{g_2+g_3,2}
\end{array}$$

Hence, we get that $\xi_\Gamma^*(I_{g,3,\beta}(\gamma_1,\gamma_2,\gamma_3)) = (\mathsf{Id}\times\xi_{\Gamma_1})^* \circ \xi_{\Gamma_2}^*(I_{g,3,\beta}(\gamma_1,\gamma_2,\gamma_3))$. Now, we can apply the splitting axiom to ξ_{Γ_2} first, and then to ξ_{Γ_1} to get

$$\begin{aligned}
\xi_\Gamma^*(I_{g,3,\beta}(\gamma_1,\gamma_2,\gamma_3)) &= \sum_{\beta=\beta_1+\beta_2} \sum_{i,j} g^{i,j} \left(\mathsf{Id}\times\xi_{\Gamma_1}\right)^* \left(I_{g_1,3,\beta_1}(\gamma_1,\gamma_2,T_i) \otimes I_{g_2+g_3,2,\beta_2}(\gamma_3,T_j)\right) \\
&= \sum_{\beta=\beta_1+\beta_2+\beta_3} \sum_{i,j} \sum_{l,k} a_{i,j,k,l} g^{i,j} g^{k,l} I_{g_1,3,\beta_1}(\gamma_1,\gamma_2,T_i) \otimes I_{g_2,2,\beta_2}(T_j,T_k) \otimes I_{g_3,2,\beta_3}(\gamma_3,T_l)
\end{aligned}$$

where $a_{i,j,k,l}$ is the sign coming from the multiplication

$$T_i \otimes T_j \otimes 1 \cdot 1 \otimes T_k \otimes T_l = a_{i,j,k,l} T_i \otimes (T_j T_k) \otimes T_l.$$

Recall from the Künneth that this sign arises from the multiplication structure:

$$(a_1 \otimes a_2)\cdot(b_1 \otimes b_2) = (-1)^{|a_2|\,|b_1|}(a_1 b_1)\otimes(a_2 b_2).$$

Thus, one has that

$$\begin{aligned}
\int_{\overline{\mathcal{M}}_{g,3}} [\Gamma,\alpha]\, I_{g,3,\beta}(\gamma_1,\gamma_2,\gamma_3) = \sum_{\beta=\beta_1+\beta_2+\beta_3} \sum_{i,j} \sum_{l,k} a_{i,j,k,l} g^{i,j} g^{k,l} \\
\int_{\overline{\mathcal{M}}_{g_1,3}} \alpha_1 I_{g_1,3,\beta_1}(\gamma_1,\gamma_2,T_i) \int_{\overline{\mathcal{M}}_{g_2,2}} \alpha_2 I_{g_2,2,\beta_2}(T_j,T_k) \int_{\overline{\mathcal{M}}_{g_3,2}} \alpha_3 I_{g_3,2,\beta_3}(\gamma_3,T_l)
\end{aligned}$$

In order to solve this example we have studied the pullback of the Gromov-Witten class through a gluing morphism of a certain tree. We found an analogous to the splitting axiom but for a tree with two edges. We are interested in deducing a formula for any tree generalizing the splitting axiom. In the next result we prove such formula.

Theorem 2.5.2. *Let Γ be a stable graph of genus g and n legs which is a tree, and let $\gamma_1,\ldots,\gamma_n \in H^*(E)$. Then,*

$$\begin{aligned}
\xi_\Gamma^*\left(I_{g,n,\beta}(\gamma_1,\ldots,\gamma_n)\right) = (-1)^a \sum_{\beta=\sum_{v\in V(\Gamma)}\beta_v} \sum_{\substack{i_h, j_{h'} \\ \{h,h'\}\in E(\Gamma)}} \operatorname{sign}((i_h, j_{h'})_{\{h,h'\}\in E(\Gamma)}) \\
\prod_{h,h'\in E(\Gamma)} g^{i_h, j_{h'}} \prod_{v\in V(\Gamma)} \pi_v^*\left(I_{g(v),n(v),\beta_v}\left((\gamma_i)_{i\in L(v)}, (T_{i_h})_{h\in E(v)}, (T_{j_h})_{h\in E(v)}\right)\right)
\end{aligned} \tag{17}$$

where the sign $(-1)^a$ *of the expression comes from ordering the classes* γ_i *such that all the evaluation classes corresponding to legs lying in the same vertex are together, and* $\mathrm{sign}((i_h, j_{h'})_{\{h,h'\}\in E(\Gamma)})$ *is the sign arising from the multiplication of the elements* $1 \otimes \cdots \otimes T_{i_h} \otimes \cdots \otimes T_{j_h} \otimes \cdots \otimes 1$ *in the cohomology ring* $H^*(E^{\#V(\Gamma)})$.

Proof. First of all, we assume that the evaluation classes are already ordered properly so that we can forget about the sign. Moreover, for simplicity in the notation we forget $\mathrm{sign}((i_h, j_{h'})_{\{h,h'\}\in E(\Gamma)})$ too. We prove the result by induction on the number of vertices of Γ. For $\#V(\Gamma) = 2$ this statement is exactly the splitting axiom. Now, assume $\#V(\Gamma) > 2$ and that the statement holds for every stable graph with less vertices. We want to factor ξ_Γ through two morphisms where we can apply the splitting axiom and induction respectively. To do this, we need to define three new stable graphs.

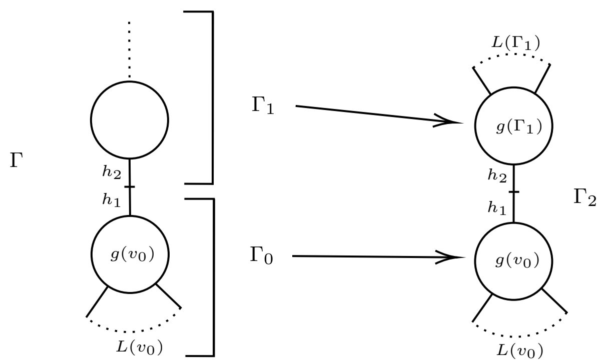


Figure 3: The stable graph Γ is the union of Γ_0 and Γ_1 through $\{h_1, h_2\}$. The stable graph Γ_2 is the result of contracting the subgraph Γ_1 inside Γ to a vertex.

Since Γ is a tree, we can choose $v_0 \in V(\Gamma)$ to be a extreme vertex of the graph. In particular, this means that there is only one edge $\{h_0, h_0'\}$ at v_0. Since Γ is a tree this edge cannot be a loop. We fix $h_0 \in H(v_0)$. Let Γ_0 be the trivial stable graph of genus $g(v_0)$ and $n(v_0)$ legs, and let Γ_1 be the stable graph resulting from erasing the vertex v_0 and the half edge h_0 from Γ. Now denote by Γ_2 the stable graph with two vertices u_0 and u_1, and one edge $\{h_0, h_0'\}$ such that $g(u_0) = g(v_0)$, $L(u_0) = L(v_0)$, $g_(u_1) = g(\Gamma_1)$ and $L(u_1) = L(\Gamma_1)$ (see Figure 3). Intuitively, Γ can be splitted in Γ_0 and Γ_1, and Γ_2 is the graph resulting from contracting Γ_1 inside Γ to a unique vertex. As a result we have that $\overline{\mathcal{M}}_\Gamma = \overline{\mathcal{M}}_{\Gamma_0} \times \overline{\mathcal{M}}_{\Gamma_1}$.

The reason for introducing these stable graphs is to factor ξ_Γ through the gluing morphisms related to these graphs as it is shown in the following commutative diagram:

$$\overline{\mathcal{M}}_\Gamma = \overline{\mathcal{M}}_{\Gamma_0} \times \overline{\mathcal{M}}_{\Gamma_1} \xrightarrow{\quad \xi_\Gamma \quad} \overline{\mathcal{M}}_{g,n}$$

$$\overline{\mathcal{M}}_{\Gamma_0} \times \overline{\mathcal{M}}_{\Gamma_1} \xrightarrow{\mathrm{Id}\times\xi_{\Gamma_1}} \overline{\mathcal{M}}_{\Gamma_2} = \overline{\mathcal{M}}_{\Gamma_0} \times \overline{\mathcal{M}}_{g(\Gamma_1),n(\Gamma_1)} \xrightarrow{\xi_{\Gamma_2}} \overline{\mathcal{M}}_{g,n}$$

Thus, we have reached the desired situation where $\xi_\Gamma = \xi_{\Gamma_2} \circ (\mathrm{Id} \times \xi_{\Gamma_1})$, and we can apply the splitting axiom to ξ_{Γ_2} and induction to ξ_{Γ_1}. More concretely, we get

$$\xi_\Gamma^* \left(I_{g,n,\beta}(\gamma_1, \dots, \gamma_n)\right) = (\mathrm{Id} \times \xi_{\Gamma_1})^* \circ \xi_{\Gamma_2}^* \left(I_{g,n,\beta}(\gamma_1, \dots, \gamma_n)\right) =$$

$$\sum_{\beta=\beta_{v_0}+\beta_2} \sum_{i_{h_0}, j_{h_0'}} g^{i_{h_0}, j_{h_0'}} I_{g(v_0),n(v_0),\beta_{v_0}} \left((\gamma_i)_{i\in L(v)}, T_{i_{h_0}}\right) \otimes \xi_{\Gamma_1}^* \left(I_{g(\Gamma_1),n(\Gamma_1)} \left((\gamma_i)_{i\in L(\Gamma_1)}, T_{i_{h_0'}}\right)\right) =$$

$$\sum_{\beta=\beta_{v_0}+\beta_2} \sum_{i_{h_0}, j_{h_0'}} \sum_{\beta_2 = \sum_{v\in V(\Gamma_1)} \beta_v} \sum_{\substack{i_h, j_{h'} \\ h, h' \in E(\Gamma_1)}} g^{i_{h_0}, j_{h_0'}} I_{g(v_0),n(v_0),\beta_{v_0}} \left((\gamma_i)_{i\in L(v)}, T_{i_{h_0}}\right) \otimes$$

$$\prod_{h,h'\in E(\Gamma_1)} g^{i_h, j_{h'}} \prod_{v\in V(\Gamma_1)} \pi_v^* \left(I_{g(v),n(v),\beta_v} \left((\gamma_i)_{i\in L(v)}, (T_{i_h})_{h\in E(v)}, (T_{j_h})_{h\in E(v)}\right)\right)$$

Joining together the indexes of the sums and products we get that this expression is exactly the expression in (17). □

Since Theorem 2.5.2 is a generalization of the splitting lemma to any stable graph which is a tree, the natural next step in to try to generalize the reduction axiom. Let $\Gamma_{g,n,k}$ be the stable graph with one vertex, k edges, n legs and genus g (see Figure 4).

$\Gamma_{g,n,k} =$

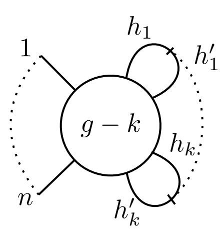


Figure 4: Stable graph of genus g, n legs, one vertex and k edges.

Then, we have a gluing morphism $\xi_{\Gamma_{g,n,k}} : \overline{\mathcal{M}}_{\Gamma_{g,n,k}} \longrightarrow \overline{\mathcal{M}}_{g,n}$. As before, we want to find a formula for the pullback of Gromov-Witten classes through this gluing morphism generalizing the reduction axiom.

Theorem 2.5.3. *Let $\Gamma = \Gamma_{g,n,k}$ and let $\gamma_1, \dots, \gamma_n \in H^*(E)$. Then*

$$\xi_\Gamma^* \left(I_{g,n,\beta}(\gamma_1, \dots, \gamma_n)\right) = \sum_{\substack{i_l, j_l \\ l \in \overline{k}}} \prod_{m=1}^{k} g^{i_m, j_m} I_{g-k,n+2k,\beta} \left(\gamma_1, \dots, \gamma_n, T_{i_1}, T_{j_1}, \dots, T_{i_k}, T_{j_k}\right) \quad (18)$$

where $\overline{k} = \{1, \dots, k\}$.

Proof. We will argue by induction on k. First of all, note that the case $k = 1$ it is exactly the reduction axiom. Thus, assume that $k > 1$ and that the theorem holds for every $k' < k$. Let $\Gamma = \Gamma_{g,n,k}$. Consider the stable graphs $\Gamma_1 = \Gamma_{g-k+1,n+2(k-1),1}$ and $\Gamma_2 = \Gamma_{g,n+2,k-1}$. Note that Γ_2 comes from splitting the last edge of Γ. Similarly, we can get Γ_1 from splitting all the edges from Γ except the last one (see Figure 5).

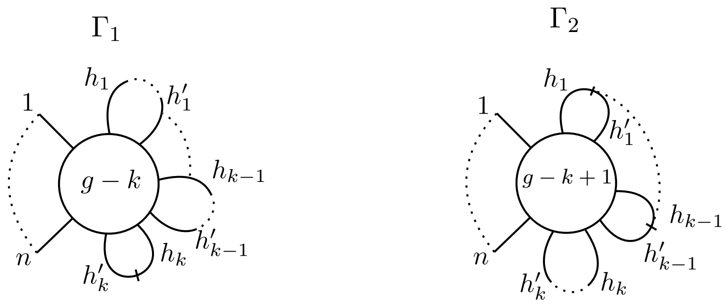
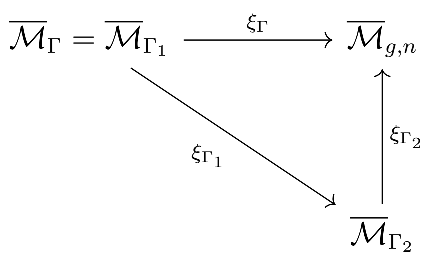


Figure 5: The stable graphs Γ_1 is the result of splitting all the edges of Γ except the last one $\{h_k, h_k'\}$. On the other hand, if we split the edge $\{h_k, h_k'\}$ from Γ we get Γ_2.

Note that we have that $\overline{\mathcal{M}}_\Gamma = \overline{\mathcal{M}}_{g-k,n+2k} = \overline{\mathcal{M}}_{\Gamma_1}$ and $\overline{\mathcal{M}}_{\Gamma_2} = \overline{\mathcal{M}}_{g-k+1,n+2k-2}$. Hence, the following diagram



commutes. As a result, ξ_Γ factors through the composition of two morphisms, ξ_{Γ_1} and ξ_{Γ_2}, where we can apply the reduction axiom and induction, respectively. More precisely, we get:

$$\xi_\Gamma^* \left(I_{g,n,\beta}(\gamma_1, \dots, \gamma_n)\right) =$$

$$\xi_{\Gamma_1}^* \circ \xi_{\Gamma_2}^* \left(I_{g,n,\beta}(\gamma_1, \dots, \gamma_n)\right) =$$

$$\sum_{\substack{i_l, j_l \\ l \in \overline{k}\setminus\{k\}}} \prod_{m=1}^{k-1} g^{i_m,j_m} \xi_{\Gamma_1}^* \left(I_{g-k+1,n+2k-2,\beta}\left(\gamma_1, \dots, \gamma_n, T_{i_1}, T_{j_1}, \dots, T_{i_{k-1}}, T_{j_{k-1}}\right)\right) =$$

$$\sum_{\substack{i_l, j_l \\ l \in \overline{k}\setminus\{k\}}} \sum_{i_k, j_k} g^{i_k,j_k} \prod_{m=1}^{k-1} g^{i_m,j_m} \xi_{\Gamma_1}^* \left(I_{g-k,n+2k,\beta}\left(\gamma_1, \dots, \gamma_n, T_{i_1}, T_{j_1}, \dots, T_{i_{k-1}}, T_{j_{k-1}}, T_{i_k}, T_{j_k}\right)\right).$$

Joining together the indexes of the sums and the product we get exactly (18). □

So far, we have seen in Theorems 2.5.1, 2.5.2, and 2.5.3 how to compute invariants where the stable graph of the decorated stratum class is either a trivial graph, or a tree or a stable graph of the form $\Gamma_{g,n,k}$ for some k. The computation for a general stable graph follows from these three cases. To do so, let first study the behavior of Gromov-Witten classes through a general gluing morphism. The formula for these pullbacks is a consequence of Theorems 2.5.2 and 2.5.3.

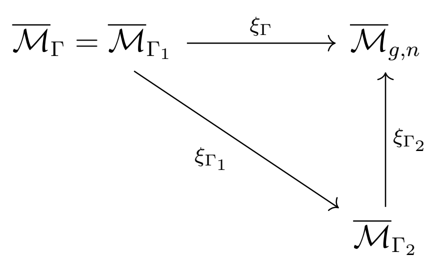
Proposition 2.5.5. *Let Γ be a stable graph of genus g and n legs, and let $\gamma_1, \ldots, \gamma_n \in H^*(E)$. Then,*

$$\xi_\Gamma^* \left(I_{g,n,\beta}(\gamma_1, \ldots, \gamma_n)\right) = (-1)^a \sum_{\beta = \sum_{v \in V(\Gamma)} \beta_v} \sum_{\substack{i_h, j_{h'} \\ h, h' \in E(\Gamma)}} \operatorname{sign}((i_h, j_{h'})_{\{h,h'\} \in E(\Gamma)})$$
$$\prod_{h,h' \in E(\Gamma)} g^{i_h, j_{h'}} \prod_{v \in V(\Gamma)} \pi_v^* \left(I_{g(v),n(v),\beta_v}\left((\gamma_i)_{i \in L(v)}, (T_{i_h})_{h \in E(v)}, (T_{j_h})_{h \in E(v)}\right)\right) \tag{19}$$

where the sign $(-1)^a$ of the expression comes from ordering the classes γ_i, and $\operatorname{sign}((i_h, j_{h'})_{\{h,h'\} \in E(\Gamma)})$ arises from the multiplication of the classes coming from the diagonal class in $H^*(E^{\# V(\Gamma)})$.

Proof. The idea of the proof consists in, as before, factor our gluing morphism through two gluing morphisms where we can apply Theorems 2.5.2 and 2.5.3. First of all, if Γ is a tree, it follows from Theorem 2.5.2. So, we assume that Γ is not a tree. Let $L = \{\{h_1, h_1'\}, \ldots, \{h_k, h_k'\}\}$ be a minimum (not unique) set of edges of Γ such that without them Γ is a tree. Let Γ_1 be the stable graph resulting from splitting all the edges in L. This implies that Γ_1 is a tree with $n + 2k$ legs and $\overline{\mathcal{M}}_\Gamma = \overline{\mathcal{M}}_{\Gamma_1}$.

We consider now the stable graph $\Gamma_2 = \Gamma_{g,n,k}$. This means that $\overline{\mathcal{M}}_{\Gamma_2} = \overline{\mathcal{M}}_{g,n+2k}$. This graph results from joining all vertices of Γ together, erasing all the edges outside L. Now, the relation among the stable graphs Γ, Γ_1, and Γ_2 lies in the following commutative diagram:



Since Γ_1 is a tree and $\Gamma_2 = \Gamma_{g,n,k}$, we can apply Theorems 2.5.2 and 2.5.3 to $\xi_\Gamma^* \left(I_{g,n,\beta}(\gamma_1, \ldots, \gamma_n)\right) = \xi_{\Gamma_1}^* \circ \xi_{\Gamma_2}^* \left(I_{g,n,\beta}(\gamma_1, \ldots, \gamma_n)\right)$ and, after joining together the sums and products of the expression we get the desired formula. $\square$

As a consequence of this result and (16), we get the general formula for our invariants.

Corollary 2.5.1. *Let Γ be a stable graph of genus g and n legs, and let $[\Gamma, \alpha]$ be a decorated stratum class on Γ with $\alpha = \prod_{v \in V(\Gamma)} \pi_v^*(\alpha_v)$. Then, for $\gamma_1, \ldots, \gamma_n \in H^*(E)$ it holds that*

$$\int_{\overline{\mathcal{M}}_{g,n}} [\Gamma, \alpha]\, I_{g,n,\beta}(\gamma_1, \ldots, \gamma_n) = (-1)^a \sum_{\beta = \sum_{v \in V(\Gamma)} \beta_v} \sum_{\substack{i_h, j_{h'} \\ h, h' \in E(\Gamma)}} \operatorname{sign}((i_h, j_{h'})_{\{h,h'\} \in E(\Gamma)})$$

$$\prod_{h,h' \in E(\Gamma)} g^{i_h j_{h'}} \prod_{v \in V(\Gamma)} \int_{\overline{\mathcal{M}}_{g(v),n(v)}} \alpha_v I_{g(v),n(v),\beta_v}\left((\gamma_i)_{i \in L(v)}, (T_{i_h})_{h \in E(v)}, (T_{j_h})_{h \in E(v)}\right)$$

where the sign $(-1)^a$ of the expression comes from ordering the classes γ_i such that all the evaluation classes corresponding to legs lying in the same vertex are together, and $\operatorname{sign}((i_h, j_{h'})_{\{h,h'\} \in E(\Gamma)})$ is the sign arising from the multiplication of the elements $1 \otimes \cdots \otimes T_{i_h} \otimes \cdots \otimes T_{j_h} \otimes \cdots \otimes 1$ in the cohomology ring $H^*(E^{\#V(\Gamma)})$.

This corollary concludes the computation of the Gromov-Witten invariants with tautological classes over the elliptic curve. Nevertheless, it is important to remark that Proposition 2.5.5 and Corollary 2.5.1 hold for every projective non-singular complex variety, and not only for E. The only difference between the general case and the elliptic curve is that, as it is shown in Proposition A.2.1, the pullback $\rho^*(\psi_i)$ generally in not equal to ψ_i in $\overline{\mathcal{M}}_{g,n}(X, \beta)$. Hence, in the argument for the trivial graph case, one must add some extra terms to the formulas.

As we mentioned in the beginning of the subsection, all these results are proven for invariants with a fixed β. However, we are interested in determining the generating series

$$\langle \mu; \gamma_1, \ldots, \gamma_n \rangle = \sum_{d \geq 0} \int_{\overline{\mathcal{M}}_{g,n}} \mu I_{g,n,d}(\gamma_1, \ldots, \gamma_n) q^d.$$

Most of the results stated in this subsection are exactly the same for these invariants except the one where sums over the partitions of β appear. In particular, all the results and formulas in the case of the trivial graph are exactly the same. However, whenever we apply the splitting lemma the sums indexed over the partitions of β appear. In these cases, the formulas are changed by erasing these sums. To proof this, we can just focus on the case of the splitting axiom. Recall that the effectivity axioms imply that β must be equal to $k\,[E]$ for some $k \in \mathbb{N}$ (otherwise the invariant is zero). Thus, we get that we can index these sums by the partitions of k and we get

$$\sum_{k = k_1 + k_2} I_{g_1,n_1,k_1}(\gamma_l, T_i) \otimes I_{g_2,n_2,k_2}(\gamma_i, T_j) = \left(\sum_{k_1 \geq 0} I_{g_1,n_1,k_1}(\gamma_l, T_i) \right) \otimes \left(\sum_{k_1 \geq 0} I_{g_2,n_2,k_2}(\gamma_i, T_j) \right).$$

Using this, Corollary 2.5.1 can be translated in to the following corollary.

Corollary 2.5.2. *Let Γ be a stable graph of genus g and n legs, and let $[\Gamma, \alpha]$ be a decorated stratum class on Γ with $\alpha = \prod_{v \in V(\Gamma)} \pi_v^*(\alpha_v)$. Then, for $\gamma_1, \dots, \gamma_n \in H^*(E)$ it holds that*

$$\begin{aligned} \langle [\Gamma, \alpha] \,; \gamma_1, \dots, \gamma_n \rangle = (-1)^a \sum_{\substack{i_h, j_{h'} \\ h, h' \in E(\Gamma)}} \operatorname{sign}((i_h, j_{h'})_{\{h,h'\} \in E(\Gamma)}) \\ \prod_{h,h' \in E(\Gamma)} g^{i_h, j_{h'}} \prod_{v \in V(\Gamma)} \left\langle \alpha_v; (\gamma_i)_{i \in L(v)}, (T_{i_h})_{h \in E(v)}, (T_{j_h})_{h \in E(v)} \right\rangle . \end{aligned} \tag{20}$$

Now, the invariants appearing in the right hand side of (20) can be computed as sums of invariants of the form $\langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_{n'}}(\gamma_{n'}) \rangle$ using Theorem 2.5.1. As a result, we can conclude the quasimodularity of our invariants.

Corollary 2.5.3. *For $\gamma_1, \dots, \gamma_n \in H^*(E)$ and $\mu \in RH^*(\overline{\mathcal{M}}_{g,n})$, it holds that $\langle \mu; \gamma_1, \dots, \gamma_n \rangle$ is a quasimodular form.*

2.6 Summary of the algorithm

As a conclusion of the analysis developed through the previous subsections, this subsection will be devote to formalize the algorithm for computing Gromov-Witten invariants on elliptic curves. We will outline all the reduction steps we have accomplished previously.

We recall that we are interested in computing Gromov-Witten invariants together with tautological classes. However, taking into account the degree axiom, we were able to gather together invariants in a generating series:

$$\langle \mu; \gamma_1, \dots, \gamma_n \rangle = \sum_{d \geq 0} \int_{\overline{\mathcal{M}}_{g,n}} \mu I_{g,n,da}(\gamma_1, \dots, \gamma_n) q^d.$$

Given $\gamma_1, \dots, \gamma_n \in H^*(E)$ and $\mu \in RH^*(\overline{\mathcal{M}}_{g,n})$, the algorithm should compute the generating series $\langle \mu; \gamma_1, \dots, \gamma_n \rangle$. Let us refine the input and the output of the algorithm.

The INPUT of the algorithm should be the classes $\gamma_1, \dots, \gamma_n$ and μ. By Theorem A.1.3 we can assume that μ is a decorated stratum class $[\Gamma, \alpha]$. On the other hand, recalling the structure of the cohomology of E and the linearity axiom, we can assume that $\gamma_i \in \mathcal{B} = \{1, \alpha, \beta, \omega\}$. Thus, the input of the algorithm should be a decorated stratum class that fixes the genus and the number of markings, and the evaluation classes γ_i as elements of $\mathcal{B}$.

Focusing now on the OUTPUT, and using Corollary 2.5.3, we get that our invariants are quasimodular forms. Thus, we are interested in computing the answer as a polynomial in the G_2, G_4, and G_6. As a result, the output of our algorithm should be a polynomial in these three quasimodular form.

Now that we have stated what the input and output of the algorithm are, we can outline the algorithm in the following steps.

1. Use Corollary 2.5.2 to reduce the computation to invariants where the stable graph of the decorated stratum class is trivial.

2. Use the procedure developed in the proof of Theorem 2.5.1 to express invariants with trivial stable graph as invariants of the form $\langle \tau_{k_1}(\gamma'_1) \cdots \tau_{k_m}(\gamma'_m) \rangle$.

3. Using Proposition 2.1.1 we can reduce the computation to disconnected invariants. We could also use Corollary 2.3.2 to reduce to stationary connected invariants and then apply Proposition 2.1.1.

4. Use the Virasoro operators or Proposition 2.3.1 to reduce the computation of these invariants to stationary invariants.

5. Compute stationary invariants using Theorem 2.2.1.

6. Compute the n–point correlation function in terms of Θ using the Theorems 2.2.3 or 2.2.2. Compute the power expansion of Θ using the Propositions 2.2.1 or 2.2.2.

3 Gromov-Witten Theory of K3 surfaces

Throughout this section we carry out the theoretical study of the algorithm for computing Gromov-Witten invariants on $K3$ surfaces. We recall that a $K3$ surface is a nonsingular proper surface with trivial canonical bundle and simply connected. During this section S will denote a $K3$ surface. As a motivation for studying these invariants, we recall from the introduction, the enumerative geometry applicability behind Gromov-Witten invariants. In particular, we recall the definition of the integers $N_g(h)$ as the number of genus g curves on a $K3$ surfaces with h nodes passing through g generic points. It turns out that these numbers coincide with the following Gromov-Witten invariants (see [9])

$$\langle 1; \mathbf{p}, \ldots, \mathbf{p} \rangle^S_{g,g,\beta} := \int_{\left[\overline{\mathcal{M}}_{g,g}(S,\beta)\right]^{\mathrm{red}}} \prod_{i=1}^{g} \mathrm{ev}_i^*(\mathbf{p})$$

where $\beta \in H_2(S)$ is a primitive class with $\langle \beta, \beta \rangle = 2h - 2$ for $h \geq 0$, and $\mathbf{p}$ denotes the class of a point. These invariants were initially computed in [9]. Note that in the above invariant we are not using the virtual fundamental class but the reduced virtual fundamental class. The reason why these new class is needed is due to the fact that for $K3$ surfaces the virtual fundamental class for $\beta \neq 0$ vanishes.

The section is structured in seven subsections. In Subsection 3.1 we briefly introduce some notions related to $K3$ surfaces as its Hodge structure, elliptic $K3$ surfaces, and some properties about its cohomology group. The main result of this subsection is Theorem 3.1.1 that states the existence of a deformation between any two complex $K3$ surfaces. This will allow us to focus on elliptic $K3$ surfaces with section. We denote by s and f the classes of the section and the fiber inside $H_2(S)$. In Subsection 3.2 we introduce the Gromov-Witten invariants we are interested in. In particular, we introduce the reduce virtual class as the tool for overcoming the vanishing of the virtual fundamental class. Using the reduced virtual class we can state the main goal of this subsection, namely, the computation of Gromov-Witten invariants of the form:

$$\langle \alpha; \gamma_1, \ldots, \gamma_n \rangle^S_{g,n,\beta} := \int_{\left[\overline{\mathcal{M}}_{g,n}(S,\beta)\right]^{\mathrm{red}}} \rho^*(\alpha) \prod_{i=1}^{n} \mathrm{ev}_i^*(\gamma_i)$$

where $\beta \in H_2(S)$ is a primitive effective curve class, $\alpha \in RH^*(\overline{\mathcal{M}}_{g,n})$, and $\gamma_1, \ldots, \gamma_n \in H^*(S)$. We will see that we can assume that $\beta = s + hf$ for $h \geq 0$. This allows us to focus our attention on the generating series:

$$\langle \alpha; \gamma_1, \ldots, \gamma_n \rangle^S_{g,n} := \sum_{h \geq 0} \left(\int_{\left[\overline{\mathcal{M}}_{g,n}(S,s+hf)\right]^{\mathrm{red}}} \rho^*(\alpha) \prod_{i=1}^{n} \mathrm{ev}_i^*(\gamma_i) \right) q^{h-1}$$

To compute these invariants we will argue by induction on the pair (g, n). During Subsection 3.2 we prove the base cases of the induction and the case where no evaluation class is the class of a point. Subsections 3.3, 3.4, 3.5, and 3.6 will focus on the remaining case where there exists a evaluation class which is the class of a point. In particular, in Subsection 3.3 we construct a degeneration from S to $S \cup_E (\mathbb{P}^1 \times E)$ and, as a consequence of the degeneration formula, we reduce the computation of the invariants to relative invariants on $\mathbb{P}^1 \times E$ relative to E. Afterwards, in Subsection 3.4, we apply the product formula to compute these relative invariants using invariants on E, i.e. invariants studied on the previous section. To achieve this reduction step, one has to compute the expression as tautological of the class

$$\mathcal{J}_g(k,n) := \rho^* \left(\left[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1/0,(1),1) \right]^{\text{vir}} \frown \prod_{i=1}^{k} \text{ev}_i^*(\mathbf{p}) \right).$$

The task of Subsections 3.5 and 3.6 is to accomplish this by first applying localization to reduced the problem to the study of the classes ψ_0^k in $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1/0,\infty,(1,1),1)^{\sim}$. As a conclusion, we deduce that the generating series $\langle \alpha; \gamma_1, \ldots, \gamma_n \rangle^S_{g,n,\beta}$ lies in $\frac{1}{\Delta(q)}\mathbf{QM}$, where $\Delta(q)$ denotes the discriminant form.

Finally, let us mention that, as a consequence of the degeneration formula and the localization formula, the moduli space of relative stable maps, the moduli space of stable map to nonsingular varieties, and the moduli space of non rigid maps play a fundamental role in the algorithm. Appendix A.3 briefly introducea these spaces and some of its most important properties. So, this appendix will play an important role throughout this section.

3.1 K3 surfaces and elliptically fibered K3 fibers

In this subsection, we briefly introduce some notions, and results related to the $K3$ surface, that will allow us to define in Subsection 3.2 the Gromov-Witten invariants we are interested in. Our main reference for this subsection is [23].

Definition 3.1.1. *A complex $K3$ surface is a non-singular proper variety S of dimension 2 such that $H^1(S, \mathcal{O}_S) = 0$ and $\omega_S \simeq \mathcal{O}_S$.*

From this definition follows that $\mathcal{T}_S \simeq \Omega_S$, and that the euler characteristic of S is 24. In particular, the cohomology groups of S are:

$$H^0(S) = H^4(S) = \mathbb{Q}, \quad H^1(S) = H^3(S) = 0, \quad \text{and} \quad H^2(S) = \mathbb{Q}^{22}.$$

Moreover, the Hodge structure of the second cohomology group is $H^{2,0}(S) = \mathbb{C}$, $H^{0,2}(S) = \mathbb{C}$, and $H^{1,1}(S) = \mathbb{C}^{20}$. In the case of the elliptic curve, we were able to identify the possible choice of class $\beta \in H_2(E)$ with $\mathbb{N}$ and gather all the invariants in a generating series. However, for the $K3$ surface, we have that $H_2(S) = \mathbb{Z}^{22}$. So, a priory we can not gather the Gromov-Witten invariants in a generating series. Moreover, one can check that the algebraic curve classes correspond to $H^{1,1}(S,\mathbb{C}) \cap H_2(S,\mathbb{Z})$. The idea will be to focus on the invariants on $\overline{\mathcal{M}}_{g,n}(S,\beta)$ for β a primitive effective curve class. Recall that a class is primitive if it is not divisible. To do so, we first introduce the notion of elliptic $K3$ surface (see Chapter 11 in [23]).

Definition 3.1.2. *An elliptic $K3$ surface is a $K3$ surface S together with a surjective morphism $\pi : S \to \mathbb{P}^1$ whose generic geometric fibers are elliptic curves.*

From the definition, one can check that the morphism $\pi : S \to \mathbb{P}^1$ is flat and thus every geometric fiber will have arithmetic genus 1. However, not every fiber will be smooth (see [23] Chapter 11.1. for a classification of the fibers). The idea is to reduce our analysis to elliptic $K3$ surface. For this purpose, we will use the degeneration axiom and the following Theorem (see Chapter 7, Theorem 1.1 in [23]).

Theorem 3.1.1. *For any two complex $K3$ surfaces S_1 and S_2, there exists a smooth proper morphism of $\mathbb{C}$–schemes $X \to Y$ where Y is connected with two geometric points $t_1, t_2 \in B$ such that the fibers $X_{t_1} \simeq S_1$ and $X_{t_2} \simeq S_2$.*

Thus, we can assume that S is an elliptic $K3$ surface. Moreover, using Chapter 11 Remark 1.4., in [23], we can assume that $\pi : S \to \mathbb{P}^1$ has a section $s : \mathbb{P}^1 \to S$. In addition, we can assume that S has only 24 non-singular fibers and they are rational nodal curves (see [23] Chapter 11.1.). Since π is proper, s is a closed immersion and, thus, we can consider two special homology classes in $H_2(S)$, the fiber class f and the section class s, corresponding to an elliptic curve E and $\mathbb{P}^1$ respectively. In this situation, we have that

$$\mathrm{Pic}(S) \simeq \mathbb{Z}f \oplus \mathbb{Z}s$$

Moreover, we have that $\langle s, s\rangle = -2$, $\langle s, f\rangle = 0$, and $< f, f >= 0$.

As commented above, we are interested in Gromov-Witten invariants on $\overline{\mathcal{M}}_{g,n}(S, \beta)$ with β a primitive effective curve class. In this sense, we focus our attention on $\overline{\mathcal{M}}_{g,n}(S, s + hf)$ for $h \geq 0$ with S an elliptic $K3$ surface with section. By the global Torelli theorem (see [23]) we have the following:

Proposition 3.1.1. *Given two $K3$ surfaces S and S' together with primitive classes β and β' of Hodge type $(1, 1)$, there exists a deformation from S to S' taking β to β' if and only if $\langle\beta, \beta\rangle = \langle\beta', \beta'\rangle$.*

Note that $\langle s + hf, s + hf\rangle = 2h - 2$ and if β is a primitive effective curve class it must be of the form $\beta = m_1 s + m_2 f$ for $m_i \geq 0$. Thus, $\langle\beta, \beta\rangle = -2m_1^2 + 2m_1 m_2$. As a result, Proposition 3.1.1 implies that one can directly work with $\overline{\mathcal{M}}_{g,n}(S, s + hf)$. Moreover, this will allow, in the next subsection, to define a generating series as we did for the elliptic curve case. The task of this section is to compute these invariants.

However, before moving to the next section, let us introduce some notation about the cohomology of S. We will see in the next subsection that again, in the $K3$ surface algorithm, the splitting and reduction axioms play a fundamental role. Thus, we will need to fix a suitable basis of the cohomology of S. In [23], Chapter 3 Proposition 3.5., an explicit expression of the intersection form of S is presented. However, we are interested in a simpler expression of the matrix $(g_{i,j})$. For the rest of the section, let $\mathbf{p}$ denote the class of a point. We want to extend the set $\{1, s, f, \mathbf{p}\}$ to get a proper basis of $H^*(S)$. Using some linear algebra arguments, we can find cohomology classes $\delta_1, \ldots, \delta_n \in H^2(S)$ such that $\{1, s, f, \delta_1, \ldots, \delta_n, \mathbf{p}\}$ is a basis of $H^*(S)$ with $< \delta_i, \delta_j >= \delta_{i,j}$, $< \delta_i, s >= 0$, and $< \delta_i, f >= 0$ for all i, j. We will denote this basis by $\mathcal{B}$. In particular, this means that the matrices $(g_{i,j})$ and $(g^{i,j})$ appearing in the expression of the diagonal classes are:

$$(g_{i,j}) = \left(\begin{array}{c|cc|c|c} 0 & 0 & 0 & \cdots\ 0\ \cdots & 1 \\ \hline 0 & -2 & 1 & \cdots\ 0\ \cdots & 0 \\ 0 & 1 & 0 & \cdots\ 0\ \cdots & 0 \\ \hline \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \mathrm{Id}_{20} & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \hline 1 & 0 & 0 & \cdots\ 0\ \cdots & 0 \end{array}\right) \quad \text{and} \quad (g^{i,j}) = \left(\begin{array}{c|cc|c|c} 0 & 0 & 0 & \cdots\ 0\ \cdots & 1 \\ \hline 0 & 0 & 1 & \cdots\ 0\ \cdots & 0 \\ 0 & 1 & 2 & \cdots\ 0\ \cdots & 0 \\ \hline \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \mathrm{Id}_{20} & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \hline 1 & 0 & 0 & \cdots\ 0\ \cdots & 0 \end{array}\right).$$

3.2 Gromov-Witten invariants on K3 surfaces

In this subsection we begin the study of Gromov-Witten invariants over $K3$ surfaces. In particular, we define a generating series as we did for the elliptic curve. The idea is to compute these series by means of some quasimodular forms. However, a problem arises in the frame of $K3$ surfaces. The virtual class vanishes for $\beta \neq 0$. This difficulty is overcome by the notion of reduced virtual class. We will define the Gromov-Witten invariants over $K3$ surfaces using this new class.

Once we have introduced the invariants we deal with their computation on the K3 surface. The idea is to reason by induction over the tuple (g, n). In particular, in this subsection we check the base cases of the induction. The general case is split in two cases: (1) No evaluation class is the class of a point; (2) There exists a evaluation class that is the class of a point.

The first case is solved at the end of this subsection while the second case will be studied in Subsections 3.3, 3.4, 3.5, and 3.6. In particular, as a conclusion of this algorithm, we will see in Corollary 3.4.2 that all these invariants are elements in $\frac{1}{\Delta(q)}\mathbf{QM}$ where $\Delta(q)$ is the discriminant quasimodular form. Concurrently to this study, we show how to reduce descendent invariants, i.e. invariant of the form

$$\langle \alpha; \tau_{k_1}(\gamma_1) \cdots \tau_{k_n}(\gamma_n) \rangle^S_{g,n,h} := \int_{[\overline{\mathcal{M}}_{g,n}(S,s+hf)]^{\mathrm{red}}} \rho^*(\alpha) \prod_{i=1}^n \tau_{k_i}(\gamma_i)$$

to the regular invariants. The main reference for this subsection is [36]

As spoiled above, let us first deal with the vanishing of the invariants on $\overline{\mathcal{M}}_{g,n}(S, \beta)$ with $\beta \neq 0$.

Proposition 3.2.1. *For $\beta \neq 0$, and $\gamma_1, \dots, \gamma_n \in H^*(S)$ and $\alpha \in RH^*(\overline{\mathcal{M}}_{g,n})$, the Gromov-Witten invariant $\langle \alpha; \gamma_1, \dots, \gamma_n \rangle^S_{g,n,\beta}$ vanishes. Moreover, it holds that $\left[\overline{\mathcal{M}}_{g,n}(S,\beta)\right]^{\text{vir}} = 0$ for $\beta \neq 0$.*

Before seeing how to overcome this difficulty, let us deal with the case $\beta = 0$. As it is shown in Subsection A.2, for $\beta = 0$, it holds that $\overline{\mathcal{M}}_{g,n}(S,0) \simeq \overline{\mathcal{M}}_{g,n} \times S$. In this situation a natural question about the virtual fundamental class and the fundamental class arises: Do they coincide? What is the relation between the virtual fundamental class and the fundamental class? The answer to the first question is negative. More concretely, [7] Proposition 5.5. shows that in case of having both a fundamental class and a virtual fundamental class, both coincide if the virtual dimension is equal to the dimension. However, one can check that the virtual dimension does not coincide with the dimension if $g > 0$. For $g > 0$ the relation among these classes is stated in the "mapping to a point" axiom (see Appendix A.2). In particular, we have that

$$\left[\overline{\mathcal{M}}_{g,n}(S,0)\right]^{\text{vir}} = [\overline{\mathcal{M}}_{g,n} \times S] \frown e(\mathbb{E} \otimes \mathcal{T}_S) \quad \text{for } g \geq 1.$$

In [18], Section 2, the authors compute $e(\mathbb{E} \otimes \mathcal{T}_S)$ for the case of a surface and get

$$e(\mathbb{E} \otimes \mathcal{T}_S) = \begin{cases} c_2(\mathcal{T}_S) - c_1(\mathcal{T}_S)\lambda_1 & \text{if } g = 1 \\ -c_1(\mathcal{T}_S)\lambda_g\lambda_{g-1} + c_1(\mathcal{T}_S)^2\lambda_g\lambda_{g-2} & \text{if } g \geq 2 \end{cases}$$

Furthermore, taking into account that for $K3$ surfaces $c_1(\mathcal{T}_S) = 0$, we get that

$$\left[\overline{\mathcal{M}}_{g,n}(S,0)\right]^{\text{vir}} = \begin{cases} \left[\overline{\mathcal{M}}_{g,n} \times S\right] & \text{if } g = 0 \\ \left[\overline{\mathcal{M}}_{g,n} \times S\right] \frown c_2(\mathcal{T}_S) & \text{if } g = 1 \\ 0 & \text{if } g \geq 2 \end{cases} \tag{21}$$

Finally, using Hirzebruch's Riemann-Roch formula (see [13] Theorem 14.4) we get that $c_2(\mathcal{T}_S) = 24\mathbf{p}$.

Returning to the case $\beta \neq 0$, to deal with the vanishing of the virtual fundamental class, the idea is to use the reduced virtual class $\left[\overline{\mathcal{M}}_{g,n}(S,\beta)\right]^{\text{red}}$. This class has dimension $g + n$. For the construction of this class we refer to [9] and [38]. Using the reduced virtual class we can define reduced Gromov-Witten invariants as

$$\begin{aligned} \langle \alpha; \gamma_1, \dots, \gamma_n \rangle^S_{g,n,h} &:= \int_{\overline{\mathcal{M}}_{g,n}} \alpha \rho_* \left(\left[\overline{\mathcal{M}}_{g,n}(S, s + hf)\right]^{\text{vir}} \frown \prod_i \text{ev}_i^*(\gamma_i) \right) \\ &= \int_{\left[\overline{\mathcal{M}}_{g,n}(S,s+hf)\right]^{\text{vir}}} \rho^*(\alpha) \prod_i \text{ev}_i^*(\gamma_i) \end{aligned}$$

for $\gamma_1, \dots, \gamma_n \in H^*(S)$ and $h \geq 0$. During the rest of the section we will forget about the reduced notation and we will refer to these intersection numbers as invariants.

Now, using the deformation axiom for reduced invariants, and the deformation results introduced in the previous subsection, we focus our study on the case S elliptic $K3$ surface and $\beta = s + hf$. In this situation, as we did for the elliptic curve, we can gather these invariants in a generating series, namely,

$$\langle \alpha; \gamma_1, \dots, \gamma_n \rangle_{g,n}^S := \sum_{h \geq 0} \langle \alpha; \gamma_1, \dots, \gamma_n \rangle_{g,n,h}^S q^{h-1}.$$

Note that again, this generating series is compatible with the degree axiom since $\omega_S \simeq \mathcal{O}_S$ and thus, $\int_{s+hf} c_1(\mathcal{O}_S) = 0$. Moreover, we can generalize these invariants a bit more. Recall from Section 2 that for the elliptic curve case $\rho^*(\psi_i) = \psi_i$. However, Proposition A.2.1 states that, in general, $\rho^*(\psi_i) = \psi_i - \overline{\rho}^*([\Gamma_i])$, where Γ_i is as in Proposition A.1.3. In the frame of the $K3$ surface, the class $\overline{\rho}^*([\Gamma_i])$ does not vanish. As a result, we can consider descendent classes inside the invariant. In this sense, we also take care about invariants of the form

$$\langle \alpha; \tau_{k_1}(\gamma_1) \cdots \tau_{k_n}(\gamma_n) \rangle_{g,n}^S := \sum_{h \geq 0} \left(\int_{\left[\overline{\mathcal{M}}_{g,n}(S, s+hf)\right]^{\mathrm{red}}} \rho^*(\alpha) \prod_i \tau_{k_i}(\gamma_i) \right) q^{h-1}.$$

We refer to these invariants as descendent invariants. Once we have introduced these notions, we can properly state the goal of this section: deriving an algorithm for determining these invariants. Moreover, we will see in Corollary 3.4.2 that these invariants lie on $\frac{1}{\Delta(q)}\mathbf{QM}$, where Δ is the discriminant modular form defined as:

$$\Delta = \frac{1}{1728}(E_4^3 - E_6^2).$$

Note that the power expansion of $\frac{1}{\Delta}$ has one pole of order 1 at zero. As in Section 2, we are interested in finding an expression of the invariants as elements in $\frac{1}{\Delta(q)}\mathbb{Q}[G_2, G_4, G_6]$. About the INPUT of the algorithm, by the linearity axiom, we can assume that $\gamma_i \in \mathcal{B}$. Also, using Theorem A.1.3 we can assume that α is a decorated stratum class.

The general idea is to argue by induction on the pair (g, n) using the following order: $(g, n) > (g', n')$ if and only if $g > g'$, or $g = g'$ and $n > n'$. We structure the argument as follows:

1. We see the first cases of the induction corresponding to the unstable cases $2g - 2 + n \leq 0$.

2. We reduce recursively the computation of the descendent invariants $\langle \alpha; \tau_{k_1}(\gamma_1) \cdots \tau_{k_n}(\gamma_n) \rangle_{g,n}^S$ to invariants of the form $\langle \alpha'; \gamma_1', \dots, \gamma_n' \rangle_{g',n'}^S$.

3. Then, we study the case $(g, n) > (0, 3)$. The argument is divided in two cases. First we solve the case where no evaluation class is the class of a point. The study of the algorithm until this point is done along the rest of this subsection. Finally, the case where one of the evaluation classes is the class of a point will be treated in the Subsections 3.3, 3.4, 3.5, and 3.6.

Let us begin the study of the algorithm through the analysis of the cases where $2g-2+n \leq 0$. In order words, the cases corresponding to $(g,n) \in \{(0,0),(0,1),(0,2),(1,0)\}$. Note that here we are considering the first three cases of our induction. Let us focus on the case $(g,n)=(0,0)$. This case is a concrete case of a general known formula for The Gromov-Witten invariants with a enumerative geometric meaning called the Yau Zaslow formula (see [9] Theorem 1.1.).

Theorem 3.2.1. *For $g \geq 0$,*

$$\langle 1; \mathbf{p}, \ldots, \mathbf{p} \rangle_{g,n}^{S} = \frac{1}{\Delta(q)} \left(q \frac{\partial}{\partial q} G_2 \right)^{g}$$

As we commented in the introduction, these invariants count the number of genus g curves inside S passing through g points. Therefore, for $g=0$, we get $\langle \emptyset \rangle_{0,0}^{S} = \frac{1}{\Delta(q)}$.

For the cases $(0,1)$ and $(0,2)$ the idea consists in applying the dilation, string and divisor equation to reduce the invariants to the base case. In Appendix A.2 we state these equations in the context of the virtual fundamental class. However, we now are working with the reduced virtual class. Nevertheless, the equations remain the same for the reduced frame. The reason why this happens is because these three equations are derived from the fact that for the forgetful morphism π we have

$$\pi^{*}\left(\left[\overline{\mathcal{M}}_{g,n}(X,\beta) \right]^{\text{vir}} \right) = \left[\overline{\mathcal{M}}_{g,n+1}(X,\beta) \right]^{\text{vir}}$$

and the same equality holds for the reduced virtual class (see [36]).

Let us treat now the second case of the induction, i.e. we assume that $(g,n) = (0,1)$. Let us first deal with the argument for regular invariants, and afterwards for the generating series. Consider the invariant $\langle \tau_k(\gamma) \rangle_{0,1,h}^{S}$. By the degree axiom, it must hold that $2k + \deg(\gamma) = 2(g+n) = 2$. Thus, two possibilities appear, either $\tau_k(\gamma) = \tau_1(0)$ or $\tau_k(\gamma) = \tau_0(\gamma)$ with $\gamma \in H^2(S)$.

- If $\tau_k(\gamma) = \tau_1(0)$, we apply the dilation equation to get that $\langle \tau_1(1) \rangle_{0,1,h}^{S} = -2 \langle \emptyset \rangle_{0,0,h}^{S}$ and, as a result,
$$\langle \tau_1(1) \rangle_{0,1}^{S} = -2 \langle \emptyset \rangle_{0,0}^{S} = \frac{-2}{\Delta(q)}$$
- If $\tau_k(\gamma) = \tau_0(\gamma)$ with $\gamma \in H^2(S)$, we can apply the divisor equation to get
$$\langle \tau_0(\gamma) \rangle_{0,1,h}^{S} = \left(\int_{s+hf} \gamma \right) \langle \emptyset \rangle_{0,0,h}^{S}.$$
We observe that if $\gamma \notin \{s,f\}$, then $\int_{s+hf} \gamma = 0$ and, thus, the invariant vanishes. If $\gamma = s$, then $\int_{s+hf} s = -2+h$. On the other hand, if $\gamma = f$ we get $\int_{s+hf} f = 1$.

As a result one obtains that

$$\langle\tau_0(\gamma)\rangle^S_{0,1,h} = \begin{cases} (h-2)\langle\emptyset\rangle^S_{0,0,h} & \text{if } \gamma = s \\ \langle\emptyset\rangle^S_{0,0,h} & \text{if } \gamma = f \\ 0 & \text{else} \end{cases}.$$

Note that for $\gamma = s$, the integer h appears in the invariant and thus the generating series is not directly $\frac{1}{\Delta(q)}$. However, we have that:

$$\langle\tau_0(\gamma)\rangle^S_{0,1} = \sum_{h\geq 0}(h-2)\langle\emptyset\rangle^S_{0,0,h}q^{h-1} = -\langle\emptyset\rangle^S_{0,0} + \sum_{h\geq 0}(h-1)\langle\emptyset\rangle^S_{0,0,h}q^{h-1} =$$
$$-\langle\emptyset\rangle^S_{0,0} + \sum_{h\geq 0}\langle\emptyset\rangle^S_{0,0,h}\left(q\frac{\partial}{\partial q}\right)q^{h-1} = -\frac{1}{\Delta} + q\frac{\partial}{\partial q}\frac{1}{\Delta(q)}.$$

Now, using that $q\dfrac{\partial}{\partial q}\dfrac{1}{\Delta(q)} = \dfrac{24G_2}{\Delta(q)}$, we get that

$$\langle\tau_0(\gamma)\rangle^S_{0,1} = \begin{cases} -\frac{1}{\Delta(q)} + \frac{24G_2}{\Delta(q)} & \text{if } \gamma = s \\ \frac{1}{\Delta(q)} & \text{if } \gamma = f \\ 0 & \text{else} \end{cases}.$$

After analizing the $(0,1)$ case, we assume that $(g,n) = (0,2)$. This means that the invariants are of the form $\langle\tau_{k_1}(\gamma)\tau_{k_2}(\gamma')\rangle^S_{0,2,h}$. Again, by the degree axiom, one can check that the only possibilities for k_i and γ,γ' are:

(1) $\tau_{k_1}(\gamma) = \tau_{k_2}(\gamma') = \tau_1(1)$.

(2) $\tau_{k_1}(\gamma) = \tau_0(1)$ and $\tau_{k_2}(\gamma') = \tau_1(\gamma')$ for $\gamma' \in H^2(S)$.

(3) $\tau_{k_1}(\gamma) = \tau_0(1)$ and $\tau_{k_2}(\gamma') = \tau_0(\gamma')$ for $\gamma' \in H^4(S)$.

(4) $\tau_{k_1}(\gamma) = \tau_1(1)$ and $\tau_{k_2}(\gamma') = \tau_0(\gamma')$ for $\gamma' \in H^2(S)$.

(5) $\tau_{k_1}(\gamma) = \tau_0(\gamma)$ and $\tau_{k_2}(\gamma') = \tau_0(\gamma')$ for $\gamma,\gamma' \in H^2(S)$.

For the cases (1) and (4), after applying the dilation equation, we get that $\langle\tau_1(1)\tau_{k_2}(\gamma')\rangle^S_{0,2} = -\langle\tau_{k_2}(\gamma')\rangle^S_{0,1}$ and we apply the previous case. In the cases (2) and (3) we can apply the string equation, and we get $\langle\tau_0(1)\tau_{k_2}(\gamma')\rangle^S_{0,2} = \langle\tau_{k_2-1}(\gamma')\rangle^S_{0,1}$. For the case (3), $k_2 = 0$ and so, after applying the string equation, we get that the invariant vanishes. On the other hand, case (2) is reduced to the previous induction case. Finally, for case (5) we can apply the divisor equation as we did above and get:

$$\langle\tau_0(\gamma)\tau_0(\gamma')\rangle^S_{0,2} = \begin{cases} -\langle\tau_0(\gamma')\rangle^S_{0,1} + q\frac{\partial}{\partial q}\left(\langle\tau_0(\gamma')\rangle^S_{0,1}\right) & \text{if } \gamma = s, \\ \langle\tau_0(\gamma')\rangle^S_{0,1} & \text{if } \gamma = f, \\ 0 & \text{else} . \end{cases}$$

This finishes the case $(g,n)=(0,2)$. The only missing unstable case to be studied is $(g,n)=(1,0)$. However, note that in this situation we do not have any evaluation classes, neither ψ–classes nor tautological classes in the insertion. Thus the only possibility is $\langle 1\rangle_{1,0}^S$ which vanishes by the degree axiom. Nevertheless, in this situation one can consider also invariants with λ–classes. In particular, one can consider the invariant $\langle \lambda_1\rangle_{1,0}^S$. Let us briefly comment how to compute this invariant. Following [36], we define the rational numbers $R_{g,h}$ as

$$R_{g,h}=\int_{\left[\overline{\mathcal{M}}_{g,0}(S,s+hf)\right]^{\mathrm{red}}}(-1)^g\lambda_g.$$

Then, [36] Corollary 2 shows how to compute explicitly these numbers through the following formula

$$\sum_{g\geq 0}\sum_{h\geq 0}R_{g,h}u^{2g-2}q^{h-1}=\frac{1}{u^2\Delta(q)}\exp\left(\sum_{g\geq 1}u^{2g}\frac{|B_{2g}|}{g\cdot(2g)!}E_{2g}(q)\right).$$

This formula is known as the KKV (Katz-Klemm-Vafa) formula. In particular, using this formula we get that

$$\langle\lambda_1\rangle_{1,0}^S=\frac{2G2}{\Delta}.$$

This finishes the study of the cases $2g-2+n\leq 0$. Our next step will be to reduce the computation of descendent invariants to invariants of the form $\langle\alpha;\gamma_1,\ldots,\gamma_n\rangle_{g,n}^S$. For this purpose, we deal with the class $\overline{\rho}^*([\Gamma_i])$. We will argue by using the analogous to the splitting axiom for the reduced virtual class. The reduced axiom will not change in the reduced frame. However, the reduced version of the splitting axiom differs from the original. Recall that the splitting axiom is a consequence of the behavior of the virtual fundamental class under the gluing morphisms (see Appendix A.2). In the reduced case we get the following version of Appendix A.2 equation (43):

$$\begin{aligned}\xi_\Gamma^!\left(\left[\overline{\mathcal{M}}_{g,n}(S,\beta)\right]^{\mathrm{red}}\right)=p_*\circ\Delta^!\left(\left[\overline{\mathcal{M}}_{g_1,n_1+1}(S,\beta)\right]^{\mathrm{red}}\times\left[\overline{\mathcal{M}}_{g_2,n_2+1}(S,0)\right]^{\mathrm{vir}}\right)+\\ p_*\circ\Delta^!\left(\left[\overline{\mathcal{M}}_{g_1,n_1+1}(S,0)\right]^{\mathrm{vir}}\times\left[\overline{\mathcal{M}}_{g_2,n_2+1}(S,\beta)\right]^{\mathrm{red}}\right)\end{aligned}$$

where $\Delta^!$ denotes the Gysin map of Δ (see [36] page 62). As a consequence of the above equation and equation (21), we can state the reduced version of the splitting axiom (see [31] Proposition 1.1).

Proposition 3.2.2. *Let Γ be a stable graph as in the splitting axiom (see Appendix A.2 Figure 16) and let $[\Gamma,\alpha]$ be a decorated stratum class over Γ, with $\alpha=\alpha_1\otimes\alpha_2$. Let $\gamma_1,\ldots,\gamma_n\in H^*(S)$, then*

$$\begin{aligned}\langle[\Gamma,\alpha];\gamma_1,\ldots,\gamma_n\rangle_{g,n,s+hf}^S=\sum_{k,l}g^{k,l}\left(\langle\alpha_1;(\gamma_i)_{i\in S_1},I_k\rangle_{g_1,|S_1|+1,\beta}^S\ I_2\right.\\ \left.+\ I_1\ \langle\alpha_2;(\gamma_i)_{i\in S_2},I_l\rangle_{g_2,|S_2|+1,\beta}^S\right)\end{aligned}$$

where I_1 *(similarly* I_2*) is defined as*

$$I_1 = \begin{cases} \left(\int_{\overline{\mathcal{M}}_{g_1,|S_1|+1}} \alpha_1\right)\left(\int_S \prod_{i\in S_1} \gamma_i T_k\right) & \text{if } g_1 = 0, \\ \left(\int_{\overline{\mathcal{M}}_{g_1,|S_1|+1}} \alpha_1\right)\left(\int_S 24\mathbf{p} \prod_{i\in S_1} \gamma_i T_k\right) & \text{if } g_1 = 1, \\ 0 & \text{if } g_1 \geq 0, \end{cases}.$$

Now, using this axiom, not only for stable graphs, but also for unstable graphs, we can reduce recursively the computation of the descendent invariants $\langle \alpha; \tau_{k_1}(\gamma_1)\cdots\tau_{k_n}(\gamma_n)\rangle_{g,n}^S$ to invariants with $k_i = 0$. The idea is to recursively move the ψ classes on $\overline{\mathcal{M}}_{g,n}(S,\beta)$ to their tautological analogous on $\overline{\mathcal{M}}_{g,n}$. This recursion is stated in the following result.

Proposition 3.2.3. *Let* $g, n \geq 0$ *with* $2g-2+n>0$, $\alpha \in RH^*(\overline{\mathcal{M}}_{g,n})$, $\gamma_1,\ldots,\gamma_n \in H^*(S)$, *and* $k_1,\ldots,k_n \geq 0$ *with* $k_1 > 0$. *Then,*

$$\langle \alpha; \tau_{k_1}(\gamma_1)\cdots\tau_{k_n}(\gamma_n)\rangle_{g,n}^S = \langle \alpha\psi_1; \tau_{k_1-1}(\gamma_1)\cdots\tau_{k_n}(\gamma_n)\rangle_{g,n}^S + \Lambda$$

where Λ *is defined as*

$$\Lambda = \begin{cases} \sum_{k,l} g^{k,l}\left(\int_{\overline{\mathcal{M}}_{0,n}} \alpha \prod_{i=2}^n \psi_i^{k_i}\right)\left(\int_S \prod_{i=2}^n \gamma_i T_l\right)\langle \tau_{k_1-1}(\gamma_1)\tau_0(T_k)\rangle_{0,2}^S & \text{if } g = 0 \\ 0 & \text{if } g \geq 1 \end{cases}$$

Proof. As commented above, the idea is to use the splitting axiom for the prestable graph Γ_i as in Appendix A.1 Figure 15. We prove the statement for the invariant $\langle \alpha; \tau_{k_1}(\gamma_1)\cdots\tau_{k_n}(\gamma_n)\rangle_{g,n,h}^S$ for a particular h rather for the generating series. As a consequence of Proposition A.1.3 we get that $\psi_i = \rho^*(\psi_i) + \overline{\rho}^*([\Gamma_i])$, and thus we have

$$\langle \alpha; \tau_{k_1}(\gamma_1)\cdots\tau_{k_n}(\gamma_n)\rangle_{g,n}^S = \int_{\left[\overline{\mathcal{M}}_{g,n}(S,s+hf)\right]^{\mathrm{red}}} \rho^*(\alpha)\prod_{i=1}^n \psi_i^{k_i}\mathrm{ev}_i^*(\gamma_i) =$$

$$\int_{\left[\overline{\mathcal{M}}_{g,n}(S,s+hf)\right]^{\mathrm{red}}} \rho^*(\alpha)(\rho^*(\psi_i) + \overline{\rho}^*([\Gamma_i]))\psi_i^{k_i-1}\mathrm{ev}_i^*(\gamma_1)\prod_{i=2}^n \psi_i^{k_i}\mathrm{ev}_i^*(\gamma_i) =$$

$$\langle \alpha\psi_1; \tau_{k_1-1}(\gamma_1)\cdots\tau_{k_n}(\gamma_n)\rangle_{g,n}^S + \int_{\left[\overline{\mathcal{M}}_{g,n}(S,s+hf)\right]^{\mathrm{red}}} \rho^*(\alpha)\overline{\rho}^*([\Gamma_i])\,\psi_i^{k_i-1}\mathrm{ev}_i^*(\gamma_1)\prod_{i=2}^n \psi_i^{k_i}\mathrm{ev}_i^*(\gamma_i).$$

From the above equality, we just need to check that the second term is equal to Λ. To do so, we can apply the reduced splitting axiom to Γ_1 to get that the second term is equal to

$$\sum_{k,l} g^{k,l}\left(\langle \tau_{k_1-1}(\gamma_1)\tau_0(T_k)\rangle_{0,2,s+hf}^S\, I_2 + I_1\,\langle \alpha; \tau_{k_2}(\gamma_2)\cdots\tau_{k_n}(\gamma_n)\tau_0(T_l)\rangle_{g,n,s+hf}^S\right)$$

with I_i defined as in Proposition 3.2.2. However, in this case, $\overline{\mathcal{M}}_{0,2}(S,0)$ is empty because of the stability conditions; so $I_1 = 0$. Thus, we can focus on I_2. It will be enough to check that

$$\sum_{k,l} g^{k,l} \langle \tau_{k_1-1}(\gamma_1)\tau_0(T_k)\rangle^S_{0,2,s+hf}\, I_2 = \Lambda.$$

Note that, for $g = 0$ or $g = 2$ the equality holds. For $g = 1$ we need to check that I_2 vanishes. In this case, we get that

$$I_2 = \left(\int_{\overline{\mathcal{M}}_{1,n}} \alpha \prod_{i=2}^{n} \psi_i^{k_i}\right)\left(\int_S 24\mathbf{p} \prod_{i=2}^{n} \gamma_i T_l\right).$$

As a result, the only term of I_2 that does not vanish is the corresponding to $(T_k, T_l) = (p, 1)$. However, in this case the invariant $\langle \tau_{k_1-1}(\gamma_1)\tau_0(p)\rangle^S_{0,2,s+hf}$ vanishes as we saw in the above study of the basic cases of the induction. Thus, for $g = 1$, we also get the desired equality. □

Now, using this result, we can recursively erase the descendent classes from our invariant. Thus, we can write the descendent invariants by means of invariants of the form $\langle \alpha; \gamma_1, \dots, \gamma_n\rangle^S_{g,n}$. Moreover, note that in most of the cases the term Λ will vanish.

Let us illustrate by an example how this recursion works. We consider the invariant $\langle 1; \tau_0(1)\tau_0(s)\tau_1(f)\rangle^S_{0,3}$. Using Proposition 3.2.3 we get that

$$\langle 1; \tau_0(1)\tau_0(s)\tau_1(f)\rangle^S_{0,3} = \langle \psi_3; \tau_0(1)\tau_0(s)\tau_0(f)\rangle^S_{0,3} + \\ \sum_{k,l} g^{k,l}\left(\int_{\overline{\mathcal{M}}_{03}} 1\right)\left(\int_S sT_l\right)\langle \tau_0(f)\tau_0(T_k)\rangle^S_{0,2}.$$

We observe that $\langle \psi_3; \tau_0(1)\tau_0(s)\tau_0(f)\rangle^S_{0,3} = 0$, since $\psi_3 = 0$ in $\overline{\mathcal{M}}_{0,3}$. Now, using the previous study of the basic cases of the induction, we deduce that the possible choices of (T_k, T_l) are (s, f), (f, s), and (f, f). Therefore, we get that

$$\langle 1; \tau_0(1)\tau_0(s)\tau_1(f)\rangle^S_{0,3} = -2\langle \tau_0(f)\tau_0(f)\rangle^S_{0,2} + \langle \tau_0(f)\tau_0(s)\rangle^S_{0,2} + 2\langle \tau_0(f)\tau_0(f)\rangle^S_{0,2} = \\ \langle \tau_0(f)\tau_0(s)\rangle^S_{0,2}.$$

Note that this is exactly the same equality we get by applying the string equation.

We will focus now on the invariants $\langle \alpha; \gamma_1, \dots, \gamma_n\rangle^S_{g,n}$. Recall that we have already studied the cases $(g, n) = (0, 0), (0, 1), (0, 2)$, and $(1, 0)$. So we can assume that $(g, n) > (0, 2)$ and $2g-2+n > 0$. As commented above, we will distinguish two cases depending on whether one of the evaluation classes is equal to $\mathbf{p}$ or not. The rest of this subsection will be devoted to study the case where no evaluation class is the class of a point. The case where one of the evaluation classes is the class of a point will be solved in the next subsections. Thus, let us assume that $\gamma_i \neq \mathbf{p}$ for all i. In particular, this means that $\deg(\gamma_i) \leq 1$. By the degree axiom we get that

$$g + n = \deg(\alpha) + \sum_{i=1}^{n} \deg(\gamma_i) \leq \deg(\alpha) + n.$$

As a result, we get that $\deg(\alpha) \geq g$, and the Ionel-Getzler vanishing can be applied (see [14] Proposition 2).

Theorem 3.2.2. *For $\alpha \in RH^d(\overline{\mathcal{M}}_{g,n})$, if $d \geq g + \delta_{0,g} - \delta_{0,n}$, then*

$$\alpha \in \mathrm{Im} RH^*(\partial\overline{\mathcal{M}}_{g,n}) \subset RH^*(\overline{\mathcal{M}}_{g,n}).$$

Note that the only case where we might not be able to apply this result is when $\deg(\alpha) = 0$. However, in this situation we get that $\deg(\gamma_i) = 1$ for all i, and we can apply the divisor equation, as we did in the basic cases of the induction, to write the invariant by means of lower invariants in the induction. We assume now that $\deg(\alpha) > 0$. Using the Ionel-Getzler vanishing, we can assume that α is a decorated stratum class with non trivial stable graph. In other words, we assume that the tautological class inside the invariant is of the form $[\Gamma, \alpha]$ for Γ non trivial stable graph. To compute this invariant we can apply the reduced versions of the splitting and reduction axiom. In particular, we have the reduced analogous to Corollary 2.5.2.

Proposition 3.2.4. *Let Γ be a non trivial stable graph and let $[\Gamma, \alpha]$ be a decorated stratum class over Γ. Then, for $\gamma_1, \ldots, \gamma_n \in H^*(S)$, it holds that*

$$\begin{aligned}\langle [\Gamma, \alpha]; \gamma_1, \ldots, \gamma_n \rangle^S_{g,n} \;=\; & \sum_{v \in V(\Gamma)} \sum_{\substack{i_h, i_{h'} \\ (h,h') \in E(\Gamma)}} \prod_{(h,h') \in E(\Gamma)} g^{i_h, i_{h'}} \\ & \langle \alpha_v; (\gamma_i)_{i \in L(v)}, (T_h)_{h \in E(v)} \rangle^S_{g(v),n(v)} \prod_{u \in V(\Gamma)\setminus\{v\}} I_u,\end{aligned}$$

where I_v is defined as

$$I_u = \begin{cases} \left(\int_{\overline{\mathcal{M}}_{g(u),n(u)}} \alpha_u\right) \left(\int_S \prod_{i \in L(u)} \gamma_i \prod_{h \in E(u)} T_{i_h}\right) & \text{if } g(u) = 0 \\ \left(\int_{\overline{\mathcal{M}}_{g(u),n(u)}} \alpha_u\right) \left(\int_S 24\mathbf{p} \prod_{i \in L(u)} \gamma_i \prod_{h \in E(u)} T_{i_h}\right) & \text{if } g(u) = 1 \\ 0 & \text{if } g(u) \geq 2. \end{cases}$$

Now notice that, since the stable graph Γ is not trivial, all the invariants appearing on the right hand side of the above formula are lower in the induction. As a result, this finishes the case where all the evaluation classes have degree lower than 2. Before ending with this case, let us briefly comment the above formula. First of all, we observe that if the stable graph Γ has two or more vertices, if genus greater or equal than 2, then the invariant is directly zero. However, we can refine more our arguments in this vanishing case. For $v \in V(\Gamma)$, we define the weight of v as

$$w(v) = \begin{cases} g(v) & \text{if } g(v) \in \{0,1\} \\ 2 & \text{if } g(v) \geq 2. \end{cases}$$

We define the weight of a path inside Γ as the sum of weights of the vertices of the path (paths inside Γ are required not to repeat vertices). Thus, one deduces that the invariant $\langle [\Gamma, \alpha]; \gamma_1, \ldots, \gamma_n \rangle^S_{g,n}$ vanishes whenever Γ is a non-trivial stable graph with a path of weight greater or equal to 4. The argument for proving this fact lies in the proofs presented in Subsection 2.5. The idea is as follows: if such a path exists, one can factor the gluing morphisms related to Γ through the gluing morphism of a stable graph of the form of Appendix A.2 Figure 16, with both g_1 and g_2 greater or equal than 2. Now, the vanishing of these morphisms is a consequence of the reduced version of the splitting axiom.

Once we have concluded with the case where no evaluation class is the class of a point, we focus our attention on the case where at least one evaluation class is equal to $\mathbf{p}$. Therefore, we can assume without loss of generality that $\gamma_1 = \mathbf{p}$. The rest of the section will be fully devoted to this case. The argument will be divided in four steps. In Subsection 3.3, the degeneration formula and the fact that $\gamma_1 = \mathbf{p}$ will allow us to rewrite the invariants in terms of relative invariants on $\mathbb{P}^1 \times E$ relative to E, and invariants where we can apply induction. Secondly, in Subsection 3.4 we reduce the computation of these relative invariants to invariants over elliptic curves using the product formula. To do so we will express the cohomology class $\mathcal{J}_g(k,n)$ (see (29)) in term of tautological classes. Subsections 3.5 and 3.6 will focus on finding this expression. First we will apply the localization formula in Subsection 3.5. As a result of applying localization, ψ_0 classes on the moduli space of non-rigid maps will appear. In Subsection 3.6 we will see how to deal with these classes.

3.3 Degeneration Formula

As stated above, we are interested in computing the invariant

$$\langle \alpha; \mathbf{p}, \gamma_2, \ldots, \gamma_n \rangle^S_{g,n}.$$

For this purpose, the first step will be to apply the degeneration formula. This subsection will be fully devoted to understand how the degeneration formula is applied in our case. In this formula the moduli space of relative stable maps and the moduli space of stable maps to a singular variety play a crucial role; Appendix A.3 provides a brief introduction to these spaces. The main references, that we will follow, are [16], [32], and [33], [34].

Let Y be a projective $\mathbb{C}$–scheme which is the union of two projective smooth schemes Y_1 and Y_2 whose intersection is a smooth divisor D. Let $\overline{\mathcal{M}}_{g,n}(Y,\beta)$ be the moduli space of stable maps to Y (see Appendix A.3). The degeneration formula allows to express invariant on this space in terms of relative invariants over the moduli spaces of relative stable maps to Y_1 and Y_2 relative to D_1 and D_2, respectively. In order to reach this situation, the first step will be to find a proper degeneration of S to a suitable scheme Y as before. More precisely, we will apply the normal cone degeneration to $E \hookrightarrow S$ and we will get a degeneration

$$S \rightsquigarrow \left(\mathbb{P}^1 \times E\right) \cup_E S.$$

Thus, the degeneration formula will allow us to express invariants on S by means of relative invariants on S and $\mathbb{P}^1 \times E$ relative to E and $0 \times E$, respectively. Moreover, we will see how to apply induction to compute the relative invariants on S relative to E. The main references, that we will follow, are [16], [32], [33], [34], and [36]. For the construction of the desired degeneration we will follow [16]. We follow [32], [33], and [34] for the introduction of the degeneration formula. On the other hand, in [36] Section 7, it is shown how to apply the degeneration formula to our problem.

As commented above, our first step is to build a suitable degeneration for S. In our case, this degeneration will be the normal cone degeneration. We will do the construction for the concrete case of our elliptically fibered $K3$ surface (see Chapter 5 in [16] for the general construction). Recall that $S \to \mathbb{P}^1$ is an elliptically fibered $K3$ surface with a section. Let $E \subseteq S$ be an elliptic fiber. The idea is to construct a flat morphism $\mathcal{W} \longrightarrow \mathbb{P}^1$ such that for every $t \in \mathbb{P}^1 \setminus \{0\}$ geometric point, the fiber $\mathcal{W}_t = S$ and $\mathcal{W}_0 = S \cup_E \mathbb{P}^1$.

Recall that, for a closed immersion $X \hookrightarrow Y$ with corresponding sheaf of ideals $\mathcal{I}$, the normal cone $C_X Y$ is defined as:

$$C_X Y := \underline{\operatorname{Spec}}_X \left(\bigoplus \mathcal{I}/\mathcal{I}^2 \right).$$

We are interested in computing $C_E S$. To do so, we recall first that we have the following cartesian diagram:

$$\begin{array}{ccc} E & \xrightarrow{\pi'} & 0 \\ \big\downarrow & & \big\downarrow \\ S & \xrightarrow{\pi} & \mathbb{P}^1 \end{array} \tag{22}$$

Recall that $S \to \mathbb{P}^1$ is flat and that regular embeddings are stable under flat base change. Using that $t \hookrightarrow \mathbb{P}^1$ is regular, we get that the embedding $E \hookrightarrow S$ is regular of codimension 1. Thus, by result B.6.2. of [16], we have that $C_E S$ coincides with the normal bundle $N_E S$. In particular, one gets that $C_E S$ is a line bundle. Now, using again the fibered diagram (22), we get a closed immersion $C_E S \hookrightarrow \pi'^* (C_0\mathbb{P}^1) = \mathbb{A}^1 \times E$. Since $C_E S$ is also a line bundle we get that $C_E S = \mathbb{A}^1 \times E$. In particular, as line bundles we have $C_E S = \mathcal{O}_E$.

After this computation, we can construct our degeneration. We consider the blow up $W = \mathrm{Bl}_{E\times 0}(S \times \mathbb{P}^1) \to S \times \mathbb{P}^1$. For $t \in \mathbb{P}^1$ geometric point, we denote by W_t the respective fiber through the composition $\pi : W \to S \times \mathbb{P}^1 \to \mathbb{P}^1$. Using this notation we get that $W_t = S$ for $t \neq 0$ and $W_0 = \mathbb{P}(C_E S \oplus \mathcal{O}_E) \cup_E S$. By the previous computation, $C_E S = \mathcal{O}_E$ and thus, $W_0 = (\mathbb{P}^1 \times E) \cup_E S$. Moreover, using result B.6.7. of [16], we get that the morphism $\pi : W \to \mathbb{P}^1$ is flat. As a result, we get the desired degeneration.

As in Appendix A.3, we consider the moduli space $\overline{\mathcal{M}}_{g,n}(\mathcal{W}, s+hf)$ related to our family W. Recall that for $t \in \mathbb{P}^1$ geometric point we have that $\overline{\mathcal{M}}_{g,n}(\mathcal{W}, s+hf)_t = \overline{\mathcal{M}}_{g,n}(W_t, s+hf)$. The idea now is to use an analogous to the degeneration axiom for this family. Note that, we cannot use directly the degeneration axiom since W_0 is singular. Nevertheless, in [32] equation (6.1), the analogous result for our family is stated. Thus, we only need to see how the evaluation classes and the class $s+hf$ behave under our family.

Let us begin with the evaluation classes. Let $\gamma \in H^*(S)$. We need to find a locally constant section $\gamma_t \in H^*(W_t)$ for all $t \in \mathbb{P}^1$. Clearly, for $t \neq 0$, $\gamma_t = \gamma$. The idea is to find a cohomology class $\overline{\gamma} \in H^*(W)$ such that, for every $t \neq 0$, the pullback to W_t is γ. First of all, we consider the following cartesian diagrams:

$$\begin{array}{ccccc}
 & & W & \xleftarrow{\iota'_t} & W_t \\
 & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_t} \\
S & \xleftarrow{p_1} & S \times \mathbb{P}^1 & \xleftarrow{\iota_t} & S \times t \\
\downarrow & & \downarrow{\scriptstyle p_2} & & \downarrow \\
\mathrm{spec}(\mathbb{C}) & \longleftarrow & \mathbb{P}^1 & \longleftarrow & t
\end{array}$$

Note that $p_1 \circ \iota_t = \mathrm{Id}_S$ and $\pi_t = \mathrm{Id}_S$ for every $t \neq 0$. Let $\overline{\gamma} = \pi^* \circ p_1^*(\gamma)$. Then, for $t \neq 0$

$$\gamma_t := \iota'_t(\overline{\gamma}) = \pi_t^* \circ \iota_t^* \circ p_1^*(\gamma) = \gamma.$$

As a result, $\overline{\gamma}$ is the class we are looking for. In particular, we are interested in $\gamma_0 = \pi_0^*(\gamma) \in H^*(W_0)$ for $\gamma \in \mathcal{B}$. Note that from the surjection

$$p : \big(\mathbb{P}^1 \times E\big) \sqcup S \longrightarrow W_0$$

and using a Mayer Vietoris argument we get that $H^*(W_0) \subseteq H^*(\mathbb{P}^1 \times E) \oplus H^*(S)$. Clearly, for $\gamma = 1$ we have $\gamma_0 = 1$. Let focus on the case $\gamma = s$. We denote by s the classes of the sections of both S and $\mathbb{P}^1 \times E$. Note that the morphism π_0 is not flat and hence the pullback does note behave properly with fundamental classes. However, we can consider the composition $p' := \pi_0 \circ p$. This morphism is neither flat, but now it is between two nonsingular varieties. Then, the pullback of $s = [\mathbb{P}^1]$ through this morphisms is $[(p')^{-1}(E)] = (s, s)$. Moreover, one can check that, for $\gamma \in \{\delta_1, \ldots, \delta_{20}\}$, $\gamma_0 = (\gamma, 0) \in H^*(W_0)$.

The only left cases to be studied are $\gamma \in \{f, \mathbf{p}\}$. Let us show the argument for $\mathbf{p}$. The case $\gamma = f$ will derive from this. We want to choose γ_0 to be $(\mathbf{p}, 0)$, $(\mathbf{p}, \mathbf{p})$, or $(0, \mathbf{p})$. Recall that our elliptic $K3$ surface has a section $s : \mathbb{P}^1 \hookrightarrow S$. This means that we have a closed immersion $(s, \mathrm{Id}) : \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow S \times \mathbb{P}^1$. Using [22] Corollary 7.15 we get a closed immersion $\mathrm{BL}_{(0,0)}(\mathbb{P}^1 \times \mathbb{P}^1) \hookrightarrow W$. Now, we have an open immersion $\mathbb{A}^2 \subset \mathbb{P}^1 \times \mathbb{P}^1$ around $(0, 0)$. Using that open immersions are flat, we get the following open immersion follwed by a closed immersion

$$\mathrm{BL}_0(\mathbb{A}^2) \hookrightarrow \mathrm{BL}_{(0,0)}(\mathbb{P}^1 \times \mathbb{P}^1) \hookrightarrow W$$

Inside $\mathbb{P}^1 \times \mathbb{P}^1$ we can take the line $L_1 := \{x\} \times \mathbb{P}^1$ with $x \in \mathbb{P}^1 \backslash \{0\}$. Since $0 \notin L_1$, it lifts to a line $\overline{L}_1 := \pi^{-1}(L_1)$ in W. In particular, $\overline{L}_1 \times_W W_t = \{x\}$ for every $t \in \mathbb{P}^1$. Moreover, taking $\overline{\gamma} = [\overline{L}_1]$ we get that $\gamma_0 = (0, [x]) = (0, \mathbf{p})$.

On the other hand, let L_2 be the diagonal $\Delta \subset \mathbb{P}^1 \times \mathbb{P}^1$. We get that $(0, 0)$ lies in L_2. Thus, L_2 lifts to a line $\overline{L}_2$ in W. Note that $\overline{L}_2 \subset \mathrm{BL}_0(\mathbb{P}^1 \times \mathbb{P}^1) \subset W$ and $\overline{L}_2$ intersects the exceptional divisor of $\mathrm{BL}_0(\mathbb{P}^1 \times \mathbb{P}^1)$ (which is isomorphic to $\mathbb{P}^1$) in the point $[1 : 1]$. Note that this holds since we can work locally around $(0, 0)$ and we can assume that we are in $\mathrm{BL}_0(\mathbb{A}^2)$. As a result, since the closed immersion $\mathrm{BL}_0(\mathbb{P}^1 \times \mathbb{P}^1) \hookrightarrow W$ leads to a closed immersion among the exceptional divisors, we get that $\overline{L}_2$ intersects the exceptional divisor of W, $\mathbb{P}^1 \times E$, at the point $([1 : 1], 0)$. Hence, $\overline{L}_2 \times_W W_0 = ([1 : 1], 0)$, and choosing $\overline{\gamma} = [\overline{L_2}]$ we get $\gamma_0 = (\mathbf{p}, 0)$. Similarly, choosing L_3 to be the line $\{0\} \times \mathbb{P}^1$, we get that for $\overline{\gamma} = [\overline{L}_3]$, $\gamma_0 = (\mathbf{p}, \mathbf{p})$, for $\overline{L}_3$ the respective lift of L_3.

Note that for $\gamma = f$ we can apply the same argument as before but choosing the class of $L_i \times_{\mathbb{P}^1 \times \mathbb{P}^1} \mathbb{P}^1 \times S \subset \mathbb{P}^1 \times S$. Summarizing we get that:

- If $\gamma = 1$, $\gamma_0 = (1, 1)$.
- If $\gamma = s$, $\gamma_0 = s = (s_1, s_2)$.
- If $\gamma = f$, we can choose γ_0 to be (f, f), $(f, 0)$ or $(0, f)$,
- If $\gamma = \mathcal{B} \backslash \{1, s, f, \mathbf{p}\}$, $\gamma_0 = (0, \gamma)$.
- If $\gamma = \mathbf{p}$, we can choose γ_0 to be $(\mathbf{p}, \mathbf{p})$, $(\mathbf{p}, 0)$ or $(0, \mathbf{p})$.

Similarly, for the homology class $s + hf$, we can set for $t = 0$ the respective class $s + hf$ in W_0. As a result of this study, we can state the analogous to the degeneration axiom for our family.

Proposition 3.3.1. *For $\gamma_1, \ldots, \gamma_n \in H^*(S)$ and $\alpha \in RH^*(\overline{\mathcal{M}}_{g,n})$, it holds*

$$\langle \alpha; \gamma_1, \ldots, \gamma_n \rangle^S_{g,n,h} = \langle \alpha; \gamma_{1,0}, \ldots, \gamma_{n,0} \rangle^{W_0}_{g,n,h}.$$

In particular, this implies that the invariants on S can be computed through invariants on W_0. The next step consists in determining the invariants on W_0. To do so, the main tool is the degeneration formula. In the following, we will roughly introduce the idea behind this formula. For the proofs, we refer to [34]. We recall (see Subsection A.3) that a stable map to W_0 is a tuple $(f : C \to W_0[k], p_1, \dots, p_n)$ for some k. In particular, $W_0[k]$ splits in two components $S[i]$ and $(\mathbb{P}^1 \times E)[k-i]$ for $i \in \{0, \dots, k\}$. This means that we can split the data of the stable map to W_0 in two relative stable maps to S and $\mathbb{P}^1 \times E$ respectively, with same relative profile (see Figure 6). The idea is to first study the different possible choices for this splitting and construct a morphism from the product of respective moduli spaces of relative stable maps to $\overline{\mathcal{M}}_{g,n}(S, s+hf)$. Then, the degeneration formula will be a consequence of the behavior of the virtual fundamental classes under these morphisms.

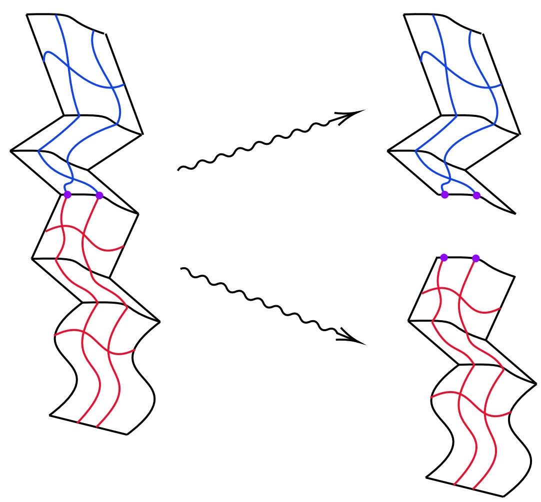


Figure 6: Splitting of a stable map to W_0 into two relative stable maps to S and $\mathbb{P}^1 \times E$.

The first choices to take in this splitting are about the genus, the markings and the class $s + hf$. There are no restrictions on the possible choices of these splittings. Let $g = g_1 + g_2$, $\{1, \dots, n\} = S_1 \sqcup S_2$ and $h = h_1 + h_2$. Thus, we can split $(f : C \to W_0[k], p_1, \dots, p_n)$ as two relative stable maps

$$(f_1 : C_1 \to S[k_1], p_i : i \in S_1, q_1, \dots, q_r), \quad (f_2 : C_2 \to \mathbb{P}^1 \times E[k_1], p_i : i \in S_2, q'_1, \dots, q'_r)$$

lying in $\overline{\mathcal{M}}_{g_1,|S_1|}(S, \mu, s+h_1 f)$ and $\overline{\mathcal{M}}_{g_2,|S_2|}(S, \mu, s+h_2 f)$ respectively, for some partition μ of length r. Note that the relative conditions of both spaces must be the same by the definition of stable map to W_0.

Our next step is to prove that μ must be the partition (1) of 1. This will be a consequence of $\varepsilon_* \circ f_*([C]) = s + hf$, where ε is the projection from $W_0[k]$ to W_0. First of all, we recall that we checked at the beginning of the section that $N_{S/E} = \mathcal{O}_E$ and, as a result, $\Delta = \mathbb{P}^1 \times E$. Now, we split the morphsim f into $k+2$ components as follows. Let $f_0 : C_0 \to S$ be the restriction of f to the components mapping to S. Similarly we consider $f_i : C_i \to \Delta = \mathbb{P}^1 \times E$ be the restriction of f to the components mapping to the i–copy of Δ, for $i \in \{1, \ldots, n\}$. Finally, we denote by $f_{k+1} : C_{k+1} \to \mathbb{P}^1 \times E$ the restriction of f to $\mathbb{P}^1 \times E$ respectively (see Figure 7).

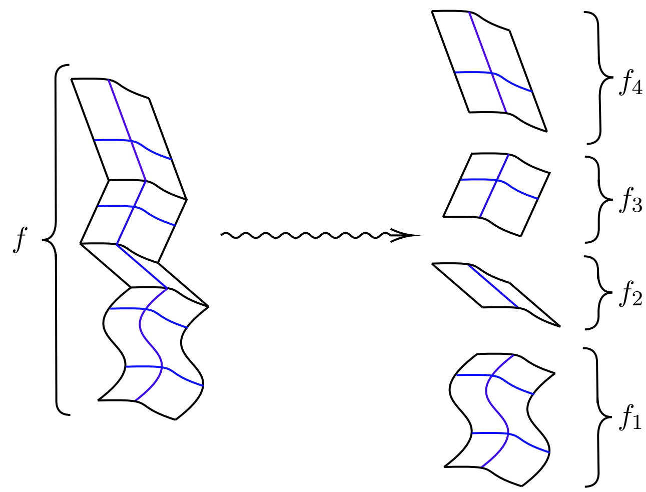


Figure 7: Splitting of a stable map f to W_0 in the morphisms f_i to each component of $W_0[k]$.

We observe that f_{k+1} corresponds to two morphisms to $\mathbb{P}^1$ and E respectively. Since $f_{k+1*}([C_{k+1}]) = s + h'f$ for some h', the morphism to $\mathbb{P}^1$ must be constant for every irreducible component of C_{k+1} apart from one component whose restriction must have degree 1 and, hence, it must be an isomorphism. On the other hand, by definition, only nodes can be mapped to $0 \times E$. As a result the order of contact of C_{k+1} with $0 \times E$ must be 1 corresponding to the transversal intersection of the divisor with the component mapping to $\mathbb{P}^1$ with degree 1. Again, by definition, the order of contact of C_k and C_{k+1} with E must be the same. This means that, again, $f_{k*}([C_k]) = s + h'f$ for some h' (if not, the order of contact will be different). Applying this argument recursively to every morphism f_i we can conclude that μ must be a partition of 1.

Let us introduce the set gathering all the possible splittings as

$$\Omega_{g,n,h} = \{(g_1, g_2, S_1, S_2, h_1, h_2) : g = g_1 + g_2,\ S_1 \sqcup S_2 = \{1, \ldots, n\} \text{ and } h = h_1 + h_2\}.$$

For every $\Gamma = (g_1, g_2, S_1, S_2, h_1, h_2) \in \Omega_{g,n,h}$, we will denote by $\overline{\mathcal{M}}_{\Gamma_1}(\mathbb{P}^1 \times E/E)$ and $\overline{\mathcal{M}}_{\Gamma_2}(S/E)$ the moduli spaces

$$\begin{aligned} \overline{\mathcal{M}}_{\Gamma_1}(\mathbb{P}^1 \times E/E) &:= \overline{\mathcal{M}}_{g_1,|S_1|}(\mathbb{P}^1 \times E/E, (1), s + h_1 f) \\ \overline{\mathcal{M}}_{\Gamma_2}(S/E) &:= \overline{\mathcal{M}}_{g_2,|S_2|}(S/E, (1), s + h_2 f). \end{aligned}$$

Using this notation, we get a morphism

$$\Phi_\Gamma : \ \overline{\mathcal{M}}_{\Gamma_1}(\mathbb{P}^1 \times E/E) \times_E \overline{\mathcal{M}}_{\Gamma_2}(S/E) \ \longrightarrow \ \overline{\mathcal{M}}_{g,n}(W_0, s + hf)$$

by gluing relative stable maps $(f : C \to \mathbb{P}^1 \times E[k], p_i : i \in S_1, q_1)$ and $(f' : C' \to S[k], p'_i : i \in S_2, q'_1)$ through the E_∞ divisor. Note that this is possible since $f(q_1) = f'(q'_1)$ in E_∞. The reason why this happens lies in the fiber diagram:

$$\begin{array}{ccc} \overline{\mathcal{M}}_{\Gamma_1}(\mathbb{P}^1 \times E/E) \times_E \overline{\mathcal{M}}_{\Gamma_2}(S/E) & \longrightarrow & \overline{\mathcal{M}}_{\Gamma_2}(S/E) \\ \downarrow & & \downarrow \mathrm{ev}_1^E \\ \overline{\mathcal{M}}_{\Gamma_1}(\mathbb{P}^1 \times E/E) & \xrightarrow[\mathrm{ev}_1^E]{} & E \end{array}$$

We recall that ev_1^E is the evaluation map corresponding to the unique relative marked point. As a result, the fiber product over E is asking that $f(q_1) = f'(q'_1)$. Note that the above cartesian diagram can be rewriten as

$$\begin{array}{ccc} \overline{\mathcal{M}}_{\Gamma_1}(\mathbb{P}^1 \times E/E) \times_E \overline{\mathcal{M}}_{\Gamma_2}(S/E) & \hookrightarrow & \overline{\mathcal{M}}_{\Gamma_1}(\mathbb{P}^1 \times E/E) \times \overline{\mathcal{M}}_{\Gamma_2}(S/E) \\ \downarrow & & \downarrow \mathrm{ev}_1^E \times \mathrm{ev}_1^E \\ E & \xhookrightarrow[\Delta]{} & E \times E \end{array}$$

where Δ is the diagonal morphism.

Once all the required tools for understanding the main result of this subsection have been introduced, the next theorem states how the virtual classes behave under the pushforward of the morphisms Φ_Γ (see [34] Theorem 3.15. or [32] Theorem 17.).

Theorem 3.3.1. *For $g, n, h \geq 0$, it holds that*

$$\left[\overline{\mathcal{M}}_{g,n}(W_0, s + hf)\right]^{\mathrm{vir}} = \sum_{\Gamma \in \Omega_{g,n,h}} \Phi_{\Gamma *} \circ \Delta^! \left(\left[\overline{\mathcal{M}}_{\Gamma_1}(\mathbb{P}^1 \times E/E)\right]^{\mathrm{vir}} \times \left[\overline{\mathcal{M}}_{\Gamma_2}(S/E)\right]^{\mathrm{vir}} \right)$$

where $\Delta^!$ is the Gysin map of Δ.

As a consequence of this theorem we can prove the degeneration formula for invariants on W_0 that will allow us to express invariants on W_0 by means of relative invariants on S and $\mathbb{P}^1 \times E$ relative to E. However, we will use the analogous result in the frame of reduced virtual class. As for $\overline{\mathcal{M}}_{g,n}(S, s + hf)$, one can construct the reduced virtual class of $\overline{\mathcal{M}}_{g,n}(S/E, (1), s + hf)$. In this sense, the reduced version of the above theorem will allow us to express the reduced invariants on S in terms of reduced relative invariants on S and relative invariants on $\mathbb{P}^1 \times E$. To do so, let us first briefly study how the tautological classes and evaluation classes behave under Φ.

Let us begin with the tautological classes. Let $\Gamma \in \Omega_{g,n,h}$, then we recall the following morphisms

$$\begin{aligned} \rho &: \overline{\mathcal{M}}_{g,n}(W_0, s+hf) \longrightarrow \overline{\mathcal{M}}_{g,n} \\ \rho_{\Gamma_1} &: \overline{\mathcal{M}}_{\Gamma_1}(\mathbb{P}^1 \times E/E) \longrightarrow \overline{\mathcal{M}}_{g_1,|S_1|+1} \\ \rho_{\Gamma_2} &: \overline{\mathcal{M}}_{\Gamma_2}(S/E) \longrightarrow \overline{\mathcal{M}}_{g_2,|S_2|+1} \end{aligned}$$

Now, we can associate to Γ a stable graph, that we will denote also by Γ. The association is as follows. Let Γ be the stable graph of genus g and n legs with two vertices of genus g_1 and g_2, and legs S_1 and S_2 respectively, and just one edge. In other words, it is a stable graph of the same type as the ones appearing in the splitting axiom (see Appendix A.2 Figure 16). Now, we consider the corresponding gluing morphism

$$\xi_\Gamma : \overline{\mathcal{M}}_\Gamma = \overline{\mathcal{M}}_{g_1,|S_1|+1} \times \overline{\mathcal{M}}_{g_2,|S_2|+1} \longrightarrow \overline{\mathcal{M}}_{g,n}.$$

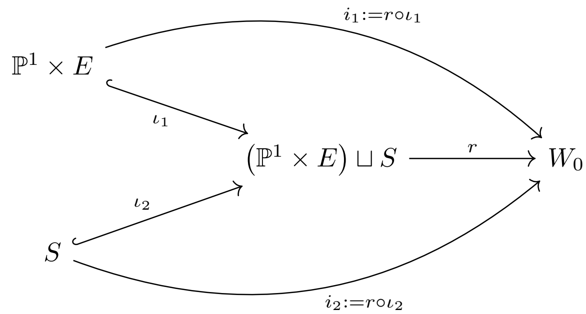
Using this morphism one can check that the following diagram commutes

$$\begin{array}{ccc} \overline{\mathcal{M}}_{g,n}(W_0, s+hf) & \xleftarrow{\Phi_\Gamma} & \overline{\mathcal{M}}_{\Gamma_1}(\mathbb{P}^1 \times E/E) \times_E \overline{\mathcal{M}}_{\Gamma_2}(S/E) \\ & & \downarrow \Delta \\ \downarrow \rho & & \overline{\mathcal{M}}_{\Gamma_1}(\mathbb{P}^1 \times E/E) \times \overline{\mathcal{M}}_{\Gamma_2}(S/E) \\ & & \downarrow \rho_{\Gamma_1} \times \rho_{\Gamma_2} \\ \overline{\mathcal{M}}_{g,n} & \xleftarrow[\xi_\Gamma]{} & : \overline{\mathcal{M}}_{g_1,|S_1|+1} \times \overline{\mathcal{M}}_{g_2,|S_2|+1} \end{array} \tag{23}$$

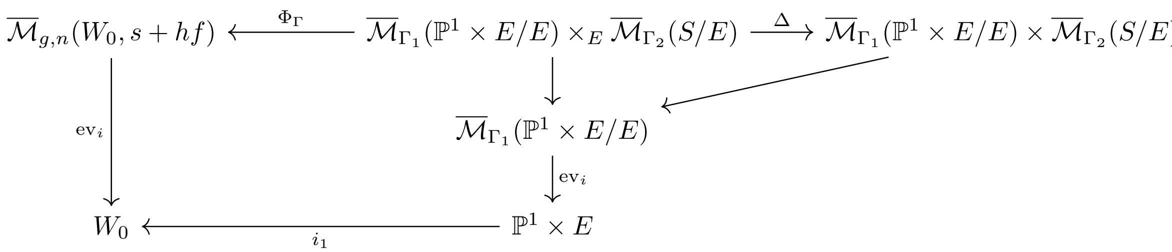
The corresponding diagram for the cohomology groups will allow us to study how the tautological classes will behave in the degeneration formula. However, before studying the evaluation classes, it is important to highlight the unstable cases. If $2g_1 - 2 + |S_1| + 1 \leq 0$ or $2g_2 - 2 + |S_2| + 1 \leq 0$ the gluing morphism ξ_Γ is not defined. First of all, we apply the degeneration formula to $(g, n) > (0, 3)$ and $(g, n) \neq (1, 0)$. This means that both unstable cases can not happen simultaneously. Assume that $2g_1 - 2 + |S_1| + 1 \leq 0$ (the treatment of the case $2g_2 - 2 + |S_2| + 1 \leq 0$ is similar). We have two possibilities:

- If $(g_1, |S_1|) = (0, 0)$, then in Diagram (23), instead of the gluing morphism ξ_Γ, we get the forgetful morphism.
- If $(g_1, |S_1|) = (0, 1)$, then instead of the gluing morphism in Diagram (23) we get the identity morphism.

Let us proceed analogously with the study of the evaluation morphisms. First, note that we have morphisms



Now, let $i \in \{1, \ldots, n\}$. We assume that $i \in S_1$, then we have a commutative diagram



(24)

If $i \in S_2$ we get the analogous commutative diagram for S instead that for $\mathbb{P}^1 \times E$. Using the corresponding commutative diagrams in cohomology we will be able to see how the evaluation classes behave in the degeneration formula. Now we are ready to state the degeneration formula for W_0 in the following proposition.

Proposition 3.3.2. *(Degeneration Formula) Let $g, n, h \geq 0$, $\gamma_1, \ldots, \gamma_n \in H^*(W_0)$, and $\alpha \in RH^*(\overline{\mathcal{M}}_{g,n})$, then*

$$\langle \alpha; \gamma_1, \ldots, \gamma_n \rangle^{W_0}_{g,n,h} = \sum_{\Gamma \in \Omega_{g,n,h}} \langle \alpha_{\Gamma_1}; \gamma_{i,1} : i \in S_1 | \omega \rangle^{\mathbb{P}^1 \times E/E}_{g_1, |S_1|, h_1} \; \langle \alpha_{\Gamma_2} : \gamma_{i,2} : i \in S_2 | 1 \rangle^{S/E}_{g_2, |S_2|, h_2}$$

where $\alpha_{\Gamma_1} \otimes \alpha_{\Gamma_2} = \xi^*_\Gamma(\alpha)$ and $\gamma_i = (\gamma_{i,1}, \gamma_{i,2}) \in H^*(W_0) \subset H^*(\mathbb{P}^1 \times E) \oplus H^*(S)$, where the relative invariants on the left hand side have effective curve class $s + h_i f$ respectively.

Proof. By Theorem 3.3.1 we get that

$$\langle \alpha; \gamma_1, \ldots, \gamma_n \rangle^{W_0}_{g,n,h} =$$
$$\sum_{\Gamma \in \Omega_{g,n,h}} \Phi_{\Gamma *} \circ \Delta^! \left(\left[\overline{\mathcal{M}}_{\Gamma_1}(\mathbb{P}^1 \times E/E) \right]^{\mathrm{vir}} \times \left[\overline{\mathcal{M}}_{\Gamma_2}(S/E) \right]^{\mathrm{vir}} \right) \frown \left(\rho^*(\alpha) \prod_{i=1}^n \mathrm{ev}_i^*(\gamma_i) \right) =$$
$$\sum_{\Gamma \in \Omega_{g,n,h}} \Delta^! \left(\left[\overline{\mathcal{M}}_{\Gamma_1}(\mathbb{P}^1 \times E/E) \right]^{\mathrm{vir}} \times \left[\overline{\mathcal{M}}_{\Gamma_2}(S/E) \right]^{\mathrm{vir}} \right) \frown \Phi^*_\Gamma \left(\rho^*(\alpha) \prod_{i=1}^n \mathrm{ev}_i^*(\gamma_i) \right). \tag{25}$$

Now, using the commutative diagrams (23) and (24) we get that

- $\Phi_\Gamma^* \circ \rho^*(\alpha) = \left(\Delta^* \circ (\rho_{\Gamma_1} \times \rho_{\Gamma_2})^* \circ \xi_\Gamma^*\right)(\alpha) = \Delta^*\left(\rho_{\Gamma_1}^*(\alpha_{\Gamma_1}) \otimes \rho_{\Gamma_2}^*(\alpha_{\Gamma_2})\right)$.
- For $i \in S_1$, $\Phi_\Gamma^* \circ \mathrm{ev}_i^*(\gamma_i) = \left(\Delta^* \circ \mathrm{ev}_i^* \circ i_1^*\right)(\gamma_i) \otimes 1 = \Delta^*\left(\mathrm{ev}_i^*(\gamma_{i,1}) \otimes 1\right)$.
- For $i \in S_2$, $\Phi_\Gamma^* \circ \mathrm{ev}_i^*(\gamma_i) = 1 \otimes \left(\Delta^* \circ \mathrm{ev}_i^* \circ i_2^*\right)(\gamma_i) = \Delta^*\left(1 \otimes \mathrm{ev}_i^*(\gamma_{i,2})\right)$.

Now, substituting these three equations in equation (25) we get that the invariant $\langle \alpha; \gamma_1, \ldots, \gamma_n \rangle_{g,n,h}^{W_0}$ is equal to

$$\sum_{\Gamma \in \Omega_{g,n,h}} \Delta^! \left(\left[\overline{\mathcal{M}}_{\Gamma_1}(\mathbb{P}^1 \times E/E)\right]^{\mathrm{vir}} \times \left[\overline{\mathcal{M}}_{\Gamma_2}(S/E)\right]^{\mathrm{vir}} \right) \frown$$
$$\frown \Delta^* \left(\rho_{\Gamma_1}^*(\alpha_{\Gamma_1}) \otimes \rho_{\Gamma_2}^*(\alpha_{\Gamma_2}) \prod_{i \in S_1} \mathrm{ev}_i^*(\gamma_{i,1}) \otimes 1 \prod_{i \in S_2} 1 \otimes \mathrm{ev}_i^*(\gamma_{i,2}) \right) =$$
$$\sum_{\Gamma \in \Omega_{g,n,h}} \Delta_* \circ \Delta^! \left(\left[\overline{\mathcal{M}}_{\Gamma_1}(\mathbb{P}^1 \times E/E)\right]^{\mathrm{vir}} \times \left[\overline{\mathcal{M}}_{\Gamma_2}(S/E)\right]^{\mathrm{vir}} \right) \frown$$
$$\frown \left(\rho_{\Gamma_1}^*(\alpha_{\Gamma_1}) \otimes \rho_{\Gamma_2}^*(\alpha_{\Gamma_2}) \prod_{i \in S_1} \mathrm{ev}_i^*(\gamma_{i,1}) \otimes 1 \prod_{i \in S_2} 1 \otimes \mathrm{ev}_i^*(\gamma_{i,2}) \right).$$

Now, taking into account the splitting axiom, we recall that the class of the diagonal in $E \times E$ is $\sum_{i,j} g^{i,j} T_i \otimes T_j$ where the sum runs over the basis of the cohomology of E, $\{1, \alpha, \beta, \omega\}$, accordingly to the vector space basis introduced in Section 2. Then we have that

$$\Delta_* \circ \Delta^! \left(\left[\overline{\mathcal{M}}_{\Gamma_1}(\mathbb{P}^1 \times E/E)\right]^{\mathrm{vir}} \times \left[\overline{\mathcal{M}}_{\Gamma_2}(S/E)\right]^{\mathrm{vir}} \right) =$$
$$\left[\overline{\mathcal{M}}_{\Gamma_1}(\mathbb{P}^1 \times E/E)\right]^{\mathrm{vir}} \times \left[\overline{\mathcal{M}}_{\Gamma_2}(S/E)\right]^{\mathrm{vir}} \frown \left(\sum_{i,j} g^{i,j} \mathrm{ev}_1^{E*}(T_i) \otimes \mathrm{ev}_1^{E*}(T_j) \right).$$

Combining the last two equations we get that

$$\begin{aligned} \langle \alpha; \gamma_1, \ldots, \gamma_n \rangle_{g,n,h}^{W_0} = & \sum_{\Gamma \in \Omega_{g,n,h}} \sum_{i,j} g^{i,j} \langle \alpha_{\Gamma_1}; \gamma_{i,1} : i \in S_1 | T_i \rangle_{g_1, |S_1|, h_1}^{\mathbb{P}^1 \times E/E} \\ & \langle \alpha_{\Gamma_2} : \gamma_{i,2} : i \in S_2 | T_j \rangle_{g_2, |S_2|, h_2}^{S/E}. \end{aligned} \tag{26}$$

Note that equation (26) differs to the one we want to proof only by the classes related to the diagonal class of E. Recall that the possible choices of (T_i, T_j) are $(1, \omega)$, $(\omega, 1)$, (α, ω), and (β, α). As a result, it will be enough to check that the invariants corresponding to $(T_i, T_j) \in \{(1, \omega), (\alpha, \beta), (\beta, \alpha)\}$ vanish. First of all, if T_j has odd degree, one can argue using the degree axiom or the vanishing of the odd cohomology groups of S to conclude the vanishing of the respective terms. As a result, the terms of equation (26) corresponding to (T_i, T_j) equal to (α, β) or (β, α) vanish.

Thus, it only remains to check that these relative invariants on S also vanish for $T_j = \omega$. To do so, we argue analogously as we did in the study of the possible choices of Γ. In particular, we recall that when we proved that the relative profile condition of the spliting of the stable maps was $\mu = (1)$, we in particular showed that for S, $\mathbb{P}^1 \times E$ and each degeneration Δ, the stable map must map to the $\mathbb{P}^1$ factor with degree 1 one component and with degree 0 the rest. This means that the relative evaluation map of $\overline{\mathcal{M}}_{\Gamma_2}(S/E)$, ev_1^E must factor through the intersection point between the section and the E of S. The reason why this happens is because in S the section s is fixed. However, this argument does not work for $\mathbb{P}^1 \times E$ since in this case all the sections over the factor $\mathbb{P}^1$ are equivalent. Therefore, $\mathrm{ev}_1^{E*}(\omega) = 0$. This concludes the proof since, now, equation (26) is equal to the desired expression. □

Thus, Proposition 3.3.2 shows how to determine invariants for a concrete h by computing invariants on S and $\mathbb{P}^1 \times E$ relative to E. However, we are interested in the generating series $\langle \alpha; \gamma_1, \dots, \gamma_n \rangle_{g,n}^S$. Nevertheless, similarly as we did at the end of Subsection 2.5, Proposition 3.3.2 can be translated for generating series.

Corollary 3.3.1. *For $g, n \geq 0$, $\gamma_1, \dots, \gamma_n \in H^*(W_0)$ and $\alpha \in RH^*(\overline{\mathcal{M}}_{g,n})$, it holds*

$$\langle \alpha; \gamma_1, \dots, \gamma_n \rangle_{g,n}^{W_0} = \sum_{\Gamma \in \Omega_{g,n}} \langle \alpha_{\Gamma_1}; \gamma_{i,1} : i \in S_1 | \omega \rangle_{g_1, |S_1|}^{\mathbb{P}^1 \times E/E} \langle \alpha_{\Gamma_2} : \gamma_{i,2} : i \in S_2 | 1 \rangle_{g_2, |S_2|}^{S/E}$$

where the invariants on the left hand side are the generating series

$$\langle \alpha_{\Gamma_1}; \gamma_{i,1} : i \in S_1 | \omega \rangle_{g_1, |S_1|}^{\mathbb{P}^1 \times E/E} = \sum_{h \geq 0} \langle \alpha_{\Gamma_1}; \gamma_{i,1} : i \in S_1 | \omega \rangle_{g_1, |S_1|, h}^{\mathbb{P}^1 \times E/E} q^h$$
$$\langle \alpha_{\Gamma_2}; \gamma_{i,2} : i \in S_2 | \omega \rangle_{g_2, |S_2|}^{S/E} = \sum_{h \geq 0} \langle \alpha_{\Gamma_2}; \gamma_{i,2} : i \in S_2 | \omega \rangle_{g_2, |S_2|, h}^{S/E} q^h.$$

Moreover, combining Corollary 3.3.1 and Proposition 3.3.1 we get that

$$\langle \alpha; \gamma_1, \dots, \gamma_n \rangle_{g,n}^{S} = \sum_{\Gamma \in \Omega_{g,n}} \langle \alpha_{\Gamma_1}; \gamma_{i,1} : i \in S_1 | \omega \rangle_{g_1, |S_1|}^{\mathbb{P}^1 \times E/E} \langle \alpha_{\Gamma_2} : \gamma_{i,2} : i \in S_2 | 1 \rangle_{g_2, |S_2|}^{S/E} \quad (27)$$

for $\gamma_i \in S$. In this situation $\gamma_{i,1}$ and $\gamma_{i,2}$ are defined by $\gamma_{i,0} = (\gamma_{i,1}, \gamma_{i,2})$. Now, taking into account the previous study in this subsection, we recall that

- If $\gamma_i = 1$, $\gamma_{i,0} = (1, 1)$.
- If $\gamma_i = \mathbf{p}$, then we can choose $\gamma_{i,0}$ to be $(\mathbf{p}, 0)$.
- Similarly, if $\gamma_i = f$, we can choose $\gamma_{i,0}$ to be $(f, 0)$.
- If $\gamma_i = s$, then $\gamma_i = (s, s)$.
- Finally, if γ_i is not one of the above, we have that $\gamma_{i,0} = (0, \gamma_i)$.

This simplifies significantly formula (27) since we can assume $i \in S_1$ if $\gamma_i \in \{f, \mathbf{p}\}$ and $i \in S_2$ if $\gamma_i \in \mathcal{B}\backslash\{1, s, f, \mathbf{p}\}$.

Let us use now the degeneration formula to continue with our induction. We recall that we are computing invariants on S by induction on (g, n), so we can assume that we have already computed all the invariants for $(g', n') < (g, n)$. Moreover in Subsection 3.2, we saw how to compute the case (g, n) when no evaluation class is $\mathbf{p}$. Now, to the case where $\gamma_1 = \mathbf{p}$ we can apply the degeneration formula, taking into account that 1 must lie in S_1 and we get that

$$\langle \alpha; \mathbf{p}, \gamma_2, \ldots, \gamma_n \rangle_{g,n}^{S} = \sum_{\Gamma \in \Omega_{g,n}} \langle \alpha_{\Gamma_1}; \mathbf{p}, \gamma_i : i \in S_1 | \omega \rangle_{g_1, |S_1|}^{\mathbb{P}^1 \times E/E} \langle \alpha_{\Gamma_2} : \gamma_i : i \in S_2 | 1 \rangle_{g_2, |S_2|}^{S/E}.$$

In particular, since $S_1 \neq \emptyset$, all the relative invariants over S on the left hand side of the equation have $(g_2, |S_2|) < (g, n)$. Moreover, we can assume that $\gamma_i \notin \{\mathbf{p}, f\}$ for all $i \in S_2$. The rest of this subsection will be devoted to express these relative invariants by means of relative invariants on $\mathbb{P}^1 \times E$ relative to E and invariants on S where we can apply the induction. Then, the only remaining step will be to compute the relative invariants on $\mathbb{P}^1 \times E$. This will be the task of the rest of the section.

The idea for computing the relative invariants on S relative to E is to see the degeneration formula from another perspective. To do so, we have to first study the basic relative invariants on $\mathbb{P}^1 \times E$ relative to E. In particular we will compute these invariants for $(g, n) = (0, 0)$ or $(g, n) = (0, 1)$. Note that these cases coincide with the unstable cases $2g - 2 + n + 1 \leq 0$. To deal with these cases, we will find a suitable expression of the respective moduli spaces of relative stable maps. This is done in next two propositions.

Proposition 3.3.3. *For $h \geq 0$, it holds*

$$\overline{\mathcal{M}}_{0,0}(\mathbb{P}^1 \times E/E, (1), s + hf) = \begin{cases} E & \text{if } h = 0 \\ \emptyset & \text{if } h > 0 \end{cases}$$

and

$$\overline{\mathcal{M}}_{0,0}(\mathbb{P}^1/0, (1), [\mathbb{P}^1]) = \begin{cases} \{\mathrm{pt}\} & \text{if } h = 0 \\ \emptyset & \text{if } h > 0 \end{cases}$$

Moreover, the virtual fundamental classes of these spaces coincide with their fundamental classes.

Proof. We present the proof for $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^1 \times E, (1), [\mathbb{P}^1])$; the same reasoning can be applied to the case $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^1/0, (1), s + hf)$. Moreover, we will present the argument for the complex point of the stack. The argument for general families derives from this. Let $(f : C \to \mathbb{P}^1 \times E[k], q_1)$ be a relative stable map to $\mathbb{P}^1 \times E$. First of all, note that the genus of C is 0 by assumption. This means that every morphism from C to E must be constant. As a result, if $h > 0$, the condition $\varepsilon_* \circ f_*([C]) = s + hf$ is never satisfied, and thus, $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^1 \times E/E, (1), s + hf) = \emptyset$.

Now, assume that $h = 0$, we then claim that k must be equal to 0. Suppose, on the contrary that $k > 0$, we will reach a contradiction with the stability. We split the relative stable map f in $k+1$ morphisms $f_i : C_i \to \mathbb{P}^1 \times E$ corresponding to the restriction f to the components that maps to $\mathbb{P}^1 \times E$ and each copy of Δ in $\mathbb{P}^1 \times E[k]$. We note that $\Delta = \mathbb{P}^1 \times E$ and, thus, each f_i corresponds to morphisms $f_{i,1} : C_i \to \mathbb{P}^1$ and $f_{i,2} : C_i \to E$. Since $f_{i,2}$ restricted to each irreducible component is constant, we will care only about $f_{i,1}$. Since $f_{i,1*}([C_i])$ must be equal to $[\mathbb{P}^1]$, we have that it must contract every irreducible component of C_i except one that must be mapped to $\mathbb{P}^1$ with degree 1, i.e. the restriction will be an isomorphism. Now, if C_i has a contracted component, it must have a extreme contracted component with only one special point coming from a node. This is a contradiction with the stability condition of f.

As a result, we may assume that C has no contracted component. This means that $C_i = (\mathbb{P}^1, p_1, p_2)$ and f_i is an isomorphism between (C_i, p_1, p_2) and $(\mathbb{P}^1, 0, \infty)$; without lost of generality we assume that $f_i = \mathrm{Id}$. However, $\mathrm{Aut}(\mathbb{P}^1, 0, \infty)$ is isomorphic to $\mathbb{C}^*$. Thus we can construct infinite automorphisms (g_1, g_2) of f by setting g_1 to be the identity on C_1, and C_i for $i > 1$, and the action of an element of $\lambda \in \mathbb{C}^*$ on C_1. Analogously, we set g_2 to be the identity everywhere except on the first copy of Δ where g_2 will be the action of λ^{-1}. Then $f \circ g_1 = g_2 \circ f$, and thus the automorphism group of f is not finite.

We can assume then that $k = 0$. This implies that $C = \mathbb{P}^1$ and $f = (f_1, f_2) : C \to \mathbb{P}^1 \times E$ with f_1 an isomorphism and f_2 constant. Now, let $(f' = (f'_1, f'_2) : \mathbb{P}^1 \to \mathbb{P}^1 \times E, q_1)$ be other relative stable map. Then, $(f'^{-1} \circ f, \mathrm{Id})$ is an isomorphism between f and f'. We can conclude thus that the data of a relative stable map is equivalent to the constant morphism $f_2 : \mathbb{P}^1 \to E$ and, as a result, $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^1 \times E/E, (1), s) \simeq E$. Note that this isomorphism is indeed the relative evaluation morphism ev_1^E.

To prove that the virtual fundamental classes of these spaces coincide with their fundamental classes, it is enough to check that the virtual dimension coincides with the dimension (see [7] Proposition 5.5). We will check this for $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^1 \times E/E, (1), s)$. The result for $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^1/0, (1), [\mathbb{P}^1])$ follows from this case. Using Appendix A.3 equation (45) we get that

$$\mathrm{vdim}(\overline{\mathcal{M}}_{0,0}(\mathbb{P}^1 \times E/E, (1), s)) = \mathrm{vdim}(\overline{\mathcal{M}}_{0,0}(\mathbb{P}^1 \times E, s)) = -1 - \int_s c_1(\omega_{\mathbb{P}^1 \times E}).$$

It is enough to check that $\int_s c_1(\omega_{\mathbb{P}^1 \times E}) = -2$. Using [22], Exercise 8.3.(b), we get that $\omega_{\mathbb{P}^1 \times E} = \pi_1^*(\omega_{\mathbb{P}^1}) \otimes \pi_2^*(\omega_E)$ where π are the projection of the product. Now, from [16], Remark 3.2.3.(b), we get that

$$c_1(\omega_{\mathbb{P}^1 \times E}) = c_1(\pi_1^*(\omega_{\mathbb{P}^1})) + c_1(\pi_2^*(\omega_E)) = \pi_1^*(c_1(\omega_{\mathbb{P}^1})) + \pi_2^*(c_2(\omega_E)).$$

Finally, using the fact that, for a genus g smooth curve C $\int_C c_1(\omega_C) = 2g - 2$ (see [16], Example 3.2.14.), we get that

$$\int_s c_1(\omega_{\mathbb{P}^1 \times E}) = \int_{\mathbb{P}^1} c_1(\omega_{\mathbb{P}^1}) + \int_E c_1(\omega_E) = -2. \qquad \square$$

Proposition 3.3.4. *For $h \geq 0$, it holds*

$$\overline{\mathcal{M}}_{0,1}(\mathbb{P}^1 \times E/E, (1), s + hf) = \begin{cases} \mathbb{P}^1 \times E & \text{if } h = 0 \\ \emptyset & \text{if } h > 0 \end{cases}$$

and

$$\overline{\mathcal{M}}_{0,0}(\mathbb{P}^1/E, (1), [\mathbb{P}^1]) = \begin{cases} \mathbb{P}^1 & \text{if } h = 0 \\ \emptyset & \text{if } h > 0 \end{cases}$$

Moreover, in both cases, the virtual fundamental class and the fundamental class coincide.

Proof. We present the proof for $\overline{\mathcal{M}}_{0,1}(\mathbb{P}^1 \times E/E, (1), 1)$. The statement for $\mathbb{P}^1$ follows similarly. Again, as before, we argue for the complex points of the stack; the argument can be extended to families of relative stable maps. The same argument as in the proof of Proposition 3.3.3 shows that, when $h > 0$, the moduli space must be empty. Now, let $(f : C \to \mathbb{P}^1 \times E[k], p_1, q_1)$ be a relative stable map. Exactly as we did in Proposition 3.3.3, we can split f into f_i for $i \in \{0, \dots, n\}$ corresponding to the component of C maping to the i–th copy of $\Delta \simeq \mathbb{P}^1 \times E$, with f_0 corresponding to $\mathbb{P}^1 \times E$. The same argument used there shows that, if for $i > 0$ C_i does not have the marked point p_1, then the relative stable map is not stable. Thus, we have two cases: either $k = 0$ and there is no degeneration, or $k = 1$ and p_1 lies in C_1. One can check that all the relative stable maps with $p_1 \in C_1$ are isomorphic if and only if they are mapped to the same point through the respective constant map to E.

On the other hand, if $k = 0$, then $C \simeq \mathbb{P}^1$ and two different relative stable maps with $k = 0$, $(f; \mathbb{P}^1 \to \mathbb{P}^1 \times E, p_1, q_1)$ and $(f'; \mathbb{P}^1 \to \mathbb{P}^1 \times E, p_1', q_1')$ will be isomorphic if and only if $f(p_1) = f(p_1')$. As result of all these considerations, the evaluation map

$$\mathrm{ev}_1 : \overline{\mathcal{M}}_{0,1}(\mathbb{P}^1 \times E, (1), 1) \longrightarrow \mathbb{P}^1 \times E$$

is an isomorphism. Finally, the same computation performed in the proof of Proposition 3.3.3 shows that, for both spaces, the virtual dimension and the dimension coincide. As a result, the virtual fundamental class and the fundamental class coincide too. □

Due to these two results, we can directly compute the respective relative invariants on $\mathbb{P}^1 \times E$ relative to E

$$\langle \emptyset | \omega \rangle_{0,0}^{\mathbb{P}^1 \times E/E} = \int_E \omega = 1, \quad \text{and} \quad \langle \gamma | \omega \rangle_{0,1}^{\mathbb{P}^1 \times E/E} = \int_{\mathbb{P}^1 \times E} \gamma \, (1 \otimes \omega) = \begin{cases} 1 & \text{if } \gamma = f, \\ 0 & \text{otherwise.} \end{cases} \tag{28}$$

Let us recall that we are interested in computing relative invariants of the form $\langle \alpha; \gamma_1', \dots, \gamma_{n'}' | 1 \rangle_{g',n'}^{S/E}$ with $\gamma_i' \notin \{f, \mathbf{p}\}$ and $(g', n') < (g, n)$. The idea for computing these invariants is to use the degeneration formula to the invariant $\langle \alpha; \gamma_1', \dots, \gamma_{n'}, f \rangle_{g',n'+1}^{S}$. Note that $(g', n' + 1) \leq (g, n)$. However, by assumption no evaluation class is $\mathbf{p}$. Thus, we have seen how to compute this invariant using the Ionel-Getzler vanishing and the splitting and reduction axioms. Moreover, note that in the splitting inside the degeneration formula, the only marking already fixed in the $\mathbb{P}^1 \times E$ contribution is the $n+1$ marking whose evaluation class is f. Thus, we get that

$$\begin{aligned}\langle \alpha; \gamma'_1, \ldots, \gamma'_{n'}, f\rangle^S_{g',n'+1} = \;& \langle \alpha; \gamma'_1, \ldots, \gamma'_{n'} | 1\rangle^{S/E}_{g,n} \langle f|\omega\rangle^{\mathbb{P}^1\times E/E}_{0,1} + \\ & \sum_{\substack{\Gamma \in \Omega_{g',n'+1} \\ \{f\} \subsetneq S_1}} \langle \alpha_{\Gamma_1}; \gamma'_i : i \in S'_1 | \omega\rangle^{\mathbb{P}^1\times E/E}_{g'_1,|S'_1|} \langle \alpha_{\Gamma_2} : \gamma'_i : i \in S'_2 | 1\rangle^{S/E}_{g'_2,|S'_2|}.\end{aligned}$$

Now, using the above computation of $\langle f|\omega\rangle^{\mathbb{P}^1\times E/E}_{0,1}$, we get that

$$\begin{aligned}\langle \alpha; \gamma'_1, \ldots, \gamma'_{n'} | 1\rangle^{S/E}_{g,n} = \;& \langle \alpha; \gamma'_1, \ldots, \gamma'_{n'}, f\rangle^S_{g',n'+1} - \\ & \sum_{\Gamma \in \Omega_{g',n'+1} : \{f\} \subsetneq S_1} \langle \alpha_{\Gamma_1}; \gamma'_i : i \in S'_1 | \omega\rangle^{\mathbb{P}^1\times E/E}_{g'_1,|S'_1|} \langle \alpha_{\Gamma_2} : \gamma'_i : i \in S'_2 | 1\rangle^{S/E}_{g'_2,|S'_2|}.\end{aligned}$$

As a result, we have written the relative invariant on S relative to E by means of an invariant on S lower in the induction or of the type studied in the first case, relative invariants on $\mathbb{P}^1 \times E$ relative to E, and relative invariant on S relative to E with lower genus and marking. As a consequence, arguing by induction we can recursively compute the relative invariants once we know how to compute the relative invariants on $\mathbb{P}^1 \times E$. Note that the base case of the recursion $(g, n) = (0, 0)$ can be done directly from the degeneration formula and get:

$$\langle \emptyset\rangle^S_{0,0} = \langle \emptyset|1\rangle^{S/E}_{0,0} \langle \emptyset|\omega\rangle^{\mathbb{P}^1\times E/E}_{0,0} = \langle \emptyset|1\rangle^{S/E}_{0,0}.$$

Summarizing, in this subsection we have used the degeneration formula to write invariants on S by means of relative invariants on S and $\mathbb{P}^1 \times E$ relative to E. Then, using the degeneration formula again, we reduced the computation of relative invariants on S to relative invariants on $\mathbb{P}^1 \times E$. As a result, once we see how to compute these relative invariants, our algorithm will be completed. This will be the task of the following subsections.

However, before moving to the next section, let us briefly comment on the relative invariants on S relative to E. We have shown how to compute them in the case where no evaluation class is f neither $\mathbf{p}$, and these invariants are the only ones required for computing absolute invariants on S. In order to compute all these relative invariants we need to assume that we know how to compute absolute invariants on S. We recall that while applying the degeneration formula, we assumed that the evaluation classes equal to f or $\mathbf{p}$ lied in the factor $\mathbb{P}^1 \times E$. These was a consequence of the behaviour of the evaluation classes through the normal cone degeneration. We made the choice of moving these classes to $(0, f)$ and $(0, \mathbf{p})$. Nevertheless, we saw that (f, f) and $(\mathbf{p}, \mathbf{p})$ were feasible options too. Using this selection we get that the evaluation classes f and $\mathbf{p}$ are not fixed anymore to the factor $\mathbb{P}^1 \times E$ and using the same argument as above, we can write the relative invariants on S by means of absolute invariants, relative invariants on $\mathbb{P}^1 \times E$, and relative invariants on S with lower genus or markings. Finally, arguing by recursion we can compute all the relative invariants on S.

3.4 Product formula

In the previous subsection, as a consequence of the degeneration formula, we reduced the computation of absolute invariants on S to relative invariants on $\mathbb{P}^1 \times E$. This subsection begins the study of these invariants. In particular, we will reduce the computation of these invariants to invariants on E. The main tool to accomplish this will be the product formula. Roughly speaking, the product formula allows us to relate invariants over a product with the invariants over each factor of the product. The references followed in this subsection are [5] and [30]. In [5] the author presents a proof of the product formula for the absolute case. In [30] Corollary 4.1. one can find the analogous relative version of the product formula.

Let us introduce the general statement to afterwards apply it to $\mathbb{P}^1 \times E$. Let $X \times Y$ the product of two nonsingular projective varieties and let D be a smooth divisor of Y. Moreover, assume that $H^1(Y) = 0$. This implies that

$$H_2(X \times Y) = (H_2(X) \otimes H_0(Y)) \oplus (H_0(X) \otimes H_2(Y)).$$

Thus, the effective algebraic curve classes of $X \times Y$ are of the form $\beta = (\beta_X, \beta_Y) \in H_2(X \times Y)$ for $\beta_X \in H_2(X)$ and $\beta_Y \in H_2(Y)$ effective algebraic curve classes on X and Y respectively. We consider the moduli spaces $\overline{\mathcal{M}}_{g,n}(X \times Y/X \times D) := \overline{\mathcal{M}}_{g,n}(X \times Y/X \times D, \mu, \beta)$, $\overline{\mathcal{M}}_{g,n}(X, \beta_X)$, and $\overline{\mathcal{M}}_{g,n}(Y/D) := \overline{\mathcal{M}}_{g,n}(Y/D, \mu, \beta_Y)$. Then, the product formula relates the relative invariants on $\overline{\mathcal{M}}_{g,n}(X \times Y/X \times D)$ with invariants on $\overline{\mathcal{M}}_{g,n}(X, \beta_X)$ and $\overline{\mathcal{M}}_{g,n}(Y/D)$. Let m denote the length of the partition μ.

Let us explain the idea behind this relation. Let $(f : C \to X \times Y[k], p_i, q_j)$ be a relative stable map to $X \times Y$ relative to $X \times D$. Then, one can check that in this situation, $\Delta = X \times \mathbb{P}(N_{Y/D} \oplus \mathcal{O}_D)$ and, hence, we will be able to split f into a stable map to X with class β_X and a relative stable map to Y relative to D with class β_Y. This allows us to construct a comutative diagram

$$\begin{array}{ccccc}
\overline{\mathcal{M}}_{g,n}(X \times Y/X \times D) & \xrightarrow{\Phi} & \overline{\mathcal{M}}_{g,n+m}(X, \beta_X) \times_{\overline{\mathcal{M}}_{g,n+m}} \overline{\mathcal{M}}_{g,n}(Y/D) & \longrightarrow & \overline{\mathcal{M}}_{g,n+m}(X, \beta_X) \times \overline{\mathcal{M}}_{g,n}(Y/D) \\
& \searrow^{\rho_{X \times Y}} & \downarrow & & \downarrow{\scriptstyle \rho_X \times \rho_Y} \\
& & \overline{\mathcal{M}}_{g,n+m} & \xrightarrow{\Delta} & \overline{\mathcal{M}}_{g,n+m} \times \overline{\mathcal{M}}_{g,n+m}
\end{array}$$

We start with the following theorem (see [30], Theorem 2.2.).

Theorem 3.4.1. *In the above situation it holds that*

$$\Phi_* \left([\overline{\mathcal{M}}_{g,n}(X \times Y/X \times D)]^{\mathrm{vir}} \right) = \Delta^! \left([\overline{\mathcal{M}}_{g,n}(X, \beta_x)]^{\mathrm{vir}} \times [\overline{\mathcal{M}}_{g,n}(Y/D)]^{\mathrm{vir}} \right)$$

The product formula is a consequence of this result. However, before seeing how to apply the result, let us introduce some notation. For $\gamma_1 \otimes \gamma'_1, \ldots, \gamma_n \otimes \gamma'_n \in H^*(X \times Y)$ and $\delta_1 \otimes \delta'_1, \ldots, \delta_m \otimes \delta'_m \in H^*(X \times D)$, we consider the relative Gromov-Witten class

$$I_{g,n,\beta}^{X\times Y/X\times D}(\gamma_1 \otimes \gamma'_1, \ldots, \gamma_n \otimes \gamma'_n | \delta_1 \otimes \delta'_1, \ldots, \delta_m \otimes \delta'_m) :=$$
$$\rho_{X\times Y*}\left([\overline{\mathcal{M}}_{g,n}(X \times Y/X \times D)]^{\mathrm{vir}} \frown \prod_{i=1}^n \mathrm{ev}_i^*(\gamma_i \otimes \gamma'_i) \prod_{j=1}^m \mathrm{ev}_j^{X\times D*}(\delta_j \otimes \delta'_j)\right).$$

Analogously, we denote by

$$I_{g,n+m,\beta_X}^X(\gamma_1, \ldots, \gamma_n, \delta_1, \ldots, \delta_m) \text{ and } I_{g,n,\beta_Y}^{Y/D}(\gamma'_1, \ldots, \gamma'_n | \delta'_1, \ldots, \delta'_m)$$

the respective Gromov-Witten classes on $\overline{\mathcal{M}}_{g,n+m}(X, \beta_X)$ and $\overline{\mathcal{M}}_{g,n}(Y/D)$. Using this notation, we can state the product formula (see [30] Corollary 4.1).

Corollary 3.4.1. *(Product Formula) For $\gamma_1 \otimes \gamma'_1, \ldots, \gamma_n \otimes \gamma'_n \in H^*(X \times Y)$ and $\delta_1 \otimes \delta'_1, \ldots, \delta_m \otimes \delta'_m \in H^*(X \times D)$, it holds that*

$$I_{g,n,\beta}^{X\times Y/X\times D}(\gamma_1 \otimes \gamma'_1, \ldots, \gamma_n \otimes \gamma'_n | \delta_1 \otimes \delta'_1, \ldots, \delta_m \otimes \delta'_m) =$$
$$I_{g,n+m,\beta_X}^X(\gamma_1, \ldots, \gamma_n, \delta_1, \ldots, \delta_m) I_{g,n,\beta_Y}^{Y/D}(\gamma'_1, \ldots, \gamma'_n | \delta'_1, \ldots, \delta'_m).$$

Let us apply Corollary 3.4.1 to our relative invariants on $\mathbb{P}^1 \times E$. To do this, we fix $X = E$, $Y = \mathbb{P}^1$ and $D = \{0\}$. Note that $H^1(\mathbb{P}^1) = 0$. So we can apply the product formula and we get that

$$\langle \alpha; \gamma_1 \otimes \gamma'_1, \ldots, \gamma_n \otimes \gamma'_n | 1 \otimes \omega \rangle_{g,n,h}^{\mathbb{P}^1\times E/E} = \int_{\overline{\mathcal{M}}_{g,n}} \alpha I_{g,n,1}^{\mathbb{P}^1/0}(\gamma_1, \ldots, \gamma_n | 1) I_{g,n+1,h}^E(\gamma'_1, \ldots, \gamma'_n, \omega).$$

We observe that $\gamma_i \in H^*(\mathbb{P}^1)$ is either 1 or $\mathbf{p}$. We denote $I_{g,n,1}^{\mathbb{P}^1/0}(\gamma_1, \ldots, \gamma_n | 1)$, with $\gamma_1 = \cdots = \gamma_k = \mathbf{p}$ and $\gamma_{k+1} = \cdots = \gamma_n = 1$, by $\mathcal{J}_g(k, n)$. In other words,

$$\mathcal{J}_g(k, n) := \rho_*\left(\left[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1/0, (1), 1)\right]^{\mathrm{vir}} \frown \prod_{i=1}^k \mathrm{ev}_i^*(\mathbf{p})\right) \in RH^{2(g+k-2)}(\overline{\mathcal{M}}_{g,n+1}) \quad (29)$$

If we prove that $\mathcal{J}(k, n)$ is a tautological class, the invariant on the left hand side of the above equation will be exactly of the same type of the ones studied in Section 2. Thus, our algorithm would be completed. Subsections 3.5 and 3.6 will be fully devoted to deal with this problem.

Assume now that $\mathcal{J}(k, n)$ is tautological for all k and n. Then, we can rewrite the above equation in terms of the generating series:

$$\langle \alpha; \gamma_1 \otimes \gamma'_1, \ldots, \gamma_n \otimes \gamma'_n | 1 \otimes \omega \rangle_{g,n}^{\mathbb{P}^1\times E/E} = \langle \alpha \mathcal{J}(k, n); \gamma'_1, \ldots, \gamma'_n, \omega \rangle_{g,n+1}^E \quad (30)$$

In particular, we will get the following Corollary:

Corollary 3.4.2.

(1) For $\gamma_1 \otimes \gamma'_1, \dots, \gamma_n \otimes \gamma'_n \in H^*(\mathbb{P}^1 \times E)$ *and* $\alpha \in RH^*(\overline{\mathcal{M}}_{g,n+1})$*, the generating series* $\langle \alpha; \gamma_1 \otimes \gamma'_1, \dots, \gamma_n \otimes \gamma'_n | 1 \otimes \omega \rangle_{g,n}^{\mathbb{P}^1 \times E/E}$ *is a quasimodular form.*

(2) For $\gamma_1, \dots, \gamma_n$ *and* $\alpha \in RH^*(\overline{\mathcal{M}}_{g,n})$*,* $\langle \alpha; \gamma_1, \dots, \gamma_n \rangle_{g,n}^S$ *lies in* $\dfrac{1}{\Delta(q)}\mathbf{QM}$.

Before finishing this subsection, we have to deal with a technical detail. We observe that in order to apply the product formula the stability condition on $\overline{\mathcal{M}}_{g,n+1}$ must be satisfied. In order words, we have to check what happen in the cases where $2g - 2 + n + 1 \geq 0$. However, the only possible unstable cases are $(0, 0)$ and $(0, 1)$. Both of these cases were computed in the previous subsection as a consequence of Propositions 3.3.3 and 3.3.4. As a consequence, we can now focus on the computation of $\mathcal{J}_g(k, n)$ as tautological class. This will be the task of the next two subsections.

3.5 Localization on $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1/0, \infty, 1)$

As a result of the study above, we have reduced the computation of the invariants on S to compute the cohomology class $\mathcal{J}_g(k, n)$ as a tautological class in $RH^*(\overline{\mathcal{M}}_{g,n})$. To do so, the main tool we will be the virtual localization. The idea behind of the virtual localization formula is to, given a $\mathbb{C}^*$–action over $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1/0, (1), 1)$, express the equivariant virtual fundamental class in terms of the equivariant virtual classes of the $\mathbb{C}^*$–fixed locus. In this sense, after stating the virtual localization formula, the first step to be accomplished in order to apply it will be to define a $\mathbb{C}^*$–action on $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1/0, (1), 1)$ and compute the fixed locus. In Proposition 3.5.1 the $\mathbb{C}^*$–fixed locus is stated. In particular, the moduli space of non-rigid maps will appear as a part of these fixed locus. In Appendix A.3 a brief introduction to this space is given. For further details we refer to [21] Section 2.4 or [17] Section 5.1. Afterwards, we will study the particular expression of the localization formula in our case.

For an introduction for equivariant cohomology in algebraic geometry we refer to [1] in the frame of algebraic varieties. There, the localization formula for varieties is introduced. For the development of the notion of equivariant cohomology in the frame of algebraic stacks we refer to [35]. There, in Section 2.8. the virtual localization formula is stated. The virtual localization formula was proven in [19] by T. Graber and R. Pandharipande. Finally, [14] will be our main reference, where the authors study our problem in a more general frame by considering all possible general conditions, and the analogous disconnected moduli space. We will follow this paper for the application of the localization formula. However, once we have localized, the argument we will differ to the one developed in [14].

Let us start introducing the virtual localization for Deligne-Mumford stack (see [19] or [35]).

Theorem 3.5.1. *(Virtual Localization Formula) Let M be a complex Deligne-Mumford stack with an action of the torus $G = (\mathbb{C}^*)^k$. Let $F_1, \dots, F_N$ be the connected components of the substack of G–fixed points. Moreover, assume that there exists a G–equivariant embedding of M into a smooth Deligne-Mumford stack. Then,*

$$[M]_G^{\mathrm{vir}} = \sum_{j=1}^{N} (\iota_j)_* \left(\frac{[F_j]_G^{\mathrm{vir}}}{e^G(N_j^{\mathrm{vir}})} \right) \tag{31}$$

where ι_j is the embedding of F_j in M.

Before studying how to apply the above result to the computation of $\mathcal{J}_g(k,n)$, let us comment equation (31). We observe that, the left hand side of the equation lies in the equivariant homology ring of M while each term of the right hand side of the equation lies in the localization of the equivariant homology ring. Thus, the localization formula holds in the localization of the equivariant homology group. On the other hand, note that in Theorem 3.5.1 M must admit an equivariant embedding into a smooth Deligne-Mumford stack. In the case of $\overline{\mathcal{M}}_{g,n}(X,\beta)$, with X a nonsingular porjective variety with a $(\mathbb{C}^*)^k$ action, this embedding is constructed in [19] Appendix A.

Now, the idea is to use the virtual localization formula to reduce the problem of computing the class $\mathcal{J}_g(k,n)$, as a tautological class from $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1/0,(1),1)$, to the fixed locus of a $\mathbb{C}^*$–action. To do so, the first step is to define a $\mathbb{C}^*$–action on $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1/0,(1),1)$ and compute its fixed points. For simplicity, we denote $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1/0,(1),1)$ by $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1/0)$. We recall that, from Subsection A.3, a relative stable map to $\mathbb{P}^1$ is a $(f : C \to \mathbb{P}^1[k], p_1, \dots, p_n, q_1)$ satisfying certain properties. Now, we observe that the divisor of $\mathbb{P}^1$, considered for the relative condition, is the point 0. So, $\Delta = \mathbb{P}^1$. This means that $\mathbb{P}^1[k]$ is a chain of copies of $\mathbb{P}^1$ as its shown in Figure 8

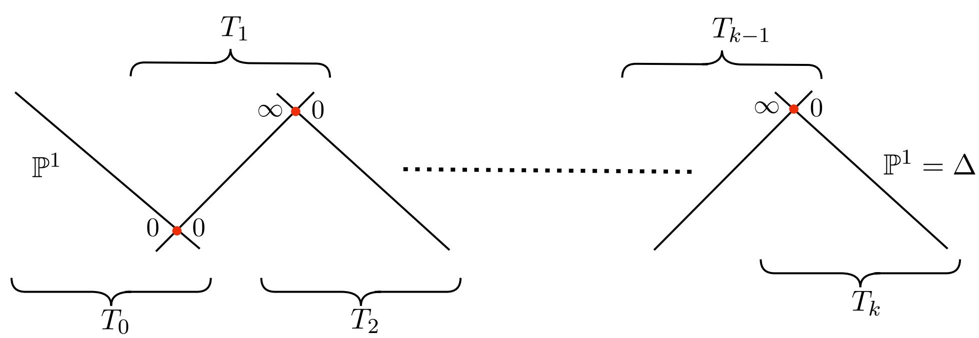


Figure 8: Ilustration of $\mathbb{P}^1[k]$.

We denote by T_0 the $\mathbb{P}^1$ component of $\mathbb{P}^1[k]$ and by $T_1, \dots, T_k$ the k glued copies of $\mathbb{P}^1$. Then, we can define an $\mathbb{C}^*$–action on $\mathbb{P}^1[k]$ by fixing $T_1, \dots, T_k$ and acting on T_0 by

$$\lambda[a : b] = [a : \lambda b] \text{ for } [a : b] \in \mathbb{P}^1 \text{ and } \lambda \in \mathbb{C}^*.$$

This is well defined since the fixed points of the action on T_0 are 0 and ∞. Then, we define the action of $\lambda \in \mathbb{C}^*$ on f as the composition of f with the action of λ on $\mathbb{P}^1[k]$. Note that the resulting map is still a relative stable map since the action of $\mathbb{C}^*$ on $\mathbb{P}^1[k]$ fixes the k–th degenerations and it acts by automorphisms on T_0. So

$$\varepsilon_* \circ f_*([C]) = \varepsilon_* \circ \lambda_* \circ f_*([C])[\mathbb{P}^1].$$

We remark that the $\mathbb{C}^*$–action on $\mathbb{P}^1[k]$ could have been defined by the same action on each $\mathbb{P}^1$ fixing 0 and ∞. However, by the definition of morphisms of relative stable maps, this action defines exactly the same action as the one explained above on $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1/0)$. Once we have defined the $\mathbb{C}^*$–action, the next step for applying the localization formula is to compute the $\mathbb{C}^*$–fixed locus. This is recorded in the next result.

Proposition 3.5.1. *The $\mathbb{C}^*$–fixed locus of the above action on $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1/0)$ is*

$$\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1/0)^{\mathbb{C}^*} = \bigsqcup_{\substack{g=g_1+g_2 \\ S_1 \sqcup S_2 = \mathbf{n}}} \overline{\mathcal{M}}_{g_1,|S_1|}(\mathbb{P}^1/0,\infty)^{\sim} \times \overline{\mathcal{M}}_{g_2,|S_2|+1}$$

where $\mathbf{n} = \{1,\ldots,n\}$ *and* $\overline{\mathcal{M}}_{g_1,|S_1|}(\mathbb{P}^1/0,\infty)^{\sim} := \overline{\mathcal{M}}_{g_1,|S_1|}(\mathbb{P}^1/0,\infty,(1,1),1)^{\sim}$

Proof. We present the proof for the complex point. The argument for families derives from this. Let us begin by studying how the relative stable maps of $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1/0)$ look like. Let $(f : C \to \mathbb{P}^1[k], p_1,\ldots,p_n,q_1)$ be relative stable maps to $\mathbb{P}^1$, i.e. a complex point of $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1/E)$. Using the above notation, we can split the morphisms f in $f_i : C_i \to \mathbb{P}^1$ for $i \in \{0,\ldots,k\}$ where f_i is the restriction of f to the components of C mapping to T_i; we denote by C_i each of these components. Then, $\varepsilon_* \circ f_*([C]) = [\mathbb{P}^1]$ is equivalent to $f_{0*}([C_0]) = [\mathbb{P}^1]$. In particular, this means that C_0 must have one genus zero irreducible component mapping with degree 1 to $\mathbb{P}^1$ (and hence isomorphic), and the rest of the irreducible components are contracted. Since by definition the orders of contacts at the singularities of $\mathbb{P}^1[k]$ must be equal on each component, the same situation holds for f_1. We observe that, recursively, the same holds for all f_i.

Now, let $(f : C \to \mathbb{P}^1[k], p_1,\ldots,p_n,q_1)$ be in the $\mathbb{C}^*$–fixed locus. Let us prove first the following statements:

1. $\varepsilon \circ f$ must map every marked point of C to the $\mathbb{C}^*$–fixed points of $\mathbb{P}^1$, i.e. to 0 or ∞.

2. $\varepsilon \circ f$ must map every node of C to 0 or ∞.

3. $\varepsilon \circ f$ must map every contracted irreducible component of C to 0 or ∞.

The proof of the first claim is a consequence of the definition of morphisms of relative stable maps. First of all, if p_i lies in the component C_i for $i > 0$, $\varepsilon \circ f(p_i)$ is directly zero. In particular, this works for q_1. So, we may assume that $p_i \in C_0$. If f is a fixed point, we have that for all $\lambda \in \mathbb{C}^*$ $f \simeq \lambda f$ must hold. This means that $\forall\lambda \in \mathbb{C}^*$ there exists a isomorphism (g_λ, h_λ) between f and λf. In particular, this implies that

$$\lambda \circ f(p_i) = \lambda \circ f \circ g_\lambda(p_i) = h_\lambda \circ f(p_i) = f(p_i)$$

where we have used that $g_\lambda(p_i) = p_i$ and $h_\lambda|_{T_0}$ must be the identity. Thus $\lambda \circ f(p_i) = f(p_i)$ for all $\lambda \in \mathbb{C}^*$. As a result, $f(p_i)$ must be a $\mathbb{C}^*$–fixed point of $\mathbb{P}^1$.

Let us prove the second statement. Let N be a node of C. As above, we can assume that N lies in C_0. For every $\lambda \in \mathbb{C}^*$, let (g_λ, h_λ) be an isomorphism between λf and f. We recall that g_λ is an automorphism of the prestable curve (C, p_i, q_1). In particular, this means that $g_\lambda(N)$ must be a node of C. Since the number of nodes of C is finite, we have that, for all $\lambda \in \mathbb{C}^*$, $g_\lambda(N)$ must lie in the finite set of nodes $\{N_1, \ldots, N_l\}$. Therefore for all $\lambda \in \mathbb{C}^*$ it holds:

$$\lambda \circ f(N) = h_\lambda \circ \lambda \circ f(N) = f \circ g_\lambda(N) \in \{f(N_1), \ldots, f(N_l)\}.$$

However, for every $x \in \mathbb{P}^1 \backslash \{0, \infty\}$ there exists a $\lambda_x \in \mathbb{C}^*$ such that $\lambda_x f(N) = x$. Thus, if $f(N) \notin \{0, \infty\}$ we reach a contradiction.

The argument of the proof of the third statement is similar to the one used for the second point. Let C' be a contracted component of C. We may assume that C' is mapped to T_0. For all $\lambda \in \mathbb{C}^*$, let (g_λ, h_λ) be the isomorphism between λf and f. Then g_λ must map C' to another contracted component of C. The set of contracted components of C is finite. Denote this set by H. For all $\lambda \in \mathbb{C}^*$ we have

$$\lambda \circ f(C') = h_\lambda \circ \lambda \circ f(C') = f \circ g_\lambda(C') \in f(H).$$

As above, if $f(C') \notin \{0, \infty\}$ we reach a contradiction.

Let us recall all we know so far about f. The composition $\varepsilon \circ f$ maps marked points, nodes, and contracted components to $\{0, \infty\}$, and f contracts every component of C except $k+1$ genus zero irreducible components that are mapped to each T_i with degree 1 and thus the restriction is an isomorphism. All together means that f is equivalent to the map f_0 and f_∞ where f_∞ is the restriction of f to the components mapped to $\Delta[k]$, i.e. $f_\infty = f|_{C_1 \cup \cdots \cup C_k}$. Moreover, we can split the morphisms f_0 in $f_{0,0}$ and $f_{0,1}$, where $f_{0,0}$ is the restriction of f_0 to the contracted components of C_0 mapped to 0, and $f_{0,1}$ is the restriction of f to the component of C_0 mapped with degree 1 to T_0. In other words, $f_{0,1}$ is an isomorphisms from $(\mathbb{P}^1, N_1, N_2)$ to $(\mathbb{P}^1, 0, \infty)$. Note that, the choice of $f_{0,1}$ corresponds to choosing $\lambda \in \mathbb{C}^*$. Since f is assumed to be a $\mathbb{C}^*$–fixed point, then all choices of $f_{0,1}$ give the same relative stable curve. As a result, f is equivalent to the morphisms f_∞ and $f_{0,0}$. Finally, notice that f_∞ coincides with the definition to non-rigid stable map to $\mathbb{P}^1$ relative to 0 and ∞. Moreover, $f_{0,0}$ corresponds to a stable map to a point. Finally, since $\overline{\mathcal{M}}_{g,n}(\mathrm{pt}, \beta) = \overline{\mathcal{M}}_{g,n}$, we get the desired result. □

Let $\Omega_{g,n}$ be the set $\{(g_1, g_2, S_1, S_2) : g = g_1 + g_2,\ S_1 \sqcup S_2 = \mathbf{n}\}$ and for $\Gamma \in \Omega_{g,n}$, let the tuple (g_i, S_i) be denoted by Γ_i. Then, the localization formula for $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1/0)$ can be written as

$$\left[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1/0)\right]^{\mathrm{vir},\mathbb{C}^*} = \sum_{\Gamma\in\Omega_{g,n}} \iota^{\mathbb{C}^*}_{\Gamma *}\left([\overline{\mathcal{M}}^{\sim}_{\Gamma_1}]^{vir} \times [\overline{\mathcal{M}}_{\Gamma_2}]^{\mathrm{vir}} \frown \frac{1}{e^{\mathbb{C}^*}(N^{\mathrm{vir}}_{\Gamma})}\right) \tag{32}$$

where $\overline{\mathcal{M}}^{\sim}_{\Gamma_1}$ and $\overline{\mathcal{M}}_{\Gamma_2}$ denote the spaces $\overline{\mathcal{M}}_{g_1,|S_1|}(\mathbb{P}^1/0,\infty)^{\sim}$ and $\overline{\mathcal{M}}_{g_2,|S_2|+1}$ respectively. Note that in the above sum, it appears terms where $\overline{\mathcal{M}}^{\sim}_{\Gamma_1}$ or $\overline{\mathcal{M}}_{\Gamma_2}$ are empty due to stability. For example, $\overline{\mathcal{M}}_{0,1}$ or $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^1/0,\infty)^{\sim}$. In such cases, the unstable spaces are treated as a point.

Let us now explain the meaning of the localization formula and how we will use it. First of all, let us specify where the equation (32) holds. Recall that the $\mathbb{C}^*$–equivariant cohomology of a point is $\mathbb{Q}[t]$ (see [35] Example 5.). As a result, all the $\mathbb{C}^*$–equivariant cohomology/homology rings that will be utilized are $\mathbb{Q}[t]$ modules. As commented above, the localization formula holds in the localization of the equivariant homology rings. In our case, the localization is performed by the multiplicative closed subset of $\mathbb{Q}[t]$ generated by t. Then, from the inclusion $\iota_\Gamma : \overline{\mathcal{M}}_\Gamma \hookrightarrow \overline{\mathcal{M}}_{g,n}(\mathbb{P}^1/0)$, we have the following commutative diagram:

$$\begin{array}{ccc}
H_*(\overline{\mathcal{M}}_\Gamma)\otimes\mathbb{Q}[t] = H_{\mathbb{C}^* *}(\overline{\mathcal{M}}_\Gamma) & \xrightarrow{\ \iota^{\mathbb{C}^*}_{\Gamma *}\ } & H_{\mathbb{C}^* *}(\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1/0)) \\
\downarrow & & \downarrow \\
H_*(\overline{\mathcal{M}}_\Gamma)\otimes\mathbb{Q}[t,t^{-1}] = H_{\mathbb{C}^* *}(\overline{\mathcal{M}}_\Gamma)[t^{-1}] & \xrightarrow[\ \iota^{\mathbb{C}^*}_{\Gamma *}\ ]{} & H_{\mathbb{C}^* *}(\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1/0))[t^{-1}]
\end{array}$$

where the second row is the localization of the first row by t. Hence, the localization formula holds in $H_{\mathbb{C}^* *}(\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1/0))[t^{-1}]$. However, despite the fact that the terms of right hand side of (32) are element in this localization, the left hand side not. This means that all the "denominators" of these terms are cancelled in the sum.

Now, to apply the localization formula to compute $\mathcal{J}_g(k,n)$ we have to lift this class to the $\mathbb{C}^*$–equivariant frame. First of all, using [35] Example 46, we get that $H^*_{\mathbb{C}^*}(\mathbb{P}^1) = \mathbb{Q}[x,t]/\langle x(x+t)\rangle$ and the morphism $H^*_{\mathbb{C}^*}(\mathbb{P}^1) \to H^*(\mathbb{P}^1)$ is the surjection given by $t=0$. For every class $\gamma_i \in H^*(\mathbb{P}^1)$, we denote by $\overline{\gamma} \in H^*_{\mathbb{C}^*}(\mathbb{P}^1)$ a lift through this morphism. We get the following commutative diagram

$$\begin{array}{ccccccc}
 & & H^*(\mathbb{P}^1) & \xrightarrow{\ \mathrm{ev}_i^*\ } & H^*(\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1)) & \xrightarrow{\ \rho_*\ } & H^*(\overline{\mathcal{M}}_{g,n+1}) \\
 & \nearrow & & \nearrow & & \nearrow{\scriptstyle \pi_{\mathbb{C}^*}} & \\
H^*_{\mathbb{C}^*}(\mathbb{P}^1) & \xrightarrow{\ \mathrm{ev}_i^{\mathbb{C}^* *}\ } & H^*_{\mathbb{C}^*}(\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1)) & \xrightarrow{\ \rho_*^{\mathbb{C}^*}\ } & H^*_{\mathbb{C}^*}(\overline{\mathcal{M}}_{g.n+1}) = H^*(\overline{\mathcal{M}}_{g,n+1})\otimes\mathbb{Q}[t] & & \\
 & & \downarrow & & \downarrow\uparrow{\scriptstyle PD} & & \\
 & & H_{\mathbb{C}^* *}(\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1)) & \longrightarrow & H_{\mathbb{C}^* *}(\overline{\mathcal{M}}_{g.n+1}) = H_*(\overline{\mathcal{M}}_{g,n+1})\otimes\mathbb{Q}[t] & &
\end{array}$$

Note that $\pi_{\mathbb{C}^*}$ is the morphism substituting $t=0$. Using this diagram we get that for

$\gamma_1, \ldots, \gamma_n \in H^*(\mathbb{P}^1)$ with $\gamma_i = \mathbf{p}$ for $i \leq k$ and $\gamma_i = 1$ else, we have

$$\begin{aligned} \mathcal{J}_g(k,n) &= PD \circ \rho^* \left([\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1/0)]^{\mathrm{vir}} \frown \textstyle\prod_{i=1}^n \mathrm{ev}_i^*(\gamma_i) \right) \\ & \quad \pi_{\mathbb{C}^*} \circ PD \circ \rho_*^{\mathbb{C}^*} \left([\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1/0)]^{\mathrm{vir},\mathbb{C}^*} \frown \prod_{i=1}^n \mathrm{ev}_i^*(\overline{\gamma}_i) \right). \end{aligned}$$

Now, we can apply the localization formula to $[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1/0)]^{\mathrm{vir},\mathbb{C}^*}$, and we get

$$\mathcal{J}_g(k,n) = \pi_{\mathbb{C}^*} \circ PD \circ \rho_*^{\mathbb{C}^*} \left(\left(\sum_{\Gamma \in \Omega_{g,n}} \iota_{\Gamma *}^{\mathbb{C}^*} \left([\overline{\mathcal{M}}_{\Gamma_1}^{\sim} \times \overline{\mathcal{M}}_{\Gamma_2}]^{\mathrm{vir}} \right) \frown \frac{1}{e^{\mathbb{C}^*}(N_\Gamma^{\mathrm{vir}})} \right) \frown \prod_{i=1}^n \mathrm{ev}_i^*(\overline{\gamma}_i) \right).$$

We now must take care of some technical details. We note that some of the terms of the sum indexed in $\Omega_{g,n}$ are in the localization of the equivariant homology group. So we have to map the evaluation classes to the localization. This allows us to move "$\frown$" inside of this sum. Furthermore, we note that the pushforward $\iota_*^{\mathbb{C}^*}$ is taken among the localization of the respective rings, while $\rho_*^{\mathbb{C}^*}$ is not in the localization. However, we can use the respective morphisms in the localization as it is shown in the following commutative diagram

$$\begin{array}{ccccc} H_*(\overline{\mathcal{M}}_\Gamma) \otimes \mathbb{Q}[t] & \xrightarrow{\iota_*^{\mathbb{C}^*}} & H_{\mathbb{C}^* *}(\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1/0)) & \xrightarrow{\rho_*^{\mathbb{C}^*}} & H_*(\overline{\mathcal{M}}_{g,n}) \otimes \mathbb{Q}[t] \\ \downarrow & & \downarrow & & \downarrow \\ H_*(\overline{\mathcal{M}}_\Gamma) \otimes \mathbb{Q}[t,t^{-1}] & \xrightarrow[\iota_*^{\mathbb{C}^*}]{} & H_{\mathbb{C}^* *}(\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1/0))[t^{-1}] & \xrightarrow[\rho_*^{\mathbb{C}^*}]{} & H_*(\overline{\mathcal{M}}_{g,n}) \otimes \mathbb{Q}[t,t^{-1}] \end{array}$$

Thus, we get that

$$\begin{aligned} \mathcal{J}_g(k,n) = & \quad \pi_{\mathbb{C}^*} \left(\sum_{\Gamma \in \Omega_{g,n}} PD \circ \rho_*^{\mathbb{C}^*} \left(\iota_{\Gamma *}^{\mathbb{C}^*} \left([\overline{\mathcal{M}}_{\Gamma_1}^{\sim} \times \overline{\mathcal{M}}_{\Gamma_2}]^{\mathrm{vir}} \frown \frac{1}{e^{\mathbb{C}^*}(N_\Gamma^{\mathrm{vir}})} \right) \frown \prod_{i=1}^n \mathrm{ev}_i^*(\overline{\gamma}_i) \right) \right) \\ & \quad \pi_{\mathbb{C}^*} \left(\sum_{\Gamma \in \Omega_{g,n}} PD \circ \rho_*^{\mathbb{C}^*} \circ \iota_{\Gamma *}^{\mathbb{C}^*} \left([\overline{\mathcal{M}}_{\Gamma_1}^{\sim} \times \overline{\mathcal{M}}_{\Gamma_2}]^{\mathrm{vir}} \frown \frac{1}{e^{\mathbb{C}^*}(N_\Gamma^{\mathrm{vir}})} \prod_{i=1}^n (\iota^{\mathbb{C}^*})^* \circ \mathrm{ev}_i^*(\overline{\gamma}_i) \right) \right). \end{aligned}$$

In order to simplify this expression the next step is to study the compositions $\rho_*^{\mathbb{C}^*} \circ \iota_{\Gamma *}^{\mathbb{C}^*}$ and $(\iota^{\mathbb{C}^*})^* \circ \mathrm{ev}_i^{\mathbb{C}^* *}$. To deal with the second composition, note that if $\gamma_i = 1$, $\overline{\gamma}_i = 1$, and hence $(\iota^{\mathbb{C}^*})^* \circ \mathrm{ev}_i^*(\overline{\gamma}_i) = 1$. So we can restrict to the case $\gamma_i = \mathbf{p}$. We assume first that $i \in S_2$. Recall that, from the proof of Proposition 3.5.1, $\overline{\mathcal{M}}_{\Gamma_2}$ arises from the part of the relative stable maps mapping to $\infty \in \mathbb{P}^1$. As a result we get that the composition $\mathrm{ev}_i \circ \iota_\Gamma$ factors through $\{\infty\}$, and we get the following commutative diagram in equivariant cohomology

$$\begin{array}{ccc}
H^*_{\mathbb{C}^*}(\mathrm{pt}) = \mathbb{Q}[t] & \longleftarrow & H^*_{\mathbb{C}^*}(\mathbb{P}^1) \\
\downarrow & & \downarrow \mathrm{ev}_i^* \\
H^*(\overline{\mathcal{M}}_\Gamma) \otimes \mathbb{Q}[t] & \xleftarrow[(\iota_\Gamma^{\mathbb{C}^*})^*]{} & H^*_{\mathbb{C}^*}(\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1/0)) \\
\downarrow & & \downarrow \\
H^*(\overline{\mathcal{M}}_\Gamma) \otimes \mathbb{Q}[t, t^{-1}] & \longleftarrow & H^*_{\mathbb{C}^*}(\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1/0))[t^{-1}]
\end{array}$$

Note that the morphism $H^*_{\mathbb{C}^*}(\{\mathrm{pt}\}) \to H^*(\overline{\mathcal{M}}_\Gamma) \otimes \mathbb{Q}[t]$ maps $f(t)$ to $1 \otimes f(t)$. Using the excess intersection formula one can check that the pullback from $\mathbb{P}^1$ to $\{\infty\}$ of $\mathbf{p}$ in the equivariant cohomology is $-t$. As a consequence, we get that $(\iota^{\mathbb{C}^*})^* \circ \mathrm{ev}_i^{\mathbb{C}^* *}(\mathbf{p}) = -t$. Similarly, if i lies in S_1, $\mathrm{ev}_i \circ \iota_\Gamma$ will factor through $\{0\}$. Therefore, we get $(\iota^{\mathbb{C}^*})^* \circ \mathrm{ev}_i^{\mathbb{C}^* *}(\mathbf{p}) = t$. However, this case is not needed, since, if $\gamma_i = \mathbf{p}$ we can choose it to be the class $[\infty]$, and in equivariant frame we get that the pullback of this equivariant class, though the equivariant inclusion of 0 in $\mathbb{P}^1$, is zero. Thus, if $\gamma_i = \mathbf{p}$ and $i \in S_1$ the term will directly vanish. Thus, we will assume that $\{1, \ldots, k\} \subseteq S_2$. Denote by $\Omega_{g,k,n}$ the set of tuples (g_1, g_2, S_1, S_2) such that $g = g_1 + g_2$ and $S_1 \sqcup S_2 = \{1, \ldots, n\}$ with $\{1, \ldots, k\} \subseteq S_2$. As a consequence of the above considerations we have that

$$\mathcal{J}_g(k, n) = \pi_{\mathbb{C}^*}\left(\sum_{\Gamma \in \Omega_{g,n,k}} PD \circ \rho_*^{\mathbb{C}^*} \circ \iota_{\Gamma *}^{\mathbb{C}^*}\left(\left[\overline{\mathcal{M}}^{\sim}_{\Gamma_1} \times \overline{\mathcal{M}}_{\Gamma_2}\right]^{\mathrm{vir}} \frown \frac{(-t)^k}{e^{\mathbb{C}^*}(N_\Gamma^{\mathrm{vir}})}\right)\right).$$

Let us now focus on the composition $PD \circ \rho_*^{\mathbb{C}^*} \circ (\iota_\Gamma^{\mathbb{C}^*})_*$. We recall that

$$\overline{\mathcal{M}}_\Gamma = \overline{\mathcal{M}}^{\sim}_{\Gamma_1} \times \overline{\mathcal{M}}_{\Gamma_2} = \overline{\mathcal{M}}_{g_1,|S_1|}(\mathbb{P}^1/0, \infty, (1), 1) \times \overline{\mathcal{M}}_{g_2,|S_2|+1}.$$

So, we get morphisms ρ_i from $\overline{\mathcal{M}}_{\Gamma_i}$ to the respective moduli spaces of stable maps. Using these morphisms we get the following commutative diagram:

$$\begin{array}{ccc}
\overline{\mathcal{M}}_\Gamma = \overline{\mathcal{M}}^{\sim}_{\Gamma_1} \times \overline{\mathcal{M}}_{\Gamma_2} & \xrightarrow{\iota_\Gamma} & \overline{\mathcal{M}}(\mathbb{P}^1/0) \\
\rho_1 \times \rho_2 \downarrow & & \downarrow \rho \\
\overline{\mathcal{M}}_{g_1,|S_1|+2} \times \overline{\mathcal{M}}_{g_2,|S_2|+1} & \xrightarrow[\xi]{} & \overline{\mathcal{M}}_{g,n+1}
\end{array}$$

where ξ is the gluing morphism of the splitting lemma type. In particular, note that $\rho_2 = \mathrm{Id}$. As a result, we get the following commutative diagram in the equivariant cohomology frame:

$$\begin{array}{ccc} H_{\mathbb{C}^**}(\overline{\mathcal{M}}_\Gamma) = H_*(\overline{\mathcal{M}}^\sim_{\Gamma_1}) \otimes H_*(\overline{\mathcal{M}}_{\Gamma_2}) \otimes \mathbb{Q}[t] & \xrightarrow{(\iota_\Gamma^{\mathbb{C}^*})_*} & H_{\mathbb{C}^**}(\overline{\mathcal{M}}(\mathbb{P}^1/0)) \\ \downarrow{\scriptstyle \rho_{1*}\otimes \mathrm{Id}\otimes \mathrm{Id}} & & \downarrow{\scriptstyle (\rho^{\mathbb{C}^*})_*} \\ H^*(\overline{\mathcal{M}}_{g_1,|S_1|+2}) \otimes H^*(\overline{\mathcal{M}}_{g_2,|S_2|+1}) \otimes \mathbb{Q}[t] & \xrightarrow{\xi_*\otimes \mathrm{Id}} & H^*(\overline{\mathcal{M}}_{g,n+1}) \otimes \mathbb{Q}[t] \end{array}$$

and the analogous diagram in the localization. Considering the localization morphisms of $\pi_{\mathbb{C}^*}$ we can now express $\mathcal{J}_g(k,n)$ as

$$\mathcal{J}_g(k,n) = \pi_{\mathbb{C}^*}\left(\sum_{\Gamma\in\Omega_{g,n,k}} (\xi_*\otimes \mathrm{Id})\circ PD \circ (\rho_{1*}\otimes \mathrm{Id}\otimes \mathrm{Id})\left(\left[\overline{\mathcal{M}}^\sim_{\Gamma_1}\times \overline{\mathcal{M}}_{\Gamma_2}\right]^{\mathrm{vir}} \frown \frac{(-t)^k}{e^{\mathbb{C}^*}(N_\Gamma^{\mathrm{vir}})}\right)\right).$$

With this expression we can now fully understand the localization formula. The expression where we are applying $\pi_{\mathbb{C}^*}$ to is a polynomial in t, t^{-1} with coefficients in $H^*(\overline{\mathcal{M}}_{g,n+1})$. Applying $\pi_{\mathbb{C}^*}$ means to take the independent term, i.e. the term t^0. But now, all the compositions of morphisms among the localized equivariant rings are the identity on the $\mathbb{Q}[t,t^{-1}]$ parts of the tensor products. Denote by μ_Γ the independent term of $\frac{(-t)^k}{e^{\mathbb{C}^*}(N_\Gamma^{\mathrm{vir}})}$ seen as a polynomial in t, t^{-1} with coefficients in the cohomology rings. Then, the above considerations lead us to the following expression of $\mathcal{J}_g(k,n)$

$$\mathcal{J}_g(k,n) = \sum_{\Gamma\in\Omega_{g,n,k}} (\xi_*\otimes \mathrm{Id})\circ PD \circ (\rho_{1*}\otimes \mathrm{Id})\left(\left[\overline{\mathcal{M}}^\sim_{\Gamma_1}\times \overline{\mathcal{M}}_{\Gamma_2}\right]^{\mathrm{vir}} \frown \mu_\Gamma\right). \tag{33}$$

As a consequence of this expression, we focus our attention on the class μ_Γ. We need to find a suitable expression of this class. To do so, we need to compute the equivariant Euler class of N_Γ^{vir}. This computation is performed in [14] Section 1.3. Following this last paper we get that

$$\frac{1}{e^{\mathbb{C}^*}(N_\Gamma^{\mathrm{vir}})} = \begin{cases} \dfrac{1}{(t-\psi_0)}\dfrac{(-1)^{g_2}}{t(t+\psi_{|S_2|+1})}\displaystyle\sum_{j=0}^{g_2}\lambda_j t^{g-j} & \left\{\begin{array}{l}\text{if } 2g_2-2+|S_2|+1>0\\ \text{and } (g_1,|S_1|)\neq(0,0).\end{array}\right. \\ \dfrac{1}{(t-\psi_0)}\dfrac{-1}{t} & \left\{\begin{array}{l}\text{if } (g_2,|S_2|)=(0,1)\\ \text{and } (g_1,|S_1|)\neq(0,0).\end{array}\right. \\ \dfrac{1}{(t-\psi_0)} & \left\{\begin{array}{l}\text{if } (g_2,|S_2|)=(0,0)\\ \text{and } ((g_1,|S_1|)\neq(0,0).\end{array}\right. \\ \dfrac{(-1)^{g_2}}{t(t+\psi_{|S_2|+1})}\displaystyle\sum_{j=0}^{g_2}\lambda_j t^{g-j} & \left\{\begin{array}{l}\text{if } 2g_2-2+|S_2|+1>0\\ \text{and}(g_1,|S_1|)=(0,0).\end{array}\right. \\ \dfrac{-1}{t} & \left\{\begin{array}{l}\text{if } (g_2,|S_2|)=(0,1)\\ \text{and } (g_1,|S_1|)=(0,0).\end{array}\right. \\ 1 & \left\{\begin{array}{l}\text{if } (g_2,|S_2|)=(0,0)\\ \text{and } (g_1,|S_1|)=(0,0).\end{array}\right. \end{cases}$$

In the formula above $1/(e^{\mathbb{C}^*}(N_\Gamma^{\mathrm{vir}}))$ is an element in $H^*(\overline{\mathcal{M}}_\Gamma)\otimes\mathbb{Q}[t,t^{-1}]$. So, why do elements as $1/(t-\psi_0)$ or $1/(t+\psi_{|S_2|+1})$ appear? The reason is because these quotients can be expanded as the power series as follows:

- $\dfrac{1}{t-\psi_0}=\dfrac{1}{t}\dfrac{1}{1-\frac{\psi_0}{t}}=\dfrac{1}{t}\left(\displaystyle\sum_{k\geq 0}\frac{\psi_0^k}{t^k}\right).$
- $\dfrac{1}{t+\psi_{|S_2|+1}}=\dfrac{1}{t}\dfrac{1}{1+\frac{\psi_{|S_2|+1}}{t}}=\dfrac{1}{t}\left(\displaystyle\sum_{l\geq 0}\frac{(-\psi_{|S_2|+1})^l}{t^l}\right).$

Note that the above power series are finite sums since, by dimension reasons, ψ_0^k and $\psi_{|S_2|+1}^k$ will vanish for k and l big enough. As a consequence, we have that μ_Γ must be a class in $H^*(\overline{\mathcal{M}}_{\Gamma_1}^\sim\times\overline{\mathcal{M}}_{\Gamma_2})$ that is sum of classes of the form $\psi_0^k\otimes\psi_{|S_2+1|}^l\lambda_j$ or zero. Using this last expression of μ_Γ, together with equation (33), we can restrict the computation of $\mathcal{J}_g(k,n)$ to the study of the classes ψ_0^k and their pullback to $\overline{\mathcal{M}}_{g_1,|S_1|+2}$. More precisely, we can rewrite (33) as

$$\begin{aligned}\mathcal{J}_g(k,n)&=\sum_{\Gamma\in\Omega_{g,n}}(\xi_*\otimes\mathrm{Id})\circ PD\circ(\rho_{1*}\otimes\mathrm{Id})\left(\left[\overline{\mathcal{M}}_{\Gamma_1}^\sim\times\overline{\mathcal{M}}_{\Gamma_2}\right]^{\mathrm{vir}}\frown\left(\sum_{k,l,j}\psi_0^k\otimes\psi_{|S_2|+1}^l\lambda_j\right)\right)\\&=\sum_{\Gamma\in\Omega_{g,n}}\sum_{k,l,j}\xi_*\left(\left(PD\circ\rho_{1*}(\left[\overline{\mathcal{M}}_{\Gamma_1}^\sim\right]^{\mathrm{vir}}\frown\psi_0^k)\right)\otimes\psi_{|S_2|+1}^l\lambda_j\right).\end{aligned}$$

By definition, the pushforward of a tautological class through a gluing morphism is tautological. As a result, the above expression reduces the computation of $\mathcal{J}_g(k,n)$ to the computation as tautological classes of cohomology classes of the form:

$$\Psi_{g,n}(k):=\rho_*\left(\left[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1/0\infty)^\sim\right]^{\mathrm{vir}}\frown\psi_0^k\right)$$

for $k\geq 0$, where $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1/0,\infty)^\sim=\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1/0\infty,(1),1)^\sim$. The next subsection will be fully devoted to this task. However, let us first present a simplification of our algorithm. Let π_a be the forgetful morphisms among moduli spaces of stable maps forgetting a marked points. Using these morphisms, the following holds:

$$\pi_a^*(\mathcal{J}_g(k,n))=\mathcal{J}_g(k,n+a). \tag{34}$$

Since the pullback of tautological classes through the forgetful morphisms is again tautological, it is enough to focus on the classes $\mathcal{J}_g(k,k)$ for $k\geq 0$ with $2g-2+k+1>0$. Recall that, in our previous study of the localizacion formula, we set that for all $\Gamma\in\Omega_{g,n,k}$, $\{1,\ldots,k\}\subseteq S_2$. Thus, since we are assuming $k=n$, we get that $S_1=\emptyset$. Therefore, we conclude that it is enough to express the class $\Psi_{g,n}(k)$ by means of tautological classes in the case of $n=0$.

Before moving to the next subsection, let us illustrate by an example the computation of $\mathcal{J}_g(k, n+a)$. Let us determine the class $\mathcal{J}_1(2,2)$. Note that the only possible terms of the localization formula are (g_1, S_1, g_2, S_2) equal to $(0,0,1,2)$ or $(1,0,0,2)$. Let us focus on the first case. There the equivariant Euler class of the virtual normal bundle is

$$\frac{-1}{t(t+\psi_3)}\sum_{j=0}^{1}\lambda_j t^{1-j}$$

As a result, the corresponding term of the localization formula is

$$(-t)^2\frac{-1}{t^2}\left(\sum_{m\geq 0}(-1)^m\frac{\psi_3^m}{t^m}\right)\left(\sum_{j=0}^{1}\lambda_j t^{1-j}\right)=\sum_{m\geq 0}\sum_{j=0}^{2}(-1)^{m+1}\psi_3^m\lambda_j t^{1-j-m}.$$

The next step is to extract the t^0 term of this series. The contributions to the coefficient of t^0 happens when (j,m) are either $(1,0)$ or $(0,1)$. As a result. this first term of the localization formula gives us $\psi_3-\lambda_1$. Similarly, for the second term the Euler class is

$$\frac{1}{(t-\psi_0)}\frac{(-1)^0}{t(t+\psi_3)}\sum_{j=0}^{0}\lambda_j t^{g-j}=\frac{1}{t^3}\left(\sum_{l\geq 0}\frac{\psi_0^l}{t^l}\right)\left(\sum_{m\geq 0}\frac{(-\psi_3)^m}{t^m}\right).$$

Therefore, the term inside the localization formula correspond to the series

$$(-t)^2\frac{1}{t^3}\left(\sum_{l\geq 0}\frac{\psi_0^l}{t^l}\right)\left(\sum_{m\geq 0}\frac{(-\psi_3)^m}{t^m}\right)=\sum_{l\geq 0}\sum_{m\geq 0}(-1)^m\psi_0^l\otimes\psi_3^m\, t^{-1-l-m}.$$

Note that in the above series there is no term t^0. Hence the contribution of this term to $\mathcal{J}_1(2,2)$ is zero and we get $\mathcal{J}_1(2,2)=\psi_3-\lambda_1$.

3.6 Rubber calculus

As a conclusion of the study developed in the previous subsection, we have reduced the computation of relative invariants on $\mathbb{P}^1\times E$ to determine the class

$$\Psi_{g,0}(k):=\rho_*\left(\left[\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1/0,\infty)^{\sim}\right]^{\mathrm{vir}}\frown\psi_0^k\right)$$

as a tautological class. We recall that $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1/0,\infty)^{\sim}$ is the moduli space of non rigid stable maps $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1/0,\infty,(1)1)^{\sim}$ and $\rho:\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1/0,\infty)^{\sim}\to\overline{\mathcal{M}}_{g,n+2}$ is the map forgetting about the stable map and stabilizing the resulting prestable curve. In [14], the authors present the computation of these classes as elements in the tautological ring in a more general frame. However, here, we will present an alternative argument also based on the ideas presented in [37] Section 1.5.

We will argue by induction on k, so our first step is to study the case $k = 0$. This case is precisely solved by the double ramification cycle. Because of this, let us first briefly introduce this cycle and how to compute it. To do so, we will follow the paper [26]. Let $\mu = (\mu_1, \ldots, \mu_m) \in \mathbb{Z}^m_{\leq 0}$ and $\nu = (\nu_1, \ldots, \nu_l) \in \mathbb{Z}^l_{\leq 0}$ be such that

$$D := \sum_{i=1}^{m} \mu_i = \sum_{j=1}^{l} \mu_j.$$

We fix $n \geq 0$, and we denote by A the tuple $A = (\mu_1, \ldots, \mu_m, -\nu_1, \ldots, -\nu_l, 0, \ldots, 0) \in \mathbb{Z}^{m+l+n}$ where the number of zeros at the end part is n. We consider the moduli space

$$\overline{\mathcal{M}}_{g,A}(\mathbb{P}^1/0,\infty)^{\sim} := \overline{\mathcal{M}}_{g,n}(\mathbb{P}^1/0,\infty,\mu,\nu,D[\mathbb{P}^1])^{\sim}.$$

Let ρ the respective map to $\overline{\mathcal{M}}_{g,n+m+l}$. Then, we introduce the notion of double ramification cycle.

Definition 3.6.1. *Let A be as above and let $g \geq 0$ be such that $2g-2+n+l+m > 0$. The* double ramification cycle *for g and A is defined as*

$$\mathrm{DR}_g(A) := PD \circ \rho_* \left(\left[\overline{\mathcal{M}}_{g,A}(\mathbb{P}^1/0,\infty)^{\sim} \right]^{\mathrm{vir}} \right).$$

In particular, this means that in our situation $k = 0$, $\Psi_{g,0}(0) = \mathrm{DR}_g((1,-1))$. So we need to find an expression of this double ramification function in terms of tautological classes. This can be done through a more general statement for $\mathrm{DR}_g(A)$. In particular, we have the following result (see [26] Proposition 2).

Theorem 3.6.1. *The double ramification cycle is tautological. In particular, $\mathrm{DR}_g(A) \in RH^g(\overline{\mathcal{M}}_{g,n+m+l})$.*

This statement was first proven in [14] as a particular case of Theorem 1 (this result also states that both $\mathcal{J}_g(k,n)$ and $\Psi_{g,n}(k)$ are tautological) providing an algorithm by recursion to compute the expression of the double ramification cycle as tautological class. However, in [26] Theorem 1, an explicit formula of the double ramification cycle as tautological class is given. This finishes the case $k = 0$. However, before dealing with the general case, let us state a result that we will use in our argument. This is the analogous to the dilation equation in the non rigid frame.

Proposition 3.6.1. *Let $\pi^{\sim} : \overline{\mathcal{M}}_{g,1}(\mathbb{P}^1/0,\infty)^{\sim} \to \overline{\mathcal{M}}_{g,0}(\mathbb{P}^1/0,\infty)^{\sim}$ be the forgetful morphism. The following equatility holds*

$$(\pi^{\sim})_* \left(\left[\overline{\mathcal{M}}_{g,1}(\mathbb{P}^1/0,\infty)^{\sim} \right]^{\mathrm{vir}} \frown \psi_1 \right) = (2g-2+2) \left[\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1/0,\infty)^{\sim} \right]^{\mathrm{vir}}.$$

As the dilation equation in the absolute frame, this result is a consequence of the fact that the pullback of the virtual fundamental class of $\left[\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1/0,\infty)^{\sim} \right]^{\mathrm{vir}}$ is $\left[\overline{\mathcal{M}}_{g,1}(\mathbb{P}^1/0,\infty)^{\sim} \right]^{\mathrm{vir}}$. We can now state and prove the main theorem of this subsection.

Theorem 3.6.2. *For $g \geq 1$ and $k \geq 0$, the class $\Psi_{g,0}(k)$ is tautological and can be computed via the following recursion:*

(1) For $k = 0$, $\Psi_{g,0}(0) = \mathrm{DR}_g(1, -1)$.

(2) For $k \geq 1$, it holds

$$\Psi_{g,0}(k) = \sum_{g=g_1+g_2, g_1, g_2 > 0} \frac{g_1}{g} \xi_{\Gamma_{g_1}*} \left(\mathrm{DR}_{g_1}(1, -1) \otimes \Psi_{g_2,0}(k-1) \right)$$

where Γ_{g_1} is the genus g stable graph with one edge and two vertices, one of genus g_1 and one leg, and the other of genus g_2 and one leg (see Figure 9).

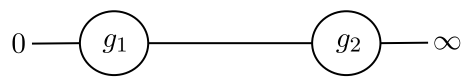


Figure 9: Genus g stable graph with one edge and two vertices. The first vertex has genus g_1 and one leg labelled by 0. The second vertex has genus ∞ and one leg labelled by ∞.

Proof. We argue by recursion on k. As commented above, the base case $k = 0$ is solved through the above remarks on the double ramification cycle. Assume that $k \geq 1$. Using the dilation equation, we get that

$$\begin{aligned} \left[\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1/0,\infty)^{\sim}\right]^{\mathrm{vir}} \frown \psi_0^k &= \frac{1}{2g}(\pi^{\sim})_*(\left[\overline{\mathcal{M}}_{g,1}(\mathbb{P}^1/0,\infty)^{\sim}\right]^{\mathrm{vir}} \frown \psi_1) \frown \psi_0^k \\ &= \frac{1}{2g}(\pi^{\sim})_* \left(\left[\overline{\mathcal{M}}_{g,1}(\mathbb{P}^1/0,\infty)^{\sim}\right]^{\mathrm{vir}} \frown \psi_1 \, (\pi^{\sim})^*(\psi_0^k) \right). \end{aligned} \tag{35}$$

Our next step is to compute $(\pi^{\sim})^*(\psi_0)$. To do so, we recall that we have a morphism $\overline{\rho} : \overline{\mathcal{M}}_{g,n}(\mathbb{P}^1/0,\infty,(1,1),1) \to \mathfrak{M}_{g,n+2}$ and in particular $\overline{\rho}^*(\psi_i) = \psi_i$ for $i \in \{1,\ldots,n,0,\infty\}$. Moreover, note that $\overline{\rho}$ commutes with the respective forgetful morphisms. As a result, we get that

$$(\pi^{\sim})^*(\psi_0) = (\pi^{\sim})^* \circ \overline{\rho}^*(\psi_0) = \overline{\rho}^* \circ \pi^*(\psi_0). \tag{36}$$

Let Γ_0 be the prestable graph of genus g and three markings with two vertices of genus 0 and 1, one edge, and 3 legs $0, 1, \infty$ such that $0, 1$ lies in the genus 0 vertex and ∞ in the other vertex (see Figure 10).

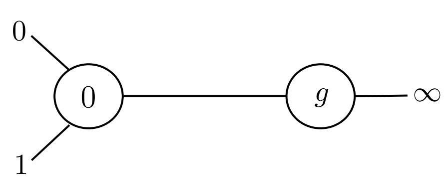


Figure 10: Genus g stable graph with 3 legs, 1 edge, and 2 vertices. The first vertex has genus 0 and two legs labelled by 0 and 1. The second vertex has genus g and one leg labelled by ∞.

We denote by $[\Gamma_0]$ the pushforward of $[\mathfrak{M}_{\Gamma_0}]$ through the gluing morphism, i.e. the class of the substack of $\mathfrak{M}_{g,3}$ of prestable curves with graph Γ_0. Then, using [4] Proposition 3.10, we get that

$$\pi^*(\psi_0) = \psi_0 - [\Gamma_0]$$

in $\mathfrak{M}_{g,3}$. Using this result together with equation (36) we can rewrite equation (35) as

$$\frac{1}{2g}(\pi^{\sim})_* \left(\left[\overline{\mathcal{M}}_{g,1}(\mathbb{P}^1/0,\infty)^{\sim} \right]^{\mathrm{vir}} \frown \psi_1 \left(\psi_0 - \overline{\rho}^*([\Gamma_0]) \right)^k \right). \tag{37}$$

The next step in our proof will be to state that $\psi_1 \overline{\rho}^*([\Gamma_0]) = 0$. This will simplify significantly the above expression. More precisely, we have the following claim.

Claim 1. $\psi_1 \overline{\rho}^*([\Gamma_0]) = 0$.

Proof. First of all, we note that $\psi_1 \overline{\rho}^*([\Gamma_0]) = \overline{\rho}^*(\psi_1[\Gamma_0])$. *Let focus for now on* $\psi_1[\Gamma_0]$. *Recall that* $[\Gamma_0] = \xi_{\Gamma_0 *}([\mathfrak{M}_{\Gamma_0}])$, *and thus we get that*

$$\psi_1[\Gamma_0] = \psi_1 \xi_{\Gamma_0 *}([\mathfrak{M}_{\Gamma_0}]) = \xi_{\Gamma_0 *} \circ \xi^*_{\Gamma_0}(\psi_1).$$

Moreover, it holds that $\xi^*_{\Gamma_0}(\psi_1) = 1 \otimes \psi_1$ *(see [4] Section 3.2.). The idea is to find a nice expression of* ψ_1 *in* $\mathfrak{M}_{0,3}$. *From Propotition A.1.3, we get that* $\psi_1 = \mathrm{st}^*(\psi_1) + [\Gamma_1]$ *for* Γ_1 *prestable graph as in Appedix A.1 Figure 15. Now,* $\overline{\mathcal{M}}_{0,3} = \{\mathrm{pt}\}$, *and hence,* $\mathrm{st}^*(\psi_1) = 0$ *and* $\psi_1 = [\Gamma_1]$. *Therefore, we get that*

$$\psi_1[\Gamma_0] = \xi_{\Gamma_0 *}(1 \otimes [\Gamma_1]) = \xi_{\Gamma_0 *}(1 \otimes \xi_{\Gamma_1 *}([\mathfrak{M}_{\Gamma_1}])).$$

We can rewrite this expression using an additional prestable graph. Let Γ_2 *be the prestable graph resulting from replacing in* Γ_0 *the genus* 0 *vertex by* Γ_1 *making the leg* ∞ *of* Γ_1 *the other half edge of the unique edge of* Γ_0 *(see Figure 11).*

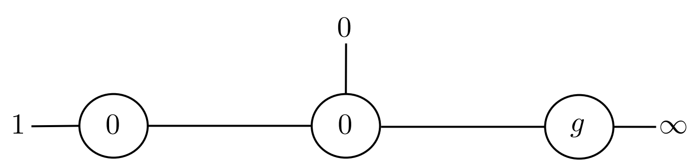


Figure 11: Genus g prestable graph with 3 legs, 2 edge, and 3 vertices. The first vertex has gens 0 and one leg labelled by 1. The second vertex has genus 0 and one leg labelled by 0. The third vertex has genus g and two legs labelled by 0 and ∞.

In particular, we have that $\mathfrak{M}_{\Gamma_2} = \mathfrak{M}_{g,2} \times \mathfrak{M}_{\Gamma_1}$, *and we get that the following diagram commutes:*

$$\begin{array}{ccc} \mathfrak{M}_{\Gamma_2} & \xrightarrow{\xi_{\Gamma_2}} & \mathfrak{M}_{g,3} \\ {\scriptstyle \mathrm{Id}\times\xi_{\Gamma_1}} \downarrow & \nearrow {\scriptstyle \xi_{\Gamma_0}} & \\ \mathfrak{M}_{\Gamma_0} = \mathfrak{M}_{g,2} \times \mathfrak{M}_{\Gamma_1} & & \end{array}$$

Thus, we deduce that $\psi_1[\Gamma_0] = [\Gamma_2]$. *The argument finishes arguing that* $\overline{\rho}^*([\Gamma_2]) = 0$. *The reason why this happens is because the preimage by* $\overline{\rho}$ *of the divisor of* $\mathfrak{M}_{g,3}$ *of prestable curves with graph* Γ_2 *is empty. Note that there is no non-rigid stable map with such a graph. More specificaly, let* $(f : C \to \mathbb{P}^1[k], p_1, q_0, q_\infty)$ *be a non rigid stable map whose curve has such a graph. As we argued in Proposition 3.5.1, one can check that the restriction of* f *to the preimage of each copy of* $\mathbb{P}^1$ *inside* $\mathbb{P}^1[k]$ *must have degree* 1. *This will lead to the existence of a genus* 0 *contracted component with at most* 2 *special points corresponding to the extreme vertex of genus* 0 *and leg* 1 *in* Γ_2. *This contradicts the stability condition of* f. *As a consequence,* $\psi_1\overline{\rho}^*([\Gamma_0]) = \overline{\rho}^*([\Gamma_2]) = 0$. ☐

Applying this claim to equation (37) we get that

$$\left[\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1/0,\infty)^\sim\right]^{\text{vir}} \frown \psi_0^k = \frac{1}{2g}(\pi^\sim)_*\left(\left[\overline{\mathcal{M}}_{g,1}(\mathbb{P}^1/0,\infty)^\sim\right]^{\text{vir}} \frown \psi_1\psi_0\right).$$

We have reached the situation where we can work with $\overline{\mathcal{M}}_{g,1}(\mathbb{P}^1/0,\infty)^\sim$. Following [37] Section 1.5.5., we can build a morphism from $\overline{\mathcal{M}}_{g,1}(\mathbb{P}^1/0,\infty)^\sim$ to $\mathfrak{M}_{0,3}$ as follows. Let $(f : C \to \mathbb{P}^1[k], p_1, q_0, q_\infty)$ be a non-rigid stable map. Note that in this situation, $\Delta = \mathbb{P}^1$ and hence $\mathbb{P}^1[k]$ is a genus 0 nodal curve (see Fig. 8). Thus, we get a morphism

$$\begin{array}{rccc} \varepsilon : & \overline{\mathcal{M}}_{g,1}(\mathbb{P}^1/0,\infty)^\sim & \longrightarrow & \mathfrak{M}_{0,3} \\ & (f : C \to \mathbb{P}^1[k], p_1, q_0, q_\infty) & \longmapsto & (\mathbb{P}^1[k], f(p_1), f(q_0), f(q_\infty)). \end{array}$$

We denote the three marked points of $\mathfrak{M}_{0,3}$ by 1, 0, and ∞. Now, as it is stated in [37] Section 1.5.5., we get that $\psi_0 = \varepsilon^*(\psi_0)$. Applying the same argument as before we get that $\psi_0 = \text{st}^*(\psi_0) + [\Gamma_0]$ for Γ_0 as in Proposition A.1.3 and, since $\overline{\mathcal{M}}_{0,3} = \{\text{pt}\}$, we get that $\psi_0 = \varepsilon^*([\Gamma_0])$. The idea now is to use the splitting of this graph, and observing that a non rigid stable map $(f : C \to \mathbb{P}^1[k], p_1, q_0, q_\infty)$ lying in the preimage of the divisor corresponding to Γ_0 can be split in to two components corresponding to each vertex of the graph. In particular, we get a commutative diagram

$$\begin{array}{ccc} \displaystyle\bigsqcup_{\substack{g = g_1 + g_2 \\ g_2 > 0}} \overline{\mathcal{M}}_{g_1,1}(\mathbb{P}^1/0,\infty)^\sim \times \overline{\mathcal{M}}_{g_2,0}(\mathbb{P}^1/0,\infty)^\sim & \xrightarrow{\xi^\sim_{\Gamma_0}} & \overline{\mathcal{M}}_{g,1}(\mathbb{P}^1/0,\infty)^\sim \\ \Big\downarrow{\scriptstyle \varepsilon_1\times\varepsilon_2} & & \Big\downarrow{\scriptstyle \varepsilon} \\ \mathfrak{M}_{\Gamma_0} = \mathfrak{M}_{0,3} \times \mathfrak{M}_{0,2} & \xrightarrow{\xi_{\Gamma_0}} & \mathfrak{M}_{0,3} \end{array}$$

For each g_1, g_2, we denote $\overline{\mathcal{M}}_{g_1,1}(\mathbb{P}^1/0,\infty)^\sim$ and $\overline{\mathcal{M}}_{g_2,0}(\mathbb{P}^1/0,\infty)^\sim$ by $\overline{\mathcal{M}}^\sim_{g_1}$ and $\overline{\mathcal{M}}^\sim_{g_2}$, respectively. Furthermore, we denote by $\xi^\sim_{g_1}$ the restriction of ξ_{Γ_0} to $\overline{\mathcal{M}}^\sim_{g_1} \times \overline{\mathcal{M}}^\sim_{g_2}$.

Moreover, we use the analogous of the splitting axiom in our non rigid case (see [37] Section 1.5.5.) and, in combination with the above study, we get

$$\begin{aligned}\left[\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1/0,\infty)^{\sim}\right]^{\text{vir}} \frown \psi_0^k &= \frac{1}{2g}(\pi^{\sim})_*\left(\left[\overline{\mathcal{M}}_{g,1}(\mathbb{P}^1/0,\infty)^{\sim}\right]^{\text{vir}} \frown \psi_0^k\psi_1\right) = \\ &\frac{1}{2g}(\pi^{\sim})_*\left(\left[\overline{\mathcal{M}}_{g,1}(\mathbb{P}^1/0,\infty)^{\sim}\right]^{\text{vir}} \frown \psi_0^{k-1}\psi_1\varepsilon^*([\Gamma_0])\right) = \\ &\frac{1}{2g}(\pi^{\sim})_*\circ\xi^{\sim}_{\Gamma_0 *}\left(\sum_{g=g_1+g_2,\, g_2>0}\left[\overline{\mathcal{M}}^{\sim}_{g_1}\times\overline{\mathcal{M}}^{\sim}_{g_2}\right]^{\text{vir}} \frown (\xi^{\sim}_{g_1})^*(\psi_0^{k-1}\psi_1)\right).\end{aligned} \tag{38}$$

Now, since for $\mathfrak{M}_{g,n}$ the pullback through gluing morphisms of ψ–classes is the ψ–class corresponding to the factor of the product where the marking lie (see [4] Section 3.2.), the same holds in the non-rigid case. In particular, we get that

$$(\xi^{\sim}_{g_1})^*(\psi_0^{k-1}\psi_1) = \psi_1\otimes\psi_0^{k-1}.$$

Applying this equality to (38), one gets that $\left[\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1/0,\infty)^{\sim}\right]^{\text{vir}}\frown \psi_0^k$ is equal to

$$\frac{1}{2g}(\pi^{\sim})_*\circ\xi^{\sim}_{\Gamma_0 *}\left(\sum_{g=g_1+g_2,\, g_2>0}\left(\left[\overline{\mathcal{M}}^{\sim}_{g_1}\right]^{\text{vir}}\frown\psi_1\right)\otimes\left(\left[\overline{\mathcal{M}}^{\sim}_{g_2}\right]^{\text{vir}}\frown\psi_0^{k-1}\right)\right)$$

In this situation, the idea is to apply the dilation equation again to get rid of the ψ_1 class. To do so, we will use the following commutative diagram

$$\begin{array}{ccc}\overline{\mathcal{M}}^{\sim}_{g_1}\times\overline{\mathcal{M}}^{\sim}_{g_2} & \xrightarrow{\ \xi^{\sim}_{g_1}\ } & \overline{\mathcal{M}}_{g,1}(\mathbb{P}^1/0,\infty)^{\sim} \\ \Big\downarrow{\scriptstyle \pi^{\sim}\times\mathrm{Id}} & & \Big\downarrow{\scriptstyle \pi^{\sim}} \\ \overline{\mathcal{M}}_{g_1,0}(\mathbb{P}^1/0,\infty)^{\sim}\times\overline{\mathcal{M}}^{\sim}_{g_2} & \xrightarrow{\ \xi^{\sim}_{\Gamma_{g_1}}\ } & \overline{\mathcal{M}}_{g,0}(\mathbb{P}^1/0,\infty)^{\sim}\end{array}$$

where Γ_{g_1} is the prestable graph resulted from erasing the leg 1 from Γ_0 and changing the genus of its two vertices by g_1 and g_2 respectively, and $\xi^{\sim}_{\Gamma_{g_1}}$ is the corresponding gluing morphism in the non-rigid setting. From this diagram we get that

$$\begin{aligned}&\left[\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1/0,\infty)^{\sim}\right]^{\text{vir}}\frown\psi_0^k = \\ &\frac{1}{2g}\sum_{g=g_1+g_2,\, g_2>0}(\xi_{\Gamma_{g_1}})^{\sim}_*\left(((\pi^{\sim})_*\otimes\mathrm{Id})\left(\left(\left[\overline{\mathcal{M}}^{\sim}_{g_1}\right]^{\text{vir}}\frown\psi_1\right)\otimes\left(\left[\overline{\mathcal{M}}^{\sim}_{g_2}\right]^{\text{vir}}\frown\psi_0^{k-1}\right)\right)\right) = \\ &\sum_{g=g_1+g_2,\, g_2>0}\frac{g_1}{g}(\xi_{\Gamma_{g_1}})^{\sim}_*\left(\left[\overline{\mathcal{M}}_{g_1,0}(\mathbb{P}^1/0,\infty)^{\sim}\right]^{\text{vir}}\otimes\left(\left[\overline{\mathcal{M}}^{\sim}_{g_2}\right]^{\text{vir}}\frown\psi_0^{k-1}\right)\right).\end{aligned}$$

The last step will be to pushforward the above expression through ρ. In particular, we will need the following commutative diagram:

$$\begin{CD}
\overline{\mathcal{M}}_{g_1,0}(\mathbb{P}^1/0,\infty)^{\sim} \times \overline{\mathcal{M}}^{\sim}_{g_2} @>{\xi^{\sim}_{\Gamma_{g_1}}}>> \overline{\mathcal{M}}_{g,0}(\mathbb{P}^1/0,\infty)^{\sim} \\
@V{\rho\times\rho}VV @VV{\rho}V \\
\overline{\mathcal{M}}_{\Gamma_{g_1}} @>{\xi_{\Gamma_{g_1}}}>> \overline{\mathcal{M}}_{g,2}
\end{CD}$$

Using this diagram we finally get that

$$\Psi_{g,0}(k) = \rho_*\left(\left[\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1/0,\infty)^{\sim}\right]^{\mathrm{vir}} \frown \psi_0^k\right) =$$

$$\sum_{g=g_1+g_2,\, g_2>0} \frac{g_1}{g}(\xi_{\Gamma_{g_1}})_*\left(\rho_*\left(\left[\overline{\mathcal{M}}_{g_1,0}(\mathbb{P}^1/0,\infty)^{\sim}\right]^{\mathrm{vir}}\right) \otimes \rho_*\left(\left[\overline{\mathcal{M}}^{\sim}_{g_2}\right]^{\mathrm{vir}} \frown \psi_0^{k-1}\right)\right) =$$

$$\sum_{g=g_1+g_2,\, g_2>0} \frac{g_1}{g}\,(\xi_{\Gamma_{g_1}})_*\left(\mathrm{DR}_{g_1}(1,-1) \otimes \Psi_{g_2,0}(k-1)\right).$$

□

Thus, Theorem 3.6.2 provides the method for the computation of $\Psi_{g,0}(k)$ for $g \geq 1$ and $k \geq 0$. We recall that this was the last remaining step to compute the class $\mathcal{J}_g(k,n)$ as tautological class and, as a result, this completes the algorithm. In particular, Corollary 3.4.2 derives from the study carry out above, concluding with the quasimodularity of the invariants on $K3$ surfaces.

3.7 Summary of the algorithm

As we did for the elliptic curve case in Subsection 2.6, the task of this subsection is to present the structure of the algorithm studied during this whole section. We specify the INPUT and OUTPUT of the algorithm, its structure, and its reduction steps.

We recall that our goal is to compute invariants of the form

$$\langle \alpha; \gamma_1, \ldots, \gamma_n \rangle^S_{g,n} \quad \text{and} \quad \langle \alpha; \tau_{k_1}(\gamma_1)\cdots\tau_{k_n}(\gamma_n)\rangle^S_{g,n}$$

for $\alpha \in RH^*(\overline{\mathcal{M}}_{g,n})$, $\gamma_1, \ldots, \gamma_n \in H^*(S)$, and $k_1, \ldots, k_n, g, n \geq 0$. By the linearity axiom, we assume that the classes γ_i lie in the basis $\mathcal{B}$ of $H^*(S)$. Moreover, using Theorem A.1.3, we also assume that the tautological class inside the invariant is given by a decorated stratum class $[\Gamma, \alpha]$. So, the INPUT of the algorithm consists in:

- The genus g and the number of markings n.
- A decorated stratum class $[\Gamma, \alpha]$.

- Evaluation classes $\gamma_1, \dots, \gamma_n \in \mathcal{B}$.
- Non-negative integers $k_1, \dots, k_n$.

On the other hand, using Corollary 3.4.2, our invariants lies in the ring $\frac{1}{\Delta(q)}\mathbf{QM} = \frac{1}{\Delta(q)}\mathbb{Q}[G_2, G_4, G_6]$. Thus, as we did in the elliptic curve case, the OUTPUT of the algorithm consists in polynomials in G_2, G_4, and G_6, times $\frac{1}{\Delta}$.

After stating the INPUT and OUTPUT of the algorithm, let us outline the procedure and reduction steps studied for computing the invariants. Recall that the main idea is to use recursion on the tuple (g, n):

1. The first step is to deal separately with the unstable cases (g, n) with $2g-2+n$, i.e., $(g, n) \in \{(0,0), (0,1), (0,2), (1,0)\}$. Using the divisor, string, and dilation equation we are able to compute these cases.

2. Secondly, we divide the general case in two cases depending on whether there is a evaluation class equal to $\mathbf{p}$ or not. If $\gamma_i \neq \mathbf{p}$, we saw that, either $[\Gamma, \alpha]$ is the class of a point and we can apply the divisor equation, or we can apply the Ionel-Getzler vanishing and assume that Γ is not the trivial graph. Under this assumption we can apply Proposition 3.2.4 to write the invariant by means of invariants lower in the recursion.

3. If there exists $\gamma_i = \mathbf{p}$ we can apply the degeneration formula to write the invariant by means of relative invariants on S and $\mathbb{P}^1 \times E$ relative to E.

4. To compute relative invariants on S we apply again recursion on (g', n'). All these relative invariants coming from the degeneration formula have $(g', n') < (g, n)$. Then applying again the degeneration formula, we can write the relative invariant in terms of an absolute invariant on S lower in the recursion, relative invariants on $\mathbb{P}^1 \times E$ relative to E, and relative invariants on S relative to E lower in the recursion. This recursion ends when reached the case $(0, 0)$, where the relative invariant is $\frac{1}{\Delta(q)}$.

5. From the above reduction steps, the last ingredients to be computed are the relative invariants on $\mathbb{P}^1 \times E$. We distinguish two cases depending on whether $2g-2+n+1 \leq 0$ or not. The first case corresponds to the cases $(0,0)$ and $(0,1)$ which are solved via Propositions 3.3.3 and 3.3.4.

6. If $2g-2+n+1 > 0$, we can apply the product formula to reduce the computation of the relative invariant on $\mathbb{P}^1 \times E$ to the invariant on E of the form

$$\langle \alpha \mathcal{J}_g(k, n); \gamma_1, \dots, \gamma_n, \omega \rangle^E_{g,n+1}.$$

   After computing the class $\mathcal{J}_g(k, n)$ as a tautological class we can apply the algorithm presented in Section 2 to compute the invariant.

7. Using equation (34), we can compute $\mathcal{J}_g(k,n)$ as the pullback through the forgetful morphisms of $\mathcal{J}_g(k,k)$. Applying the localization formula studied in Subsection 3.5, we reduce the computation of this class to the computation of $\rho_*(\psi_0^k)$ as tautological class.

8. Finally, the computation of this class is performed through the recursion proven in Theorem 3.6.2.

Before finishing the section let us comment other possible procedures for computing the invariants on S. Note that the main idea of our algorithm is based on reducing the computation of invariants on S to relative invariants on $\mathbb{P}^1 \times E$ with the final goal of ending up in invariants on E which we already know how to compute. To arrive to the invariants on E we applied first the product formula and afterwards localization. However, in the same way we defined the $\mathbb{C}^*$–action on $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1/0,(1),1)$, we could have defined a $\mathbb{C}^*$–action on $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1 \times E/E,(1),s+hf)$. As a result, we could have applied localization formula first to $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1 \times E/E,(1),s+hf)$ and afterwards the product formula. This is the procedure followed in proof of [10] Theorem 2.

Another possible path we could have taken, in the above algorithm, is in the computation of the virtual pushforward of ψ_0^k as tautological class. As commented at the beginning of Subsections 3.5 and 3.6, in [14] Theorem 1 the authors prove, in a more general setting, that this pushforward is tautological. Moreover, the proof of this result provides an algorithm to compute the expression of the class as tautological class. The idea behind this algorithm is to argue again by induction on (g,n) and using some tautological relations to reduce the current step of the recursion to lower cases.

4 Guide for the code and examples of computations

As commented in the introduction, concurrently to the theoretical study of the above algorithms for computing Gromov-Witten invariants on elliptic curves and $K3$ surfaces, one of the goals of this master thesis is to implement a SageMath code for computing these invariants. This section is fully devoted to introduce the code and to show how to use it. At the same time, we expose some examples and computations using the code.

The section is structured in two subsections. Subsection 4.1 will be devoted to introduce the code related to the algorithm for the elliptic curve. Among some of the important functions this branch of the code has, we highlight the ones for the computing the n–point correlation function, connected/disconnected descendent invariants on E, or Gromov-Witten invariants on E. On the other hand, the second subsection will be focused on the algorithm for $K3$ surfaces. We introduce the functions of the program computing invariants on S, relative invariants on S, and relative invariants on $\mathbb{P}^1 \times E$.

As it has been shown all along the previous sections, in both algorithms, the operations on the tautological ring as pullbacks and pushforwards through gluing and forgetful morphisms need to be performed. In this sense, the presented code is implemented using the package `admcycles` developed by Vincent Delecroix, Johannes Schmitt, and Jason van Zelm. This package allows us to work with tautological classes and with the tautological ring. Some of the tools provided by `admcycles` are the computation of operations on the tautological ring (as multiplications, pullbacks and pushforwards through gluing and forgetful morphisms), computing the integral over $\overline{\mathcal{M}}_{g,n}$, or computing the double ramification cycle as tautological class, etc. For an introduction to this package, how to use it and how to download it, we refer to [12].

The code is available at https://gitlab.com/jo314schmitt/admcycles/-/tree/GWK3. After following the installation instruction there, and before any of the computations below, the next lines should be executed:

```
sage: from admcycles import*
sage: from admcycles.gromovwitten import*
```

4.1 Guide for the elliptic curve algorithm

As mentioned above, this subsection focuses on the branch of the code dedicated to compute elliptic curve invariants. We show some of the main functions of the program and we see some examples of the computations.

Let us begin with the n–point correlation function. We recall from Subsection 2.2 the importance of these functions while computing stationary invariants. In particular, we saw in Theorems 2.2.2 and 2.2.3 two different ways of computing F_n. Both formulas have been implemented in the following function:

- `point_correlation_function(n,t,method = None)`: Returns $z_1 \cdots z_n F_n$ as an element in $\mathbf{QM}[[z_1, \ldots, z_n]] = \mathbb{Q}[G_2, G_4, G_6][[z_1, \ldots, z_n]]$ truncated at degree $t+1$. If `method` is not specified, it computes F_n using the recursive formula stated in Theorem 2.2.2. On the other hand if one set `method = 'det'`, it computes F_n using Theorem 2.2.2.

Let us show an example of how to use this function:

```
sage: point_correlation_function(2,5)
1 + G2*Z0^2 + G2*Z1^2 + (1/2*G2^2 + 1/12*G4)*Z0^4 + (-G2^2 + 5/6*G4)*Z0^2
*Z1^2 + (1/2*G2^2 + 1/12*G4)*Z1^4 + O(Z0, Z1)^6

sage: point_correlation_function(2,5,'det')
1 + G2*Z0^2 + G2*Z1^2 + (1/2*G2^2 + 1/12*G4)*Z0^4 + (-G2^2 + 5/6*G4)*Z0^2
*Z1^2 + (1/2*G2^2 + 1/12*G4)*Z1^4 + O(Z0, Z1)^6
```

Derived from the above function, the computation of stationary invariants is implemented in the following function:

- `stationary_invariants(n,K)`: Given n non negative integer and $K = [k_1, \ldots, k_n]$ list of non negative integers of length n, it computes the stationary invariant $\langle \tau_{k_1}(\omega) \cdots \tau_{k_n}(\omega) \rangle^{\bullet}$.

Let us compute the stationary invariants $\langle \tau_4(\omega) \rangle^{\bullet}$, $\langle \tau_1(\omega)\tau_1(\omega) \rangle^{\bullet}$, and $\langle \tau_0(\omega)\tau_2(\omega)\tau_0(\omega) \rangle^{\bullet}$ using the above function:

```
sage: stationary_invariants(1,[4])
1/6*G2^3 + 1/12*G2*G4 + 1/360*G6

sage: stationary_invariants(2,[1,1])
-8/3*G2^3 + 2/3*G2*G4 + 7/180*G6

sage: stationary_invariants(3,[0,2,0])
15/2*G2^4 - 6*G2^2*G4 + 85/24*G4^2 - 7/15*G2*G6
```

Let us move to the non-stationary invariants studied during subsections 2.3 and 2.4. To compute these invariants, the first step is to define the respective descendent Gromov-Witten class. An important part of these classes are the evaluation classes. These evaluation classes are supposed to be in the basis $\{1, \alpha, \beta, \omega\}$ of the cohomology of E. These classes will be represented in the code by the symbolic variables `alpha, beta, omega` and the unit will be represented by the integer 1. The following SageMath class allow us to define the descendent Gromov-Witten class:

- `descendent_GW_class_E(Λ,Ch,descendents)`: Given:
  - Λ: List of nonnegative integers $[i_1, \ldots, i_l]$,
  - `Ch`: List of nonnegative integers $[j_1, \ldots, j_m]$,
  - `descendents`: List whose elements of the form $[[k_1, \gamma_1], \ldots, [k_n, \gamma_n]]$, where $k_1, \ldots, k_n$ are nonnegative integers, and $\gamma_1, \ldots, \gamma_n$ are 1 or one of the symbolic variables `alpha`, `beta`, or `omega`,

it defines the cohomology class

$$\lambda_{i_1} \cdots \lambda_{i_l} \mathrm{ch}_{j_1} \cdots \mathrm{ch}_{j_m} \tau_{k_1}(\gamma_1) \cdots \tau_{k_n}(\gamma_n)$$

Let us show some examples on how to define such classes:

```
sage: I1 = descendent_GW_class_E([],[],[[1,omega],[1,alpha],[1,beta],[1,1]])
sage: I1
Descendent classes:[[1, omega], [1, alpha], [1, beta], [1, 1]]
Lambda classes:[]
Chern characters:[]

sage: I2 = descendent_GW_class_E([1,1],[],[[1,omega],[1,omega]]); I2
Descendent classes:[[1, omega], [1, omega]]
Lambda classes:[1, 1]
Chern characters:[]
```

Once we have defined the descendent invariant, we can compute, both, the connected or the disconnected invariants:

- `descendent_GW_class_E.connected_invariant()`: Computes the connected invariant as an element in $\mathbb{Q}[G_2, G_4, G_6]$. It assumes that no Chern character is given.
- `descendent_GW_class_E.disconnected_invariant()`: Computes the disconnected invariant of the given class as an element of $\mathbb{Q}[G_2, G_4, G_6]$.

Let us show some examples through the computation of the invariants of the classes `I1` and `I2` defined in the previous example:

```
sage: I1.connected_invariant()
-80/3*G2^3 + 20/3*G2*G4 + 7/18*G6

sage:I2.connected_invariant()
50/63*G4^2 - 1/6*G2*G6

sage: I2.disconnected_invariant()
407/504*G4^2 - 1/6*G2*G6
```

In [45] Section 4.4, the author conjectured that for $g \geq 1$ it holds

$$\langle \lambda_{g-1} \tau_{g-1}(\omega) \rangle = \frac{g!}{2^{g-1}} C_{2g}.$$

This conjecture has been proved by G. Oberdieck and A. Pixton in [42]. We can check the basic cases of this statement using the function `check_conjecture_lambda(g)`:

```
sage: check_conjecture_lambda(2)
True

sage: check_conjecture_lambda(3)
True
```

However, the computation of λ classes quickly leads to stationary invariants with $n \geq 5$ and the computation of the n–point correlation function for these cases requires a considerably amount of time. In these sense, to check the conjecture for genus bigger equal than 4 takes lot of time. However, we have checked it for $g \leq 5$.

Now, we deal with the most general kind of invariants we have considered for the elliptic curves, the invariants of the form

$$\langle \mu; \gamma_1, \dots, \gamma_m \rangle_{g,n}^{E}$$

for $\mu \in RH^*(\overline{\mathcal{M}}_{g,n})$ and $\gamma_i \in H^*(E)$. As before, the first step to compute these invariants using the exposed code is to create the respective cohomology class. This is done through the following SageMath class:

- `GW_class_E(g,n,Ev,tautclass)`: Given the following OUTPUT:
  - `g`: Non negative integer,
  - `n`: Non negative integer,
  - `Ev`: List $[\gamma_1, \dots, \gamma_n]$ whose elements are either 1 or one of the symbolic variables `alpha`, `beta`, or `omega`.
  - `tautclass`: Tautological class μ defined through `admcycles` (see [12] Section 3 for its construction). If it is not given, it is fixed to be the unit.

  it creates the Gromov-Witten class

$$\mu\, \rho_* \left(\prod_{i=0}^{n} \mathrm{ev}_i^*(\gamma_i) \right) \in RH^*(\overline{\mathcal{M}}_{g,n})$$

Again, before seeing how to compute the respective invariants, we illustrate by examples how to compute such classes.

```
sage: G = StableGraph([0,1,1],[[1,2,4,5],[3,6,7],[8,9]],[(4,6),(5,8),(7,9)])
sage: t = psiclass(1,3,3)*psiclass(3,3,3)*G.boundary_pushforward()
sage: I1 = GW_class_E(3,3,[1,omega,omega],t);I1
Evaluation classes: [1, omega, omega]
Tautological class:
Graph : [0, 1, 1] [[1, 2, 4, 5], [3, 6, 7], [8, 9]] [(4, 6), (5, 8), (7, 9)]
Polynomial : 1*psi_1^1 psi_3^1
```

Once the Gromov-Witten invariants have been created, we focus on the respective invariants computation. The function `GW_class_E.invariant(h=None)` provides this computation. If h is not given, it computes the respective generating series as an element in $\mathbb{Q}[G_2, G_4, G_6]$. On the other hand, if h is given (it must be a nonnegative integer), `GW_class_E.invariant(h)` will compute the rational number corresponding to the invariant with effective curve class $h[E]$. Let us show the computation of some invariants using the Gromov-Witten classes defined in the previous example

```
sage: G = StableGraph([0,1,1],[[1,2,4,5],[3,6,7],[8,9]],[(4,6),(5,8),(7,9)])
sage: t = psiclass(1,3,3)*psiclass(3,3,3)*G.boundary_pushforward()
sage: I2 = GW_class_E(3,3,[1,omega,omega],t)

sage: I2.invariant()
4*G2^4 - 10/3*G2^2*G4 + 25/36*G4^2

sage: I2.invariant(5)
200
```

Recall that the pullback of a λ class from $\overline{\mathcal{M}}_{g,n}$ to $\overline{\mathcal{M}}_{g,n}(E,h)$ through the morphism ρ is the respective λ on $\overline{\mathcal{M}}_{g,n}(E,h)$. As a result, we can compute the invariants whose tautological class is a λ class through either the procedure studied in Subsection 2.4 or the one studied in Subsection 2.5. Let us show through some examples that both procedures lead to the same answer.

```
sage: t = lambdaclass(1,3,1)*psiclass(1,3,1)**3
sage: I3 = GW_class_E(3,1,[omega],t)
sage: I3.invariant()
1/12*G2*G4 + 1/144*G6

sage: I4 = descendent_GW_class_E([1],[],[[3,omega]])
sage: I4.connected_invariant()
1/12*G2*G4 + 1/144*G6
```

However, due to the complicated expression of the λ classes as tautological classes, the algorithm developed in Subsection 2.4 is more efficient. For instance, in the example above, for the descendent invariants, we computed the connected and disconnected invariants of `I2`. Below we show the computation of the same invariants using the algorithm for invariants with tautological class. One can check that the answer is the same but the time required for determining the invariant increases remarkable.

```
sage: t = lambdaclass(1,3,2)**2*psiclass(1,3,2)*psiclass(2,3,2)
sage: I5 = GW_class_E(3,2,[omega,omega],t)
sage: I5.invariant()
50/63*G4^2 - 1/6*G2*G6
```

Finally, to conclude this subsection, one can also ask for the compatibility of the invariants with the Faber-Zagier relations (see [46]). In [12] Section 3.3. it is explained how to use `admcycles` to compute generators and the relations among them for a given genus g, number of markings n, and degree r of the tautological ring. Let T be the set of generators and R one of those relations. Then, for a given $\gamma_1, \ldots, \gamma_n \in H^*(E)$ and $\alpha \in RH^*(\overline{\mathcal{M}}_{g,n})$, it holds that

$$\sum_{\mu \in T} a_\mu \langle \mu\, \alpha; \gamma_1, \ldots, \gamma_n \rangle_{g,n}^E = 0$$

where a_α is the coefficient of α inside the relation R. We can check that these vanishings holds through the function `GW_class_E.check_tau_relation(r)`. This function returns `True` or `False` depending whether the relations are satisfied or not. For instance,

```
sage: I = GW_class_E(2,1,[omega])
sage: I.check_tau_relation(2)
True

sage: I = GW_class_E(2,2,[omega,omega])
sage: I.check_tau_relation(2)
True
```

Before passing to the next subsection let us comment how to work with quasimodular forms with the program. In Subsection A.2, we introduced the Eisenstein series E_k and different normalizations as C_k and G_k. As shown in the above examples, the OUTPUT of the program, while computing invariants, is the respective quasimodular form as polynomial in $\mathbb{Q}[G_2, G_4, G_6]$. However, one can be interested in finding such a expression for the generators C_k or E_k. This and other useful tools about quasimodular forms can be used through the functions:

- `normalization(p, To)`: Given `p` a quasimodular form and `To` is a string among `'G'`, `'C'`, and `'E'` denoting the normalization we want to change `p` to.
- `G_poly(k)`: Given a positive integer `k` it returns the Eisenstein series G_k as an element in $\mathbb{Q}[G_2, G_4, G_6]$.
- `G(k,h)`: Returns the power expansion of the quasimodular form G_k truncated at q^{h+1}.
- `coeff_series(h,p)`: Returns the coefficient of the term q^h of the power expansion of the quasimodular form `p`.

We finish this subsection illustrating by some example how to use some of the functions above.

```
sage: I = GW_class_E(2,2,[omega,omega],psiclass(1,2,2)**2)
sage: i = I.invariant(); i
-2*G2^3 + 1/6*G2*G4 + 7/120*G6

sage: normalization(i,'C')
-2*C2^3 + 2*C2*C4 + 21*C6

sage: normalization(i,'E')
1/6912*E2^3 - 1/34560*E2*E4 - 1/8640*E6

sage: coeff_series(9,i)
13791/8
```

4.2 Guide for the $K3$ surface algorithm

Similarly as the previous subsection, the task of this subsection is to show how to use the program developed as a part of this thesis for computing Gromov-Witten invariants on $K3$ surfaces. We recall that in the procedure for computing these invariants, the relative invariants on S and $\mathbb{P}^1 \times E$ relative to E are also computed. So, the program will allow us to compute these invariants too. Moreover, an important case of the induction used to compute invariants on S is based on the Ionel-Getzler vanishing (Theorem 3.2.2). As a result, we will see how to, under the assumptions of the theorem, express tautological classes in terms of the pushforward of classes through the boundary.

Let us begin with the relative invariants on $\mathbb{P}^1 \times E$ relative to E. We recall that, as a consequence of the product formula, the computation of these invariants is reduced to the computation of the class $\mathcal{J}_g(k,n)$ as tautological class. The function `J_class(g,k,n)` computes the expression this class. As a result of the localization formula, the λ classes arise quickly in the expression of these classes. This means that the expression of the class gets complicated soon as the following example illustrates (`tautclass.toTautbasis()` compute the vector of a tautological class in terms of the basis of the tautological ring, see [12] Section 3.3. for the details).

```
sage: J = J_class(1,2,2)
sage: J.toTautbasis()
(-1, 1, 1, 1, -1)

sage: J = J_class(1,3,3)
sage: J.toTautbasis()
(-37/4, 19/4, -2, -6, -6, -5, -1, 1, 3, 5, 7, 5/2, 3/2, 7, 3/2, 5, 5/2, -7,
2, 5/2, -3, 5/2, -3)

sage: J = J_class(2,2,2)
sage: J.toTautbasis()
(-3/2, 1/2, -2, -1, -1/2, 5, 2, 3/2, 2, 1/2, 1/2, 1/2, -3/2, -1/2, 1, -1,
-2, 3/2, -1, -7/2, 3/2, 1, -2, 0, 1/2, 0, 0, 0, 1/2, 1/2, -3/2, 1/2, -1/2,
-1/2, 0, 0, 0, 1/2, 0, -1/2, 1/2, 1/2, 1/2, 0)
```

After seeing how to compute these invariants, we focus on the relative invariants on $\mathbb{P}^! \times E$. As we did for the invariants on E in the previous subsection, the first step is to define the respective class. This can be accomplished as follows:

- `rel_class_P1xE`: Given
  - `g`: non-negative integer,
  - `n`: non-negative integer,
  - `ev_P1`: list $[\gamma_1, \dots, \gamma_n]$ where γ_i is `1` or the symbolic variable `p`,
  - `ev_E`: list $[\gamma_1', \dots, \gamma_n']$ where γ_i' is `1` or one of the symbolic variable `alpha`, `beta`, or `omega`,
  - `tau_class`: tautological class μ defined through `admcycles` (note that the tautological class must lie in $\overline{\mathcal{M}}_{g,n+1}$),

  it defines the relative Gromov-Witten class

  $$\mu\, \rho_* \left(1 \otimes \omega \prod_{i=1}^{n} \mathrm{ev}_i^*(\gamma_i \otimes \gamma_i') \right) \in RH^*(\overline{\mathcal{M}}_{g,n+1}).$$

  Note that it fixes the relative evaluation class to be ω.

Let us see an example where these classes are defined.

```
sage: I1 = rel_class_P1xE(1,3,[p,1,p],[omega,1,1],psiclass(1,1,4)); I1
Evaluation classes: [(p, omega), (1, 1), (p, 1)]
Tautological class: Graph :      [1] [[1, 2, 3, 4]] []
Polynomial : 1*psi_1^1
```

Once the class is defined, one can compute the respective invariant through the function `rel_class_P1xE.invariant(h = None)`:

```
sage: I1.invariant()
-2*G2^2 + 5/6*G4

sage: I1.invariant(7)
2540160
```

Now that we have seen how to compute the relative invariants on $\mathbb{P}^1 \times E$, our next step is the invariants over the $K3$ surfaces. As commented above, an important part of the algorithm uses the Ionel-Getzler vanishing to write tautological classes with trivial graphs by means of decorated stratum classes with non trivial graphs, where we can use the splitting and reduction axioms. This is done through the function `express_as_boundary(tau_class)` where `tau_class` is a tautological class defined through `admcycles` and it assumes that it has degree bigger or equal than the genus. Let us see some examples.

```
sage: express_as_boundary(psiclass(1,1,1))
Graph :      [0] [[3, 4, 1]] [(3, 4)]
Polynomial : 1/24*

sage: express_as_boundary(kappaclass(1,1,2))
Graph :      [0, 1] [[1, 2, 4], [5]] [(4, 5)]
Polynomial : 1*
Graph :      [0] [[4, 5, 1, 2]] [(4, 5)]
Polynomial : 1/12*
```

After this little parenthesis on tautological classes, let us show how to define the Gromov-Witten classes on $K3$ surfaces. This is done via the following SageMath object:

- `GW_class_K3`: Given
  - `g`: non-negative integer,
  - `n`: non-negative integer,
  - `descendent_classes`: list $[[\gamma_1, k_1], \dots, [\gamma_n, k_n]]$ where γ_i' is `1` or one of the symbolic variable `s`, `f`, `delta1`, ... , `delta20`, or `p`, and k_i are non-negative integers,
  - `tau_class`: tautological class μ defined through `admcycles` (note that the tautological class must lie in $\overline{\mathcal{M}}_{g,n}$),

  it defines the relative Gromov-Witten class

$$\mu\, \rho_* \left(\prod_{i=1}^{n} \tau_{k_i}(\gamma_i) \right) \in RH^*(\overline{\mathcal{M}}_{g,n}).$$

The following example shows how to define the above class:

```
sage: I2 = GW_class_K3(1,1,[[s,0]],kappaclass(1,1,1)); I2
Descendent classes: [[s, 0]]
Tautological class: Graph :      [1] [[1]] []
Polynomial : 1*(kappa_1^1 )_0

sage: t = psiclass(2,1,4)*kappaclass(1,1,4)
sage: I3 = GW_class_K3(1,4,[[s,0],[s,0],[f,0],[1,0]],t); I3
Descendent classes: [[s, 0], [s, 0], [f, 0], [1, 0]]
Tautological class: Graph :      [1] [[1, 2, 3, 4]] []
Polynomial : 1*(kappa_1^1 )_0 psi_2^1

sage: I4 = GW_class_K3(2,1,[[p,0]],psiclass(1,2,1)); I4
Descendent classes: [[p, 0]]
Tautological class: Graph :      [2] [[1]] []
Polynomial : 1*psi_1^1
```

Once we have defined the Gromov-Witten class, we can compute the invariant through the function `GW_class_K3.invariant(h = None)` as follows:

```
sage: I2.invariant()
44*G2^2*Delta - 2*G2*Delta + 5/3*G4*Delta

sage: I3.invariant()
5080*G2^3*Delta_inv - 502*G2^2*Delta_inv + 650*G2*G4*Delta_inv +
12*G2*Delta_inv - 185/6*G4*Delta_inv + 119/12*G6*Delta_inv

sage: I3.invariant(4)
164820

sage: I4.invariant()
-8/3*G2^3*Delta_inv + 4/3*G2*G4*Delta_inv - 7/360*G6*Delta_inv
```

In the OUTPUT given by the program, for the invariants over $K3$ surfaces, `Delta_inv` denotes $\frac{1}{\Delta(q)}$. Now, using this function we can compute the integers $N_g(h)$ of genus g curves on a $K3$ surfaces passing through g generic points and with h nodes. Recall that these integers coincide with the invariants

$$\langle 1; \mathbf{p}, \dots, \mathbf{p} \rangle^S_{g,g,s+hf}.$$

For example, we get

```
sage: N1 = GW_class_K3(1,1,[[p,0]])
sage: N1.invariant(3)
480

sage: N1.invariant(6)
378420

sage: N2 = GW_class_K3(2,2,[[p,0],[p,0]])
sage: N2.invariant(3)
36

sage: N2.invariant(5)
8728

sage: N3 = GW_class_K3(3,3,[[p,0],[p,0],[p,0]])
sage: N3.invariant(4)
42
```

In Subsection 3.2 we introduced the KKV (Katz-Klemm-Vafa) formula:

$$\sum_{g\geq 0}\sum_{h\geq 0} R_{g,h}u^{2g-2}q^{h-1} = \frac{1}{u^2\Delta(q)}\exp\left(\sum_{g\geq 1} u^{2g}\frac{|B_{2g}|}{g\cdot(2g)!}E_{2g}(q).\right) =$$

$$\frac{1}{u^2\Delta(q)}\exp\left(\sum_{g\geq 1}(-1)^g u^{2g}\frac{4}{(2g)!}G_{2g}(q).\right)$$

where $R_{g,h} = \langle(-1)^g\lambda_g\rangle^S_{g,0,h}$; this was proven by D. Maulik, R. Pandharipande, and R. P. Thomas in [36]. For example, from this formula one can check that

$$\langle\lambda_1\rangle^S_{1,0} = \frac{1}{\Delta(q)}2G_2,$$
$$\langle\lambda_2\rangle^S_{2,0} = \frac{1}{\Delta(q)}(2G_2^2 + \frac{1}{6}G_4),$$
$$\langle\lambda_3\rangle^S_{3,0} = \frac{1}{\Delta(q)}(\frac{1}{3}G_2^2 + \frac{1}{3}G_2G_4 + \frac{1}{180}G_6).$$

Note that, a priory, the invariant $\langle\lambda_1\rangle^S_{1,0}$ can not be computed through our algorithm since λ_1 does not come from a tautological class ($2g-2+n \leq 0$ in this case). We will see how to overcome this difficulty with the degeneration formula later. Now, lets check with the program that, for $g=2$ and $g=3$, the KKV formula holds:

```
sage: R2 = GW_class_K3(2,0,[],lambdaclass(2,2,0))
sage: R2.invariant()
2*G2^2*Delta_inv + 1/6*G4*Delta_inv

sage: t = lambdaclass(3,3,0)
sage: R3 = GW_class_K3(3,0,[],t)
sage: R3.invariant()
4/3*G2^3*Delta_inv + 1/3*G2*G4*Delta_inv + 1/180*G6*Delta_inv
```

Before dealing with the relative invariants on S, as we did with the elliptic curve in the previous subsection, we can check whether the invariants satisfy or not the Faber-Zagier relations. Let us show some examples:

```
sage: I5 = GW_class_K3(1,2,[[s,0],[f,0]])
sage: I5.check_tau_relation(1)
True

sage: I6 = GW_class_K3(2,2,[[p,0],[s,0]])
sage: I6.check_tau_relation(1)
True

sage: I7 = GW_class_K3(1,4,[[s,0],[s,0],[f,0],[1,0])
sage: I7.check_tau_relation(2)
True
```

Let us see now how to compute relative invariants on S relative to E. These invariants appear in our algorithm as a result of the degeneration formula. Moreover, we can compute them when no evaluation class is f or $\mathbf{p}$. Before computing these invariants, we have to define again the respective class:

- `rel_GW_class_K3`: Given
  - `g`: non-negative integer,
  - `n`: non-negative integer,
  - `Ev`: list $[\gamma_1, \dots, \gamma_n]$ where γ_i is `1` or the symbolic variable `p`,
  - `rel_ev`: The relative evaluation class δ represented by `1` or the symbolic variables `s`, `f`, `delta1`,..., `delta20`, `p`,
  - `tau_class`: tautological class μ defined through `admcycles` (note that the tautological class must lie in $\overline{\mathcal{M}}_{g,n+1}$),

  it defines the relative Gromov-Witten class

$$\mu\,\rho_*\left(\mathrm{ev}_i^{E*}(\delta)\prod_{i=1}^{n}\mathrm{ev}_i^*(\gamma_i)\right)\in RH^*(\overline{\mathcal{M}}_{g,n+1}).$$

The next example shows how to define these classes:

```
sage: I5 = rel_GW_class_K3(1,1,[s],1,psiclass(1,1,2)); I5
evaluation classes: [s]
relative evaluation class: 1
Tautological class: Graph :      [1] [[1, 2]] []
Polynomial : 1*psi_1^1

sage: I6 = rel_GW_class_K3(1,2,[s,s],1,psiclass(1,1,3));I6
evaluation classes: [s, s]
relative evaluation class: 1
Tautological class: Graph :      [1] [[1, 2, 3]] []
Polynomial : 1*psi_1^1
```

The function `rel_GW_class_K3.rel_invariant(h = None)` computes the relative invariants as follows:

```
sage: I5.rel_invariant()
20*G2^2*Delta_inv - G2*Delta_inv + 5/3*G4*Delta_inv

sage: I6.rel_invariant()
448*G2^3*Delta_inv - 44*G2^2*Delta_inv + 40*G2*G4*Delta_inv + G2*Delta_inv -
5/3*G4*Delta_inv + 7/6*G6*Delta_inv
```

Let us briefly comment again on the KKV formula. As mentioned above, using the exposed algorithm we cannot compute the invariant $\langle\lambda_1\rangle^S_{1,0}$, since in this case $2g-2+n\le 0$ and thus we can not see λ_1 as the pullback of a tautological class. However, we can apply the degeneration formula to the invariant to get

$$\langle\lambda_1\rangle^S_{1,0}=\langle\emptyset\rangle^{S/E}_{0,0}\langle\lambda_1|\omega\rangle^{\mathbb{P}^1\times E/E}_{1,0}+\langle\lambda_1\rangle^{S/E}_{1,0}\langle\emptyset|\omega\rangle^{\mathbb{P}^1\times E/E}_{0,0}=\frac{1}{\Delta(q)}\langle\lambda_1|\omega\rangle^{\mathbb{P}^1\times E/E}_{1,0}+\langle\lambda_1\rangle^{S/E}_{1,0}.$$

Now, all the λ classes appearing in the the above equation come from the respective λ classes in the tautological ring. Thus, we can compute the invariants:

```
sage: I7 = rel_GW_class_K3(1,0,[],1,lambdaclass(1,1,1))
sage: I7.rel_invariant()
2*G2*Delta_inv

sage: I8 = rel_class_P1xE(1,0,[],[],lambdaclass(1,1,1))
sage: I8.invariant()
0
```

As a result, we get that

$$\langle \lambda_1 \rangle_{1,0}^S = \frac{1}{\Delta(q)} 2G_2.$$

Note that this invariant coincides with the one computed above using the KKV formula.

5 Conclusions and further considerations

Let us finish this master thesis with a short section devoted to the conclusions and further considerations. Concerning to the conclusions,

- We have analyzed the required theory for computing the Gromov-Witten invariants of elliptic curves. We have seen that one can gather the Gromov-Witten invariants in generating series, that are quasimodular forms, to afterwards reduce the computation to the case descendent invariants where no tautological class appears.
- We have analyzed the required theory for computing the Gromov-Witten invariants of $K3$ surfaces for primitive classes. Using the degeneration formula, the product formula, and the virtual localization formula, we have seen how the computation of the invariants on $K3$ surfaces is reduced to invariants on elliptic curves. In addition, we have see how to compute descendent invariants by means of the previous invariants.
- We have presented the algorithm for computing Gromov-Witten invariants on elliptic curves and, in the case of $K3$ surfaces, for primitive classes. As a consequence of these algorithms we have derived quasimodularity properties of the respective invariants. Moreover, we have developed an SageMath implementation of both algorithms and we have given a brief introduction of how to use it.

Concerning further considerations, from the theoretical point of view, for $K3$ surfaces we have only study the algorithm for primitive classes. In [36] Section 7.5. the quasimodulariy properties of these invariants in the non primitive setting are conjectured. Moreover, in [41] Conjecture C2, G. Oberdieck and R. Pandharipande conjectured how to write the non primivite invariants by means of the primivite ones. In addition, in [3] Y. Bae and T.-H. Buelles accomplished the study for non primitive classes with divisibility 2.

From the programming point of view, further worklines can be mentioned. For example, the improvement of the running time of some parts of the code. However, the most important considerations are the implementation of the computation of other Gromov-Witten invariants. For example, we have implemented the computation of relative invariants on $\mathbb{P}^1 \times E$ relative to E. However, the absolute invariants on this product can also be computed in terms of invariants on the elliptic curve. Moreover, a possible application of the program could be the search of patterns among the invariants for deriving Virasoro operators for the K3 surface analogous to the case of elliptic curves. Another possible application would be to test [40] Conjecture H for genus 4 and 5.

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A Introduction to Gromov-Witten Theory

This Appendix is devoted to introduce the required theoretical machinery for this master thesis. In particular, the first step is to define Gromov-Witten invariants and discuss their properties. These invariants are just intersection numbers on the moduli space of stable curves. However, invariants over other moduli spaces play a crucial role in the $K3$ surface case. Therefore, we give a brief introduction to the moduli spaces we are interested in, mainly:

- the moduli stack of prestable curves $\mathfrak{M}_{g,n}$,
- the moduli space of stable curves $\overline{\mathcal{M}}_{g,n}$,
- the moduli space of stable maps $\overline{\mathcal{M}}_{g,n}(X, \beta)$,
- the moduli space of relative stable maps $\overline{\mathcal{M}}_{g,n}(X/D, \beta, \mu_1, \ldots, \mu_n)$,
- and the moduli space of relative stable maps to non-rigid targets $\overline{\mathcal{M}}_{g,n}(X/D, \beta, \mu_1, \ldots, \mu_n)$.

The first three moduli spaces are vital for defining Gromov-Witten invariants. The moduli space of relative stable maps leads to relative Gromov-Witten invariants. These invariants appear in the K3 surfaces case after applying the degeneration formula.

The appendix is structured in 3 sub-appendixes. The first one is fully devoted to the moduli stack of prestable curves $\mathfrak{M}_{g,n}$ and the moduli space of stable curves $\overline{\mathcal{M}}_{g,n}$. We briefly give the definition of both spaces and some important morphisms, namely the forgetful morphism and the gluing morphism. Using this notion we introduce the tautological ring $RH^*(\overline{\mathcal{M}}_{g,n}) \subseteq H^*(\overline{\mathcal{M}}_{g,n})$ and some of its most important properties. The elements of $RH^*(\overline{\mathcal{M}}_{g,n})$ are called tautological classes and they are part of the input of the invariants.

The second sub-appendix focuses on the moduli space of stable maps, the central object of study in the field of the Gromov-Witten theory. There, we introduce this moduli space and some of its most important properties, as well as give an introduction to Gromov-Witten theory by defining Gromov-Witten classes, Gromov-Witten invariants, and the Gromov-Witten axioms.

In the last part of the appendix wel give a brief introduction to the moduli space of relative stable maps and the moduli space of relative stable maps to non-rigid targets as well as the definition of Gromov-Witten invariants over these two spaces.

A.1 The moduli space of stable curves and the tautological ring

We start introducing the concepts of prestable n–pointed curve (see Def. A.1.1), of stable n–pointed curve (see Def. A.1.2), and of n–pointed family of stable genus g curves (see Def. A.1.3).

Definition A.1.1. *A* prestable n–pointed curve $(C, p_1, \ldots, p_n)$ *is a connected projective complex curve* C *with at worst nodal singularities together with* n *distinct smooth points* $p_1, \ldots, p_n \in C$.

An isomorphism between prestable curves $(C, p_1, \ldots, p_n)$ and $(C', p'_1, \ldots, p'_n)$ is an isomorphism of curves $\varphi : C \to C'$ such that $\varphi(p_i) = p'_i$ for all $i \in \{1, \ldots, n\}$.

Definition A.1.2. *A* stable n–pointed curve *is a prestable* n*–pointed curve* $(C, p_1, \ldots, p_n)$ *such that the automorphism group* $\mathrm{Aut}((C, p_1, \ldots, p_n))$ *is finite. This condition is equivalent to asking that every irreducible component* C_v *of the normalization of* C *satisfies one of the following stability conditions:*

1. C_v *has genus* 0 *and contains at least* 3 *special points,*
2. C_v *has genus* 1 *and contains at least* 1 *special point,*
3. C_v *has genus at least* 2,

where the special points of C_v *are the preimages of nodes and the marked points* p_i *through the normalization morphism.*

Definition A.1.3. *An* n–pointed family of stable genus g curves *over a* $\mathbb{C}$*–scheme* S *is a flat proper surjective finitely presented morphism of schemes* $\pi : C \to S$*, together with* n *sections* $p_1, \ldots, p_n : S \to C$*, such that:*

1. $p_1, \ldots, p_n$ *are pairwise disjoint sections lying in the smooth locus of* π.
2. *For every geometric point* $s \in S$*, the fiber* C_s *together with the images of* $p_i(s)$ *is a stable (prestable) curve of arithmetic genus* g *and* n *marked point.*

A morphisms between two n–pointed families of stable genus g curves $(C \to S, p_1, \ldots, p_n)$ and $(C' \to S', p'_1, \ldots, p'_n)$ is a pair (f_1, f_2) where f_1 and f_2 are morphisms $f_1 : C \to C'$ and $f_2 : S \to S'$ with $f_1 \circ p_i = p'_i \circ f_2$ such that the following diagram is cartesian:

$$\begin{array}{ccc} C & \xrightarrow{f_1} & C' \\ \downarrow & & \downarrow \\ S & \xrightarrow{f_2} & S' \end{array}$$

Similarly, one can consider n–pointed families of stable genus g curves. However, in this case, in the definition above we will require C to be not a scheme but an algebraic space (see [25] Section 2.3.).

For $g, n \geq 0$ with $2g - 2 + n > 0$, we define the category $\overline{\mathcal{M}}_{g,n}$ whose objects are n–pointed families of stable genus g curves over $\mathbb{C}$–schemes and whose morphisms are the morphisms among these families defined above. The reason why we need g and n to satisfy $2g-2+n$ is because of stability conditions. Now, we can define a functor

$$\overline{\mathcal{M}}_{g,n}: \longrightarrow \mathbb{C}\text{–Schemes}^{\mathrm{op}}$$

mapping a families of curves $(C \to S, p_1, \dots, p_m)$ to S. Then, it is know that $\overline{\mathcal{M}}_{g,n}$ is an algebraic stack. Some fundamental properties of $\overline{\mathcal{M}}_{g,n}$ appear in the next theorem (see [47] Theorem 5.1.)

Theorem A.1.1. *$\overline{\mathcal{M}}_{g,n}$ is a Deligne-Mumford stack called the moduli space of stable curves. Moreover, $\overline{\mathcal{M}}_{g,n}$ is irreducible, proper and smooth of dimension $3g-3+n$.*

Analogously, we can define the category $\mathfrak{M}_{g,n}$ of n–pointed families of prestable genus g curves. As before, this category ends up being an algebraic stack. The following theorem states some of the most important properties of $\mathfrak{M}_{g,n}$ (see [4] Section 2).

Theorem A.1.2. *For $g, n \geq 0$, $\mathfrak{M}_{g,n}$ is a quasi-separated, smooth, locally of finite type stack of dimension $3g-3+n$ called the moduli stack of prestable maps.*

The relation between $\overline{\mathcal{M}}_{g,n}$ and $\mathfrak{M}_{g,n}$ is that $\overline{\mathcal{M}}_{g,n}$ is an open substack of $\mathfrak{M}_{g,n}$. Later in this subsection we will see how a morphism $\mathrm{st} : \mathfrak{M}_{g,n} \to \overline{\mathcal{M}}_{g,n}$, that stabilizes prestable curves by contracting non stable components, can be introduced.

We will be interested in performing intersection theory on $\overline{\mathcal{M}}_{g,n}$. More precisely, we will focus our attention on the tautological ring. To introduce this notion, first we have to define the gluing and forgetful morphisms. We will see how these maps relate moduli spaces of stable curves with different g and n by gluing stable curves or forgetting some marked points, respectively. Moreover, apart from being required for defining the tautological ring, they will play a crucial role in our algorithm since they are part of the data inside the Gromov-Witten axioms.

The main geometric idea behind the gluing morphism is to glue together stable curves through some of their marked points to get a new stable curve. In order to do this, we need to find a combinatoric tool to bring together all the gluing data. The notion of stable graph will accomplish this task. In particular, we see how every stable curve has an associated stable graph. As a result, the gluing morphisms parametrize the boundary strata of $\overline{\mathcal{M}}_{g,n}$. We start with the concept of prestable graph.

Definition A.1.4. *Given $g, n \geq 0$, a prestable graph of genus g and n markings is a tuple*

$$\Gamma\,(V, H, L,\, g: V \to \mathbb{N},\, v: H \to V,\, \iota: H \to H,\, l: L \to \{1, \dots, n\}) \tag{39}$$

where

1. *V is the finite set of vertices. Each vertex has assigned a genus through the map $g: V \to \mathbb{N}$.*

2. *H is the finite set of half edges. Each half edge is associated to a vertex through the map $v : H \to V$. The map $\iota : H \to H$ is an involution on H encoding which half edges are joined together to create an edge, i.e.,*

$$H = E \sqcup L$$

where E the set of pairs of half edges $\{h, h'\}$ such that $\iota(h) = h'$, and $L = \{h \in H : \iota(h) = h\}$. The elements of L are called the legs of Γ

3. *The graph Γ with vertices V and edges E is connected.*

4. *$l : L \to \{1, \dots, n\}$ is a bijection that labels the legs in L.*

5. *The total genus of the graph, which is defined as*

$$g(\Gamma) = \sum_{v \in V} g(v) + 1 + \#E - \#V,$$

must be equal to g.

Let Γ be a prestable graph as in (39). We introduce the following notation: for $v \in V$, we denote $H(v) = \{h \in H : v(h) = v\}$, $n(v) = \#H(v)$, $L(v) = \{h \in L : v(h) = v\}$ and $E(v) = H(v) \backslash L(v)$. Moreover, in order to simplify the notation, we will denote by $V(\Gamma)$ to the set of vertices of Γ (similarly with H, L and E).

Definition A.1.5. *Given a prestable graph Γ, we say that Γ is* stable *if every $v \in V$ satisfies the stability condition $2g(v) - 2 + n(v) > 0$.*

The reason why prestable graphs are the combinatorial object we are looking for is because every prestable curve has a prestable graph associated called the dual graph. Moreover, the prestable curve is stable if and only if the associated stable graph is stable. More precisely, we have the following definition:

Definition A.1.6. *Let $(C, p_1, \dots, p_n)$ be a genus g prestable curve. The* dual graph *Γ_C of C is defined as the following prestable graph:*

1. *The set of vertices $V(\Gamma_C)$ is in bijection with the set of irreducible components of the normalization of C. For $v \in V$, let us denote by C_v the corresponding component of the normalization. Then, set the genus map is defined as $g(v) = g(C_v)$ for $v \in V$.*

2. *The set of half edges $H(\Gamma_C)$ is in bijection with the set of all special points in the normalization of C, i.e., the preimages of nodes and marked points. We denote by h_p the half edge for the special point p.*

3. *The map $v : H \to V$ maps the half edge h_p to the vertex v with $p \in C_v$.*

4. *For $h_p \in H$, if $p = p_i$ for some $i \in \{1, \dots, n\}$, the involution ι fixes h_p. Otherwise, $p \in \{q', q''\}$ where $\{q', q''\}$ is the preimage of a node q in C. In this case $\iota(h_{q'}) = h_{q''}$. Thus, $L = \{h_{p_1}, \dots, h_{h_n}\}$ and E is the set of preimages of nodes through the normalization.*

5. *Finally, we define $l : L \to \{1, \dots, n\}$ by $l(h_{p_i}) = i$.*

In this situation, one has that $g(\Gamma_C) = g(C) = g$. Moreover, one can check that the stability condition of C Γ_C is equivalent to the stability condition of the dual graph. As a result, Γ_C is indeed a stable graph if and only if its dual graph is stable. In Fig. 12 appears an example of a dual graph.

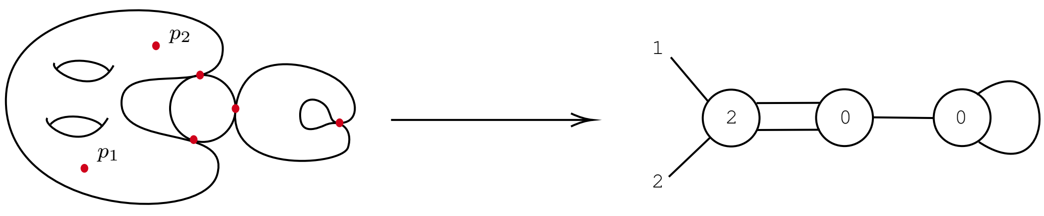

Figure 12: Given the stable curve (C, p_1, p_2), Γ_C is the corresponding dual graph.

After introducing the notion of stable graph, we can move to the construction of the gluing morphisms. For a stable graph Γ of genus g and n legs, let

$$\overline{\mathcal{M}}_\Gamma := \prod_{v \in V(\Gamma)} \overline{\mathcal{M}}_{g(v),n(v)}.$$

Then, a $\mathbb{C}$–point of this space will be a tuple $(C, (q_h)_{h \in H(v)})_{v \in V(\Gamma)}$. The geometric idea is to glue this tuple of curves into a stable curve $(C, p_1, \dots, p_n) \in \overline{\mathcal{M}}_{g,n}$ by gluing the marked points q_h and $q_{h'}$ together for every edge $\{h, h‘\}$ in Γ. The dual graph of the resulting curve is again Γ. More precisely, we have the following proposition (see [47] Theorem 5.1. or [2] Chapter XII.10.).

Proposition A.1.1. *For every stable graph Γ of genus g and n marked points, there exists a morphism, called* gluing morphism,

$$\xi_\Gamma : \overline{\mathcal{M}}_\Gamma \longrightarrow \overline{\mathcal{M}}_{g,n}$$

that restricted to the $\mathbb{C}$–points behaves as the gluing of curves described above.

The other morphism between moduli spaces of stable curves that is vital for our study is the forgetful morphism. This will be a morphism from $\overline{\mathcal{M}}_{g,n+1}$ to $\overline{\mathcal{M}}_{g,n}$ whenever $\overline{\mathcal{M}}_{g,n}$ is defined, i.e. we require $2g - 2 + n > 0$. As before, let us first describe the geometric intuition behind the construction through the $\mathbb{C}$–points. Let

$(C, p_1, \ldots, p_n, p_{n+1})$ be a stable curve of genus g and $n+1$ marked points. As the name of the morphism suggests, the idea is to forget about the $(n+1)$–th marked point. This leads us to a n–pointed curve $(C, p_1, \ldots, p_n)$. However, it might happen that $(C, p_1, \ldots, p_n)$ is not stable. Assuming that $2g - 2 + n > 0$, there are only two cases where the resulting curve $(C, p_1, \ldots, p_n)$ is not stable. Both cases arise when p_{n+1} is on a genus 0 component C_v: (1) The special points of C_v are p_{n+1} and two nodes; (2) The special points of C_v are p_{n+1}, other marked point p_i and a node. To fix these two unstable cases, we contract the genus 0 component to a node in the first case and to a marked point p_i in the second case (see Figure 13). As a result, we get a stable curve in $\overline{\mathcal{M}}_{g,n}$.

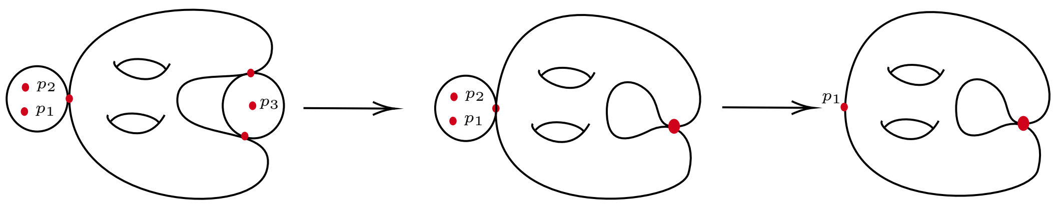

Figure 13: The first forgetful morphism forgets the marking p_3 and contracts the respective genus 0 unstable component. The second forgetful morphism forgets the marking p_2 and contracts the respective genus 0 unstable component to the marking p_1.

More precisely, we have the following proposition (see [47] Theorem 5.1. or [2] Chapter XII.10):

Proposition A.1.2. *For every g and n satisfying $2g - 2 + n > 0$, there exists a morphism $\pi : \overline{\mathcal{M}}_{g,n+1} \longrightarrow \overline{\mathcal{M}}_{g,n}$, called the forgetful morphism, acting on the complex points as described above. Moreover, the universal curve of $\overline{\mathcal{M}}_{g,n}$ is $\overline{\mathcal{M}}_{g,n+1}$ together with the forgetful morphism. The sections p_i of the universal curve coincide with the gluing morphism of the stable graphs given by Fig. 14.*

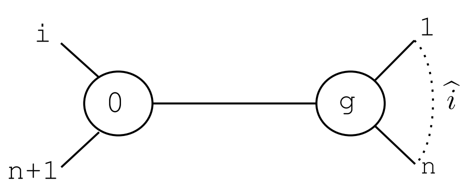

Figure 14: Stable graph whose gluing morphism corresponds to the i-th section of the forgetful morphism.

Let us comment about the analogous constructions for $\mathfrak{M}_{g,n}$. In this case, given a prestable graph Γ of genus g and n legs, we can construct as before a gluing morphism

$$\xi_\Gamma : \mathfrak{M}_\Gamma := \prod_{v \in V(\Gamma)} \mathfrak{M}_{g(v),n(v)} \longrightarrow \mathfrak{M}_{g,n}.$$

Also, we can build the forgetful morphism $\pi : \mathfrak{M}_{g,n+1} \to \mathfrak{M}_{g,n}$, this time without the need of stabilizing curves. Finally, we can construct a morphism

$$\mathrm{st} : \mathfrak{M}_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,n}$$

that stabilizes the prestable curve contracting non stable components as we did for the forgetful morphism. In particular, the forgetful morphism on $\overline{\mathcal{M}}_{g,n}$ factors as follows:

$$\begin{array}{ccc} \overline{\mathcal{M}}_{g,n+1} & \xrightarrow{\pi} & \overline{\mathcal{M}}_{g,n} \\ \big\downarrow & & \big\uparrow \mathrm{st} \\ \mathfrak{M}_{g,n+1} & \xrightarrow{\pi} & \mathfrak{M}_{g,n} \end{array}$$

After this brief parethesis, we define the tautological ring.

Definition A.1.7. *The* tautological ring $(RH^*(\overline{\mathcal{M}}_{g,n}))_{g,n}$ *is the smallest system of* $\mathbb{Q}$*–algebras* $RH^*(\overline{\mathcal{M}}_{g,n}) \subseteq H^*(\overline{\mathcal{M}}_{g,n})$ *containing all the fundamental classes* $\left[\overline{\mathcal{M}}_{g,n}\right]$ *and which is closed under the pushforward by all gluing and forgetful morphisms. A* tautological class *is a cohomology class lying in a tautological ring.*

We are interested in a more explicit description of the tautological ring. The idea is to give a finite list of generators that we can work with. To do so, first we introduce some special cohomology classes that have a crucial roll inside the tautological ring.

Definition A.1.8. *Let* $\pi : \overline{\mathcal{M}}_{g,n+1} \longrightarrow \overline{\mathcal{M}}_{g,n}$ *be the forgetful morphisms and* $p_i : \overline{\mathcal{M}}_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,n+1}$ *be the sections of* π *corresponding to the marked points. Then, for* $i \in \{1, \ldots, n\}$*, we define the* i–th cotangent line bundle $\mathbb{L}_i$ *as*

$$\mathbb{L}_i = p_i^* \Omega_\pi.$$

We define the i–th ψ–class *as the first Chern class of* $\mathbb{L}_i$

$$\psi_i := c_1(\mathbb{L}_i) \in H^2(\overline{\mathcal{M}}_{g,n})$$

Note that, using the universal curve of $\mathfrak{M}_{g,n}$ one can define ψ–classes on $\mathfrak{M}_{g,n}$ too. However, in the prestable frame, the forgetful morphism is not anymore the universal curve (see [4] Corollary 2.7.). One natural question in this situation is to know what the relation between the ψ–classes on $\overline{\mathcal{M}}_{g,n}$ and the ψ classes on $\mathfrak{M}_{g,n}$ is. The answer lies in the following proposition (see [4] Proposition 3.14.).

Proposition A.1.3. *Let $g, n \geq 0$ with $2g-2+n > 0$. For $i \in \{1, \ldots, n\}$, let us denote by Γ_i the genus g and n legs prestable graph with one edge and two vertices, one of genus g and $n-1$ legs corresponding to $\{1, \ldots, n\} \backslash \{i\}$, and one with genus 0 and one marking corresponding to i (see Figure 15). Then, for $i \in \{1, \ldots, n\}$,*

$$\psi_i = \mathrm{st}^*(\psi_i) + [\Gamma_i]$$

where $[\Gamma_i]$ is the pushforward of $[\mathfrak{M}_{\Gamma_i}]$ through the corresponding gluing morphism.

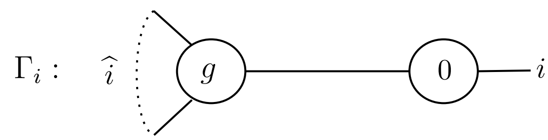

Figure 15: Genus g & n legs prestable graph Γ_i with 1 edge and 2 vertices. One vertex has genus g and $\{1, \ldots, n\} \backslash \{i\}$ legs. The other vertex has genus 0 and 1 leg corresponding to i.

Recall that the tautological ring is closed under the pushforward by the gluing morphisms. We will see that the ψ–classes are tautological, so we will care about the pushforward of these classes under the forgetful morphism. This is recorded in the following definition.

Definition A.1.9. *For $a \geq 0$, the a–th κ–class κ_a is defined as the pushforward*

$$\kappa_a := \pi_* \left(\psi_{n+1}^{a+1}\right) \in H^{2a}(\overline{\mathcal{M}}_{g,n})$$

Proposition A.1.4. *The ψ–classes and κ–classes are tautological.*

Proof. Taking into account the definition of the tautological ring and the κ–classes, it is enough to check the result for ψ_i. This statement is classical, for the proof see e.g. [47], Theorem 6.25. □

Now, we will combine the ψ and κ classes with the gluing morphism to define the following cohomology classes that will generate the tautological ring.

Definition A.1.10. *Given a stable graph Γ, a decorated stratum class $[\Gamma, \alpha]$ is the pushforward*

$$[\Gamma, \alpha] := (\xi_\Gamma)_* (\alpha)$$

where α is a cohomology class in $\overline{\mathcal{M}}_\Gamma$ of the form

$$\alpha = \prod_{v \in V(\Gamma)} \pi_v^*(\alpha_v)$$

with $\alpha_v \in RH^*(\overline{\mathcal{M}}_{g(v),n(v)})$ a product of ψ and κ classes.

The following theorem states that the decorated stratum classes forms a generating set of the tautological ring (see [47] Theorem 6.28. or [20] Proposition 11.).

Theorem A.1.3. *The decorated stratum classes $[\Gamma, \alpha]$ forms a finite generating set of the tautological ring as a $\mathbb{Q}$–vector space.*

By this theorem we have a explicit list of generators and we can reduce our study of invariants with tautological classes to invariants with decorated stratum classes. Let us introduce another type of tautological classes.

Definition A.1.11. *Let $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ be the forgetful morphism. We define the* Hodge bundle *as the rank g vector bundle over $\overline{\mathcal{M}}_{g,n}$ given gy $\mathbb{E} := \pi_*\omega_\pi$ where ω_π is the relative dualizing sheaf of the forgetful morphism.*

*We define the λ–*classes *as the Chern classes $\lambda_k = c_k(\mathbb{E}) \in H^{2k}(\overline{\mathcal{M}}_{g,n})$ for $k \in \{1, \dots, g\}$.*

In [39], page 307, it is proven that for $\overline{\mathcal{M}}_{g,0}$ the λ classes are tautological. Then, the pullback of λ_j through the forgetful morphism is again λ_j and we get the next result:

Proposition A.1.5. *For $k \in \{1, \dots, g\}$, λ_k is tautological.*

The reason to introduce the notion of λ–classes is that they appear when applying localization in Section 3. Moreover, we will see in the next subsection that they play an important role in the "mapping to a point" axiom of Gromov-Witten theory.

A.2 The moduli spaces of stable maps and Gromov-Witten Theory

So far, we have introduced the moduli space of stable curves and some important notions related with the final goal of defining the notion of Gromov-Witten invariants. These invariants are intersection numbers on the moduli space of stable maps. In some base cases, we expect these invariants to answer enumerative questions on a projective nonsingular complex variety X. However, we do not have a direct way of relating $\overline{\mathcal{M}}_{g,n}$ to X. We need to find an intermediate space that performs as a link between both spaces. This connection is done through the moduli space of stable maps $\overline{\mathcal{M}}_{g,n}(X, \beta)$. This subsection will be fully devoted to the study of this moduli space with the final goal of defining Gromov-Witten invariants.

The first step is to provide a brief introduction to the moduli space of stable maps and some of their properties. Once we are familiarized with this space, we will be able to define Gromov-Witten classes and Gromov-Witten invariants. We will list the Gromov-Witten axioms that play a fundamental role in our algorithm. Finally, we will introduce the disconnected variant of the spaces $\overline{\mathcal{M}}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}(X, \beta)$ as well as the respective disconnected invariants. In order to reach the concept of moduli space of stable maps, as we did for stable curves, we have to first introduce the notions of stable maps and family of stable maps. During the rest of this section let X be a nonsingular complex projective variety and let $\beta \in H_2(X)$.

Definition A.2.1. *An n–**pointed stable map** is a tuple $(C, p_1, \dots, p_n, f)$ where*

1. *$(C, p_1, \dots, p_n)$ is a prestable curve.*
2. *$f : C \longrightarrow X$ is a morphism such that every irreducible component C_v of C satisfies one of the following stability conditions:*
   (a) *C_v has genus 0, is contracted (i.e. $f|_{C_v}$ is constant), and C_v contains at least 3 special points.*
   (b) *C_v has genus 1, is contracted, and C_v contains at least 1 special point.*
   (c) *C_v has at least genus 2 or it is not contracted.*

*A **genus g stable map with class $\beta \in H_2(X)$** is a stable map $(C, p_1, \dots, p_n, f)$ such that C has arithmetic genus g and $f_*([C]) = \beta$ in $H_2(X)$.*

Suppose that β is the class of a curve C' inside X, i.e. $\beta = [C'] \in H_*(X)$. Then, the condition $f_*([C]) = \beta$ for a stable map (C, p_i, f) with f generically injective means that the map f is a parametrization of the curve $C' = f(C)$. This is the clue for understanding stable maps as the link between stable curves and the curves inside X and the reason why some Gromov-Witten invariants will count curves inside X.

A morphism between stable maps $(C, p_1, \dots, p_n, f)$ and $(C', p'_1, \dots, p'_n, f')$ is a morphism of curves $\varphi : C \to C'$ such that $\varphi(p_i) = p'_i$ and $f = f' \circ \varphi$. The reason why we ask the stable map to satisfy the stability condition is because this stability condition is again equivalent to $\mathrm{Aut}((C, p_1, \dots, p_n, f))$ being finite.

Remark A.2.1.

1. *Note that if $(C, p_1, \dots, p_n, f)$ is a stable map and we forget about the morphism f, then the resulting n–pointed prestable curve $(C, p_1, \dots, p_n)$ might not be stable. For example, this happens when C has a genus zero component with at most two special points where f is not constant. However, we will see how to fix this problem similarly as we did with the forgetful morphism of $\overline{\mathcal{M}}_{g,n}$.*
2. *If X is a point then the definition of stable map coincides with the definition of stable curve.*

Definition A.2.2. *Let S be a scheme over $\mathbb{C}$. An n–**pointed family of stable map of genus g over S with class β** consists in an n–pointed family of prestable genus g curves over S $(C \to S, p_1, \dots, p_n)$, together with a morphism $f : C \longrightarrow X$ such that for each geometric point $s \in S$, the restriction $f_s : C_s \to X$ of f to the geometric fiber of C over s together with the images of the sections p_i is a stable map, i.e. $(C_s, p_1(s), \dots, p_n(s), f_s)$ is a stable map of genus g and class β.*

A morphisms n–pointed families of genus g stable maps $(C \to S, f : C \to X, p_1, \ldots, p_n)$ and $(C' \to S', f : C' \to X, p'_1, \ldots, p'_n)$ is morphism between prestable curves (g_1, g_2) between $(C \to S, p_1, \ldots, p_n)$ and $(C' \to S', p'_1, \ldots, p'_n)$ such that the following diagram computes

$$\begin{array}{ccc} C & \xrightarrow{\quad g_1 \quad} & C' \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f'} & \\ & X & \end{array}$$

Now, as we did in the previous subsection, we can define the category $\overline{\mathcal{M}}_{g,n}(X, \beta)$ of n–pointed families of genus g stable maps with class β over $\mathbb{C}$–schemes. As expected, $\overline{\mathcal{M}}_{g,n}(X, \beta)$ is an algebraic stack The following result states some important properties of $\overline{\mathcal{M}}_{g,n}(X, \beta)$ (see [11] Theorem 7.1.4.).

Theorem A.2.1. *$\overline{\mathcal{M}}_{g,n}(X, \beta)$ is a proper Deligne-Mumford stack called the moduli space of stable maps.*

As a consequence of Remark A.2.1 (2), it holds that $\overline{\mathcal{M}}_{g,n}(\{\mathrm{pt}\}, 0) = \overline{\mathcal{M}}_{g,n}$ and hence, $\overline{\mathcal{M}}_{g,n}(X, \beta)$ is a generalization of $\overline{\mathcal{M}}_{g,n}$; in the following, with $\{\mathrm{pt}\}$ we mean $\mathrm{spec}(\mathbb{C})$. Moreover, if $\beta = 0$, one has that every stable map must be constant. This implies that $\overline{\mathcal{M}}_{g,n}(X, 0) = \overline{\mathcal{M}}_{g,n} \times X$.

Returning to the general case X and β, one observes that the new stack $\overline{\mathcal{M}}_{g,n}(X, \beta)$ does not behave as well as $\overline{\mathcal{M}}_{g,n}$. In general, it is non-reduced, possibly reducible and of impure dimension. This will lead to some problems due to the lack of a fundamental class. However, this difficulty can be solved by defining a virtual fundamental class $\left[\overline{\mathcal{M}}_{g,n}(X, \beta)\right]^{\mathrm{vir}} \in H_{2e}(\overline{\mathcal{M}}_{g,n}(X, \beta))$ where e is the expected dimension defined as:

$$e = (1 - g)(\dim(X) - 3) - \int_{\beta} c_1(\omega_X) + n.$$

This virtual fundamental class will play the role of the fundamental class. For the construction and analysis of the properties of this class we refer to [7] and [**?**]. For example, if $X = \{\mathrm{pt}\}$, one has that $\left[\overline{\mathcal{M}}_{g,n}(X, \beta)\right]^{\mathrm{vir}} = \left[\overline{\mathcal{M}}_{g,n}\right]$. In general, if $\overline{\mathcal{M}}_{g,n}(X, \beta)$ is of pure dimension, and the dimension is equal to the virtual dimension, the virtual fundamental class coincides with the fundamental class (see [7] Proposition 5.5).

This virtual class will allow us, as we will see bellow, to construct Gromov-Witten invariants are rational numbers of the form

$$\int_{\left[\overline{\mathcal{M}}_{g,n}(X,\beta)\right]^{\mathrm{vir}}} \alpha$$

for α being a certain type of cohomology classes in $\overline{\mathcal{M}}_{g,n}(X, \beta)$. So the next natural question is which classes will we be interesting for our purposes. As we commented in the motivation, for building $\overline{\mathcal{M}}_{g,n}(X, \beta)$ the idea is to link our three objects

$\overline{\mathcal{M}}_{g,n}$, $\overline{\mathcal{M}}_{g,n}(X,\beta)$, and X. To achieve this goal, we will construct morphisms between $\overline{\mathcal{M}}_{g,n}(X,\beta)$ to the other two spaces. We will see how the classes to insert in our integrals are going to be product of pullbacks and pushforwards through these morphisms.

The idea behind the morphism between $\overline{\mathcal{M}}_{g,n}(X,\beta)$ and X lies on the morphism $f : C \to X$ inside the data of a family of stable maps. For every $i \in \{1,\dots,n\}$ we can define a morphism

$$\mathrm{ev}_i : \overline{\mathcal{M}}_{g,n}(X,\beta) \longrightarrow X$$

mapping a family of stable maps $(C \to S, p_1,\dots,p_n : S \to C, f : C \to X)$ to the morphism $f \circ p_i : S \to X$. On complex points, ev_i maps a stable map $(C, p_1 \dots, p_n, f)$ to $f(p_i) \in X$. These morphisms are called the evaluation maps.

On the other hand, for building the morphism between $\overline{\mathcal{M}}_{g,n}(X,\beta)$ and $\overline{\mathcal{M}}_{g,n}$ we will follow the construction of the forgetful morphism for $\overline{\mathcal{M}}_{g,n}$. This time, instead of forgetting the $(n+1)$–th marked point, we will erase the morphism f from the data of a stable map $(C, p_1,\dots,p_n, f)$ to get a n–pointed prestable curve $(C, p_1,\dots,p_n)$. However, as we mentioned in Remark A.2.1 (1), $(C, p_1,\dots,p_n)$ might not be stable. This difficulty is solved exactly in the same way as we did with the forgetful morphism by contracting the non stable irreducible component to get a stable curve $(C', p_1,\dots,p_n)$ in $\overline{\mathcal{M}}_{g,n}$. For g and n with $2g-2+n>0$, we get a projection morphism

$$\rho : \overline{\mathcal{M}}_{g,n}(X,\beta) \longrightarrow \overline{\mathcal{M}}_{g,n}$$

that on the complex points forgets the morphism f and stabilizes the resulting n–pointed curve. Note that this morphism, as the forgetful morphism of $\overline{\mathcal{M}}_{g,n}$ factor through $\mathfrak{M}_{g,n}$. More concretely, there is a morphism $\overline{\rho} : \overline{\mathcal{M}}_{g,n}(X,\beta) \to \mathfrak{M}_{g,n}$ that forgets the map f from the data of stable maps. Then $\rho = \mathrm{st} \circ \overline{\rho}$

Apart from these two morphisms, one can also define, exactly in the same way as we did for $\overline{\mathcal{M}}_{g,n}$, a forgetful morphism

$$\pi : \overline{\mathcal{M}}_{g,n+1}(X,\beta) \longrightarrow \overline{\mathcal{M}}_{g,n}(X,\beta).$$

Again, the forgetful morphism acts on the complex point by forgetting a marked point and possibly contracting non stable components. Also, for $\overline{\mathcal{M}}_{g,n}(X,\beta)$ we have that the forgetful morphism also coincides with the universal curve. Geometrically, the sections $p_i : \overline{\mathcal{M}}_{g,n}(X,\beta) \longrightarrow \overline{\mathcal{M}}_{g,n+1}(X,\beta)$ glue to the i–th marked point a genus 0 contracted component with markings p_i and p_{n+1}.

After this brief introduction to $\overline{\mathcal{M}}_{g,n}(X,\beta)$, we are ready to formally introduce Gromov-Witten classes.

Definition A.2.3. *For $\gamma_1,\dots,\gamma_n \in H^*(X)$ the Gromov-Witten class $I_{g,n,\beta}(\gamma_1,\dots,\gamma_n)$ is defined as the cohomology class in $\overline{\mathcal{M}}_{g,n}$ given by*

$$I_{g,n,\beta}(\gamma_1,\dots,\gamma_n) := \rho_*(\mathrm{ev}_1^*(\gamma_1)\cdots\mathrm{ev}_n^*(\gamma_n)),$$

where
$$\rho_* : H^*\left(\overline{\mathcal{M}}_{g,n}(X,\beta)\right) \to H^*(\overline{\mathcal{M}}_{g,n})$$
is defined as the Poincaré dual of $\rho_*(\left[\overline{\mathcal{M}}_{g,n}(X,\beta)\right]^{vir} \frown \alpha)$ *for* $\alpha \in H^*(\overline{\mathcal{M}}_{g,n}(X,\beta))$*.*

Definition A.2.4. *A* Gromov-Witten invariant *of a Gromov-Witten class* $I_{g,n,\beta}(\gamma_1,\ldots,\gamma_n)$ *is the rational number*

$$< I_{g,n,\beta}(\gamma_1,\ldots,\gamma_n) >:= \int_{[\overline{\mathcal{M}}_{g,n}]} I_{g,n,\beta}(\gamma_1,\ldots,\gamma_n) \tag{40}$$

In this situation, we call the classes $\gamma_1,\ldots,\gamma_n$ *the* evaluation classes*.*

However, we will be interested in a more general kind of invariants. Mainly, we will focus our study on the invariants of the form

$$< \mu I_{g,n,\beta}(\gamma_1,\ldots,\gamma_n) >:= \int_{[\overline{\mathcal{M}}_{g,n}]} \mu I_{g,n,\beta}(\gamma_1,\ldots,\gamma_n) \tag{41}$$

for $\mu \in RH^*(\overline{\mathcal{M}}_{g,n})$. By definition, these invariants vanish if the cohomological degree of $\mu I_{g,n,\beta}(\gamma_1,\ldots,\gamma_n)$ does not coincide with the homological degree of $\left[\overline{\mathcal{M}}_{g,n}\right]$, namely $2(3g-3+n)$.

Remark A.2.2. *These invariants are defined as intersection numbers in* $\overline{\mathcal{M}}_{g,n}$*, however we could have analogously introduced them as*

$$< \mu I_{g,n,\beta}(\gamma_1,\ldots,\gamma_n) >:= \int_{\left[\overline{\mathcal{M}}_{g,n}(X,\beta)\right]^{vir}} \rho^*(\mu)\mathrm{ev}_1^*(\gamma_1)\cdots\mathrm{ev}_n^*(\gamma_n).$$

One can check that both definitions are equivalent using the following algebraic topology fact from the projection formula derived: For $f : X \to Y$ *continuous,* $a \in H_*(X)$*, and* $\alpha \in H^*(Y)$*, then*
$$f_*(a \frown f^*(\alpha)) = f_*(a) \frown \alpha. \tag{42}$$

Before describing the main properties of these invariants, let us introduce some important cohomology classes on $\overline{\mathcal{M}}_{g,n}(X,\beta)$. More concretely, we are interested in defining ψ and λ classes, as we did in $\overline{\mathcal{M}}_{g,n}$. The construction of both classes follows the same structure as before. Considering the forgerful morphism as the universal curve of $\overline{\mathcal{M}}_{g,n}(X,\beta)$, one can define the i–cotangent line bundle $\mathbb{L}_i := p_i^*(\omega_\pi)$. The i–th ψ class is defined as $\psi_i := c_1(\mathbb{L})$. By abuse of notation, we will denote by ψ_i the ψ classes in both $\overline{\mathcal{M}}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}(X,\beta)$.

The relevance of these classes lies in the definition of descendent classes and their invariants. A descendent class is a cohomology class in $\overline{\mathcal{M}}_{g,n}(X,\beta)$ of the form $\tau_{k_i}(\gamma_i) = \psi_i^{k_i}\mathrm{ev}_i(\gamma_i)$ for $i \in \{1,\ldots,n\}$, $k_i \geq 0$, and $\gamma_i \in H^*(X)$. Using these classes we can define the following invariants

$$\langle \tau_{k_1}(\gamma_1)\cdots\tau_{k_n}(\gamma_n)\rangle_{g,n,\beta}^X := \int_{\left[\overline{\mathcal{M}}_{g,n}(X,\beta)\right]^{\mathrm{vir}}} \tau_{k_1}(\gamma_1)\cdots\tau_{k_n}(\gamma_n),$$

that will play a crucial role in the algorithm for the elliptic case. We will denote these invariants by descendent invariants.

The natural question to be ask is which is the relation between these invariants and the ones defined in (41). One can rephrase the question by asking how the ψ classes from $\overline{\mathcal{M}}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}(X,\beta)$ are related. Does it hold that $\psi_i = \rho^*(\psi_i)$? Unfortunately the answer to this question is generally negative. The ψ classes does not behave nicely under pullbacks through ρ. However, they are related through the following proposition (see [27] Section 6.2.1).

Proposition A.2.1. *Let X be a projective non-singular complex variety and $\beta \in H^2(X)$. Let $\rho : \overline{\mathcal{M}}_{g,n}(X,\beta) \to \overline{\mathcal{M}}_{g,n}$ be the projection morphism. Then, it holds that*

$$\rho^*(\psi_i) = \psi_i - [E_i]$$

where E_i is the sum of all boundary divisors such that the marked point p_i lies in a rational component whose special points are a node and p_i.

Note that the proof of this result follows from Proposition A.1.3 and the fact that $\psi_i = \overline{\rho}^*(\psi_i)$. Note that this last equality make sense since the morphism $\overline{\rho} : \overline{\mathcal{M}}_{g,n}(X,\beta) \to \mathfrak{M}_{g,n}$ just forgets about the stable morphisms and keeps the curve intact.

Moving to the λ classes, as we did for $\overline{\mathcal{M}}_{g,n}$, we can define the Hodge bundle $\mathbb{E}$ on $\overline{\mathcal{M}}_{g,n}(X,\beta)$ which will be again a rank g vector bundle. We define the λ–classes on $\overline{\mathcal{M}}_{g,n}(X,\beta)$ as $\lambda_k = c_k(\mathbb{E})$ for $k \in \{1,\dots,g\}$. This time, conversely to the ψ classes, it can be checked that $\rho^*(\mathbb{E}) = \mathbb{E}$ and hence we get

$$\lambda_k = \rho^*(\lambda_k)$$

for $k \in \{1,\dots,g\}$. Since the λ–classes in $\overline{\mathcal{M}}_{g,n}$ are tautological, the λ–classes on $\overline{\mathcal{M}}_{g,n}(X,\beta)$ will appear in our invariants. Moreover, these classes will be crucial while applying localization.

To deal with the invariants with λ classes, it will be useful to work with the Chern character of the Hodge bundle that we will denote by $\mathrm{ch}_k = \mathrm{ch}_k(\mathbb{E})$. For the definition of Chern characters we refer to [13] Chapter 14.2.1. The Chern characters ch_k, are related to the λ classes by the following equation:

$$1 + \lambda_1 t + \cdots + \lambda_g t^g = e^{\sum_{k\geq 1}(k-1)!\mathrm{ch}_k t^k}.$$

Using Corollary 5.3. of [39] we get that $\mathrm{ch}_k = 0$ for k even. The relevance of the Chern classes will lie in the elliptic case where we will study how to deal with invariants with these classes inside.

After this brief parenthesis introducing ψ and λ classes we can again center our attention on the Gromov-Witten invariants. We want to study some of their main properties. The idea is to analyze the properties of the Gromov-Witten classes and the study of the invariants will derive from this. These properties are recorded in the Gromov-Witten axioms. Thus, our next goal is to introduce these axioms. However, we do not give a complete list of the axiom; for a complete list see [11] Section 7.3.

- Linearity Axiom: $I_{g,n,\beta}(\gamma_1, \ldots, \gamma_n)$ is linear in each of the evaluation classes.

- Effectivity Axiom: $I_{g,n,\beta}(\gamma_1, \ldots, \gamma_n) = 0$ if β is not an effective class. Recall that β is effective if $\beta = \sum n_i \beta_i$ where $n_i \in \mathbb{N}$ and β_i is the class of a curve in X.

- Degree axiom: The cohomology degree of $I_{g,n,\beta}(\gamma_1, \ldots, \gamma_n)$ is

$$2(g-1)\dim(X) + 2\int_\beta \omega_X + \sum \deg(\gamma_i)$$

- Equivariant Axiom: Let σ be a permutation of the symmetric group S_n. Then

$$I_{g,n,\beta}(\gamma_1, \ldots, \gamma_n) = (-1)^a I_{g,n,\beta}(\gamma_{\sigma(1)}, \ldots, \gamma_{\sigma(n)}).$$

  when the sign arises from the permutation of odd evaluation classes.

- Fundamental class axiom: Let $\pi : \overline{\mathcal{M}}_{g,n+1} \longrightarrow \overline{\mathcal{M}}_{g,n}$ be the forgetful morphism. Then,

$$\pi^* \left(I_{g,n,\beta}(\gamma_1, \ldots, \gamma_n) \right) = I_{g,n+1,\beta}(\gamma_1, \ldots, \gamma_n, 1).$$

- Deformation axiom: Let $\mathcal{W} \to C$ be a smooth projective map over a connected scheme C. For every geometric point $t \in C$ denote $\mathcal{W}_t$ the respective fiber, and consider locally constant section $\beta_t \in H_2(\mathcal{W}_t)$ and $\gamma_{1,t}, \ldots, \gamma_{n,t} \in H^*(\mathcal{W}_t)$. Then, the family of Gromov-Witten classes $\{I_{g,n,\beta_t}(\gamma_{1,t}, \ldots, \gamma_{n,t})\}_t$ is constant, i.e., for every $s, t \in C$ geometric points we have

$$I_{g,n,\beta_t}(\gamma_{1,t}, \ldots, \gamma_{n,t}) = I_{g,n,\beta_s}(\gamma_{1,s}, \ldots, \gamma_{n,s})$$

The next axioms deals with the behavior of the virtual fundamental class in the case $\beta = 0$. As commentted above, in this situation we have $\overline{\mathcal{M}}_{g,n}(X,0) = \overline{\mathcal{M}}_{g,n} \times X$. However, the dimension and the virtual dimension of $\overline{\mathcal{M}}_{g,n}(X,0)$ does not coincide unless $g = 0$ or X is a point. As a result, the fundamental class and the virtual fundamental class does not coincide in general. The "mapping to a point" axiom related both classes under these assumptions

- Mapping to a point: For $\beta = 0$ we have

$$\begin{cases} \left[\overline{\mathcal{M}}_{g,n} \times X\right] & \text{if } g = 0 \text{ or } X = \{\text{pt}\}, \\ \left[\overline{\mathcal{M}}_{g,n} \times X\right] \frown c_{\text{top}}(\mathbb{E} \otimes \mathcal{T}_X) & \text{otherwise}, \end{cases}$$

As a result, for $g > 0$ or $\dim(X) > 0$ we have that

$$I_{g,n,0}(\gamma_1, \ldots, \gamma_n) = \rho_* \left(\prod_i \mathrm{ev}_i^*(\gamma_i) c_{\mathrm{top}}(\mathbb{E} \otimes \mathcal{T}_X) \right)$$

where ρ_* is the composition $\mathrm{PD} \circ \rho_*(\left[\overline{\mathcal{M}}_{g,n} \times X\right] \frown -)$.

Before introducing our last two axioms, let fix some notation. Let $\mathcal{B} = \{T_i\}$ be a basis of the cohomology of X and set $g_{i,j} = \int_X T_i \smile T_j$. Consider the matrix $g = (g_{i,j})_{i,j}$ and let $g^{i,j}$ be the respective entries of the matrix g^{-1}. One can check that the fundamental class of the diagonal inside the cohomology of $X \times X$ is $\sum_{i,j} g^{i,j} T_i \otimes T_j$. In this situation, the last two axioms are introduced as follows.

- Splitting axiom: Let $S_1 \sqcup S_2 = \{1, \ldots, n\}$ and $g = g_1 + g_2$ with $2g_i - 2 + |S_i| > 0$ for $i = 1, 2$. Consider the stable graph Γ of genus g and n markings with one edge and two edges v_1 and v_2 with legs S_1 and S_2, and genus g_1 and g_2 respectively (see Fig. 16).

  Let $\xi : \overline{\mathcal{M}}_{g,|S_1|+1} \times \overline{\mathcal{M}}_{g,|S_2|+1} \longrightarrow \overline{\mathcal{M}}_{g,n}$ be the gluing morphisms corresponding to Γ, and let σ be the permutation mapping $(1, \ldots, n)$ to (S_1, S_2). Then,

$$\xi^* \left(I_{g,n,\beta}(\gamma_1, \ldots, \gamma_n) \right) =$$

$$= \mathrm{sgn}(\sigma) \sum_{\beta = \beta_1 + \beta_2} \sum_{i,j} g^{i,j} I_{g_1, |S_1|+1, \beta_1} \left((\gamma_k)_{k \in S_1}, T_i \right) \otimes I_{g_2, |S_2|+1, \beta_2} \left((\gamma_k)_{k \in S_2}, T_j \right),$$

  where the first sum is indexed by all the length two partitions of β and the second sum is taken over $\mathcal{B}^2$. Note that thanks to the effectivity axiom, the first sum is finite.

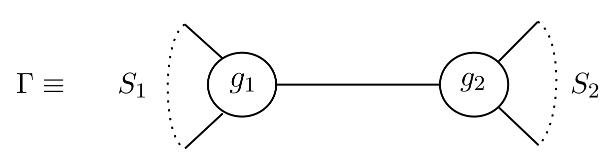

Figure 16: Stable graph of genus g and n markings with one edge and two vertex with genus g_i and S_i as the set of legs respectively.

- Reduction axiom: Consider the stable graph Γ of genus g and n markings with one vertex and one edge. Let $\xi : \overline{\mathcal{M}}_{g-1,n+2} \longrightarrow \overline{\mathcal{M}}_{g,n}$ be the corresponding gluing morphism. Then,

$$\xi^* \left(I_{g,n,\beta}(\gamma_1, \ldots, \gamma_n) \right) = \sum_{i,j} g^{i,j} I_{g-1,n+2,\beta}(\gamma_1, \ldots, \gamma_n, T_i, T_j).$$

These last two axioms are a consequence of the behave of the virtual fundamental class under the respective gluing morphisms. More concretely, for $g = g_1 + g_2$, $\{1, \ldots, n\} = S_1 \sqcup S_2$ with $|S_i| = n_i$, $\beta = \beta_1 + \beta_2$, and Γ the corresponding stable graph as in Figure 16, we can consider the following cartesian diagrams:

$$\begin{array}{ccc} P & \longrightarrow & \overline{\mathcal{M}}_{g,n}(X,\beta) \\ \downarrow & & \downarrow \rho \\ \overline{\mathcal{M}}_{\Gamma} & \xrightarrow{\xi_\Gamma} & \overline{\mathcal{M}}_{g,n} \end{array}$$

$$\begin{array}{ccc} \overline{\mathcal{M}}_{g_1,n_1+1}(X,\beta_1) \times_X \overline{\mathcal{M}}_{g_2,n_2+1}(X,\beta_2) & \longrightarrow & \overline{\mathcal{M}}_{g_1,n_1+1}(X,\beta_1) \times \overline{\mathcal{M}}_{g_2,n_2+1}(X,\beta_2) \\ \downarrow & & \downarrow \\ X & \xrightarrow{\Delta} & X \times X \end{array}$$

Then, we have a morphism $p : \overline{\mathcal{M}}_{g_1,n_1+1}(X,\beta_1) \times_S \overline{\mathcal{M}}_{g_2,n_2+1}(X,\beta_2) \to \overline{\mathcal{M}}_{g,n}(X,\beta)$ by gluing stable maps through the markings corresponding to the edge of Γ. As a result we get a morphism

$$p : \overline{\mathcal{M}}_{g_1,n_1+1}(X,\beta_1) \times_S \overline{\mathcal{M}}_{g_2,n_2+1}(X,\beta_2) \longrightarrow P.$$

Then, the splitting axiom is a consequence of the following equality between virtual classes:

$$\xi_\Gamma^! \left([\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}} \right) = \sum_{\beta=\beta_1+\beta_2} p_* \circ \Delta^! \left([\overline{\mathcal{M}}_{g_1,n_1+1}(X,\beta_1)]^{\mathrm{vir}} \times [\overline{\mathcal{M}}_{g_2,n_2+1}(X,\beta_2)]^{\mathrm{vir}} \right) \quad (43)$$

where $\xi^!$ is the Gysin map of ξ_Γ. A similar study can be done for the reduction axiom. See [6] for the details of these constructions.

One can check that the splitting reduction and equivariant axioms coincide with the ones required in cohomological field theories and, as a consequence, one can see Gromov-Witten theory as a cohomological field theory (see [29] Section 6.). However, we will not use this point of view of Gromov-Witten theory.

All these axioms state properties of Gromov-Witten classes that can be easily translated to the analogous properties for our invariants. For example, the Degree axiom implies that

$$\langle \mu I_{g,n,\beta}(\gamma_1, \ldots, \gamma_n) \rangle = 0$$

if $2(3g-3+n) \neq 2(g-1)\dim(X) + 2\int_\beta \omega_X + \sum \deg(\gamma_i) + \deg(\mu)$. Analogously, the reduction, splitting and fundamental class axioms gives the same formulas for the respective invariants.

In addition to the Gromov Wittem axioms, we will need three more properties of Gromov-Witten invariants, mainly, the divisor, dilation, and string equations (see [24] Section 26.3.). All of these equations arise from the fact that the pullback of $[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}$ through the forgetful map is $[\overline{\mathcal{M}}_{g,n+1}(X,\beta)]^{\mathrm{vir}}$ (see [6]).

- String equation:

$$\langle\tau_{k_1}(\gamma_1)\cdots\tau_{k_n}(\gamma_n)\tau_0(1)\rangle^X_{g,n+1,\beta}=\sum_{i=1}^n\langle\tau_1(\gamma_1)\cdots\tau_{k_i-1}(\gamma_i)\cdots\tau_{k_n}(\gamma_n)\rangle^X_{g,n,\beta}$$

- Dilation equation:

$$\langle\tau_{k_1}(\gamma_1)\cdots\tau_{k_n}(\gamma_n)\tau_1(1)\rangle^X_{g,n+1,\beta}=(2g-2+n)\langle\tau_{k_1}(\gamma_1)\cdots\tau_{k_n}(\gamma_n)\rangle^X_{g,n,\beta}$$

- Divisor equation: Let $H\in H^*(X)$. Then, we have

$$\langle\tau_{k_1}(\gamma_1)\cdots\tau_{k_n}(\gamma_n)\tau_0(H)\rangle^X_{g,n+1,\beta}=\left(\int_\beta H\right)\langle\tau_{k_1}(\gamma_1)\cdots\tau_{k_n}(\gamma_n)\rangle^X_{g,n,\beta}+$$
$$\sum_{i=1}^n\langle\tau_{k_1}(\gamma_1)\cdots\tau_{k_{i-1}}(\gamma_{i-1})\tau_{k_i-1}(\gamma_iH)\tau_{k_{i+1}}(\gamma_{i+1})\cdots\tau_{k_n}(\gamma_n)\rangle^X_{g,n,\beta}$$

It is important to remark that these equations only hold whenever the forgetful morphisms is defined. For example, in the case treated in Section 2 where X is an elliptic curve, one can check that $\overline{\mathcal{M}}_{0,0}(X,\beta)$, $\overline{\mathcal{M}}_{0,1}(X,\beta)$, and $\overline{\mathcal{M}}_{0,2}(X,\beta)$ are empty (every morphisms from $\mathbb{P}^1$ to an elliptic curve is constant). As a result one can not apply these equations for $(g,n+1)=(0,3)$. More concretely, using the string equation the invariant $\langle\tau_0(1)\tau_0(1)\tau_0(\omega)\rangle^X_{0,3}$ should vanish. However, as a consequence of Subsection 2.3 it can be check that this invariant is indeed 1.

So far, we have worked with the moduli spaces of stable curves and stable maps and we have defined the Gromov-Witten invariants as intersection numbers over them. Recall that in the definition of these two spaces the source curves are required to be connected. However, this condition may be omitted in the definition of stable curve, and stable map, leading us to the moduli space of possible disconnected stable curves and moduli space of possible disconnected stable maps denoted by $\overline{\mathcal{M}}^\bullet_{g,n}$ and $\overline{\mathcal{M}}_{g,n}(X,\beta)^\bullet$, respectively. This disconnected version of $\overline{\mathcal{M}}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}(X,\beta)$ will be fundamental in the algorithm for the elliptic curve, where we will reduce the connected case to the disconnected one.

We want to define, as in the connected case, Gromov-Witten invariants on $\overline{\mathcal{M}}^\bullet_{g,n}$. For this purpose, first we have to study the relation between $\overline{\mathcal{M}}_{g,n}$ and $\overline{\mathcal{M}}^\bullet_{g,n}$. Let $(C,p_1,\dots,p_n)$ be a genus g, possibly disconnected stable curve, let $C_1,\dots,C_k$ be the connected components of C, and let P_j be the set of marked points lying on the j–th component. Note that $(C,p_1,\dots,p_n)$ is equivalent to the data $((C_j,P_j))_{j=1,\dots,k}$.

Denote by $g_i=g(C_i)$ the genus of the i–th connected component, and by $n_i=\#P_i$ the number of markings of the component. This means that $(C_i,P_i)\in\overline{\mathcal{M}}_{g_i,n_i}$, and hence

$$((C_i,P_i))_{i=1,\dots,k}\in\prod_{i=1}^k\overline{\mathcal{M}}_{g_i,n_i}.$$

As a result we get that $\overline{\mathcal{M}}^{\bullet}_{g,n}$ is a quotient of a disjoint union of products of moduli spaces of stable curves. The disjoint union is indexed on the possible choices of connected components and distributions of the genus and the markings across the components. Each product is indexed on the number of connected components, and each term of the product will correspond to the moduli space of stable curves of the respective genus and markings associated to the respective connected component. Then, in each product we have to quotient by the group of automorphisms permuting the terms of the products with same genus and no markings.

However, we also want the disjoint union to be finite in order to have a nice fundamental class. To check this, we need to check that the choices of number of components and the possible distributions of the genus and the markings is finite. First of all, note that once we fix the number of connected components, the amount choices of possible distributions of the genus and the markings is finite. Thus, we can focus our attention on the possible number of components. Using the fact that the genus of a disconnected curve is equal to the sum of the genus of each component minus the number of component, we get that

$$g = \sum_{i=1}^{k}(g_i - 1) + 1. \tag{44}$$

Now, let k_0 and k_1 be the number of components of genus 0 and 1, respectively. Denoting by r the reminder of n by 3, by the stability condition we have that $k_1 < n$ and $j_0 < r$. In (44) these components are the only ones with non positive contribution to the total genus. As a result, we get that $k < 2k_0 + k_1 + g < 2r + n + g$. All together implies that the possible choices of number of components is finite. Hence, we have that $\overline{\mathcal{M}}^{\bullet}_{g,n}$ is the desire quotient of a finite disjoint union of finite product of moduli spaces of stable curves. Consequently, one can check that $\overline{\mathcal{M}}^{\bullet}_{g,n}$ is again a proper smooth Deligne Mumford stack of pure dimension $3g-3+n$, whose fundamental class $[\overline{\mathcal{M}}^{\bullet}_{g,n}]$ is a finite sum of products of fundamental classes of the form $[\overline{\mathcal{M}}_{g',n'}]$.

A similar argument can be used to prove that $\overline{\mathcal{M}}_{g,n}(X,\beta)^{\bullet}$ is quotient of a finite disjoint union of finite products of moduli spaces of connected stable maps. The only difference in the argument is that the stability condition for stable maps allows genus 0 or 1 irreducible components with no markings if the components are not contracted. As before, if we check that the amount of these component is upper bounded we will have that the disjoint union will be finite. This follows from the effectivity axiom. Indeed, let $(C, p_1, \ldots, p_n, f)$ be a possibly disconnected stable map, and let $C_1, \ldots, C_k$ be the non contracted irreducible components of C. We get that

$$f_*([C]) = \sum_{i=1}^{k} f_*([C_i]) = \beta.$$

Thus, the non contracted components correspond to partitions of β and, by the effectivity axiom, the possible partitions are finite. As a result, $\overline{\mathcal{M}}_{g,n}(X,\beta)^{\bullet}$ will be a finite disjoint union of finite products of moduli spaces of connected stable maps. Hence, we have that $\overline{\mathcal{M}}_{g,n}(X,\beta)^\bullet$ is a proper Deligne-Mumford stack with same virtual dimension as $\overline{\mathcal{M}}_{g,n}(X,\beta)$ and with virtual fundamental class $\left[\overline{\mathcal{M}}_{g,n}(X,\beta)^\bullet\right]^{\text{vir}}$. This virtual fundamental class is the sum of the virtual fundamental class of each disjoint component. Each of these component is a finite product of moduli spaces of stable curves. Hence, we get that $\left[\overline{\mathcal{M}}_{g,n}(X,\beta)^\bullet\right]^{\text{vir}}$ is a finite sum of finite product of virtual fundamental classes of some $\overline{\mathcal{M}}_{g',n'}(X,\beta')$.

Using this virtual fundamental class, we can define Gromov-Witten invariants for the disconnected case. As before, we have the morphisms

$$\begin{array}{rccc} \mathrm{ev}_i: & \overline{\mathcal{M}}_{g,n}(X,\beta)^\bullet & \longrightarrow & X \\ \rho: & \overline{\mathcal{M}}_{g,n}(X,\beta)^\bullet & \longrightarrow & \overline{\mathcal{M}}^\bullet_{g,n} \end{array}$$

which allow us to define the disconnected Gromov-Witten class as

$$I_{g,n,\beta}(\gamma_1,\ldots,\gamma_n)^\bullet := PD\circ\rho^*\left(\left[\overline{\mathcal{M}}_{g,n}(X,\beta)^\bullet\right]^{\text{vir}} \frown \mathrm{ev}_1(\gamma_1)\cdots\mathrm{ev}_n(\gamma_n)\right)$$

for $\gamma_1,\ldots,\gamma_n \in H^*(X)$. We will denote the corresponding disconnected Gromov-Witten invariant as

$$\langle\gamma_1,\ldots,\gamma_n\rangle^{X,\bullet}_{g,n,\beta} = \int_{\overline{\mathcal{M}}^\bullet_{g,n}} I_{g,n,\beta}(\gamma_1,\ldots,\gamma_n)^\bullet.$$

Similarly, we can define descendent classes and invariants of the form $\langle\tau_{k_1}(\gamma_1)\cdots\tau_{k_n}(\gamma_n)\rangle^{X,\bullet}_{g,n,\beta}$ as we did in the connected case.

The reason why we have introduced these disconnected invariants is because, as we will see in Section 2, an important part of the algorithm for the elliptic curve will be developed for disconnected invariants. In Proposition 2.1.1, we will see how to write the connected invariants in terms of the disconnected ones using the fact that the disconnected spaces are disjoint unions of product of the connected spaces.

A.3 The moduli space of relative stable maps and non-rigid targets

We can define also invariants over moduli spaces with virtual fundamental class. In particular, in this subsection we will focus on three new moduli spaces:

- The moduli space of relative stable maps.
- The moduli space of stable maps to singular varieties.
- The moduli space of stable maps to non rigid targets.

The construction of these three stacks is quite challenging. Here we will only give the description of the complex points and some basic properties. For a deeper study of these constructions we refer to [32], or [33]. For an overview description of the complex points we refer to [21] and [17].

We will see how the moduli space of relative stable maps is a generalization of $\overline{\mathcal{M}}_{g,n}(X,\beta)$. Let D be a smooth divisor of X. The idea behind this moduli space is to parametrize stable maps that intersect this divisor with a fix profile. However, it turns out that if we want to parametrize directly this object we get a non proper stack. We will see how to solve this difficulty using the notion of relative stable maps (see [33] Section 4 or [32] Sections 2,3, and 4).

Following the construction of the previous moduli space we will be able to define the moduli space of stable maps to a singular variety Y. Note that in the definition of stable maps, we asked X to be a complex non-singular projective variety. However, we will now define the analogous space for varieties with a certain type of singularities. More concretely, we will assume that Y is a projective variety with two irreducible component Y_1 and Y_2 that are smooth varieties of same dimension that intersect transversally along smooth divisors $D_1 \subset Y_1$ and $D_2 \subset Y_2$ inside Y. In particular, the singular locus of Y is a divisor $D \simeq D_i$. Again, we will only describe the complex point of this moduli space. For the general construction we refer to [33] Sections 1,2, and 3, or [32] Sections 5 and 6.

In Subsection 3.3 one can see that these two spaces will play a crucial role in the degeneration formula. This formula allows to compute absolute Gromov-Witten invariants in term of the relative ones, that we define below.

Finally, the construction of the moduli space of stable maps to non-rigid targets will derive from the moduli space of relative stable maps. The invariants over this space will be fundamental in the algorithm for the $K3$ surface. As a consequence of the localization formula that is used in Subsection 3.5, this moduli space will form part of the data of the $\mathbb{C}^*$–fixed locus.

Our first goal in this subsection is to define the moduli space of relative stable maps. Let Y be a non-singular projective variety and let D be a smooth divisor of Y. We want to parametrize stable maps intersecting the divisor D with a fix profile. Let us see a first approach of this notion of stable map.

Definition A.3.1. *Let $g, n, m \geq 0$, $\beta \in H_2(Y)$ and $\mu = (\mu_1, \dots, \mu_m)$ a positive partition of $\int_\beta [D]$. An n–pointed genus g regular prestable maps to Y relative to D with profile μ is a tuple*

$$(f : C \to Y; p_1, \dots, p_n, q_1, \dots, q_m)$$

where $(C, p_1, \dots, p_n, q_1, \dots, q_m)$ is a genus g prestable curve with $n + m$ marked points satisying

- $f_*([C]) = \beta$.

- $f^{-1}(D) = \{q_1, \ldots, q_m\}$ *and* $f^*(D) = \sum_{i=1}^{m} \mu_i q_i$ *(see Figure 17).*

An automorphism of a regular prestable map $(f : C \to Y, p_i, q_j)$ is an automorphisms of C fixing the points p_i and q_j, and commuting with f. A regular prestable map is stable if the automorphism group is finite.

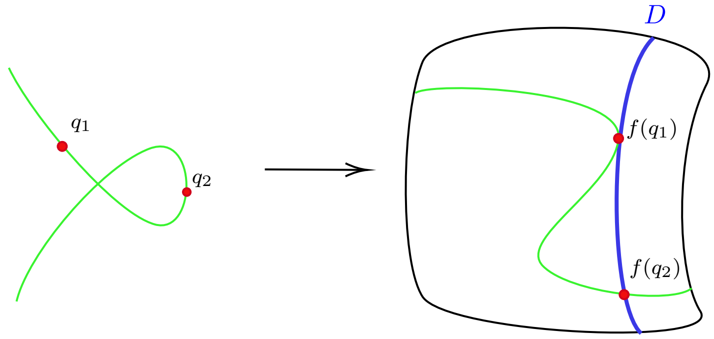

Figure 17: Regular stable map relative to D with profile $(2, 1)$.

The problem arising from this definition is that the respective moduli space parametrizing relative stable morphisms is a separated Deligne-Mumford stack but not proper. The moduli stack of stable maps will be a nice compactification of this space. Let $\Delta = \mathbb{P}(N^\vee_{Y/D} \oplus \mathcal{O}_Y)$. The main idea is to glue copies of Δ through D and consider regular relative stable maps to this new space.

Recall that Δ has two divisors isomorphic to D, mainly $D_0 := \mathbb{P}(0 \oplus \mathcal{O}_Y)$ and $D_\infty := \mathbb{P}(N^\vee_{Y/D} \oplus 0)$. Let Y_1 be the scheme resulting from gluing Y and Δ through D and D_0 respectively. Analogously, Y_2 is constructed by gluing Y_1 and Δ by the divisor D_∞ of Y_1 and D_0 of Δ. Inductively, define Y_k by gluing transversally Y_{k-1} and Δ through the D_∞ divisor of the last copy of Δ inside Y_{k-1} and the divisor D_0 of Δ (see Figure 18). The scheme Y_k is called the k–degeneration of Y. Fix $Y_0 = Y$.

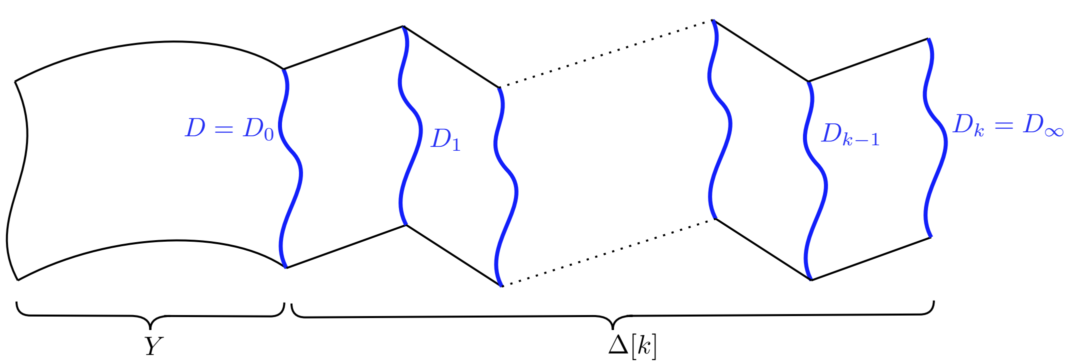

Figure 18: Ilustration of Y_k.

In particular, we have a filtration $Y_0 \subset Y_1 \subset \cdots \subset Y_k$. Denote by D_∞ the divisor of Y_k corresponding to the divisor D_∞ of the last glued copy Δ of Y_k. For $i \in \{0, \dots, k\}$ denote by D_i the divisor given by the closed immersion $D_\infty \subset Y_i \subset Y_k$. In particular $D_\infty = D_k$. Note that the singular locus of Y_k is

$$\operatorname{sing}(Y_k) = \bigcup_{i=0}^{k-1} D_i.$$

Finally, denote by $\Delta[k] \subset Y_k$ the k glued copies of Δ (See Figure 18). Since we have a projection $\Delta \to D_0$, we get a morphism $\Delta[k] \to D_0$ contracting $\Delta[k]$ to D_0. Using this morphism we can construct a projection

$$\varepsilon : Y_k \to Y.$$

Now that we have introduced this notation, we can properly define the notion of relative stable maps. The idea is to consider stable maps to Y_k instead of to Y for some k. To do so, we first introduce the notion of prestable relative map as follows

Definition A.3.2. *Let $g, n, m \geq 0$, $\beta \in H_2(Y)$ and let $\mu = (\mu_1, \dots, \mu_m)$ be a partition of $\int_\beta D$. An n–pointed genus g prestable relative map of profile $\mu = (\mu_1, \dots, \mu_m)$ and class β is an n–pointed genus g regular relative prestable map to Y_k relative to D_∞*

$$(f : C \to Y_k, p_1, \dots, p_n, q_1, \dots, q_m)$$

for some $k \geq 0$ satisfying:

- $\varepsilon_* \circ f_* ([C]) = \beta$.
- *The preimage of* $\operatorname{sing}(Y_k)$ *are nodes of C.*
- *For every node mapped to the singular locus of Y_k, f maps the two local branches of the node to two different irreducible components of Y_k and the orders of contact to the singular locus of the two local branches are the same.*

Let us now define the isomorphisms among prestable relative maps. For this purpose, note that we have a $\mathbb{C}^*$–action on Δ fiberwise whose fixed locus is D_0 and D_∞. This defines an $(\mathbb{C}^*)^k$–action on $\Delta[k]$. This action acts by automorphisms of $\Delta[k]$. A isomorphism between two relative stable maps $(f : C \to, Y_k, p_i, q_i)$ and $(f' : C' \to, Y_k, p_i', q_i')$ is a tuple (g_1, g_2) where g_1 is an isomorphism between the prestable curves (C, p_i, q_i) and (C', p_i', q_i'), g_2 is the automorphism of Y_k given by the identity on Y_0 and the action of an element of $(\mathbb{C}^*)^k$ on $\Delta[k]$, such that the next diagram commutes:

$$\begin{array}{ccc} C & \xrightarrow{g_1} & C' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y[k] & \xrightarrow[g_2]{} & Y[k] \end{array}$$

We say that a relative stable map is stable if the automorphism group is finite. One can check that being stable corresponds to the following stability conditions:

- Every genus 0 contracted component has at least 3 special points.
- Every genus 1 contracted component has at least 1 special point.
- For each copy Δ_i of Δ inside Y_k, let C_i be the corresponding components of C mapping to Δ_i. Then, C_i can not be a disjoint union of genus 0 components with two special points corresponding to nodes, such that f restricted to each component is an isomorphism to the fibers of Δ_i mapping the two special points to D_0 and D_∞ respectively.

In the previous subsections, we saw how to extend the notions of stable curves and maps to families over complex schemes. Using these notions, we were able to define the stacks $\overline{\mathcal{M}}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}(X,\beta)$. However, for the case of relative stable maps, this construction is quite challenging. Roughly speaking, the idea is to first construct a stack $\mathcal{Y}^{\mathrm{rel}}$ gathering all the k–th degenerations of Y for all k, and then consider flat families of stable curves mapping to this stack. For the details see [33] Sections 1,2, and 3, or [32] Sections 5 and 6. The resulting stack is called the moduli space of relative stable maps that we will denote by $\overline{\mathcal{M}}_{g,n}(Y/D,\mu,\beta)$. More concretely, we have the following result (see [33] Theorem 4.10).

Theorem A.3.1. *$\overline{\mathcal{M}}_{g,n}(Y/D,\mu,\beta)$ is a proper, separated Deligne-Mumford stack.*

The next goal is to define Gromov-Witten invariants on this new moduli space. First of all we need to have a virtual fundamental class $\left[\overline{\mathcal{M}}_{g,n}(Y/D,\mu,\beta)\right]^{\mathrm{vir}}$. [34] Section 3 provides the construction of this class. The virtual dimension of this class is

$$\mathrm{vdim}(\overline{\mathcal{M}}_{g,n}(\Delta_L,\mu,\nu,\beta)^{\sim}) = \mathrm{vdim}(\overline{\mathcal{M}}_{g,n}(\Delta_L,\beta)) - (|\mu| - \ell(\mu)), \tag{45}$$

where $|\mu| := \sum_{j=1}^{m}\mu_j$ and $\ell(\mu) = m$ are the size and the length of the partition, respectively. The next step is to construct the evaluation maps. Note that in the notion of relative stable map, there are two types of marked points. The marked points p_i for $i\in\{1,\dots,n\}$ are mapped to Y through the composition $\varepsilon\circ f$. However, the marked points $q_j\in\{1,\dots,m\}$ are mapped to D since $f(q_j)$ must lie in D_∞. This lead us to two types of evaluation maps:

$$\begin{array}{rccc} \mathrm{ev}_i: & \overline{\mathcal{M}}_{g,n}(Y/D,\mu,\beta) & \longrightarrow & Y \\ & (f,p_1,\dots,p_n,q_j) & \longmapsto & \varepsilon\circ f(p_i) \end{array}$$

for $i\in\{1,\dots,n\}$, and

$$\begin{array}{rccc} \mathrm{ev}_j^D: & \overline{\mathcal{M}}_{g,n}(Y/D,\mu,\beta) & \longrightarrow & D \\ & (f,p_i,q_1,\dots,q_m) & \longmapsto & \varepsilon\circ f(q_j) \end{array}$$

Finally, as for the moduli space of stable maps, we have a morphism

$$\rho: \ \overline{\mathcal{M}}_{g,n}(Y/D,\mu,\beta) \ \longrightarrow \ \overline{\mathcal{M}}_{g,n+m}$$

that maps a relative stable map $(f: C \to Y_k, p_i, q_j)$ to the stabilization of (C, p_i, q_j). As a result, we can define **relative Gromov-Witten invariants**. Let $\alpha \in RH^*(\overline{\mathcal{M}}_{g,n+m})$, $\gamma_1, \dots, \gamma_n \in H^*(Y)$, and $\delta_1, \dots, \delta_m \in H^*(D)$, we define these invariants as

$$\langle \alpha; \gamma_1, \dots, \gamma_n | \delta_1, \dots, \delta_m \rangle^{Y/D}_{g,n,\mu,\beta} := \int_{\overline{\mathcal{M}}_{g,n+m}} \rho_* \left(\prod_{i=1}^{n} \mathrm{ev}_i^*(\gamma_i) \prod_{j=1}^{m} \mathrm{ev}_j^D(\delta_i) \right),$$

where $\rho_* = PD \circ \rho_*([\overline{\mathcal{M}}_{g,n}(Y/D,\mu,\beta)]^{\mathrm{vir}} \frown -)$.

Now, we move to the next moduli space we want to introduce, namely, the moduli space of stable maps to a singular variety Y. However, we will only allow a specific type of singularity.

Let Y be a projective variety that is the union of two smooth varieties Y_1 and Y_2 which intersect transversally at a smooth irreducible divisor D of Y. Denote by D_i the divisor D inside Y_i for $i = 1, 2$. Note that the singular locus of Y is D. In particular we have that $N_{Y_1/D_1} \simeq N_{Y_2/D_2}$ and hence, $\Delta = \mathbb{P}(N_{Y_1/D_1} \oplus \mathcal{O}_D) \simeq \mathbb{P}(N_{Y_2/D_2} \oplus \mathcal{O}_D)$. Let $Y_{1,k}$ be the k–degeneration of Y_1 as above. Now, glue $Y_{1,k}$ and Y_2 through the divisors D_∞ of $Y_{1,k}$ and D_2 of Y_2. Denote by $Y[k]$ to the resulting scheme (see Figure 19). Note that $Y[k]$ can be split in three components Y_1, $\Delta[k]$, and Y_2 (see Figure 19).

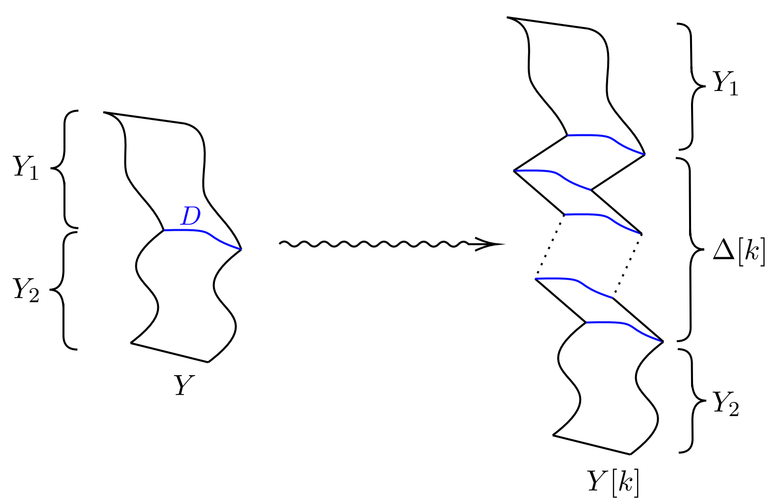

Figure 19: Ilustration of Y and $Y[k]$.

Again, using the projection $\Delta[k] \to D$, collapsing the fibers, we get a morphism

$$\varepsilon: Y[k] \longrightarrow Y.$$

For $i \leq k$ denote by $D[k]$ the divisor $D_{1,k}$ inside $Y_{1,k}$. In particular, we have that $D[1] = D_1$ and $D[k+1] = D_2$. Note that, as before, the singular locus of $Y[k]$ is precisely the disjoint union of the divisors $D[i]$. Once we have constructed $Y[k]$ we can define the notion of stable map to Y as follows.

Definition A.3.3. *Let $g, n \geq 0$ and $\beta \in H_2(Y)$, a n–pointed genus g prestable map to Y with class β is a tuple*

$$(f : C \to Y[k], p_1, \dots, p_n)$$

for some $k \geq 0$ such that

- *$(C, p_1, \dots, p_n)$ is a genus g prestable cure with n marked points.*
- *$\varepsilon_* \circ f_* ([C]) = \beta$.*
- *Every point of C mapping to the singular locus of $Y[K]$ has to be a node and the two local branches around the node must be mapped to different irreducible components of $Y[k]$ with same orders of contact to $\operatorname{sing}(Y[k])$.*

A morhism between prestable maps $(f : C \to Y[k], p_1, \dots, p_n)$ and $(f' : C' \to Y[k], p'_1, \dots, p'_n)$ is a tuple (g_1, g_2) where g_1 is a morphism between the prestable curves (C, p_i) and (C', p'_i), and g_2 is an automorphism of $Y[k]$ such that restricted to Y_1 and Y_2 is the identity and on $\Delta[k]$ is defined by the action of an element of $(\mathbb{C}^*)^k$. Then, we say that a prestable curve to Y is stable if the automorphism group is finite.

Again, the construction of the notion of family of stable maps to Y is complicated. For the details we refer to [33] Sections 1,2, and 3 or [32] Sections 5 and 6. There, the author considers a flat family $W \to C$ where C is a connected smooth curve with a fixed closed point 0 such that

- The fiber of 0, denoted by W_0, is Y.
- The fiber, W_t, of a geometric point $t \in C \setminus \{0\}$ is a projective smooth scheme.

Let $C \setminus \{0\}$ and let W^0 be the restriction to the family to C_0. The idea is to find a nice compactification of the family of moduli spaces of stable maps

$$\overline{\mathcal{M}}_{g,n}(W^0, \beta) = \bigsqcup_{t \in C^0} \overline{\mathcal{M}}_{g,n}(W_t, \beta_t) \longrightarrow C^0.$$

such that the fiber at 0 is the desire space. Roughly speaking, to construct this space, the author first defines spaces $W[k]$ by recursive blowing ups gathering the information of the fibers of the family together with all the degenerations $Y[i]$ for $i \leq k$. Then, a stack, called the stack of expanded degenerations $\mathcal{W}$, is built. This stack collects all the data from the $W[k]$ for every k. Using this stack one can define the notion of family of stable maps over $\mathcal{W}$, and define the moduli space of stable maps to $\mathcal{W}$ denoted by $\overline{\mathcal{M}}_{g,n}(\mathcal{W}, \beta)$ together with a morphism $\overline{\mathcal{M}}_{g,n}(\mathcal{W}, \beta) \to C$.The main properties of this moduli space are gathered in the following result (see [33] Theorem 3.10.).

Theorem A.3.2. *$\overline{\mathcal{M}}_{g,n}(\mathcal{W},\beta)$ is a C–proper, separated Deligne Mummford stack and for every $t \in C^0$ we have*

$$\overline{\mathcal{M}}_{g,n}(\mathcal{W},\beta)_t := \overline{\mathcal{M}}_{g,n}(\mathcal{W},\beta) \times_C t = \overline{\mathcal{M}}_{g,n}(W_t,\beta).$$

Finally, the fiber at 0 denoted by $\overline{\mathcal{M}}_{g,n}(W_0,\beta)$ is the desired moduli space parametrizing stable maps to Y. Moreover, for every geometric point $t \in C$, $\overline{\mathcal{M}}_{g,n}(\mathcal{W},\beta)$ and $\overline{\mathcal{M}}_{g,n}(W_t,\beta)$ admit virtual fundamental classes (see [34] Section 3). The next step is to define Gromov-Witten invariants on these spaces. Exactly as before, we have morphisms

$$\begin{array}{rccc} \rho : & \overline{\mathcal{M}}_{g,n}(\mathcal{W},\beta) & \longrightarrow & \overline{\mathcal{M}}_{g,n} \\ \rho_t : & \overline{\mathcal{M}}_{g,n}(W_t,\beta) & \longrightarrow & \overline{\mathcal{M}}_{g,n} \end{array}$$

and evaluation maps

$$\begin{array}{rccc} \mathrm{ev}_i : & \overline{\mathcal{M}}_{g,n}(\mathcal{W},\beta) & \longrightarrow & W \\ \mathrm{ev}_{i,t} : & \overline{\mathcal{M}}_{g,n}(W_t,\beta) & \longrightarrow & W_t. \end{array}$$

Using these maps we can define Gromov-Witten invariant on $\overline{\mathcal{M}}_{g,n}(\mathcal{W},\beta)$ and $\overline{\mathcal{M}}_{g,n}(W_t,\beta)$ for every geometric point $t \in C$. In general, we will have an analogous degeneration axiom, meaning that the Gromov-Witten invariants of the fibers of $\overline{\mathcal{M}}_{g,n}(\mathcal{W},\beta)$ will be equal (see [32] Equation (6.1.)). We will apply, in Section 3, these constructions to the K3 surface case. There, we will construct the normal cone degeneration. We will be able to apply these results to this degeneration, allowing us to compute invariants on the $K3$ surface through invariants on a scheme of the same form as Y. Finally, we will see how the invariants on Y can be computed using the degeneration formula.

Now, we can move to the last moduli space we want to introduce in this subsection, the moduli space of stable maps to non rigid target. So far, we first defined relative stable maps to a nonsingular variety Y as stable maps to Y_k. Recall that Y_k can be split as Y and $\Delta[k]$. Later, we considered $Y = Y_1 \cup_D Y_2$ and we defined the stable maps to this singular variety as stable maps to $Y[k]$ which turned to be the gluing of Y_1, $\Delta[k]$, and Y_2 through D. The idea now is to consider stable maps to $\Delta[k]$. However, we will give the more general definition.

Let D be a smooth projective variety and let L be a line bundle over D. Consider the projective bundle $\Delta_L = \mathbb{P}(L \oplus \mathcal{O}_D)$. Note that $\Delta = \Delta_{N_{Y/D}}$ in the above definitions. As before, Δ_L carries two divisors isomorphic to D, mainly $D_0 = \mathbb{P}(0 \oplus \mathcal{O}_D)$ and $D_\infty = \mathbb{P}(L \oplus 0)$. Let $\Delta_L[k]$ be the scheme resulting from gluing k–copies of Δ_L as the construction of $\Delta[k]$. Denote by D_0 the divisor of $\Delta_L[k]$ corresponding to D_0 in the first copy of Δ_L. Similarly let D_∞ denote the divisor corresponding to D_∞ in the last copy of Δ_L.

Definition A.3.4. *Let $g, n \geq 0$, $\beta \in H_2(\Delta_L)$ and let $\mu = (\mu_1, \dots, \mu_m)$ and $\nu = (\nu_1, \dots, \nu_l)$ be partitions of $\int_\beta D_\infty$ and $\int_\beta D_0$ respectively. An n–pointed prestable non-rigid map to Δ_L with class β and multiplicities μ and ν is a tuple*

$$(f : C \to \Delta_L[k], p_1, \dots, p_n, q_1, \dots, q_m, q'_1, \dots, q'_l)$$

for some $k \geq 0$*, such that:*

- $(C, p_1, \ldots, p_n, q_1, \ldots, q_m, q'_1, \ldots, q'_l)$ *is a genus* g *prestable curve with* $n + m + l$ *marked points.*
- $f^{-1}(D_0) = \{q_1, \ldots, q_m\}$ *and* $f^{-1}(D_\infty) = \{q'_1, \ldots, q'_l\}$*. Moreover,* f *must satisfy:*

$$f^*(D_\infty) = \sum_{j=1}^{m} \mu_i q_j \quad \textit{and} \quad f^*(D_0) = \sum_{k=1}^{l} \nu_i q'_k$$

- *The only points of* C *mapping the singular locus of* $\Delta_L[K]$ *are nodes. Moreover, for every such a node,* f *maps the two local branches of the node to different irreducible components of* $\Delta_L[k]$ *with same orders of contact to the singular locus of* $\Delta_L[k]$*.*

As before, we have an action of $(\mathbb{C}^*)^k$ on $\Delta_L[k]$. An isomorphism between two prestable non-rigid maps $(f : C \to \Delta_L[k], p_i, q_j, q'_k)$ and $(\overline{f} : \overline{C} \to \Delta_L[k], \overline{p}_i, \overline{q}_j, \overline{q}'_k)$ is a tuple (g_1, g_2) where g_1 is an isomorphism between the prestable maps (C, p_i, q_j, q'_k) and $(\overline{C}, \overline{p}_i, \overline{q}_j, \overline{q}'_k)$, and g_2 is an automorphism of $\Delta_L[k]$ given by the action of an element of $(\mathbb{C}^*)^k$, such that the following diagram commutes:

$$\begin{array}{ccc} C & \xrightarrow{g_1} & \overline{C} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \overline{f}} \\ \Delta_L[k] & \xrightarrow[g_2]{} & \Delta_L[k] \end{array}$$

We say that a prestable non-rigid map is stable if its automorphism group is finite. Denote by $\overline{\mathcal{M}}_{g,n}(\Delta_L, \mu, \nu, \beta)^\sim$ the moduli space stable of non-rigid maps. The following theorem records some of its most important properties (see [17] Theorem 5.1.16.)

Theorem A.3.3. $\overline{\mathcal{M}}_{g,n}(\Delta_L, \mu, \nu, \beta)$ *is a proper, separated Deligne–Mumford stack with virtual class* $\left[\overline{\mathcal{M}}_{g,n}(\Delta_L, \mu, \nu, \beta)^\sim\right]^{\mathrm{vir}}$ *with virtual dimension*

$$\mathrm{vdim}(\overline{\mathcal{M}}_{g,n}(\Delta_L, \mu, \nu, \beta)^\sim) = \mathrm{vdim}(\overline{\mathcal{M}}_{g,n}(\Delta_L, \beta)) - (|\mu| + |\nu| - \ell(\mu) - \ell(\nu)) - 1.$$

In this space, we have two special ψ–classes. The divisors D_0 and D_∞ of Δ_L allow us to construct cotangent line bundles $\mathbb{L}_0$ and $\mathbb{L}_\infty$ on $\overline{\mathcal{M}}_{g,n}(\Delta_L, \mu, \nu, \beta)^\sim$ (see [37] Section 1.5.2.). We can define the ψ classes $\psi_0 := c_1(\mathbb{L}_0)$ and $\psi_\infty := c_1(\mathbb{L}_\infty)$.

As before, we can construct evaluation morphisms and a projection to the moduli space of stable curves:

$$
\begin{array}{rlcll}
\mathrm{ev}_i: & \overline{\mathcal{M}}_{g,n}(\Delta_L,\mu,\nu,\beta)^{\sim} & \longrightarrow & \Delta_L & \forall i \in \{1,\dots,n\} \\
\mathrm{ev}_j^{\infty}: & \overline{\mathcal{M}}_{g,n}(\Delta_L,\mu,\nu,\beta)^{\sim} & \longrightarrow & D & \forall j \in \{1,\dots,m\} \\
\mathrm{ev}_k^{0}: & \overline{\mathcal{M}}_{g,n}(\Delta_L,\mu,\nu,\beta)^{\sim} & \longrightarrow & D & \forall k \in \{1,\dots,k\} \\
\rho: & \overline{\mathcal{M}}_{g,n}(\Delta_L,\mu,\nu,\beta)^{\sim} & \longrightarrow & \overline{\mathcal{M}}_{g,n+m+l} &
\end{array}
$$

Using thse morphisms we can define Gromov-Witten classes as

$$
I^{\sim}_{g,n,\mu,\nu,\beta}(\gamma^0_1,\dots\gamma^0_k|\psi^{k_1}_{\infty},\psi^{k_2}_0,\gamma_1,\dots\gamma_n|\gamma^{\infty}_1,\dots\gamma^{\infty}_m) =
$$
$$
PD\circ\rho_*\left(\left[\overline{\mathcal{M}}_{g,n}(\Delta_L,\mu,\nu,\beta)^{\sim}\right]^{\mathrm{vir}} \frown \psi^{k_1}_{\infty}\psi^{k_2}_0\prod_i \mathrm{ev}^*_i(\gamma_i)\prod_j(\mathrm{ev}^{\infty}_j)^*(\gamma^{\infty}_i)\prod_i(\mathrm{ev}^0_k)^*(\gamma^0_i)\right)
$$

for $\alpha \in RH^*(\overline{\mathcal{M}}_{g,n+m+k})$, $\gamma_i \in H^*(\Delta_L)$, and $\gamma^{\infty}_j,\gamma^0_k \in H^*(D)$. Analogously, we define the respective Gromow Witten invariants by

$$
\langle\gamma^0_1,\dots\gamma^0_k|\psi^{k_1}_{\infty},\psi^{k_2}_0,\alpha;\gamma_1,\dots\gamma_n|\gamma^{\infty}_1,\dots\gamma^{\infty}_m\rangle^{\sim} :=
$$
$$
\int_{\overline{\mathcal{M}}_{g,n+m+k}}\alpha I^{\sim}_{g,n,\mu,\nu,\beta}(\gamma^0_1,\dots\gamma^0_k|\psi^{k_1}_{\infty},\psi^{k_2}_0,\gamma_1,\dots\gamma_n|\gamma^{\infty}_1,\dots\gamma^{\infty}_m)
$$

This moduli space and these Gromov-Witten classes play a fundamental role in the algorithm for the $K3$ surface in Subsection 3.5. These stack will appear as a part of the data inside the fix locus of the localization formula. More concretely, we will be interested in applying this construction to $\mathbb{P}^1$.