Chow Rings of Hilbert Schemes of Points of a K3 Surface and the Hyper-Kähler Resolution Conjecture

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1 Introduction

1.1 Overview

The Hilbert scheme $X^{[n]}$ of n points on a variety X is the scheme parametrizing zero dimensional length-n subschemes of X. If X is a smooth projective surface, $X^{[n]}$ is smooth, and it is a (hyper-Kähler) crepant resolution of the symmetric product $X^{(n)} = X^n/S_n$ via the Hilbert-Chow morphism.

In this thesis, we consider the case that X is a $K3$ surface, or more generally a smooth projective surface with trivial canonical class.

The Chow groups of $X^{[n]}$ can be described by Nakajima operators, which define a Heisenberg algebra action on and give a basis of the Chow rings

$$A^*(\mathrm{Hilb}) = \bigoplus_{n\geq 0} A^*(X^{[n]}),$$

where $\mathrm{Hilb} = \coprod_{n\geq 0} X^{[n]}$. While the cup products of cohomological classes given in terms of these operators can be computed using the work of Lehn and Sorger [13], the product of such classes in the Chow ring has not been fully computed. The purpose of this thesis is to obtain a formula for this product by proving that the Chow ring of $X^{[n]}$ is isomorphic to the orbifold Chow ring of $[X^n/S_n]$. This isomorphism is a case of the *Motivic Hyper-Kähler Resolution Conjecture* proposed by Fu, Tian and Vial [8, Conjecture 1.3], one of several conjectures about crepant resolutions of singular spaces of orbifolds relating their algebraic or topological invariants to certain invariants of the orbifold, based on Ruan's [20] cohomological Crepant Resolution Conjecture. For further discussion of these conjectures, we refer to [8, §3].

Lehn and Sorger [13] proved that the cohomology ring $H^*(X^{[n]}, \mathbb{Q})$ is isomorphic to a ring $H^{[n]}$ they constructed from the cohomology $H = H^*(X, \mathbb{Q})$ of X, namely the S_n invariants of

$$\bigoplus_{g\in S_n} H^{\otimes [n]/\langle g\rangle} \cdot g$$

with action by conjugation, and a product [13, (2.9)] defined naturally by the relationship of the orbits of g, h, gh and $\langle g, h\rangle$, with a correction factor involving the Euler class. Fantechi and Göttsche [5] proved that $H^{[n]}$ is isomorphic to the orbifold cohomology ring of $X^{(n)}$ (up to a sign change in the definition of the product), originally defined by Chen and Ruan [4], [20], which Fantechi-Göttsche expressed as the invariants of

$$H(X^n, S_n) = \bigoplus_{g\in S_n} H^*((X^n)^g) \cdot g$$

with the natural S_n action and a product with correction factor coming from the top Chern class of an obstruction bundle. While the absence of a Künneth decomposition for the Chow ring means we cannot give an algebraic construction of $A^*(X^{[n]})$ in the manner of Lehn-Sorger, we can still describe its structure by obtaining an isomorphism with the *orbifold Chow ring* of $[X^n/S_n]$. This was defined by Abramovich, Graber and Vistoli [1] in their work giving algebraic counterparts for Chen and Ruan's orbifold cohomology and Gromov-Witten theory, and Jarvis, Kaufmann and Kimura [11] constructed it as the ring of invariants of a "stringy Chow ring" in a similar manner to Fantechi-Göttsche's description of orbifold cohomology. We define the orbifold Chow ring in Section 3, and examine its structure for $[X^n/S_n]$ in Section 4.

The work of De Cataldo and Migliorini [3] provides an additive isomorphism $A^*(X^{[n]}) \cong A^*_{\mathrm{orb}}(X^{(n)})$ for all surfaces X, as proven by Fu, Tian and Vial [8, 5.2]. This isomorphism is closely related to the Nakajima operators, as we shall see in Section 4. In particular, proving it is a ring isomorphism is equivalent to proving that products of the Najakima basis of $A^*(X^{[n]})$ can be computed by a certain formula, given in (18).

In their preprint [7], Fu and Tian give a proof that this is a ring isomorphism, based on Voisin's announced theory of *universally defined cycles*.

In this work I present an alternate proof, following the techniques of Lehn-Sorger and adapting them in a more geometric context. The main new input is the recent work of Maulik and Neguț [17], which relates the operators of multiplication by certain Chern characters to the Nakajima operators, as we recall in Theorem 2.4.

Theorem 1.1. *Let X be a smooth projective complex surface with trivial canonical class (in particular, a K3 surface). Then there is an isomorphism of graded $\mathbb{C}$-algebras*

$$A^*\big(X^{[n]}\big)_{\mathbb{C}} \cong A^*_{orb}\big([X^n/S_n]\big)_{\mathbb{C}}.$$

Indeed, we have an isomorphism in the category of complex Chow motives

$$\mathfrak{h}\big(X^{[n]}\big) \cong \mathfrak{h}_{orb}\big([X^n/S_n]\big)$$

where $\mathfrak{h}_{orb}([X^n/S_n])$ is the orbifold Chow motive defined by Fu-Tian-Vial [8, Definition 2.5]

For non-$K3$ surfaces with trivial canonical class (namely abelian surfaces), this result has already been obtained by Fu-Tian-Vial, [8, Theorem 1.4]. Our proof works for both types of surface. The use of complex coefficients is only required to fix a sign disparity, and the theorem remains true with rational coefficients if we introduce a sign change in the definition of the orbifold Chow ring.

The first step in proving the ring isomorphism will be proving it preserves multiplication by certain Chern characters, which we shall describe in section 5. In 6, we describe orbifold correspondences and operators, in order to transfer multiplication operators to the orbifold Chow ring. We put the pieces together and complete the proof in 7

1.2 Conventions and notation

For the remainder of this thesis, all Chow rings will be taken with coefficients in the complex numbers (or more restrictively they can be taken with $\mathbb{Q}[i]$). Chern classes and characters are always taken in the Chow ring with rational or complex coefficients, and for any $x \in A^*(X)$ we denote by $x_d \in A^d(X)$ the homogeneous component in degree d of x.

We will work with the language of operators between Chow rings and correspondences interchangeably: an operator $\mathfrak{a} : A^*(X) \to A^*(Y)$ will here always be induced by a correspondence $\alpha \in A^*(X \times Y)$ such that $\mathfrak{a}x = \pi_{2*}(\alpha \cdot \pi_1^* x)$ for $x \in A^*(X)$, where π_1, π_1 are the projections

$$X \xleftarrow{\ \pi_1\ } X \times Y \xrightarrow{\ \pi_2\ } Y$$

Let $\alpha^t \in A^*(Y \times X)$ be the transpose obtained from the isomorphism $A^*(Y \times X) \cong A^*(X \times Y)$, which determines an operator $A^*(Y) \to A^*(X)$.

Given another correspondence $\beta \in A^*(Y \times Z)$, we can take the composition

$$\beta \circ \alpha = \pi_{13*}(\pi_{23}^*\beta \cdot \pi_{12}^*\alpha) \in A^*(X \times Z)$$

where π_{ij} are the projections

$$X \times Y \xleftarrow{\ \pi_{12}\ } X \times Y \times Z \xrightarrow{\ \pi_{23}\ } Y \times Z, \qquad X \times Y \times Z \xrightarrow{\ \pi_{13}\ } X \times Z$$

For any two correspondences $\alpha \in A^*(X \times Y)$, $\gamma \in A^*(Z \times W)$, we can take the product

$$\alpha \times \gamma \in A^*(X \times Y \times Z \times W) \cong A^*((X \times Z) \times (Y \times W))$$

as a correspondence defining an operator $A^*(X \times Z) \to A^*(Y \times W)$.

We often work with operators of the form $\mathfrak{f} : A^*(X^{[n]}) \to A^*(X^{[m]} \times X^r)$, which we can consider as operators $A^*(X^{[n]}) \to A^*(X^{[m]})$ parametrized by $A^*(X^r)$. That is, for any $\alpha \in A^*(X^r)$ we have an operator

$$\mathfrak{f}(\alpha) : A^*\left(X^{[n]}\right) \to A^*\left(X^{[m]}\right)$$
$$\mathfrak{f}(\alpha)x = \pi_{1*}\left(\pi_2^*(\alpha) \cdot \mathfrak{f}x\right)$$

$$\begin{array}{ccc} & X^{[m]} \times X^t & \\ \swarrow^{\pi_1} & & \searrow^{\pi_2} \\ X^{[m]} & & X^t \end{array}$$

If $\mathfrak{g} : A^*\left(X^{[m]}\right) \to A^*\left(X^{[l]} \times X^s\right)$ is another such operator, we can take their composition

$$\mathfrak{g}\mathfrak{f} : A^*\left(X^{[n]}\right) \to A^*\left(X^{[l]} \times X^{s+r}\right)$$

where $\mathfrak{g}$ goes to the first s factors of X^{s+r} and $\mathfrak{f}$ to the last r, i.e.

$$\mathfrak{g}\mathfrak{f} := (\mathfrak{g} \times \mathrm{id}_{X^s *}) \circ \mathfrak{f}.$$

Thus if $\alpha \in A^*(X^r),\ \beta \in A^*(X^s)$ then $\mathfrak{g}\mathfrak{f}(\beta \times \alpha) = \mathfrak{g}(\beta)\mathfrak{f}(\alpha)$.

We sometimes need to reorder the indices of X^r in these operators, so if some permutation $\sigma \in S_r$ acts on X^r by permuting the factors, we denote

$$\mathfrak{f}^{\sigma(1)\dots\sigma(n)} := (\mathrm{id}_{X^{[m]}*} \times \sigma_*) \circ \mathfrak{f} : A^*\left(X^{[n]}\right) \to A^*\left(X^{[m]} \times X^s\right).$$

In particular, when we take the commutator of two operators

$$\mathfrak{p} : A^*(\mathrm{Hilb}) \to A^*(\mathrm{Hilb} \times X^r)$$
$$\mathfrak{q} : A^*(\mathrm{Hilb}) \to A^*(\mathrm{Hilb} \times X^s),$$

where $\mathrm{Hilb} = \coprod_{n=0}^{\infty} X^{[n]}$, we will implicitly reorder these indices:

$$[\mathfrak{p}, \mathfrak{q}] := \mathfrak{p}\mathfrak{q} - (\mathfrak{q}\mathfrak{p})^{s+1\dots s+t,1\dots s} : A^*(\mathrm{Hilb}) \to A^*(\mathrm{Hilb} \times X^{s+t})$$

If $\alpha \in A^*(X^r)$, we shall use the notation

$$\alpha_{i_1\dots i_r} := \pi^*_{i_1\dots i_r}(\alpha) \in A^*(X^s)$$

for any s and distinct integers $1 \le i_j \le s$. It is useful to extend this notation to the case that i_j are not distinct, to mean we intersect the corresponding indices, i.e.

$$\alpha_{i_1 i_1\dots i_{r-1}} = \pi_{1\dots s,*}(\Delta_{i_1,s+1} \cdot \pi^*_{s+1,i_1\dots i_{r-1}}(\alpha))$$

and so on.

We will need to apply similar transformations of operators: if $\mathfrak{f} : A^*(X^{[n]}) \to A^*(X^{[m]} \times X)$ and $\mathfrak{g} : A^*(X^{[n]}) \to A^*(X^{[m]} \times X^t)$ are operators, then we define operators

$$\Delta_*(\mathfrak{f}) : A^*(X^{[n]}) \to A^*(X^{[m]} \times X^2)$$
$$\Delta_*(\mathfrak{f})x = (\mathrm{id}_{X^{[n]}} \times \Delta)_*(\mathfrak{f}x)$$
$$\mathfrak{g}|_{\Delta_{1\dots t}} : A^*(X^{[n]}) \to A^*(X^{[m]} \times X)$$
$$\mathfrak{g}|_{\Delta_{1\dots t}} x = (\mathrm{id}_{X^{[n]}} \times \Delta_{1\dots t})^*(\mathfrak{g}x)$$

where $\Delta : X \to X^2$ in the second line is the diagonal embedding, and $\Delta_{1\dots t} : X \to X^t$ is the small diagonal embedding. Inserting parameters $\alpha \in A^*(X^2),\ \beta \in A^*(X)$ gives

$$\Delta_*(\mathfrak{f})(\alpha) = \mathfrak{f}(\Delta^*(\alpha)) = \mathfrak{f}(\alpha_{11})$$

$$\mathfrak{g}|_{\Delta_{1\dots t}}(\beta) = \mathfrak{g}(\Delta_*(\beta)) = \mathfrak{g}(\beta_1\Delta_{1\dots t})$$

Finally, we often work with projections from $X^{[n]} \times X^s$ onto the various factors, so denote by

$$\pi_{i_1,\dots,i_r} : X^{[n]} \times X^s \to X^r$$

the projection onto the factors $i_1, \dots i_r$ of X^r, and by

$$\pi_{[n],i_1,\dots,i_r} : X^{[n]} \times X^s \to X^{[n]} \times X^r$$

the projection onto both $X^{[n]}$ and the factors $i_1, \dots i_r$ of X^r. In particular we denote the pushforward along the projection away from an index i by $\int_i$, i.e. for $x \in X^{[n]} \times X^s$ we have

$$\int_i x = \pi_{[n],1,\dots\hat{i}\dots s,*}x$$

2 Hilbert Schemes of Surfaces

2.1 K3 Surfaces

For our purposes, a K3 surface is a connected smooth projective complex algebraic surface with trivial canonical bundle $\omega_X \simeq \mathcal{O}_X$ (or equivalently $c_1(TX) = 0$), and $H^1(X, \mathcal{O}_X) = 0$. The only surfaces with trivial canonical class are $K3$ surfaces and tori, so the latter condition is equivalent to not being a torus.

Our proof will work for any surface with trivial canonical class, but the result for Abelian surfaces has already been proven by Fu-Tian-Vial [8, Theorem 1.5]. Let $e = c_2(TX)$ be the Euler class; for $K3$ surfaces Beauville and Voisin [2] proved that $e = 24c$ where $c \in A^2(X)$ represents any closed point on a rational curve in X.

2.2 Hilbert Schemes of Points

Let X be a complex projective variety. The Hilbert scheme of n points on X is the parameter space of closed subschemes of length n, often represented by their ideal sheaves:

$$X^{[n]} = \{Z \subset X \mid \dim H^0(Z, \mathcal{O}_Z) = n\} = \{I \subset \mathcal{O}_X \mid \dim H^0(X, \mathcal{O}_X/I) = n\}$$

This is a projective variety, as the simplest case of a Hilbert scheme $\mathrm{Hilb}^P(X)$ constructed by Grothendieck [10] which parametrize closed subschemes of X with Hilbert polynomial P.

In our case X is always a smooth projective surface, so $X^{[n]}$ is smooth of dimension $2n$ as proven by Fogarty [6].

The cohomology and Chow groups of Hilbert schemes of points of surfaces can be described using operators constructed by Nakajima [18] (see also Grojnowski [9]) as follows. Let $n \geq 0$ and $k \geq 1$, and consider the closed subscheme

$$X^{[n,n+k]} = \{(I \supset J) \mid \mathrm{Supp}\,(I/J) = \{x\} \text{ for some } x \in X\} \subset X^{[n]} \times X^{[n+k]} \tag{1}$$

with projections

$$\begin{array}{ccccc} & & X^{[n,n+k]} & & \\ & \swarrow^{p_-} & \downarrow^{p_X} & \searrow^{p_+} & \\ X^{[n]} & & X & & X^{[n+k]} \end{array} \tag{2}$$

As a correspondence this defines the Nakajima operators $\mathfrak{q}_{\pm k} : A^*\big(X^{[n]}\big) \to A^*\big(X^{[n\pm k]} \times X\big)$ by

$$\mathfrak{q}_k = (p_+ \times p_X)_* \circ p_-^* : A^*\big(X^{[n]}\big) \to A^*\big(X^{[n+k]} \times X\big)$$
$$\mathfrak{q}_{-k} = (-1)^k\,(p_- \times p_X)_* \circ p_+^* : A^*\big(X^{[n+k]}\big) \to A^*\big(X^{[n]} \times X\big)$$

These operators can be combined for each n giving operators

$$\mathfrak{q}_k : A^*(\mathrm{Hilb}) \to A^*(\mathrm{Hilb} \times X)$$

for any $0 \neq k \in \mathbb{Z}$. They operators satisfy relations

$$[\mathfrak{q}_k, \mathfrak{q}_l] = k\delta^0_{k+l}\,\mathrm{Id}_{X^{[*]}} \times \Delta$$

as correspondences $A^*(\mathrm{Hilb}) \to A^*(\mathrm{Hilb} \times X^2)$, or if $\alpha, \beta \in A^*(X)$

$$[\mathfrak{q}_k(\alpha), \mathfrak{q}_l(\beta)] = k\delta^0_{k+l} \int_X (\alpha \cdot \beta)\,\mathrm{id}_{\mathrm{Hilb}}$$

as correspondences $A^*(\mathrm{Hilb}) \to A^*(\mathrm{Hilb})$

Note that our Nakajima operators $\mathfrak{q}_k, \mathfrak{q}_{-k}$ correspond to Lehn-Sorger's $\mathfrak{p}_{-k}, -\mathfrak{p}_k$.

Definition 2.1.
1. If $\lambda = (\lambda_1, \ldots, \lambda_d)$ is a partition, denote $\mathfrak{q}_\lambda := \mathfrak{q}_{\lambda_1} \cdots \mathfrak{q}_{\lambda_d}$.
2. Let v be the identity in $A^*(X^{[0]}) \simeq \mathbb{C}$, sometimes called the vacuum vector.

Thus for any $\alpha \in A^*(X^d)$ and any partition $\lambda_1 + \cdots + \lambda_d = n$, we have an element

$$\mathfrak{q}_{\lambda_1} \cdots \mathfrak{q}_{\lambda_d}(\alpha) v \in A^*(X^{[n]}).$$

These elements can be used to describe a basis; the precise statement will be formulated in Theorem 4.5 after giving more context

2.3 Chern characters

Let $\Xi_n \subset X^{[n]} \times X$ be the universal subscheme, which is characterized by the property

$$\Xi_n \cap (\{I\} \times X) = \{I\} \times Z$$

for any $Z \subset X$ of length n cut out by the ideal sheaf I.

The Chow ring of $X^{[n]}$ can also be described in terms of operators $\mathfrak{G}_d : A^*\big(X^{[n]}\big) \to A^*\big(X^{[n]} \times X\big)$ defined by multiplication by the d'th Chern character of the tautological sheaf $\mathcal{O}_{\Xi_n}$, i.e.

$$\mathfrak{G}_d x = \mathrm{ch}_d\left(\mathcal{O}_{\Xi_n}\right) \cdot \pi_{[n]}^* x$$

Based on the work of Markmann [16], Neguț-Oberdieck-Yin showed

Theorem 2.2 ([19, Theorem 2.3]). *Any element of $A^*(X^{[n]})$ can be expressed as a sum of elements of the form*

$$\mathfrak{G}_{d_1} \dots \mathfrak{G}_{d_k}(\alpha) \cdot 1_{X^{[n]}}$$

with $\alpha \in A^*(X^k)$

In order to prove our main theorem, we need an expression for the operators $\underline{\mathfrak{G}}_d$ in terms of Nakajima operators, which is provided by the work of Maulik and Negut [17] (the cohomological version of which is due to Li, Qin and Wang [14]).

Definition 2.3.

1. Let $n \in \mathbb{Z}$ and $d \geq 0$. A *generalized partition* of n with length d is a partition consisting of (possibly negative) nonzero integers

$$(\lambda_1, \dots, \lambda_d) = \lambda = (\dots (-2)^{m_{-2}}(-1)^{m_{-1}} 1^{m_1} 2^{m_2} \dots)$$

with $\sum_{j=1}^d \lambda_j = n = \sum_{i \in \mathbb{Z}} i m_i$. Define

$$|\lambda| = n, \quad l(\lambda) = d, \quad s(\lambda) = \sum_i i^2 m_i, \quad \text{and} \quad \lambda! = \prod_i m_i!.$$

and let $\mathcal{P}_{\mathbb{Z}}(n,d)$ be the set of all such generalized partitions. We sometimes use $\lambda \vdash n$ to mean λ is a partition of n.

2. If $n > 0$ let $\mathcal{P}(n,d)$ be the set of partitions of n into d parts in the usual sense, i.e. each $\lambda_j > 0$ for all j (or equivalently $m_i = 0$ for $i < 0$).

Now we can state the theorem of Maulik-Negut:

Theorem 2.4 ([17, Thm 1.7]). *If $c_1(TX) = 0$ and $c_2(TX) = e$, there are operators*

$$\{\mathfrak{I}_n^k : A^*(\mathrm{Hilb}) \to A^*(\mathrm{Hilb} \times X)\}_{n \in \mathbb{Z}}^{k \geq 0},$$

such that

$$\mathfrak{I}_n^0 = -\mathfrak{q}_n$$
$$\mathfrak{I}_0^k = k!\left(\mathfrak{G}_{k+1} + \frac{1}{12}\pi_X^*(e)\,\mathfrak{G}_{k-1}\right),$$

which are given in terms of Nakajima operators as [17, (3.16)]

$$\mathfrak{I}_n^d = d!\left(-\sum_{\lambda \in \mathcal{P}_{\mathbb{Z}}(n,d+1)} \frac{1}{\lambda!}\mathfrak{q}_\lambda|_{\Delta_{1\dots d+1}} + \sum_{\lambda \in \mathcal{P}_{\mathbb{Z}}(n,d-1)} \frac{s(\lambda)+n^2-2}{24\lambda!}\pi_X^*(e)\,\mathfrak{q}_\lambda|_{\Delta_{1\dots d-1}}\right) \tag{3}$$

The commutator of these operators is given by, if $k + k' \geq 3$,

$$[\mathfrak{I}_n^k, \mathfrak{I}_{n'}^{k'}] = (kn' - k'n)\Delta_*(\mathfrak{I}_{n+n'}^{k+k'-1}) + \Omega_{n,n'}^{k,k'}\Delta_*\left(\frac{\pi_X^*(e)}{12}\mathfrak{I}_{n+n'}^{k+k'-3}\right) \tag{4}$$

for some integers $\Omega_{n,n'}^{k,k'}$ which are given in [14, 5.2] (using a different sign convention $k \leftrightarrow -k$)

Thus we have an expression for $\mathfrak{G}_d$ in terms of Nakajima operators:

$$\mathfrak{G}_d = \frac{\mathfrak{J}_0^{d-1}}{(d-1)!} - \frac{\pi_X^*(e)\,\mathfrak{J}_0^{d-1}}{12\,(d-1)!} = -\sum_{\lambda\in\mathcal{P}_{\mathbb{Z}}(0,d)} \frac{1}{\lambda!}\mathfrak{q}_\lambda|_{\Delta_{1\ldots d}} + \sum_{\lambda\in\mathcal{P}_{\mathbb{Z}}(0,d-2)} \frac{s(\lambda)+n^2}{24\lambda!}\pi_X^*(e)\,\mathfrak{q}_\lambda|_{\Delta_{1\ldots d-2}} \tag{5}$$

The tautological sheaf $\mathcal{O}^{[n]} = \pi_{[n]*}(\mathcal{O}_\Xi)$ is a vector bundle of rank n on $X^{[n]}$ (as the projection $\Xi \to X^{[n]}$ is flat and finite of degree n), and by Grothendieck-Riemann-Roch we have

$$\mathrm{ch}\left(\mathcal{O}^{[n]}\right) = \pi_{[n]*}\left(\mathrm{ch}\left(\mathcal{O}_{\Xi_2}\right)\pi_X^*\left(\mathrm{td}\,X\right)\right).$$

Indeed, we also consider Chern character operators that relate more closely to $\mathcal{O}^{[n]}$

$$\underline{\mathfrak{G}}_d x = \left(\mathrm{ch}\left(\mathcal{O}_{\Xi_n}\right)\cdot\pi_X^*(\mathrm{td}\,X)\right)_d \pi_{[n]}^*(x),$$

where $(-)_d$ denotes taking the homogeneous component of degree d. These can be expressed as

$$\underline{\mathfrak{G}}_d = \mathfrak{G}_d + 2\pi_X^*(c)\mathfrak{G}_{d-2} = \frac{\mathfrak{J}_0^{d-1}}{(d-1)!} \tag{6}$$

These $\underline{\mathfrak{G}}_d$ are denoted as $\mathfrak{G}_d$ in cohomology by Li-Qin-Wang [15, 5.1]. We can take the total Chern character operator

$$\underline{\mathfrak{G}} := \sum_{d\geq 0}\underline{\mathfrak{G}}_d$$

with Lehn-Sorger's tautological classes [13, 3.6] being $\alpha^{[n]} = \underline{\mathfrak{G}}(\alpha)$ In particular, $\underline{\mathfrak{G}}(1) = \mathrm{ch}(\mathcal{O}^{[n]})$.

The Chern character operators described above give a way to describe multiplication in $A^*(X^{[n]})$ using Nakajima operators; in principle computing the product $\mathfrak{q}_\lambda(\alpha)v\cdot\mathfrak{q}_\nu(\beta)$ can be done by obtaining an expression for $\mathfrak{q}_\lambda(\alpha)v$ in terms of Chern character operators, applying (5) to obtain an expression for the multiplication operation and using the commutation relations to reduce the result to the Nakajima basis. To obtain a general formula for the product, however, it is more reasonable to look to the orbifold Chow ring of $[X^n/S_n]$ where a product formula already exists, and work back to prove these rings are isomorphic by comparing the action of operators $\underline{\mathfrak{G}}_{d_1}\ldots\underline{\mathfrak{G}}_{d_r}(\alpha)$ to the orbifold multiplication by the image of $\underline{\mathfrak{G}}_{d_1}\ldots\underline{\mathfrak{G}}_{d_r}(\alpha)\cdot 1_{X^{[n]}}$.

2.4 Virasoro operators

We also consider the Virasoro operators

$$\mathfrak{L}_d = \frac{1}{2}\sum_{\substack{a+b=d\\ a,b\in\mathbb{Z}}} {:}\mathfrak{q}_a\mathfrak{q}_b{:}\big|_\Delta = -\mathfrak{J}_d^1,$$

originally constructed in cohomology by Lehn, where :−: is the normal ordered product which rearranges the factors $\mathfrak{q}_{i_1}\ldots\mathfrak{q}_{i_r}$ in descending order of the indices $\{i_j\}$.

Lemma 2.5. *These operators satisfy*

$$[\mathfrak{L}_n, \mathfrak{q}_k] = -k\Delta_*(q_{n+k})$$

Proof. All terms of

$$\frac{1}{2}\sum_{\substack{a+b=d\\ a,b\in\mathbb{Z}}}[{:}\mathfrak{q}_a\mathfrak{q}_b{:},\mathfrak{q}_k]$$

vanish except those with $a = -k$ or $b = -k$. The case $2k = -n$ has to be considered separately, so assume $-k \neq n+k$. Since ${:}\mathfrak{q}_k\mathfrak{q}_{n-k}{:} = {:}\mathfrak{q}_{n-k}\mathfrak{q}_k{:} = \mathfrak{q}_k\mathfrak{q}_{n-k}$,

$$\begin{aligned}\frac{1}{2}\sum_{\substack{a+b=d\\ a,b\in\mathbb{Z}}}[{:}\mathfrak{q}_a\mathfrak{q}_b{:},\mathfrak{q}_k] &= [\mathfrak{q}_{n+k}\mathfrak{q}_{-k},\mathfrak{q}_k]\\ &= \mathfrak{q}_{n+k}[\mathfrak{q}_{-k},\mathfrak{q}_k]\\ &= -k\mathfrak{q}_{n+k}\times\Delta.\end{aligned}$$

Then we have

$$\begin{aligned}
[\mathfrak{L}_n, \mathfrak{q}_k] &= -k(\mathfrak{q}_{n+k} \times \Delta)|_{\Delta_{12}} \\
&= -k \int_2 \left(\Delta_{12}\Delta_{23} \cdot \left(\mathfrak{q}_{n+k} \times X^2\right)\right) \\
&= -k \int_2 \left(\Delta_{123} \cdot \left(\mathfrak{q}_{n+k} \times X^2\right)\right) \\
&= -k\Delta_{12} \cdot (\mathfrak{q}_{n+k} \times X) \\
&= -k\Delta_*(\mathfrak{q}_{n+k})
\end{aligned}$$

as required.

Meanwhile if $n = -2k$, we get

$$\begin{aligned}
\frac{1}{2} \sum_{\substack{a+b=d \\ a,b\in\mathbb{Z}}} [:\mathfrak{q}_a\mathfrak{q}_b:, \mathfrak{q}_k] &= \frac{1}{2}[\mathfrak{q}_{-k}\mathfrak{q}_{-k}, \mathfrak{q}_k] \\
&= \frac{1}{2}\left(\mathfrak{q}_{-k}[\mathfrak{q}_{-k}, \mathfrak{q}_k] + ([\mathfrak{q}_{-k}, \mathfrak{q}_k]\mathfrak{q}_{-k})^{132}\right) \\
&= \frac{1}{2}\left(\mathfrak{q}_{-k} \times \Delta + (\mathfrak{q}_{-k} \times \Delta)^{132}\right).
\end{aligned}$$

which is $\Delta_*(\mathfrak{q}_{-k}) = \Delta_*(\mathfrak{q}_{n+k})$ by similar arguments to the first case for each term. □

Under the assumption that $t = c_1(TX)$ vanishes, we have

$$[\partial, q_k] = \left[-\frac{1}{2}\mathfrak{J}^2_0(1), \mathfrak{J}^0_k\right] = -k\mathfrak{J}^1_k = k\mathfrak{L}_k$$

If $t \neq 0$, we can still compute this commutator using [17, Theorem 1.6], which gives the expression

$$\partial = \underline{\mathfrak{G}}_3(1) = -\frac{1}{6} \sum_{n_1+n_2+n_3=0} :\mathfrak{q}_{n_1}\mathfrak{q}_{n_2}\mathfrak{q}_{n_3}:(\Delta_{123}) - \frac{t}{2}\sum_{k=1}^{\infty} k\mathfrak{q}_k\mathfrak{q}_{-k}(\Delta_{12}).$$

Applying the Nakajima relations one can, assuming $k > 0$, compute the commutator

$$[\partial, q_k] = k\mathfrak{L}_k + \frac{k(k-1)}{2}t\mathfrak{q}_k \tag{7}$$

In particular, if $k, l > 0$ we have

$$[[\partial, \mathfrak{q}_k], \mathfrak{q}_l] = -kl\Delta_*\mathfrak{q}_{k+l} \tag{8}$$

This allows us to construct the operators $\mathfrak{q}_k$ from only $\mathfrak{q}_1$ and ∂, a crucial reduction step in our later results.

Lemma 2.6.

$$\mathfrak{q}_k = (-1)^{k-1}\frac{1}{(k-1)!}\,\mathrm{ad}([\partial, \mathfrak{q}_1(1)])^{n-1}\mathfrak{q}_1$$

Proof. This follows from repeatedly applying the relation

$$[[\partial, \mathfrak{q}_1(1)], \mathfrak{q}_k] = -k\mathfrak{q}_{k+1}.$$

□

3 Orbifolds

3.1 Orbifold Chow Ring

The total orbifold Chow group is abstractly defined [1, 4.4] as follows.

Definition 3.1. Let $\mathcal{X}$ be an orbifold. Consider the inertia stack

$$\mathcal{X}_1 = \{(x,g) \mid x \in \mathrm{Ob}(\mathcal{X}), g \in \mathrm{Aut}(x)\} = \mathcal{X} \times_{\mathcal{X}\times\mathcal{X}} \mathcal{X}$$

and X_1 be its coarse moduli space. The *total orbifold Chow group* of $\mathcal{X}$ is defined as the total Chow group of X_1

$$A^*_{\mathrm{orb}}(\mathcal{X}) := A^*(X_1).$$

If $\mathcal{X} = [M/G]$ is a global quotient, we can construct this by first defining the *stringy Chow group*

$$A^*(M,G) = \bigoplus_{g\in G} A^*(M^g)\cdot g, \tag{9}$$

which is the Chow group of the inertia variety

$$I_G(M) := \coprod_{g\in G} M^g.$$

There is a natural G action $g : M^h \to M^{ghg^{-1}}$ on $I_G(M)$, with the inertia stack being the quotient by this action. We denote the induced action on $A^*(M,G)$ by $x \mapsto g \bullet x$, and the orbifold Chow group is given by the invariants of this action:

$$A^*_{\mathrm{orb}}([M/G]) = A^*(M,G)^G \tag{10}$$

If $x \in A^*(M,G)$, we will use the notation $x_g \in A^*(M^g)$ for the projection onto the summand $A^*(M^g)\cdot g$.

The total stringy and orbifold Chow groups have a grading that is shifted from the original grading of the direct summands.

Definition 3.2. Let $g \in G$ be of order r, and x be a point of M fixed by g. The *age* or *degree shifting number* of an element g at x is

$$a(g,x) = \sum_{j=1}^{m} \frac{\alpha_j}{r}$$

where $e^{2\pi i \frac{\alpha_j}{r}}$, $j = 1,\dots m$, $0 \le \alpha_i < 1$ are the eigenvalues (repeated with multiplicity) of the action of g on the tangent space T_xM at x. In the case that M^g is connected, we say the age $a(g)$ of g is the constant value $a(g,x)$ for any x.

Note that $a(g) + a(g^{-1}) = \mathrm{codim}(M^g \subset M)$.

Then the grading on the stringy Chow group is given by

$$A^k(M,G) = \bigoplus_{g\in G} A^{k-a(g)}(M^g) \tag{11}$$

assuming M^g is connected (otherwise the grading is shifted separately on each component). As the age is invariant under conjugation, this descends to a grading on $A^*_{\mathrm{orb}}(M)$.

3.2 The orbifold product

The ring structures on $A^*(M,G)$ and $A^*_{\text{orb}}(\mathcal{X})$ are complicated to define, in particular the appearance of an obstruction bundle on $M^{\langle g,h\rangle}$ and its top Chern class. Abramovich-Graber-Vistoli define [1, 4.7] orbifold Gromov-Witten operations, of which the ordinary product is the 3 point case with homology class 0. Meanwhile Fantechi-Göttsche defined the product in $H^*(M,G)$ explicitly using an obstruction bundle constructed from a Galois cover, while Jarvis-Kaufmann-Kimura define [11, 1.6] the product in $A^*(M,G)$ the same way except the obstruction bundle is given as an explicit expression in $K(M^{\langle g,h\rangle})$, which is the simplest version thus the one we use.

Consider the decomposition

$$TM|_{M^g} = \bigoplus_{j=0}^{r-1} W_{g,j},$$

where $W_{g,j}$ is the eigen-subbundle associated to the eigenvalue $e^{2\pi i \frac{j}{r}}$ Define the virtual bundle

$$S_g = \bigoplus_{j=0}^{r-1} \frac{j}{r} W_{g,j} \in K_0\left(M^g\right) \tag{12}$$

For $g,h \in G$, the obstruction bundle $F_{g,h}$ on $M^{g,h} := M^{\langle g,h\rangle} = M^g \cap M^h$ can be defined in the K group as

$$F_{g,h} = S_g|_{M^{g,h}} + S_h|_{M^{g,h}} + S_{(gh)^{-1}}|_{M^{g,h}} + TM^{g,h} - TM|_{M^{g,h}} \tag{13}$$

This is shown in [11] to be an actual vector bundle; its rank is

$$a(g) + a(h) - a(gh) - \operatorname{codim}\left(M^{g,h} \subset M^{gh}\right)$$

Lemma 3.3. *For any g,h,G, let $H = \langle g,h\rangle$*

1. *The vector bundles $F_{g,h}$ and $F_{h,g}$ on M^H are isomorphic*

2. *Let $v \in G$, then the isomorphism $v : M^H \to M^{vHv^{-1}}$ gives an isomorphism $v^* F_{vgv^{-1},vhv^{-1}} \cong F_{g,h}$.*

Proof. 1. This follows from the fact that, if $l \in H$, $S_g|_{M^H} \cong S_{lgl^{-1}}|_{M^H}$. Indeed, if $W_{g,j}|_{M^H}$ is the eigen-subbundle of TM_{M^H} with respect to the action of g with eigenvalue $e^{2\pi\sqrt{-1}\frac{j}{r}}$, then $W_{lgl^{-1},j}|_{M^H} = l(W_{g,j}|_{M^H})$ is the corresponding eigenbundle of the action of lgl^{-1}. Thus the summands (12) of S_g and $S_{lgl^{-1}}$ with the same coefficients are isomorphic when restricted to M^H, so $S_g|_{M^H} \cong S_{lgl^{-1}}|_{M^H}$.

Since $(hg)^{-1} = g^{-1}(gh)^{-1}g$ is conjugate to $(gh)^{-1}$ in H, we have $F_{g,h} = F_{h,g}$ as they only differ in the terms $S_{(gh)^{-1}}|_{M^H} \cong S_{(hg)^{-1}}|_{M^H}$.

2. By definition, $W_{vgv^{-1},j} = v \cdot W_{g,j}$, so $S_g = v^* S_{vgv^{-1}}$ and the claim follows.

□

Let $g,h \in S_n$ and consider the diagram

$$\begin{array}{ccc} M^g & & M^h \\ & \nwarrow \quad \nearrow & \\ & M^{g,h} & \\ & \downarrow \iota^{gh} & \\ & M^{gh} & \end{array}$$

The orbifold intersection product is defined for the summands

$$\star : A^{i-a(g)}\left(M^g\right) \otimes A^{j-a(h)}\left(M^h\right) \to A^{i+j-a(gh)}\left(M^{gh}\right)$$

by

$$\alpha g \star \beta h = \iota^{gh}_* \left(\alpha|_{M^{g,h}} \cdot \beta|_{M^{g,h}} \cdot c_{\text{top}}\left(F_{g,h}\right)\right) \cdot (gh)$$

This product is associative [11, Lemma 5.4]. By Lemma 3.3.2, it is compatible with the G action: $(g \bullet x) \star (g \bullet y) = g \bullet (x \star y)$, so descends to a product on $A^*_{\text{orb}}([M/G])$, which [11, 8] is equivalent to the other definitions of the orbifold intersection product.

When restricted to $A^*(M,G)^G$, it is also commutative; this is immediate from the symmetry of Abramovich-Graber-Vistoli's definition, but in the stringy Chow ring there is a stronger statement that the invariant subring is central:

Lemma 3.4. *If $x \in A^*(M,G)^G$ and $y \in A^*(M,G)$, $x \star y = y \star x$. In particular, the ring $A^*_{orb}([M/G])$ is commutative.*

The proof is the same as that of [5, Thm 1.29], using Lemma 3.3.

4 The Hilbert Scheme and the symmetric product

4.1 The symmetric group

Let S_n be the symmetric group on n letters $[n] := \{1, \dots, n\}$. Each element of S_n has a disjoint cycle decomposition whose cycle type is represented by a partition $\lambda = (\lambda_1, \dots, \lambda_r) \vdash n$ which also describes the lengths of its orbits. If $\lambda = (1^{m_1} 2^{m_2} 3^{m_3} \dots)$, the symmetric group of lambda is defined as

$$S_\lambda = S_{m_1} \times S_{m_2} \times S_{m_3} \times \dots.$$

We can label the orbits of g by $1, \dots, r$ such that λ_j is the length of orbit j, and S_λ acts simply transitively on the set of such labellings (or on the orbits if a labelling is fixed).

Let $S_{n,\lambda} \subset S_n$ be the conjugacy class of permutations with cycle type λ, and $C_\lambda := |S_{n,\lambda}|$ be their number. We will often abbreviate $g \in S_{n,\lambda}$ as $g \in \lambda$.

Let $Z(g) = \{h \in S_n \mid hgh^{-1} = g\}$ be the centralizer of g, and $Z_\lambda = |Z(g_\lambda)|$ be the order of the centralizer of any $g_\lambda \in \lambda$. $Z(g)$ is isomorphic to the semidirect product $(\mathbb{Z}/(\lambda_1) \times \dots \times \mathbb{Z}/(\lambda_r)) \rtimes S_\lambda$, where the normal subgroup $\mathbb{Z}/(\lambda_1) \times \dots \times \mathbb{Z}/(\lambda_r)$ is generated by each cycle of the disjoint cycle decomposition, so

$$Z_\lambda = \prod_{j \geq 1} j^{m_j} m_j!, \quad C_\lambda = \frac{n!}{Z_\lambda} = \frac{n!}{\prod_{j \geq 1} j^{m_j} m_j!}$$

4.2 The symmetric product orbifold

Now consider the orbifold $[X^n/S_n]$, where S_n acts on X^n by permuting the factors.

The age of an element $g \in S_n$ is given by $a(g) = n - |S_n/\langle g \rangle| = l(g)$, the length of the shortest expression of g as a product of transpositions. We also define the age of a partition $\lambda \in \mathcal{P}(n,d)$ to be $n - d$, the age of a permutation of cycle type λ.

The coarse moduli space of $[X^n/S_n]$ is the symmetric product $X^{(n)} = X^n/S_n$. This quotient variety is singular, with a resolution given by the Hilbert-Chow morphism

$$\rho : X^{[n]} \to X^{(n)}$$

which takes a length-n subscheme $Z \in X$ to the collection $\sum_{x \in Z} l(\mathcal{O}_{Z,x})$ of its points counted with multiplicity. This is a crepant (or equivalently hyper-Kähler) resolution, meaning it preserves the canonical bundle: $\rho^* \omega_{X^{(n)}} = \omega_{X^{[n]}}$.

As discussed in the introduction, the motivic hyper-Kähler resolution conjecture [8] posits an isomorphism of algebra objects $\mathfrak{h}\left(X^{[n]}\right) \cong \mathfrak{h}^*_{\text{orb}}\left(X^{(n)}\right)$ in the category of complex Chow motives, which we shall go on to prove in this case.

Remark 4.1. In the remainder of this thesis we shall slightly abuse notation by referring to the orbifold $[X^n/S_n]$ as $X^{(n)}$; there should be no ambiguity as we will not need to refer to the quotient variety again.

We now examine the structure of $A^*_{\rm orb}(X^{(n)})$. The following lemma was proven in [5, Thm 3.8] but we shall prove it directly using the construction (13).

Lemma 4.2. *If $g, h \in S_n$, the top Chern class of $F_{g,h}$ is*

$$c_{top}(F_{g,h}) = \prod_{o \in O(g,h)} \pi_o^*(c_2(X)^{\mathrm{gr}(h,l)(o)}) \tag{14}$$

where $O(g,h)$ is the set of orbits of $\langle g, h\rangle$, $\pi_o : (X^n)^{\langle g,h\rangle} \cong X^{O(g,h)} \to X$ is the projection to the o'th component, and $\mathrm{gr}(h,l) : O(g,h) \to \mathbb{N}$ is the graph defect function [13, 2.7]

$$\mathrm{gr}(h,l)(o) = \frac{1}{2}(|o| + 2 - |o/\langle l\rangle| - |o/\langle h\rangle| - |o/\langle lh\rangle|)$$

Proof. We can decompose $X^n = \prod_{o\in O(g,h)} X^{|o|}$ with H acting separately on each factor, and the tangent bundle decomposes as

$$TX^n = \bigoplus_{o\in O(g,h)} \bar{\pi}_o^* TX^{|o|},$$

where $\bar{\pi}_o : X^n \to X^{|o|}$ is the projection. Then every term in (13) splits this way so we have

$$F_{g,h} = \bigoplus_{o\in O(g,h)} \bar{\pi}_o^* F_{o;g,h},$$

where $F_{o;g,h}$ is (13) for $M = X^{|o|}$. Hence we may consider each orbit seperately, regarding H as a subgroup of $S_{|o|}$ with a single orbit on $\{1, \dots, |o|\}$ acting on $X^{|o|}$ for each $o \in O(g,h)$, and have to prove that $c_{\rm top}(F_{o;g,h}) = c_2(X)^{\mathrm{gr}(h,l)(o)}$. Then $M^H = \Delta$ is the diagonal in X^n.

If g has cycle type $\lambda = (\lambda_1, \dots, \lambda_r)$, then $(X^n)^g \cong X^r$ with

$$TX^n|_{(X^n)^g} \cong \bigoplus_{j=1}^r \pi_j^*(TX^{\lambda_j}|_{\Delta_j})$$

where $\Delta_j \cong X$ is the small diagonal in X^{λ_j} and π_j are the projections onto the factors $(X^n)^g \cong X^r \to X$. Each TX^{λ_j} decomposes into eigenbundles isomorphic to $T\Delta_i$ having eigenvalues $e^{\sqrt{-1}\pi k/\lambda_j}$ for $k = 0, \dots, \lambda_j - 1$. Restricting to $(X^n)^H = \Delta \cong X$, which in the above description is the small diagonal in X^r, we get

$$S_g|_\Delta = \sum_{i=1}^{j} \sum_{k=0}^{\lambda_j - 1} \frac{k}{\lambda_j} T\Delta$$

But $\sum_{k=0}^{\lambda_j-1} k = \frac{1}{2}\lambda_j(\lambda_j - 1)$, so the coefficient is $\frac{1}{2}\sum_{i=1}^{j}(\lambda_j - 1) = \frac{1}{2}l(g) = \frac{1}{2}(n - |O(g)|)$. Since $TX^n_\Delta \cong T\Delta^{\bigoplus n}$, we get

$$F_{g,h} = \left(\frac{1}{2}(n - |O(g)| + n - |O(h)| + n - |O((gh)^{-1})|) + 1 - n\right) T\Delta = \mathrm{gr}(g,h)(o) \cdot T\Delta$$

in $K(\Delta)$ (using $O((gh)^{-1}) = O(gh)$), which proves the claim. □

If g is of cycle type $\lambda = (\lambda_1, \dots, \lambda_r)$, and we label the orbits of g with $1, \dots, r$ according to their lengths, then $(X^n)^g \cong X^r$. Define $X^\lambda := X^r$, and we use the notation $(X^n)^g \cong_\lambda X^\lambda$ to mean an isomorphism determined by such a labelling.

A choice of $g \in \nu$ and $(X^n)^g \cong_\nu X^\nu$ determines an isomorphism of quotient varieties

$$\left(\coprod_{h\in\nu} (X^n)^h\right) / S_n \cong X^\nu / S_\nu =: X^{(\nu)},$$

so choosing representatives of each cycle type gives an isomorphism

$$\left(\bigoplus_{h\in\nu} A^*((X^n)^h)\right)^{S_n} \cong A^*(X^\nu)^{S_\nu} \cong A^*(X^{(\nu)}),$$

hence

$$A^*_{\text{orb}}\left([X^n/S_n]\right) \cong \bigoplus_{\nu\vdash n} A^*\left(X^\nu\right)^{S_\nu} \cong A^*\Big(\coprod_{\nu\vdash n} X^{(\nu)}\Big) \tag{15}$$

as shown in [8, Lemma 5.3].

4.3 The additive isomorphism

Following [7, 4], for a partition ν of n consider correspondences

$$\Gamma_\nu = \left(X^{[n]} \times_{X^{(n)}} X^\nu\right)_{\text{red}} = \{(Z, x_1, \ldots, x_k) \mid \rho(Z) = \nu_1 x_1 + \cdots + \nu_k x_k\}$$

This correspondence is equal to the Nakajima correspondence $\mathfrak{q}_\nu \cdot v \in A^*\left(X^{[n]} \times X^\nu\right)$, see [19, Thm 2.4].

For any $g \in S_n$ of cycle type ν, we denote $\Gamma_g \in A^*\left(X^{[n]} \times (X^n)^g\right)$ for the image of Γ_ν under any identification $(X^n)^g \cong_\nu X^\nu$ (Γ_ν is symmetric under S_ν so the choice of identification doesn't matter).

Define the morphism

$$\Gamma = \sum_{g\in G} \frac{1}{\sqrt{-1}^{a(g)}} \Gamma_g : A^*\left(X^{[n]}\right) \to \bigoplus A^{*-a(g)}\left((X^n)^g\right) = A^*\left(X^n, S_n\right).$$

The factor of $\sqrt{-1}^{-a(g)}$ is required to make signs compatible; one could remove it and define the morphism for rational coefficients by adding a sign change in the definition of $\star_{orb}$.

Let $\iota : A^*_{\text{orb}}\left([X^n/S_n]\right) \to A^*\left(X^n, S_n\right)$ and $p : A^*\left(X^n, S_n\right) \to A^*_{\text{orb}}\left([X^n/S_n]\right)$ be the inclusion of and projection onto the S_n-invariant component; in particular p is the symmetrizer

$$p(x) = \operatorname{Sym}(x) := \frac{1}{n!} \sum_{g\in S_n} g \bullet x.$$

Then

Theorem 4.3 ([7, 4.2])**.**

$$\Phi = p \circ \Gamma : A^*\big(X^{[n]}\big) \to A^*_{orb}\big(X^{(n)}\big)$$
$$\Psi = \frac{1}{n!}\Gamma^t \circ \iota : A^*_{orb}\big(X^{(n)}\big) \to A^*\big(X^{[n]}\big)$$

are a pair of inverse isomorphisms of vector spaces. Moreover, they are isomorphisms in the category of complex Chow motives

$$\mathfrak{h}\big(X^{[n]}\big) \cong \mathfrak{h}^*_{orb}\big(X^{(n)}\big)$$

Using (15), this reduces to the main result [3, Thm. 5.4.1] of De Cataldo and Migliorini, which provides an isomorphism

$$\bigoplus_{\nu\vdash n} A^*(X^{(\nu)}) \cong A^*(X^{[n]})$$

using the quotient of the correspondences Γ_ν by S_ν.

In particular, if $g \in \lambda$ and $\alpha_g \in A^*((X^n)^g) \subset A^*(X^n, S_n)$ in the is the image of $\alpha \in A^*(X^\lambda)$ under any choice of $X^\lambda \cong_\lambda (X^n)^g$, then we have

$$\Gamma_g^t\left(\alpha_g\right) = \mathfrak{q}_\lambda\left(\alpha\right) v$$
$$\sum_{h \in S_n} (-1)^{a(g)}\, \Gamma_g\left(\mathfrak{q}_\lambda\left(\alpha\right) v\right) = n!\, \mathrm{Sym}\left(\alpha_g\right)$$

If $\alpha \in A^*(X^\lambda)^{S_\lambda}$, then for each $h \in \lambda$ there is a unique $\alpha_h \in A^*((X^n)^h)$ coming from any choice of $X^\lambda \cong_\lambda (X^n)^h$, with

$$\mathrm{Sym}(\alpha_g) = \frac{1}{C_\lambda} \sum_{h \in \lambda} \alpha_h,$$

and thus

$$\Phi(\mathfrak{q}_\lambda(\alpha) v) = \sqrt{-1}^{a(g)} Z_\lambda \sum_{h \in \lambda} \alpha_h. \tag{16}$$

Remark 4.4. The morphism $\Psi = \frac{1}{n!}\Gamma^t$ is defined on all of $A^*(X^n, S_n)$. By the symmetry of Γ, it satisfies

$$\Psi(x) = \Psi(\mathrm{Sym}(x))$$

for any $x \in A^*(X^n, S_n)$.

Similarly, the projection p is in the definition of Φ for purely formal reasons; the image of Γ is already contained in the invariant subring of $A^*(X^n, S_n)$.

A closely related consequence of [3] is the the existence of a basis of $A^*(X^{[n]})$ using Nakajima operators:

Theorem 4.5 ([19, Theorem 2.4])**.** *There is a decomposition*

$$A^*(X^{[n]}) = \bigoplus_{\lambda \vdash n} \mathfrak{q}_\lambda\left(A^*(X^\lambda)^{S_\lambda}\right)$$

i.e. a basis $\{\mathfrak{q}_\lambda(\alpha)\}_{\lambda \vdash n}$ *where* α *ranges over a basis of the symmetric part* $A^*(X^\lambda)^{S_\lambda}$ *of* $A^*(X^\lambda)$

The main result of this thesis is the folowing theorem

Theorem 4.6 ($\implies$ Theorem 1.1)**.** *The maps* Φ, Ψ *defined above are inverse isomorphisms of rings*

$$A^*\left(X^{[n]}\right) \cong A^*_{orb}\left(X^{(n)}\right)$$

Using this we can compute the intersection products of the Nakajima basis $\mathfrak{q}_\nu(\alpha)v \cdot \mathfrak{q}_\lambda(\beta)v$. Indeed, assuming $\alpha \in A^*(X^\lambda)^{S_\lambda}$ and $\beta \in A^*(X^\nu)^{S_\nu}$ are already symmetric,

$$\begin{aligned}
\mathfrak{q}_\nu(\alpha)v \cdot \mathfrak{q}_\lambda(\beta)v &= \Psi(\Phi(\mathfrak{q}_\lambda(\alpha)v) \star \Phi(\mathfrak{q}_\nu(\beta)v)) \\
&= \Psi\left(\frac{(-1)^{-\frac{3}{2}(a(\lambda)+a(\nu))}(n!)^2}{C_\nu C_\lambda} \sum_{\substack{g \in \nu \\ h \in \lambda}} \iota_*^{g,h}\left(\alpha_h|_{X^{g,h}} \cdot \beta_h|_{X^{g,h}} \cdot \prod_{o \in O(g,h)} \pi_o^*(e)^{\mathrm{gr}(g,h)(o)}\right)\right)
\end{aligned}$$

$$\mathfrak{q}_\nu(\alpha)v \cdot \mathfrak{q}_\lambda(\beta)v = \frac{n!}{C_\nu C_\lambda} \sum_{\substack{g \in \nu \\ h \in \lambda}} (-1)^{\frac{1}{2}(a(g,h))} \mathfrak{q}_{\kappa_{gh}}\left(\iota_*^{g,h}\left(\alpha_g|_{X^{g,h}} \cdot \beta_h|_{X^{g,h}} \cdot \prod_{o \in O(g,h)} \pi_o^*(e)^{\mathrm{gr}(g,h)(o)}\right)\right) v \tag{17}$$

where we abbreviate $a(g,h) := a(g) + a(h) - a(g,h)$ (Lehn-Sorger call this quantity the degree defect), $e = 24c$ is the Euler class of X, $\iota^{h,l} : X^{h,l} \hookrightarrow X^{hl}$ is the inclusion and κ_{hl} is the cycle type of gh. We can fix a certain $g \in \nu$ to simplify

$$\mathfrak{q}_\nu(\alpha)v \cdot \mathfrak{q}_\lambda(\beta)v = \frac{n!}{C_\lambda} \sum_{h \in \lambda} (-1)^{\frac{1}{2}(a(g,h))} \mathfrak{q}_{\kappa_{gh}}\left(\iota_*^{g,h}\left(\alpha_g|_{X^{g,h}} \cdot \beta_h|_{X^{g,h}} \cdot \prod_{o \in O(g,h)} \pi_o^*(e)^{\mathrm{gr}(g,h)(o)}\right)\right) v \tag{18}$$

The proof of Theorem 4.6 follows the method Lehn and Sorger [13] used to prove a cohomological version. We prove that

$$\Phi(x) \star \Phi(y) = \Phi(x \cdot y) \tag{$*$}$$

by the following reduction steps:

- Prove $(*)$ in the case $x = c_1(\mathcal{O}^{[n]})$
- Develop the notion of orbifold correspondences, letting us consider elements and operators of orbifold Chow rings parametrized by $A^*(X^s)$, and transfer operators from the Hilbert scheme side to the orbifold side
- Reduce to the case that $y = \mathfrak{q}_1 z \in A^*_{\text{orb}}(X^{[n]} \times X^s)$
- Prove $(*)$ in the case $x = c(\mathcal{O}^{[n]})$ (and hence $x = \mathrm{ch}_d(\mathcal{O}^{[n]})$) by using the commutator $[c(\mathcal{O}^{[*]}), \mathfrak{q}_1]$ to reduce to $c_1(\mathcal{O}^{[n]})$
- Prove that the operators $\underline{\mathfrak{G}}_d$ transfer properly from the Hilbert scheme to the orbifold side by looking at their commutator with $\mathfrak{q}_1$ and using the transfer property to reduce to $\underline{\mathfrak{G}}_d(1)$
- Prove the case $x = \underline{\mathfrak{G}}_{d_1} \dots \underline{\mathfrak{G}}_{d_s}(\alpha)$, which gives the claim for all $x \in A^*(X^{[n]})$.

5 Properties of tautological Chern classes and Chern character operators

5.1 Chern classes and their commutators

In this section we obtain some properties of Chern classes and characters of the tautological sheaf and multiplication by them, in order to prove the isomorphisms Φ and Ψ are multiplicative for $c_1(\mathcal{O}^{[n]})$.Let X be any smooth projective surface. The work of Lehn [12] gives a description of the Chern classes of tautological sheaves on $X^{[n]}$ in terms of Nakajima operators.

Proposition 5.1. *We have the following commutators*

1. $c\left(\mathcal{O}^{[n+1]}\right)\mathfrak{q}_1 - \mathfrak{q}_1 c\left(\mathcal{O}^{[n]}\right) = [\partial, \mathfrak{q}_1] c\left(\mathcal{O}^{[n]}\right)$
2. $[\underline{\mathfrak{G}}, q_1] = \Delta_*(\exp(\operatorname{ad}\partial) q_1)$

Proof. This proposition and proof is a modification of Lehn's result in cohomology [12, Thm 4.2], with the first claim being more specific and the second more general as required in this context. Consider the diagram (2) used to define the operator $\mathfrak{q}_1$:

$$\begin{array}{ccccc} X^{[n]} & \xleftarrow{\phi} & X^{[n,n+1]} & \xrightarrow{\psi} & X^{[n+1]} \\ & & \big\downarrow{\scriptstyle\sigma} & & \\ & & X & & \end{array}$$

where $X^{[n,n+1]}$ is the nested Hilbert scheme (1). By previous results of Lehn's paper, there is an exact sequence on $X^{[n,n+1]} \times X$

$$0 \to \sigma_X^* \mathcal{O}_{\Delta_X} \otimes \pi_1^* \Lambda \to \psi_X^*(\mathcal{O}_{\Xi_{n+1}}) \to \phi_X^*(\mathcal{O}_{\Xi_n}) \to 0 \tag{19}$$

where $\Lambda = \mathcal{O}_{X^{[n,n+1]}}(-E)$ is a line bundle defined by the exceptional divisor E of $X^{[n,n+1]}$ considered as a blow-up of $X^{[n]} \times X$, but its identity is irrelevant aside from being a line bundle. Let $\lambda = c_1(\Lambda)$.

1. The exact sequence (19) pushes forward to

$$0 \to \Lambda \to \psi^*(\mathcal{O}^{[n+1]}) \to \phi^*(\mathcal{O}^{[n]}) \to 0$$

so $\psi^* c(\mathcal{O}^{[n+1]}) = \phi^* c(\mathcal{O}^{[n]})(1+\lambda)$. Then

$$\begin{aligned} c(\mathcal{O}^{[n+1]})q_1 x &= \pi^*_{X^{[n+1]}}(c(\mathcal{O}^{[n+1]}))(\psi\times\sigma)_*(\phi^*(x)) \\ &= (\psi\times\sigma)_*(\psi^*(c(\mathcal{O}^{[n+1]}))\phi^*(x)) \\ &= (\psi\times\sigma)_*((1+\lambda)\phi^*(c(\mathcal{O}^{[n+1]})x)) \\ &= q_1 c(\mathcal{O}^{[n]})x + (\psi\times\sigma)_*(\lambda\phi^*(c(\mathcal{O}^{[n+1]})x)). \end{aligned}$$

If we look at the component with degree 1 specifically we find

$$\begin{aligned} [\partial, q_1]x = c_1(\mathcal{O}^{[n+1]})q_1 x - q_1 c_1(\mathcal{O}^{[n]})x &= (\psi\times\sigma)_*(\lambda\phi^*(c_0(\mathcal{O}^{[n+1]})x)) \\ &= (\psi\times\sigma)_*(\lambda\phi^*(x)), \end{aligned}$$

hence the operator $x \mapsto (\psi\times\sigma)_*(\lambda\phi^*(x))$ is just $[\partial, q_1]$. We obtain

$$c(\mathcal{O}^{[n+1]})q_1 x - q_1 c(\mathcal{O}^{[n]})x = [\partial, q_1]c(\mathcal{O}^{[n+1]})x$$

as required.

2. Now consider the Chern characters of (19)

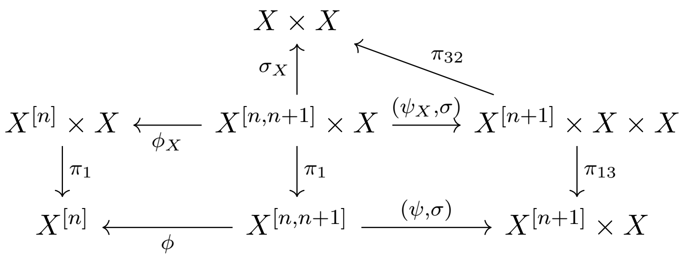
$$\psi_X^* \operatorname{ch}(\mathcal{O}_{\Xi_{n+1}}) - \psi_X^* \operatorname{ch}(\mathcal{O}_{\Xi_{n+1}}) = \sigma_X^* \operatorname{ch}(\mathcal{O}_\Delta)\pi_1^* \exp(\lambda) \tag{20}$$

There are two morphisms

$$(\psi,\sigma)_X, (\psi_X, \sigma) : X^{[n,n+1]} \times X \to X^{[n+1]} \times X \times X$$

where $\sigma : X^{[n,n+1]} \to X$ goes to the second and third factor respectively. They are related by $(\psi,\sigma)_X = (\mathrm{id}_{X^{[n+1]}} \times \mathrm{tw})(\psi_X, \sigma)$ where $\mathrm{tw} : X\times X \to X \times X$ exchanges the factors.

There is a commutative diagram with cartesian squares



Then we have

$$\begin{aligned} \underline{\mathfrak{G}}\mathfrak{q}_1 x &= \pi_{12}^*(\operatorname{ch}(\mathcal{O}_{\Xi_{n+1}}))\pi_2^*(\operatorname{td} X)\cdot \pi_{13}^*(\psi,\sigma)_*(\phi^*(x)) \\ &= \pi_{12}^*(\operatorname{ch}(\mathcal{O}_{\Xi_{n+1}}))\pi_2^*(\operatorname{td} X)\cdot (\psi_X,\sigma)_*(\pi_1^*\phi^*(x)) \\ &= (\psi_X,\sigma)_*((\psi_X,\sigma)^*(\pi_{12}^*\operatorname{ch}(\mathcal{O}_{\Xi_{n+1}})\pi_2^*(\operatorname{td} X))\cdot \pi_1^*\phi^*(x)) \\ &= (\psi_X,\sigma)_*(\psi_X^*(\operatorname{ch}(\mathcal{O}_{\Xi_{n+1}}))\pi_2^*(\operatorname{td} X)\pi_1^*\phi^*(x)). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \mathfrak{q}_1\underline{\mathfrak{G}}x &= ((\psi,\sigma)_X)_*((\phi_X)^*(\operatorname{ch}(\mathcal{O}_{\Xi_n})\pi_2^*(\operatorname{td} X)\bar{\pi}_1^* x)) \\ &= ((\psi,\sigma)_X)_*(\phi_X^*(\operatorname{ch}(\mathcal{O}_{\Xi_n}))\pi_2^*(\operatorname{td} X)\pi_1^*\phi^* x). \end{aligned}$$

Taking the commutator $[\underline{\mathfrak{G}}, \mathfrak{q}_1]x = \underline{\mathfrak{G}}\mathfrak{q}_1 x - (\mathfrak{q}_1\underline{\mathfrak{G}})^{21}x$, observe that after applying the twist the expressions above only differ by replacing $\phi_X^*(\operatorname{ch}(\mathcal{O}_{\Xi_n}))$ with $\psi_X^*(\operatorname{ch}(\mathcal{O}_{\Xi_{n+1}}))$, so applying (20) we get

$$\begin{aligned} \underline{\mathfrak{G}}\mathfrak{q}_1 x - (\mathfrak{q}_1\underline{\mathfrak{G}})^{21}x &= (\psi_X,\sigma)_*(\sigma_X^*\operatorname{ch}(\mathcal{O}_\Delta)\pi_1^*\exp(\lambda)\pi_2^*(\operatorname{td} X)\pi_1^*\phi^* x) \\ &= (\psi_X,\sigma)_*(\sigma_X^*(\operatorname{ch}(\mathcal{O}_\Delta)\pi_2^*(\operatorname{td} X))\pi_1^*(\exp(\lambda)\phi^* x)) \\ &= \pi_{32}^*(\operatorname{ch}(\mathcal{O}_\Delta)\pi_2^*(\operatorname{td} X))\cdot(\psi_X,\sigma)_*(\pi_1^*(\exp(\lambda)\phi^* x)) \\ &= \pi_{32}^*(\operatorname{ch}(\mathcal{O}_\Delta)\pi_2^*(\operatorname{td} X))\cdot \pi_{13}^*(\psi,\sigma)_*(\exp(\lambda)\phi^* x) \end{aligned}$$

Applying the Grothendieck-Riemann-Roch theorem to the diagonal embedding $\Delta : X \to X \times X$, we find $\mathrm{ch}(\mathcal{O}_\Delta)\pi_1^* \operatorname{td}(X) = \Delta_*(X) = \Delta$, so

$$[\underline{\mathfrak{G}}, q_1] = \pi_{2,3}^*\Delta \cdot \pi_{13}^*(\exp(\operatorname{ad}\partial)q_1) = \Delta_*(\exp(\operatorname{ad}\partial)q_1)$$

□

From the second part of this lemma, we obtain what Li, Qin and Wang called the transfer property:

Corollary 5.2.

$$[\underline{\mathfrak{G}}_d, \mathfrak{q}_1] = \Delta_*[\underline{\mathfrak{G}}_d(1), \mathfrak{q}_1]$$

Proof. If $x \in A^*(X^{[n]})$, we have

$$\Delta_*[\underline{\mathfrak{G}}_d(1), \mathfrak{q}_1]x = \Delta_*(\pi_{[n]2,*}(\Delta_*(\exp(\operatorname{ad}\partial)q_1))) = \Delta_*(\exp(\operatorname{ad}\partial)q_1)$$

since the following diagram with the diagonal embedding and the projection commutes, which remains true after taking the product with $X^{[n]}$.

$$\begin{array}{ccc} X & \xrightarrow{\Delta} & X \times X \\ & \searrow^{\mathrm{id}} & \downarrow \pi_2 \\ & & X \end{array}$$

□

Next we verify in Chow Lehn's description of the total Chern class $c\left(\mathcal{O}^{[n]}\right)$ of the tautological sheaf.

Proposition 5.3 ([12, Thm 4.6])**.**

$$\sum_{n\geq 0} c\big(\mathcal{O}^{[n]}\big) = \exp\left(\sum_{k\geq 1} \frac{(-1)^{k-1}}{k}\mathfrak{q}_k(1)\right) v$$

Proof. Let $\mathfrak{c}(1)$ be the operator of multiplication by $c(\mathcal{O}^{[n]})$ on $A^*(X^{[n]})$, and $\mathfrak{C}(1) = \mathfrak{c}(1)\mathfrak{q}_1(1)\mathfrak{c}(1)^{-1}$. By the previous proposition, we have $\mathfrak{C}(1) = \mathfrak{q}_1(1) + \mathfrak{q}_1'(1)$ where $\mathfrak{q}' := [\partial, \mathfrak{q}_1]$. The result follows formally from the relations

$$[\mathfrak{q}_1'(1), \mathfrak{q}_k] = -k\mathfrak{q}_{k+1} \tag{21}$$
$$\mathfrak{q}_1'(1)v = 0 \tag{22}$$

(which are immediate consequences of (8) and (7)) by Lehn's proof of a more general version of the statement in cohomology, which we reproduce in this context.

$$\begin{aligned}\sum_{n\geq 0} c\big(\mathcal{O}^{[n]}\big) &= \mathfrak{c}(1)\sum_{n\geq 0} 1_{X^{[n]}} \\ &= \mathfrak{c}(1)\exp(\mathfrak{q}_1(1))v \\ &= \mathfrak{c}(1)\exp(\mathfrak{q}_1(1))\mathfrak{c}(1)^{-1}v \\ &= \exp(\mathfrak{c}(1)\mathfrak{q}_1(1)\mathfrak{c}(1)^{-1})v \\ &= \exp(\mathfrak{C}(1))v\end{aligned}$$

To compute

$$\frac{1}{n!}(\mathfrak{q}_1(1) + \mathfrak{q}_1'(1))^n v$$

we use the relation $[\mathfrak{q}_1'(1), \mathfrak{q}_k] = -k\mathfrak{q}_{k+1}$ to commute the $\mathfrak{q}_1'(1)$ to the right and apply $\mathfrak{q}_1'v = 0$, obtaining terms of the form

$$(\nu_1 - 1)! \ldots (\nu_s - 1)!(-1)^{\nu_1 - 1}\mathfrak{q}_{\nu_1}(1) \ldots (-1)^{\nu_s - 1}\mathfrak{q}_{\nu_s}(1) = \alpha! \prod_{i\geq 1}\left(\frac{(-1)^{i-1}\mathfrak{q}_i(1)}{i}\right)^{\alpha_i}$$

where $\alpha = (1^{\alpha_1}2^{\alpha_2}\dots) = (\nu_1, \nu_2, \dots)$ is some partition of n, with $\alpha! = \nu_1!\nu_2!\dots$. Such a term arises from commuting $\nu_j - 1$ operators $\mathfrak{q}_1'(1)$ from the left with one $\mathfrak{q}_1(1)$ for each ν_j, so the number of terms with a given partition is the number of ways to arrange ν_j identical items for each ν_j (with the rightmost one being taken as $\mathfrak{q}_1(1)$ and the others as $\mathfrak{q}_1'(1)$) into a string of length n. This is just the number of ways to partition a set of n elements according to the partition α, which is given by

$$\frac{1}{\alpha_1!\alpha_2!\dots}\frac{n!}{\alpha!}.$$

Hence

$$\begin{aligned}\exp(\mathfrak{C}(1))v &= \sum_{\alpha}\prod_{i\geq 1}\frac{1}{\alpha_i!}\left(\frac{(-1)^{i-1}\mathfrak{q}_i(1)}{i}\right)^{\alpha_i} v \\ &= \prod_{i\geq 1}\sum_{\alpha_i\geq 0}\frac{1}{\alpha_i!}\left(\frac{(-1)^{i-1}\mathfrak{q}_i(1)}{i}\right)^{\alpha_i} v \\ &= \prod_{i\geq 1}\exp\left(\frac{(-1)^{i-1}\mathfrak{q}_i(1)}{i}\right) v \\ &= \exp\left(\sum_{i\geq 1}\frac{(-1)^{i-1}\mathfrak{q}_i(1)}{i}\right) v.\end{aligned}$$

□

5.2 Chern classes in the orbifold Chow ring

Remark 5.4. While the results of the previous subsection hold for any smooth projective surface X, henceforth we shall require that $c_1(X) = 0$.

The next step is to determine the image of $c(\mathcal{O}^{[n]})$ in $A^*_{\text{orb}}(X^{(n)})$ and then prove our isomorphism respects multiplication by it, starting with the first Chern class.

Define the element

$$\varepsilon^n = \sum_{g\in S_n}\sqrt{-1}^{a(g)}\,\text{sgn}\,(g)\cdot g \in A^*_{\text{orb}}\left(X^{(n)}\right).$$

Using the previous description of $c(\mathcal{O}^{[n]})$, we check that

Proposition 5.5 ([13, Prop. 4.3])**.**

$$\Psi\left(\varepsilon^n\right) = c(\mathcal{O}^{[n]})$$

Proof. The proof is identical to that of Lehn-Sorger except for the factors of $\sqrt{-1}^{a(g)}$ which cancel out; we reproduce it here for completeness.

Recall that the number of permutations $g \in S_n$ of cycle type $\lambda = (\lambda_1, \dots, \lambda_s) = (1^{m_1}2^{m_2}\dots)$ is given by

$$C_\lambda = \frac{n!}{\prod_{k\geq 1} k^{m_k} m_k!},$$

which have sign $\operatorname{sgn}(\lambda) := \operatorname{sgn}(g) = (-1)^{\sum_{j=1}^{s}(\lambda_j - 1)}$. Thus

$$\begin{aligned}
\Psi\left(\varepsilon^n\right) &= \frac{1}{n!}\sum_{g\in G}\frac{1}{\sqrt{-1}^{a(g)}}\Gamma_g^t(\sqrt{-1}^{a(g)}\operatorname{sgn}(g)\cdot g) \\
&= \frac{1}{n!}\sum_{\lambda\vdash n} C_\lambda \operatorname{sgn}(\lambda) q_{\lambda_1}\dots q_{\lambda_s}(1)v \\
&= \sum_{\lambda\vdash n}\frac{(-1)^{\sum_{j=1}^{s}(\lambda_j-1)}}{\prod_{k\geq 1}k^{m_k}m_k!} q_{\lambda_1}(1)\dots q_{\lambda_s}(1)v \\
&= \sum_{\substack{m\in\mathbb{N}_+^{\mathbb{N}} \\ \sum_k km_k = n}} \prod_k \frac{1}{m_k!}\left(\frac{(-1)^{k-1}}{k}q_k(1)\right)^{m_k} v \\
&= \left(\exp\left(\sum_k \frac{(-1)^{k-1}}{k}q_k(1)\right)v\right)_n
\end{aligned}$$

meaning the part of $\exp\left(\sum_k \frac{(-1)^{k-1}}{k}q_k(1)\right)v \in A^*(\mathrm{Hilb})$ in the direct summand $A^*(X^{[n]})$. By Proposition 5.3, this is $c(\mathcal{O}^{[n]})$ □

To prove that $c_k\left(\mathcal{O}^{[n]}\right)$ is multiplicative, we start with $c_1\left(\mathcal{O}^{[n]}\right) = \Psi\left(\varepsilon_2^n\right)$, where ε_2^n is the degree-2 component of ε^n:

Proposition 5.6 (following [13, 4.4]). *We have*

$$\Psi\left(\varepsilon_2^n \star y\right) = c_1\left(\mathcal{O}^{[n]}\right)\Psi\left(y\right)$$

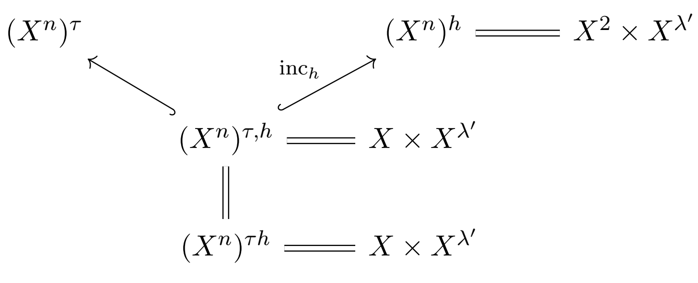
for all $y \in A^*_{orb}\left(X^{(n)}\right)$.

Proof. The proof is a small modification of Lehn-Sorger's argument to account for the fact that we cannot decompose $\mathfrak{q}_\lambda(\alpha)$ as a composition of individual $\mathfrak{q}_{\lambda_i}(\alpha_i)$, and also include the sign changes.

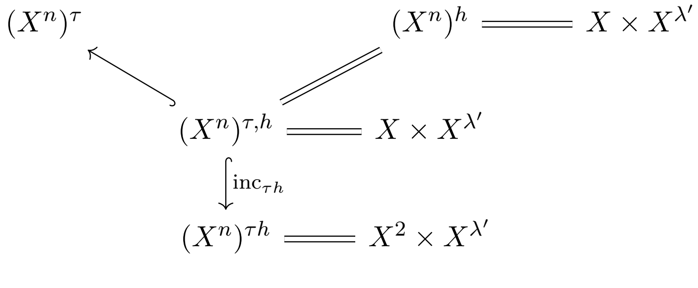
As Ψ is linear and defined on all of $A^*(X, S_n)$, we may assume without loss of generality that $y = \alpha\cdot h \in A^*(X^n, S_n)$ for some $h\in S_n, \alpha \in A^*((X^n)^g)$.

Consider multiplication of y by some transposition $\tau = (i\ j)$, considered as both an element of S_n and the element $1\cdot\tau$ Let h be of cycle type $\lambda = (\lambda_1,\dots,\lambda_s)$. There are two cases to consider: either the indices i and j appear in the same cycle of h or different cycles.

In the first case, $h = (ix_2\dots x_l)(jz_2\dots z_m)h'$ and $\tau h = (ix_2\dots x_l jz_2\dots z_m)h'$. Then $a(\tau h) = a(h)+1$ and the multiplication is given by $\tau\star y = \operatorname{inc}_h^*(\alpha)\cdot\tau h$, as the graph defect at the relevant orbit is $0 = \frac{1}{2}(l+m+2-(l+m-1)-2-1)$. As inc_h is the diagonal embedding of the first component, this can be written as $\alpha_{112\dots s-1}\cdot\tau h$.



In the second case, $h = (ix_2\dots x_l jz_2\dots z_m)h'$ and $\tau h = (ix_2\dots x_l)(jz_2\dots z_m)h'$, so $a(\tau h) = a(h)-1$. The graph defect is again 0 and the multiplication is given by $\tau\star y = \operatorname{inc}_{\tau h,*}(\alpha)\cdot\tau h = (\Delta_{12}\alpha_{13\dots s+1})\cdot\tau h$.



Considering multiplication by the whole of $\varepsilon_2^n = -\sqrt{-1}\sum_{i<j}(i\ j)$, we see that the first case occurs $\lambda_{\bar{i}}\lambda_{\bar{j}}$ times with i, j occurring in the $\bar{i}$'th and $\bar{j}$'th cycles. The second occurs $\lambda_{\bar{i}}$ times for each partition of $\lambda_{\bar{i}}$ into the ordered pair (l, m) up to double counting. Thus

$$\begin{aligned} n!\Psi(\varepsilon_2^n y) = &-\sqrt{-1}^{1-(a(h)+1)}\sum_{j<k}\lambda_j\lambda_k\mathfrak{q}_{\lambda_1}\cdots\mathfrak{q}_{\lambda_{j-1}}\mathfrak{q}_{\lambda_j+\lambda_k}\mathfrak{q}_{\lambda_{j+1}}\cdots\widehat{\mathfrak{q}_{\lambda_k}}\cdots\mathfrak{q}_{\lambda_s}(\alpha_{1\ldots j-1,j,j+1\ldots k-1,j,k\ldots s-1})v \\ &-\frac{\sqrt{-1}^{1-(a(h)-1)}}{2}\sum_j\lambda_j\sum_{l+m=\lambda_j}\mathfrak{q}_{\lambda_1}\cdots\mathfrak{q}_{\lambda_{j-1}}\mathfrak{q}_l\mathfrak{q}_{\lambda_{j+1}}\cdots\cdots\mathfrak{q}_{\lambda_s}\mathfrak{q}_m(\alpha_{1\ldots s}\Delta_{j,s+1})v \\ = &-\sqrt{-1}^{a(h)}\sum_{j<k}\lambda_j\lambda_k\mathfrak{q}_{\lambda_1}\cdots\mathfrak{q}_{\lambda_{j-1}}\mathfrak{q}_{\lambda_j+\lambda_k}\mathfrak{q}_{\lambda_{j+1}}\cdots\widehat{\mathfrak{q}_{\lambda_k}}\cdots\mathfrak{q}_{\lambda_s}(\alpha_{1\ldots j-1,j,j+1\ldots k-1,j,k\ldots s-1})v \\ &+\frac{\sqrt{-1}^{a(h)}}{2}\sum_j\lambda_j\sum_{l+m=\lambda_j}\mathfrak{q}_{\lambda_1}\cdots\mathfrak{q}_{\lambda_{j-1}}\mathfrak{q}_l\mathfrak{q}_{\lambda_{j+1}}\cdots\cdots\mathfrak{q}_{\lambda_s}\mathfrak{q}_m(\alpha_{1\ldots s}\Delta_{j,s+1})v \end{aligned}$$

On the other hand,

$$n!c_1\big(\mathcal{O}^{[n]}\big)\Psi(y) = i^{-a(h)}\partial\mathfrak{q}_{\lambda_1}\cdots\mathfrak{q}_{\lambda_s}(\alpha)v$$

so we apply the relations

$$\begin{aligned} [\partial, \mathfrak{q}_k] &= k\mathfrak{L}_k \\ [\mathfrak{L}_d, \mathfrak{q}_k] &= -k\Delta_*(q_{d+k}) \\ \partial v &= 0 \\ \mathfrak{L}_n v &= \frac{1}{2}\sum_{\substack{k+l=n \\ k,l>0}} q_k q_l|_\Delta \end{aligned}$$

Commuting ∂ to the end, we get

$$c_1\big(\mathcal{O}^{[n]}\big)\Psi\,(y) = \sqrt{-1}^{-a(h)}(\lambda_1\mathfrak{L}_{\lambda_1}\mathfrak{q}_{\lambda_2}\cdots\mathfrak{q}_{\lambda_s}(\alpha) + \cdots + \lambda_s\mathfrak{q}_{\lambda_1}\cdots\mathfrak{q}_{\lambda_{s-1}}\mathfrak{L}_{\lambda_s}(\alpha)).$$

Commuting the operators $\mathfrak{L}_{\lambda_i}$ to the end gives a term

$$-\sqrt{-1}^{-a(h)}\lambda_i\lambda_j\mathfrak{q}_{\lambda_1}\cdots\mathfrak{q}_{\lambda_{i-1}}\mathfrak{q}_{\lambda_i+\lambda_j}\mathfrak{q}_{\lambda_{i+1}}\cdots\widehat{\mathfrak{q}_{\lambda_j}}\cdots\mathfrak{q}_{\lambda_s}(\alpha_{1\ldots i-1,i,i+1\ldots j-1,i,j\ldots s-1})$$

arising from each $[\mathfrak{L}_{\lambda_i}, \mathfrak{q}_{\lambda_j}]$, and terms

$$\sqrt{-1}^{-a(h)}\frac{1}{2}\sum_i\lambda_i\sum_{l+m=\lambda_i}\mathfrak{q}_{\lambda_1}\cdots\mathfrak{q}_{\lambda_{i-1}}\mathfrak{q}_l\mathfrak{q}_{\lambda_{i+1}}\cdots\cdots\mathfrak{q}_{\lambda_s}\mathfrak{q}_m(\alpha_{1\ldots s}\Delta_{i,s+1})$$

arising from $\mathfrak{L}_{\lambda_i}\cdot v$, which proves the equality. □

Remark 5.7. In the proof of the last proposition we have seen that the graph defect of a transposition with any element is zero, which will be useful to recall later.

6 Orbifold Correspondences

6.1 Orbifold Nakajima operators

In order to proceed with the proof, we need to work with orbifold correspondences of the form

$$\alpha \in A^*_{\mathrm{orb}}(X^{(n)} \times X^s),$$

where $X^{(n)} \times X^s = [(X^n \times X^s)/S_n]$ is an orbifold with S_n acting trivially on the X^s factor. Then we can work with operators $A^*_{\mathrm{orb}}(X^{(n)}) \to A^*_{\mathrm{orb}}(X^{(m)} \times X^s)$ parametrized by X^s similar to the Nakajima and Chern character operators on the Hilbert scheme side, such that intersecting with the pullback of a class in $A^*(X^s)$ and pushing down to $A^*_{\mathrm{orb}}(X^{(m)})$ gives an operator $A^*_{\mathrm{orb}}(X^{(n)}) \to A^*_{\mathrm{orb}}(X^{(m)})$

In particular, we need to define an analogue $\mathfrak{r}_1 : A^*_{\text{orb}}(X^{(n)}) \to A^*_{\text{orb}}(X^{(n+1)} \times X)$ of the Nakajima operator $\mathfrak{q}_1$. Indeed, for any $k \geq 0$ we can define $\mathfrak{r}_k : A^*_{\text{orb}}(X^{(n)}) \to A^*_{\text{orb}}(X^{(n+k)} \times X)$ as the linear extension of

$$\mathfrak{r}_k \cdot \alpha g = \frac{1}{(n)!} \sum_{\phi:[n]\hookrightarrow[n+k]} (\alpha_{\phi(1)\ldots\phi(n)} \Delta_{\bar{\phi},n+k+1})\ \phi_* g$$

where for any injection $\phi : [n] \hookrightarrow [n+k]$, $\phi_* g \in S_{n+k}$ is the permutation that has the induced action of g on $\phi([n])$ and the trivial action on $[n+k] \setminus \phi([n])$, and $\Delta_{\bar{\phi},n+k+1}$ is the diagonal along the indices $[n+k] \setminus \phi([n])$ and $n+k+1$.

This is well defined as if $\alpha \in A^*((X^n)^g)$, then

$$\alpha_{\phi(1)\ldots\phi(n)} \Delta_{\bar{\phi},n+k+1} \in A^*((X^{n+k})^{\phi_* g} \times X),$$

and by summing over all ϕ we ensure $\mathfrak{r}_k \alpha_g$ lies in the symmetric part $A^*_{\text{orb}}(X^{(n+k)} \times X)$, so indeed defines an operator between the orbifold Chow rings. We only require the simplest version $\mathfrak{r}_1$, which can be expressed as the operator induced by the map

$$\begin{aligned} A^*(X^n, S_n) &\to A^*\Big(X^{n+1} \times X, S_{n+1}\Big) \\ \alpha g &\mapsto \sum_{i=1}^{n+1} (\alpha_{1\ldots i-1,n+1,i+1,\ldots n} \Delta_{i,n+2}) \phi_{i,*} g \\ &= \sum_{i=1}^{n+1} (i\ n+1) \bullet ((\alpha_{1\ldots\ldots n} \Delta_{n+1,n+2}) g) \end{aligned}$$

where $\bullet$ denotes the action of S_{n+1} on the stringy Chow ring and $\phi_i : [n] \hookrightarrow [n+1]$ is the increasing injection that avoids i.

Define the morphism

$$\begin{aligned} p_n^{n+1} : A^*(X^n, S_n) &\to A^*(X^{n+1}, S_{n+1}) \\ p_n^{n+1}(\alpha g) &= \pi^*_{1\ldots n}(\alpha) \iota(g) \end{aligned} \tag{23}$$

where $\pi_{1\ldots n} : (X^{n+1})^{\iota(g)} \to (X^n)^g$ is the projection $X^{n+1} \to X^n$ onto the first n factors, restricted to the g-invariant diagonals.

Then $\mathfrak{r}_1$ is given by

$$\mathfrak{r}_1(x) = \sum_{i=1}^{n+1} (i\ n+1) \bullet (\pi^*_{(n+1)} p_n^{n+1}(x) \star \Delta_{n+1,n+2}) \tag{24}$$

where we abbreviate $\Delta_{n+1,n+2} = \Delta_{n+1,n+2} \cdot 1_{S_{n+1}}$.

The map p_n^{n+1} does not preserve the invariant subring, but it does respect the products:

Lemma 6.1.

$$p_n^{n+1}(x \star y) = p_n^{n+1}(x) \star p_n^{n+1}(y)$$

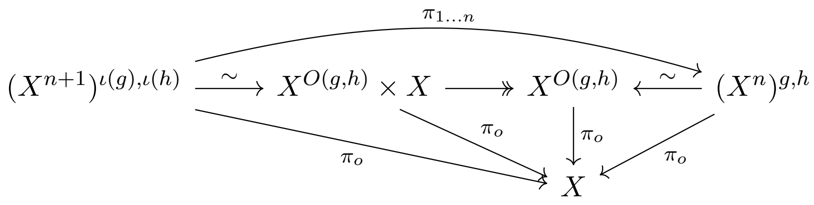
Proof. For any $h, l \in S_n$, $\iota(hl) = \iota(h)\iota(l)$ and $\iota(\langle h, l\rangle) = \langle \iota(h), \iota(l)\rangle$, so

$$(X^{n+1})^{\iota(h)} = (X^n)^h \times X, \quad (X^{n+1})^{\iota(h),\iota(l)} = (X^n)^{h,l} \times X$$

Recall that the obstruction class can be characterized as

$$c_{\text{top}}\big(F_{\iota(h),\iota(l)}\big) = \prod_{o \in O(\iota(h),\iota(l))} \pi_o^*\big(c_2(X)^{\text{gr}(\iota(h),\iota(l))(o)}\big),$$

with the orbits $O(\iota(h), \iota(l))$ consisting of the orbits $O(h, l)$ (as subsets of $\{1, \ldots, n\}$) as well as $\{n+1\}$. The graph defect of $o \in O(h, l)$ remains the same, and is zero for $\{n+1\}$). Hence $c_{\text{top}}(F_{\iota(h),\iota(l)}) = \pi_{1\ldots n}^* c_{\text{top}}(F_{h,l})$ by the commutativity of the diagram



As the diagram

$$\begin{array}{ccccc}
(X^{n+1})^{\iota(g)} & \xleftarrow{\text{inc}_{\iota(g)}} & (X^{n+1})^{\iota(g),\iota(h)} & \xrightarrow{\text{inc}_{\iota(gh)}} & (X^{n+1})^{\iota(gh)} \\
\downarrow{\scriptstyle \pi_{1\ldots n}} & & \downarrow{\scriptstyle \pi_{1\ldots n}} & & \downarrow{\scriptstyle \pi_{1\ldots n}} \\
(X^n)^g & \xleftarrow[\text{inc}_g]{} & (X^n)^{g,h} & \xrightarrow[\text{inc}_{gh}]{} & (X^n)^{gh}
\end{array}$$

is fibred (i.e. commutes with both squares being cartesian), we have

$$\begin{aligned}
p_n^{n+1}(\alpha_g \star \beta_h) &= \pi_{1\ldots n}^* \text{inc}_{gh,*} \Big(\alpha|_{X^{g,h}} \cdot \beta|_{X^{g,h}} \cdot c_{\text{top}}\left(F_{g,h}\right)\Big) \\
&= \text{inc}_{gh,*} \Big(\pi_{1\ldots n}^*(\alpha|_{X^{g,h}}) \cdot \pi_{1\ldots n}^*(\beta|_{X^{g,h}}) \cdot \pi_{1\ldots n}^*\big(c_{\text{top}}\big(F_{g,h}\big)\big)\Big) \\
&= \text{inc}_{gh,*} \Big(\pi_{1\ldots n}^*(\alpha)|_{X^{\iota(g),\iota(h)}} \cdot \pi_{1\ldots n}^*(\beta)|_{X^{\iota(g),\iota(h)}} \cdot c_{\text{top}}\big(F_{\iota(g),\iota(h)}\big)\Big) \\
&= p_n^{n+1}(\alpha_g) \star p_n^{n+1}(\beta_h)
\end{aligned}$$

□

6.2 Projection operators

We also have to work with projections of the form

$$\begin{aligned}
\pi_{(n),i_1\ldots i_t} &: X^{(n)} \times X^s \to X^{(n)} \times X^t \\
\pi_{i_1\ldots i_t} &: X^{(n)} \times X^s \to X^t
\end{aligned}$$

for $t \leq s$, and distinct $1 \leq i_1 \ldots i_t \leq s$ and the induced pullback/pushforward morphisms

$$\begin{aligned}
\pi_{(n),i_1\ldots i_t,*} &: A^*(X^{(n)} \times X^s, S_n) \to A^*(X^{(n)} \times X^t, S_n) \\
& \qquad \alpha g \mapsto \Big(\pi_{(X^n)^g,i_1\ldots i_t,*}(\alpha)\Big)\, g \\
\pi_{(n),i_1\ldots i_t}^* &: A^*(X^{(n)} \times X^t, S_n) \to A^*(X^{(n)} \times X^s, S_n) \\
& \qquad \alpha g \mapsto \Big(\pi_{(X^n)^g,i_1\ldots i_t}^*(\alpha)\Big)\, g \\
\pi_{i_1\ldots i_t}^* &: A^*(X^t) \to A^*(X^{(n)} \times X^s, S_n) \\
& \qquad \beta \mapsto \Big(\pi_{i_1,\ldots i_t}^*(\beta)\Big)\, 1_{S_n}
\end{aligned}$$

where $\pi_{(X^n)^g,i_1\ldots i_t} : (X^n)^g \times X^s \to (X^n)^g \times X^t$ is the obvious projection.

These satisfy the following important properties:

Lemma 6.2. *The pullbacks $\pi_{(n),i_1\ldots i_t}^*$ and $\pi_{i_1\ldots i_t}^*$ are ring homomorphisms*

$$\begin{aligned}
\pi_{(n),i_1\ldots i_t}^*(x) \star \pi_{(n),i_1\ldots i_t}^*(y) &= \pi_{(n),i_1\ldots i_t}^*(x \star y) \\
\pi_{i_1\ldots i_t}^* \alpha \star \pi_{i_1\ldots i_t}^* \beta &= \pi_{i_1\ldots i_t}^*(\alpha \cdot \beta),
\end{aligned}$$

and the pushforward and pullback along the map $\pi_{(n),i_1\ldots i_t}$ satisfy the projection formula

$$x \star \pi_{(n),i_1\ldots i_t,*}(y) = \pi_{(n),i_1\ldots i_t,*}(\pi_{(n),i_1\ldots i_t}^* x \star y)$$

Proof. The first two claims are straightforward:

1. If $F_{g,h}$ and $F'_{g,h}$ are the obstruction bundles on $(X^n)^{g,h} \times X^t$ and $(X^n)^{g,h} \times X^s$ respectively, then

$$F'_{g,h} = \pi^*_{(X^n)^g, i_1 \dots i_t}(F_{g,h}), \tag{25}$$

   and

2. The obstruction bundle $F_{1,1}$ is trivial.

For the third claim, let $g, h \in S_n$, $\alpha \in A^*((X^n)^g), \beta \in A^*((X^n)^g)$ and consider the product

$$\begin{aligned}
\alpha g \star \pi_{(n),i_1\dots i_t,*}(\beta h) &= \mathrm{inc}_{gh,*}\left(\alpha|_{X^{g,h}\times X^t} \cdot (\pi_{(X^n)^h, i_1\dots i_t,*}(\beta)|_{X^{g,h}\times X^t}) \cdot c_{\mathrm{top}}(F_{g,h})\right) gh \\
&= \mathrm{inc}_{gh,*}\left(\alpha|_{X^{g,h}\times X^t} \cdot \pi_{(X^n)^{g,h}, i_1\dots i_t,*}(\beta|_{X^{g,h}\times X^s}) \cdot c_{\mathrm{top}}(F_{g,h})\right) gh \\
&\qquad \text{and applying the projection formula for } \pi_{(X^n)^{g,h}, i_1\dots i_t}, \\
&= \mathrm{inc}_{gh,*}\left(\pi_{(X^n)^{g,h}, i_1\dots i_t,*}(\pi^*_{(X^n)^{g,h}, i_1\dots i_t}(\alpha|_{X^{g,h}\times X^t} \cdot c_{\mathrm{top}}(F_{g,h})) \cdot \beta|_{X^{g,h}\times X^s})\right) gh \\
&= \pi_{(X^n)^{g,h}, i_1\dots i_t,*}\left(\mathrm{inc}_{gh,*}(\pi^*_{(X^n)^{g,h}, i_1\dots i_t}(\alpha|_{X^{g,h}\times X^t} \cdot c_{\mathrm{top}}(F_{g,h})) \cdot \beta|_{X^{g,h}\times X^s})\right) gh \\
&\qquad \text{(by push-pull)} \\
&= \pi_{(n), i_1\dots i_t,*}\left(\mathrm{inc}_{gh,*}(\pi^*_{(X^n)^{g,h}, i_1\dots i_t}(\alpha)|_{X^{g,h}\times X^s} \cdot \beta|_{X^{g,h}\times X^s}) \cdot c_{\mathrm{top}}(F'_{g,h}) gh\right) \\
&\qquad \text{(by (25))} \\
&= \pi_{(n), i_1\dots i_t,*}\left(\pi^*_{(n), i_1\dots i_t}(\alpha g) \star \beta h\right)
\end{aligned}$$

□

As $\pi_{(n),i_1\dots i_t,*}$ and $\pi^*_{(n),i_1\dots i_t}$ are S_n-equivariant and the image of $\pi^*_{i_1\dots i_t}$ is S_n-invariant, they restrict to morphisms between orbifold Chow rings

$$A^*_{\mathrm{orb}}(X^{(n)} \times X^t) \underset{\pi_{(n),i_1\dots i_t,*}}{\overset{\pi^*_{(n),i_1\dots i_t}}{\rightleftarrows}} A^*_{\mathrm{orb}}(X^{(n)} \times X^s) \xleftarrow[\pi^*_{i_1\dots i_t}]{} A^*(X^t)$$

These let us safely work with classes and operators parametrized by X^s:

Definition 6.3. If $x \in A^*_{\mathrm{orb}}(X^{(n)} \times X^s)$ and $\beta \in X^s$, then

$$x(\beta) := \pi_{(n),*}\left(x \star \pi^*_{1\dots s}\beta\right) \in A^*_{\mathrm{orb}}(X^{(n)})$$

Similarly, if $\mathfrak{p} : A^*_{\mathrm{orb}}(X^{(m)}) \to A^*_{\mathrm{orb}}(X^{(n)} \times X^s)$ is an operator, then $\mathfrak{p}(\beta) : A^*_{\mathrm{orb}}(X^{(m)}) \to A^*_{\mathrm{orb}}(X^{(n)})$ is the operator

$$y \mapsto \pi_{(n),*}(\mathfrak{p}y \star \pi^*_{1\dots s}\beta)$$

6.3 Transferring correspondences

To transfer these kinds correspondence between the orbifold and Hilbert scheme sides, we define

$$\Phi_{X^s} : A^*(X^{[n]} \times X^s) \rightleftarrows A^*_{\mathrm{orb}}(X^{(n)} \times X^s) : \Psi_{X^s}$$

by replacing in the definition of Φ and Ψ the correspondences

$$\Gamma_g : A^*(X^{[n]}) \rightleftarrows A^*((X^n)^g) : \Gamma_g^t$$

with the products of correspondences

$$(\Gamma_g \times \mathrm{id}_{X^s*}) : A^*(X^{[n]} \times X^s) \rightleftarrows A^*((X^n)^g \times X^s) : (\Gamma_g^t \times \mathrm{id}_{X^s*}),$$

Then

Proposition 6.4. *Φ_{X^s} and Ψ_{X^s} define inverse isomorphisms of vector spaces.*

Proof. All the pieces of Theorem 4.3 still go through after taking the product with the correspondence $\mathrm{id}_{X^s *} = \Delta_{X^s} \in A^*(X^s \times X^s)$. In particular we still have isomorphisms

$$\Big(\coprod_{g\in\nu}(X^n)^g \times X^s\Big)/S_n \cong X^{(\nu)} \times X^s.$$

and if

$$\hat{\Gamma} : A^*\Big(\Big(\coprod_{\nu\vdash n} X^{(\nu)}\Big)\Big) \to A^*(X^{[n]})$$

is the correspondence giving the isomorphism of de Cataldo-Migliorini, then the product

$$\hat{\Gamma} \times \Delta_{X^s} : A^*\Big(\Big(\coprod_{\nu\vdash n} X^{(\nu)}\Big) \times X^s\Big) \to A^*(X^{[n]} \times X^s)$$

remains an isomorphism. This follows directly from the fact that the product of correspondences is compatible with composition:

$$(\Gamma \times \Lambda) \circ (\Gamma' \times \Lambda') = (\Gamma \circ \Gamma') \times (\Lambda \circ \Lambda'),$$

for any correspondences

$$X \xrightarrow{\Gamma'} Y \xrightarrow{\Gamma} X'$$

$$Z \xrightarrow{\Lambda'} W \xrightarrow{\Lambda} Z'$$

□

In order to transfer operator correspondences between the orbifold and Hilbert scheme side, we need the following properties

Lemma 6.5.

$$\begin{aligned}
\Phi_{X^s}\pi^*_{[n],i_1,\dots i_t} &= \pi^*_{(n),i_1,\dots i_t}\Phi_{X^t} \\
\Phi_{X^t}\pi_{[n],i_1,\dots i_t,*} &= \pi_{(n),i_1,\dots i_t,*}\Phi_{X^s} \\
\Phi_{X^s}(x \cdot \pi^*_{i_1,\dots i_t}\alpha) &= \Phi_{X^s}(x) \star \pi^*_{i_1,\dots i_t}\alpha
\end{aligned}$$

Proof. For the first two claims, we have to prove equalities of correspondences

$$\begin{aligned}
(\Gamma_g \times \mathrm{id}_{X^s *}) \circ (\mathrm{id}_{X^{[n]}*} \times \pi^*_{i_1,\dots i_t}) &= (\mathrm{id}_{X^{(n)}*} \times \pi^*_{i_1,\dots i_t}) \circ (\Gamma_g \times \mathrm{id}_{X^t *}), \\
(\Gamma_g \times \mathrm{id}_{X^t *}) \circ (\mathrm{id}_{X^{[n]}*} \times \pi_{i_1,\dots i_t,*}) &= (\mathrm{id}_{X^{(n)}*} \times \pi_{i_1,\dots i_t,*}) \circ (\Gamma_g \times \mathrm{id}_{X^s *})
\end{aligned}$$

which both follow from basic properties of correspondences: for any $\Lambda \in A^*(Y \times Z)$, then

$$(\Gamma_g \times \mathrm{id}_{Z*}) \circ (\mathrm{id}_{X^{(n)}} \times \Lambda) = \Gamma_g \times \Lambda = (\mathrm{id}_{X^{[n]}} \times \Lambda) \circ (\Gamma_g \times \mathrm{id}_{Y*})$$

For the third claim, we have $\pi^*_{i_1,\dots i_t}\alpha = (X^n \times \alpha_{i_1,\dots i_t})1_{S_n}$ by definition. Thus for any $\beta \in A^*((X^n)^g)$, we have the product

$$\beta g \star \pi^*_{i_1,\dots i_t}\alpha = (\beta \cdot ([(X^n)^g] \times \alpha_{i_1,\dots i_t}))g. \tag{26}$$

Then if

$$X^{[n]} \times X^s \xleftarrow[\psi_{X^s}]{} (X^n)^g \times X^{[n]} \times X^s \xrightarrow[\phi_{X^s}]{} (X^n)^g \times X^s$$

are the projections, we compute

$$\begin{aligned}
\Phi_{X^s}(x) \star \pi^*_{i_1,\dots i_t}\alpha &= \sum_{g\in G} \frac{1}{\sqrt{-1}^{a(g)}} \phi_{X^s,*}((\Gamma_g \times X^s)\cdot \psi^*_{X^s}x) g \star \pi^*_{i_1,\dots i_t}\alpha \\
&\quad \text{applying (26) gives} \\
&= \sum_{g\in G} \frac{1}{\sqrt{-1}^{a(g)}} \left(\phi_{X^s,*}((\Gamma_g \times X^s)\cdot \psi^*_{X^s}x)\cdot ([(X^n)^g] \times \alpha_{i_1,\dots i_t})\right) g \\
&= \sum_{g\in G} \frac{1}{\sqrt{-1}^{a(g)}} \phi_{X^s,*}((\Gamma_g \times X^s)\cdot \psi^*_{X^s}x \cdot ([(X^n)^g \times X^{[n]}] \times \alpha_{i_1,\dots i_t})) g \\
&\quad \text{(by the projection formula)} \\
&= \sum_{g\in G} \frac{1}{\sqrt{-1}^{a(g)}} \phi_{X^s,*}((\Gamma_g \times X^s)\cdot \psi^*_{X^s}(x\cdot (X^{[n]} \times \alpha_{i_1,\dots i_t}))) g \\
&= \sum_{g\in G} \frac{1}{\sqrt{-1}^{a(g)}} \phi_{X^s,*}((\Gamma_g \times X^s)\cdot \psi^*_{X^s}(x\cdot \pi_{i_1,\dots i_t}\alpha)) g \\
&= \Phi_{X^s}(x\cdot \pi^*_{i_1,\dots i_t}\alpha)
\end{aligned}$$

□

It follows from these relations that Φ_{X^s} is compatible with operator correspondences parametrized by X^s:

Corollary 6.6. *If $y \in A^*(X^{[n]} \times X^s)$ and $\beta \in A^*(X^s)$, then*

$$(\Phi_{X^s}(y))(\beta) = \Phi(y(\beta))$$

Now we verify that $\mathfrak{r}_1$ is indeed the orbifold equivalent of $\mathfrak{q}_1$, compatible with the isomorphisms Φ, Ψ.

Proposition 6.7.

$$\Psi_X \mathfrak{r}_1 = \mathfrak{q}_1 \Psi : A^*_{orb}(X^{(n)}) \to A^*\left(X^{[n+1]} \times X\right)$$

and equivalently

$$\mathfrak{r}_1 \Phi = \Phi_X \mathfrak{q}_1 : A^*(X^{[n]}) \to A^*_{orb}\left(X^{(n+1)} \times X\right)$$

Proof. We prove the first statement directly, from which the second follows by Proposition 6.4.

Let λ be a partition of n with length l, let $g \in S_n$ be of cycle type λ, and let $\alpha \in A^*((X^n)^g)$. As we have defined Ψ and $\mathfrak{r}_1$ on all of $A^*(X^n, S_n)$, we only need to prove $\Psi_X(\mathfrak{r}_1 x) = \mathfrak{q}_1\Psi(x)$ for $x = \alpha g$. Let $\alpha_\lambda \in A^*(X^\lambda)$ be the image of α under some choice of isomorphism $f : (X^n)^g \cong_\lambda X^\lambda$ so that $\Gamma^t_g(\alpha) = \mathfrak{q}_\lambda(\alpha_\lambda)v$. Hence $\mathfrak{q}_1 \cdot \Gamma^t_g(\alpha) = \mathfrak{q}_1 \cdot \mathfrak{q}_\lambda(\alpha_\lambda)v = \mathfrak{q}_\lambda(\alpha_\lambda)\mathfrak{q}_1 v$, and thus $\mathfrak{q}_1 \cdot \Psi$ is given by:

$$\mathfrak{q}_1 \cdot \Psi(\alpha g) = \frac{1}{\sqrt{-1}^{a(g)} n!} \mathfrak{q}_\lambda(\alpha_\lambda)\mathfrak{q}_1 v.$$

Meanwhile if $g_i = (i\ n+1)\iota(g)(i\ n+1)$ for $1 \le i \le n+1$, the action of $\mathfrak{r}_1$ is given by

$$\mathfrak{r}_1 \cdot \alpha g = \sum_{i=1}^{n+1} (\alpha_{1\dots i-1,n+1,i+1\dots n}\Delta_{i,n+2}) g_i.$$

To compute $\Psi_X(\mathfrak{r}_1 \cdot \alpha g)$, we first apply the correspondence $\Gamma^t_{g_i} \times \mathrm{id}_{X*}$ to the summand $(\mathfrak{r}_1 \cdot \alpha g)_{g_i}$ corresponding to g_i:

$$\begin{aligned}
\left(\Gamma^t_{g_i} \times \mathrm{id}_{X*}\right)\left((\mathfrak{r}_1 \cdot \alpha g)_{g_i}\right) &= \left(\Gamma^t_{g_i} \times \mathrm{id}_{X*}\right)(\alpha_{1\dots i-1,n+1,i+1\dots n}\Delta_{i,n+2}) \\
&= \psi_{X,*}((\Gamma_{g_i} \times X)\cdot \phi^*_X(\alpha_{1\dots i-1,n+1,i+1\dots n}\Delta_{i,n+2}))
\end{aligned}$$

Consider the isomorphism $(X^{n+1})^{g_i} \cong X^\lambda \times X$ which identifies the i'th factor with X and the rest with X^λ using the chosen isomorphism $f : (X^n)^g \cong X^\lambda$. We can describe its effect on the above expression:

$$\begin{array}{ccccccc}
\Gamma_{g_i} \times X & \subset & X^{[n]} \times (X^{n+1})^{g_i} \times X & \xrightarrow{\phi_X} & (X^{n+1})^{g_i} \times X & \supset & \alpha_{1\dots i-1,n+1,i+1\dots n}\Delta_{i,n+2} \\
\Big\downarrow & & \Big\downarrow{\scriptstyle\sim} & & \Big\downarrow{\scriptstyle\sim} & & \Big\downarrow \\
\mathfrak{q}_\lambda \mathfrak{q}_1 v \times X & \subset & X^{[n]} \times X^\lambda \times X \times X & \xrightarrow{\pi_{1\dots l+2}} & X^\lambda \times X \times X & \supset & \alpha_\lambda \times \Delta
\end{array}$$

(abusing notation on the right side by using the $\subset$ symbol to mean elements of the Chow ring).

Hence

$$\begin{aligned}
\big(\Gamma_{g_i}^t \times \mathrm{id}_{X*}\big)\big((\mathfrak{r}_1 \cdot \alpha g)_{g_i}\big) &= \pi_{X^{[n]},l+2}\big((\mathfrak{q}_\lambda \mathfrak{q}_1 v \times X) \cdot \pi^*_{1\dots l+2}(\alpha_\lambda \times \Delta)\big) \\
&= \pi_{X^{[n]},l+2}\big((\mathfrak{q}_\lambda \mathfrak{q}_1 v \times X) \cdot \pi^*_{1\dots l}(\alpha_\lambda) \cdot (X^{[n]} \times \Delta_{l+1,l+2})\big)
\end{aligned}$$

Due to the factor of $\Delta_{l+1,l+2}$, projecting onto the $l+2$ factor is equivalent to projecting onto the $l+1$ factor, with the result being

$$\big(\Gamma_{g_i}^t \times \mathrm{id}_{X*}\big)\big((\mathfrak{r}_1 \alpha g)_{g_i}\big) = \pi_{X^{[n]},l+1}(\mathfrak{q}_\lambda \mathfrak{q}_1 v \cdot \pi^*_{1\dots l}\alpha_\lambda) = \mathfrak{q}_\lambda(\alpha_\lambda)\mathfrak{q}_1 v.$$

Hence we obtain $\Psi_X(\mathfrak{r}_1 \cdot \alpha g)$:

$$\Psi_X(\mathfrak{r}_1 \cdot \alpha g) = \frac{1}{(n+1)!} \sum_{i=1}^{n+1} \frac{1}{\sqrt{-1}^{a(g_i)}} \big(\Gamma_{g_i}^t \times \mathrm{id}_{X*}\big)\big((\mathfrak{r}_1 \alpha g)_{g_i}\big) = \frac{1}{\sqrt{-1}^{a(\iota(g))} n!} \mathfrak{q}_\lambda(\alpha_\lambda)\mathfrak{q}_1 v$$

which equals $\mathfrak{q}_1 \Psi(\alpha g)$ as required. □

7 Multiplication by Chern characters

7.1 Chern classes continued

We now pick up where we left off in proving the isomorphisms Φ, Ψ are multiplicative, starting with the case of Chern classes of the tautological sheaf.

Let $\iota : S_n \to S_{n+1}$ be the inclusion induced by $[n] \hookrightarrow [n+1]$, and τ_i be the transposition $(i\ n+1)$.

Lemma 7.1. *Recall the morphism* $p_n^{n+1} : A^*(X^n, S_n) \to A^*(X^{n+1}, S_{n+1})$ (23). *We have*

$$\varepsilon^{n+1} - p_n^{n+1}(\varepsilon^n) = -\sqrt{-1} \sum_{i=1}^{n+1} \tau_i \star p_n^{n+1}(\varepsilon^n) = (\varepsilon_2^{n+1} - p_n^{n+1}(\varepsilon_2^n)) \star p_n^{n+1}(\varepsilon^n)$$

Proof. That the second and third expression are equal is immediate. We have

$$p_n^{n+1}(\varepsilon^n) = \sum_{\sigma \in S_n} \big(\sqrt{-1}^{a(g)} \operatorname{sgn}(\sigma)\big) \cdot \iota(\sigma)$$

The product $\tau_i \star \iota(\sigma)$ is just $\tau_i \cdot \iota(\sigma)$, as the graph defect $g(\tau, h)$ is always 0 when τ is a transposition and $(X^{n+1})^{\tau_i, \iota(\sigma)} = (X^{n+1})^{\tau_i \cdot \iota(\sigma)}$ so $\mathrm{inc}_{\tau_i \iota(\sigma)}$ is trivial. Indeed if i is in the disjoint cycle $(i\ i_2 \dots i_s)$ of σ, then the invariant subscheme of $\tau_i(i\ i_2 \dots i_s) = (n+1\ i\ i_2 \dots i_s)$ is the diagonal $\Delta_{ii_1 \dots i_s, n+1}$, which is intersection of the invariant subschemes $\Delta_{i,n+1}$ of τ_i and $\Delta_{ii_1 \dots i_s}$ of $(n+1\ i\ i_2 \dots i_s)$.

As $l(\tau_i \iota(\sigma)) = l(\sigma) + 1$, the RHS of the claim is

$$-\sum_{i=1}^{n} \sum_{\sigma \in S_n} \big(\sqrt{-1}^{a(g)+1} \operatorname{sgn}(\sigma)\big) \cdot (\tau_i \iota(\sigma)) = \sum_{i=1}^{n} \sum_{\sigma \in S_n} \big(\sqrt{-1}^{a(\tau_i \iota(g))} \operatorname{sgn}(\tau_i \iota(\sigma))\big) \cdot (\tau_i \iota(\sigma)),$$

so all that remains to prove is that every element of $S_{n+1} \setminus \iota(S_n)$ can be written uniquely in the form $(i\ n+1)\iota(\sigma)$ for some $1 \le i \le n, \sigma \in S_n$. Such an expression exists because any cycle containing $n+1$ can be decomposed as $(n+1\ i_2 \dots i_s) = (i_2\ n+1)(i_2 \dots i_s)$. This expression is unique as if $(i\ n+1)\iota(\sigma) = (j\ n+1)\iota(\sigma')$ with $i \neq j$, then $\iota(\sigma') = (i\ j\ n+1)\iota(\sigma)$, but this permutes $\sigma^{-1}(i) \mapsto n+1$ so cannot lie in $\iota(S_n)$. If $(i\ n+1)\iota(\sigma) = (i\ n+1)\iota(\sigma')$ then $\sigma = \sigma'$ as ι is injective, so the decomposition is indeed unique. □

Using this we obtain our version of [13, Prop. 4.6], to match Proposition 5.1 on the orbifold side.

Proposition 7.2. $\varepsilon^{n+1}\mathfrak{r}_1 - \mathfrak{r}_1\varepsilon^n = \big(\varepsilon_2^{n+1}\mathfrak{r}_1 - \mathfrak{r}_1\varepsilon_2^n\big)\varepsilon^n$

Proof. We have

$$\begin{aligned}\varepsilon^{n+1} \star \mathfrak{r}_1 y - \mathfrak{r}_1 \cdot (\varepsilon^n \star y) =& \pi^*_{(n+1)}\varepsilon^{n+1} \star \left(\sum_{i=1}^{n+1}(i\ n+1) \bullet (\pi^*_{(n+1)}p_n^{n+1} y \star \Delta_{n+1,n+2})\right)\\ &- \sum_{i=1}^{n+1}(i\ n+1) \bullet (\pi^*_{(n+1)}p_n^{n+1}(\varepsilon^n \star y) \star \Delta_{n+1,n+2}).\end{aligned}$$

As $\pi^*_{(n+1)}\varepsilon^{n+1}$ is symmetric, we can move it inside the action of $\sum_{i=1}^{n+1}(i\ n+1)$:

$$\begin{aligned}\varepsilon^{n+1} \star \mathfrak{r}_1 y - \mathfrak{r}_1 \cdot (\varepsilon^n \star y) &= \sum_{i=1}^{n+1}(i\ n+1) \bullet \Big(\pi^*_{(n+1)}((\varepsilon^{n+1} - p_n^{n+1}(\varepsilon^n)) \star p_n^{n+1} y) \star \Delta_{n+1,n+2}\Big)\\ &= \sum_{i=1}^{n+1}(i\ n+1) \bullet \Big(\pi^*_{(n+1)}((\varepsilon_2^{n+1} - p_n^{n+1}(\varepsilon_2^n)) \star p_n^{n+1}\varepsilon^n \star p_n^{n+1} y) \star \Delta_{n+1,n+2}\Big)\\ &\qquad \text{(by the previous lemma)}\\ &= \sum_{i=1}^{n+1}(i\ n+1) \bullet \Big(\pi^*_{(n+1)}(\varepsilon_2^{n+1} \star p_n^{n+1}(\varepsilon^n \star y)) \star \Delta_{n+1,n+2}\Big)\\ &\quad - \sum_{i=1}^{n+1}(i\ n+1) \bullet \Big(\pi^*_{(n+1)}p_n^{n+1}(\varepsilon_2^n \star \varepsilon^n \star y) \star \Delta_{n+1,n+2}\Big)\\ &= \varepsilon_2^{n+1} \star \mathfrak{r}_1(\varepsilon^n \star y) - \mathfrak{r}_1(\varepsilon_2^n \star \varepsilon^n \star y),\end{aligned}$$

as required. □

This allows us to prove inductively that Ψ preserves multiplication by the elements ε_d^d and the Chern classes $c_d(\mathcal{O}^{[n]})$ of the tautological sheaf, using the ideas in [13, Prop. 4.7].

Proposition 7.3.

$$\Phi(c(\mathcal{O}^{[n]})y) = \varepsilon^n \star \Phi\,(y)$$

for all $y \in A^*\big(X^{[n]}\big)$.

Proof. We prove a slightly stronger claim: that for any partition $\lambda \vdash n$ of length s,

$$\Phi_{X^s}(\pi^*_{X^{[n]}} c(\mathcal{O}^{[n]}) \cdot \mathfrak{q}_\lambda v) = \pi^*_{(n)}\varepsilon^n \star \Phi_{X^s}(\mathfrak{q}_\lambda v).$$

Lemma 2.6 allows us to reduce to the claim that, if $y \in A^*(X^{[n]} \times X^s)$ is given by any composition of the operators $\mathfrak{q}_1$ and ∂ applied to the vacuum vector, then

$$\Phi_{X^s}(\pi^*_{[n]} c(\mathcal{O}^{[n]}) \cdot y) = \pi^*_{(n)}\varepsilon^n \star \Phi_{X^s}(y).$$

We can prove this by induction on the length of y as an expression in $\partial, \mathfrak{q}_1$. The base case $\Phi(c(\mathcal{O}^{[0]})v) = \Phi(v) = 1_{S_0} \star \Phi\,(v)$ is trivial.

Case 1: $y = \partial z$ with $z \in A^*(X^{[n]} \times X^s)$ satisfying the induction hypothesis

$$\Phi_{X^s}\big(\pi^*_{[n]} c(\mathcal{O}^{[n]}) \cdot z\big) = \pi^*_{(n)}(\varepsilon^n) \star \Phi_{X^s}(z).$$

Then by Proposition 5.6, we have

$$\Phi_{X^s}(y) = \Phi_{X^s}\big(\pi^*_{[n]} c_2(\mathcal{O}^{[n]}) \cdot z\big) = \pi^*_{(n)}(\varepsilon_2^n) \star \Phi_{X^s}(z)$$

and

$$\Phi_{X^s}\big(\pi_{[n]}^*c(\mathcal{O}^{[n]})y\big) = \Phi_{X^s}\big(\pi_{[n]}^*\big(c_2(\mathcal{O}^{[n]})c(\mathcal{O}^{[n]})\big)\cdot z\big) = \pi_{(n)}^*(\varepsilon_2^n) \star \Phi_{X^s}\big(\pi_{[n]}^*c(\mathcal{O}^{[n]})\cdot z\big).$$

Using the induction hypothesis, the above expression becomes $\pi_{(n)}^*(\varepsilon_2^n \star \varepsilon^n) \star \Phi_{X^s}(z)$. As the product $\star$ is commutative, we can move ε_2^n back inside Φ to get

$$\Phi_{X^s}\big(\pi_{[n]}^*c(\mathcal{O}^{[n]})y\big) = \pi_{(n)}^*\varepsilon^n \star \Phi_{X^s}\big(\pi_{[n]}^*c_2(\mathcal{O}^{[n]})z\big) = \pi_{(n)}^*\varepsilon^n \star \Phi_{X^s}(y)$$

as required.

Case 2: $y = \mathfrak{q}_1 z$, with $z \in A^*(X^{[n-1]} \times X^{s-1})$ satisfying the induction hypothesis

$$\Phi_{X^{s-1}}\big(\pi_{[n-1]}^*c(\mathcal{O}^{[n-1]})\cdot z\big) = \pi_{(n-1)}^*\varepsilon^{n-1} \star \Phi_{X^{s-1}}(z).$$

Then $\Phi_{X^s}(y) = \mathfrak{r}_1\Phi_{X^{s-1}}(z)$ by Proposition 6.7, so

$$\pi_{(n)}^*(\varepsilon^n) \star \Phi_{X^s}(y) = \pi_{(n)}^*(\varepsilon^n) \star \mathfrak{r}_1\Phi_{X^{s-1}}(z) = [\varepsilon^\bullet, \mathfrak{r}_1]\Phi_{X^{s-1}}(z) + \mathfrak{r}_1\cdot(\pi_{(n-1)}^*(\varepsilon^n) \star \Phi_{X^{s-1}}(z))$$

By the induction hypothesis, the second term equals

$$\mathfrak{r}_1\Phi_{X^{s-1}}\big(\pi_{[n-1]}^*c(\mathcal{O}^{[n-1]})\cdot z\big) = \Phi_{X^s}\big(\mathfrak{q}_1\cdot(\pi_{[n-1]}^*c(\mathcal{O}^{[n-1]})\cdot z)\big).$$

Meanwhile by the previous proposition, the first term is:

$$\begin{aligned}
[\varepsilon^\bullet, \mathfrak{r}_1]\Phi_{X^s}^n(z) &= \pi_{(n)}^*(\varepsilon_2^n) \star \mathfrak{r}_1\left(\pi_{(n-1)}^*(\varepsilon^{n-1}) \star \Phi_{X^{s-1}}(z)\right) - \mathfrak{r}_1\left(\pi_{(n-1)}^*(\varepsilon_2^{n-1} \star \varepsilon^{n-1}) \star \Phi_{X^{s-1}}(z)\right)\\
&= \pi_{(n)}^*(\varepsilon_2^n) \star \mathfrak{r}_1\left(\Phi_{X^{s-1}}(\pi_{[n-1]}^*c(\mathcal{O}^{[n-1]})z)\right) - \mathfrak{r}_1\left(\Phi_{X^{s-1}}(\pi_{[n-1]}^*(c_2(\mathcal{O}^{[n-1]})c(\mathcal{O}^{[n-1]}))\cdot z)\right)\\
&\quad \text{(by the induction hypothesis)}\\
&= \pi_{(n)}^*(\varepsilon_2^n) \star \Phi_{X^s}(\mathfrak{q}_1\cdot(\pi_{[n-1]}^*c(\mathcal{O}^{[n-1]})z)) - \Phi_{X^s}(\mathfrak{q}_1\cdot(\pi_{[n-1]}^*(c_2(\mathcal{O}^{[n-1]})c(\mathcal{O}^{[n-1]}))\cdot z))\\
&= \Phi_{X^s}\left(\pi_{[n]}^*c_2(\mathcal{O}^{[n]})\mathfrak{q}_1\cdot(\pi_{[n-1]}^*c(\mathcal{O}^{[n-1]})z) - \mathfrak{q}_1\cdot(\pi_{[n-1]}^*(c_2(\mathcal{O}^{[n-1]})c(\mathcal{O}^{[n-1]}))\cdot z)\right)\\
&= \Phi_{X^s}([\partial, \mathfrak{q}_1]\cdot(\pi_{[n-1]}^*c(\mathcal{O}^{[n-1]})z))\\
&\quad \text{which by Proposition 5.1 is}\\
&= \Phi_{X^s}(\pi_{[n]}^*c(\mathcal{O}^{[n]})\mathfrak{q}_1 z - \mathfrak{q}_1\cdot(\pi_{[n-1]}^*c(\mathcal{O}^{[n-1]})z))
\end{aligned}$$

Thus adding the terms together gives

$$\pi_{(n)}^*(\varepsilon^n) \star \Phi_{X^s}(y) = \Phi_{X^s}(\pi_{[n]}^*c(\mathcal{O}^{[n]})\mathfrak{q}_1 z) = \Phi_{X^s}(\pi_{[n]}^*c(\mathcal{O}^{[n]})y)$$

as required. □

As Φ respects multiplication by the total Chern class of $\mathcal{O}^{[n]}$, it must also by the individual Chern classes $c_d(\mathcal{O}^{[n]})$, as we can see by taking homogeneous $y \in A^*(X^{[n]})$ and looking at the components of $\Phi(c(\mathcal{O}^{[n]})y)$ in each degree. It follows that the same holds true for components $\mathrm{ch}_d(\mathcal{O}^{[n]}) = \underline{\mathfrak{G}}_d(1)$ of the Chern character

Corollary 7.4. $\Phi(\underline{\mathfrak{G}}_d(1)y) = \Phi(\underline{\mathfrak{G}}_d(1)1_{[n]}) \star \Phi(y)$

7.2 Orbifold Chern character operators

The next step is to define analogues of the Chern character operators $\underline{\mathfrak{G}}_d$ for $A^*_{\mathrm{orb}}(X^{(n)})$, then prove they also satisfy the transfer property. Using this we will prove that Φ_{X^s} preserves multiplication by Chern character operators by the same method as the previous proposition.

Define operators $\mathfrak{F}_d : A^*_{\mathrm{orb}}(X^{(n)}) \to A^*_{\mathrm{orb}}(X^{(n)} \times X)$ for $d \geq 0$ as:

$$\mathfrak{F}_d x := \Phi_X(\underline{\mathfrak{G}}_d 1_{X^{[n]}}) \star \pi_{(n)}^* x$$

or more explicitly, $\mathfrak{F}_d x = \Phi_X(\mathrm{ch}_d(\mathcal{O}_{\Xi_n})\pi_1^*(\mathrm{td}\, X)) \star \pi_{(n)}^* x$

It follows that

$$\mathfrak{F}_d(\beta)x = \Phi(\underline{\mathfrak{G}}_d(\beta)1_{X^{[n]}}) \star x. \tag{27}$$

The following lemma is our version of [13, (4.8)], but where Lehn-Sorger had to derive an expression for $a^{[n]}$ using vertex algebra calculus, we are provided with the analogous expression for $\underline{\mathfrak{G}}_d$ by the work of Maulik-Negut [17] as given in Theorem 2.4.

Lemma 7.5. *There is an expression*

$$\underline{\mathfrak{G}}_d 1_{X^{[n]}} = \sum_{|\lambda| \leq n} \frac{\mathfrak{q}_1(1)^{n-|\lambda|}}{(n-|\lambda|)!} \frac{\pi_X^*(c_{d,\lambda})\mathfrak{q}_\lambda|_\Delta}{|\lambda|!} v$$

where the sum is over all partitions $\lambda \vdash m$ for $m \leq n$, with coefficients $c_{d,\lambda} = c'_{d,\lambda} + c''_{d,\lambda} \cdot e$, for $c'_{d,\lambda}, c''_{d,\lambda} \in \mathbb{Q}$ which depend only on d and λ; in particular they are independent of n.

Proof. Putting together (3) and (6), we have the following expression for $\underline{\mathfrak{G}}_d$:

$$\underline{\mathfrak{G}}_d = - \sum_{\lambda \in \mathcal{P}_{\mathbb{Z}}(0,d)} \frac{1}{\lambda!} \mathfrak{q}_\lambda|_{\Delta_{1\ldots d}} + \sum_{\lambda \in \mathcal{P}_{\mathbb{Z}}(0,d-2)} \frac{s(\lambda) - 2}{24\lambda!} \pi_X^*(e)\, \mathfrak{q}_\lambda|_{\Delta_{1\ldots d-2}}$$

Hence

$$\begin{aligned} \underline{\mathfrak{G}}_d 1_{X^{[n]}} &= \left(- \sum_{\substack{k \leq \min(n,d-1) \\ \lambda \in \mathcal{P}(k,d-k)}} \frac{1}{\lambda!} \mathfrak{q}_\lambda \mathfrak{q}_{-1}^k|_{\Delta_{1\ldots d}} + \sum_{\substack{k \leq \min(n,d-3) \\ \lambda \in \mathcal{P}(k,d-2-k)}} \frac{s(\lambda)-2}{24\lambda!} \pi_X^*(e)\, \mathfrak{q}_\lambda \mathfrak{q}_{-1}^k|_{\Delta_{1\ldots d-2}} \right) \cdot \frac{1}{n!} \mathfrak{q}_1(1)^n v \\ &= - \sum_{\substack{k \leq n, d-1 \\ \lambda \in \mathcal{P}(k,d-k)}} \frac{1}{\lambda!(n-k)!} \mathfrak{q}_\lambda|_\Delta \mathfrak{q}_1(1)^{n-k} v + \sum_{\substack{k \leq n, d-3 \\ \lambda \in \mathcal{P}(k,d-2-k)}} \frac{s(\lambda)-2}{24\lambda!(n-k)!} \pi_X^*(e)\, \mathfrak{q}_\lambda|_\Delta \mathfrak{q}_1(1)^{n-k} v \end{aligned}$$

which has the required form. □

Following the ideas of [13, 4.11], let $\lambda \vdash m$ for some $m \leq n$, choose some $g_\lambda \in S_m$ of cycle type λ and define

$$B_\lambda^n = \sqrt{-1}^{a(g_\lambda)} \binom{n}{|\lambda|} \mathrm{Sym}(\Delta_{1\ldots m,n+1}^{g_\lambda} \cdot g_\lambda) \in A_{\mathrm{orb}}^*(X^{(n)} \times X),$$

where we identify g_λ with its image in S^n under the inclusion $\{1, \ldots, m\} \hookrightarrow \{1, \ldots, n\}$, and $\Delta_{1\ldots m,n+1}^{g_\lambda}$ is the diagonal with the given indices in $(X^n)^{g_\lambda} \times X = (X^m)^{g_\lambda} \times X^{n-m} \times X$.

Proposition 7.6.

$$\Psi_X(B_\lambda^n) = \frac{\mathfrak{q}_1(1)^{n-|\lambda|}}{(n-|\lambda|)!} \frac{\mathfrak{q}_\lambda|_\Delta}{|\lambda!|} \cdot v$$

Proof. If l is the length of $\lambda \vdash m$, the right hand side can be rewritten as

$$\begin{aligned} &\pi_{[n],1,*} \left(\pi_{1\ldots l}^*(\Delta_{1\ldots l}) \frac{\mathfrak{q}_1(1)^{n-m}}{(n-m)!} \frac{\mathfrak{q}_\lambda}{m!} v \right) \\ &= \frac{1}{|\lambda|!(n-m)!} \pi_{[n],1,*} \left(\pi_{1\ldots,l}^*(\Delta_{1\ldots l}) \mathfrak{q}_\lambda \mathfrak{q}_1^{n-m} v \right) \end{aligned}$$

Meanwhile

$$\begin{aligned} (\Gamma_{g_\lambda}^t \times \mathrm{id}_{X*})(\Delta_{1\ldots m,n+1}^{g_\lambda}) &= \psi_{X,*}((\Gamma_{g_\lambda} \times X) \cdot \phi_X^* \Delta_{1\ldots m,n+1}^{g_\lambda}) \\ &= \psi_{X,*}((\Gamma_{g_\lambda} \times X) \cdot (X^{[n]} \times \Delta_{1\ldots m,n+1}^{g_\lambda})) \end{aligned}$$

We have an isomorphism $(X^n)^{g_\lambda} \cong X^\lambda \times X^{n-m} = X^{l+n-m}$ such that, if $L = l + n - m + 1$ we get the following identifications:

$$\begin{array}{ccccccc} \Gamma_{g_\lambda} \times X & \subset & X^{[n]} \times (X^n)^{g_\lambda} \times X & \xrightarrow{\phi_X} & (X^n)^{g_\lambda} \times X & \supset & \Delta_{1\ldots m,n+1}^{g_\lambda} \\ \big\downarrow & & \big\downarrow\wr & & \big\downarrow\wr & & \big\downarrow \\ \mathfrak{q}_\lambda \mathfrak{q}_1^{n-m} v \times X & \subset & X^{[n]} \times X^\lambda \times X^{n-m} \times X & \xrightarrow{\pi_{1\ldots L}} & X^\lambda \times X^{n-m} \times X & \supset & \Delta_{1\ldots l,L} \end{array}$$

Thus we have

$$(\Gamma^t_{g_\lambda} \times \mathrm{id})(\Delta^{g_\lambda}_{1\ldots m,n+1}) = \pi_{[n],1,*}((\mathfrak{q}_\lambda \mathfrak{q}_1^{n-m} v \times X) \cdot (X^{[n]} \times \Delta_{1\ldots l,L})),$$

which is equal to $\pi_{[n],1,*}(\mathfrak{q}_\lambda \mathfrak{q}_1^{n-m} v \cdot (X^{[n]} \times \Delta_{1\ldots l}))$ because projecting away from the L'th factor gives:

$$\begin{aligned}\int_L (\mathfrak{q}_\lambda \mathfrak{q}_1^{n-m} v \times X) \cdot (X^{[n]} \times \Delta_{1\ldots l,L}) &= \mathfrak{q}_\lambda \mathfrak{q}_1^{n-m} v \cdot \int_L (X^{[n]} \times \Delta_{1\ldots l,L}) \\ &= \mathfrak{q}_\lambda \mathfrak{q}_1^{n-m} v \cdot (X^{[n]} \times \Delta_{1\ldots l}).\end{aligned}$$

Hence we compute

$$\begin{aligned}\Psi_X(B^n_\lambda) &= \frac{1}{n!}\binom{n}{m}\left(\sum_{g\in G}(\Gamma^t_{g_\lambda} \times \mathrm{id}_{X*})\right)(\mathrm{Sym}(\Delta^{g_\lambda}_{1\ldots m,n+1} g_\lambda)) \\ &= \frac{1}{n!}\binom{n}{m}\mathfrak{q}_\lambda \mathfrak{q}_1(1)^{n-m} v,\end{aligned}$$

as each of the $n!$ terms in

$$\mathrm{Sym}(\Delta^{g_\lambda}_{1\ldots m,n+1} g_\lambda) = \frac{1}{n!}\sum_{\sigma\in S_n} \Delta^{\sigma g_\lambda \sigma^{-1}}_{1\ldots m,n+1} \cdot (\sigma g_\lambda \sigma^{-1})$$

contributes $\frac{1}{n!}\mathfrak{q}_\lambda \mathfrak{q}_1(1)^{n-m}$ when $\Gamma^t_{\sigma g_\lambda \sigma^{-1}} \times \mathrm{id}_{X*}$ is applied. Expanding the binomial coefficient and cancelling gives

$$\Psi_X(B^n_\lambda) = \frac{1}{|\lambda|!(n-|\lambda|)!}\mathfrak{q}_\lambda \mathfrak{q}_1(1)^{n-m}$$

□

Combining the the previous two results, we obtain:

Corollary 7.7. *In $A^*_{orb}(X^{(n)} \times X)$, we have the equality:*

$$\Phi_X(\underline{\mathfrak{G}}_d 1_{X^{[n]}}) = \sum_{|\lambda|\le n} \pi^*_X(c_{d,\lambda}) B^n_\lambda. \tag{28}$$

Now we can derive the transfer property for $\mathfrak{F}_d$.

Proposition 7.8 (based on [13, Prop. 4.12])**.** *Let $B^\bullet_\lambda : A^*_{orb}(X^{(n)}) \to A^*_{orb}(X^{(n)} \times X)$ for all n be the family of operators given by $B^\bullet_\lambda x = B^n_\lambda \star \pi^*_{(n)} x$ for $x \in A^*_{orb}(X^{(n)})$, with $B^n_\lambda = 0$ if $n < |\lambda|$. Then we have the following equality of operators:*

$$[B^\bullet_\lambda, \mathfrak{r}_1] = \Delta_*[B^\bullet_\lambda(1), \mathfrak{r}_1] : A^*_{orb}(X^{(n)}) \to A^*_{orb}(X^{(n+1)} \times X^2).$$

Consequentially, the operators $\mathfrak{F}_d$ satisfy the transfer property:

$$[\mathfrak{F}_d, \mathfrak{r}_1] = \Delta_*[\mathfrak{F}_d(1), \mathfrak{r}_1]$$

Proof. For clarity we refer to the factors of X in e.g. $X^{(n)} \times X^2$ with the indices a, b. Let $x \in A^*_{\mathrm{orb}}(X^{(n)})$ and consider the action of $[B^\bullet_\lambda, \mathfrak{r}_1]$ on x. We may assume $|\lambda| \le n+1$ otherwise the result is zero.

We compute the first term:

$$B^\bullet_\lambda \mathfrak{r}_1 x = \sqrt{-1}^{a(g_\lambda)}\binom{n+1}{|\lambda|}\mathrm{Sym}(\Delta_{1\ldots m,a} g_\lambda) \star \left(\sum_{i=1}^{n}(i\ n+1) \bullet (\pi^*_{(n+1)} p^{n+1}_n(x) \star \Delta_{n+1,b})\right),$$

(with the expression in brackets already symmetric, so it can be moved inside Sym)

$$= \sqrt{-1}^{a(g_\lambda)}\binom{n+1}{|\lambda|}\sum_{i=1}^{n}\mathrm{Sym}\left((\Delta_{1\ldots m,a} g_\lambda) \star \left((i\ n+1) \bullet (\pi^*_{(n+1)} p^{n+1}_n(x) \star \Delta_{n+1,b})\right)\right)$$

$$B^\bullet_\lambda \mathfrak{r}_1 x = \sqrt{-1}^{a(g_\lambda)}\binom{n+1}{|\lambda|}\sum_{i=1}^{n+1}\mathrm{Sym}\left(((i\ n+1) \bullet (\Delta^{g_\lambda}_{1\ldots m,a} g_\lambda)) \star (\pi^*_{(n+1)} p^{n+1}_n(x) \star \Delta_{n+1,b})\right), \tag{29}$$

using the fact that $\mathrm{Sym}((i\ n+1)\bullet y)=\mathrm{Sym}(y)$. For $|\lambda|<i\leq n+1$, the action of $(i\ n+1)$ has no effect on $(\Delta_{1\ldots m,a})_{g_\lambda}$, so that part of the sum gives

$$
\begin{aligned}
&\sqrt{-1}^{a(g_\lambda)}\binom{n+1}{|\lambda|}(n+1-|\lambda|)\,\mathrm{Sym}\left((\Delta^{g_\lambda}_{1\ldots m,a}g_\lambda)\star(\pi^*_{(n+1)}p^{n+1}_n(x)\star\Delta_{n+1,b})\right)\\
&=\sqrt{-1}^{a(g_\lambda)}\binom{n+1}{|\lambda|}(n+1-|\lambda|)\,\mathrm{Sym}\left(((\Delta^{g_\lambda}_{1\ldots m,a}g_\lambda)\star\pi^*_{(n+1)}p^{n+1}_n(x))\star\Delta_{n+1,b}\right)\\
&=\sqrt{-1}^{a(g_\lambda)}\binom{n+1}{|\lambda|}(n+1-|\lambda|)\,\mathrm{Sym}\left(p^{n+1}_{n,X^2}((\Delta^{g_\lambda}_{1\ldots m,a}g_\lambda)\star\pi^*_{(n)}x)\star\Delta_{n+1,b}\right)\\
&=\sqrt{-1}^{a(g_\lambda)}\binom{n+1}{|\lambda|}(n+1-|\lambda|)\,\mathrm{Sym}\left(p^{n+1}_{n,X^2}(\mathrm{Sym}(\Delta^{g_\lambda}_{1\ldots m,a}g_\lambda)\star\pi^*_{(n)}x)\star\Delta_{n+1,b}\right)
\end{aligned}
$$

where p^{n+1}_{n,X^2} is the operator $p^{n+1}_n\times\mathrm{id}_{X^2*}: A^*(X^n\times X^2,S_n)\to A^*(X^{n+1}\times X^2,S_{n+1})$. As everything inside the outer Sym is already symmetric in the indices $1,\ldots,n$, we can simplify the symmetrizer as $\mathrm{Sym}(-)=\frac{1}{n+1}\sum_{i=1}^{n+1}(i\ n+1)\bullet(-)$. Thus the above expression becomes:

$$
\begin{aligned}
&\sqrt{-1}^{a(g_\lambda)}\binom{n+1}{|\lambda|}\frac{(n+1-|\lambda|)}{n+1}\sum_{i=1}^{n}(i\ n+1)\bullet\left(p^{n+1}_{n,X^2}(\mathrm{Sym}(\Delta^{g_\lambda}_{1\ldots m,a}g_\lambda)\star\pi^*_{(n)}x)\star\Delta_{n+1,b}\right)\\
&=\sum_{i=1}^{n}(i\ n+1)\bullet\left(p^{n+1}_{n,X^2}\left(\sqrt{-1}^{a(g_\lambda)}\binom{n}{|\lambda|}\mathrm{Sym}(\Delta^{g_\lambda}_{1\ldots m,a}g_\lambda)\star\pi^*_{(n)}x\right)\star\Delta_{n+1,b}\right)\\
&=(\mathfrak{r}_1B^\bullet_\lambda)^{b,a}x,
\end{aligned}
$$

i.e. $\mathfrak{r}_1B^\bullet_\lambda$ with the indices a,b swapped. Not in particular that this is zero if $|\lambda|=n+1$, so nothing goes wrong in the edge case.

Thus taking the commutator $[B^\bullet_\lambda,\mathfrak{r}_1]$ leaves the terms of (29) with $i\leq|\lambda|$:

$$
\begin{aligned}
[B^\bullet_\lambda,\mathfrak{r}_1]x&=\binom{n+1}{|\lambda|}\sum_{i=1}^{|\lambda|}\mathrm{Sym}\left(((i\ n+1)\bullet(\Delta^{g_\lambda}_{1\ldots m,a}g_\lambda))\star(\pi^*_{(n+1)}p^{n+1}_n(x)\star\Delta_{n+1,b})\right)\\
&=\binom{n+1}{|\lambda|}\sum_{i=1}^{|\lambda|}\mathrm{Sym}\left((\Delta^{g_{\lambda,i}}_{1\ldots i-1,n+1,i+1,\ldots m,a}g_{\lambda,i})\star\pi^*_{(n+1)}p^{n+1}_n(x)\star\Delta_{n+1,b}\right)
\end{aligned}
$$

where $g_{\lambda,i}=(i\ n+1)g_\lambda(i\ n+1)$. We have the product

$$
\begin{aligned}
(\Delta^{g_{\lambda,i}}_{1\ldots i-1,n+1,i+1,\ldots m,a}g_{\lambda,i})\star(\Delta_{n+1,b}1_{S_{n+1}})&=(\Delta^{g_{\lambda,i}}_{1\ldots i-1,n+1,i+1,\ldots m,a}\cdot\Delta_{n+1,b})g_{\lambda,i}\\
&=(\Delta^{g_{\lambda,i}}_{1\ldots i-1,n+1,i+1,\ldots m,a,b})g_{\lambda,i}
\end{aligned}
$$

so we can simplify:

$$
[B^\bullet_\lambda,\mathfrak{r}_1]x=\binom{n+1}{|\lambda|}\sum_{i=1}^{|\lambda|}\mathrm{Sym}\left((\Delta^{g_{\lambda,i}}_{1\ldots i-1,n+1,i+1,\ldots m,a,b}g_{\lambda,i})\star\pi^*_{(n+1)}p^{n+1}_n(x)\right)
$$

We hence obtain

$$\begin{aligned}
\Delta_*[B^\bullet_\lambda(1),\mathfrak{r}_1]x &= \binom{n+1}{|\lambda|}\sum_{i=1}^{|\lambda|}\int_c \mathrm{Sym}\left((\Delta^{g_{\lambda,i}}_{1\ldots i-1,n+1,i+1,\ldots m,c,a}g_{\lambda,i})\star\pi^*_{(n+1)}p^{n+1}_n(x)\star\Delta_{ab}\right)\\
&= \binom{n+1}{|\lambda|}\sum_{i=1}^{|\lambda|}\mathrm{Sym}\left(\left(\left(\int_c\Delta^{g_{\lambda,i}}_{1\ldots i-1,n+1,i+1,\ldots m,c,a}g_{\lambda,i}\right)\star\Delta_{ab}\star\pi^*_{(n+1)}p^{n+1}_n(x)\right)\right.\\
&= \binom{n+1}{|\lambda|}\sum_{i=1}^{|\lambda|}\mathrm{Sym}\left((\Delta^{g_{\lambda,i}}_{1\ldots i-1,n+1,i+1,\ldots m,a,b}g_{\lambda,i})\star\pi^*_{(n+1)}p^{n+1}_n(x)\right)\\
&= [B^\bullet_\lambda,\mathfrak{r}_1]x,
\end{aligned}$$

So the first claim is proved.

The second claim reduces to the first using (28). In particular, we have

$$\mathfrak{F}_d=\sum_\lambda \pi_1^*(c_{d,\lambda})B^\bullet_\lambda,$$

where we sum over all partitions λ (but $B^n_\lambda=0$ for $n<|\lambda|$). Thus the commutator is given by

$$[\mathfrak{F}_d,\mathfrak{r}_1]=\sum_\lambda\pi_1^*(c_{d,\lambda})[B^\bullet_\lambda,\mathfrak{r}_1]=\sum_\lambda\pi_1^*(c_{d,\lambda})\Delta_*[B^\bullet_\lambda(1),\mathfrak{r}_1]=\sum_\lambda\Delta_*\big(\pi_1^*(c_{d,\lambda})[B^\bullet_\lambda(1),\mathfrak{r}_1]\big),$$

and on the other hand we have

$$\Delta_*[\mathfrak{F}_d(1),\mathfrak{r}_1]=\sum_\lambda\Delta_*[B^\bullet_\lambda(c_{d,\lambda}),\mathfrak{r}_1]=\sum_\lambda\Delta_*\left(\int_1\pi_1^*(c_{d,\lambda})\Delta_*[B^\bullet_\lambda(1),\mathfrak{r}_1]\right).$$

The last expression can be rewritten as

$$\sum_\lambda\Delta_*\left(\int_1\Delta_*\big(\pi_1^*(c_{d,\lambda})[B^\bullet_\lambda(1),\mathfrak{r}_1]\big)\right)=\sum_\lambda\Delta_*\big(\pi_1^*(c_{d,\lambda})[B^\bullet_\lambda(1),\mathfrak{r}_1]\big)$$

which proves the claim. □

7.3 Proof of the main theorem

Finally we prove that Φ transforms the operators $\underline{\mathfrak{G}}_d$ into $\mathfrak{F}_d$ using similar methods as [13, 4.13], before deriving Theorem 4.6 as a consequence.

Proposition 7.9. *Let $y\in A^k(X^{[n]})$. Then*

$$\mathfrak{F}_d\Phi(y)=\Phi_X(\underline{\mathfrak{G}}_d y)$$

Proof. As in the proof of Proposition 7.3, we actually prove that $\mathfrak{F}_d\Phi_{X^s}(y)=\Phi_{X^{s+1}}(\underline{\mathfrak{G}}_d y)$ when $y=\mathfrak{q}_\lambda v$ for any $\lambda\in\mathcal{P}(n,s)$, by reducing to the case that y is any composed string of the operators ∂ and $\mathfrak{q}_1$ and using induction on the length of the string.

Case 1: $y=\partial z$, where $z\in A^*(X^{[n]}\times X^s)$ satisfies the induction hypothesis $\Phi_{X^{s+1}}(\underline{\mathfrak{G}}_d z)=\mathfrak{F}_d\Phi_{X^s}(z)$. Then by Proposition 5.6, $\Phi_{X^s}(y)=\pi^*_{(n)}\varepsilon^n_2\star\Phi_{X^s}(z)$ and

$$\Phi_{X^{s+1}}(\underline{\mathfrak{G}}_d y)=\Phi_{X^{s+1}}(\partial\underline{\mathfrak{G}}_d z)=\pi^*_{(n)}\varepsilon^n_2\star\Phi_X(\underline{\mathfrak{G}}_d z)$$

since the multiplication operators $\partial,\underline{\mathfrak{G}}_d$ commute. Using the induction hypothesis and the fact that the multiplication operator $\mathfrak{F}_d$ commutes with multiplication by ε^n_2, we get:

$$\mathfrak{F}_d\Phi_{X^s}(y)=\mathfrak{F}_d(\pi^*_{(n)}\varepsilon^n_2\star\Phi_{X^s}(z))=\pi^*_{(n)}\varepsilon^n_2\star\mathfrak{F}_d\Phi_{X^s}(z)=\pi^*_{(n)}\varepsilon^n_2\star\Phi_{X^{s+1}}(\underline{\mathfrak{G}}_d z)=\Phi_{X^{s+1}}(\underline{\mathfrak{G}}_d y),$$

as required.

Case 2: $y = \mathfrak{q}_1 z$, where $z \in A^*(X^{[n-1]} \times X^{s-1})$ satisfies the induction hypothesis $\Phi_{X^s}(\underline{\mathfrak{G}}_d z) = \mathfrak{F}_d \Phi_{X^{s-1}}(z)$. Then $\Phi_{X^s}(y) = \Phi_{X^s}(\mathfrak{q}_1 z) = \mathfrak{r}_1 \Phi_{X^{s-1}}(z)$ by Proposition 6.7, so we obtain

$$\mathfrak{F}_d \Phi_{X^s}(y) = \mathfrak{F}_d \mathfrak{r}_1 \Phi_{X^{s-1}}(z) = [\mathfrak{F}_d, \mathfrak{r}_1]\Phi_{X^{s-1}}(z) + (\mathfrak{r}_1 \mathfrak{F}_d \Phi_{X^{s-1}}(z))^{21}$$

By the induction hypothesis, the second term is

$$(\mathfrak{r}_1 \Phi_{X^s}(\underline{\mathfrak{G}}_d z))^{21} = (\Phi_{X^{s+1}}(\mathfrak{q}_1 \underline{\mathfrak{G}}_d z))^{21}$$

while by the transfer property the first term is

$$\begin{aligned}(\Delta_*[\mathfrak{F}_d(1), \mathfrak{r}_1])\Phi_{X^{s-1}}(z) = \pi_{12}^*\Delta_{12} \star \pi_{(n),2\ldots s+1}^*(\mathfrak{F}_d(1)\mathfrak{r}_1\Phi_{X^{s-1}}(z)) \\ - \pi_{12}^*\Delta_{12} \star \pi_{(n),2\ldots s+2}^*(\mathfrak{r}_1\mathfrak{F}_d(1)\Phi_{X^{s-1}}(z)).\end{aligned}$$

We know from (27) and Corollary 7.4 that $\mathfrak{F}_d(1)\Phi(x) = \Phi(\underline{\mathfrak{G}}_d(1)) \star \Phi(x) = \Phi(\underline{\mathfrak{G}}_d(1)x)$, so this becomes

$$\begin{aligned}\Delta_*[\mathfrak{F}_d(1), \mathfrak{r}_1]\Phi_{X^{s-1}}(z) &= \pi_{12}^*\Delta_{12} \star \pi_{(n),2\ldots s+2}^*(\Phi_{X^s}(\underline{\mathfrak{G}}_d(1)\mathfrak{q}_1 z))) \\ &\quad - \pi_{12}^*\Delta_{12} \star \pi_{(n),2\ldots s+2}^*(\Phi_{X^s}^n(\mathfrak{q}_1\underline{\mathfrak{G}}_d(1)z)) \\ &= \Phi_{X^{s+1}}(\pi_{12}^*\Delta_{12} \cdot \pi_{(n),2\ldots s+2}^*([\underline{\mathfrak{G}}_d(1), \mathfrak{q}_1]z)) \\ &= \Phi_{X^{s+1}}(\Delta_*[\underline{\mathfrak{G}}_d(1), \mathfrak{q}_1]z) \\ &= \Phi_{X^{s+1}}([\underline{\mathfrak{G}}_d, \mathfrak{q}_1]z).\end{aligned}$$

Adding the terms together, we get

$$\begin{aligned}\mathfrak{F}_d\Phi_{X^s}^n(y) &= \Phi_{X^{s+1}}^n(([\underline{\mathfrak{G}}_d, \mathfrak{q}_1]z) + \Phi_{X^{s+1}}^n(\mathfrak{q}_1\underline{\mathfrak{G}}_d z) \\ &= \Phi_{X^{s+1}}^n(\underline{\mathfrak{G}}_d\mathfrak{q}_1 z) \\ &= \Phi_{X^{s+1}}^n(\underline{\mathfrak{G}}_d y)\end{aligned}$$

as required. □

Now we obtain the main result, Theorem 4.6, by proving that Φ is a ring isomorphism.

Proof of Theorem 4.6. By Theorem 2.2, we only need to prove that $\Phi(x \cdot y) = \Phi(x) \star \Phi(y)$ for $x = \underline{\mathfrak{G}}_{d_1} \ldots \underline{\mathfrak{G}}_{d_s}(\beta)1_{X^{[n]}}$, i.e. that

$$\Phi(\underline{\mathfrak{G}}_{d_1} \ldots \underline{\mathfrak{G}}_{d_s}(\beta)1_{X^{[n]}}) \star \Phi(x) = \Phi(\underline{\mathfrak{G}}_{d_1} \ldots \underline{\mathfrak{G}}_{d_s}(\beta)x)$$

From the previous result, we have

$$\mathfrak{F}_{d_1} \ldots \mathfrak{F}_{d_s}(\beta)\Phi(x) = \Phi(\underline{\mathfrak{G}}_{d_1} \ldots \underline{\mathfrak{G}}_{d_s}(\beta)x),$$

but $\mathfrak{F}_{d_1} \ldots \mathfrak{F}_{d_s}(\beta)$ is just the operator of multiplication by $\mathfrak{F}_{d_1} \ldots \mathfrak{F}_{d_s}(\beta)1_{X^{(n)}}$:

$$\begin{aligned}\mathfrak{F}_{d_1} \ldots \mathfrak{F}_{d_s}(\beta)y &= \pi_{(n),*}\left(\pi_{(n),1}^*\Phi_X(\underline{\mathfrak{G}}_{d_1}1_{X^{[n]}}) \star \cdots \star \pi_{(n),s}^*\Phi_X(\underline{\mathfrak{G}}_{d_s}1_{X^{[n]}}) \star \pi_{(n)}^* y \star \pi_{1\ldots s}^*\beta\right) \\ &= \pi_{(n),*}\left(\pi_{(n),1}^*\Phi_X(\underline{\mathfrak{G}}_{d_1}1_{X^{[n]}}) \star \cdots \star \pi_{(n),s}^*\Phi_X(\underline{\mathfrak{G}}_{d_s}1_{X^{[n]}}) \star \pi_{1\ldots s}^*\beta\right) \star y \\ &= \pi_{(n),*}\left(\pi_{(n),1}^*\Phi_X(\underline{\mathfrak{G}}_{d_1}1_{X^{[n]}}) \star \cdots \star \pi_{(n),s}^*\Phi_X(\underline{\mathfrak{G}}_{d_s}1_{X^{[n]}}) \star \pi_{(n)}^*1_{X^{(n)}} \star \pi_{1\ldots s}^*\beta\right) \star y \\ &= (\mathfrak{F}_{d_1} \ldots \mathfrak{F}_{d_s}(\beta)1_{X^{(n)}}) \star y.\end{aligned}$$

Hence we have

$$\mathfrak{F}_{d_1} \ldots \mathfrak{F}_{d_s}(\beta)\Phi(x) = (\mathfrak{F}_{d_1} \ldots \mathfrak{F}_{d_s}(\beta)\Phi(1_{X^{[n]}})) \star \Phi(x) = \Phi(\underline{\mathfrak{G}}_{d_1} \ldots \underline{\mathfrak{G}}_{d_s}(\beta)1_{X^{[n]}}) \star \Phi(x)$$

which proves the theorem. □

8 Examples

We now apply (18)

$$\mathfrak{q}_\nu(\alpha)v \cdot \mathfrak{q}_\lambda(\beta)v = \frac{n!}{C_\lambda}\sum_{h\in\lambda}(-1)^{\frac{1}{2}(a(g,h))}\mathfrak{q}_{\kappa_{gh}}\left(\iota_*^{g,h}\left(\alpha_h|_{X^{g,h}}\cdot\beta_l|_{X^{g,h}}\cdot\prod_{o\in O(g,h)}\pi_o^*(e_X)^{\mathrm{gr}(g,h)(o)}\right)\right)v$$

to give a multiplication table for $A^*(X^{[n]})$, by first computing the products $\alpha_g \star \beta_h$ for each pair (g,h) up to simultaneous conjugation. As the inclusions $S_n \hookrightarrow S_m$ for $m > n$ just add singleton orbits of $\langle g,h\rangle$ with no graph defect, the product in these factors is the ordinary intersection product so we only need to compute $\alpha_g \star \beta_h$ in the smallest S_n where a given conjugacy class of pairs (g,h) appears.

8.1 $n = 2$

The Chow ring of $X^{[2]}$ has a basis $\{\mathfrak{q}_2(\alpha), \mathfrak{q}_1\mathfrak{q}_1(\Gamma)\}$ where α ranges over a basis of $A^*(X)$ and Γ over a basis of $A^*\left(X^2\right)^{S_2}$. In particular,

$$\begin{aligned}
A^0\left(X^{[2]}\right) &= \mathfrak{q}_1\mathfrak{q}_1\left(\mathbb{C}\cdot 1_{X^2}\right)v\\
A^1\left(X^{[2]}\right) &= \mathfrak{q}_1\mathfrak{q}_1\left(A^1\left(X^2\right)\right)v \oplus \mathfrak{q}_2\left(\mathbb{C}\cdot 1_X\right)v\\
A^2\left(X^{[2]}\right) &= \mathfrak{q}_1\mathfrak{q}_1\left(A^2\left(X^2\right)\right)v \oplus \mathfrak{q}_2\left(A^1(X)\right)v\\
A^3\left(X^{[2]}\right) &= \mathfrak{q}_1\mathfrak{q}_1\left(A^3\left(X^2\right)\right)v \oplus \mathfrak{q}_2\left(A^2(X)\right)v\\
A^4\left(X^{[2]}\right) &= \mathfrak{q}_1\mathfrak{q}_1\left(A^4\left(X^2\right)\right)v
\end{aligned}$$

The identity is given by $\frac{1}{2}\mathfrak{q}_1\mathfrak{q}_1(1)v$. The elements of S_n are classified as $S_{2,(1,1)} = \{1\}$ with age 0, $S_{2,(2)} = \{(12)\}$ with age 1, and all graph defects vanish. The products in $A^*(X^2, S_2)$ are

$$\begin{aligned}
\alpha_{12}1_{S_2} \star \alpha_{12}1_{S_2} &= (\alpha_{12}\beta_{12})1_{S_2}\\
\alpha_{12}1_{S_2} \star \beta_2(12) &= (\alpha_{11}\beta_1)(12)\\
\alpha_1(12) \star \beta_1(12) &= \Delta_*(\alpha_1\beta_1)1_{S_2} && a((12),(12)) = 2\\
&= (\alpha_1\beta_1\Delta_{12})1_{S_2}
\end{aligned}$$

Thus the products in $A^*(X^{[3]})$ are given by:

$$\begin{aligned}
\mathfrak{q}_1\mathfrak{q}_1(\Gamma)v \cdot \mathfrak{q}_1\mathfrak{q}_1(\Lambda)v &= 2\mathfrak{q}_1\mathfrak{q}_1(\Gamma\cdot\Lambda)\\
\mathfrak{q}_1\mathfrak{q}_1(\Gamma)v \cdot \mathfrak{q}_2(\beta)v &= 2\mathfrak{q}_2(\Gamma|_\Delta\cdot\beta)\\
&= 2\mathfrak{q}_2(\Gamma_{11}\beta)\\
\mathfrak{q}_2(\alpha)v \cdot \mathfrak{q}_2(\beta)v &= -2\mathfrak{q}_1\mathfrak{q}_1(\Delta_*(\alpha\beta)).
\end{aligned}$$

8.2 $n=3$

The Chow ring of $X^{[3]}$ has a basis $\{\mathfrak{q}_3(\alpha), \mathfrak{q}_2\mathfrak{q}_1(\Gamma), \mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_1(\Lambda)\}$ where α ranges over a basis of $A^*(X)$, Γ over a basis of $A^*(X^2)$ and Γ over a basis of $A^*(X^3)^{S_3}$. In particular,

$$\begin{aligned}
A^0\left(X^{[2]}\right) &= \mathfrak{q}_1^3(\mathbb{C}\cdot 1_{X^3})v\\
A^1\left(X^{[2]}\right) &= \mathfrak{q}_1^3\left(A^1\left(X^3\right)\right)v \oplus \mathfrak{q}_2\mathfrak{q}_1(\mathbb{C}\cdot 1_{X^2})v\\
A^2\left(X^{[2]}\right) &= \mathfrak{q}_1^3\left(A^2\left(X^3\right)\right)v \oplus \mathfrak{q}_2\mathfrak{q}_1\left(A^1\left(X^2\right)\right)v \oplus \mathfrak{q}_3(\mathbb{C}\cdot 1_{X^2})v\\
A^3\left(X^{[2]}\right) &= \mathfrak{q}_1^3\left(A^3\left(X^3\right)\right)v \oplus \mathfrak{q}_2\mathfrak{q}_1\left(A^2\left(X^2\right)\right)v \oplus \mathfrak{q}_3\left(A^1(X)\right)v\\
A^4\left(X^{[2]}\right) &= \mathfrak{q}_1^3\left(A^4\left(X^3\right)\right)v \oplus \mathfrak{q}_2\mathfrak{q}_1\left(A^3\left(X^2\right)\right)v \oplus \mathfrak{q}_3\left(A^2(X)\right)v\\
A^5\left(X^{[2]}\right) &= \mathfrak{q}_1^3\left(A^5\left(X^3\right)\right)v \oplus \mathfrak{q}_2\mathfrak{q}_1\left(A^4\left(X^2\right)\right)v\\
A^6\left(X^{[2]}\right) &= \mathfrak{q}_1^3\left(A^6\left(X^3\right)\right)v
\end{aligned}$$

The identity is given by $\frac{1}{6}\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_1(1)v$. The elements of S_3 are classified as $1 \in (1,1,1)$ with age 0, $(12),(13),(23) \in (2,1)$ with age 1, and $(123),(132) \in (3)$ with age 2. The nontrivial graph defects are $\mathrm{gr}((123),(123))$ which is $\frac{1}{2}(3+2-1-1-1) = 1$ on the only orbit $\{1,2,3\}$, and $\mathrm{gr}((123),(132))$ which is $\frac{1}{2}(3+2-1-1-3) = 0$ on $\{1,2,3\}$.

The new products in $A^*(X^3, S_3)$ are as follows, where for the product gh we list other $g' \in S_3$ such that the pair (g',h) is conjugate to (g,h) in the third column:

$\alpha_{123}1_{S_3} \star \beta_1(123) = (\alpha_{111}\beta_1)(123)$		
$\alpha_{12}(12) \star \beta_1(123) = (\alpha_{11}\beta_1\Delta_{12})(23)$	$a((12),(123)) = 2$	$(23),(13)$
$\alpha_1(123) \star \beta_1(123) = (\alpha_1\beta_1 e_1)(132)$	$a((123),(123)) = 2$	
$\alpha_1(132) \star \beta_1(123) = (\alpha_1\beta_1\Delta_{123})1_{S_3}$	$a((132),(123)) = 4$	

Thus the products in $A^*(X^{[3]})$ are given by:

$$\begin{aligned}
\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_1(\Gamma)v \cdot \mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_1(\Lambda)v &= 6\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_1(\Gamma\cdot\Lambda)\\
\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_1(\Gamma)v \cdot \mathfrak{q}_2\mathfrak{q}_1(\Lambda)v &= 6\mathfrak{q}_2\mathfrak{q}_1(\Gamma_{112}\cdot\Lambda)\\
\mathfrak{q}_2\mathfrak{q}_1(\Gamma)v \cdot \mathfrak{q}_2\mathfrak{q}_1(\Lambda)v &= \frac{6}{3}(-\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_1(\Gamma_{13}\Lambda_{13}\Delta_{12})v + 2\mathfrak{q}_3(\Gamma_{11}\Lambda_{11})v)\\
&= 4\mathfrak{q}_3(\Gamma_{11}\Lambda_{11})v - 2\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_1(\Gamma_{13}\Lambda_{13}\Delta_{12})v\\
\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_1(\Gamma)v \cdot \mathfrak{q}_3(\Lambda)v &= 6\mathfrak{q}_3(\Gamma_{111}\cdot\Lambda)\\
\mathfrak{q}_2\mathfrak{q}_1(\Gamma)v \cdot \mathfrak{q}_3(\Lambda)v &= -6\mathfrak{q}_2\mathfrak{q}_1(\Gamma_{11}\Lambda_1\Delta_{12})v\\
\mathfrak{q}_3(\alpha)v \cdot \mathfrak{q}_3(\beta)v &= 3(\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_1(\Delta_*(\alpha\beta))v - \mathfrak{q}_3(\alpha\beta e)v).
\end{aligned}$$

8.3 $n=4$

The Chow ring of $X^{[4]}$ has a basis

$$\begin{aligned}
&\mathfrak{q}_4(\alpha),\ \alpha \in A^*(X)\\
&\mathfrak{q}_3\mathfrak{q}_1(\alpha),\ \alpha \in A^*(X^2)\\
&\mathfrak{q}_2\mathfrak{q}_2(\alpha),\ \alpha \in A^*(X^2)^{S_2}\\
&\mathfrak{q}_2\mathfrak{q}_1\mathfrak{q}_1(\alpha),\ \alpha \in A^*(X^3)^{1\times S_2}\\
&\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_1(\alpha),\ \alpha \in A^*(X^4)^{S_4}
\end{aligned}$$

The identity is given by $\frac{1}{24}\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_1\,(1)\,v$. The elements of S_4 are classified as $S_{4,(1,1,1,1)} = \{1_{S_4}\}$ with age 0, $S_{4,(2,1,1)} = \{(12),(13),(14),(23),(24),(34)\}$ with age 1,

$$S_{4,(3,1)} = \{(123),(132),(124),(142),(134),(143),(234),(243)\}$$

with age 2, $S_{4,(2,2)} = \{(12)(34),(13)(24),(14)(23)\}$ with age 2, and

$$S_{4,(2,2)} = \{(1234),(1243),(1324),(1342),(1423),(1432)\}$$

with age 3.

The new products in $A^*(X^4, S_4)$ are

$\alpha_{123}(14) \star \beta_{12}(123) = (\alpha_{111}\beta_{11})(1234)$	$a(g,h) = 0$	$(24),(34)$
$\alpha_{12}(124) \star \beta_{12}(123) = (\alpha_{11}\beta_{11}\Delta_{12})(14)(23)$	$a(g,h) = 2$	$(234),(143)$
$\alpha_{12}(134) \star \beta_{12}(123) = (\alpha_{11}\beta_{11}\Delta_{12})(124)$	$a(g,h) = 2$	$(142),(243)$
$\alpha_{1234}1_{S_4} \star \beta_1(1234) = (\alpha_{1111}\beta_1)(1234)$	$a(g,h) = 0$	
$\alpha_{123}(12) \star \beta_1(1234) = (\alpha_{111}\beta_1\Delta_{12})(234)$	$a(g,h) = 2$	$(23),(34),(14)$
$\alpha_{123}(13) \star \beta_1(1234) = (\alpha_{111}\beta_1\Delta_{12})(12)(34)$	$a(g,h) = 2$	(24)
$\alpha_{12}(123) \star \beta_1(1234) = (\alpha_{11}\beta_1 e_1)(1342)$	$a(g,h) = 2$	$(234),(134),(124)$
$\alpha_{12}(132) \star \beta_1(1234) = (\alpha_{11}\beta_1\Delta_{123})(34)$	$a(g,h) = 4$	$(243),(143),(142)$
$\alpha_{1234}1_{S_4} \star \beta_{12}(12)(34) = (\alpha_{1122}\beta_{12})(12)(34)$	$a(g,h) = 0$	
$\alpha_{123}(12) \star \beta_{12}(12)(34) = (\alpha_{133}\beta_{13}\Delta_{12})(34)$	$a(g,h) = 2$	(34)
$\alpha_{123}(13) \star \beta_{12}(12)(34) = (\alpha_{111}\beta_{11})(1234)$	$a(g,h) = 0$	$(23),(14),(34)$
$\alpha_{12}(123) \star \beta_{12}(12)(34) = (\alpha_{11}\beta_{11}\Delta_{12})(134)$	$a(g,h) = 2$	$(132),(124),(142),(134),$ $(143),(234),(243)$
$\alpha_1(1234) \star \beta_1(1234) = (\alpha_1\beta_1 e_1\Delta_{12})(13)(24)$	$a(g,h) = 4$	
$\alpha_1(1243) \star \beta_1(1234) = (\alpha_1\beta_1 e_1\Delta_{12})(142)$	$a(g,h) = 4$	$(1423),(1342),(1324)$
$\alpha_1(1432) \star \beta_1(1234) = (\alpha_1\beta_1\Delta_{1234})1_{S_4}$	$a(g,h) = 6$	
$\alpha_{12}(12)(34) \star \beta_1(1234) = (\alpha_{11}\beta_1\Delta_{123})(24)$	$a(g,h) = 4$	$(14)(23)$
$\alpha_{12}(13)(24) \star \beta_1(1234) = (\alpha_{11}\beta_1 e_1)(1432)$	$a(g,h) = 2$	

Thus the products in $A^*(X^{[4]})$ are given by:

$$\begin{aligned}
\mathfrak{q}_1^4(\alpha)v \cdot \mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_1(\beta)v &= 24\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_1(\alpha\cdot\beta) \\
\mathfrak{q}_1^4(\alpha)v \cdot \mathfrak{q}_2\mathfrak{q}_1\mathfrak{q}_1(\beta)v &= 24\mathfrak{q}_2\mathfrak{q}_1(\alpha_{1123}\cdot\beta) \\
\mathfrak{q}_1^4(\alpha)v \cdot \mathfrak{q}_2\mathfrak{q}_2(\beta)v &= 24\mathfrak{q}_2\mathfrak{q}_2(\alpha_{1122}\cdot\beta) \\
\mathfrak{q}_1^4(\alpha)v \cdot \mathfrak{q}_3\mathfrak{q}_1(\beta)v &= 24\mathfrak{q}_3\mathfrak{q}_1(\alpha_{1112}\cdot\beta) \\
\mathfrak{q}_1^4(\alpha)v \cdot \mathfrak{q}_4(\beta)v &= 24\mathfrak{q}_4(\alpha_{1111}\cdot\beta) \\
\mathfrak{q}_2\mathfrak{q}_1\mathfrak{q}_1(\alpha)v \cdot \mathfrak{q}_2\mathfrak{q}_1\mathfrak{q}_1(\beta)v &= \frac{24}{6}(-\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_1(\alpha_{134}\beta_{134}\Delta_{12})v + 5\mathfrak{q}_3\mathfrak{q}_1(\alpha_{112}\beta_{112})v) \\
&= 20\mathfrak{q}_3\mathfrak{q}_1(\alpha_{112}\beta_{112})v - 4\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_1(\alpha_{134}\beta_{134}\Delta_{12})v \\
\mathfrak{q}_2\mathfrak{q}_1\mathfrak{q}_1(\alpha)v \cdot \mathfrak{q}_3\mathfrak{q}_1(\beta)v &= \frac{24}{6}(-3\mathfrak{q}_2\mathfrak{q}_1\mathfrak{q}_1(\alpha_{113}\beta_{13}\Delta_{12})v + 3\mathfrak{q}_4(\alpha_{111}\beta_{11})v) \\
&= 12\mathfrak{q}_4(\alpha_{111}\beta_{11})v - 12\mathfrak{q}_2\mathfrak{q}_1\mathfrak{q}_1(\alpha_{113}\beta_{13}\Delta_{12})v \\
\mathfrak{q}_2\mathfrak{q}_1\mathfrak{q}_1(\alpha)v \cdot \mathfrak{q}_2\mathfrak{q}_2(\beta)v &= \frac{24}{6}(2\mathfrak{q}_2\mathfrak{q}_1\mathfrak{q}_1(\alpha_{133}\beta_{13}\Delta_{12})v + 4\mathfrak{q}_4(\alpha_{111}\beta_{11})v) \\
&= 16\mathfrak{q}_4(\alpha_{111}\beta_{11})v + 8\mathfrak{q}_2\mathfrak{q}_1\mathfrak{q}_1(\alpha_{133}\beta_{13}\Delta_{12})v
\end{aligned}$$

$$
\begin{aligned}
\mathfrak{q}_3\mathfrak{q}_1(\alpha)v\cdot\mathfrak{q}_3\mathfrak{q}_1(\beta)v &= \frac{24}{8}(\mathfrak{q}_1^4(\alpha_{14}\beta_{14}\Delta_{123})v - \mathfrak{q}_3\mathfrak{q}_1(\alpha\beta e_1)v - 3\mathfrak{q}_3\mathfrak{q}_1(\alpha_{11}\beta_{11}\Delta_{12})v - 3\mathfrak{q}_2\mathfrak{q}_2(\alpha_{11}\beta_{11}\Delta_{12})v)\\
&= 3\mathfrak{q}_1^4(\alpha_{14}\beta_{14}\Delta_{123})v - 3\mathfrak{q}_3\mathfrak{q}_1(\alpha\beta e_1)v - 9\mathfrak{q}_3\mathfrak{q}_1(\alpha_{11}\beta_{11}\Delta_{12})v - 9\mathfrak{q}_2\mathfrak{q}_2(\alpha_{11}\beta_{11}\Delta_{12})v)\\
\mathfrak{q}_3\mathfrak{q}_1(\alpha)v\cdot\mathfrak{q}_2\mathfrak{q}_2(\beta)v &= -24\mathfrak{q}_3\mathfrak{q}_1(\alpha_{11}\beta_{11}\Delta_{12})v\\
\mathfrak{q}_2\mathfrak{q}_1\mathfrak{q}_1(\alpha)v\cdot\mathfrak{q}_4(\beta)v &= \frac{24}{6}(-4\mathfrak{q}_3\mathfrak{q}_1(\alpha_{111}\beta_1\Delta_{12})v - 2\mathfrak{q}_2\mathfrak{q}_2(\alpha_{111}\beta_1\Delta_{12})v)\\
&= -16\mathfrak{q}_3\mathfrak{q}_1(\alpha_{111}\beta_1\Delta_{12})v - 8\mathfrak{q}_2\mathfrak{q}_2(\alpha_{111}\beta_1\Delta_{12})v\\
\mathfrak{q}_3\mathfrak{q}_1(\alpha)v\cdot\mathfrak{q}_4(\beta)v &= \frac{24}{8}(-4\mathfrak{q}_4(\alpha_{11}\beta_1 e_1)v + 4\mathfrak{q}_2\mathfrak{q}_1\mathfrak{q}_1(\alpha_{11}\beta_1\Delta_{123})v)\\
&= 12\mathfrak{q}_2\mathfrak{q}_1\mathfrak{q}_1(\alpha_{11}\beta_1\Delta_{123})v - 12\mathfrak{q}_4(\alpha_{11}\beta_1 e_1)v\\
\mathfrak{q}_2\mathfrak{q}_2(\alpha)v\cdot\mathfrak{q}_4(\beta)v &= \frac{24}{3}(2\mathfrak{q}_2\mathfrak{q}_1\mathfrak{q}_1(\alpha_{11}\beta_1\Delta_{123})v - \mathfrak{q}_4(\alpha_{11}\beta_1 e_1)v)\\
&= 16\mathfrak{q}_2\mathfrak{q}_1\mathfrak{q}_1(\alpha_{11}\beta_1\Delta_{123})v - 8\mathfrak{q}_4(\alpha_{11}\beta_1 e_1)v\\
\mathfrak{q}_4(\alpha)v\cdot\mathfrak{q}_4(\beta)v &= \frac{24}{6}(\mathfrak{q}_2\mathfrak{q}_2(\alpha_1\beta_1 e_1\Delta_{12})v + 4\mathfrak{q}_3\mathfrak{q}_1(\alpha_1\beta_1 e_1\Delta_{12})v - \mathfrak{q}_1^4(\alpha_1\beta_1\Delta_{1234})v)\\
&= 4\mathfrak{q}_2\mathfrak{q}_2(\alpha_1\beta_1 e_1\Delta_{12})v + 16\mathfrak{q}_3\mathfrak{q}_1(\alpha_1\beta_1 e_1\Delta_{12})v - 4\mathfrak{q}_1^4(\alpha_1\beta_1\Delta_{1234})v
\end{aligned}
$$

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