The Chow Ring of the Fano Variety of Lines of a Cubic Fourfold via Lefschetz Actions

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Author's Declarations

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Contents

1 Introduction

Let $S^{[n]}$ be the Hilbert scheme of n points of a projective $K3$ surface S. The main theorem of [23] states that the action on the cohomology ring of the Neron–Severi Lie algebra $\mathfrak{g}_{\mathrm{NS}}$, which is the Lie subalgebra of $\mathrm{End}_{\mathbb{Q}}(H^*(S^{[n]},\mathbb{Q}))$ generated by all algebraic Lefschetz triples (e_a, f_a, h) (see Section 2), can be lifted to the Chow ring, and explicit formulas for the lifts of e_a, f_a and h in terms of Nakajima operators are provided. The goal of this master's thesis is to investigate the question whether this theorem is equally true for the Fano variety of lines $F(Y)$ of a smooth cubic hypersurface $Y \subseteq \mathbb{P}^5$. An analog for the Fano varieties $F(Y)$ could be expected because they are examples of hyperkähler varieties deformation equivalent to the Hilbert schemes of points $S^{[2]}$ of suitable $K3$ surfaces S [2], which form the most well-known class of hyperkähler varieties other than $K3$ surfaces themselves.

Let X be a hyperkähler variety deformation equivalent to $S^{[2]}$ for a $K3$ surface S. In particular, X is of complex dimension 4. As a matter of notation, for any cycle class $Z \in \mathrm{CH}^*(X)$ denote by Z_1 and Z_2 the pullbacks to $X \times X$ via the two projections. Denote by q_X the Beauville–Bogomolov quadratic form on $H^2(X,\mathbb{Q})$ and by $\mathfrak{B} \in H^4(X\times X,\mathbb{Q})$ its associated cohomology class. Let $a \in H^{1,1}(X,\mathbb{Q})$ such that $q_X(a) \neq 0$. Recall the Lefschetz operator e_a which is given by the cup product with the class a as well as the Lefschetz dual operator f_a, both endomorphisms of $H^*(X,\mathbb{Q})$. Moreover, denote by h the grading operator given on $H^k(X,\mathbb{Q})$ by multiplication by $k-4$. The first result of this work is then Theorem 5.8, providing an explicit lift F_a of the Lefschetz dual f_a to the Chow ring of correspondences in terms of a lift $L \in \mathrm{CH}^2(X \times X)$ of $\mathfrak{B}$. With some additional work, we obtain in Proposition 5.14 a lift H of h depending only on L.

Conjecture 1.1 (Conjecture 5.21)**.** *Let X be a hyperkähler variety of $K3^{[2]}$-type. Let $L \in \mathrm{CH}^2(X \times X)$ be a lift of $\mathfrak{B}$ and let l be its pullback along the diagonal embedding $\Delta : X \hookrightarrow X \times X$. For any divisor class $a \in \mathrm{CH}^1(X)$ with $q_X(a) \neq 0$ we define*

$$F_a := \frac{4}{25q_X(a)}(l_1a_1 + l_2a_2) + \frac{2}{q_X(a)}L(a_1 + a_2) \in \mathrm{CH}^3(X \times X),$$
$$H := \frac{4}{23\cdot 25}(l_2^2 - l_1^2) + \frac{2}{25}L(l_2 - l_1) \in \mathrm{CH}^4(X \times X).$$

Then there exists a lift L as above such that the linear map

$$\varphi : \mathfrak{g}_{\mathrm{NS}}(X) \longrightarrow \mathrm{CH}^*(X \times X)$$

given by $\varphi(e_a) = \Delta_*(a)$, $\varphi(f_a) = F_a$ and $\varphi(h) = H$ is a well-defined Lie algebra homomorphism.

This is analogous to the main theorem of [23]. For any hyperkähler variety X of $K3^{[n]}$-type there exists a lift $L \in \mathrm{CH}^2(X \times X)$ of $\mathfrak{B}$ coming from Markman's twisted sheaves [19], see also [26, Theorem 9.15]. In the case $X = S^{[2]}$ for a projective

$K3$ surface S, it is shown in [26, Proposition 16.1] that the latter agrees with the explicit construction of a lift L in [26], stated in Theorem 4.21. We use this explicit lift in Section 6 in order to prove that our formulas for the lifts F_a of f_a and H of h agree with the canonical lifts provided in [23] in terms of Nakajima operators. This is the content of Theorems 6.7 and 6.8. Hence, the main theorem of *loc. cit.* shows that the conjecture is true if $X = S^{[2]}$. In the case of the Fano variety of lines $F = F(Y)$ of a smooth cubic fourfold Y, the explicit construction of L in [26], given in Theorem 4.14, is not rigorously proven to coincide with the cycle class obtained in [26, Theorem 9.15], but Shen and Vial expect it to be true as well [26, p. 4].

Theorem 1.2 (Theorem 5.19)**.** Let $F = F(Y)$ be the Fano variety of lines of a smooth cubic fourfold Y and $g \in \mathrm{CH}^1(F)$ the Plücker polarization class. Let $L \in \mathrm{CH}^2(F \times F)$ be the explicit lift of $\mathfrak{B}$ from Theorem 4.14. Then there is an $\mathfrak{sl}_2(\mathbb{Q})$-action on $\mathrm{CH}^*(F)$ given by the Lie algebra homomorphism

$$\begin{aligned} \mathfrak{sl}_2(\mathbb{Q}) &\to \mathrm{CH}^*(F \times F), \\ e &\mapsto \Delta_*(g), \\ f &\mapsto F_g, \\ h &\mapsto H, \end{aligned}$$

lifting the $\mathfrak{sl}_2(\mathbb{Q})$-action on the cohomology ring $H^*(F, \mathbb{Q})$ given by e_g, f_g and h.

Whenever $F(Y)$ has Picard rank 1, i.e., $\mathrm{CH}^1(F(Y)) = \langle g \rangle$, which is true for very general cubic fourfolds Y, Conjecture 1.1 indeed reduces to Theorem 1.2. Even more evidence for the conjecture is provided by Proposition 5.29, where we are able to prove the additional commutation relation $[F_a, F_b] = 0$ in full generality.
We propose two new relations in Conjecture 5.15 which hold in cohomology and would yield a generalization of Theorem 1.2 to the case of divisor classes different from g, see Proposition 5.20. Conjecture 1.1, however, would still be out of reach. The reason we do not succeed in proving the latter in the case of the Fano variety of lines $F(Y)$ is that we neither have at our disposal an analog of the machinery of Nakajima operators nor do we have sufficiently strong injectivity results for the cycle class map yet, extending results such as [6, Proposition 6.4], stated below as Theorem 4.15, which is the main geometrical input used in the proof of Theorem 1.2.
One of the reasons why it is desirable to have explicit canonical lifts of f_a and especially h available, is that the eigenspace decomposition of the lift H of h, if it is indeed diagonalizable, could be expected to be multiplicative with respect to the intersection product. This is because the eigenspace decomposition of H can be viewed as an analog of the Beauville decomposition in the abelian variety case, as discussed in the introduction of [22]. The following result establishes such an eigenspace decomposition in particular in the cases $X = F(Y)$ and $X = S^{[2]}$, mildly upgraded by Theorem 5.22.

Theorem 1.3 (Theorem 5.22)**.** Let X be a hyperkähler variety of $K3^{[2]}$-type endowed with a lift $L \in \mathrm{CH}^2(X \times X)$ of $\mathfrak{B}$ satisfying all the relations (4.6)-(4.9) from Section 4.1, e.g., X can be the Fano variety of lines of a smooth cubic fourfold or the Hilbert scheme of two points of a projective $K3$ surface endowed with the explicit lift L of Theorems 4.14 and 4.21, respectively. Let $\Lambda^i_\lambda \subseteq \mathrm{CH}^i(X)$ be the eigenspace for the eigenvalue λ of H_*. The operator $H_* \in \mathrm{End}_{\mathbb{Q}}(\mathrm{CH}^*(X))$ is diagonalizable with eigenspace decomposition

$$\begin{aligned}
\mathrm{CH}^0(X) &= \Lambda^0_{-4},\\
\mathrm{CH}^1(X) &= \Lambda^1_{-2},\\
\mathrm{CH}^2(X) &= \Lambda^2_0 \oplus \Lambda^2_{-2},\\
\mathrm{CH}^3(X) &= \Lambda^3_2 \oplus \Lambda^3_0,\\
\mathrm{CH}^4(X) &= \Lambda^4_4 \oplus \Lambda^4_2 \oplus \Lambda^4_0,
\end{aligned}$$

where all direct summands are non-trivial and those not sitting in the leftmost column belong to the homologically trivial cycle classes $\mathrm{CH}^*(X)_{\mathrm{hom}}$. Conversely, the cycle class map is injective on Λ^3_2 as well as on $\Lambda^4_4 = \langle l^2 \rangle$.

This decomposition agrees with the Fourier decomposition of [26, Theorem 2] by Theorem 5.26, implying in particular the following multiplicativity conjecture in the case of the Fano variety of lines of a very general cubic fourfold. We note also that the Fourier decomposition requires all the mentioned relations (4.6)-(4.9) from Section 4.1 while the last one of these (4.9) is not actually needed for Theorem 1.3. We introduce the following notation. For $s \in \mathbb{Z}$ let

$$\mathrm{CH}^i(X)_s := \big\{ Z \in \mathrm{CH}^i(X) : H_*(Z) = (2i - 4 - 2s)Z \big\} = \Lambda^i_{2i-4-2s}.$$

In here, s should be seen as a sort of defect. The terms with $s = 0$ give the expected eigenvalue $2i - 4$ of H_* and correspond to the leftmost column in the eigenspace decomposition of H_* above. We can see that only terms with $s \geq 0$ occur. In fact, there is a conjecture by Beauville in the abelian variety case which predicts this behavior, see again the introduction of [22]. It also predicts the injectivity of the cycle class map on $\mathrm{CH}^*(X)_0$, and Theorem 1.3 confirms this in all codimensions except 2. We can now state the conjecture as follows.

Conjecture 1.4 (Conjecture 5.23)**.** Let X be as in Theorem 1.3. Then for all occurring $s, t \in \mathbb{Z}$ we conjecture that the intersection product gives a well-defined map

$$\mathrm{CH}^i(X)_s \times \mathrm{CH}^j(X)_t \overset{\cdot}{\longrightarrow} \mathrm{CH}^{i+j}(X)_{s+t}.$$

This can be rewritten in equivalent ways, see [22] and also our short discussion after Conjecture 5.23. By Theorem 1.4 of *loc. cit.* and our Theorem 6.8, the above conjecture is true for $X = S^{[2]}$. An independent proof is obtained through Theorem 5.26, stating that the decomposition of Theorem 1.3 agrees with the Fourier decomposition of [26, Theorem 2], and using the multiplicativity of the latter [26, Theorem 3]. In the case of the Fano variety $X = F(Y)$ we show the simplest case of divisors directly in Proposition 5.24. If Y is very general, the conjecture is true by the corresponding result for the Fourier decomposition, see Corollary 5.27 and Remark 5.28. In fact, using [6, Proposition A.7], only the case $i = j = 2$, $s = t = 0$ is not yet known for arbitrary smooth cubic fourfolds Y. In the paragraph preceding [26, Theorem 3], Shen and Vial remark that these results actually provide strong evidence for the Fourier version of Conjecture 1.4 which they, however, do not state explicitly. It agrees, of course, with Conjecture 1.4 because the decompositions agree.

Theorem 1.5 (Corollary 5.27)**.** Conjecture 1.4 is true if $X = S^{[2]}$ for a projective $K3$ surface S and if $X = F(Y)$ for a *very general* cubic fourfold Y.

The text is structured as follows: We will begin in Section 2 by recalling the Grothendieck standard conjecture of Lefschetz type in order to provide some general motivation. In Section 3 we introduce the Hilbert and Quot functor and state their representability. In Section 4 we then concentrate on those Hilbert schemes especially relevant to us, the Fano variety of lines of smooth cubic fourfolds and the Hilbert scheme of points of $K3$ surfaces which at the same time provide examples of hyperkähler varieties. We introduce the Beauville–Bogomolov form q_X and its associated cohomology class $\mathfrak{B}$ in Subsection 4.1. In the two following subsections we in particular introduce the explicit lifts of $\mathfrak{B}$ to the Chow ring constructed in [26]. Sections 5 and 6 contain our new results, outlined above, and we end with a short appendix, collecting a few lemmas.

In this master's thesis we work over $\mathbb{C}$, and the Chow ring is always with coefficients in $\mathbb{Q}$. For basics on the Chow ring and intersection theory we refer to [5, 7]. We will also make heavy use of the language of algebraic correspondences, as explained, e.g., in [15]. Finally, by a variety we mean a separated, integral scheme of finite type over a field, usually over $\mathbb{C}$.

2 The Grothendieck standard conjecture of Lefschetz type

We start by recalling very briefly some basic facts from Kähler geometry. These can be found, e.g., in [12, Section 1.2 and Chapter 3]. On a compact complex Kähler manifold X of complex dimension n, the operator of cup product with a given Kähler class a in $H^{1,1}(X,\mathbb{C})$ is called the *Lefschetz operator*, denoted

$$e_a : H^\bullet(X,\mathbb{C}) \to H^{\bullet+2}(X,\mathbb{C}),\ \beta \mapsto \beta \cup a.$$

More generally, we will always denote by e_a the cup product operator with a given class $a \in H^2(X)$, even if a is not a Kähler class and the field of coefficients is different from $\mathbb{C}$. In this particular case, however, the hard Lefschetz theorem gives the remarkable result that k-fold iteration is an isomorphism

$$e_a^k : H^{n-k}(X,\mathbb{C}) \xrightarrow{\cong} H^{n+k}(X,\mathbb{C}). \tag{2.1}$$

Let next $h \in \mathrm{End}_{\mathbb{C}}(H^*(X,\mathbb{C}))$ be the *grading operator*, acting on $H^k(X,\mathbb{C})$ by multiplication with the integer $k-n$. Instead of coefficients in $\mathbb{C}$ we can equally consider coefficients in $\mathbb{Q}$.

Lemma 2.1. Let $a \in H^2XF,\mathbb{Q})$ such that (2.1) holds for all $1 \le k \le n$. Then there exists a unique endomorphism $f_a \in \mathrm{End}_{\mathbb{Q}}(H^*(X,\mathbb{Q}))$, called the *Lefschetz dual operator*, which decreases the degree by 2 and satisfies $[e_a, f_a] = h$. In particular (e_a, f_a, h) is an $\mathfrak{sl}_2$-triple, i.e., the commutation relations

$$[e_a, f_a] = h,\ [h, e_a] = 2e_a,\ [h, f_a] = -2f_a$$

hold.

Proof. Existence can be shown using the Hodge star operator in a linear algebra version, see the proof of [12, Proposition 1.2.26]. Another proof, using the primitive decomposition, can be found in [14, Proposition 1.4.6].
Uniqueness is rather simple. We abbreviate $f := f_a$, $e := e_a$ and $H^i := H^i(X,\mathbb{Q})$. First observe that f is uniquely determined on H^0 and H^1 because it decreases the degree by 2. Now, let f be determined on H^i. Let $\beta \in H^{2n-i}$ and write $\beta = e^{n-i}(\widehat{\beta})$ for a unique $\widehat{\beta} \in H^i$. Then using $[e,f] = h$ several times we can express $f(\beta) = f(e^{n-i}(\widehat{\beta}))$ as a polynomial in a and $\widehat{\beta}$ with the additional summand $e^{n-i}(f(\widehat{\beta}))$ which is determined already. Hence, if f is determined on H^i, then it is determined on H^{2n-i} as well. Next, if f is determined on H^{2n-i+4}, then it is determined also on H^i. Indeed, for $\beta \in H^i$ we have $f(\beta) \in H^{i-2}$, and therefore $e^{n-i+2}(f(\beta))$ can be expressed as a polynomial in a and β with the addtional summand $f(e^{n-i+2}(\beta))$. Here, $e^{n-i+2}(\beta) \in H^{2n-i+4}$ and e^{n-i+2} is an isomorphism $H^{i-2} \cong H^{2n-i+2}$, concluding the proof. □

Remark 2.2. One can also use the commutation relations to express f_a directly as a sum of compositions of powers of e_a and the inverses $(e_a^k)^{-1}$ coming from the hard Lefschetz theorem. For example, if $n = 4$, we have $f_a = 4(e_a^4)^{-1} \circ e_a^3$ on $H^2(X)$ and $f_a = 4e_a \circ (e_a^4)^{-1} \circ e_a^2 + 2(e_a^2)^{-1} \circ e_a$ on $H^4(X)$. It is possible to derive a general formula of this sort but it is not overly helpful for our purposes.

If we now change the setting slightly and let X be a smooth projective variety over $\mathbb{C}$ and a an ample divisor with cohomology class in $H^2(X,\mathbb{Q}) \cap H^{1,1}(X,\mathbb{C})$, again denoted a, then in $H^2(X,\mathbb{C})$ the class a can be represented by a Kähler form, so that the e_a^k are isomorphisms by the hard Lefschetz theorem, and Lemma 2.1 gives a unique formal Lefschetz dual f_a. By construction, (e_a, f_a, h) is then an $\mathfrak{sl}_2$-triple, here with rational coefficients. What can be observed about the operator e_a, moreover, is that it is algebraic in the sense that it lifts to the algebraic correspondence $\Delta_*(a) \in \mathrm{CH}^{n+1}(X \times X)$ in the Chow ring, where $\Delta : X \hookrightarrow X \times X$ denotes the diagonal embedding. The Grothendieck standard conjectures now ask whether f_a is algebraic as well [15].

Conjecture 2.3 (Grothendieck's standard conjecture of Lefschetz type)**.** Let a be an ample divisor class on a smooth projective variety X of dimension n. Then the Lefschetz dual f_a lifts to a correspondence $F_a \in \mathrm{CH}^{n-1}(X \times X)$.

As a consequence, the commutator of algebraic correspondences

$$H_a := [\Delta_*(a), F_a]$$

is automatically a lift of the grading operator h. Note that H_a, contrary to h, might depend on a.

Remark 2.4. In the literature two different definitions for the Lefschetz dual operator can be found in the algebraic setting. The Grothendieck standard conjectures of Lefschetz type with respect to either one, however, are equivalent [14, 15]. The other Lefschetz dual f'_a is a one-sided inverse of the Lefschetz operator and can be defined via the following diagrams (which are actually the same), the first for $i \leq n$, the second for $i > n$:

$$\begin{array}{ccc} H^i(X) & \xrightarrow[\cong]{e_a^{n-i}} & H^{2n-i}(X) \\ \downarrow{\scriptstyle f'_a} & & \downarrow{\scriptstyle e_a} \\ H^{i-2}(X) & \xrightarrow[e_a^{n-(i-2)}]{\cong} & H^{2n-(i-2)}(X), \end{array} \qquad \begin{array}{ccc} H^{2n-i}(X) & \xrightarrow[\cong]{e_a^{i-n}} & H^i(X) \\ \downarrow{\scriptstyle e_a} & & \downarrow{\scriptstyle f'_a} \\ H^{2n-(i-2)}(X) & \xrightarrow[e_a^{i-2-n}]{\cong} & H^{i-2}(X). \end{array}$$

However, we will only be interested in the classical definition from before.

Let $a \in H^2(X,\mathbb{Q})$ be *Lefschetz*, i.e., (2.1) holds for all $1 \leq k \leq n$. By Lemma 2.1, a unique operator f_a exists such that e_a, f_a and h satisfy the $\mathfrak{sl}_2$-commutation relations. The triple (e_a, f_a, h) is then called a *Lefschetz triple*. Looijenga and Lunts [18] and Verbitsky [28] introduced the *total Lie algebra*

$$\mathfrak{g}(X) \subseteq \mathrm{End}_{\mathbb{Q}}(H^*(X,\mathbb{Q}))$$

of a smooth projective variety X which is generated by all Lefschetz triples (e_a, f_a, h). The *Neron–Severi Lie algebra* of X is the Lie subalgebra

$$\mathfrak{g}_{\mathrm{NS}}(X) \subseteq \mathfrak{g}(X)$$

generated by only those Lefschetz triples where a is algebraic, i.e., $a \in H^{1,1}(X,\mathbb{Q})$. In view of the Grothendieck standard conjecture of Lefschetz type, a natural question to ask, then, is whether the Neron–Severi Lie algebra action on the cohomology ring $H^*(X,\mathbb{Q})$ can be lifted to an action on the Chow ring $\mathrm{CH}^*(X)$ or even whether there is a Lie algebra homomorphism

$$\mathfrak{g}_{\mathrm{NS}}(X) \longrightarrow \mathrm{CH}^*(X \times X)$$

to the ring of correspondences $\mathrm{CH}^*(X \times X)$ such that

$$\begin{array}{ccc} \mathfrak{g}_{\mathrm{NS}}(X) & \longrightarrow & \mathrm{CH}^*(X \times X) \\ & \searrow & \downarrow {\scriptstyle \mathrm{cl}} \\ & & \mathrm{End}_{\mathbb{Q}}(H^*(X,\mathbb{Q})) \end{array}$$

commutes, where cl denotes the cycle class map. By the main theorem of [23], this is the case if X is the Hilbert scheme of points of a projective $K3$ surface. We will investigate this question for the Fano variety of lines of a smooth cubic fourfold in Section 5, obtaining Theorem 5.19 as a partial analog. For more context we also refer back to the introduction.

3 Hilbert schemes

This part freely follows [27]. We give some foundational definitions regarding the Hilbert and Quot functor and introduce notions such as that of a fine moduli scheme. We do not prove the existence of the Hilbert or Quot scheme as this would take us too far, but we deal with a few basic questions such as well-definedness of the functors.
Let S be a locally noetherian base scheme, and denote by $\mathbf{Sch}_S$ the category of locally noetherian schemes over S. By an S-scheme (or scheme over S) we will always mean an object of this category. The n-dimensional projective space over S is defined to be $\mathbb{P}^n_S := \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} S$, and a projective S-scheme X is a closed embedding $i : X \hookrightarrow \mathbb{P}^n_S$ for some fixed n. Moreover, let $\mathbf{Set}$ be the category of sets.

Definition 3.1. Let X and T be locally noetherian S-schemes. A *family of closed subschemes of X over S parametrized by T* is a closed subscheme $Z \subseteq X \times_S T$ which is flat over T with respect to the projection $X \times_S T \to T$.

Definition 3.2. Let X be a projective S-scheme. The *Hilbert functor*

$$\mathcal{H}ilb_{X/S} : (\mathbf{Sch}_S)^{\mathrm{op}} \to \mathbf{Set}$$

associates to an object T the set of all families Z of closed subschemes of X over S parametrized by T. For a morphism $f : T' \to T$ of schemes over S the map of sets $\mathcal{H}ilb_{X/S}(f)$ sends a family Z parametrized by T to the family $Z \times_T T'$ parametrized by T', viewed as a family of X over S via the natural isomorphism $X \times_S T' \cong (X \times_S T) \times_T T'$.

Lemma 3.3. The Hilbert functor is well-defined.

Proof. We first show for any morphism $f : T' \to T$ of S-schemes that $Z \times_T T'$ is a closed subscheme of $(X \times_S T) \times_T T' \cong X \times_S T'$ which is flat over T'. For this, consider

$$\begin{array}{ccccc} Z \times_T T' & \xrightarrow{i \times \mathrm{id}_{T'}} & (X \times_S T) \times_T T' & \longrightarrow & T' \\ \downarrow & & \downarrow & \square & \downarrow f \\ Z & \xhookrightarrow[i]{\text{cl. emb.}} & X \times_S T & \longrightarrow & T. \end{array}$$

The right square is cartesian by definition, and so is the entire rectangle, hence so is the left square. As closed embeddings are stable under base change, $i \times \mathrm{id}_{T'}$ is a closed embedding as well. The compositon of the two lower horizontal arrows is flat by assumption, and because flat morphisms are stable under base change, so is the composition of the upper two horizontal arrows, so $Z \times_T T'$ is flat over T', as desired.
It remains to consider functoriality for compositions $T'' \xrightarrow{g} T' \xrightarrow{f} T$ of morphisms of S-schemes. Consider the following diagram:

$$\begin{array}{ccc}
(Z \times_T T') \times_{T'} T'' & \xrightarrow{\cong} & Z \times_T T'' \\
\Big\downarrow{\scriptstyle (i\times \mathrm{id}_{T'})\times \mathrm{id}_{T''}} & & \Big\downarrow{\scriptstyle i\times \mathrm{id}_{T''}} \\
((X \times_S T) \times_T T') \times_{T'} T'' & \xrightarrow{\cong} & (X \times_S T) \times_T T'' \\
\Big\downarrow{\scriptstyle \cong} & & \Big\downarrow{\scriptstyle \cong} \\
(X \times_S T') \times_{T'} T'' & \xrightarrow[\cong]{} & X \times_S T''.
\end{array}$$

The upper square commutes by the naturality of the isomorphism $(Y \times_T T') \times_{T'} T'' \cong Y \times_T T''$ in Y. The lower square commutes by the uniqueness part of the universal property of the fiber product. □

We now introduce moduli schemes from the categorical point of view where the category of schemes over S could often be replaced by different categories, e.g., by full subcategories like (smooth, projective) varieties over some field.

Definition 3.4. The *functor of points* of an S-scheme X is the contravariant Hom-functor

$$h_X : (\mathbf{Sch}_S)^{\mathrm{op}} \to \mathbf{Set},\ T \mapsto \mathrm{Hom}_S(T, X),$$

which is given on morphisms by precomposition. A morphism $t : T \to X$ of S-schemes in $h_X(T)$ is called a *T-valued point* of X.

Definition 3.5. A contravariant functor $h : (\mathbf{Sch}_S)^{\mathrm{op}} \to \mathbf{Set}$ is called *representable* if it is naturally isomorphic to the functor of points of some S-scheme M, i.e., there is a natural isomorphism $\eta : h_M \xrightarrow{\cong} h$. In this case, the element $\xi := \eta_M(\mathrm{id}_M) \in h(M)$ is called the *universal family*, and the pair (M, ξ) (or simply M itself) is called a *fine moduli scheme.*

Here, ξ determines η uniquely as a natural transformation by the Yoneda Lemma. Moreover, if (M', ξ') is another fine moduli space for h then there is a unique isomorphism $f : M \to M'$ such that $h(f)(\xi') = \xi$.

Definition 3.6. Let k be any field and X a projective k-scheme with respect to a fixed closed embedding $i : X \hookrightarrow \mathbb{P}^n_k$. Let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules on X. Then the *Hilbert polynomial* of $\mathcal{F}$ with respect to i is the unique polynomial $P_{\mathcal{F}} \in \mathbb{Q}[x]$ satisfying

$$P_{\mathcal{F}}(m) = \chi(X, \mathcal{F}(m)) = \chi(\mathbb{P}^n_k, i_*\mathcal{F}(m)),$$

for all $m \in \mathbb{Z}$. Here, χ denotes the Euler characteristic of the respective sheaf cohomology groups. Moreover, if $\mathcal{F} = \mathcal{O}_X$ we also write $P_X := P_{\mathcal{O}_X}$ and call it the *Hilbert polynomial of X.*

The Euler characteristic χ is well-defined because the sheaf cohomology groups are finite-dimensional k-vector spaces, and at most those of degree $\leq \dim(X)$ are non-zero by Grothendieck's vanishing theorem. Moreover, for m large enough, $\chi(X, \mathcal{F}(m)) = \dim_k H^0(X, \mathcal{F}(m))$ because all cohomology groups of higher degree vanish for m large enough by Serre's vanishing theorem. The Hilbert polynomial always exists by [11, Exc. II.5.2], and it really depends on the closed embedding i. It is by definition an integer-valued polynomial, i.e., $P_{\mathcal{F}}(m)$ is an integer whenever m is – although not all coefficients of $P_{\mathcal{F}}$ are necessarily integers. An important result is the following:

Theorem 3.7 ([11, Theorem III.9.9])**.** Let T be an integral noetherian scheme and X a projective T-scheme with respect to a fixed closed embedding $i : X \hookrightarrow \mathbb{P}^n_T$. Denote by $f : X \to T$ the induced morphism. For each element $t \in T$ denote by $X_t = X \times_T \operatorname{Spec}(\kappa(t))$ the scheme-theoretic fiber of f over t which is naturally a closed subscheme of $\mathbb{P}^n_{\kappa(t)}$. Then f is flat if and only if the Hilbert polynomial P_{X_t} of X_t is independent of t.

Here, X_t is naturally a closed subscheme of $\mathbb{P}^n_{\kappa(t)} \cong \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} \operatorname{Spec}(\kappa(t))$ via

$$\begin{array}{ccccc} X & \overset{i}{\hookrightarrow} & \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} T & \overset{p_2}{\longrightarrow} & T \\ \uparrow & & \uparrow & \square & \uparrow \\ X_t & \longrightarrow & \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} \operatorname{Spec}(\kappa(t)) & \longrightarrow & \operatorname{Spec}(\kappa(t)). \end{array} \qquad (f = p_2 \circ i : X \to T)$$

In here, the entire rectangle and the right square are cartesian, hence so is the left square. As closed embeddings are stable under base change, the lower left horizontal arrow is a closed embedding.
One implication of Theorem 3.7, that if f is flat then P_{X_t} is constant in t, can be extended to the case where T is not necessarily integral. In fact, if T is some locally noetherian scheme and f is flat, then the map $t \mapsto P_{X_t}$ is locally constant. This is the main motivation to introduce the following functors.

Definition 3.8. Let $P \in \mathbb{Q}[x]$ be an integer-valued polynomial and X a projective S-scheme. The functor

$$\mathcal{H}ilb^P_{X/S} : (\mathbf{Sch}_S)^{\mathrm{op}} \to \mathbf{Set}$$

associates to an S-scheme T the set of all families Z of closed subschemes of X over S parametrized by T such that $P_{Z_t} = P$ for all $t \in T$. It is also called the *Hilbert functor* if P is understood.

Lemma 3.9. The functor $\mathcal{H}ilb^P_{X/S}$ is well-defined.

Proof. We need to show that for any morphism $g : T' \to T$ of S-schemes, the image of $\mathcal{H}ilb^P_{X/S}(g)$ actually lies in $\mathcal{H}ilb^P_{X/S}(T')$, i.e., for any $Z \in \mathcal{H}ilb^P_{X/S}(T)$ and any $t' \in T'$ the Hilbert polynomial of $(Z \times_T T')_{t'}$ agrees with that of Z_t where $t := g(t')$. For this, consider the diagram

$$\begin{array}{ccccc} (Z \times_T T')_{t'} = Z \times_T \operatorname{Spec}(\kappa(t')) & \overset{i}{\hookrightarrow} & \mathbb{P}^n_{\kappa(t')} & \longrightarrow & \operatorname{Spec}(\kappa(t')) \\ \downarrow & & \downarrow b & \square & \downarrow \text{flat} \\ Z_t = Z \times_T \operatorname{Spec}(\kappa(t)) & \underset{j}{\hookrightarrow} & \mathbb{P}^n_{\kappa(t)} & \longrightarrow & \operatorname{Spec}(\kappa(t)). \end{array}$$

In here, the entire rectangle and the right square are cartesian, hence so is the left square. Also, all vertical arrows are flat because the rightmost arrow is induced by a field extension (which is always flat). Let $\mathcal{O}_{t'}$ be the structure sheaf of $(Z \times_T T')_{t'}$ and $\mathcal{O}_t$ that of Z_t. By flat base extension [11, Proposition III.9.3], we have

$$i_* \mathcal{O}_{t'} \cong b^* j_* \mathcal{O}_t.$$

Moreover, the Hilbert polynomial of $(Z \times_T T')_{t'}$ at $m \in \mathbb{Z}$ is just $\chi(\mathbb{P}^n_{\kappa(t')}, i_* \mathcal{O}_{t'}(m))$, hence it suffices to see that $H^i(\mathbb{P}^n_{\kappa(t)}, \mathcal{F}) \otimes_{\kappa(t)} \kappa(t') \cong H^i(\mathbb{P}^n_{\kappa(t')}, b^* \mathcal{F})$ for all coherent sheaves $\mathcal{F}$ on $\mathbb{P}^n_{\kappa(t)}$. This can be seen by calculating the cohomology groups via Čech cohomology and using that field extensions are flat, and hence tensoring with the field extension commutes with taking cohomology, see also the proof of [11, Proposition III.9.3]. □

Theorem 3.10 ([27, Theorem 8.1])**.** Let S be a locally noetherian scheme and X a projective S-scheme. Then for each integer-valued polynomial P the functor $\mathcal{H}ilb^P_{X/S}$ is represented by a projective S-scheme $\mathrm{Hilb}^P_{X/S}$. In particular, the Hilbert functor is representable by an S-scheme $\mathrm{Hilb}_{X/S}$.

Remark 3.11. Hartshorne [10] showed that the schemes $\mathrm{Hilb}^P_{\mathbb{P}^n_S/S}$ are connected if S is noetherian and connected, but not much about the irreducible components is known in general, and it appears to be a very difficult question.

The scheme $\mathrm{Hilb}^P_{X/S}$ for a fixed integer-valued polynomial P is usually referred to as the *Hilbert scheme* if P is understood from the context. We now introduce the Quot functor and the Grassmannian functor.

Definition 3.12. Let S be a locally noetherian scheme and X a projective S-scheme. Let $\mathcal{F}$ be a coherent sheaf on X. The *Quot functor*

$$\mathcal{Q}uot^P_{X/S}(\mathcal{F}) : (\mathbf{Sch}_S)^{\mathrm{op}} \to \mathbf{Set}$$

associates to an S-scheme T the set of equivalence classes of all coherent quotients $\mathcal{F}_T \to \mathcal{Q} \to 0$ on $X \times_S T$ having Hilbert polynomial P over every point of T and which are flat over T via the the projection $X \times_S T \to T$. Here, $\mathcal{F}_T$ is the pullback of $\mathcal{F}$ along the projection $X \times_S T \to X$. Moreover, two such quotients $\mathcal{F}_T \to \mathcal{Q} \to 0$ and $\mathcal{F}_T \to \mathcal{Q}' \to 0$ are equivalent if there is an isomorphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{F}_T & \longrightarrow & \mathcal{Q} & \longrightarrow & 0 \\ & & \downarrow\cong & & \| & & \downarrow\cong & & \\ 0 & \longrightarrow & \mathcal{K}' & \longrightarrow & \mathcal{F}_T & \longrightarrow & \mathcal{Q}' & \longrightarrow & 0. \end{array}$$

Such an isomorphism, if it exists, is obviously unique. To a morphism $f : T' \to T$ of S-schemes the Quot functor associates the map of sets sending the class of $\mathcal{F}_T \to \mathcal{Q} \to 0$ to that of the pullback $\mathcal{F}_{T'} \to (\mathrm{id}_X \times f)^*(\mathcal{Q}) \to 0$.

The Hilbert functor $\mathcal{H}ilb^P_{X/S}$ is a special case of the Quot functor letting $\mathcal{F} := \mathcal{O}_X$ be the structure sheaf. Indeed, closed subschemes of $X \times_S T$ which are flat over T correspond to those ideal sheaves of $\mathcal{O}_{X\times_S T}$ with flat cokernel over T. The following is therefore a generalization of Theorem 3.10.

Theorem 3.13 ([27, Theorem 9.1])**.** The Quot functor $\mathcal{Q}uot^P_{X/S}(\mathcal{F})$ is representable by a projective S-scheme $\mathrm{Quot}^P_{X/S}(\mathcal{F})$.

If we consider the Quot functor for $X = S = \mathbb{P}^0_S$ we can see immediately that only the constant integer-valued polynomials P give a non-trivial functor. This leads us to the definition of the Grassmannian.

Definition 3.14. The *Grassmannian functor*

$$\mathcal{G}r_S(r, n) : (\mathbf{Sch}_S)^{\mathrm{op}} \to \mathbf{Set}$$

is defined by $\mathcal{G}r_S(r, n) = \mathcal{Q}uot^r_{S/S}(\mathcal{O}^n_S)$. Its representing projective S-scheme is denoted $\mathrm{Gr}_S(r, n)$ or simply $\mathrm{Gr}(r, n)$ if the base scheme S is understood. It is called the *Grassmannian scheme*.

The elements of $\mathcal{G}r_S(r, n)(T)$ are the equivalence classes of locally free quotients of $\mathcal{O}^n_T$ of rank r. This can be seen stalk-wise because every finitely generated flat module over a noetherian local ring is free, and its rank is precisely the Hilbert polynomial. It should be mentioned that the Grassmannian $\mathrm{Gr}_S(r, n)$ can also be realized as the Hilbert scheme $\mathrm{Hilb}^P_{\mathbb{P}^{n-1}_S/S}$ for the Hilbert polynomial $P(x) = \binom{x+r-1}{r-1}$, the key reason being that a closed subscheme of a projective space over some field is a linear subspace of dimension $r - 1$ if and only if it has precisely this Hilbert polynomial, see [5, Proposition 6.7].
For simplicity we now restrict to the case $S = \mathrm{Spec}(A)$ for some noetherian ring A. The representability of the Grassmannian can be seen more directly, and its construction gives more specific information about the representing Grassmannian scheme such as (depending on the ring A) irreducibility, reducedness and dimension.

Theorem 3.15 ([27, Proposition 2.18])**.** The Grassmannian functor over $S = \mathrm{Spec}(A)$, A a noetherian ring, is representable by a projective A-scheme. It has an open cover by finitely many affine spaces over A of relative dimension $r(n - r)$. In particular, if $A = k$ is a field, then $\mathrm{Gr}_k(r, n)$ is a smooth projective variety of dimension $r(n - r)$.

4 Hyperkähler varieties and some special Hilbert schemes

In this section we collect the foundational definitions and results for the later sections. We do not give any full proofs.

4.1 Hyperkähler varieties

We introduce the notion of a hyperkähler variety and one of the most important tools for studying them, the Beauville–Bogomolov form.

Definition 4.1. A compact complex Kähler manifold X is called *hyperkähler* if it is simply connected and the global sections of its sheaf of holomorphic 2-forms $\Omega^2(X)$ is generated by a symplectic form, i.e., a 2-form which is non-degenerate at every point of X. If a hyperkähler manifold is projective, then it is called a *hyperkähler variety.*

Remark 4.2. The definition immediately implies that the Hodge number $h^{2,0}(X) = 1$, and hence $h^{0,2}(X) = 1$ as well. It is also worthwhile to remark that hyperkähler manifolds always have even complex dimension. This is because alternating non-degenerate bilinear forms only exist on vector spaces of even dimension. Also note that a hyperkähler variety of complex dimension 2 is the same as a projective $K3$ surface.

The main family of examples of hyperkähler varieties is provided by the Hilbert schemes of points $\mathrm{Hilb}^n_{S/\mathbb{C}} =: S^{[n]}$ of a projective $K3$ surface S, see Theorem 4.20 below. Another set of examples, in complex dimension 4, is given by the Fano varieties of lines of a smooth cubic fourfold, see Theorem 4.11. Note, however, that each of these is deformation equivalent to $S^{[2]}$ for a suitable $K3$ surface S [2, Proposition 2]. Some more examples of hyperkähler varieties are known but they will not be relevant to us here. One of the main motivations for studying hyperkähler varieties is the following theorem, see [1, Théorème 1].

Theorem 4.3 (Beauville–Bogomolov decomposition)**.** Every compact complex Kähler manifold with trivial canonical class admits a finite étale cover by the product of a complex torus, simply connected irreducible Calabi–Yau manifolds and hyperkähler manifolds.

We now introduce the Beauville–Bogomolov form which is one of the main tools for studying hyperkähler manifolds.

Theorem/Definition 4.4 ([24, Section 4.2])**.** Let X be a hyperkähler manifold of complex dimension $2n$. There is an integral indivisible quadratic form

$$q_X : H^2(X, \mathbb{C}) \to \mathbb{C}$$

and a positive rational number c_X such that

$$\int_X \alpha^{2n} = c_X \frac{(2n)!}{n!2^n} q_X(\alpha)^n \tag{4.1}$$

for all $\alpha \in H^2(X,\mathbb{C})$. This determines q_X uniquely if n is odd and up to sign if n is even. In the latter case, we choose it such that $q_X(\sigma+\overline{\sigma})>0$ where σ is a generator of $H^{2,0}(X)$. The associated symmetric bilinear form $(-,-)=q_X(-,-)$ is non-degenerate. We usually consider cohomology with coefficients in $\mathbb{Q}$, in which case, by integrality, $(-,-)$ takes rational values and is still non-degenerate. This is what we call the *Beauville–Bogomolov form* in this master's thesis. The constant $c_X\frac{(2n)!}{n!2^n}$ is called the *Fujiki constant*, and c_X equals 1 if X is deformation equivalent to the Hilbert scheme of points of a $K3$ surface [24].

By an induction argument it can be shown that (4.1) implies

$$\int_X \alpha_1\cdots\alpha_{2n} = \frac{c_X}{n!2^n}\sum_{\sigma\in S_{2n}}(\alpha_{\sigma(1)},\alpha_{\sigma(2)})\cdots(\alpha_{\sigma(2n-1)},\alpha_{\sigma(2n)}) \tag{4.2}$$

for all $\alpha_1,\ldots,\alpha_{2n}\in H^2(X,\mathbb{C})$, and the same is true for rational coefficients. The combinatorial factor in front of the sum simply means that we do not repeat products which are formally equal after some reordering of the n factors and using the symmetry of $(-,-)$. As an example, if X has complex dimension 4, then

$$\int_X \alpha_1\cdots\alpha_4 = c_X((\alpha_1,\alpha_2)(\alpha_3,\alpha_4)+(\alpha_1,\alpha_3)(\alpha_2,\alpha_4)+(\alpha_1,\alpha_4)(\alpha_2,\alpha_3)). \tag{4.3}$$

The Beauville–Bogomolov form can be made into a natural cohomology class

$$\mathfrak{B}\in H^4(X\times X,\mathbb{Q})$$

as follows: Let $r:=\dim_{\mathbb{Q}}H^2(X,\mathbb{Q})$. We can regard $q_X(-,-)$ as a $\mathbb{Q}$-linear map

$$H^2(X,\mathbb{Q})\otimes H^2(X,\mathbb{Q})\to\mathbb{Q},$$

hence as an element of $H^2(X,\mathbb{Q})^\vee\otimes H^2(X,\mathbb{Q})^\vee$. Now, as q_X is non-degenerate, it induces an isomorphism $H^2(X,\mathbb{Q})\cong H^2(X,\mathbb{Q})^\vee$. Under this identification we can view q_X as an element of $H^2(X,\mathbb{Q})\otimes H^2(X,\mathbb{Q})$, namely

$$q_X^{-1} := \sum_{i,j=1}^r q_{ij}v_i\otimes v_j\in H^2(X,\mathbb{Q})\otimes H^2(X,\mathbb{Q}),$$

for a basis (v_i) of $H^2(X,\mathbb{Q})$ where (q_{ij}) is the inverse matrix of $(q_X(v_i,v_j))$. This formula for q_X^{-1} does not depend on the choice of basis. Via the Künneth isomorphism we then obtain an element $\mathfrak{B}\in H^4(X\times X,\mathbb{Q})$ corresponding to the element

$$0\oplus q_X^{-1}\oplus 0\in (H^0(X)\otimes H^4(X))\oplus(H^2(X)\otimes H^2(X))\oplus(H^4(X)\otimes H^0(X)),$$

and it is this element $\mathfrak{B}$ that is called the *Beauville-Bogomolov class* of X and plays a major role in the study of hyperkähler varieties. By construction, for an orthonormal basis (e_i) of $H^2(X)$ with respect to q_X we have

$$\mathfrak{B}=\sum_{i=1}^r e_i\otimes e_i, \tag{4.4}$$

up to the Künneth isomorphism. Initially, one has to be careful because an orthonormal basis need not exist over $\mathbb{Q}$. Nonetheless, we can always find an *orthogonal* basis over $\mathbb{Q}$ with respect to the Beauville–Bogomolov form, and even more conveniently, we can just extend the field of coefficients to $\mathbb{C}$ where all square roots exist, and hence so do orthonormal bases with respect to any non-degenerate symmetric bilinear form (by a Gram–Schmidt process). The natural map

$$H^*(X,\mathbb{Q}) \to H^*(X,\mathbb{C}) = H^*(X,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$$

is injective, so that every relation among elements in $H^*(X,\mathbb{Q})$ which can be proved in $H^*(X,\mathbb{C})$, e.g., by choosing an orthonormal basis, actually holds in $H^*(X,\mathbb{Q})$ already. For this reason, we will write $H^*(X)$ most of the time, meaning coefficients in $\mathbb{Q}$ except when proving relations in the cohomology ring by usage of an orthonormal basis of $H^2(X)$, which will happen frequently, in which case we will tacitly work in $H^*(X,\mathbb{C})$.
Let now $\mathfrak{b} := \Delta^*(\mathfrak{B}) \in H^4(X)$ be the pullback via the diagonal embedding $\Delta : X \hookrightarrow X \times X$ and $\mathfrak{b}_1$ and $\mathfrak{b}_2$ the pullbacks via the two projections $p_i : X \times X \to X$. With the above expression for $\mathfrak{B}$, we then have

$$\mathfrak{b} = \sum_{i=1}^{r} e_i^2. \tag{4.5}$$

Theorem 4.5 ([19])**.** Let X be a hyperkähler variety deformation equivalent to the Hilbert scheme of $n \geq 2$ points of a $K3$ surface. Then there exists a lift $L \in \mathrm{CH}^2(X \times X)$ of the Beauville–Bogomolov form $\mathfrak{B}$.

Theorem 4.6 ([26, Proposition 1.3])**.** Let X be a hyperkähler variety of complex dimension 4 such that $H^4(X)$ is generated by the cup products of elements in $H^2(X)$. Then $\mathfrak{B}$ satisfies the quadratic equation

$$\mathfrak{B}^2 = 2c_X[\Delta] - \frac{2}{r+2}\mathfrak{B}(\mathfrak{b}_1 + \mathfrak{b}_2) - \frac{1}{r(r+2)}(2\mathfrak{b}_1^2 - r\mathfrak{b}_1\mathfrak{b}_2 + 2\mathfrak{b}_2^2).$$

Here, $r := \mathrm{dim} H^2(X)$. Moreover, $\mathfrak{B}$ is uniquely determined up to sign by this equation.

Let now X be a hyperkähler variety of $K3^{[2]}$-type and let $L \in \mathrm{CH}^2(X \times X)$ be a lift of $\mathfrak{B}$. Denote by $\Delta : X \hookrightarrow X \times X$ the diagonal embedding as well as the cycle class of its image. Let $l := \Delta^*(L) \in \mathrm{CH}^2(X)$ and l_1, l_2 the pullbacks to $X \times X$ via the projections. In this situation, $r = \mathrm{dim} H^2(X) = 23$. We consider the following relations involving L in the Chow ring of X (resp. $X \times X$). The first is referred to as the *quadratic equation for L*, analogous to the cohomological equation of Theorem 4.6:

$$L^2 = 2\Delta - \frac{2}{r+2}L(l_1 + l_2) - \frac{1}{r(r+2)}(2l_1^2 - rl_1l_2 + 2l_2^2). \tag{4.6}$$

Additionally, we consider the following three relations for all $\sigma \in \mathrm{CH}^4(X)$ and all $\tau \in \mathrm{CH}^2(X)$:

$$L_*(l^2) = 0, \tag{4.7}$$

$$L_*(l \cdot L_*(\sigma)) = (r+2)L_*(\sigma), \tag{4.8}$$

$$(L^2)_*(l \cdot (L^2)_*(\tau)) = 0. \tag{4.9}$$

These are the relations which Shen and Vial in [26] established to be the core relations necessary to obtain a Fourier decomposition as in Theorem 2 of *loc. cit.* They indeed constructed an explicit lift L satisfying these relations for both the Fano variety of lines of a smooth cubic fourfold and the Hilbert scheme of two points of a projective $K3$ surface, see Theorem 1 and the paragraphs after Theorem 2 of *loc. cit.*

Theorem 4.7. Let F be the Fano variety of lines of a smooth cubic fourfold or the Hilbert scheme of two points of a projective $K3$ surface. There exists a lift $L \in \mathrm{CH}^2(F \times F)$ of $\mathfrak{B}$ such that all the relations (4.6)-(4.9) are satisfied.

For concrete constructions of such a lift L see Theorems 4.14 and 4.21 below.

4.2 Fano varieties of lines of smooth cubic fourfolds

Definition 4.8. Let $Y \subseteq \mathbb{P}^5$ be a cubic hypersurface. The Hilbert scheme

$$F(Y) := \mathrm{Hilb}^{x+1}_{Y/\mathbb{C}}$$

for the Hilbert polynomial $x+1 \in \mathbb{Q}[x]$ is called the *Fano variety of lines of* Y.

A closed subscheme of $\mathbb{P}^n$ is a line if and only if it has Hilbert polynomial $x+1$; see [5, Proposition 6.7] for a generalization. Moreover, the definition yields that $F(Y)$ is projective by the general result Theorem 3.10 on Hilbert schemes.
The case $n = 5$ and $d = 3$ of [5, Corollary 6.33] gives the dimension and smoothness in the following theorem, [5, Theorem 6.34] shows reducedness, and the proof of [13, Proposition 2.3] indicates two approaches of how to show irreducibility.

Theorem 4.9. If $Y \subseteq \mathbb{P}^5$ is a smooth cubic hypersurface then the Fano variety $F(Y)$ of lines of Y is a smooth projective variety of dimension 4.

There is an obivious inclusion of $F(Y) = \mathrm{Hilb}^{x+1}_{Y/\mathbb{C}}$ into $\mathrm{Hilb}^{x+1}_{\mathbb{P}^5/\mathbb{C}}$ which is a closed embedding, and the latter agrees with $\mathrm{Gr}_{\mathbb{C}}(2,6)$. This is because equivalence classes of locally free quotients $\mathcal{O}_T^6 \twoheadrightarrow Q$ of rank 2 correspond to closed embeddings $\mathbb{P}(Q) \hookrightarrow \mathbb{P}(\mathcal{O}_T^6) = \mathbb{P}^5 \times_{\mathbb{C}} T$ of T-schemes which are flat over T, and the fiber over every $t \in T$ is the embedding of a line $\mathbb{P}^1_{\kappa(t)} \hookrightarrow \mathbb{P}^5_{\kappa(t)}$, thus having Hilbert polynomial $x+1$. So we have:

Proposition 4.10. $F(Y)$ is naturally a closed subvariety of $\mathrm{Gr}(2,6)$.

See also the paragraphs of [13, Chapter 3] following Theorem 1.2 for an explicit way of realizing Fano schemes as the vanishing set of a global section of some symmetric power of the dual tautological bundle on the Grassmannian.

Theorem 4.11 ([2, Propositions 1 and 2])**.** Let $Y \subseteq \mathbb{P}^5$ be a smooth cubic fourfold. Then $F(Y)$ is a hyperkähler variety deformation equivalent to the Hilbert scheme of two points of some $K3$ surface.

Definition 4.12. Let $\mathcal{E}$ be the tautological bundle on $\mathrm{Gr}(2,6)$. We fix the following two notations. Let $g \in \mathrm{CH}^1(F(Y))$ be the first Chern class $c_1(\mathcal{E}|^\vee_{F(Y)})$. Then g is called the *Plücker polarization class*. Moreover, we denote by $c \in \mathrm{CH}^2(F(Y))$ the second Chern class $c_2(\mathcal{E}|^\vee_{F(Y)})$.

Let $\mathcal{Z} \subseteq F(Y) \times Y$ be the universal family of the Hilbert scheme $F(Y)$. It is given on closed points as the set of incident pairs, i.e., pairs of the form $([l], x)$ with $x \in l$ where $l \subseteq Y$ is a line and $[l] \in F(Y)$ the corresponding closed point. This is essentially by definition: A closed point $t \in F(Y)$ is denoted $[l]$ and said to correspond to the line $l \subseteq Y$ if and only if the fiber of $\mathcal{Z}$ over t via the projection is precisely the line $l \subseteq Y$. Therefore, as a set, the closed points of $F(Y)$ really correspond to the lines in Y. Formally, this correspondence is given by $\eta_{\mathrm{Spec}(\mathbb{C})}$ where $\eta : \mathcal{H}ilb^{x+1}_{Y/\mathbb{C}} \cong h_{F(Y)}$ is the natural isomorphism coming from representability.

Definition 4.13. Let Y be a smooth cubic fourfold and $F = F(Y)$ its Fano variety of lines. The *incidence subscheme* is the closed subset $I \subseteq F \times F$ with the reduced subscheme structure, given by the set of pairs of intersecting lines inside Y. Its cycle class in $\mathrm{CH}^2(F \times F)$, also denoted I, is called the *incidence correspondence*. The *tautological subring* $R^*(F \times F) \subseteq \mathrm{CH}^*(F \times F)$ is the $\mathbb{Q}$-subalgebra generated by $I, \Delta, c_1, c_2, g_1, g_2$ where $\Delta \subseteq F \times F$ denotes the diagonal and g_i, c_i for $i = 1, 2$ are the pullbacks via the two projections $F \times F \to F$.

By [26, Lemma 17.2], we have $I = {}^t\mathcal{Z} \circ \mathcal{Z}$ which also yields the codimension of I because $\mathcal{Z}$ has codimension 3 in $F \times Y$.

Theorem 4.14 ([26, p. 81])**.** Let Y be a smooth cubic fourfold and $F = F(Y)$ its Fano variety of lines. An explicit lift L of the Beauville–Bogomolov class $\mathfrak{B}$, satisfying Theorem 4.7, is given by

$$L = \frac{1}{3}(g_1^2 + \frac{3}{2}g_1 g_2 + g_2^2 - c_1 - c_2) - I, \tag{4.10}$$

where I is the incidence correspondence. Moreover, we have $l = \frac{5}{6}c_2(T_F)$, where $c_2(T_F)$ is the second Chern class of the tangent bundle, and $c_2(T_F) = 5g^2 - 8c$. Hence, the tautological subring $R^*(F \times F)$ contains L, l_1, l_2. By the quadratic equation (4.6) for L, it then automatically contains Δ.

This lift L is from now on taken to be the canonical lift of $\mathfrak{B}$ in the Fano variety setting. As mentioned in the introduction, it is not rigorously proven that L coincides with the cycle class obtained from Markman's twisted sheaves [19] in [26, Theorem 9.15], but Shen and Vial expect it to be the case [26, p. 4].
The following injectivity result enables us to prove certain relations in the Chow ring via computations in cohomology. We make heavy use of it in the proof of Theorem 5.19 which is one of our main results.

Theorem 4.15 ([6, Proposition 6.4]). *Let $F = F(Y)$ be the Fano variety of lines of a smooth cubic fourfold Y. Then the restriction of the cycle class map to*

$$\mathrm{cl} : R^{\bullet}(F \times F) \to H^{2\bullet}(F \times F)$$

is injective.

Idea of proof. It suffices to consider the case where Y is very general in which case the restriction of the cycle class map is surjective onto the rational (p, p)-classes, and because the tautological subring is also a finite-dimensional $\mathbb{Q}$-vector space, it now suffices to show that in each degree the vector space dimensions of $R^p(F \times F)$ and $H^{p,p}(F \times F, \mathbb{Q})$ agree. This is relatively straight-forward for degrees $\neq 5, 6$ but requires an additional relation in the Chow ring in order to deal with these two cases, which is provided by the appendix of [6]. □

4.3 Hilbert schemes of points of $K3$ surfaces

Two excellent introductions to Hilbert schemes of points of surfaces and Nakajima operators are Lehn's [17, 16]. See also [23, Section 2.3] for a brief overview and the most important facts. Original sources are [21, 8]. Let S be a smooth projective surface and $S^{[n]} := \mathrm{Hilb}^n_{S/\mathbb{C}}$ the associated Hilbert scheme of n points of S. We already know that $S^{[n]}$ is projective. If $n < 0$ then $S^{[n]}$ is of course the empty scheme. The Chow ring of the empty scheme is the zero ring. Moreover, $S^{[0]} = \mathrm{Spec}(\mathbb{C})$ and its Chow ring, concentrated in degree 0, equals $\mathbb{Q}$.

Theorem 4.16 ([17, Theorem 3.3 and Lemma 3.7]). *Let S be a smooth projective surface. Then for all $n \geq 0$ the Hilbert scheme of points $S^{[n]}$ is connected and smooth of dimension $2n$.*

Theorem/Definition 4.17 ([17, Section 3.2]). *The Hilbert–Chow morphism is a morphism of schemes over $\mathbb{C}$ which topologically is given by*

$$S^{[n]} \to S^n/\Sigma_n,\ \mathcal{I} \mapsto \sum_{x \in S} \mathrm{length}(\mathcal{O}_{S,x}/\mathcal{I}_x)[x].$$

Here, S^n/Σ_n denotes the n-th symmetric power of S, the points of which are formal sums $\sum_i n_i[x_i]$ of distinct points $x_i \in S$ such that $\sum_i n_i = n$, and the $\mathcal{I} \subseteq \mathcal{O}_S$ are the ideal sheaves with quotients of length n. For $n = 1$ the symmetric power is just S itself and the Hilbert–Chow morphism is an isomorphism, as expected. For arbitrary n, the Hilbert–Chow morphism is in fact still an isomorphism between the open subsets of n distinct points.

A central idea for understanding Hilbert schemes of n points on a surface S is to consider all n simultaneously.

Definition 4.18 ([23, Section 2.3]). Let S be a smooth projective surface. For $n \geq 0$ and $i \geq 1$ the closed subset

$$Z_{n,n+i} := \{(\xi, x, \eta) \in S^{[n]} \times S \times S^{[n+i]} : \mathrm{supp}(\mathcal{I}_\xi/\mathcal{I}_\eta) = \{x\}\}$$

of the product $S^{[n]} \times S \times S^{[n+i]}$ is endowed with the reduced induced subscheme structure. Define the *Nakajima operators*

$$\mathfrak{q}_{\pm i} : \bigoplus_{n \in \mathbb{Z}} \mathrm{CH}^*(S^{[n]}) \to \bigoplus_{n \in \mathbb{Z}} \mathrm{CH}^*(S^{[n \pm i]} \times S)$$

to be given up to sign by the correspondence $[Z_{n,n+i}]$ respectively its transpose. In terms of the induced maps, for $W \in \mathrm{CH}^*(S^{[n]})$ we set

$$\mathfrak{q}_i.W = {}^t(p_{S \times S^{[n+i]}})_* \left([Z_{n,n+i}] \cdot p^*_{S^{[n]}}(W)\right).$$

Similarly, on $S^{[n+i]}$ and for $W \in \mathrm{CH}^*(S^{[n+i]})$ we set

$$\mathfrak{q}_{-i}.W = (-1)^i (p_{S^{[n]} \times S})_* \left([Z_{n,n+i}] \cdot p^*_{S^{[n+i]}}(W)\right).$$

Moreover, $\mathfrak{q}_0$ is defined to be 0. For $\alpha \in \mathrm{CH}^*(S)$ we define the correspondence

$$\mathfrak{q}_{\pm i}(\alpha) : \bigoplus_{n \in \mathbb{Z}} \mathrm{CH}^*(S^{[n]}) \to \bigoplus_{n \in \mathbb{Z}} \mathrm{CH}^*(S^{[n \pm i]})$$

by substituting $[Z_{n,n+i}]$ by $[Z_{n,n+i}] \cdot p_S^*(\alpha)$. The composition of Nakajima operators $\mathfrak{q}_{i_1} \cdots \mathfrak{q}_{i_k}$ is here understood to be a correspondence from $S^{[n]}$ to $S^{[n+i_1+\ldots+i_k]} \times S^k$, inducing a linear map

$$\bigoplus_{n \in \mathbb{Z}} \mathrm{CH}^*(S^{[n]}) \to \bigoplus_{n \in \mathbb{Z}} \mathrm{CH}^*(S^{[n+i_1+\ldots+i_k]} \times S^k),$$

where the convention is that $\mathfrak{q}_{i_1}$ contributes the leftmost S-factor. Moreover, for $\Gamma \in \mathrm{CH}^*(S^k)$ we let $\mathfrak{q}_{i_1} \cdots \mathfrak{q}_{i_k}(\Gamma)$ be the correspondence from $S^{[n]}$ to $S^{[n+i_1+\ldots+i_k]}$ given on $W \in \mathrm{CH}^*(S^{[n]})$ by

$$(p_{S^{[n+i_1+\ldots+i_k]}})_* \left((\mathfrak{q}_{i_1} \cdots \mathfrak{q}_{i_k}.W) \cdot p^*_{S^k}(\Gamma)\right).$$

Theorem 4.19 ([23, Section 2.3]). The Nakajima operators satisfy the commutation relation

$$[\mathfrak{q}_m, \mathfrak{q}_n] = m\delta_{m+n,0} \cdot \mathrm{id} \times \Delta_S,$$

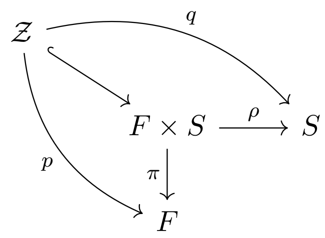
in particular

$$[\mathfrak{q}_m(\alpha), \mathfrak{q}_n(\beta)] = m\delta_{m+n,0}\langle \alpha, \beta \rangle \cdot \mathrm{id},$$

where $\langle -, - \rangle$ is the intersection pairing on S.

Theorem 4.20 ([1, Théorème 6.3]). If S is a projective $K3$ surface then $S^{[n]}$ is a hyperkähler variety of dimension $2n$.

We now consider the case $n = 2$ and set $F := S^{[2]}$ for a projective $K3$ surface S. We fix several notations taken from [26, Part 2] that will be used again in Section 6. Let $\mathcal{Z} \subseteq F \times S$ be the universal family. Its set of closed points consists of the pairs (η, x) where $x \in \mathrm{supp}(\eta)$. This is a codimension 2 closed subscheme of the product, and we denote by



the projections. The projection p is a double cover. Let $c \in \mathrm{CH}_0(S)$ be the canonical 0-cycle, represented by any point on a rational curve in S [3, Theorem 1]. We let

$$S_c := p_* q^*(c) \in \mathrm{CH}^2(F). \tag{4.11}$$

Moreover, denote by $\Delta_{\mathrm{Hilb}} \in \mathrm{CH}^1(F)$ the divisor class on F parametrizing the non-reduced length 2 subschemes of S and set

$$\delta := \frac{1}{2}\Delta_{\mathrm{Hilb}} \in \mathrm{CH}^1(F). \tag{4.12}$$

This agrees with the convention of [26] which differs in the sign from [23]. Finally, let $I \subseteq F \times F$ be the subset of pairs of length 2 subschemes which share a common support point. This is closed and irreducible (see the proof of [26, Lemma 11.2]), and endowed with the reduced induced subscheme structure it gives a closed subvariety of codimension 2, called the *incidence subscheme*. Its cycle class in $\mathrm{CH}^2(F \times F)$ is also denoted I and called the *incidence correspondence*. By *loc. cit.* we have $I = {}^t\mathcal{Z} \circ \mathcal{Z}$.

Theorem 4.21 ([26, p. 67])**.** Let $F = S^{[2]}$ for a projective $K3$ surface S. An explicit lift L of the Beauville–Bogomolov class $\mathfrak{B}$, satisfying Theorem 4.7, is given by

$$L = I - 2(S_c)_1 - 2(S_c)_2 - \frac{1}{2}\delta_1\delta_2. \tag{4.13}$$

Moreover, by [26, Proposition 16.1], L agrees with the lift of $\mathfrak{B}$ obtained from Markman's twisted sheaves [19] in [26, Theorem 9.15]. For its pullback along the diagonal embedding $i_\Delta : F \hookrightarrow F \times F$ we have

$$l := i_\Delta^*(L) = 20S_c - \frac{5}{2}\delta^2. \tag{4.14}$$

In fact, $l = \frac{5}{6}c_2(T_F)$ where $c_2(T_F)$ is the second Chern class of the tangent bundle.

5 An $\mathfrak{sl}_2(\mathbb{Q})$-action on $\mathrm{CH}^*(F(Y))$

5.1 The Lefschetz duals made explicit

Let X be a hyperkähler variety of complex dimension 4 without odd cohomology over $\mathbb{Q}$. As hyperkähler varieties are simply connected by definition, $H^1(X) = 0$ is automatic, and by Poincaré duality we have $H^7(X) = 0$ as well. So the condition of having vanishing odd cohomology over $\mathbb{Q}$ is then equivalent to demanding only $H^3(X) = 0$. With this assumption, the cup product is really commutative and we always write $\alpha\beta$ instead of $\alpha \cup \beta$. Moreover, we denote the second Betti number $r := \dim(H^2(X))$.
Another standing assumption we make is $c_X = 1$ in the Fujiki constant of Section 4.1. This is true for both the Fano variety of lines of a smooth cubic fourfold and the Hilbert scheme of two points of a $K3$ surface and more generally for all hyperkähler varieties which are deformation equivalent to a Hilbert scheme of points of a $K3$ surface [24]. We make this restriction mainly for simplicity and in order to ease the notation although it is not technically important. In fact, the basic and general results of this section, foremost Proposition 5.1 and Theorem 5.8, can be easily adapted to the case $c_X \neq 1$ by dividing $\widetilde{F}_a$ by c_X, see Remark 5.9.
We do not require throughout this section that the cup product map

$$\cup : H^2(X) \otimes H^2(X) \to H^4(X)$$

is surjective. This becomes a necessary assumption only if a cohomological decomposition of the diagonal is needed, which is the case in particular whenever the quadratic equation of Theorem 4.6 is used.
Let $a \in H^2(X, \mathbb{Q})$ and denote by e_a the endomorphism of $H^*(X)$ given by the cup product with a. Let h be the grading operator from Section 2,

$$h : H^k(X) \to H^k(X),\ \beta \mapsto (k-4)\beta.$$

Finally, denote by $\mathbf{1} \in H^8(X)$ the generator of $H^8(X)$ with integral 1.

Proposition 5.1. Let $a \in H^2(X, \mathbb{Q})$ be arbitrary. Define a map

$$\widetilde{f}_a : H^\bullet(X) \to H^{\bullet-2}(X)$$

by

$$\widetilde{f}_a(\beta) := \begin{cases} 0 & \beta \in H^0(X), \\ 4(a,\beta)[X] & \beta \in H^2(X), \\ 2\mathfrak{B}_*(a\beta) & \beta \in H^4(X), \\ 2a\mathfrak{B}_*(\beta) & \beta \in H^6(X), \\ \frac{4}{r+2}\left(\int_X \beta\right) \mathfrak{b}a & \beta \in H^8(X). \end{cases}$$

Then $\widetilde{f}_a$ satisfies $[e_a, \widetilde{f}_a] = (a,a)h$ and $[h, \widetilde{f}_a] = -2\widetilde{f}_a$. Moreover, $[h, e_a] = 2e_a$.

If $(a,a) \neq 0$, we set

$$f_a := \frac{1}{(a,a)} \widetilde{f}_a$$

to obtain the usual $\mathfrak{sl}_2(\mathbb{Q})$-commutation relations. By Lemma 5.7 below, $(a,a) \neq 0$ is equivalent to a being Lefschetz in the sense of Section 2.

Remark 5.2. In the latter case, f_a is uniquely determined by the commutation relations by Lemma 2.1. It should be emphasized that $\widetilde{f}_a$ is linear in $a \in H^2(X)$. This is quite remarkable because for one, f_a is not, and neither is it clear from the abstract description of f_a in terms of the inverses $(e_a^2)^{-1}$ and $(e_a^4)^{-1}$ that it would suffice to multiply f_a by some quadratic form $q_X(a)$ in a in order to make it linear in a.
Note moreover that for $(a,a) = 0$ the commutation relations do not determine $\widetilde{f}_a$ uniquely. Indeed, the zero map also satisfies them while $\widetilde{f}_a$ is non-zero on $H^2(X)$ whenever $a \neq 0$ because the Beauville–Bogomolov form is non-degenerate.

Before proving Proposition 5.1, we need a series of lemmas.

Lemma 5.3. For arbitrary $\gamma, \gamma' \in H^2(X)$ we have $\int_X \mathfrak{b}\gamma\gamma' = (r+2)(\gamma,\gamma')$.

Proof. Let (e_i) be an orthonormal basis of $H^2(X)$. A direct computation gives

$$\begin{aligned}
\int_X \mathfrak{b}\gamma\gamma' &= \sum_{i=1}^{r} \int_X e_i^2\gamma\gamma' \\
&= \sum_{i=1}^{r} \big((\gamma,\gamma') + 2(e_i,\gamma)(e_i,\gamma')\big) \\
&= (r+2)(\gamma,\gamma'),
\end{aligned}$$

where we used $\sum_{i=1}^{r}(e_i,\gamma)(e_i,\gamma') = (\gamma,\gamma')$. $\square$

Lemma 5.4. The linear map $H^2(X) \to H^6(X)$ given by the cup product with $\frac{1}{r+2}\mathfrak{b}$ is an isomorphism with inverse $\mathfrak{B}_*$.

Proof. First, we show that the cup product with the element $\frac{1}{r+2}\mathfrak{b}$ gives an isomorphism $H^2(X) \to H^6(X)$. By Lemma 5.3, for arbitrary $\gamma, \gamma' \in H^2(X)$ we have

$$\int_X \left(\frac{1}{r+2}\mathfrak{b}\gamma\right)\gamma' = (\gamma,\gamma').$$

Hence, $\frac{1}{r+2}\mathfrak{b}\gamma \neq 0$ whenever $\gamma \neq 0$ as $(-,-)$ is non-degenerate. Now, by Poincaré duality, $\dim(H^2(X)) = \dim(H^6(X))$, hence multiplication by $\frac{1}{r+2}\mathfrak{b}$ is indeed an isomorphism.

Next, we compute

$$\begin{aligned}\mathfrak{B}_*(\mathfrak{b}\gamma) = (p_2)_*\left(\sum_{i,j=1}^r (\gamma e_i e_j^2)\otimes e_i\right) &= \sum_{i,j=1}^r \left(\int_X e_i e_j^2 \gamma\right) e_i \\ &= \sum_{i,j=1}^r \big((\gamma, e_i) + 2\delta_{ij}(\gamma, e_j)\big)e_i \\ &= \sum_{j=1}^r\sum_{i=1}^r (\gamma, e_i)e_i + 2\sum_{i=1}^r (\gamma, e_i)e_i \\ &= (r+2)\gamma,\end{aligned}$$

where we used again an orthonormal basis (e_i). So, $\mathfrak{B}_*$ is indeed the inverse. □

Lemma 5.5. For all $a, \beta \in H^2(X)$ we have the relation

$$\frac{(a,a)^2}{r+2}\mathfrak{b}\beta = (a,a)a^2\beta - \frac{2(a,\beta)}{3}a^3.$$

Proof. By Poincaré duality, the equation holds if and only if it holds after multiplying by an arbitrary $\gamma \in H^2(X)$ and integrating. By Lemma 5.3, the left hand side then equals $(a,a)^2(\beta,\gamma)$ while the right hand side becomes

$$(a,a)((a,a)(\beta,\gamma) + 2(a,\beta)(a,\gamma)) - 2(a,\beta)(a,a)(a,\gamma),$$

and they agree. □

Remark 5.6. In the case $(a,a) \neq 0$, Lemma 5.5 can also be interpreted as giving the relation between the isomorphisms $e_a^2 : H^2(X) \to H^6(X)$ and the cup product with $\frac{1}{r+2}\mathfrak{b}$. Moreover, setting $\beta = a$ for $(a,a) \neq 0$ gives

$$\frac{1}{r+2}\mathfrak{b}a = \frac{1}{3(a,a)}a^3.$$

Observe here that the left hand side is defined even in the case $(a,a) = 0$ and is actually non-zero if $a \neq 0$. Additionally, if $X = F$ is the Fano variety of lines of a smooth cubic fourfold or the Hilbert scheme of two points of a projective $K3$ surface, then by [29, Theorem 1.4], this relation lifts to the Chow ring if a and β are interpreted as arbitrary divisor classes and $\mathfrak{b}$ is replaced by $l = \frac{5}{6}c_2(T_F)$ where $c_2(T_F)$ is the second Chern class of the tangent bundle.

Lemma 5.7. Let $a \in H^2(X)$. Then a is Lefschetz in the sense of Section 2 if and only if $(a,a) \neq 0$ if and only if either one of the two maps $e_a^2 : H^2(X) \to H^6(X)$ and $e_a^4 : H^0(X) \to H^8(X)$ is an isomorphism.

Proof. The equation $\int_X a^4 = 3(a,a)^2$ proves that $(a,a) \neq 0$ if and only if e_a^4 is an isomorphism, using that $H^0(X)$ and $H^8(X)$ are both of dimension 1. Next, for $(a,a) \neq 0$ let $\beta \in H^2(X)$ be in the kernel of e_a^2, i.e., $a^2\beta = 0$. It suffices to show $\beta = 0$ as by Poincaré duality we have $\dim(H^2(X)) = \dim(H^6(X))$. Indeed, if $a^2\beta = 0$ then $a^2\beta\gamma = 0$ for every $\gamma \in H^2(X)$. Setting $\gamma = a$ gives

$$0 = \int_X a^3\beta = 3(a,a)(a,\beta),$$

hence $(a,\beta) = 0$. Then for arbitrary γ we get

$$0 = \int_X a^2\beta\gamma = (a,a)(\beta,\gamma) + 2(a,\beta)(a,\gamma) = (a,a)(\beta,\gamma),$$

so $\beta = 0$ as $(-,-)$ is non-degenerate. Conversely, let e_a^2 be an isomorphism. Then certainly $a \neq 0$ and so $0 \neq e_a^2(a) = a^3$. By Poincaré duality again, there is some $\beta \in H^2(X)$ with $a^3\beta \neq 0$, so

$$0 \neq \int_X a^3\beta = 3(a,a)(a,\beta),$$

in particular $(a,a) \neq 0$. □

Proof of Proposition 5.1. We first show $[e_a, \widetilde{f}_a] = (a,a)h$ case by case considering each $H^i(X)$ separately. For $\beta \in H^0(X)$ we can assume by linearity $\beta = [X]$, in which case indeed

$$[e_a, \widetilde{f}_a]([X]) = 0 - \widetilde{f}_a(a) = -4(a,a)[X] = (a,a)h([X]).$$

For $\beta \in H^2(X)$, we first note

$$\begin{aligned}
\mathfrak{B}_*(a^2\beta) &= (p_2)_*\left((p_1)^*\left(a^2\beta\right)\sum_{i=1}^r e_i \otimes e_i\right) \\
&= \sum_{i=1}^r \left(\int_X a^2\beta e_i\right) e_i \\
&= \sum_{i=1}^r \big((a,a)(\beta,e_i) + 2(a,\beta)(a,e_i)\big)e_i \\
&= (a,a)\beta + 2(a,\beta)a.
\end{aligned}$$

Hence,

$$[e_a, \widetilde{f}_a](\beta) = a\widetilde{f}_a(\beta) - \widetilde{f}_a(a\beta) = 4(a,\beta)a - 2\mathfrak{B}_*(a^2\beta) = -2(a,a)\beta = (a,a)h(\beta),$$

as desired. Next, for $\beta \in H^4(X)$,

$$[e_a, \widetilde{f}_a](\beta) = 2a\mathfrak{B}_*(a\beta) - 2a\mathfrak{B}_*(a\beta) = 0 = (a,a)h(\beta).$$

In the case $\beta \in H^6(X)$, by Lemma 5.4, we can write $\beta = \frac{1}{r+2}\mathfrak{b}\widehat{\beta}$ for a unique $\widehat{\beta} \in H^2(X)$. As $h(\beta) = 2\beta$ we need to show $[e_a, \widetilde{f}_a](\beta) - 2(a,a)\beta = 0$, which, by Poincaré duality, is equivalent to the vanishing of

$$\int_X \Big([e_a, \widetilde{f}_a](\beta) - 2(a,a)\beta\Big)\,\gamma$$

for all $\gamma \in H^2(X)$. For the latter, we indeed get

$$\begin{aligned}
\int_X \Big([e_a, \widetilde{f}_a](\beta) - 2(a,a)\beta\Big)\,\gamma &= \int_X \Big(2a^2\widehat{\beta}\gamma\Big) \\
&\quad - \int_X \left(\frac{4}{(r+2)^2}\left(\int_X \mathfrak{b}\widehat{\beta}a\right)\mathfrak{b}a\gamma\right) - \int_X \left(\frac{2(a,a)}{r+2}\mathfrak{b}\widehat{\beta}\gamma\right) \\
&= \Big(2(a,a)(\widehat{\beta},\gamma) + 4(a,\widehat{\beta})(a,\gamma)\Big) \\
&\quad - 4(a,\widehat{\beta})(a,\gamma) - 2(a,a)(\widehat{\beta},\gamma) \\
&= 0.
\end{aligned}$$

Finally, let $\beta \in H^8(X)$. In this case we have

$$[e_a, \widetilde{f}_a](\beta) = a\widetilde{f}_a(\beta) - 0 = \frac{4}{r+2}\left(\int_X \beta\right)\mathfrak{b}a^2 = 4(a,a)\beta = (a,a)h(\beta),$$

using $\int_X \frac{1}{r+2}\mathfrak{b}a^2 = (a,a)$ by Lemma 5.3 and the fact that $H^8(X)$ is of dimension 1. The relations $[h, \widetilde{f}_a] = -2\widetilde{f}_a$ and $[h, e_a] = 2e_a$ follow directly from (and are, in fact, equivalent to) the fact that $\widetilde{f}_a$ decreases and e_a increases the degree by 2. □

The following is basically a corollary of Proposition 5.1 and our first main result.

Theorem 5.8. Let X be a hyperkähler variety of complex dimension 4 with $c_X = 1$ and $H^3(X,\mathbb{Q}) = 0$. Let the Beauville–Bogomolov cohomology class $\mathfrak{B}$ admit a lift $L \in \mathrm{CH}^2(X \times X)$. Then the cycle

$$\widetilde{F}_a := \frac{4}{r+2}(l_1a_1 + l_2a_2) + 2L(a_1 + a_2) \in \mathrm{CH}^3(X \times X)$$

is a lift of $\widetilde{f}_a$, where $a_i := p_i^*(a)$ are the pullbacks via the two projections $p_i : X \times X \to X$. If $(a,a) \neq 0$, we again set $F_a := \frac{1}{(a,a)}\widetilde{F}_a$.

Proof. The cohomology class of $\widetilde{F}_a$ is

$$[\widetilde{F}_a] = \frac{4}{r+2}(\mathfrak{b}_1a_1 + \mathfrak{b}_2a_2) + 2\mathfrak{B}(a_1 + a_2) \in H^6(X \times X, \mathbb{Q}),$$

and we want to show that the action of this cohomology class, viewed as a topological correspondence, coincides with the map $\widetilde{f}_a$. First, we observe that every summand acts trivially on $H^i(X)$ for all but one i. Precisely, the $\mathfrak{b}_1 a_1$-summand contributes the action on $H^2(X)$, i.e., for all $\beta \in H^2(X)$,

$$(\mathfrak{b}_1 a_1)_*(\beta) = (p_2)_* \left(\sum_{i=1}^{r} (e_i^2 a\beta) \otimes [X] \right) = \sum_{i=1}^{r} \left(\int_X e_i^2 a\beta \right) [X] = (r+2)(a,\beta),$$

and $(\mathfrak{b}_1 a_1)_*$ obviously acts as 0 on all $H^i(X)$ for $i \neq 2$. Similarly, the $\mathfrak{b}_2 a_2$-summand gives the action on $H^8(X)$, as for $\beta \in H^8(X)$ we have

$$(\mathfrak{b}_2 a_2)_*(\beta) = (p_2)_* (\beta \otimes (\mathfrak{b}a)) = \left(\int_X \beta \right) \mathfrak{b}a.$$

The $\mathfrak{B}a_1$-summand contributes the action on $H^4(X)$,

$$(\mathfrak{B}a_1)_*(\beta) = (p_2)_* (a_1 \beta_1 \mathfrak{B}) = \mathfrak{B}_*(a\beta).$$

Finally, the $\mathfrak{B}a_2$-summand gives the action on $H^6(X)$,

$$(\mathfrak{B}a_2)_*(\beta) = (p_2)_* (\beta_1 a_2 \mathfrak{B}) = a(p_2)_* (\beta_1 \mathfrak{B}) = a\mathfrak{B}_*(\beta),$$

where the second equation uses the projection formula. □

Remark 5.9. In particular, this re-proves the Grothendieck standard conjecture of Lefschetz type under the assumptions of Theorem 5.8. The latter are known to be satisfied if X is deformation equivalent to the Hilbert scheme $S^{[2]}$ of two points of a $K3$ surface S by Theorem 4.5, in particular also for the Fano variety of lines of a smooth cubic fourfold. Also note again that $\widetilde{F}_a$ is linear in $a \in \mathrm{CH}^1(X)$.
If $c_X \neq 1$, then multiplying $\widetilde{F}_a$ by c_X^{-1} is the desired lift of the modified Lefschetz dual $\widetilde{f}_a$. Indeed, for $c_X \neq 1$ Lemma 5.3 only changes by the factor c_X, and in Lemma 5.4 multiplication by $\frac{1}{r+2}\mathfrak{b}$ must be replaced by $\frac{1}{(r+2)c_X}\mathfrak{b}$. From the proof of Proposition 5.1 it then becomes clear that $\widetilde{f}_a$ essentially needs to be divided by c_X with a minor exception coming from observing

$$4(a,\beta)[X] = \frac{4}{(r+2)c_X} \left(\int_X \mathfrak{b}a\beta \right) [X] = \frac{4}{(r+2)c_X} (\mathfrak{b}_1 a_1)_*(\beta)$$

for $\beta \in H^2(X)$. Hence we can see that the $H^2(X)$ case of the case distinction in the definition of $\widetilde{f}_a$ remains unchanged while all the other cases have to be divided by c_X.

For simplicity, we now assume again $c_X = 1$.

Definition 5.10. Under the assumptions of Theorem 5.8, and for a fixed lift L of $\mathfrak{B}$, we denote the commutator by

$$\widetilde{H}_a := [\Delta_*(a), \widetilde{F}_a] = (a_2 - a_1)\widetilde{F}_a$$

and $H_a := \frac{1}{(a,a)}\widetilde{H}_a$ for $(a,a) \neq 0$. Automatically, $\widetilde{H}_a$ is a lift of $(a,a)h$ where h is the cohomological grading operator.

Proposition 5.11. Let $a \in H^2(X, \mathbb{Q})$ be arbitrary. Then, in $H^8(X \times X, \mathbb{Q})$, the following relation holds:

$$(a,a)\mathfrak{B}\mathfrak{b}_1 = (r+2)\mathfrak{B}a_1^2 - 2\mathfrak{b}_1 a_1 a_2.$$

Proof. It is *not* enough to merely show the equation for a running through a basis of $H^2(X)$ as the equation is not linear in a. Instead, we fist observe that it suffices to show the equation

$$\big((r+2)\mathfrak{B}a_1^2 - 2\mathfrak{b}_1 a_1 a_2 - (a,a)\mathfrak{B}\mathfrak{b}_1\big)\,\gamma_1 = 0$$

for all $\gamma \in H^2(X)$. Indeed, denote the entire term surrounded by the bigger bracket B. Then B lies in $H^6(X) \otimes H^2(X)$ (up to the Künneth isomorphism). Now, by Poincaré duality, $H^6(X)$ has a basis $e_i^\vee$ for $i \in \{1, \ldots, r\}$ satisfying $\int_X e_i^\vee e_j = \delta_{ij}$. We write $B = \sum_{i=1}^r e_i^\vee \otimes \beta_i$ for some $\beta_i \in H^2(X)$, and these summands are linearly independent in $H^6(X) \otimes H^2(X)$ in the sense that $B = 0$ if and only if $\beta_i = 0$ for all $i \in \{1, \ldots, r\}$. Thus, if $B \neq 0$, there is some k such that $\beta_k \neq 0$. Then letting $\gamma := e_k$ yields

$$B\gamma_1 = \sum_{i=1}^r (e_i^\vee e_k) \otimes \beta_i = \mathbf{1} \otimes \beta_k \neq 0.$$

A direct computation now shows

$$\begin{aligned}
B\gamma_1 &= (r+2)\mathfrak{B}a_1^2\gamma_1 - 2\mathfrak{b}_1 a_1 a_2 \gamma_1 - (a,a)\mathfrak{B}\mathfrak{b}_1\gamma_1 \\
&= (r+2)\sum_{i=1}^r (e_i a^2 \gamma) \otimes e_i - 2\sum_{i=1}^r (e_i^2 a\gamma) \otimes a - (a,a)\sum_{i,j=1}^r (e_i e_j^2 \gamma) \otimes e_i \\
&= (r+2) \cdot \mathbf{1} \otimes \left(\sum_{i=1}^r e_i\big(2(a,\gamma)(a,e_i) + (a,a)(\gamma,e_i)\big)\right) \\
&\quad - 2 \cdot \mathbf{1} \otimes a \cdot \sum_{i=1}^r \big((a,\gamma) + 2(a,e_i)(\gamma,e_i)\big) \\
&\quad - (a,a) \cdot \mathbf{1} \otimes \left(\sum_{i,j=1}^r e_i\big((\gamma,e_i) + 2\delta_{ij}(\gamma,e_j)\big)\right) \\
&= 2(r+2)(a,\gamma) \cdot \mathbf{1} \otimes a + (r+2)(a,a) \cdot \mathbf{1} \otimes \gamma \\
&\quad - 2(r+2)(a,\gamma) \cdot \mathbf{1} \otimes a \\
&\quad - (r+2)(a,a) \cdot \mathbf{1} \otimes \gamma \\
&= 0,
\end{aligned}$$

as desired. □

Proposition 5.12. The following relation in $H^{10}(X \times X, \mathbb{Q})$ holds for all $a \in H^2(X, \mathbb{Q})$:

$$r\mathfrak{B}\mathfrak{b}_1 a_1 = \mathfrak{b}_1^2 a_2.$$

Proof. Recall that $\mathfrak{b}^2 = r(r+2)\cdot\mathbf{1}$. We have

$$\begin{aligned}\mathfrak{B}\mathfrak{b}_1 a_1 &= \sum_{i,j=1}^{r} (e_i e_j^2 a)\otimes e_i = \sum_{i,j=1}^{r} \left(\int_X e_i e_j^2 a\right)\cdot \mathbf{1}\otimes e_i \\ &= \sum_{i,j=1}^{r} (2\delta_{ij}(a,e_j) + (a,e_i))\cdot \mathbf{1}\otimes e_i \\ &= (r+2)\cdot \mathbf{1}\otimes a,\end{aligned}$$

as desired. □

Proposition 5.13. For all $a \in H^2(X,\mathbb{Q})$ the following relation holds in $H^{10}(X\times X,\mathbb{Q})$:

$$(r+2)\mathfrak{B}^2 a_1 = 2\mathfrak{B}\mathfrak{b}_1 a_2 + \mathfrak{b}_1\mathfrak{b}_2 a_1.$$

Proof. Both sides lie in $H^6(X)\otimes H^4(X)$, and similar to the proof of Proposition 5.11 it suffices to show the equation after multiplying by γ_1 for an arbitrary $\gamma \in H^2(X)$. We calculate all three terms separately: First note $\mathfrak{b}a\gamma = (r+2)(a,\gamma)\cdot\mathbf{1}$. Thus, $\mathfrak{b}_1\mathfrak{b}_2 a_1\gamma_1 = (r+2)(a,\gamma)\cdot\mathbf{1}\otimes\mathfrak{b}$. Next, for the left hand side we have

$$\begin{aligned}\mathfrak{B}^2 a_1\gamma_1 &= \sum_{i,j=1}^{r} (e_i e_j a\gamma)\otimes(e_i e_j) \\ &= \sum_{i,j=1}^{r} \left(\int_X e_i e_j a\gamma\right)\cdot\mathbf{1}\otimes(e_i e_j) \\ &= \sum_{i,j=1}^{r} \big(\delta_{ij}(a,\gamma) + (a,e_i)(\gamma,e_j) + (a,e_j)(\gamma,e_i)\big)\cdot\mathbf{1}\otimes(e_i e_j) \\ &= (a,\gamma)\cdot\mathbf{1}\otimes\mathfrak{b} + 2\cdot\mathbf{1}\otimes(a\gamma).\end{aligned}$$

Finally,

$$\begin{aligned}\mathfrak{B}\mathfrak{b}_1 a_2\gamma_1 &= \sum_{i,j=1}^{r} (e_i e_j^2\gamma)\otimes(e_i a) \\ &= \sum_{i,j=1}^{r} \left(\int_X e_i e_j^2\gamma\right)\cdot\mathbf{1}\otimes(e_i a) \\ &= \sum_{i,j=1}^{r} \big((\gamma,e_i) + 2\delta_{ij}(\gamma,e_j)\big)\cdot\mathbf{1}\otimes(e_i a) \\ &= (r+2)\cdot\mathbf{1}\otimes(a\gamma),\end{aligned}$$

and everything fits. □

Proposition 5.14. Let X be a hyperkähler variety of complex dimension 4 with vanishing $H^3(X, \mathbb{Q})$. Whenever $L \in \mathrm{CH}^2(X \times X)$ is a lift of $\mathfrak{B}$ to the Chow ring and $l = \Delta^*(L)$ is the pullback along the diagonal embedding $\Delta : X \hookrightarrow X \times X$, then the cycle class

$$H := \frac{4}{r(r+2)}(l_2^2 - l_1^2) + \frac{2}{r+2}(l_2 - l_1)L \in \mathrm{CH}^4(X \times X)$$

is a lift of the grading operator $h \in \mathrm{End}_{\mathbb{Q}}(H^*(X))$ only depending on the lift L. Moreover, if $F = F(Y)$ is the Fano variety of lines of a smooth cubic fourfold Y and L is the canonical lift from Theorem 4.14, then $H_g = H$ in the Chow ring, where g is the Plücker polarization class.

First proof. We check the claim case by case. The action of $(\mathfrak{b}_1^2)_*$ is trivial except on $H^0(X)$, and similarly $(\mathfrak{b}_2^2)_*$ acts trivially except on $H^8(X)$. Furthermore, $(\mathfrak{b}_1\mathfrak{B})_*$ acts non-trivially only on $H^2(X)$ while $(\mathfrak{b}_2\mathfrak{B})_*$ acts non-trivially only on $H^6(X)$. We directly infer that $[H]_*$ is trivial on $H^4(X)$, hence agrees there with h. Now, on $H^0(X)$,

$$(\mathfrak{b}_1^2)_*([X]) = \left(\int_X \mathfrak{b}^2\right)[X] = r(r+2)[X],$$

so $[H]_*([X]) = -4[X] = h([X])$, as desired. Similarily, for $\mathbf{1} \in H^8(X)$ we get

$$(\mathfrak{b}_2^2)_*(\mathbf{1}) = \mathfrak{b}^2 = r(r+2)\mathbf{1},$$

so $[H]_*(\mathbf{1}) = 4 \cdot \mathbf{1} = h(\mathbf{1})$ as well. For $\beta \in H^2(X)$ we have

$$\left(\frac{1}{r+2}\mathfrak{b}_1\mathfrak{B}\right)_*(\beta) = \mathfrak{B}_*\left(\frac{1}{r+2}\mathfrak{b}\cdot\beta\right) = \beta$$

by Lemma 5.4, hence $[H]_*(\beta) = -2\beta = h(\beta)$. At last, let $\beta \in H^6(X)$. Again using Lemma 5.4, we obtain

$$\left(\frac{1}{r+2}\mathfrak{b}_2\mathfrak{B}\right)_*(\beta) = \frac{1}{r+2}\mathfrak{b}\cdot\mathfrak{B}_*(\beta) = \beta,$$

so $[H]_*(\beta) = 2\beta = h(\beta)$, as desired.
This implies $(a,a)[H] = [\widetilde{H}_a]$ in cohomology for all divisor classes a. If $X = F$ is the Fano variety and a is a rational multiple of the Plücker polarization class g, then both $(a,a)H$ and $\widetilde{H}_a$ lie in the tautological subring $R^*(F \times F)$ and thus agree in the Chow ring by the injectivity result of Theorem 4.15. Finally, $(g,g) \neq 0$ by [25, Section 2]. Precisely, we have $g^4 = 108$, so $(g,g) = \pm 6$, and in fact $(g,g) = 6$ by [24, Remark 4.3 (2)] because g is very ample. □

Second proof. Transposing the relation from Proposition 5.11, i.e., pulling back via the automorphism of $X \times X$ which swaps the factors, we get

$$(a,a)\mathfrak{B}\mathfrak{b}_2 = (r+2)\mathfrak{B}a_2^2 - 2\mathfrak{b}_2 a_1 a_2.$$

Subtracting the two equations yields

$$(a,a)\mathfrak{B}(\mathfrak{b}_2 - \mathfrak{b}_1) = (r+2)\mathfrak{B}(a_2^2 - a_1^2) - 2a_1a_2(\mathfrak{b}_2 - \mathfrak{b}_1).$$

Now,

$$\begin{aligned}
\frac{r+2}{2}[\widetilde{H}_a] &= \frac{r+2}{2}(a_2 - a_1)[\widetilde{F}_a] \\
&= 2(a_2 - a_1)(\mathfrak{b}_1 a_1 + \mathfrak{b}_2 a_2) + (r+2)\mathfrak{B}(a_2^2 - a_1^2) \\
&= (r+2)\mathfrak{B}(a_2^2 - a_1^2) - 2a_1a_2(\mathfrak{b}_2 - \mathfrak{b}_1) + 2(\mathfrak{b}_2 a_2^2 - \mathfrak{b}_1 a_1^2) \\
&= (a,a)\mathfrak{B}(\mathfrak{b}_2 - \mathfrak{b}_1) + 2(\mathfrak{b}_2 a_2^2 - \mathfrak{b}_1 a_1^2) \\
&= (a,a)\mathfrak{B}(\mathfrak{b}_2 - \mathfrak{b}_1) + 2(r+2)(a,a)(\mathbf{1}_2 - \mathbf{1}_1) \\
&= (a,a)\mathfrak{B}(\mathfrak{b}_2 - \mathfrak{b}_1) + \frac{2}{r}(a,a)(\mathfrak{b}_2^2 - \mathfrak{b}_1^2) \\
&= \frac{r+2}{2}(a,a)[H].
\end{aligned}$$

Here, we used the above equation in line four, Lemma 5.3 in line five and $\int_X \mathfrak{b}^2 = r(r+2)$ in line six. Hence, $\widetilde{H}_a$ and $(a,a)H$ agree in cohomology, and therefore H is a lift of h as long as there exists some a with $(a,a) \neq 0$ which is always the case for formal reasons, and also $a = g$ always works. The rest is identical to the first proof. □

Conjecture 5.15. Let F be the Fano variety of lines of a smooth cubic fourfold or the Hilbert scheme of two points of a projective $K3$ surface and L the canonical lift of Theorem 4.14 in the first and of Theorem 4.21 in the second case. We conjecture that for all divisor classes $a \in \mathrm{CH}^1(F)$ with $(a,a) \neq 0$ the following relations hold in the Chow ring:

$$(a,a)Ll_1 = (r+2)La_1^2 - 2l_1a_1a_2, \tag{5.1}$$

$$rLl_1a_1 = l_1^2a_2, \tag{5.2}$$

$$(r+2)L^2a_1 = 2Ll_1a_2 + l_1l_2a_1. \tag{5.3}$$

Remark 5.16. The second relation follows from the first one whenever it is known that a^3 is a multiple of la and la^2 is a multiple of l^2 in the Chow ring of F. In our setting both are known by [29, Theorem 1.4]. Moreover, the first relation and its transpose would imply the important equality $H = H_a$ by the second proof of Proposition 5.14 carried out in the Chow ring, replacing $\mathfrak{B}$ by L and $\mathfrak{b}$ by l. This equality is necessary for establishing Conjecture 5.21 below. Note also that we indirectly show $H = H_a$ in the Hilbert scheme setting in Section 6, see Remark 6.9.

Proposition 5.17. Conjecture 5.15 is true in the case of the Fano variety if a is a multiple of the Plücker polarization class g. In particular, $H = H_g$.

Proof. By the aforegoing Propositions 5.11–5.13 the relations are true in cohomology for all a. If a is a multiple of g, then by Theorem 4.15 they hold in the Chow ring as well because all occurring terms then lie in the tautological subring $R^*(F \times F)$. □

Proposition 5.18. Let $F = F(Y)$ be the Fano variety of lines of a smooth cubic fourfold Y. Let $R^*(F \times F)$ be the tautological subring of Theorem 4.15. Then the latter is closed under composition of correspondences,

$$R^*(F \times F) \circ R^*(F \times F) \subseteq R^*(F \times F).$$

Proof. This is a consequence of [6, Proposition 6.3]. Following the notation of *loc. cit.*, denote by B the moduli space of cubic hypersurfaces in $\mathbb{P}^5$ (which is just a projective space), and by B° the Zariski open subscheme of smooth cubic hypersurfaces. Denote by $\mathcal{X} \to B$ the universal family and by $\mathcal{X}^\circ \to B^\circ$ its base change. Let $\mathcal{F} \to B^\circ$ be the universal Fano variety of lines of the fibers of $\mathcal{X}^\circ \to B^\circ$. Denote by F_b the fiber of $\mathcal{F}$ over $b \in B^\circ$. Then the cited proposition states that for all $b \in B^\circ$ we have

$$\operatorname{im}\Big(\mathrm{CH}^*(\mathcal{F} \times_{B^\circ} \mathcal{F}) \to \mathrm{CH}^*(F_b \times F_b)\Big) = R^*(F_b \times F_b),$$

where the map on the left hand side is just the restriction to the fiber. In particular, if we have two tautological cycle classes $Z, W \in R^*(F_b \times F_b)$ they have preimages $\mathcal{Z}, \mathcal{W}$ in $\mathrm{CH}^*(\mathcal{F} \times_{B^\circ} \mathcal{F})$. But now we can simply form their composition $\mathcal{Z} \circ \mathcal{W} \in \mathrm{CH}^*(\mathcal{F} \times_{B^\circ} \mathcal{F})$, and its image in $\mathrm{CH}^*(F_b \times F_b)$ equals $Z \circ W$. By the proposition, therefore, $Z \circ W$ lies in $R^*(F_b \times F_b)$ again. □

In the case $\mathrm{CH}^1(F(Y)) = \langle g \rangle$, which holds for very general cubic fourfolds Y, the following theorem is a full analog of [23, Theorem 1.1]. Even under Conjecture 5.15, however, for arbitrary smooth cubic fourfolds we would not achieve a full analog in the sense of lifting the action of the *entire* Neron–Severi Lie algebra to the Chow ring. Despite these technical difficulties, we make Conjecture 5.21 at the end of this subsection.

Theorem 5.19 (Theorem 1.2)**.** Let $F = F(Y)$ be the Fano variety of lines of any smooth cubic fourfold Y. Let $L \in \mathrm{CH}^2(F \times F)$ be the lift of $\mathfrak{B}$ from Theorem 4.14. Then there is an $\mathfrak{sl}_2(\mathbb{Q})$-action on $\mathrm{CH}^*(F)$ given by the Lie algebra homomorphism

$$\begin{aligned} \mathfrak{sl}_2(\mathbb{Q}) &\to \mathrm{CH}^*(F \times F), \\ e &\mapsto \Delta_*(g), \\ f &\mapsto F_g, \\ h &\mapsto H = H_g, \end{aligned}$$

lifting the $\mathfrak{sl}_2(\mathbb{Q})$-action on the cohomology ring $H^*(F)$ given by e_g, f_g and h.

Proof. We have already seen that F_g lifts f_g and H lifts h. Moreover, it is clear that $\Delta_*(g)$ lifts e_g. By the explicit formulas for $\Delta_*(g) = \Delta g_1$, F_g and H we know that all of them lie in the tautological subring $R^*(F \times F)$ of Theorem 4.15. By Proposition 5.18, so do their compositions. Hence all the commutators lie in $R^*(F \times F)$, and as the commutation relations are true in cohomology, they are true in the Chow ring as well. □

A natural question to ask is whether for the Fano variety F of some special smooth cubic fourfold Y we can replace the Plücker polarization class g in the theorem by some linearly independent divisor class a. Indeed, under Conjecture 5.15, this is true.

Proposition 5.20. Let X be a hyperkähler variety of $K3^{[2]}$-type endowed with a symmetric lift $L \in \mathrm{CH}^2(X \times X)$ satisfying the relations (4.6) and (4.7). Let $a \in \mathrm{CH}^1(X)$ be a divisor class satisfying the relations of Conjecture 5.15 with respect to L. Then the analogous version of Theorem 5.19 holds with respect to L and for g replaced by a.

Proof. We need to show the three necessary commutation relations

$$[H, \Delta_*(a)] = 2\Delta_*(a), \quad [H, F_a] = -2F_a, \quad \text{and} \quad [\Delta_*(a), F_a] = H.$$

For the last of these we already indicated this in Remark 5.16, and it uses the first relation of Conjecture 5.15 and its transpose. Let us show now the first two commutation relations. For the first one, observe that the left hand side equals

$$(a_1 - a_2)H = \frac{4}{r(r+2)}(l_1^2 a_2 + l_2^2 a_1) + \frac{2}{r+2}L(l_1 a_2 + l_2 a_1) - \frac{2}{r+2}L(l_1 a_1 + l_2 a_2).$$

For the right hand side use the quadratic equation for L and the second and third relation of the conjecture. Then

$$2\Delta_*(a) = \frac{2}{r+2}L(l_1 a_2 + l_2 a_1) + \frac{2}{r(r+2)}(l_1^2 a_1 + l_2^2 a_2).$$

Hence the difference equals

$$\frac{2}{r(r+2)}(l_1^2 a_2 + l_2^2 a_1) - \frac{2}{r+2}L(l_1 a_1 + l_2 a_2) = 0$$

by another application of the second relation of the conjecture and its transpose.
For the second commutation relation we show without loss of generality $[H, \widetilde{F}_a] = -2\widetilde{F}_a$. Here, consider first the composition $H \circ \widetilde{F}_a$. The only summand of the latter which is not easily computed using the projection formula, $L_*(l^2) = 0$ and the second relation of Conjecture 5.15 is

$$-\frac{4}{r+2}(p_{13})_*\Big((p_{13})^*(Ll_2 a_1) \cdot (p_{23})^*(L)\Big),$$

where $p_{ij} : X \times X \times X \to X \times X$ denote the projections to the factors according to the indices. Here, we use the transpose of the third relation of the conjecture in order to write $Ll_2 a_1 = \frac{r+2}{2}L^2 a_2 - \frac{1}{2}l_1 l_2 a_2$. From this we deduce that the above summand equals

$$2l_1 a_2 - 2(p_{13})_*\Big((p_{13})^*(L^2) \cdot (p_{23})^*(La_1)\Big).$$

Next, using the quadratic equation for L^2 we obtain that the latter agrees with $-4La_1$. All together we obtain

$$H \circ \widetilde{F}_a = -\frac{16}{r+2}l_1 a_1 + \frac{8}{r+2}l_2 a_2 - 4La_1.$$

Now, ${}^tH = -H$ and ${}^t\widetilde{F}_a = \widetilde{F}_a$, so

$$[H, \widetilde{F}_a] = -4L(a_1 + a_2) - \frac{8}{r+2}(l_1 a_1 + l_2 a_2) = -2\widetilde{F}_a,$$

as desired. □

The following would be an analog of the main theorem of [23] for hyperkähler varieties of $K3^{[2]}$-type.

Conjecture 5.21 (Conjecture 1.1)**.** Let X be a hyperkähler variety of $K3^{[2]}$-type. Let $L \in \mathrm{CH}^2(X \times X)$ be a lift of $\mathfrak{B}$ and $r := \dim_{\mathbb{Q}}(H^2(X, \mathbb{Q})) = 23$. For any divisor class $a \in \mathrm{CH}^1(X)$ with $(a, a) \neq 0$ we define

$$\begin{aligned} F_a &:= \frac{4}{(r+2)(a,a)}(l_1 a_1 + l_2 a_2) + \frac{2}{(a,a)} L(a_1 + a_2) \in \mathrm{CH}^3(X \times X), \\ H &:= \frac{4}{r(r+2)}(l_2^2 - l_1^2) + \frac{2}{r+2} L(l_2 - l_1) \in \mathrm{CH}^4(X \times X), \end{aligned}$$

according to Theorem 5.8 and Proposition 5.14. Then there exists a lift L as above such that the linear map φ in the commutative diagram

$$\begin{array}{ccc} \mathfrak{g}_{\mathrm{NS}}(X) & \xrightarrow{\ \varphi\ } & \mathrm{CH}^*(X \times X) \\ & \searrow & \downarrow \mathrm{cl} \\ & & \mathrm{End}_{\mathbb{Q}}(H^*(X, \mathbb{Q})), \end{array}$$

given by $\varphi(e_a) = \Delta_*(a)$, $\varphi(f_a) = F_a$ and $\varphi(h) = H$ is a well-defined Lie algebra homomorphism.

As mentioned already in the introduction, the conjecture is true in the case of the Hilbert scheme $S^{[2]}$ of a projective $K3$ surface S by the main theorem of [23] and our Section 6 where we show that our formulas for the lifts of e_a, f_a and h agree with those of [23]. In the case of the Fano variety of lines $F = F(Y)$ of a smooth cubic fourfold Y, if L is the explicit cycle class of Theorem 4.14 and F has Picard rank 1, i.e., $\mathrm{CH}^1(F(Y)) = \langle g \rangle$, which is true for very general cubic fourfolds Y, then the conjecture reduces to Theorem 5.19. If Y is arbitrary, we lack injectivity results for the cycle class map in order to lift the entire action of the Neron–Severi Lie algebra $\mathfrak{g}_{\mathrm{NS}}$. Theorems 6.7 and 6.8 in the Hilbert scheme case, however, suggest that, if a conjecture of this type is possible, then our formulas for the lifts of f_a and h should be the correct ones.

5.2 Eigenspace decomposition of the correspondence H

In this section, we consider the action of the induced map H_* of the correspondence H obtained in Proposition 5.14. First, we observe that $H \in \mathrm{CH}^4(X \times X)$, so that $H_*(\mathrm{CH}^i(X)) \subseteq \mathrm{CH}^i(X)$ for all i. For the action on $\mathrm{CH}^i(X)$ we know that in cohomology h acts by multiplication with $2i - 4$, hence we expect the eigenspace decomposition of $\mathrm{CH}^i(X)$ under H_*, if indeed diagonalizable, to contain a direct summand for the eigenvalue $2i - 4$. This is indeed the case but other eigenvalues occur as well. The eigenvalue $2i - 4$ is called the *expected* eigenvalue and in the decomposition of Theorem 5.22 below corresponds to the leftmost column.

Theorem 5.22 (Theorem 1.3)**.** Let X be a hyperkähler variety of $K3^{[2]}$-type endowed with a lift $L \in \mathrm{CH}^2(X \times X)$ of $\mathfrak{B}$ satisfying the relations (4.6)-(4.8) from Section 4.1. Note that (4.9) is not needed here. E.g., X can be the Fano variety of lines of a smooth cubic fourfold or the Hilbert scheme of two points of a $K3$ surface. Let $\Lambda^i_\lambda \subseteq \mathrm{CH}^i(X)$ be the eigenspace for the eigenvalue λ of H_*. The operator $H_* \in \mathrm{End}_{\mathbb{Q}}(\mathrm{CH}^*(X))$ is diagonalizable with eigenspace decomposition

$$\begin{aligned} \mathrm{CH}^0(X) &= \Lambda^0_{-4}, \\ \mathrm{CH}^1(X) &= \Lambda^1_{-2}, \\ \mathrm{CH}^2(X) &= \Lambda^2_0 \oplus \Lambda^2_{-2}, \\ \mathrm{CH}^3(X) &= \Lambda^3_2 \oplus \Lambda^3_0, \\ \mathrm{CH}^4(X) &= \Lambda^4_4 \oplus \Lambda^4_2 \oplus \Lambda^4_0. \end{aligned}$$

All direct summands other than those in the leftmost column belong to the homologically trivial cycle classes $\mathrm{CH}^*(X)_{\mathrm{hom}}$. We have

$$\Lambda^3_0 = \mathrm{CH}^3(X)_{\mathrm{hom}}, \;\; \Lambda^4_4 = \langle l^2 \rangle, \;\; \Lambda^4_2 = l \cdot L_*(\mathrm{CH}^4(X)),$$

so that the cycle class map is injective on the leftmost column except maybe on Λ^2_0. Moreover, all elements of $\Lambda^4_4 \oplus \Lambda^4_2$ are multiples of l, and multiplication by l gives an injective map $\mathrm{CH}^1(X) \to \Lambda^3_2$. Furthermore, $L_*(\Lambda^4_0) = 0$.

The last statement even shows $\Lambda^4_2 = l \cdot L_*(\Lambda^4_2)$ using the assumed relation $L_*(l^2) = 0$. The following proof is inspired by the proof of [26, Theorem 2.2].

Proof. By Lemma A.3, it suffices to show that suitable products of linear polynomials in H_* with distinct zeros vanish. The result on $\mathrm{CH}^i(X)$ for $i \in \{0, 1\}$ is clear because the cycle class map injects into cohomology in these cases. Note $L_*(\mathrm{CH}^i(X)) \subseteq \mathrm{CH}^{i-2}(X)$ and that $(l_1^2)_*$ is trivial except on $\mathrm{CH}^0(X)$ while $(l_2^2)_*$ is trivial except on $\mathrm{CH}^4(X)$.
Now, for $Z \in \mathrm{CH}^2(X)$ we have $L_*(Z) = 0$. Indeed, $L_*(Z) \in \mathrm{CH}^0(X)$ and the equation holds true in cohomology because the pushforward map $(p_2)_*$ of the second projection $p_2 : X \times X \to X$ acts trivially on the direct summand $H^6(X) \otimes H^2(X)$ of $H^8(X \times X, \mathbb{Q})$. Hence,

$$H_*(Z) = \frac{-2}{r+2} L_*(lZ).$$

Now, our hypothesis includes the relation $L_*(l \cdot L_*(\sigma)) = (r+2)L_*(\sigma)$ for all $\sigma \in \mathrm{CH}^4(X)$. We obtain

$$\begin{aligned} ((H_* + 2\mathrm{id}) \circ H_*)(Z) &= (H_* + 2\mathrm{id}) \left(\frac{-2}{r+2} L_*(lZ) \right) \\ &= \frac{4}{(r+2)^2} L_*(l \cdot L_*(lZ)) - \frac{4}{r+2} L_*(lZ) \\ &= 0, \end{aligned}$$

giving the decomposition of $\mathrm{CH}^2(X)$.
Next, let $Z \in \mathrm{CH}^3(X)$. Then

$$H_*(Z) = \frac{2}{r+2} l L_*(Z),$$

and $L_*(Z)$ is a divisor class. Therefore and by Lemma 5.4, $L_*(Z) = 0$ if and only if $Z \in \mathrm{CH}^3(X)_{\mathrm{hom}}$ is homologically trivial. Again by Lemma 5.4, in cohomology the cup product by $\frac{1}{r+2}[l]$ is the inverse isomorphism of $[L]_* : H^6(X) \to H^2(X)$. Hence, if $H_*(Z) = 0$, then in cohomology

$$0 = [H_*(Z)] = \frac{2}{r+2}[l] \cup [L_*(Z)] = 2[Z],$$

hence $Z \in \mathrm{CH}^3(X)_{\mathrm{hom}}$ (and thus $L_*(Z) = 0$). We have shown $\Lambda_0^3 = \mathrm{CH}^3(X)_{\mathrm{hom}}$. Next, by the quadratic equation for L,

$$((L^2)_* - 2\mathrm{id})(Z) = \frac{-2}{r+2} l L_*(Z) = -H_*(Z),$$

or, equivalently,

$$(H_* - 2\mathrm{id})(Z) = -(L^2)_*(Z).$$

But here, $(L^2)_*(Z) \in \mathrm{CH}^3(X)_{\mathrm{hom}} = \Lambda_0^3$ because in cohomology

$$(\mathfrak{B}^2)_*([Z]) = (p_2)_* \left(\sum_{i,j=1}^{r} (e_i e_j [Z]) \otimes (e_i e_j) \right),$$

and already $e_i e_j [Z] = 0$ for degree reasons. Therefore, $H_* \circ (H_* - 2\mathrm{id}) = 0$ on $\mathrm{CH}^3(X)$, giving the desired decomposition. In order to see that $l \cdot D$ is an element of Λ_2^3 for every divisor class D, just note $H_*(l \cdot D) = \frac{2}{r+2} l \cdot L_*(lD)$, and $L_*(lD)$ is a divisor which in cohomology agrees with $(r+2)D$.
At last, let $Z \in \mathrm{CH}^4(X)$. Then

$$H_*(Z) = \frac{4}{r(r+2)} \left(\int_X [Z] \right) l^2 + \frac{2}{r+2} l L_*(Z).$$

Now, $\int_X [H_*(Z)] = 4 \int_X [Z]$, implying

$$H_*(H_*(Z)) = \frac{16}{r(r+2)} \left(\int_X [Z] \right) l^2 + \frac{4}{r+2} l L_*(Z),$$

in particular $(H_*^2 - 4H_*)(Z) = \frac{-4}{r+2} l L_*(Z)$. Applying H_* once again yields

$$(H_* \circ (H_*^2 - 4H_*))(Z) = \frac{-8}{r+2} l L_*(Z) = 2(H_*^2 - 4H_*)(Z),$$

using that $\int_X [lL_*(Z)] = 0$. This is because $L_*(Z) \in \mathrm{CH}^2(X)_{\mathrm{hom}}$ for degree reasons. We have seen, then, that $H_* \circ (H_*^2 - 4H_*) = 2(H_*^2 - 4H_*)$ or, equivalently,

$$H_* \circ (H_* - 2\mathrm{id}) \circ (H_* - 4\mathrm{id}) = 0,$$

giving the desired decomposition for $\mathrm{CH}^4(X)$. Finally, let $Z \in \Lambda_4^4$. We want to show that Z is a multiple of l^2. Indeed,

$$4Z = H_*(Z) = l^2 \cdot \frac{4}{r(r+2)} \int_X [Z] + \frac{2}{r+2} l \cdot L_*(Z).$$

Applying L_* to the equation and using again the relation $L_*(l \cdot L_*(\sigma)) = (r+2)L_*(\sigma)$ for all $\sigma \in \mathrm{CH}^4(X)$ from the hypothesis, we get

$$4L_*(Z) = L_*(l^2) \cdot \frac{4}{r(r+2)} \int_X [Z] + \frac{2}{r+2} L_*(l \cdot L_*(Z)) = 2L_*(Z),$$

hence $L_*(Z) = 0$. But then, the previous equation implies that Z is a multiple of l^2. A similar argument shows $L_*(\Lambda_0^4) = 0$. The equation $\Lambda_2^4 = l \cdot L_*(\mathrm{CH}^4(X))$ is immediate from the explicit formula for H after observing $\Lambda_2^4 \subseteq \mathrm{CH}^4(X)_{\mathrm{hom}}$. $\square$

In order to state our multiplicativity conjecture, we introduce the following notation. For $s \in \mathbb{Z}$ let

$$\mathrm{CH}^i(X)_s := \left\{ Z \in \mathrm{CH}^i(X) : H_*(Z) = (2i - 4 - 2s)Z \right\} = \Lambda^i_{2i-4-2s}.$$

In here, s should be seen as a sort of defect. The terms with $s = 0$ give the expected eigenvalue $2i - 4$ of H_*. The eigenspace decomposition of H_* above shows that only terms with $s \geq 0$ occur. In fact, there is a conjecture by Beauville in the abelian variety case which predicts this behavior, see the introduction of [22]. It also predicts the injectivity of the cycle class map on $\mathrm{CH}^*(X)_0$, and Theorem 5.22 confirms this in all codimensions except 2. We can then rewrite the eigenspace decomposition of Theorem 5.22 as

$$\begin{aligned}
\mathrm{CH}^0(X) &= \mathrm{CH}^0(X)_0, \\
\mathrm{CH}^1(X) &= \mathrm{CH}^1(X)_0, \\
\mathrm{CH}^2(X) &= \mathrm{CH}^2(X)_0 \oplus \mathrm{CH}^2(X)_1, \\
\mathrm{CH}^3(X) &= \mathrm{CH}^3(X)_0 \oplus \mathrm{CH}^3(X)_1, \\
\mathrm{CH}^4(X) &= \mathrm{CH}^4(X)_0 \oplus \mathrm{CH}^4(X)_1 \oplus \mathrm{CH}^4(X)_2.
\end{aligned} \tag{5.4}$$

We can now state the multiplicativity conjecture more naturally.

Conjecture 5.23 (Conjecture 1.4)**.** Let X be as in Theorem 5.22 such that L additionally satisfies (4.9). For all occurring $s, t \in \mathbb{Z}$ we conjecture that the intersection product gives a well-defined map

$$\mathrm{CH}^i(X)_s \times \mathrm{CH}^j(X)_t \overset{\cdot}{\longrightarrow} \mathrm{CH}^{i+j}(X)_{s+t}.$$

An equivalent way of stating the conjecture is as follows: Let

$$\widetilde{H} := \frac{1}{2}H + 2\Delta \in \mathrm{CH}^4(X \times X). \tag{5.5}$$

Then the conjecture is equivalent to $\widetilde{H}_*$ acting as a derivation, i.e.,

$$\widetilde{H}_*(Z \cdot W) = \widetilde{H}_*(Z) \cdot W + Z \cdot \widetilde{H}_*(W) \tag{5.6}$$

for all $Z, W \in \mathrm{CH}^*(X)$. Indeed, $\mathrm{CH}^i(X)_s$ is precisely the eigenspace of $\widetilde{H}_*$ for the eigenvalue $i - s$. The conjecture is then equivalent to saying that the product of the eigenspaces of $\widetilde{H}$ for the eigenvalues $i - s$ and $j - t$ lands in the one for the eigenvalue $i + j - (s + t)$ which is obviously satisfied if (5.6) holds. Conversely, if the conjecture is true, then (5.6) holds on a generating subset of the Chow ring.
Let now mult_Z denote the multiplication map with a cycle class Z. Then (5.6) can again be rewritten as

$$[\widetilde{H}_*, \mathrm{mult}_Z] = \mathrm{mult}_{\widetilde{H}_*(Z)} \tag{5.7}$$

for all $Z \in \mathrm{CH}^*(X)$, and in fact it suffices to show that the left hand side is multiplication by *some* cycle class. A generalization of this in the case of Hilbert schemes of points of projective $K3$ surfaces can be found in [22].
The conjecture is in fact true if $X = S^{[2]}$ for a projective $K3$ surface S and if $X = F(Y)$ is the Fano variety of lines of a very general cubic fourfold Y, see Corollary 5.27 below. But first, we treat the simplest case of divisors for the Fano variety of lines of an arbitrary smooth cubic fourfold Y directly.

Proposition 5.24. Let $F = F(Y)$ be the Fano variety of lines of a smooth cubic fourfold Y. Then Conjecture 5.23 is true in the case of divisors, i.e.,

$$\mathrm{CH}^1(F) \cdot \mathrm{CH}^1(F) \subseteq \mathrm{CH}^2(F)_0.$$

Proof. In the case of divisors, only the defect $s = t = 0$ occurs and we need to show

$$(\Lambda^1_{-4})^{\cdot 2} \subseteq \Lambda^2_0$$

with the notation of Theorem 5.22. In other words, if D, D' are two divisor classes, we have to show $H_*(D \cdot D') = 0$. Indeed,

$$H_*(D \cdot D') = \frac{2}{r+2} l \cdot L_*(D \cdot D') - \frac{2}{r+2} L_*(l \cdot D \cdot D') = -\frac{2}{r+2} L_*(l \cdot D \cdot D'),$$

using that $L_*(D \cdot D') = 0$ as this is a multiple of the fundamental class vanishing in cohomology. We claim that $L_*(l \cdot D \cdot D') = 0$. But $l \cdot D$ is a multiple of D^3 and so $l \cdot D \cdot D'$ is a multiple of $D^3 \cdot D'$, which is itself a multiple of l^2 by [29, Theorem 1.4] using $l = \frac{5}{6} c_2(T_F)$ from Theorem 4.14. The latter also implies $L_*(l^2) = 0$, concluding the proof. □

We now compare the eigenspace decomposition of H_* in Theorem 5.22 to the Fourier decomposition of [26, Theorem 2] which needs the additional relation (4.9). The Fourier transform is given by the correspondence

$$e^L = [X \times X] + L + \frac{1}{2}L^2 + \frac{1}{6}L^3 + \frac{1}{24}L^4 \in \mathrm{CH}^*(X \times X).$$

We define the Fourier decomposition groups

$${}^{e^L}\mathrm{CH}^i(X)_s := \{Z \in \mathrm{CH}^i(X) : (e^L)_*(Z) \in \mathrm{CH}^{4-i+2s}(X)\},$$

where s is divided by 2 compared with *loc. cit.* The Fourier decomposition groups do not depend on the precise values of the coefficients of the powers L^i as long as they are non-zero.

Theorem 5.25 ([26, Theorem 2])**.** Let X be a hyperkähler variety of $K3^{[2]}$-type endowed with a lift $L \in \mathrm{CH}^2(X \times X)$ of $\mathfrak{B}$ satisfying all the relations (4.6)-(4.9) from Section 4.1. Then there is the Fourier decomposition

$$\begin{aligned}
\mathrm{CH}^0(X) &= {}^{e^L}\mathrm{CH}^0(X)_0,\\
\mathrm{CH}^1(X) &= {}^{e^L}\mathrm{CH}^1(X)_0,\\
\mathrm{CH}^2(X) &= {}^{e^L}\mathrm{CH}^2(X)_0 \oplus {}^{e^L}\mathrm{CH}^2(X)_1,\\
\mathrm{CH}^3(X) &= {}^{e^L}\mathrm{CH}^3(X)_0 \oplus {}^{e^L}\mathrm{CH}^3(X)_1,\\
\mathrm{CH}^4(X) &= {}^{e^L}\mathrm{CH}^4(X)_0 \oplus {}^{e^L}\mathrm{CH}^4(X)_1 \oplus {}^{e^L}\mathrm{CH}^4(X)_2.
\end{aligned}$$

Theorem 5.26. Let X be a hyperkähler variety of $K3^{[2]}$-type endowed with a lift $L \in \mathrm{CH}^2(X \times X)$ of $\mathfrak{B}$ satisfying all the relations (4.6)-(4.9). Then the eigenspace decomposition of H_* from Theorem 5.22, or equivalently (5.4), agrees with the Fourier decomposition of Theorem 5.25.

The only place where the relation (4.9) is needed is the existence of the Fourier decomposition of $\mathrm{CH}^2(X)$, see the proof of [26, Theorem 2.4]. The following proof will not make any other use of this relation.

Proof. In codimensions 0 and 1 there is nothing to show. For codimension 2 it suffices to show only the two inclusions

$${}^{e^L}\mathrm{CH}^2(X)_0 \subseteq \mathrm{CH}^2(X)_0 \quad\text{and}\quad {}^{e^L}\mathrm{CH}^2(X)_1 \subseteq \mathrm{CH}^2(X)_1,$$

because in both cases their sum is $\mathrm{CH}^2(X)$, and the sums are direct. Let $Z \in {}^{e^L}\mathrm{CH}^2(X)_0$, i.e., $(e^L)_*(Z) \in \mathrm{CH}^2(X)$ which is easily seen to be equivalent to $(L^3)_*(Z) = 0$. Now, applying the quadratic equation for L twice, it can be checked that

$$\begin{aligned}
L^3 = &\left(2 - \frac{4}{r+2}\right)\Delta_*(l) + \left(\frac{1}{r+2} + \frac{8}{(r+2)^2}\right) L l_1 l_2\\
&+ \left(\frac{4}{(r+2)^2} - \frac{2}{r(r+2)}\right) L(l_1^2 + l_2^2) + \left(\frac{4}{r(r+2)} - \frac{2}{r+2}\right)(l_1^2 l_2 + l_1 l_2^2),
\end{aligned}$$

where we used $L \cdot \Delta = L \cdot \Delta_*([X]) = \Delta_*(\Delta^*(L)) = \Delta_*(l)$. Applying this to Z gives

$$0 = (L^3)_*(Z) = \left(2 - \frac{4}{r+2}\right) l \cdot Z + \left(\frac{1}{r+2} + \frac{8}{(r+2)^2}\right) l \cdot L_*(l \cdot Z) + \left(\frac{4}{r(r+2)} - \frac{2}{r+2}\right)\left(\int_X [l \cdot Z]\right) l^2.$$

Applying to this the operator L_* yields then $L_*(l \cdot Z) = 0$ using the assumed relations $L_*(l^2) = 0$ and $L_*(l \cdot L_*(\sigma)) = (r+2)L_*(\sigma)$ for all $\sigma \in \mathrm{CH}^4(X)$. But now, the explicit formula for H yields $H_*(Z) = \frac{2}{r+2} l \cdot L_*(Z) - \frac{2}{r+2} L_*(l \cdot Z) = 0$ because $L_*(Z)$ is a multiple of the fundamental class vanishing in cohomology. This proves the first inclusion.
For the second inclusion let $Z \in {}^{e^L}\mathrm{CH}^2(X)_1$ which is now equivalent to $(L^2)_*(Z) = 0$. Using the quadratic equation for L we get

$$0 = (L^2)_*(Z) = 2Z - \frac{2}{r+2} L_*(l \cdot Z) + \frac{1}{r+2}\left(\int_X [l \cdot Z]\right) l.$$

Now by [26, Proposition 4.1], $l \cdot Z \in {}^{e^L}\mathrm{CH}^4(X)_1$, and the latter by [26, Theorem 4] agrees with $l \cdot L_*(\mathrm{CH}^4(X)) \subseteq \mathrm{CH}^4(X)_{\mathrm{hom}}$, hence $\int_X [l \cdot Z] = 0$. We thus obtain

$$L_*(l \cdot Z) = (r+2)Z,$$

and this is precisely equivalent to $H_*(Z) = 2Z$. This shows the second inclusion and finishes the codimension 2 case.
In codimension 3 let first $Z \in {}^{e^L}\mathrm{CH}^3(X)_1$. This is equivalent to $L_*(Z) = 0$. But now, $H_*(Z) = \frac{2}{r+2} l \cdot L_*(Z)$, so ${}^{e^L}\mathrm{CH}^3(X)_1 = \mathrm{CH}^3(X)_1$, as desired.
Next, $Z \in {}^{e^L}\mathrm{CH}^3(X)_0$ is equivalent to $(L^2)_*(Z) = 0$, i.e., by the quadratic equation for L,

$$0 = (L^2)_*(Z) = 2Z - \frac{2}{r+2} l \cdot L_*(Z),$$

hence $H_*(Z) = \frac{2}{r+2} l \cdot L_*(Z) = 2Z$, concluding the codimension 3 case.
In codimension 4, by [26, Theorem 4], we already know the equality of the direct summands

$$\begin{aligned} {}^{e^L}\mathrm{CH}^4(X)_0 &= \langle l^2 \rangle = \Lambda^4_4 = \mathrm{CH}^4(X)_0, \\ {}^{e^L}\mathrm{CH}^4(X)_1 &= l \cdot L_*(\mathrm{CH}^4(X)) = \Lambda^4_2 = \mathrm{CH}^4(X)_1. \end{aligned}$$

Finally, let $Z \in {}^{e^L}\mathrm{CH}^4(X)_2$ which is equivalent to the vanishing of both $L_*(Z)$ and $[X \times X]_*(Z) = \left(\int_X [Z]\right)[X]$, in particular $\int_X [Z] = 0$. So we get

$$H_*(Z) = \frac{4}{r(r+2)}\left(\int_X [Z]\right) l^2 + \frac{2}{r+2} l \cdot L_*(Z) = 0,$$

i.e., $Z \in \Lambda^4_0 = \mathrm{CH}^4(X)_2$, as desired. The reverse inclusion then follows automatically. □

Corollary 5.27 (Theorem 1.5)**.** In the eigenspace decomposition of Theorem 5.22, or equivalently in (5.4), all occuring direct summands are non-trivial. Moreover, Conjecture 5.23 is true if $X = S^{[2]}$ is Hilbert scheme of two points of a projective $K3$ surface S and if $X = F(Y)$ is the Fano variety of lines of a *very general* cubic fourfold Y.

Proof. For the non-vanishing of every occurring direct summand, see [26, p. 7]. For the multiplicativity in the $S^{[2]}$ case we even have two independent possibilities. One is [26, Theorem 3] and the other is [22, Theorem 1.4] together with our Theorem 6.8 showing that the two canonical lifts of the cohomological grading operator h agree. The case of a Fano variety of lines $F(Y)$ of a very general cubic foufold Y is again [26, Theorem 3]. □

Remark 5.28. It becomes clear from the proof of [26, Theorem 3] in the case of a Fano variety $F = F(Y)$ that the assumption on Y to be very general is only needed for the inclusions

$$\begin{aligned} \mathrm{CH}^1(F)_0 \cdot \mathrm{CH}^2(F)_0 &\subseteq \mathrm{CH}^3(F)_0, \\ \mathrm{CH}^2(F)_0 \cdot \mathrm{CH}^2(F)_0 &\subseteq \mathrm{CH}^4(F)_0, \end{aligned}$$

and the first one was dealt with for Y not necessarily very general in [6, Proposition A.7]. The second inclusion, however, still remains open for arbitrary smooth cubic fourfolds Y. In the very general case these inclusions are even equalities.

5.3 Addendum: some more relations on $\mathrm{CH}^*(F(Y) \times F(Y))$

Even though we cannot establish all necessary commutation relations in order to lift the entire Neron-Severi Lie algebra action for the Fano variety of lines $F(Y)$ analogous to [23, Theorem 1.1] for arbitrary smooth cubic fourfolds Y, we can at least establish the following relation in full generality:

Proposition 5.29. Let X be any hyperkähler variety of complex dimension 4 with $H^3(X, \mathbb{Q}) = 0$ and $L \in \mathrm{CH}^2(X \times X)$ any lift of $\mathfrak{B}$ with pullback $l = \Delta^*(L)$ along the diagonal embedding. Then for all divisor classes $a, b \in \mathrm{CH}^1(X)$ we have

$$[\widetilde{F}_a, \widetilde{F}_b] = 0.$$

This result is of course trivial if a and b are linearly dependent.

Proof. Without loss of generality we can assume $c_X = 1$. During this proof we use the well known "box notation" for pullbacks along the projection maps from the products $X \times X$ and $X \times X \times X$. Denote by $1 := [X] \in \mathrm{CH}^0(X)$ the fundamental class. Then, e.g., $L \boxtimes 1$ is the cycle $(p_{12})^*(L) \in \mathrm{CH}^*(X \times X \times X)$ where $p_{ij} : X \times X \times X \to X \times X$ are the projections to the factors corresponding to the indices. We set $\lambda := \frac{4}{r+2}$ to ease the notation. By definiton,

$$\begin{aligned} [\widetilde{F}_a, \widetilde{F}_b] &= \widetilde{F}_a \circ \widetilde{F}_b - \widetilde{F}_b \circ \widetilde{F}_a \\ &= (p_{13})_* \left(\widetilde{F}_b \boxtimes 1 \cdot 1 \boxtimes \widetilde{F}_a - \widetilde{F}_a \boxtimes 1 \cdot 1 \boxtimes \widetilde{F}_b \right). \end{aligned}$$

Obviously $\widetilde{F}_b \boxtimes 1 \cdot 1 \boxtimes \widetilde{F}_a$ becomes $\widetilde{F}_a \boxtimes 1 \cdot 1 \boxtimes \widetilde{F}_b$ when exchanging the roles of a and b, in particular all summands of $\widetilde{F}_b \boxtimes 1 \cdot 1 \boxtimes \widetilde{F}_a$ which are invariant under exchanging a and b cancel out after subtracting $\widetilde{F}_a \boxtimes 1 \cdot 1 \boxtimes \widetilde{F}_b$, even before taking the pushforward. We now compute

$$\begin{aligned}
\widetilde{F}_b \boxtimes 1 \cdot 1 \boxtimes \widetilde{F}_a = &\Big(\lambda\big((lb) \boxtimes 1 \boxtimes 1 + 1 \boxtimes (lb) \boxtimes 1\big) + 2 \cdot L \boxtimes 1 \cdot \big(b \boxtimes 1 \boxtimes 1 + 1 \boxtimes b \boxtimes 1\big)\Big) \\
&\cdot \Big(\lambda\big(1 \boxtimes (la) \boxtimes 1 + 1 \boxtimes 1 \boxtimes (la)\big) + 2 \cdot 1 \boxtimes L \cdot \big(1 \boxtimes a \boxtimes 1 + 1 \boxtimes 1 \boxtimes a\big)\Big) \\
= &\lambda^2 \cdot \Big((lb) \boxtimes (la) \boxtimes 1 + (lb) \boxtimes 1 \boxtimes (la) + 1 \boxtimes (l^2ab) \boxtimes 1 + 1 \boxtimes (lb) \boxtimes (la)\Big) \\
&+ 2\lambda \cdot 1 \boxtimes L \cdot \Big((lb) \boxtimes a \boxtimes 1 + (lb) \boxtimes 1 \boxtimes a + 1 \boxtimes (lab) \boxtimes 1 + 1 \boxtimes (lb) \boxtimes a\Big) \\
&+ 2\lambda \cdot L \boxtimes 1 \cdot \Big(b \boxtimes (la) \boxtimes 1 + b \boxtimes 1 \boxtimes (la) + 1 \boxtimes (lab) \boxtimes 1 + 1 \boxtimes b \boxtimes (la)\Big) \\
&+ 4 \cdot L \boxtimes 1 \cdot 1 \boxtimes L \cdot \Big(b \boxtimes a \boxtimes 1 + b \boxtimes 1 \boxtimes a + 1 \boxtimes (ab) \boxtimes 1 + 1 \boxtimes b \boxtimes a\Big).
\end{aligned}$$

In here, $l^2ab = 0$ for dimension reasons, and the summands

$$2\lambda\big(1 \boxtimes L + L \boxtimes 1\big) \cdot 1 \boxtimes (lab) \boxtimes 1 \quad \text{and} \quad 4 \cdot L \boxtimes 1 \cdot 1 \boxtimes L \cdot 1 \boxtimes (ab) \boxtimes 1$$

are symmetric in a and b, hence cancel out after subtracting $\widetilde{F}_a \boxtimes 1 \cdot 1 \boxtimes \widetilde{F}_b$. Moreover, using the projection formula, the λ^2 term vanishes entirely after taking the pushforward for dimension reasons, hence we can neglect this term. Up to the irrelevant λ^2 summand, for the difference we then obtain

$$\begin{aligned}
&2\lambda \cdot 1 \boxtimes L \cdot \Big((lb) \boxtimes a \boxtimes 1 - (la) \boxtimes b \boxtimes 1 + (lb) \boxtimes 1 \boxtimes a - (la) \boxtimes 1 \boxtimes b \\
&\qquad + 1 \boxtimes (lb) \boxtimes a - 1 \boxtimes (la) \boxtimes b\Big) \\
&+ 2\lambda \cdot L \boxtimes 1 \cdot \Big(b \boxtimes (la) \boxtimes 1 - a \boxtimes (lb) \boxtimes 1 + b \boxtimes 1 \boxtimes (la) - a \boxtimes 1 \boxtimes (lb) \\
&\qquad + 1 \boxtimes b \boxtimes (la) - 1 \boxtimes a \boxtimes (lb)\Big) \\
&+ 4 \cdot L \boxtimes 1 \cdot 1 \boxtimes L \cdot \Big(b \boxtimes a \boxtimes 1 - a \boxtimes b \boxtimes 1 + b \boxtimes 1 \boxtimes a - a \boxtimes 1 \boxtimes b \\
&\qquad + 1 \boxtimes b \boxtimes a - 1 \boxtimes a \boxtimes b\Big).
\end{aligned} \tag{5.8}$$

Here, after applying the pushforward $(p_{13})_*$ and using

$$L_*(la) = (r+2)a = ({}^tL)_*(la) \in \mathrm{CH}^1(X),$$

we get that in (5.8) the last two terms of the first big bracket as well as the first two terms of the second big bracket cancel out. All the other summands in fact vanish individually after applying the pushforward, concluding the proof. For this last claim note that any cycle of the form

$$(p_{13})_* \left(L \boxtimes 1 \cdot 1 \boxtimes L \cdot 1 \boxtimes a \boxtimes 1\right)$$

is zero because it is a divisor on $X \times X$ vanishing in cohomology. □

Let $F = F(Y)$ be the Fano variety of lines. Let $e^L = [F \times F] + L + \frac{1}{2}L^2 + \ldots$ be the formal power series in L with respect to the intersection product. In [26], this cycle e^L is studied as the kernel of the Fourier transform $\mathcal{F}$ by analogy with the abelian variety case. For abelian varieties one has, e.g., the relation $\mathcal{F}^{-1} \circ e \circ \mathcal{F} = -f$ in the Chow ring [30, p. 11]. The following result suggests that the analogy does not carry over entirely in this respect, at least for e^L as the kernel of the Fourier transform.

Proposition 5.30. Let F be the Fano variety of lines of a smooth cubic fourfold, $g \in \mathrm{CH}^1(F)$ the Plücker polarization class and L the canonical lift of $\mathfrak{B}$ of Theorem 4.14. Then the relations

$$e^{\pm L} \circ \Delta_*(g) \circ e^L = L(g_1 \pm g_2) + \frac{1}{2}(l_1 + l_2)(g_2 \pm g_1)$$

hold in $\mathrm{CH}^*(F \times F)$.

Both cycles are *not* lifts of (multiples of) f_g. Nonetheless, these cycles resemble F_g a lot. For this simple reason it could be expected that there is a similar but different kernel for the Fourier transform giving the analog of the abelian variety case.

Proof. One only needs to calculate the compositions of correspondences which have the form

$$(p_{13})_* \left(L^s \boxtimes 1 \cdot 1 \boxtimes L^t \cdot 1 \boxtimes g \boxtimes 1\right). \tag{5.9}$$

This vanishes whenever $(s, t) \notin \{(0, 3), (3, 0), (1, 2), (2, 1)\}$. This is clear if $s + t \geq 6$ or $s + t \leq 1$ for dimension reasons. Moreover, for $s + t \geq 4$ and $s + t \leq 2$ it follows in cohomology for degree reasons and from the fact that (5.9) lies in the tautological subring of Theorem 4.15 by Proposition 5.18. For the non-zero contributions consider first $s = 1,\ t = 2$. Here, we use the quadratic equation for L^2, and by similar computations get the result $2Lg_2 + l_2 g_1$. The case $s = 0,\ t = 3$ is even simpler and yields $3l_2 g_2$. By symmetry, we have thus considered all the relevant cases, and putting everything together yields the claimed formula. □

6 The cycles L and H on $K3^{[2]}$ in Nakajima operators

Let S be a smooth projective surface. We will freely use the definitions and results of Section 4.3. Let $\mathcal{Z}_n \subseteq S^{[n]} \times S$ be the universal family. With the notation of Section 4.3 we have $\mathcal{Z}_2 = \mathcal{Z}$.

Lemma 6.1. Recall the definition of $Z_{n-1,n}$ from Definition 4.18. Let $n \geq 1$ and $p_{S\times S^{[n]}} : S^{[n-1]} \times S \times S^{[n]} \to S \times S^{[n]}$ be the projection. Then in $\mathrm{CH}^*(S \times S^{[n]})$ we have

$$(p_{S\times S^{[n]}})_* (Z_{n-1,n}) = {}^t\mathcal{Z}_n.$$

Proof. First recall that $Z_{n-1,n}$ is a closed subvariety of the product. This can be seen using the projection map $Z_{n-1,n} \to S^{[n-1,n]}$ onto the nested Hilbert scheme [16, Section 1.2] and the fact that $S^{[n-1,n]}$ is irreducible by [16, Theorem 1.9]. Indeed, the projection is proper, hence closed, and it actually is a bijection on closed points. It follows that $Z_{n-1,n}$ is irreducible, hence a closed subvariety as it is endowed with the reduced induced subscheme structure.
Now, to prove the claim, note that the equation is clearly true set-theoretically. As $Z_{n-1,n}$ is a closed subvariety of the product, it suffices to show that the degree of $p_{S\times S^{[n]}}|_{Z_{n-1,n}}$ is 1. Considering a general fiber suffices, and if $(x,\eta) \in {}^t\mathcal{Z}_n$ with η supported at n distinct points $x, x_2, \ldots, x_n$ then the only preimage in $Z_{n-1,n}$ is $([x_2,\ldots,x_n], x, \eta)$ with multiplicity 1. □

The following observation greatly facilitates the task of expressing the canonical lift L of $\mathfrak{B}$ from Theorem 4.21 in Nakajima operators.

Corollary 6.2. As correspondences in $\mathrm{CH}^*(S^{[2]} \times S^{[2]})$ we have

$${}^t\mathcal{Z} \circ \mathcal{Z} = -\mathfrak{q}_1([S]) \circ \mathfrak{q}_{-1}([S])$$

in terms of Nakajima operators.

Proof. This follows from the fact that the Hilbert–Chow morphism $S^{[1]} \to S$, sending a length 1 subscheme to its support point, is an isomorphism and from the definition of $\mathfrak{q}_{\pm 1}([S])$ as correspondences in $\mathrm{CH}^*(S^{[2]} \times S^{[1]})$. Indeed,

$$\mathfrak{q}_1([S]) = (p_{S^{[1]}\times S^{[2]}})_* (Z_{1,2}),$$
$$\mathfrak{q}_{-1}([S]) = (-1) \cdot (p_{S^{[1]}\times S^{[2]}})_* (Z_{1,2}).$$

The operator $\mathfrak{q}_{-1}([S])$ is then a correspondence from $S^{[2]}$ to $S^{[1]}$ while $\mathfrak{q}_1([S])$ is a correspondence in the opposite direction. Now, using $S^{[1]} = S$ by the Hilbert–Chow morphism, Lemma 6.1 for $n = 2$ gives the desired result. □

Lemma 6.3. Let S be a projective $K3$ surface and $\Delta_S \subseteq S \times S$ the diagonal. We denote by the same symbol its cycle class in $\mathrm{CH}^*(S \times S)$. Then we have

$$\Delta_S^2 = 24 c_1 c_2. \tag{6.1}$$

Proof. We use the Beauville–Voisin formula $c_2(T_S) = 24c$ for projective $K3$ surfaces S, proved in [3]. Then by the self-intersection formula [11, Appendix A.4.C7], we have

$$\Delta_S^2 = (i_{\Delta_S})_* c_2(\mathcal{N}_{\Delta_S/S\times S}).$$

Now, by the normal sheaf sequence

$$0 \to T_{\Delta_S} \to T_{S\times S}|_{\Delta_S} \to \mathcal{N}_{\Delta_S/S\times S} \to 0,$$

by additivity of the Chern character ch_2 and using $c_1(T_S) = 0$ we eventually obtain

$$c_2(\mathcal{N}_{\Delta_S/S\times S}) = c_2(T_S) = 24c,$$

hence $(i_{\Delta_S})_* c_2(\mathcal{N}_{\Delta_S/S\times S}) = 24\Delta_S c_1 = 24c_1c_2$, as desired. □

Proposition 6.4. Let S be a projective $K3$ surface and $F = S^{[2]}$. We have the following expressions for correspondences in terms of Nakajima operators:

$$[F \times F] = \frac{1}{4}\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}([S^4]), \tag{6.2}$$

$$\mathrm{mult}_{S_c} = -\mathfrak{q}_1\mathfrak{q}_{-1}(c_1c_2) - \mathfrak{q}_2\mathfrak{q}_{-2}(c_1c_2), \tag{6.3}$$

$$\mathrm{mult}_{\delta^2} = 12\mathfrak{q}_2\mathfrak{q}_{-2}(c_1c_2) - \frac{1}{2}\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(\Delta_{1234}), \tag{6.4}$$

$$\mathrm{mult}_l = -20\mathfrak{q}_1\mathfrak{q}_{-1}(c_1c_2) - 50\mathfrak{q}_2\mathfrak{q}_{-2}(c_1c_2) + \frac{5}{4}\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(\Delta_{1234}). \tag{6.5}$$

Of course, as correspondences we interpret mult_Z to be $\Delta_*(Z)$.

Proof. The first of these equations follows immediately from Corollary A.2 and

$$[F] = \frac{1}{2}\mathfrak{q}_1([S])\mathfrak{q}_1([S]) \cdot 1_{S^{[0]}}.$$

The last equation for mult_l of course follows from the two preceeding ones together with (4.14). For the second equation we use the notation from Section 4.3 yielding

$$S_c = p_*q^*(c) = \pi_*(\mathrm{ch}_2(\mathcal{O}_{\mathcal{Z}})\rho^*(c)),$$

as $\mathcal{Z} = \mathrm{ch}_2(\mathcal{O}_{\mathcal{Z}})$ by [26, eq. (101) on p. 75]. Now, the first formula of [20, Theorem 1.6] shows the claim. Finally, for mult_{δ^2} we use the operator e_δ of multiplication by δ and its expression in Nakajima operators from [23, eq. (4)] and compute

$$\begin{aligned}
\mathrm{mult}_{\delta^2} &= e_\delta \circ e_\delta \\
&= \frac{1}{4}\Big(\mathfrak{q}_2\mathfrak{q}_{-1}\mathfrak{q}_{-1}(\Delta_{123}) + \mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-2}(\Delta_{123})\Big)^{\circ 2} \\
&= \frac{1}{4}\Big(\mathfrak{q}_2\mathfrak{q}_{-1}\mathfrak{q}_{-1}\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-2}(\Delta_{123}\Delta_{456}) + \mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-2}\mathfrak{q}_2\mathfrak{q}_{-1}\mathfrak{q}_{-1}(\Delta_{123}\Delta_{456})\Big).
\end{aligned}$$

Using the commutation relations for Nakajima operators, the second summand inside the brackets equals $-2\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(\Delta_{1234})$. For the first summand, we obtain

$$\begin{aligned}\mathfrak{q}_2\mathfrak{q}_{-1}\mathfrak{q}_{-1}\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-2}(\Delta_{123}\Delta_{456}) &= \mathfrak{q}_2\mathfrak{q}_{-1}\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_1\mathfrak{q}_{-2}(\Delta_{124}\Delta_{356}) \\ &\quad - \mathfrak{q}_2\mathfrak{q}_{-1}\mathfrak{q}_1\mathfrak{q}_{-2}\Big((\rho_{1256})_*(\Delta_{34}\Delta_{123}\Delta_{456})\Big) \\ &= -2\mathfrak{q}_2\mathfrak{q}_{-1}\mathfrak{q}_1\mathfrak{q}_{-2}(\Delta_{1234}) \\ &= 2\mathfrak{q}_2\mathfrak{q}_{-2}\Big((\rho_{14})_*(\Delta_{23}\cdot\Delta_{1234})\Big) \\ &= 48\mathfrak{q}_2\mathfrak{q}_{-2}(c_1c_2),\end{aligned}$$

where in the last step we used $\Delta_{1234} = \Delta_{12}\Delta_{23}\Delta_{34}$ and $\Delta_S^2 = 24c_1c_2$ from Lemma 6.3 as well as $\Delta_S c_1 = \Delta_S c_2 = c_1c_2$, concluding the proof. □

Proposition 6.5. Let S be a projective $K3$ surface and $F = S^{[2]}$. Then, as correspondences in $\mathrm{CH}^2(F\times F)$, we can express L in terms of Nakajima operators by

$$L = -\mathfrak{q}_1\mathfrak{q}_{-1}([S\times S]) - \frac{1}{8}\mathfrak{q}_2\mathfrak{q}_{-2}([S\times S]) - \mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(c_1+c_4). \tag{6.6}$$

Proof. We express each summand of (4.13) in terms of Nakajima operators. For I we have $I = {}^t\mathcal{Z}\circ\mathcal{Z} = -\mathfrak{q}_1\mathfrak{q}_{-1}([S\times S])$ by [26, Lemma 11.2] and Corollary 6.2. Recall that $[F] = \frac{1}{2}\mathfrak{q}_1([S])\mathfrak{q}_1([S])\cdot 1_{S^{[0]}}$. Using this, we have

$$\begin{aligned}\delta &= e_\delta([F]) \\ &= \frac{1}{2}\Big(\mathfrak{q}_2\mathfrak{q}_{-1}\mathfrak{q}_{-1}(\Delta_{123}) + \mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-2}(\Delta_{123})\Big)\circ\frac{1}{2}\mathfrak{q}_1\mathfrak{q}_1([S\times S])\cdot 1_{S^{[0]}} \\ &= \frac{1}{4}\mathfrak{q}_2\mathfrak{q}_{-1}\mathfrak{q}_{-1}\mathfrak{q}_1\mathfrak{q}_1(\Delta_{123})\cdot 1_{S^{[0]}} \\ &= -\frac{1}{2}\mathfrak{q}_2\mathfrak{q}_{-1}\mathfrak{q}_1(\Delta_{12})\cdot 1_{S^{[0]}} \\ &= \frac{1}{2}\mathfrak{q}_2([S])\cdot 1_{S^{[0]}}.\end{aligned}$$

Together with Corollary A.2 this gives

$$\begin{aligned}-\frac{1}{2}\delta_1\delta_2 &= -\frac{1}{8}\Big(\mathfrak{q}_2([S])\cdot 1_{S^{[0]}}\Big)\boxtimes\Big(\mathfrak{q}_2([S])\cdot 1_{S^{[0]}}\Big) \\ &= -\frac{1}{8}\mathfrak{q}_2([S])\mathfrak{q}_2'([S])\cdot\Delta_{S^{[0]}} \\ &= -\frac{1}{8}\mathfrak{q}_2\mathfrak{q}_{-2}([S\times S]).\end{aligned}$$

Similarly, using Proposition 6.4, we get

$$\begin{aligned}S_c &= \mathrm{mult}_{S_c}([F]) \\ &= -\frac{1}{2}\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_1\mathfrak{q}_1(c_1c_2)\cdot 1_{S^{[0]}} \\ &= \mathfrak{q}_1\mathfrak{q}_1(c_1)\cdot 1_{S^{[0]}}.\end{aligned}$$

Applying again Corollary A.2 yields

$$-2(S_c)_1 = -\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(c_4) \quad \text{and} \quad -2(S_c)_2 = -\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(c_1),$$

and putting everything together gives the claimed formula for L. □

Theorem 6.6 ([3])**.** Let S be a projective $K3$ surface. Then in $\mathrm{CH}_2(S\times S\times S)$ we have a decomposition of the small diagonal

$$\Delta_{123} = \Delta_{12}c_3 + \Delta_{13}c_2 + \Delta_{23}c_1 - c_1c_2 - c_1c_3 - c_2c_3.$$

Theorem 6.7. Let S be a projective $K3$ surface and $F = S^{[2]}$. Let a be a divisor class on F and $\widetilde{F}_a$ the canonical lift of the Lefschetz dual $\widetilde{f}_a$ from Theorem 5.8 with respect to the canonical lift L from Theorem 4.21. Then $\widetilde{F}_a$ agrees with Oberdieck's canonical lift of $\widetilde{f}_a$ in [23, eq. (6)], i.e.,

$$\widetilde{F}_a = -2\sum_{n\geq 1}\frac{1}{n^2}\mathfrak{q}_n\mathfrak{q}_{-n}(a_1+a_2) = -2\mathfrak{q}_1\mathfrak{q}_{-1}(a_1+a_2) - \frac{1}{2}\mathfrak{q}_2\mathfrak{q}_{-2}(a_1+a_2), \tag{6.7}$$

if a is in the surface part $\mathrm{CH}^1(S)\subseteq \mathrm{CH}^1(S^{[2]}) = \mathrm{CH}^1(S)\oplus\mathbb{Q}\delta$, and

$$\begin{aligned}\widetilde{F}_\delta &= -\frac{1}{3}\sum_{i+j+k=0} :\mathfrak{q}_i\mathfrak{q}_j\mathfrak{q}_k\Big(\frac{1}{k^2}\Delta_{12} + \frac{1}{j^2}\Delta_{13} + \frac{1}{i^2}\Delta_{23} + \frac{2}{j\cdot k}c_1 + \frac{2}{i\cdot k}c_2 + \frac{2}{i\cdot j}c_3\Big): \\ &= 2\mathfrak{q}_2\mathfrak{q}_{-1}\mathfrak{q}_{-1}\left(c_3 - c_1 - \Delta_{12} - \frac{1}{8}\Delta_{23}\right) + 2\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-2}\left(c_1 - c_3 - \Delta_{23} - \frac{1}{8}\Delta_{12}\right).\end{aligned} \tag{6.8}$$

Proof. First, let a be in the surface part $\mathrm{CH}^1(S)\subseteq\mathrm{CH}^1(S^{[2]})$. We begin by expressing l_1a_1 in terms of Nakajima operators, l_2a_2 being its transpose. We use

$$l_1a_1 = [F\times F]\circ \mathrm{mult}_l \circ \mathrm{mult}_a$$

as correspondences and the formulas from Proposition 6.4 as well as the formula for $e_a = \mathrm{mult}_a$ from [23, eq. (4)]. This seems to be faster here than using Corollary A.2. We have

$$\begin{aligned} l_1a_1 &= [F\times F]\circ\mathrm{mult}_l\circ e_a \\ &= \mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}([S^4])\,\circ \\ &\circ\left(5\mathfrak{q}_1\mathfrak{q}_{-1}(c_1c_2) - \frac{5}{16}\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(\Delta_{1234})\right)\circ\mathfrak{q}_1\mathfrak{q}_{-1}(\Delta_*(a)) \\ &= \mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}([S^4])\circ\left(5\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(\Delta_{14}a_1c_2c_3) + \frac{5}{8}\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(\Delta_{1234}a_1)\right).\end{aligned} \tag{6.9}$$

For the first summand in (6.9), we get

$$\begin{aligned}&5\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(\Delta_{48}a_8c_6c_7) - 5\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(c_4c_5a_6) \\ &= -5\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(\Delta_{46}a_6c_5) + 5\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(c_3a_4) \\ &= 10\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(c_3a_4).\end{aligned}$$

The second summand in (6.9) equals

$$\begin{aligned}
&\frac{5}{8}\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(\Delta_{4678}a_8) - \frac{5}{8}\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(\Delta_{456}a_6) \\
&= -\frac{5}{4}\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(\Delta_{456}a_6) \\
&= \frac{5}{4}\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(c_3a_4 + a_3c_4) \\
&= \frac{5}{2}\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(c_3a_4),
\end{aligned}$$

where we used $\Delta_{34}a_3 = a_3c_4 + c_3a_4$ (see [3]) which requires S to be a $K3$ surface. Hence,

$$l_1a_1 = \frac{25}{2}\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(c_3a_4),$$

and therefore

$$\frac{4}{25}(l_1a_1 + l_2a_2) = 2\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(a_1c_2 + c_3a_4).$$

We now calculate $La_1 = (-L)\circ(-e_a)$ employing the formula for L from Proposition 6.5. We have

$$\begin{aligned}
La_1 = &\left(\mathfrak{q}_1\mathfrak{q}_{-1}([S\times S]) + \frac{1}{8}\mathfrak{q}_2\mathfrak{q}_{-2}([S\times S]) + \mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(c_1 + c_4)\right) \\
&\circ \left(\mathfrak{q}_1\mathfrak{q}_{-1}(\Delta_*(a)) + \mathfrak{q}_2\mathfrak{q}_{-2}(\Delta_*(a))\right) \\
&= \mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_1\mathfrak{q}_{-1}(\Delta_{34}a_4) + \frac{1}{8}\mathfrak{q}_2\mathfrak{q}_{-2}\mathfrak{q}_2\mathfrak{q}_{-2}(\Delta_{34}a_4) + \mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}\mathfrak{q}_1\mathfrak{q}_{-1}((c_1 + c_4)\Delta_{56}a_6) \\
&= \mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(c_1a_4 + a_1c_4) - \mathfrak{q}_1\mathfrak{q}_{-1}(a_2) - \frac{1}{4}\mathfrak{q}_2\mathfrak{q}_{-2}(a_2) \\
&+ \mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}((c_1 + c_5)\Delta_{46}a_6) - \mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(c_1a_4) \\
&= -\mathfrak{q}_1\mathfrak{q}_{-1}(a_2) - \frac{1}{4}\mathfrak{q}_2\mathfrak{q}_{-2}(a_2) + \mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(a_1c_4 - c_1a_4 - c_3a_4).
\end{aligned}$$

After adding La_2, clearly $\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(a_1c_4 - c_1a_4)$ cancels out. We obtain

$$2L(a_1 + a_2) = -2\mathfrak{q}_1\mathfrak{q}_{-1}(a_1 + a_2) - \frac{1}{2}\mathfrak{q}_2\mathfrak{q}_{-2}(a_1 + a_2) - 2\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(a_1c_2 + c_3a_4),$$

and putting both pieces together yields the desired formula.
Let now $a = \delta$. Using the formula for e_δ from [23, eq. (4)], we can analogously compute

$$\begin{aligned}
l_1\delta_1 &= [F\times F]\circ \mathrm{mult}_l \circ e_\delta \\
&= -25\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-2}(c_3),
\end{aligned}$$

so that

$$\frac{4}{25}(l_1\delta_1 + l_2\delta_2) = -4\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-2}(c_3) - 4\mathfrak{q}_2\mathfrak{q}_{-1}\mathfrak{q}_{-1}(c_1). \tag{6.10}$$

Using the formula for L, we now consider $2L\delta_1 = 2(-L)\circ(-e_\delta)$ and get

$$\begin{aligned} 2L\delta_1 &= \left(\mathfrak{q}_1\mathfrak{q}_{-1}([S\times S]) + \frac{1}{8}\mathfrak{q}_2\mathfrak{q}_{-2}([S\times S]) + \mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(c_1+c_4)\right) \\ &\circ \left(\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-2}(\Delta_{123}) + \mathfrak{q}_2\mathfrak{q}_{-1}\mathfrak{q}_{-1}(\Delta_{123})\right) \\ &= \mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-2}(\Delta_{345}) + \frac{1}{8}\mathfrak{q}_2\mathfrak{q}_{-2}\mathfrak{q}_2\mathfrak{q}_{-1}\mathfrak{q}_{-1}(\Delta_{345}) \\ &+ \mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-2}((c_1+c_4)\Delta_{567}) \\ &= -2\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-2}(\Delta_{23}) - \frac{1}{4}\mathfrak{q}_2\mathfrak{q}_{-1}\mathfrak{q}_{-1}(\Delta_{23}) + 2\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-2}(c_1+c_3), \end{aligned}$$

where we applied the Nakajima commutation relations several times. Therefore,

$$\begin{aligned} 2L(\delta_1+\delta_2) = \mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-2}\left(2(c_1+c_3) - 2\Delta_{23} - \frac{1}{4}\Delta_{12}\right) \\ + \mathfrak{q}_2\mathfrak{q}_{-1}\mathfrak{q}_{-1}\left(2(c_1+c_3) - 2\Delta_{12} - \frac{1}{4}\Delta_{23}\right). \end{aligned} \tag{6.11}$$

Putting equations (6.10) and (6.11) together yields the claimed formula for $\widetilde{F}_\delta$. □

Theorem 6.8. Let $H = \frac{4}{r(r+2)}(l_2^2 - l_1^2) + \frac{2}{r+2}(l_2 - l_1)L$ with $r = 23$ be the canonical lift of the cohomological grading operator h from Proposition 5.14. Let S be a projective $K3$ surface and $F = S^{[2]}$. If L is the canonical lift of $\mathfrak{B}$ of (4.13) and l its pullback (4.14), then H agrees with Oberdieck's canonical lift of h [23, eq. (5)], i.e.,

$$H = 2\sum_{n\geq 1}\frac{1}{n}\mathfrak{q}_n\mathfrak{q}_{-n}(c_2-c_1) = 2\mathfrak{q}_1\mathfrak{q}_{-1}(c_2-c_1) + \mathfrak{q}_2\mathfrak{q}_{-2}(c_2-c_1). \tag{6.12}$$

Proof. Let $c \in \mathrm{CH}_0(S)$ be the canonical 0-cycle represented by any point p lying on a rational curve inside S. Consider $l^2 \in \mathrm{CH}_0(F)$. As l is a multiple of $c_2(T_F)$ by (4.14), by [29, Theorem 1.4] its square is a multiple of the 0-cycle $[c,c] = \mathfrak{q}_1(c)\mathfrak{q}_1(c)\cdot 1_{S^{[0]}}$ which is represented by a single point in F, hence of degree 1. The latter is a multiple of δ^4 so that the cited theorem is applicable. But this immediately implies

$$l^2 = r(r+2)\mathfrak{q}_1(c)\mathfrak{q}_1(c)\cdot 1_{S^{[0]}}$$

because it is true in cohomology. By Corollary A.2 therefore

$$l_1^2 = \frac{r(r+2)}{2}\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(c_3c_4) \quad \text{and} \quad l_2^2 = \frac{r(r+2)}{2}\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(c_1c_2).$$

Thus, for the first summand of H we obtain

$$\frac{4}{r(r+2)}(l_2^2 - l_1^2) = 2\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(c_1c_2 - c_3c_4).$$

In order to express the remaining summand of H in Nakajima operators, we only need to compute $Ll_1 = L \circ \mathrm{mult}_l$ as $Ll_2 = \mathrm{mult}_l \circ L$ is just its transpose. In view of Proposition 6.5 and the formula for mult_l of Proposition 6.4, the compositions of $\mathfrak{q}_2\mathfrak{q}_{-2}$ summands and $\mathfrak{q}_1\mathfrak{q}_{-1}$ or $\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}$ summands are all trivial using the Nakajima commutation relations and the fact that we work on $F = S^{[2]}$. We claim

$$\begin{aligned} L \circ \mathrm{mult}_l = &-\frac{25}{2}\mathfrak{q}_2\mathfrak{q}_{-2}(c_2) - 25\mathfrak{q}_1\mathfrak{q}_{-1}(c_2) \\ &- 25\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(c_3c_4) - 25\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(c_1c_4). \end{aligned} \tag{6.13}$$

The last term agrees with its transpose, hence cancels out in $L(l_2 - l_1)$. Clearly, if the claim is true, we obtain precisely the desired result. It remains to show (6.13). First of all, the composition of the summand $-\frac{1}{8}\mathfrak{q}_2\mathfrak{q}_{-2}([S \times S])$ of L and the summand $-50\mathfrak{q}_2\mathfrak{q}_{-2}(c_1c_2)$ of mult_l gives

$$\frac{50}{8}\mathfrak{q}_2\mathfrak{q}_{-2}\mathfrak{q}_2\mathfrak{q}_{-2}(c_3c_4) = -\frac{25}{2}\mathfrak{q}_2\mathfrak{q}_{-2}(c_2).$$

Next, the composition of $-\mathfrak{q}_1\mathfrak{q}_{-1}([S \times S])$ of L with $-20\mathfrak{q}_1\mathfrak{q}_{-1}(c_1c_2)$ of mult_l yields

$$20\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_1\mathfrak{q}_{-1}(c_3c_4) = 20\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(c_2c_4) - 20\mathfrak{q}_1\mathfrak{q}_{-1}(c_2).$$

Observe $\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(c_2c_4) = \mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(c_1c_4)$ by the commutation relations. For the composition of $-\mathfrak{q}_1\mathfrak{q}_{-1}([S \times S])$ with $\frac{5}{4}\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(\Delta_{1234})$ we have

$$-\frac{5}{4}\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(\Delta_{3456}) = \frac{5}{2}\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(\Delta_{234}).$$

Next, for the composition of $-\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(c_1 + c_4)$ with $-20\mathfrak{q}_1\mathfrak{q}_{-1}(c_1c_2)$ we get

$$20\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}\mathfrak{q}_1\mathfrak{q}_{-1}((c_1 + c_4)c_5c_6) = -20\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(c_3c_4 + 2c_1c_4).$$

Finally, the composition of $-\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(c_1 + c_4)$ and $\frac{5}{4}\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(\Delta_{1234})$ is

$$-\frac{5}{4}\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}((c_1 + c_4)\Delta_{5678}) = -\frac{5}{2}\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(c_3c_4 + c_1\Delta_{34}).$$

Summing up these five compositions yields after some cancellation

$$\begin{aligned} L \circ \mathrm{mult}_l = &-\frac{25}{2}\mathfrak{q}_2\mathfrak{q}_{-2}(c_2) - 20\mathfrak{q}_1\mathfrak{q}_{-1}(c_2) + \frac{5}{2}\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(\Delta_{234}) \\ &- 20\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(c_3c_4 + c_1c_4) - \frac{5}{2}\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(c_3c_4 + c_1\Delta_{34}). \end{aligned} \tag{6.14}$$

Now, we use the decomposition of the small diagonal in Theorem 6.6, i.e.,

$$\Delta_{234} = \Delta_{23}c_4 + \Delta_{24}c_3 + \Delta_{34}c_2 - c_2c_3 - c_2c_4 - c_3c_4.$$

Plugging this into the corresponding summand $\frac{5}{2}\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(\Delta_{234})$ gives

$$\begin{aligned}\frac{5}{2}\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(\Delta_{234}) = & \frac{5}{2}\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(\Delta_{23}c_4) + \frac{5}{2}\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(\Delta_{24}c_3) \\ & + \frac{5}{2}\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(\Delta_{34}c_2) - \frac{5}{2}\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(c_2c_3) \\ & - \frac{5}{2}\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(c_2c_4) - \frac{5}{2}\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(c_3c_4).\end{aligned}$$

In here, the commutation relations imply that the first two summands both agree with $\frac{5}{2}\mathfrak{q}_1([S])\mathfrak{q}_1\mathfrak{q}_{-1}(\Delta_S)\mathfrak{q}_{-1}(c)$. Clearly, the fourth and fifth summand agree as well, and for the third summand we have

$$\frac{5}{2}\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(\Delta_{34}c_2) = \frac{5}{2}\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(\Delta_{34}c_1)$$

by the commutation relations. We can then continue by

$$\begin{aligned}\frac{5}{2}\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(\Delta_{234}) &= 5\mathfrak{q}_1([S])\mathfrak{q}_1\mathfrak{q}_{-1}(\Delta_S)\mathfrak{q}_{-1}(c) - 5\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(c_1c_4) \\ &\quad - \frac{5}{2}\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(c_3c_4) \\ &= -5\mathfrak{q}_1\mathfrak{q}_{-1}(c_2) - 5\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(c_1c_4) \\ &\quad - \frac{5}{2}\mathfrak{q}_1\mathfrak{q}_1\mathfrak{q}_{-1}\mathfrak{q}_{-1}(c_3c_4),\end{aligned}$$

where in the final step we used

$$\mathfrak{q}_1\mathfrak{q}_{-1}(\Delta_S) = -\mathrm{id}_{S^{[1]}}$$

as correspondences on $S^{[1]} \times S^{[1]}$. Inserting this into (6.14), we obtain exactly (6.13), as desired. □

Remark 6.9. Theorems 6.7 and 6.8 further justify our claim that, given the canonical lift L of $\mathfrak{B}$, the cycles $\widetilde{F}_a$ and H of Theorem 5.8 and Proposition 5.14 are the canonical lifts of the cohomological operators $\widetilde{f}_a$ and h. They also show that the lifting of the cohomological $\mathfrak{sl}_2(\mathbb{Q})$-action obtained in Theorem 5.19 is the correct (partial) analog of the lifting on $S^{[2]}$ dealt with for arbitrary $S^{[n]}$ in [23], and so provide additional evidence for Conjecture 5.21. Finally, these two theorems together with the main theorem of [23] also show the equation $H = H_a$ (see Remark 5.16) for all divisor classes a with $(a, a) \neq 0$ in the Hilbert scheme case, and so yields further evidence that this might be true for general hyperkähler varieties X of $K3^{[2]}$-type.

Remark 6.10. Theorem 6.8 gives another proof for Conjecture 5.23 for Hilbert schemes of two points of projective $K3$ surfaces by the more general [22, Theorem 1.4].

A Auxiliary results

Lemma A.1. Let X, Y and Z be smooth projective varieties and $\Gamma \in \mathrm{CH}^*(X \times Y)$, $\widetilde{\Gamma} \in \mathrm{CH}^*(X \times Z)$ correspondences. Then as correspondences in $\mathrm{CH}^*(Y \times Z)$ we have

$$(\Gamma \times \widetilde{\Gamma})_*(\Delta_X) = \widetilde{\Gamma} \circ {}^t\Gamma.$$

Proof. Writing out the definitions of both sides of the equation we see that we only need to show

$$p^*_{X_1 Y}(\Gamma) \cdot p^*_{X_2 Z}(\widetilde{\Gamma}) \cdot p^*_{X_1 X_2}(\Delta_X) = (\mathrm{id}_Y \times \Delta_X \times \mathrm{id}_Z)_* \left(p^*_{YX}({}^t\Gamma) \cdot p^*_{XZ}(\widetilde{\Gamma})\right). \quad \text{(A.1)}$$

In here, p_{YX} and p_{XZ} are the projections from the triple product $Y \times X \times Z$ to $Y \times X$ and $X \times Z$, respectively. The projections $p_{X_1 Y}$, $p_{X_2 Z}$ and $p_{X_1 X_2}$ are the projections from the product $X \times Y \times X \times Z$ to the factors indicated by the indices. Moreover, Δ_X denotes both the diagonal embedding $X \hookrightarrow X \times X$ and the cycle class of its image. We can now rewrite the argument of the right hand side of (A.1) as

$$\begin{aligned} p^*_{YX}({}^t\Gamma) \cdot p^*_{XZ}(\widetilde{\Gamma}) &= (\mathrm{id}_Y \times \Delta_X \times \mathrm{id}_Z)^* \left(p^*_{YX_1}({}^t\Gamma)\right) \cdot (\mathrm{id}_Y \times \Delta_X \times \mathrm{id}_Z)^* \left(p^*_{X_2 Z}(\widetilde{\Gamma})\right) \\ &= (\mathrm{id}_Y \times \Delta_X \times \mathrm{id}_Z)^* \left(p^*_{YX_1}({}^t\Gamma) \cdot p^*_{X_2 Z}(\widetilde{\Gamma})\right). \end{aligned}$$

Hence, by the projection formula, the entire right hand side of (A.1) equals

$$\begin{aligned} (\mathrm{id}_Y \times \Delta_X \times \mathrm{id}_Z)_* \left(p^*_{YX}({}^t\Gamma) \cdot p^*_{XZ}(\widetilde{\Gamma})\right) &= p^*_{YX_1}({}^t\Gamma) \cdot p^*_{X_2 Z}(\widetilde{\Gamma}) \cdot p^*_{X_1 X_2}(\Delta_X) \\ &= p^*_{X_1 Y}(\Gamma) \cdot p^*_{X_2 Z}(\widetilde{\Gamma}) \cdot p^*_{X_1 X_2}(\Delta_X). \end{aligned}$$

□

This seemingly trivial result has an interesting consequence in the case $X = \mathrm{Spec}(\mathbb{C})$ where Δ_X is an isomorphism. The following was pointed out to me by my advisor Georg Oberdieck and uses only the fact that the Nakajima correspondences $\mathfrak{q}_i$ and $\mathfrak{q}_{-i}$ are the transpose of each other up to sign.

Corollary A.2. Let S be a smooth projective surface and recall the Nakajima operators from Section 4.3. Denote by $\mathfrak{q}_i$ and $\mathfrak{q}'_j$ the operators

$$\mathfrak{q}_i, \mathfrak{q}'_j : \bigoplus_{m,n} \mathrm{CH}^*(S^{[m]} \times S^{[n]}) \to \bigoplus_{m,n} \mathrm{CH}^*(S^{[m]} \times S^{[n]} \times S)$$

acting as $\mathfrak{q}_i$ on the first factor and the identity on the second factor respectively as $\mathfrak{q}_j$ on the second factor and as the identity on the first factor. Obviously, $\mathfrak{q}_i$ commutes with $\mathfrak{q}'_j$. Let $\Gamma \in \mathrm{CH}^*(S^{k+l})$. Then we have the equation of correspondences

$$\mathfrak{q}_{i_1} \cdots \mathfrak{q}_{i_k} \mathfrak{q}'_{j_1} \cdots \mathfrak{q}'_{j_l}(\Gamma) \cdot 1_{S^{[0]}} = (-1)^{i_1 + \ldots + i_k} \mathfrak{q}_{j_1} \cdots \mathfrak{q}_{j_l} \mathfrak{q}_{-i_k} \cdots \mathfrak{q}_{-i_1}(\tau_*(\Gamma)),$$

where $\tau : S^{k+l} \to S^{k+l}$ permutes the factors according to the permutation of the indices.

Proof. This follows from the definition of the Nakajima operators and Lemma A.1 on noting that $\Delta_{S^{[0]}} = 1_{S^{[0]}}$ under the identification $S^{[0]} \times S^{[0]} = S^{[0]}$ via the diagonal map. $\square$

Lemma A.3. Let V be any vector space (possibly infinite-dimensional) over any field K and $f : V \to V$ an endomorphism. Let $\lambda_1, \ldots, \lambda_r \in K$ be distinct and define the polynomial

$$p(X) := \prod_{i=1}^{r}(X - \lambda_i) \in K[X].$$

If $p(f) = 0$, we have

$$V = \bigoplus_{i=1}^{r} V_i,$$

where $V_i = \{v \in V : f(v) = \lambda_i v\}$, which of course can be trivial.

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