Algebraic Stacks (Winter 2020/2021)

Lecturer: Georg Oberdieck

Date: Wednesday 4-6, Friday 12-2.

Oral Exam: February 19 and March 26, 2021.

Exam requirement: You have to hand in solutions to at least half of all exercise problems to be admitted to the exam.

Summary: The course is an introduction to the theory of algebraic stacks. We will discuss applications to the construction of moduli spaces.

Prerequisites: Algebraic Geometry I and II (e.g. on the level of Hartshorne's book Chapter I and II plus some background on flat/etale morphisms). Some category theory (such as Vakil's Notes on Algebraic Geometry, Chapter 1).

References:

(1) Olsson, Algebraic Spaces and Stacks
(2) Stacks Project
(3) Vistoli, Notes on Grothendieck topologies, fibered categories and descent theory (available online)
(4) Halpern-Leistner, The Moduli space, `https://themoduli.space`
(5) Guide to the literature by Jarod Alper
(6) Behrend, Introduction to algebraic stacks, lecture notes.

The main reference of the coarse is Olsson and will be used throughout, in particular for many of the exercises. However, we will diverge in some details from Olsson's presentation, in particular on quasi-coherent sheaves on algebraic stacks where we follow the stacks project.

The notes of Vistoli are also highly recommended for the first part which covers aspects of Grothendieck topologies, fibered categories and descent theory. In the second part we will use the notes of Halpern-Leistner.

Exercises and student presentations: Friday's session will be used sometimes for informal discussion of the material covered in the problem sets. From time to time there will also be student presentations, see:
`http://www.math.uni-bonn.de/~georgo/stacks/presentations.pdf`

Schedule[1]

Oct 28: Motivation mostly. Definition of moduli functors. Several examples which show that some are representable by schemes and some are not. For example, $\overline{M}_{1,1}$ is not representable which one sees by considering the family $y^2z = x^3 - tz^3$. Functor parametrizing line bundles up to isomorphism as well. Need to take isomorphisms into account, hence working with

[1]As of February 12, 2021

schemes is not enough. Definition of stack as 2-sheaf on category of schemes.

Oct 30: Grothendieck topologies, site, sheaves on sites, sheafification, topos. Example: For X a scheme, h_X is a sheaf in the fpqc topology.

Nov 4: Fibered categories, 2-Yoneda lemma. Category fibered in sets is equivalent to category of presehaves. Fiber products of category fibered in groupoids. Example BG

Nov 11: Descent theory: Descent datum for a morphism, definition prestack/stack, stack of quasi-coherent sheaves (including a discussion of faithfully flat descent for modules). As a collorary we see several further examples of stacks: Stack of locally free sheaves of finite rank, stack of quasi-coherent commutative algebras, affine morphisms, closed immersions, polarized schemes, hence $\mathcal{M}_g$ for $g \neq 0, 2$.

Nov 18: Algebraic spaces and stacks. Following Olsson Section 1.4 we explain how general schemes are obtained by a functorial procedure from affine schemes in 2 steps. We apply the same procedure for the étale topology to schemes and get algebraic spaces. Definition of properties of morphisms of algebraic spaces in two cases:
If the property P is étale local on target, then for representable morphisms.
If the property P is étale local on source and target, then for arbitrary morphism.
Show independence of choice of cover in the definition.

Nov 20: (Reference Vistoli) Group object in categories. Action on object X. G-equivariant object. G-torsors. For a G-torsor $X \to Y$ equivalence between category of desent data $\mathcal{F}(X \to Y)$ and G-equivariant objects $\mathcal{F}_G(X)$. Application to $\mathbb{Z}_2$-torsor $\mathrm{Spec}\,(\mathbb{C}) \to \mathrm{Spec}\,(\mathbb{R})$.

Nov 25: Algebraic spaces via equivalence relations: Any algebraic space defines an étale equivalence relation in schemes, and vice versa, any such equivalence relation defines an algebraic space by taking the quotient functor. Theorem: An algebraic space, which is separated and locally quasi-finite over a scheme, is a scheme. Example: Quotient of a scheme X by a free action by a finite group is algebraic space. $\mathbb{A}^1/\mathbb{Z}$ not a scheme.

Nov 27: Discussion of algebraic spaces which are not schemes. Presentation by Felix Jäger on the Hironaka example.

Dec 4: Discussion exercises, presentation Mingjia Zhang on de-Rham cohomology of algebraic stacks.

Dec 9: Definition Algebraic stack. Example BG is an algebraic stack. Diagonal $\Delta_{\mathcal{X}}$ is locally of finite type. Deligne-Mumford stacks: Definition formally unramified, equivalent to $\Omega_{X/Y} = 0$. Main Theorem: An algebraic stack is Deligne-Mumford if and only if diagonal is formally unramified. Remark that this can be checked by considering stabilizer groups of geometric points. Consequences: Criterion for an algebraic stack to be an algebraic space. Quotients $[X/G]$ are DM if and only if stabilizer group is discrete. Application $\mathcal{M}_g$ is a DM stack. Proof of main Thm.

Dec 11: Group presentations of algebraic stacks: Every algebraic stack defines a smooth groupoid in algebraic spaces, and conversely any such groupoid presentations yields an algebraic space. Main reference: Stacks project.

Dec 16: Morita equivalence, description of descent data for groupoids as morphisms on associated stack. Definition of quasi-coherent sheaves on an algebraic stack. Description in terms of groupoid presentation. Quasi-compact and quasi-separated morphisms of algebriac stacks. Diagonal is representable by algebraic spaces. Pullback and pushforward of quasi-coherent sheaves. Explicit description for pushforward for quasi-compact and quasi-separated morphisms. Coherent sheaves on Noetherian stacks.

Dec 18: Cohomology for algebraic stacks. Big étale site of an algebraic stack. $\mathcal{O}_{\mathcal{X}}$-module. Pushforward and pullback of $\mathcal{O}_{\mathcal{X}}$-modules. Derived pushforward computable by Chec resolution. Relationship with quasi-coherent sheaves. Parasitic $\mathcal{O}_{\mathcal{X}}$-modules. Several examples throughout.

Dec 23: Previous lecture continued, presentation by Mingyu Ni on group cohomology and cohomology of BG.
Jan 8: Discussion problem set

Jan 13: No lecture

Jan 15: Presentation: Lefschetz trace formula by Ali Barkhordarian

Jan 20: Coarse moduli sapce, Keel-Mori theorem, M_g has a coarse moduli space, good moduli spaces, first properties.

Jan 22: Problem Set 7 (In particular, that given a group action on a variety X, the Kähler differentials have a canonical linearization. Kähler differentials of a principle G-bundle. Cotangent complex of BG.

Jan 27: Further properties of good moduli spaces (following Alper's paper). Ended stating the main theorem of Alper's paper.

Jan 29: Discussion of coarse moduli spaces (along Problem Set 8)
Feb 3: Sketch of proof for part of Alper's theorem, i.e. the properties of good moduli spaces. Discussion geometric invariant theory, and that it y ields an example of a good moduli space. Basic idea for proof of existence of good moduli spaces.

Feb 5: $\overline{M}_{1,1}$ and modular forms, by Thiago Solovera e Nery

Feb 10: Stacky valuative criteria: Θ-reductive and S-complete. Statement of the Alper-Halpern-Leistner-Heinloth existence theorem of good moduli spaces.

Feb 12: Divisors and Riemann-Roch for orbifold curves and stacky curves (following Behrend notes). More on modular forms on $\overline{M}_{1,1}$.