Problem Set 7, Algebraic Stacks
Lecturer: Georg Oberdieck

Due date: Friday, Jan 15. Weights: 20 points each.

Problem 0. (warm-up) Consider a groupoid in algebraic spaces,

$$R \times_{s,U,t} R \xrightarrow{\ m\ } R \underset{t}{\overset{s}{\rightrightarrows}} U$$

and let (E, ϕ) be an object of $\mathrm{QCoh}(R \rightrightarrows U)$, i.e. E is a quasi-coherent sheaf on U and $\varphi : s^*E \xrightarrow{\ \cong\ } t^*E$ is a gluing data satisfying the cocycle condition $m^*\varphi = \mathrm{pr}_1^*\varphi \circ \mathrm{pr}_2^*\varphi$.

Show that φ is also well-behaved with respect to the unit e and inverse morphism i. More precisely show that

$$e^*\varphi = \mathrm{id}_E, \quad i^*\varphi = \varphi^{-1}.$$

Problem 1. Let $\mathcal{X}$ be an algebraic stack. Let $\mathrm{Sch}/\mathcal{X}$ be the category of schemes over $\mathcal{X}$, whose objects are morphisms $x : T \to \mathcal{X}$ with T a scheme, and where a morphism from $(x : T \to \mathcal{X})$ to $(x' : T \to \mathcal{X})$ is a pair (g, η) of a morphism of schemes $g : T \to T'$ and 2-isomorphism $\eta : x \to x' \circ g$.

Show that $\mathrm{Sch}/\mathcal{X} \cong \mathcal{X}$. (Hint: This is essentially the 2-Yoneda lemma. So this exercise is somewhat of a reminder, and you can be brief if you like.)

Problem 2. (a) Let G be a smooth affine group scheme over a scheme S and consider the associated groupoid

$$G \times_S G \xrightarrow{\ m\ } G \underset{t}{\overset{s}{\rightrightarrows}} S$$

which we denote by $[G \rightrightarrows S]$. Show that the associated stack is equivalent to BG (as defined using principle G-bundles).

(b) More generally, assume that G acts on a scheme X and we consider the groupoid

$$G \times G \times X \xrightarrow{\ m \times \mathrm{id}_X\ } G \times X \underset{t=\sigma}{\overset{s=\mathrm{pr}_2}{\rightrightarrows}} X$$

Show that the stack associated to this groupoid is equivalent to $[X/G]$ (as defined via principle G-bundles).

Problem 3. Let $\mathcal{X}$ be an algebraic stack and let $X \to \mathcal{X}$ be a smooth surjective cover. Show that the diagonal $\Delta_{\mathcal{X}}$ is formally unramified if and only if $\Omega_X \to \Omega_{X/\mathcal{X}}$ is surjective.

Problem 4. Olsson, 9.I

Problem 5. Olsson, 9.K

Problem 6. (Optional, Some deformation theory) Olsson, 1.I and 1.J