Problem Set 6, Algebraic Stacks

Lecturer: Georg Oberdieck

Due date: Friday, Dec 18. Weights: 25 points each.

Problem 0. Let $\mathcal{X}, \mathcal{Y}$ be stacks over Sch$/S$ with respect to the étale topology. Recall that a morphism $f : \mathcal{X} \to \mathcal{Y}$ is called representable (by algebraic spaces) if for every S-scheme T and for every morphism $T \to Y$ the fiber product $\mathcal{X} \times_{\mathcal{Y}} T$ is an algebraic space.

Show that if $f : \mathcal{X} \to \mathcal{Y}$ is representable by an algebraic space, then for every algebraic space F over S and every morphism $F \to \mathcal{Y}$ the fiber product $\mathcal{X} \times_{\mathcal{Y}} F$ is an algebraic space.

Problem 1.(Frobenius on algebraic stacks) Olsson, 8.H

Problem 2.(Functoriality of stack quotients) Olsson, 10.F

Problem 3.(Root stacks) A generalized effective Cartier divisor on a scheme X is a pair (L, s) where L is a line bundle and $s : L \to \mathcal{O}_X$ is a morphism. Every effective Cartier divisor $D \subset X$ gives rise to such a pair by setting $L = \mathcal{O}(-D)$ and s the canonical inclusion. A morphism of generalized effective Cartier divisors $(L, s) \to (L', s')$ is a morphism $\varphi : L \to L'$ such that $s' \circ \varphi = s$.

Let (L, s) be a generalized effective Cartier divisor on a scheme X. The *n-th root stack* $\mathcal{X}_n$ is the stack over the category of X-schemes defined by:

objects: $(f : T \to X, (\mathcal{M}, \lambda), \varphi)$ where $(\mathcal{M}, \lambda)$ is an generalized effective Cartier divisor on T and φ is an isomorphism

$$\varphi : (\mathcal{M}^{\otimes n}, \lambda^{\otimes n}) \to (f^*\mathcal{L}, f^*s).$$

morphisms: A morphism $(f : T \to X, (\mathcal{M}, \lambda), \varphi) \to (f' : T' \to X, (\mathcal{M}', \lambda'), \varphi')$ is a pair of a morphism of X-schemes $g : T \to T'$ and an isomorphism of generalized effective Cartier-divisors $\tilde{g} : (\mathcal{M}, \lambda) \to (g^*(\mathcal{M}'), g^*(\lambda'))$ such that the following diagram commutes:

$$\begin{array}{ccc} M^{\otimes n} & \xrightarrow{\tilde{g}^{\otimes n}} & (g^*M')^{\otimes n} \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle g^*\varphi'} & \\ & f^*\mathcal{L} & \end{array}$$

Show the following:

(a) Let $a_n : [\mathbb{A}^1/\mathbb{G}_m] \to [\mathbb{A}^1/\mathbb{G}_m]$ induced by the maps $\mathbb{A}^1 \to \mathbb{A}^1, z \mapsto z^n$ and $\mathbb{G}_m \to \mathbb{G}_m, u \mapsto u^n$. Show that we have a fiber diagram

$$\begin{array}{ccc} \mathcal{X}_n & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m] \\ \downarrow{\scriptstyle \pi_n} & & \downarrow{\scriptstyle a_n} \\ X & \xrightarrow{(L,s)} & [\mathbb{A}^1/\mathbb{G}_m] \end{array}$$

Conclude that $\mathcal{X}_n$ is an algebraic stack. (Use that $[\mathbb{A}^1/\mathbb{G}_m]$ is equivalent to the stack parametrizing a line bundle with a section.)

(b) If $L = \mathcal{O}_X$ and s is given by an element $f \in \Gamma(X, \mathcal{O}_X)$, then $\mathcal{X}_n$ is the stack quotient of $\mathrm{Spec}_X(\mathcal{O}_X[t]/(t^n - f))$ by the cyclic action of μ_n (group scheme of n-th roots of unity) given by $t \mapsto \zeta t$.

(c) The map π_n is an isomorphism over the open subset $U = \{s \neq 0\} \subset X$.

(d) If X is smooth over $\mathbb{C}$ and $V(s)$ is smooth, show that $\mathcal{X}_n$ is smooth for all $n \geq 1$.

We conclude that the stack $\mathcal{X}_n$ is isomorphic to X away from the divisor D defined by s, but has 'stacky structure' along D.

Problem 4.(Optional) (Stack of vector bundles) Olsson, 8.J. You can assume that X is a scheme and that $X \to S$ is projective.