Problem Set 5, Algebraic Stacks
Lecturer: Georg Oberdieck

Due date: Friday, Dec 4. Weights: 20 points each.

Reminder: No class on Wednesday, Dec 2 (Dies Academicus)

Problem 1.(Warm-Up) Let $n \geq 0$. Let $\mathbb{G}_m$ act on $\mathbb{A}^{n+1}$ by $t \cdot (x_0, \ldots, x_n) = (tx_0, \ldots, tx_n)$. Show that the natural map

$$\mathbb{P}^n \cong [\mathbb{A}^{n+1} \setminus \{0\}/\mathbb{G}_m] \to [\mathbb{A}^{n+1}/\mathbb{G}_m]$$

is an open embedding.

Recall that we can associate to any algebraic stack $\mathcal{X}$ a topological space $|\mathcal{X}|$ as follows. The points of $|\mathcal{X}|$ are the morphisms $f : \mathrm{Spec}(k) \to \mathcal{X}$ for k a field, modulo the smallest equivalence relation that identifies 2-isomorphic maps and identifies $f : \mathrm{Spec}(k) \to \mathcal{X}$ with the composition $\mathrm{Spec}(k') \to \mathrm{Spec}(k) \to \mathcal{X}$ for any field extension $k \subset k'$. If $U \to \mathcal{X}$ is a smooth surjective morphism from a scheme U then the image of $|U \times_{\mathcal{X}} U| \to |U| \times |U|$ is an equivalence relation with quotient $|\mathcal{X}|$. We hence endow $|\mathcal{X}|$ with the quotient topological, i.e. a subset $V \subset |\mathcal{X}|$ is open if and only its preimage in $|U|$ is open.

Problem 2. Find the $\mathbb{C}$-valued points of the following stacks, their stabilizer groups and the topological space $|\mathcal{X}|$. Which points are in the closure of the others? Draw a picture of the stacks.

(a) $\Theta = [\mathbb{A}^1/\mathbb{G}_m]$ where $\mathbb{G}_m$ acts on $\mathbb{A}^1$ by scaling.
(b) $\mathrm{ST} = [\mathbb{A}^2/\mathbb{G}_m]$, where $\mathbb{G}_m$ acts by $t \cdot (x, y) = (tx, t^{-1}y)$.

Addendum (Optional): As we will see later on the stack Θ plays an important role in defining a notion of stability for points of algebraic stack. Roughly a map from Θ to an algebraic stack $\mathcal{X}$ should be thought of as an object together with a filtration. For now use the Rees construction to show the following: Let $\mathsf{Bun}(C)$ be the stack of locally free sheaves of finite rank on a smooth projective curve C. Then there is a bijective correspondence between the set of morphisms $f : \Theta \to \mathsf{Bun}(C)$ and the set of vector bundles on C and together with a filtration. (Hint: Use that such a morphism correspond to graded $\mathcal{O}_C[t]$-modules).

3. (Slogan: A Deligne-Mumford stack is more than a scheme with automorphisms attached to each point) Consider the following two $\mathbb{C}$-stacks:

(a) $\mathcal{X}_1 = [\mathbb{A}^1/\mathbb{Z}_2]$ where $\mathbb{Z}_2$ acts by $x \mapsto -x$
(b) $\mathcal{X}_2 = T/\mathbb{Z}_2$ where $T = \mathrm{Spec}(\mathbb{C}[x, y]/(xy))$ and $\mathbb{Z}_2$ acts by $x \mapsto y$.

Show that the stacks $\mathcal{X}_i$ have identical geometric points and stabilizer groups, but that $\mathcal{X}_1 \not\cong \mathcal{X}_2$.

4. (Halpern-Leistner, Exercise 5.2) A representable morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks is smooth (resp. surjective) if for any morphism $Y \to \mathcal{Y}$ from a scheme Y the fiber product $\mathcal{X} \times_{\mathcal{Y}} Y \to Y$ is smooth (resp. surjective).

Show that if $f : \mathcal{X} \to \mathcal{Y}$ is smooth and surjective, then for any $T \in \mathrm{Sch}/S$ and $y \in \mathcal{Y}(T)$ there exists an étale cover $\pi : T' \to T$ of S-schemes and an element $x \in \mathcal{X}(T)$ such that $\pi^*(y) = f(x)$.

5. Olsson, 8.A

6. (optional) Olsson, 8.B.