Problem Set 4, Algebraic Stacks
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Due date: Friday, Nov 27. Exercises are weighted by 40/15/15/15/15/20 points.

Problem 1 (Fiber products). Let $\mathcal{X}$ be a stack in groupoids over a site C.

(a) Determine $\mathcal{X} \times_{\mathcal{X}} \mathcal{X}$ (where the morphisms are the identity).

(b) (Olsson, 3.F) Let $U, V \in C$ and let $x \in \mathcal{X}(U)$ and $y \in \mathcal{X}(V)$. Show that the fiber product of the diagram

$$\begin{array}{ccc} & & V \\ & & \big\downarrow{\scriptstyle y} \\ U & \xrightarrow{x} & \mathcal{X} \end{array}$$

is the fibered category associated to the functor

$$\underline{\mathrm{Isom}}_{\mathcal{X}}(p_1^* x, p_2^* y) : (C/(U \times V))^{\mathrm{op}} \to \mathrm{Set}.$$

(c) The inertia stack $I_{\mathcal{X}}$ of $\mathcal{X}$ is defined by the fiber diagram

$$\begin{array}{ccc} I_{\mathcal{X}} & \longrightarrow & \mathcal{X} \\ \big\downarrow & & \big\downarrow{\scriptstyle \Delta_{\mathcal{X}}} \\ \mathcal{X} & \xrightarrow{\Delta_{\mathcal{X}}} & \mathcal{X} \times \mathcal{X}. \end{array}$$

Show that $I_{\mathcal{X}} \to \mathcal{X}$ is an equivalence if and only if $\mathcal{X}$ is (equivalent to) a stack fibered in sets[1].

(d) A morphism of stacks $f : \mathcal{X} \to \mathcal{Z}$ is called an *iso-fibration* if for every $S \in C$, isomorphism $\theta : \alpha \to \beta$ in $\mathcal{Z}(S)$, and lift $\alpha' \in \mathcal{X}(S)$ with $f(\alpha') = \alpha$, there exists an $\theta' : \alpha' \to \beta'$ in $\mathcal{X}(S)$ such that $f(\theta') = \theta$.

Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be stacks and let $W \subset \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ be the full-subcategory of triples (x, y, θ) where $\theta = \mathrm{id}$. If $\mathcal{X} \to \mathcal{Z}$ is an iso-fibration show that the inclusion $W \to \mathcal{X} \times_{\mathcal{X}} \mathcal{Y}$ is an equivalence of categories.

Problem 2-5. (Olsson 5.B, C, D, H)

Problem 6 (optional) (i) Work through Example 5.3.3 in Olsson and draw a picture of the algebraic space F. Is the morphism $F \to \mathbb{A}^1$ étale? Is it separated?
(ii) Can you write F as the quotient of the line with double origin by $\mathbb{Z}_2$?

Hint: For this problem you need to read up about how algebraic spaces are defined by equivalence relations (Olsson, Section 5.2). We will discuss this approach to algebraic spaces in class soon.

[1]Strictly speaking we should say 'fibered in setoids'. A setoid is a category in which every morphism is the identity. This permits us to have a class of objects in the fiber.