**Problem Set 3, Algebraic Stacks**
Lecturer: Georg Oberdieck

Due date: Friday, Nov 20. Exercises are weighted 20 points each.
***Remark:**: Starting the week of November 16, we will move the exercise sessions to the Friday class. The regular lecture will take place Wednesdays. Because of that the exercises will be due on Friday from now on.*

**Problem 1.**
(a) Show that the map $f : \mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(\mathbb{R})$ induced by the natural inclusion $\mathbb{R} \subset \mathbb{C}$ is étale.
(b) A real structure on a $\mathbb{C}$-vector $V$ is an involution $\varphi : V \to V$ which is anti-linear (i.e. $\varphi(\lambda v) = \bar{\lambda}\varphi(v)$ for all $\lambda \in \mathbb{C}$ and $v \in V$). A morphism of vector spaces with real structure $(V, \varphi) \to (W, \psi)$ is a morphism $r : V \to W$ of $\mathbb{C}$-vector spaces such that $\psi \circ r = r \circ \phi$. Show that the category of descent data of QCoh for the étale cover $\{\mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(\mathbb{R})\}$ is equivalent to the category of $\mathbb{C}$-vector spaces with real structure.

Remark: Do not use descent for quasi-coherent sheaves. Later when talking about quotient stacks we will see how to do this problem more efficiently, so it is enough if you sketch some of the details. However, explain how you can extract the anti-linear involution from the descent data.

Hint: Use $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \times \mathbb{C}$. What is the $\mathbb{C}$-module structure induced by the projection maps under this isomorphism?

**Problem 2.** Let $C$ be a site. Let $\mathrm{Sh}(C)$ be the category of pairs $(X, F)$ where $X \in C$ and $F$ is a sheaf on $C/X$, where a morphism $(X, E) \to (Y, F)$ is a pair of a morphism $f : X \to Y$ in $C$ and morphism of sheaves $\varphi : E \to f^*F$ on $C/X$. This yields a fibered category $p : \mathrm{Sh}(C) \to C$ by sending $(X, E)$ to $X$. Show that $p : \mathrm{Sh}(C) \to C$ is a stack.
(Hint: Start by considering the sheaf $\underline{\mathrm{Hom}}_{\mathrm{Sh}(C)}(F, G)$. What is it that you need to prove? Similarly, for gluing objects what do you need to prove? Everything can be done explicit. The difficulty lies mostly in understanding the definitions and how they interact. For many hints, see a sketch in [Vistoli,Ex. 4.11]. There is also a proof in Olsson, but it is rather abstract.)

**Problem 3.**(Vistoli, Example 4.19) Let $C$ be a site, let $F \to C$ be a stack, and let $G$ be a full sub-category of $F$ such that

(i) Any cartesian morphism in $F$ whose target is in $G$ is also in $G$.
(ii) Let $\{U_i \to U\}$ be a cover in $C$, let $\xi$ be an objects of $F(U)$, and let $\xi_i$ in $F(U_i)$ be pullbacks of $\xi$ to $U_i$. If $\xi_i$ is in $G(U_i)$ for all $i$, then $\xi$ is in $G(U)$.

Show that $G$ is a substack of $F$ (i.e. a fibered subcategory which is a stack).

Conclude that the full subcategory of QCoh/$S$ consisting of locally free sheaves of finite rank is a stack in the fpqc topology. (Hint: Use that a quasi-coherent sheaf is locally free of finite rank if and only if it is flat and of finite presentation, see [Stacks-Project, 05P1].)

**Problem 4.** Let $F \to C$ be a fibered category and let $F^{\mathrm{cart}} \subset F$ be the subcategory with the same objects as $F$, but whose morphisms are the cartesian morphisms in $F$. By Problem 2 of Pset 2 the natural functor $F^{\mathrm{cart}} \to C$ is a category fibered in groupoids. Show the following:

(i) If $F$ is a stack, then $F^{\mathrm{cart}}$ is a stack.
(ii) If $F$ is a pre1stack and $F^{\mathrm{cart}}$ is a stack, then $F$ is a stack.

**Problem 5.** Let $C$ be a curve over a scheme $S$. Let $\mathrm{Bun}_n(C)$ be the category whose objects are pairs $(T, \mathcal{E})$ of a scheme $T$ and a locally free vector bundle $\mathcal{E}$ on $T \times C$ of rank $n$. A morphism $(T, \mathcal{E}) \to (T', \mathcal{E}')$ is a pair of a morphism $h : T \to T'$ and an isomorphism $\mathcal{E} \xrightarrow{\cong} (h \times \mathrm{id}_C)^* \mathcal{E}'$. Consider the fibered category $p : \mathrm{Bun}_n(C) \to \mathrm{Schemes}/S$ which sends $(T, \mathcal{E})$ to $T$. Show that $p$ is a stack.
(Hint: Use that the category of quasi-coherent sheaves is a stack).

**Problem 6.**(optional) Olsson, Exercise 1.A and 4.B.